Pacific Journal of Mathematics

ORE'S THEOREM ON CYCLIC SUBFACTOR PLANAR ALGEBRAS AND BEYOND

SEBASTIEN PALCOUX

Volume 292 No. 1 January 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 292, No. 1, 2018
dx.doi.org/10.2140/pjm.2018.292.203

ORE'S THEOREM ON CYCLIC SUBFACTOR PLANAR ALGEBRAS AND BEYOND

SEBASTIEN PALCOUX

Ore proved that a finite group is cyclic if and only if its subgroup lattice is distributive. Now, since every subgroup of a cyclic group is normal, we call a subfactor planar algebra cyclic if all its biprojections are normal and form a distributive lattice. The main result generalizes one side of Ore's theorem and shows that a cyclic subfactor is singly generated in the sense that there is a minimal 2-box projection generating the identity biprojection. We conjecture that this result holds without assuming the biprojections to be normal, and we show that this is true for small lattices. We finally exhibit a dual version of another theorem of Ore and a nontrivial upper bound for the minimal number of irreducible components for a faithful complex representation of a finite group.

1. Introduction

Vaughan Jones [1983] proved that the set of possible values for the index $|M : N|$ of a subfactor $(N \subseteq M)$ is

$$\left\{4cos^2\left(\frac{\pi}{n}\right) \mid n \geq 3\right\} \sqcup [4, \infty].$$

We observe that it is the disjoint union of a discrete series and a continuous series. Moreover, $|M : N| = |M : P| \cdot |P : N|$ for a given intermediate subfactor $N \subseteq P \subseteq M$, therefore by applying a kind of Eratosthenes sieve, we get that a subfactor with

MSC2010: primary 46L37; secondary 06D10, 16W30, 20C15, 20D30.
Keywords: von Neumann algebra, subfactor, planar algebra, biprojection, distributive lattice, finite group, representation.

an index in the discrete series or in the interval (4, 8), except the countable set of numbers composed of numbers in the discrete series, can't have a nontrivial intermediate subfactor. A subfactor without nontrivial intermediate subfactor is called maximal [Bisch 1994]. For example, any subfactor of index in $(4, 3+\sqrt{5})$ is maximal; A_∞ excepted there are exactly 19 irreducible subfactor planar algebras for this interval (see [Jones et al. 2014; Afzaly et al. 2015]). The first example is the Haagerup subfactor [Peters 2010]. Thanks to Galois correspondence [Nakamura and Takeda 1960], a finite group subfactor, $(R^G \subseteq R)$ or $(R \subseteq R \rtimes G)$, is maximal if and only if it is a prime order cyclic group subfactor (i.e., $G = \mathbb{Z}/p$ with p prime). Thus we can say that the maximal subfactors are an extension of the prime numbers.

Question 1.1. *What could be the extension of the natural numbers*?

To answer this question, we need to find a natural class of subfactors, that we will call the "cyclic subfactors", satisfying the following properties:

(1) Every maximal subfactor is cyclic.

(2) A finite group subfactor is cyclic if and only if the group is cyclic.

An old and little-known theorem published in 1938 by the Norwegian mathematician Øystein Ore states that:

Theorem 1.2 [Ore 1938]. *A finite group G is cyclic if and only if its subgroup lattice $\mathcal{L}(G)$ is distributive.*

Firstly, the intermediate subfactor lattice of a maximal subfactor is obviously distributive. Next, by Galois correspondence, the intermediate subfactor lattice of a finite group subfactor is exactly the subgroup lattice (or its reversal) of the group; but distributivity is invariant under reversal, so (1) and (2) hold by Ore's theorem. Now an abelian group, and a fortiori a cyclic group, admits only normal subgroups; but T. Teruya [1998] generalized the notion of normal subgroup by the notion of normal intermediate subfactor, so:

Definition 1.3. A finite index irreducible subfactor is cyclic if all its intermediate subfactors are normal and form a distributive lattice.

Note that an irreducible finite index subfactor $(N \subseteq M)$ admits a finite lattice $\mathcal{L}(N \subseteq M)$ of intermediate subfactors by [Watatani 1996], as for the subgroup lattice of a finite group. Moreover, a finite group subfactor remembers the group by [Jones 1980]. Section 4A exhibits some examples of cyclic subfactors: of course the cyclic group subfactors and the (irreducible finite index) maximal subfactors; moreover, up to equivalence, exactly 23279 among 34503 inclusions of groups of index < 30, give a cyclic subfactor. The class of cyclic subfactors is stable under dual, intermediate, free composition and certain tensor products. Now the natural problem concerning cyclic subfactors is to understand in what sense they are "singly generated". To answer this question, we extend the following theorem of Ore.

Theorem 1.4 [Ore 1938]. *If an interval of finite groups* $[H, G]$ *is distributive, then there exists* $g \in G$ *such that* $\langle H, g\rangle = G$.

Theorem 1.5. *An irreducible subfactor planar algebra whose biprojections are central and form a distributive lattice, has a minimal* 2*-box projection generating the identity biprojection* (*i.e., w-cyclic subfactor*).

But "normal" means "bicentral", so a cyclic subfactor planar algebra is w-cyclic. The converse is false, a group subfactor ($R^G \subseteq R$) is cyclic if and only if G is cyclic, and is w-cyclic if and only if G is linearly primitive (consider $G = S_3$). That's why we have chosen the name w-cyclic (i.e., weakly cyclic). We conjecture that Theorem 1.5 holds without the assumption that the biprojections are central.

Conjecture 1.6. *An irreducible subfactor planar algebra with a distributive biprojection lattice is w-cyclic.*

This is true if the lattice has less than 32 elements (and so, at index < 32). Now the group-theoretic reformulation of Conjecture 1.6 for the planar algebra $\mathcal{P}(R^G \subseteq R^H)$, gives a dual version of Theorem 1.4.

Conjecture 1.7. *If the interval of finite groups* $[H, G]$ *is distributive then there exists an irreducible complex representation* V *of* G *such that* $G_{(V^H)} = H$.

In general, we deduce a nontrivial upper bound for the minimal number of minimal central projections generating the identity biprojection. For $\mathcal{P}(R^G \subseteq R)$, this gives a nontrivial upper bound for the minimal number of irreducible components for a faithful complex representation of G. This is a bridge linking combinatorics and representations in the theory of finite groups. This paper is a short version of [Palcoux 2015].

2. Ore's theorem on finite groups

2A. ***Basics in lattice theory.*** A lattice $(L, \wedge, \vee)$ is a poset L in which every two elements a, b have a unique supremum (or join) $a \vee b$, and a unique infimum (or meet) $a \wedge b$. Let G be a finite group. The set of subgroups $K \subseteq G$ forms a lattice, denoted by $\mathcal{L}(G)$, ordered by $\subseteq$, with $K_1 \vee K_2 = \langle K_1, K_2\rangle$ and $K_1 \wedge K_2 = K_1 \cap K_2$. A sublattice of $(L, \wedge, \vee)$ is a subset $L' \subseteq L$ such that $(L', \wedge, \vee)$ is also a lattice. Consider $a, b \in L$ with $a \leq b$, then the interval $[a, b]$ is the sublattice $\{c \in L \mid a \leq c \leq b\}$. Any finite lattice admits a minimum and a maximum, denoted by $\hat{0}$ and $\hat{1}$. An atom is a minimal element in $L \setminus \{\hat{0}\}$ and a coatom is a maximal element in $L \setminus \{\hat{1}\}$. The top interval of a finite lattice L is the interval $[t, \hat{1}]$, with t the meet of all the coatoms. The height of a finite lattice L is the greatest length of a (strict) chain. A lattice is distributive if the join and meet operations distribute over each other.

Remark 2.1. Distributivity is stable under taking a sublattice, reversal, direct product and concatenation.

A distributive lattice is called boolean if any element b admits a unique complement $b^{\complement}$ (i.e., $b \wedge b^{\complement} = \hat{0}$ and $b \vee b^{\complement} = \hat{1}$). The subset lattice of $\{1, 2, \ldots, n\}$, with union and intersection, is called the boolean lattice $\mathcal{B}_n$ of rank n. Any finite boolean lattice is isomorphic to some $\mathcal{B}_n$.

Lemma 2.2. *The top interval of a finite distributive lattice is boolean.*

Proof. See [Stanley 2012, items a–i, pages 254–255] which use Birkhoff's representation theorem, which states a finite lattice is distributive if and only if it embeds into some $\mathcal{B}_n$. □

A lattice with a boolean top interval will be called *top boolean* (and its reversal, *bottom boolean*). See [Stanley 2012] for more details on lattice basics.

2B. ***Ore's theorem on distributive intervals of finite groups.*** Øystein Ore [1938, Theorem 4, page 267] proved the following result.

Theorem 2.3. *A finite group G is cyclic if and only if its subgroup lattice $\mathcal{L}(G)$ is distributive.*

Proof. ($\Leftarrow$): This is just a particular case of Theorem 2.5 with $H = \{e\}$.

($\Rightarrow$): A finite cyclic group $G = \mathbb{Z}/n$ has exactly one subgroup of order d, denoted by $\mathbb{Z}/d$, for every divisor d of n. Now $\mathbb{Z}/d_1 \vee \mathbb{Z}/d_2 = \mathbb{Z}/\mathrm{lcm}(d_1, d_2)$ and $\mathbb{Z}/d_1 \wedge \mathbb{Z}/d_2 = \mathbb{Z}/\mathrm{gcd}(d_1, d_2)$, but the lcm and gcd distribute other each over, so the result follows. □

Definition 2.4. An interval of finite groups $[H, G]$ is said to be H-cyclic if there is $g \in G$ such that $\langle H, g\rangle = G$. Note that $\langle H, g\rangle = \langle Hg\rangle$.

Ore extended one side of Theorem 2.3 to the interval of finite groups [Ore 1938, Theorem 7] for which we will give our own proof (which is a group-theoretic reformulation of the proof of Theorem 4.26):

Theorem 2.5. *A distributive interval $[H, G]$ is H-cyclic.*

Proof. The proof follows from the claims below and Lemma 2.2.

Claim: Let M be a maximal subgroup of G. Then $[M, G]$ is M-cyclic.

Proof of claim. For $g \in G$ with $g \notin M$, we have $\langle M, g\rangle = G$ by maximality. □

Claim: A boolean interval $[H, G]$ is H-cyclic.

Proof of claim. Let M be a coatom in $[H, G]$, and $M^{\complement}$ be its complement. By the previous claim and induction on the height of the lattice, we can assume $[H, M]$ and $[H, M^{\complement}]$ both to be H-cyclic, i.e., there are $a, b \in G$ such that $\langle H, a\rangle = M$ and $\langle H, b\rangle = M^{\complement}$. For $g = ab$, $\langle H, a, g\rangle = \langle H, g, b\rangle = \langle H, a, b\rangle = M \vee M^{\complement} = G$, since $a = gb^{-1}$ and $b = a^{-1}g$. Now, $\langle H, g\rangle = \langle H, g\rangle \vee H = \langle H, g\rangle \vee (M \wedge M^{\complement})$ but by distributivity $\langle H, g\rangle \vee (M \wedge M^{\complement}) = (\langle H, g\rangle \vee M)) \wedge (\langle H, g\rangle \vee M^{\complement}\rangle)$. So $\langle H, g\rangle = \langle H, a, g\rangle \wedge \langle H, g, b\rangle = G$. The result follows. □

Claim: $[H, G]$ is H-cyclic if its top interval $[K, G]$ is K-cyclic.

Proof of claim. Consider $g \in G$ with $\langle K, g\rangle = G$. For any coatom $M \in [H, G]$, we have $K \subseteq M$ by definition, and so $g \notin M$, then a fortiori $\langle H, g\rangle \not\subseteq M$. It follows that $\langle H, g\rangle = G$. □

3. Subfactor planar algebras and biprojections

For the notions of subfactor, subfactor planar algebra and basic properties, we refer to [Jones and Sunder 1997; Jones 1999; Kodiyalam and Sunder 2004]. See also [Palcoux 2015, Section 3] for a short introduction. A subfactor planar algebra is of finite index by definition.

3A. ***Basics on the 2-box space.*** Let $(N \subseteq M)$ be a finite index irreducible subfactor. The n-box spaces $\mathcal{P}_{n,+}$ and $\mathcal{P}_{n,-}$ of the planar algebra $\mathcal{P} = \mathcal{P}(N \subseteq M)$, are $N' \cap M_{n-1}$ and $M' \cap M_n$. Let $R(a)$ be the range projection of $a \in \mathcal{P}_{2,+}$. We define the relations $a \preceq b$ by $R(a) \le R(b)$, and $a \sim b$ by $R(a) = R(b)$. Let $e_1 := e^M_N$ and $\mathrm{id} := e^M_M$ be the Jones and the identity projections in $\mathcal{P}_{2,+}$. Note that $\mathrm{tr}(e_1) = |M : N|^{-1} = \delta^{-2}$ and $\mathrm{tr}(\mathrm{id}) = 1$. Let $\mathcal{F} : \mathcal{P}_{2,\pm} \to \mathcal{P}_{2,\mp}$ be the Fourier transform (90° rotation), $\bar{a} := \mathcal{F}(\mathcal{F}(a))$ be the contragredient of $a \in \mathcal{P}_{2,\pm}$, and $a * b = \mathcal{F}(\mathcal{F}^{-1}(a) \cdot \mathcal{F}^{-1}(b))$ be the coproduct of $a, b \in \mathcal{P}_{2,\pm}$.

Lemma 3.1. *Let a, b, c, d be positive operators of $\mathcal{P}_{2,+}$. Then*

(1) *$a * b$ is also positive*,

(2) *$[a \preceq b$ and $c \preceq d] \Rightarrow a * c \preceq b * d$*,

(3) *$a \preceq b \Rightarrow \langle a\rangle \le \langle b\rangle$*,

(4) *$a \sim b \Rightarrow \langle a\rangle = \langle b\rangle$*.

Proof. This is precisely [Liu 2016, Theorem 4.1 and Lemma 4.8] for (1) and (2). Next, if $a \preceq b$, then by (2), for any integer k, $a^{*k} \preceq b^{*k}$, and hence for all n,

$$\sum_{k=1}^{n} a^{*k} \preceq \sum_{k=1}^{n} b^{*k},$$

so $\langle a\rangle \le \langle b\rangle$ by Definition 3.8. Finally, (4) is immediate from (3). □

The next lemma follows by irreducibility (i.e., $\mathcal{P}_{1,+} = \mathbb{C}$).

Lemma 3.2. *Let $p, q \in \mathcal{P}_{2,+}$ be projections. Then*

$$e_1 \preceq p * \bar{q} \Leftrightarrow pq \neq 0.$$

Note that if $p \in \mathcal{P}_{2,+}$ is a projection then $\bar{p}$ is also a projection.

Lemma 3.3. *Let $a, b, c \in \mathcal{P}_{2,+}$ be projections with $c \preceq a * b$. Then there exist $a' \preceq c * \bar{b}$ and $b' \preceq \bar{a} * c$ such that a', b' are projections and $aa', bb' \neq 0$.*

Proof. The proof follows from Lemmas 3.1 and 3.2, and

$$e_1 \preceq c * \bar{c} \preceq (a * b) * \bar{c} = a * (b * \bar{c}).$$

We can also apply [Liu 2016, Lemma 4.10]. □

3B. *On the biprojections.*

Definition 3.4 [Liu 2016, Definition 2.14]. A biprojection is a projection $b \in \mathcal{P}_{2,\pm}$ with $\mathcal{F}(b)$ a multiple of a projection.

Note that $e_1 = e_N^M$ and $\mathrm{id} = e_M^M$ are biprojections.

Theorem 3.5 [Bisch 1994, page 212]. *A projection b is a biprojection if and only if it is the Jones projection e_K^M of an intermediate subfactor $N \subseteq K \subseteq M$.*

Therefore the set of biprojections is a lattice of the form $[e_1, \mathrm{id}]$.

Theorem 3.6. *An operator b is a biprojection if and only if*

$$e_1 \leq b = b^2 = b^\star = \bar{b} = \lambda b * b, \ \text{with } \lambda^{-1} = \delta\, \mathrm{tr}(b).$$

Proof. See [Landau 2002, items 0–3, page 191] and [Liu 2016, Theorem 4.12]. □

Lemma 3.7. *Consider $a_1, a_2, b \in \mathcal{P}_{2,+}$ with b a biprojection. Then*

$$(b \cdot a_1 \cdot b) * (b \cdot a_2 \cdot b) = b \cdot (a_1 * (b \cdot a_2 \cdot b)) \cdot b = b \cdot ((b \cdot a_1 \cdot b) * a_2) \cdot b,$$
$$(b * a_1 * b) \cdot (b * a_2 * b) = b * (a_1 \cdot (b * a_2 * b)) * b = b * ((b * a_1 * b) \cdot a_2) * b.$$

Proof. By exchange relations [Landau 2002] for b and $\mathcal{F}(b)$. □

Definition 3.8. Consider $a \in \mathcal{P}_{2,+}$ positive, and let p_n be the range projection of $\sum_{k=1}^{n} a^{*k}$. By finiteness, there exists N such that for all $m \geq N,\ p_m = p_N$, which is a biprojection [Liu 2016, Lemma 4.14], denoted $\langle a \rangle$, called the biprojection generated by a. It is the smallest biprojection $b \succeq a$. For S a finite set of positive operators, let $\langle S \rangle$ be the biprojection $\left\langle \sum_{s \in S} s \right\rangle$, it is the smallest biprojection b such that $b \succeq s$, for all $s \in S$.

3C. *Intermediate planar algebras and 2-box spaces.*

Let $N \subseteq K \subseteq M$ be an intermediate subfactor. The planar algebras $\mathcal{P}(N \subseteq K)$ and $\mathcal{P}(K \subseteq M)$ can be derived from $\mathcal{P}(N \subseteq M)$, see [Bakshi 2016; Landau 1998].

Theorem 3.9. *Consider the intermediate subfactors*

$$N \subseteq P \subseteq K \subseteq Q \subseteq M.$$

Then there are two isomorphisms of von Neumann algebras

$$l_K : \mathcal{P}_{2,+}(N \subseteq K) \to e_K^M \mathcal{P}_{2,+}(N \subseteq M) e_K^M,$$
$$r_K : \mathcal{P}_{2,+}(K \subseteq M) \to e_K^M * \mathcal{P}_{2,+}(N \subseteq M) * e_K^M,$$

for the usual $+$, $\times$ *and* $()^\star$, *such that*

$$l_K(e^K_P) = e^M_P \quad \text{and} \quad r_K(e^M_Q) = e^M_Q.$$

Moreover, the coproduct $*$ *is also preserved by these maps, but up to a multiplicative constant,* $|M:K|^{1/2}$ *for* l_K *and* $|K:N|^{-1/2}$ *for* r_K. *Then, for all* $m \in \{l^{\pm 1}_K, r^{\pm 1}_K\}$, *and for all* $a_i > 0$ *in the domain of* m, $m(a_i) > 0$ *and*

$$\langle m(a_1), \dots, m(a_n)\rangle = m(\langle a_1, \dots, a_n\rangle).$$

Proof. This is clear from [Bakshi 2016] or [Landau 1998], using Lemma 3.7. □

Notations 3.10. Let $b_1 \le b \le b_2$ be the biprojections $e^M_P \le e^M_K \le e^M_Q$. We define $l_b := l_K$ and $r_b := r_K$; also $\mathcal{P}(b_1, b_2) := \mathcal{P}(P \subseteq Q)$ and

$$|b_2 : b_1| := \operatorname{tr}(b_2)/\operatorname{tr}(b_1) = |Q:P|.$$

4. Ore's theorem on subfactor planar algebras

4A. *The cyclic subfactor planar algebras.* In this subsection, we define the class of cyclic subfactor planar algebras, we show that it contains plenty of examples, and we prove that it is stable under dual, intermediate, free composition and certain tensor products. Let $\mathcal{P}$ be an irreducible subfactor planar algebra.

Definition 4.1 [Teruya 1998]. A biprojection b is normal if it is bicentral (that is, if b and $\mathcal{F}(b)$ are central).

Definition 4.2. An irreducible subfactor planar algebra is said to be

- distributive if its biprojection lattice is distributive,
- Dedekind if all its biprojections are normal,
- cyclic if it is both Dedekind and distributive.

Moreover, we call a subfactor cyclic if its planar algebra is cyclic.

Examples 4.3. A group subfactor is cyclic if and only if the group is cyclic; every maximal subfactor is cyclic, in particular every 2-supertransitive subfactor, as the Haagerup subfactor [Asaeda and Haagerup 1999; Izumi 2001; Peters 2010], is cyclic. Up to equivalence, exactly 23279 among 34503 inclusions of groups of index < 30, give a cyclic subfactor (more than 65%).

Definition 4.4. Let G be a finite group and H a subgroup. The core H_G is the largest normal subgroup of G contained in H. The subgroup H is called core-free if $H_G = \{1\}$; in this case the interval $[H, G]$ is also called core-free. Two intervals of finite groups $[A, B]$ and $[C, D]$ are called equivalent if there is a group isomorphism $\phi : B/A_B \to D/C_D$ such that $\phi(A/A_B) = C/C_D$.

Remark 4.5. A finite group subfactor remembers the group [Jones 1980], but a finite group-subgroup subfactor does not remember the equivalence class of the interval in general. A counterexample was found by V. S. Sunder and V. Kodiyalam [2000], the intervals $[\langle(1234)\rangle, S_4]$ and $[\langle(12)(34)\rangle, S_4]$ are not equivalent whereas their corresponding subfactors are isomorphic; but thanks to the complete characterization by M. Izumi [2002], it remembers the interval in the maximal case, because the intersection of a core-free maximal subgroup with an abelian normal subgroup is trivial.

Theorem 4.6. *The free composition of irreducible finite index subfactors has no extra intermediate.*

Proof. See [Liu 2016, Theorem 2.22]. □

Corollary 4.7. *The class of finite index irreducible cyclic subfactors is stable under free composition.*

Proof. By Theorem 4.6, the intermediate subfactor lattice of a free composition is the concatenation of the lattice of the two components (see also Remark 2.1). By Theorem 3.9 and Lemma 3.7, the biprojections remain normal. □

The following theorem was proved in the 2-supertransitive case by Y. Watatani [1996, Proposition 5.1]. The general case was conjectured by the author, but specified and proved after a discussion with F. Xu.

Theorem 4.8. *Let* $(N_i \subset M_i)$, $i = 1, 2$, *be irreducible finite index subfactors. Then*

$$\mathcal{L}(N_1 \subset M_1) \times \mathcal{L}(N_2 \subset M_2) \subsetneq \mathcal{L}(N_1 \otimes N_2 \subset M_1 \otimes M_2)$$

if and only if there are intermediate subfactors $N_i \subseteq P_i \subset Q_i \subseteq M_i$, $i = 1, 2$, *such that* $(P_i \subset Q_i)$ *is of depth* 2 *and isomorphic to* $(R^{\mathbb{A}_i} \subset R)$, *with* $\mathbb{A}_2 \simeq \mathbb{A}_1^{\mathrm{cop}}$ *being the* (*very simple*) *Kac algebra* $\mathbb{A}_1$ *with the opposite coproduct.*

Proof. Consider the intermediate subfactors

$$N_1 \otimes N_2 \subseteq P_1 \otimes P_2 \subset R \subset Q_1 \otimes Q_2 \subseteq M_1 \otimes M_2$$

with R not of tensor product form, $P_1 \otimes P_2$ and $Q_1 \otimes Q_2$ the closest (below and above, respectively) to R among those of tensor product form. Now using [Xu 2013, Proposition 3.5(2)], $(P_i \subseteq Q_i)$, $i = 1, 2$, are of depth 2, their corresponding Kac algebras, $\mathbb{A}_i$, $i = 1, 2$, are very simple and $\mathbb{A}_2 \simeq \mathbb{A}_1^{\mathrm{cop}}$ [Xu 2013, Definition 3.6 and Proposition 3.10]. The converse is given by [Xu 2013, Theorem 3.14]. □

Remark 4.9. By Theorem 4.8 and Remark 2.1, the class of (finite index irreducible) cyclic subfactors is stable under certain tensor products (i.e., if there is no cop-isomorphic intermediate of depth 2), and by Theorem 3.9 and Lemma 3.7, the biprojections remain normal.

Lemma 4.10. *If a subfactor is cyclic then the intermediate and dual subfactors are also cyclic.*

Proof. The proof follows from Remark 2.1, Theorem 3.9 and Lemma 3.7. ☐

A subfactor as $(R \subseteq R \rtimes G)$ or $(R^G \subseteq R)$ is called a "group subfactor". Then, the following lemma justifies the choice of the word "cyclic".

Lemma 4.11. *A cyclic "group subfactor" is a "cyclic group" subfactor.*

Proof. By Galois correspondence, if a "group subfactor" is cyclic then the subgroup lattice is distributive, and so the group is cyclic by Ore's Theorem 2.3. The normal biprojections of a group subfactor corresponds to the normal subgroups [Teruya 1998], but every subgroup of a cyclic group is normal. ☐

Problem 4.12. Is a depth 2 irreducible finite index cyclic subfactor, a cyclic group subfactor?

The answer could be "no" because the following fusion ring (first reported in [Palcoux 2013]), the first known to be simple integral and nontrivial, *could be* the Grothendieck ring of a "maximal" Kac algebra of dimension 210 and type $(1, 5, 5, 5, 6, 7, 7)$.

$$
\begin{matrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{matrix}
\quad
\begin{matrix} 0&1&0&0&0&0&0 \\ 1&1&0&1&0&1&1 \\ 0&0&1&0&1&1&1 \\ 0&1&0&0&1&1&1 \\ 0&0&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \end{matrix}
\quad
\begin{matrix} 0&0&1&0&0&0&0 \\ 0&0&1&0&1&1&1 \\ 1&1&1&0&0&1&1 \\ 0&0&0&1&1&1&1 \\ 0&1&0&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \end{matrix}
\quad
\begin{matrix} 0&0&0&1&0&0&0 \\ 0&1&0&0&1&1&1 \\ 0&0&0&1&1&1&1 \\ 1&0&1&1&0&1&1 \\ 0&1&1&0&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \end{matrix}
\quad
\begin{matrix} 0&0&0&0&1&0&0 \\ 0&0&1&1&1&1&1 \\ 0&1&0&1&1&1&1 \\ 0&1&1&0&1&1&1 \\ 1&1&1&1&1&1&1 \\ 0&1&1&1&1&2&1 \\ 0&1&1&1&1&1&2 \end{matrix}
\quad
\begin{matrix} 0&0&0&0&0&1&0 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&2&1 \\ 1&1&1&1&2&1&2 \\ 0&1&1&1&1&2&2 \end{matrix}
\quad
\begin{matrix} 0&0&0&0&0&0&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&2 \\ 0&1&1&1&1&2&2 \\ 1&1&1&1&2&2&1 \end{matrix}
$$

4B. ***The w-cyclic subfactor planar algebras.*** Let $\mathcal{P}$ be an irreducible subfactor planar algebra.

Theorem 4.13. *Let $p \in \mathcal{P}_{2,+}$ be a minimal central projection. Then there exists a minimal projection $v \leq p$ such that $\langle v \rangle = \langle p \rangle$.*

Proof. If p is a minimal projection, then the theorem clearly holds. Else, let $b_1, \ldots, b_n$ be the coatoms of $[e_1, \langle p \rangle]$ (n is finite by [Watatani 1996]). If $p \npreceq \sum_{i=1}^n b_i$, then there exists $u \leq p$, a minimal projection such that $u \npreceq b_i$ for all i, so that $\langle u \rangle = \langle p \rangle$. If not, $p \preceq \sum_{i=1}^n b_i$ (with $n > 1$, otherwise $p \leq b_1$ and $\langle p \rangle \leq b_1$, a contradiction). Let $E_i = \text{range}(b_i)$ and $F = \text{range}(p)$, then $F = \sum_i E_i \cap F$ (because p is a minimal central projection) with $1 < n < \infty$ and $E_i \cap F \subsetneq F$ for all i (otherwise there exists i with $p \leq b_i$, a contradiction), so $\dim(E_i \cap F) < \dim(F)$ and there exists $U \subseteq F$, a one-dimensional subspace such that $U \nsubseteq E_i \cap F$ for all i, and so a fortiori $U \nsubseteq E_i$ for all i. It follows that $u = p_U \leq p$ is a minimal projection such that $\langle u \rangle = \langle p \rangle$. ☐

Thanks to Theorem 4.13, we can give the following definition:

Definition 4.14. The planar algebra $\mathcal{P}$ is weakly cyclic (or w-cyclic) if it satisfies one of the following equivalent assertions:

- There exists a minimal projection $u \in \mathcal{P}_{2,+}$ such that $\langle u \rangle = \mathrm{id}$.
- There exists a minimal central projection $p \in \mathcal{P}_{2,+}$ such that $\langle p \rangle = \mathrm{id}$.

We call a subfactor w-cyclic if its planar algebra is w-cyclic.

The following remark justifies the choice of the word "w-cyclic".

Remark 4.15. By Corollary 6.12, a finite group subfactor $(R^G \subset R)$ is w-cyclic if and only if G is linearly primitive, which is strictly weaker than cyclic (see for example S_3), nevertheless the notion of w-cyclic is a singly generated notion in the sense that "there is a minimal projection generating the identity biprojection". We can also see the weakness of this assumption by the fact that the minimal projection does not necessarily generate a basis for the set of positive operators, but just the support of it, i.e., the identity.

Question 4.16. *Is a cyclic subfactor planar algebra w-cyclic*?
The answer is "yes" by Theorem 4.27.

Let $\mathcal{P} = \mathcal{P}(N \subseteq M)$ be an irreducible subfactor planar algebra. Take an intermediate subfactor $N \subseteq K \subseteq M$ and its biprojection $b = e^M_K$.

Lemma 4.17. *Let $\mathcal{A}$ be a $\star$-subalgebra of $\mathcal{P}_{2,+}$. Then any element $x \in \mathcal{A}$ is positive in $\mathcal{A}$ if and only if it is positive in $\mathcal{P}_{2,+}$.*

Proof. If x is positive in $\mathcal{A}$, it is of the form $aa^\star$, with $a \in \mathcal{A}$, but $a \in \mathcal{P}_{2,+}$ also, so x is positive in $\mathcal{P}_{2,+}$. Conversely, if x is positive in $\mathcal{P}_{2,+}$ then

$$\langle xy|y \rangle = \mathrm{tr}(y^\star xy) \geq 0,$$

for any $y \in \mathcal{P}_{2,+}$, so in particular, for any $y \in \mathcal{A}$, which means x is positive in $\mathcal{A}$. □

Note that Lemma 4.17 will be applied to $\mathcal{A} = b\mathcal{P}_{2,+}b$ or $b * \mathcal{P}_{2,+} * b$.

Proposition 4.18. *The planar algebra $\mathcal{P}(e_1, b)$ is w-cyclic if and only if there is a minimal projection $u \in \mathcal{P}_{2,+}$ such that $\langle u \rangle = b$.*

Proof. The planar algebra $\mathcal{P}(N \subseteq K)$ is w-cyclic if and only if there is a minimal projection $x \in \mathcal{P}_{2,+}(N \subseteq K)$ such that $\langle x \rangle = e^K_K$, if and only if $l_K(\langle x \rangle) = l_K(e^K_K)$, if and only if $\langle u \rangle = e^M_K$ (by Theorem 3.9), with $u = l_K(x)$ a minimal projection in $e^M_K \mathcal{P}_{2,+} e^M_K$ and in $\mathcal{P}_{2,+}$. □

Lemma 4.19. *For any minimal projection $x \in \mathcal{P}_{2,+}(b, \mathrm{id})$, $r_b(x)$ is positive and for any minimal projection $v \preceq r_b(x)$, there is $\lambda > 0$ such that $b * v * b = \lambda r_b(x)$.*

Proof. Firstly, x is positive, so by Theorem 3.9, $r_b(x)$ is also positive. For any minimal projection $v \preceq r_b(x)$, we have $b * v * b \preceq r_b(x)$, because

$$b * v * b \preceq b * r_b(x) * b = b * b * u * b * b \sim b * u * b = r_b(x),$$

by Lemma 3.1(2) and with $u \in \mathcal{P}_{2,+}$. Now by Lemma 3.1(1), $b * v * b > 0$, so $r_b^{-1}(b * v * b) > 0$ also, and by Theorem 3.9,

$$r_b^{-1}(b * v * b) \preceq x.$$

But x is a minimal projection, so by positivity, there exists $\lambda > 0$ such that

$$r_b^{-1}(b * v * b) = \lambda x.$$

It follows that $b * v * b = \lambda r_b(x)$. □

Lemma 4.20. *Consider $v \in \mathcal{P}_{2,+}$ positive. Then $\langle b * v * b\rangle = \langle b, v\rangle$.*

Proof. Firstly, by Definition 3.8, $b * v * b \preceq \langle b, v\rangle$, so $\langle b * v * b\rangle \le \langle b, v\rangle$, by Lemma 3.1(3). Next $e_1 \le b$ and $x * e_1 = e_1 * x = \delta^{-1}x$, so

$$v = \delta^2 e_1 * v * e_1 \preceq b * v * b.$$

Moreover by Theorem 3.6, $\bar{v} \preceq \langle b * v * b\rangle$, but by Lemma 3.2,

$$\bar{v} * b * v * b \succeq \bar{v} * e_1 * v * b \sim \bar{v} * v * b \succeq e_1 * b \sim b.$$

Then $b, v \le \langle b * v * b\rangle$, so we also have $\langle b, v\rangle \le \langle b * v * b\rangle$. □

Proposition 4.21. *The planar algebra $\mathcal{P}(b, \mathrm{id})$ is w-cyclic if and only if there is a minimal projection $v \in \mathcal{P}_{2,+}$ such that $\langle b, v\rangle = \mathrm{id}$ and $r_b^{-1}(b * v * b)$ is a positive multiple of a minimal projection.*

Proof. The planar algebra $\mathcal{P}(K \subseteq M)$ is w-cyclic if and only if there is a minimal projection $x \in \mathcal{P}_{2,+}(K \subseteq M)$ such that $\langle x\rangle = e_M^M$, if and only if $r_K(\langle x\rangle) = r_K(e_M^M)$, if and only if $\langle r_K(x)\rangle = e_M^M$ by Theorem 3.9. The result follows by Lemmas 4.19 and 4.20. □

4C. *The main result.*

Let $\mathcal{P}$ be an irreducible subfactor planar algebra.

Lemma 4.22. *A maximal subfactor planar algebra is w-cyclic.*

Proof. By maximality $\langle u\rangle = \mathrm{id}$ for any minimal projection $u \ne e_1$. □

Definition 4.23. The top intermediate subfactor planar algebra is the intermediate associated to the top interval of the biprojection lattice.

Lemma 4.24. *An irreducible subfactor planar algebra is w-cyclic if its top intermediate is so.*

Proof. Let $b_1, \dots, b_n$ be the coatoms in $[e_1, \mathrm{id}]$ and $t = \bigwedge_{i=1}^n b_i$. By assumption and Proposition 4.21, there is a minimal projection $v \in \mathcal{P}_{2,+}$ with $\langle t, v\rangle = \mathrm{id}$. If there exists i such that $v \le b_i$, then $\langle t, v\rangle \le b_i$, a contradiction. So $v \not\le b_i$ for all i, and then $\langle v\rangle = \mathrm{id}$. □

Definition 4.25. Let $h(\mathcal{P})$ be the height of the biprojection lattice $[e_1, \mathrm{id}]$. Note that $h(\mathcal{P}) < \infty$ because the index is finite.

Theorem 4.26. *If the biprojections in $\mathcal{P}_{2,+}$ are central and form a distributive lattice, then $\mathcal{P}$ is w-cyclic.*

Proof. By Lemma 4.10, we can make an induction on $h(\mathcal{P})$. If $h(\mathcal{P}) = 1$, then we apply Lemma 4.22. Now suppose that the theorem holds for $h(\mathcal{P}) < n$, we will prove it for $h(\mathcal{P}) = n \geq 2$. By Lemmas 2.2 and 4.24, we can assume the biprojection lattice to be boolean. For b in the open interval (e_1, id), its complement $b^{\complement}$ (see Section 2A) is also in (e_1, id). By induction and Proposition 4.18, there are minimal projections u, v such that $b = \langle u \rangle$ and $b^{\complement} = \langle v \rangle$. Take any minimal projection $c \preceq u * v$, then

$$\langle c \rangle = \langle c \rangle \vee e_1 = \langle c \rangle \vee (b \wedge b^{\complement}) = \langle c \rangle \vee (\langle u \rangle \wedge \langle v \rangle),$$

so *by distributivity*

$$\langle c \rangle = (\langle c \rangle \vee \langle u \rangle) \wedge (\langle c \rangle \vee \langle v \rangle) = \langle c, u \rangle \wedge \langle c, v \rangle.$$

Then by Lemma 3.3, $\langle c \rangle = \langle u', c, v \rangle \wedge \langle u, c, v' \rangle$ with u', v' minimal projections and uu', $vv' \neq 0$, so in particular the central support $Z(u') = Z(u)$ and $Z(v') = Z(v)$. Now by assumption, every biprojection is central, so $u \leq Z(u') \leq \langle u', c, v \rangle$ and $v \leq Z(v') \leq \langle u, c, v' \rangle$, so $\langle c \rangle = \mathrm{id}$. □

Theorem 4.27. *A cyclic subfactor planar algebra is w-cyclic.*

Proof. This is immediate by Theorem 4.26 because a normal biprojection is by definition bicentral, so a fortiori central. □

5. Extension for small distributive lattices

We extend Theorem 4.26 without assuming the biprojections to be central, but for distributive lattices with less than 32 elements. Because the top lattice of a distributive lattice is boolean (Lemma 2.2), we can reduce the proof to $\mathcal{B}_n$ with $n < 5$.

Definition 5.1. An irreducible subfactor planar algebra is said to be boolean (or $\mathcal{B}_n$) if its biprojection lattice is boolean (of rank n).

Proposition 5.2. *An irreducible subfactor planar algebra such that the coatoms $b_1, \dots, b_n \in [e_1, \mathrm{id}]$ satisfy $\sum_i \frac{1}{|\mathrm{id} : b_i|} \leq 1$, is w-cyclic.*

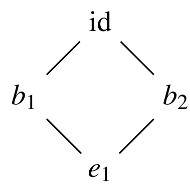
Proof. Firstly, by Lemmas 4.22 and 4.24, we can assume that $n > 1$. By definition, $|\mathrm{id} : b_i| = \mathrm{tr}(\mathrm{id})/\mathrm{tr}(b_i)$ so by assumption $\sum_i \mathrm{tr}(b_i) \leq \mathrm{tr}(\mathrm{id})$. If $\sum_i b_i \sim \mathrm{id}$ then $\sum_i b_i \geq \mathrm{id}$, but $\sum_i \mathrm{tr}(b_i) \leq \mathrm{tr}(\mathrm{id})$ so $\sum_i b_i = \mathrm{id}$. Now $e_1 \leq b_i$ for all i, therefore $ne_1 \leq \sum_i b_i = \mathrm{id}$, contradiction with $n > 1$. So $\sum_i b_i \prec id$, which implies the existence of a minimal projection $u \not\leq b_i$ for all i, which means that $\langle u \rangle = \mathrm{id}$. □

Remark 5.3. The converse is false, $(R \subset R \rtimes \mathbb{Z}/30)$ is a counterexample, because $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{31}{30} > 1$.

Corollary 5.4. *An irreducible subfactor planar algebra with at most two coatoms in* $[e_1, \mathrm{id}]$ *is w-cyclic.*

Proof. We have $\sum_i \frac{1}{|\mathrm{id}:b_i|} \le \frac{1}{2} + \frac{1}{2}$, and the result follows by Proposition 5.2. □

Examples 5.5. Every $\mathcal{B}_2$ subfactor planar algebra is w-cyclic.



Lemma 5.6. *Let* $u, v \in \mathcal{P}_{2,+}$ *be minimal projections. If* $v \not\preceq \langle u \rangle$ *then there exist minimal projections* $c \preceq u * v$ *and* $w \preceq \bar{u} * c$ *such that* $w \not\preceq \langle u \rangle$*.*

Proof. Assume that for all $c \preceq u * v$ and for all $w \preceq \bar{u} * c$, we have $w \le \langle u \rangle$. Now there are minimal projections $(c_i)_i$ and $(w_{i,j})_{i,j}$ such that $u * v \sim \sum_i c_i$ and $\bar{u} * c_i \sim \sum_j w_{i,j}$. It follows that $u * v \sim \sum_{i,j} w_{i,j} \preceq \langle u \rangle$, but

$$v \sim e_1 * v \preceq (\bar{u} * u) * v = \bar{u} * (u * v) \preceq \langle u \rangle,$$

which is in contradiction with $v \not\preceq \langle u \rangle$. □

For the distributive case, we can upgrade Proposition 5.2 as follows:

Theorem 5.7. *A distributive subfactor planar algebra with coatoms* $b_1, \dots, b_n \in [e_1, \mathrm{id}]$ *satisfying* $\sum_i \frac{1}{|\mathrm{id}:b_i|} \le 2$*, is w-cyclic.*

Proof. By Lemmas 2.2 and 4.24, we can assume the subfactor planar algebra to be boolean. If $K := \bigwedge_{i,j,i \ne j} (b_i \wedge b_j)^{\perp} \ne 0$, then consider $u \le K$ a minimal projection, and $Z(u)$ its central support. If $\langle Z(u) \rangle = \mathrm{id}$, then we are okay. Otherwise there exists i such that $\langle u \rangle = \langle Z(u) \rangle = b_i$. But $b_i^{\complement}$ is an atom in $[e_1, \mathrm{id}]$, so there is a minimal projection v such that $b_i^{\complement} = \langle v \rangle$. Recall that $b_i \wedge b_i^{\complement} = e_1$, so $v \not\preceq \langle u \rangle$, and by Lemma 5.6, there are minimal projections $c \preceq u * v$ and $w \preceq \bar{u} * c$ such that $w \not\preceq \langle u \rangle$ (and $\langle u, w \rangle = \mathrm{id}$ by maximality). By Lemma 3.3, there exists $u' \preceq c * \bar{u}$ with $Z(u') = Z(u)$ and $u' \not\le u$, but $u \le K$ so $u' \not\preceq b_i \wedge b_j$ for all $j \ne i$, and now $u' \le Z(u) \le b_i$, so $\langle u' \rangle = b_i$. Using distributivity (as for Theorem 4.26) we conclude

$$\langle c \rangle = \langle u, c \rangle \wedge \langle c, v \rangle \ge \langle u, w \rangle \wedge \langle u', v \rangle = \mathrm{id} \wedge \mathrm{id} = \mathrm{id}\,.$$

Otherwise $K = 0$, but $(b_i \wedge b_j)^{\perp} \ge b_j^{\perp}$ for all i, so $\bigwedge_{j \ne i} b_j^{\perp} = 0$ for all i. Let $p_1, \dots, p_r$ be the minimal central projections. Then $b_i = \bigoplus_{s=1}^{r} p_{i,s}$ with $p_{i,s} \le p_s$ and $p_{i,1} = p_1 = e_1$. Now $b_i^{\perp} = \bigoplus_{s=1}^{r} (p_s - p_{i,s})$, so by assumption,

$$0 = \bigwedge_{j \ne i} \bigoplus_{s=1}^{r} (p_s - p_{j,s}) = \bigoplus_{s=1}^{r} \bigwedge_{j \ne i} (p_s - p_{j,s}), \quad \text{for all } i.$$

It follows that $p_s = \bigvee_{j\neq i} p_{j,s}$ for all i and s, so $\mathrm{tr}(p_s) \le \sum_{j\neq i} \mathrm{tr}(p_{j,s})$. Now if there exists s such that $p_{i,s} < p_s$ for all i, then $\langle p_s \rangle = id$, which is okay; otherwise for all s, there exists i with $p_{i,s} = p_s$, but $\sum_{j\neq i} \mathrm{tr}(p_{j,s}) \ge \mathrm{tr}(p_s)$, so $\sum_j tr(p_{j,s}) \ge 2\,\mathrm{tr}(p_s)$. Then

$$\sum_i \mathrm{tr}(b_i) \ge n \cdot \mathrm{tr}(e_1) + 2 \sum_{s\neq 1} \mathrm{tr}(p_s) = 2\,\mathrm{tr}(\mathrm{id}) + (n-2)\,\mathrm{tr}(e_1).$$

Now $|\mathrm{id} : b_i| = \mathrm{tr}(\mathrm{id}) / \mathrm{tr}(b_i)$, so

$$\sum_i \frac{1}{|\mathrm{id} : b_i|} \ge 2 + \frac{n-2}{|\mathrm{id} : e_1|}$$

which contradicts the assumption, because we can assume $n > 2$ by Corollary 5.4. The result follows. □

Remark 5.8. The converse is false because there exist w-cyclic distributive subfactor planar algebras with $\sum_i (1/|\mathrm{id} : b_i|) > 2$. For example, the subfactor $(R \rtimes S_2^n \subset R \rtimes S_3^n)$ is w-cyclic and $\mathcal{B}_n$, but $\sum_i (1/|\mathrm{id} : b_i|) = \frac{n}{3}$.

Corollary 5.9. *Every $\mathcal{B}_n$ subfactor planar algebra with $|\mathrm{id} : b| \ge \frac{n}{2}$, for any coatom $b \in [e_1, \mathrm{id}]$, is w-cyclic. Then for all $n \le 4$, any $\mathcal{B}_n$ subfactor planar algebra is w-cyclic.*

Proof. By assumption (following the notations of Theorem 5.7)

$$\sum_i \frac{1}{|\mathrm{id} : b_i|} \le \sum_i \frac{2}{n} = 2.$$

But $|\mathrm{id} : b| \ge 2$, so any $n \le 4$ works. □

Corollary 5.10. *A distributive subfactor planar algebra having less than 32 biprojections (or of index < 32), is w-cyclic.*

Proof. In this case, the top of $[e_1, \mathrm{id}]$ is boolean of rank $n < 5$, because $32 = 2^5$; the result follows by Lemma 4.24 and Corollary 5.9. □

Conjecture 5.11. *A distributive subfactor planar algebra is w-cyclic.*

By Lemmas 2.2 and 4.24, we can reduce Conjecture 5.11 to the boolean case, and then extend it to the top boolean case.

Remark 5.12. The converse of Conjecture 5.11 is false, because the group S_3 is linearly primitive but not cyclic (see Corollary 6.12).

Problem 5.13. What is the natural additional assumption (A) such that $\mathcal{P}$ is distributive if and only if it is w-cyclic and satisfies (A)?

Assuming Conjecture 5.11 and using Remark 2.1, we get:

Conjecture 5.14. *For any distributive subfactor planar algebra $\mathcal{P}$ and any biprojection $b \in \mathcal{P}_{2,+}$, the planar algebras $\mathcal{P}(e_1, b)$, $\mathcal{P}(b, \mathrm{id})$ and their duals are w-cyclic.*

Remark 5.15. The converse is false because the interval $[S_2, S_4]$, proposed by Zhengwei Liu, gives a counterexample.

Remark 5.16. A cyclic subfactor planar algebra satisfies Conjecture 5.14 (thanks to Theorem 4.27 and Lemma 4.10).

Problem 5.17. Is a Dedekind subfactor planar algebra $\mathcal{P}$ distributive if and only if for any biprojection $b \in \mathcal{P}_{2,+}$, the planar algebras $\mathcal{P}(e_1, b)$, $\mathcal{P}(b, \mathrm{id})$ and their duals are w-cyclic?

6. Applications

6A. ***A nontrivial upper bound.*** For any irreducible subfactor planar algebra $\mathcal{P}$, we exhibit a nontrivial upper bound for the minimal number of minimal 2-box projections generating the identity biprojection. We will use the notations of Section 3C.

Lemma 6.1. *Let $b' < b$ be biprojections. If $\mathcal{P}(b', b)$ is w-cyclic, then there is a minimal projection $u \in \mathcal{P}_{2,+}$ such that $\langle b', u\rangle = b$.*

Proof. Consider the isomorphisms of von Neumann algebras

$$l_b : \mathcal{P}_{2,+}(e_1, b) \to b\mathcal{P}_{2,+}b$$

and, with $a = l_b^{-1}(b')$,

$$r_a : \mathcal{P}_{2,+}(b', b) \to a * \mathcal{P}_{2,+}(e_1, b) * a.$$

Then, by assumption, the planar algebra $\mathcal{P}(b', b)$ is w-cyclic, so by Proposition 4.21, there exists a minimal projection $u' \in \mathcal{P}_{2,+}(e_1, b)$ such that

$$\langle a, u'\rangle = l_b^{-1}(b).$$

Then by applying the map l_b and Theorem 3.9, we get

$$b = \langle l_b(a), l_b(u')\rangle = \langle b', u\rangle$$

with $u = l_b(u')$ a minimal projection in $b\mathcal{P}_{2,+}b$, so in $\mathcal{P}_{2,+}$. □

Assuming Conjecture 5.11 and using Lemma 6.1, we get a nontrivial upper bound:

Conjecture 6.2. *The minimal number r of minimal projections generating the identity biprojection (i.e., $\langle u_1, \dots, u_r\rangle = \mathrm{id}$) is at most the minimal length ℓ for an ordered chain of biprojections*

$$e_1 = b_0 < b_1 < \cdots < b_\ell = id$$

such that $[b_i, b_{i+1}]$ is distributive (or better, top boolean).

Remark 6.3. We can deduce some theorems from Conjecture 6.2, by adding some assumptions to $[b_i, b_{i+1}]$, according to Theorems 4.26 or 5.7.

Remark 6.4. Let $(N \subset M)$ be any irreducible finite index subfactor. We can deduce a nontrivial upper bound for the minimal number of (algebraic) irreducible sub-N-N-bimodules of M, generating M as a von Neumann algebra.

6B. ***Back to the finite groups theory.*** As applications, we get a dual version of Theorem 2.5, and for any finite group G, we get a nontrivial upper bound for the minimal number of irreducible components for a faithful complex representation. The action of G on the hyperfinite II_1 factor R is always assumed to be outer.

Theorem 6.5 [Burnside 1911, § 226]. *A complex representation V of a finite group G is faithful if and only if any irreducible complex representation W is equivalent to a subrepresentation of $V^{\otimes n}$, for some $n \geq 0$.*

Definition 6.6. A group G is linearly primitive if it admits a faithful irreducible complex representation.

Definition 6.7. Let W be a representation of a group G, K be a subgroup of G, and X be a subspace of W. Let the *fixed-point subspace* be

$$W^K := \{w \in W \mid kw = w, \text{ for all } k \in K\}$$

and the *pointwise stabilizer subgroup*

$$G_{(X)} := \{g \in G \mid gx = x, \text{ for all } x \in X\}.$$

Definition 6.8. An interval $[H, G]$ is said to be linearly primitive if there is an irreducible complex representation V of G with $G_{(V^H)} = H$.

The group G is linearly primitive if and only if the interval $[\{e\}, G]$ is.

Lemma 6.9. *Let H be a core-free subgroup of G. Then G is linearly primitive if $[H, G]$ is so.*

Proof. Take V as above. Now, $V^H \subset V$ so $G_{(V)} \subset G_{(V^H)}$, but $\ker(\pi_V) = G_{(V)}$, so it follows that $\ker(\pi_V) \subset H$, but H is a core-free subgroup of G, and $\ker(\pi_V)$ is a normal subgroup of G, so $\ker(\pi_V) = \{e\}$, which means that V is faithful on G, i.e., G is linearly primitive. □

Lemma 6.10. *Letting $p_x \in \mathcal{P}_{2,+}(R^G \subseteq R)$ be a minimal projection on the one-dimensional subspace $\mathbb{C}x$ and H a subgroup of G, then*

$$p_x \leq b_H := |H|^{-1} \sum_{h \in H} \pi_V(h) \Leftrightarrow H \subset G_x.$$

Proof. If $p_x \leq b_H$ then $b_H x = x$ and for every $h \in H$ we have that

$$\pi_V(h)x = \pi_V(h)[b_H x] = [\pi_V(h) \cdot b_H]x = b_H x = x$$

which means that $h \in G_x$, and so $H \subset G_x$. Conversely, if $H \subset G_x$ (i.e., $\pi_V(h)x = x$ for every $h \in H$) then $b_H x = x$, which means that $p_x \leq b_H$. □

Theorem 6.11. *Let $[H, G]$ be an interval of finite groups. Then*

- *$(R \rtimes H \subseteq R \rtimes G)$ is w-cyclic if and only if $[H, G]$ is H-cyclic.*
- *$(R^G \subseteq R^H)$ is w-cyclic if and only if $[H, G]$ is linearly primitive.*

Proof. By Proposition 4.21, $(R \rtimes H \subseteq R \rtimes G)$ is w-cyclic if and only if there exists a minimal projection u in

$$\mathcal{P}_{2,+}(R \subseteq R \rtimes G) \simeq \bigoplus_{g \in G} \mathbb{C}e_g \simeq \mathbb{C}^G$$

such that $\langle b, u\rangle = \mathrm{id}$, with $b = e^{R\rtimes G}_{R\rtimes H}$, and $r_b^{-1}(b * u * b)$ is a minimal projection if and only if there exists $g \in G$ such that $\langle H, g\rangle = G$, because u is of the form e_g and $Hg'H = HgH$ for all $g' \in HgH$.

By Proposition 4.18, $(R^G \subseteq R^H)$ is w-cyclic if and only if there exists a minimal projection u in

$$\mathcal{P}_{2,+}(R^G \subseteq R) \simeq \bigoplus_{V_i \text{ irr}} \mathrm{End}(V_i) \simeq \mathbb{C}G$$

such that $\langle u\rangle = e^R_{R^H}$, if and only if, by Lemma 6.10, $H = G_x$ with $u = p_x$ the projection on $\mathbb{C}x \subseteq V_i$ (with $Z(p_x) = p_{V_i}$). Note that $H \subset G_{(V_i^H)} \subset G_x$ so $H = G_{(V_i^H)}$. □

Corollary 6.12. *The subfactor $(R^G \subseteq R)$ (respectively, $(R \subseteq R \rtimes G)$) is w-cyclic if and only if G is linearly primitive (respectively, cyclic).*

Examples 6.13. The subfactors $(R^{S_4} \subset R^{S_2})$, its dual and $(R^{S_3} \subset R)$, are w-cyclic, but $(R \subset R \rtimes S_3)$ and $(R^{S_4} \subset R^{\langle(1,2)(3,4)\rangle})$ are not.

By Theorem 6.11, the group-theoretic reformulation of Conjecture 5.11 on $(R^G \subseteq R^H)$ is the following dual version of Theorem 2.5.

Conjecture 6.14. *Let $[H, G]$ be a distributive interval of finite groups. Then there exists an irreducible complex representation V of G such that $G_{(V^H)} = H$.*

If, moreover, H is core-free, then G is linearly primitive (Lemma 6.9).

Problem 6.15. Is a finite group G linearly primitive if and only if there is a core-free subgroup H such that the interval $[H, G]$ is bottom boolean?

By Theorem 6.5, Conjecture 6.2 on $\mathcal{P}(R^G \subseteq R)$ reformulates as follows:

Conjecture 6.16. *The minimal number of irreducible components for a faithful complex representation of a finite group G is at most the minimal length ℓ for an ordered chain of subgroups*

$$\{e\} = H_0 < H_1 < \cdots < H_\ell = G$$

such that $[H_i, H_{i+1}]$ is distributive (or better, bottom boolean).

This provides a bridge linking combinatorics and representations in the theory of finite groups.

Remark 6.17. We can upgrade Conjecture 6.16 by taking for H_0 any core-free subgroup of H_1, instead of just $\{e\}$; we can also deduce some theorems, by adding some assumptions to $[H_i, H_{i+1}]$, according to the group-theoretic reformulation of Theorems 4.27 or 5.7. Note that a normal biprojection in $\mathcal{P}(R^G \subseteq R^H)$ is given by a subgroup $K \in [H, G]$ with $HgK = KgH$ for all $g \in G$, see [Teruya 1998, Proposition 3.3].

Remark 6.18. We can also formulate results for finite quantum groups (i.e., finite-dimensional Kac algebras), where the biprojections correspond to the left coideal $\star$-subalgebras, see [Izumi et al. 1998, Theorem 4.4].

Acknowledgments

This work is supported by the Institute of Mathematical Sciences, Chennai. The author is grateful to Vaughan Jones, Dietmar Bisch, Scott Morrison and David Evans for their recommendation for a postdoc at the IMSc. Thanks to my hosts V.S. Sunder and Vijay Kodiyalam, and to Zhengwei Liu, Feng Xu, Keshab Chandra Bakshi and Mamta Balodi, for useful advice and fruitful exchanges.

References

[Afzaly et al. 2015] N. Afzaly, S. Morrison, and D. Penneys, “The classification of subfactors with index at most $5\frac{1}{4}$”, preprint, 2015. arXiv

[Asaeda and Haagerup 1999] M. Asaeda and U. Haagerup, “Exotic subfactors of finite depth with Jones indices $(5+\sqrt{13})/2$ and $(5+\sqrt{17})/2$”, *Comm. Math. Phys.* **202**:1 (1999), 1–63. MR Zbl

[Bakshi 2016] K. C. Bakshi, “Intermediate planar algebra revisited”, preprint, 2016. arXiv

[Bisch 1994] D. Bisch, “A note on intermediate subfactors”, *Pacific J. Math.* **163**:2 (1994), 201–216. MR Zbl

[Burnside 1911] W. Burnside, *Theory of groups of finite order*, 2nd ed., Cambridge Univ. Press, 1911. Zbl

[Izumi 2001] M. Izumi, “The structure of sectors associated with Longo–Rehren inclusions, II: Examples”, *Rev. Math. Phys.* **13**:5 (2001), 603–674. MR Zbl

[Izumi 2002] M. Izumi, “Characterization of isomorphic group-subgroup subfactors”, *Int. Math. Res. Not.* **2002**:34 (2002), 1791–1803. MR Zbl

[Izumi et al. 1998] M. Izumi, R. Longo, and S. Popa, “A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras”, *J. Funct. Anal.* **155**:1 (1998), 25–63. MR Zbl

[Jones 1980] V. F. R. Jones, *Actions of finite groups on the hyperfinite type* II_1 *factor*, Mem. Amer. Math. Soc. **237**, American Mathematical Society, Providence, RI, 1980. MR Zbl

[Jones 1983] V. F. R. Jones, “Index for subfactors”, *Invent. Math.* **72**:1 (1983), 1–25. MR Zbl

[Jones 1999] V. F. R. Jones, “Planar algebras, I”, 1999. To appear in *New Zealand J. Math.* Zbl arXiv

[Jones and Sunder 1997] V. F. R. Jones and V. S. Sunder, *Introduction to subfactors*, London Mathematical Society Lecture Note Series **234**, Cambridge Univ. Press, 1997. MR Zbl

[Jones et al. 2014] V. F. R. Jones, S. Morrison, and N. Snyder, "The classification of subfactors of index at most 5", *Bull. Amer. Math. Soc.* (*N.S.*) **51**:2 (2014), 277–327. MR Zbl

[Kodiyalam and Sunder 2000] V. Kodiyalam and V. S. Sunder, "The subgroup-subfactor", *Math. Scand.* **86**:1 (2000), 45–74. MR Zbl

[Kodiyalam and Sunder 2004] V. Kodiyalam and V. S. Sunder, "On Jones' planar algebras", *J. Knot Theory Ramifications* **13**:2 (2004), 219–247. MR Zbl

[Landau 1998] Z. A. Landau, *Intermediate subfactors*, Ph.D. thesis, University of California, Berkeley, 1998, Available at https://search.proquest.com/docview/304420808.

[Landau 2002] Z. A. Landau, "Exchange relation planar algebras", *Geom. Dedicata* **95** (2002), 183–214. MR

[Liu 2016] Z. Liu, "Exchange relation planar algebras of small rank", *Trans. Amer. Math. Soc.* **368**:12 (2016), 8303–8348. MR Zbl

[Nakamura and Takeda 1960] M. Nakamura and Z. Takeda, "On the fundamental theorem of the Galois theory for finite factors", *Proc. Japan Acad.* **36** (1960), 313–318. MR

[Ore 1938] O. Ore, "Structures and group theory, II", *Duke Math. J.* **4**:2 (1938), 247–269. MR Zbl

[Palcoux 2013] S. Palcoux, "Non-'weakly group theoretical' integral fusion categories?", message board post, 2013, Available at http://mathoverflow.net/q/132866.

[Palcoux 2015] S. Palcoux, "Ore's theorem for cyclic subfactor planar algebras and applications", preprint, 2015. arXiv

[Peters 2010] E. Peters, "A planar algebra construction of the Haagerup subfactor", *Internat. J. Math.* **21**:8 (2010), 987–1045. MR Zbl

[Stanley 2012] R. P. Stanley, *Enumerative combinatorics, I*, 2nd ed., Cambridge Studies in Advanced Mathematics **49**, Cambridge Univ. Press, 2012. MR Zbl

[Teruya 1998] T. Teruya, "Normal intermediate subfactors", *J. Math. Soc. Japan* **50**:2 (1998), 469–490. MR Zbl

[Watatani 1996] Y. Watatani, "Lattices of intermediate subfactors", *J. Funct. Anal.* **140**:2 (1996), 312–334. MR Zbl

[Xu 2013] F. Xu, "On a problem about tensor products of subfactors", *Adv. Math.* **246** (2013), 128–143. MR Zbl

Received February 5, 2017. Revised July 26, 2017.

SEBASTIEN PALCOUX
INSTITUTE OF MATHEMATICAL SCIENCES
CHENNAI 600113
INDIA

sebastienpalcoux@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2018 is US $475/year for the electronic version, and $640/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 292 No. 1 January 2018

0030-8730(201801)292:1;1-Z