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Let $F$ be a field with char $F \neq 2$, let $a_1, \ldots, a_n \in F^*$, and let $f \in F[y]$ be a monic polynomial of degree $2m$. Let further $S$ be an affine hypersurface over $F$ determined by the equation $f(y) = \sum_{i=1}^n a_ix_i^2$. In the first part of the paper we prove a certain version of Springer’s theorem. Namely, we show that if the form $\psi \simeq (1, -a_1, \ldots, -a_n)$ is anisotropic and $S$ has an $L$-rational point for some odd-degree extension $L/F$, then $S$ has an $L$-rational point for some odd-degree extension $L/F$ with $[L : F] \leq m$, and the last inequality is strict in general.

In the second part we consider the case where the polynomial $f$ is quartic. We show that the surface $S$ has a rational point if and only if the quadratic form $\psi \perp (-x, g(x))$ is isotropic over $F(x)$, where $g(x) \in F[x]$ is a certain polynomial of degree at most 3, whose coefficients are expressed in a polynomial way via the coefficients of $f$.

In the third part we describe all Pfister forms that belong to the Witt kernel $W(F(C)/F)$, where $C$ is the plane nonsingular curve determined by the equation $y^2 = a_4x^4 + a_2x^2 + a_1x + a_0$. In the case where the $u$-invariant of $F$ is at most 10, we describe generators of the ideal $W(F(C)/F)$.

Introduction

Let $F$ be a field of characteristic different from 2. We investigate some properties of the affine hypersurface $S$ determined by the equation $f(y) = \sum_{i=1}^n a_ix_i^2$, where $a_i \in F^*$ and $f$ is a monic polynomial of degree $2m$. In Section 1, we prove a version of Springer’s theorem for $S$ (Proposition 1.1). In particular, we show that if $m = 2$ (i.e., the polynomial $f$ is quartic), and $S$ has a $K$-rational point for some odd-degree extension $K/F$, then $S$ has an $F$-rational point. Sections 2 and 3 can be considered as generalizations of some results in [Haile and Han 2007; Shick 1994]. Namely, for the affine hyperelliptic curve $C$ with the equation $f(y) = ax^2$ over a field $F$, where $a \in F^* \setminus F^{*2}$ and $f(y)$ is a quartic polynomial, two questions have been investigated in [Haile and Han 2007]. First, it has been shown that existence

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of a rational point on $C$ is equivalent to triviality of a certain quaternion algebra over a certain quadratic extension of the rational function field $F(x)$. It is easy to see that this is equivalent to isotropicity of some 4-dimensional quadratic form over $F(x)$. In Proposition 2.1 we obtain a similar criterion for the affine hypersurface $S : f(y) = \sum_{i=1}^n a_i x_i^2$, where $a_i \in F^*$, the form $\langle 1, -a_1, \ldots, -a_n \rangle$ is anisotropic, and $f$ is a monic quartic polynomial. This proves independently of Section 1 that existence of a rational point over any odd-degree field extension $K/F$ implies existence of a rational point of $S$ over the field $F$ itself.

Another result in [Haile and Han 2007; Shick 1994] is a computation of the relative Brauer group $\text{Br}(F(C)/F)$, where $C$ is the affine hyperelliptic curve above. Obviously, this is equivalent to description of all 2-fold Pfister forms $\pi$ over $F$ such that $\pi_{F(C)} = 0$. Section 3 is devoted to investigation of the Witt kernel $W(F(C)/F)$. Applying an invertible change of variables, we may assume that the curve $C$ is determined by the equation $y^2 = a_4 x^4 + a_2 x^2 + a_1 x + a_0$, where $a_i \in F$, $a_4 \neq 0$. We will also assume that $C$ is nonsingular, for the opposite case is trivial. Let $e \in F$. Set

$$d(e) = -\det \begin{pmatrix} a_4 & 0 & \frac{1}{2}(a_2-e) \\ 0 & e & \frac{1}{2}a_1 \\ \frac{1}{2}(a_2-e) & \frac{1}{2}a_1 & a_0 \end{pmatrix}.$$ 

In Proposition 3.1 we show that if $0 \neq Q \in \text{Br}(F(C)/F)$, then either $Q = (a_4, e)$, where $e \neq 0$, $d(e) \in F^{*2} \cup \{0\}$, or $a_1 = 0$ and $Q = (a_4, a_2^2 - 4a_0a_4)$. Conversely, any quaternion algebra of the types above belongs to $\text{Br}(F(C)/F)$.

Proposition 3.1 is not new, but we give it for the convenience of the reader, and because we need its proof a bit later in Proposition 3.2. In fact, the original proof of Proposition 3.1, which is very similar to ours, is given in [Shick 1994]. However, in Proposition 3.2 and Corollary 3.3 we describe all Pfister forms $\pi$ (not necessarily 2-fold) over $F$ such that $\pi_{F(C)} = 0$. More precisely, if $\pi_{F(C)} = 0$, then either $\pi$ is divisible by a 2-fold Pfister form $\rho$ such that $\rho_{F(C)} = 0$, or there exist $e, r \in F$, $e \neq 0$, $r^2 - d(e) \neq 0$ such that $\langle a_4, e, r^2 - d(e) \rangle \subset \pi$. Conversely, $\langle a_4, e, r^2 - d(e) \rangle \in W(F(C)/F)$ for any $e, r \in F$, $e \neq 0$, $r^2 - d(e) \neq 0$. If the $u$-invariant of $F$ is at most 10, this is sufficient for the computation of the Witt kernel $W(F(C)/F)$.

A few words about the notation. Throughout all the fields have characteristic different from 2. By a form we always mean a quadratic form over a field. For $a_1, \ldots, a_n \in F^*$ we denote the Pfister form $\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$ as $\langle a_1, \ldots, a_n \rangle$ (take notice of signs!), and $D(\varphi)$ is the set of all nonzero values of the form $\varphi$. If the form $\varphi$ is considered as an element of the Witt ring $W(F)$, then $\dim \varphi$ denotes the dimension of the anisotropic part of $\varphi$.

If $\varphi$ is a regular form over the field $F$, $\dim \varphi \geq 3$, then by $F(\varphi)$ we denote the function field of the corresponding projective quadric.

Slightly abusing notation, we often identify a form with its symmetric matrix.
1. A version of Springer’s theorem

The well-known Springer’s theorem claims that if $K/F$ is an odd-degree field extension, and a projective quadric $X$ has a rational point over $K$, then it has a rational point over $F$. Below we give an affine version of this theorem for certain hypersurfaces.

**Proposition 1.1.** Let $F$ be a field, let $a_1, \ldots, a_n \in F^*$, and let $f \in F[y]$ be a monic polynomial of degree $2m$. Let $S = S(f, a_1, \ldots, a_n)$ be the affine hypersurface over $F$ determined by the equation $f(y) = \sum_{i=1}^{n} a_i x_i^2$. Suppose that $S$ has a $K$-rational point for some odd-degree extension $K/F$.

1. If the form $\langle a_1, \ldots, a_n \rangle$ is anisotropic, then $S$ has an $L$-rational point for some odd-degree extension $L/F$ with $[L : F] \leq 2m - 1$.

2. If the form $\langle 1, -a_1, \ldots, -a_n \rangle$ is anisotropic, then $S$ has an $L$-rational point for some odd-degree extension $L/F$ with $[L : F] \leq m$, and the last inequality is strict in general. In particular, if $m = 2$, i.e., $f$ is a quartic polynomial, then $S$ has an $F$-rational point.

3. If the form $\langle a_1, \ldots, a_n \rangle$ is isotropic, then $S$ has an $F$-rational point.

**Proof.** (1)–(2) Assume the form $\langle a_1, \ldots, a_n \rangle$ is anisotropic, and $K/F$ is an odd-degree field extension. Suppose $[K : F] \geq s$, where $s = m + 1$ if the form $\langle 1, -a_1, \ldots, -a_n \rangle$ is anisotropic, and $s = 2m + 1$ otherwise. Let $f(\alpha) = \sum_{i=1}^{n} a_i \beta_i^2$ for some $\alpha, \beta_i \in K$. It suffices to find an odd-degree field extension $L/F$ with $[L : F] < [K : F]$ such that $S$ has a rational $L$-point. Since $[K : F(\alpha)]$ is odd, we get by Springer’s theorem, applied to the extension $K/F(\alpha)$, that the form $\langle a_1, \ldots, a_n, -f(\alpha) \rangle$ is isotropic over $F(\alpha)$. Hence we may assume that $\beta_i \in F(\alpha)$ for each $i$. We may assume also that $[F(\alpha) : F] \geq s$, for otherwise there is nothing to be proved. Let $g$ be the minimal polynomial of $\alpha$. In particular, $\deg g = [F(\alpha) : F] \geq s$. Let $\beta_i = p_i(\alpha)$, where $p_i \in F[x]$, $\deg p_i \leq \deg g - 1$. Also $\deg f = 2m \leq 2(s - 1) \leq 2(\deg g - 1)$. We have

$$\sum_{i=1}^{n} a_i p_i^2 - f = gh \quad \text{for some } h \in F[x], \quad \text{and} \quad \deg \left( \sum_{i=1}^{n} a_i p_i^2 - f \right) \leq 2(\deg g - 1).$$

If $\deg \left( \sum_{i=1}^{n} a_i p_i^2 - f \right)$ is even, then $\deg h$ is odd, and

$$\deg h \leq 2(\deg g - 1) - \deg g = \deg g - 2 = [F(\alpha) : F] - 2 \leq [K : F] - 2.$$ 

Hence $S$ has an $L$-rational point, where $L = F[x]/p(x)$, and $p$ is an arbitrary odd-degree prime divisor of $h$. Moreover, $[L : F] < [K : F]$.

If $\deg \left( \sum_{i=1}^{n} a_i p_i^2 - f \right)$ is odd, or $h = 0$, then, since $f$ is monic of even degree, the form $\langle 1, -a_1, \ldots, -a_n \rangle$ is isotropic. Hence $s = 2m + 1$, and $\deg \left( \sum_{i=1}^{n} a_i p_i^2 \right) = \deg f = 2m$. Therefore, in this case $h = 0$, and so $S$ has an $F$-rational point.
Now let us show that in the inequality $[L:F] \leq m$ in the second part of Proposition 1.1, the number $m$ cannot be replaced by a smaller number, provided we consider all fields $F$ and all odd-degree extensions $K/F$. Consider two cases:

Case (a): $m$ is odd. Let $F$ be a field such that there exists an irreducible polynomial $p$ of degree $m$ over $F$. Consider the equation

$$p(y)^2 = \sum_{i=1}^{n} a_i x_i^2.$$ 

Clearly, it has a solution over the field $K = F[y]/p(y)$ with $x_1 = \cdots = x_n = 0$. Suppose that $L/F$ is an odd-degree extension, $\alpha, \beta_i \in L$, and $p(\alpha)^2 = \sum a_i \beta_i^2$. Since the form $\langle 1, -a_1, \ldots, -a_n \rangle$ is anisotropic, we get by Springer’s theorem applied to the odd-degree extension $K/F$ that $p(\alpha) = \beta_1 = \cdots = \beta_n = 0$. Hence $m = \deg p = [F(\alpha) : F] \leq [L:F]$.

Case (b): $m$ is even. Let $k$ be a field, let $F = k((t))$ be the Laurent series field, and let the hypersurface $S$ be determined by the equation $(y^{m-1}+t)(y^{m+1}+t) = \sum_{i=1}^{n} a_i x_i^2$. Let $L/F$ be an odd-degree extension, $[L:F] \leq m - 3$. Obviously, the field $L$ is complete with respect to a discrete valuation $v$ such that $1 \leq v(t) \leq m - 3$. It is easy to show that $(\alpha^{m-1}+t) (\alpha^{m+1}+t) \in L[2]$ for any $\alpha \in L$. Therefore, by Springer’s theorem

$$(\alpha^{m-1}+t) (\alpha^{m+1}+t) \neq \sum_{i=1}^{n} a_i \beta_i^2$$

for any $\beta_i \in L$.

(3) This is obvious, since any element of $F$ is a value of the form $\langle a_1, \ldots, a_n \rangle$. □

**Remark 1.2.** The hypothesis that the form $\langle 1, -a_1, \ldots, -a_n \rangle$ is anisotropic is essential in the second part of Proposition 1.1, at least for $m = 2$. Indeed, consider the equation $y^4 + 2 = x^2$ over $\mathbb{Q}$. Let $L = F(\delta)$, where $\delta$ is a root of the irreducible polynomial $p(u) = 2u^3 - u^2 + 2$. Obviously, $x = \delta^2 - \delta, y = \delta$ is a solution of the equation in question over $L$.

Let us prove now that this equation has no solution over $\mathbb{Q}$. It suffices to show that if $x, y, z \in \mathbb{Z}$, and $y^4 + 2z^4 = x^2$, then $z = 0$. Assume the contrary, so we may suppose that $y^4 + 2z^4 = x^2$, $z > 0$ and $z$ is as small as possible. In particular, $y$ and $z$ are coprime; hence $y$ is odd. Over $\mathbb{Q}(\sqrt{-2})$ we have $(y^2 + z^2 \sqrt{-2})(y^2 - z^2 \sqrt{-2}) = x^2$, and it is easy to see that the numbers $y^2 + z^2 \sqrt{-2}$ and $y^2 - z^2 \sqrt{-2}$ are coprime in the Euclidean ring $\mathbb{Z}[\sqrt{-2}]$. Since the group of units of the ring $\mathbb{Z}[\sqrt{-2}]$ consists of 1 and $-1$, we get that $y^2 + z^2 \sqrt{-2} = \pm (u + v \sqrt{-2})$ for some $u, v \in \mathbb{Z}$, $v > 0$. If $y^2 + z^2 \sqrt{-2} = -(u + v \sqrt{-2})^2$, then $y^2 = 2v^2 - u^2, z^2 = -2uv$. The equality $y^2 = 2v^2 - u^2$ implies that $u$ and $v$ are odd. But then, clearly, the equality $z^2 = -2uv$ is impossible.

Thus $y^2 + z^2 \sqrt{-2} = (u + v \sqrt{-2})^2$, which means that $y^2 = u^2 - 2v^2, z^2 = 2uv$. In particular, $u$ is odd. Since $(u - y)(u + y) = 2v^2$, and the numbers $\frac{1}{2}(u - y)$,
\( \frac{1}{2}(u + y) \) are, obviously, coprime, we may assume, changing if needed the sign of \( y \), that \( \frac{1}{2}(u - y) = t^2 \), \( \frac{1}{2}(u + y) = 2s^2 \) for some coprime \( s, t > 0 \). Therefore, we have

\[
\begin{align*}
  u &= 2s^2 + t^2, \\
  y &= 2s^2 - t^2, \\
  v &= 2st;
\end{align*}
\]

hence \( z^2 = 2uv = 4st(2s^2 + t^2) \), and so \( s = \alpha^2, \ t = \beta^2, \ 2s^2 + t^2 = \gamma^2 \), which implies \( \beta^4 + 2\alpha^4 = \gamma^2 \) for some positive integers \( \alpha, \beta, \gamma \). Moreover, obviously,

\[
0 < \alpha = \sqrt{s} < \sqrt{v} < z,
\]

a contradiction to the minimality of \( z \).

In fact, there are similar counterexamples for any characteristic. Namely, let \( k \) be a field, \( t \) indeterminate, and \( F = k(t) \). By an argument similar to the one for the equation \( y^4 + 2 = x^2 \) over \( \mathbb{Q} \), one can easily show that the equation \( y^4 - t = x^2 \) has no solution in \( F \). On the other hand, \( x = \alpha^2 - \alpha, \ y = \alpha \) is a solution of the same equation over the field \( F(\alpha) \), where \( \alpha \) is a root of the polynomial \( p(u) = 2u^3 - u^2 - t \).

However, we do not know if there exists a counterexample for each finite field, and for each number field.

**Proposition 1.3.** Let \( F \) be a field, \( a_1, \ldots, a_n \in F^* \), and the form \( \langle 1, -a_1, \ldots, -a_n \rangle \) be isotropic. Let further \( f \in F[y] \) be a monic polynomial of degree \( 2m \), where \( m \) is not divisible by \( \text{char} \ F \). Then the hypersurface \( S = S(f, a_1, \ldots, a_n) \) has an \( L \)-rational point for some odd-degree field extension \( L/F \) with \( [L : F] \leq 2m - 1 \).

**Proof.** Since \( 1 \in D(\langle a_1, \ldots, a_n \rangle) \), we may assume that \( n = 1 \) and \( a_1 = 1 \). Replacing if needed \( y \) by \( y + c \), where \( c \in F^* \), we may assume that the coefficient \( a \) at \( y^{2m-1} \) of the polynomial \( f(y) \) is nonzero. Then setting \( x = z + y^m \), one can see that the equation \( f(y) = x^2 \) is equivalent to the equation \( ay^{2m-1} + \sum_{i=0}^{2m-2} p_i(z)y^i = 0 \), where \( p_i(z) \in F[z] \). It is clear that the last equation has a required point. \( \square \)

**Remark 1.4.** We do not know whether Proposition 1.3 remains valid if \( m \) is divisible by \( \text{char} \ F \).

Another natural question is whether the inequality \( [L : F] \leq 2m - 1 \) in the first part of Proposition 1.1 is strict for each \( m \). In view of Remark 1.2 it is strict for \( m = 2 \).

## 2. A criterion for existence of rational points for certain affine hypersurfaces

We give a criterion in the language of quadratic forms for the existence of a rational point for the hypersurface \( S \) in the case where \( m = 2 \) (the polynomial \( f \) is quartic) and the form \( \langle 1, -a_1, \ldots, -a_n \rangle \) is anisotropic. The main ingredient in the sequel is the strong form of the Cassels–Pfister theorem [Pfister 1995, Chapter 1, Generalization 2.3 of Theorem 2.2], which reads as follows:
Theorem. Let \( \varphi(x_1, \ldots, x_n) = \sum_{1 \leq i, j \leq n} l_{ij}(t) x_i x_j \) be an anisotropic form over \( F(t) \), where \( l_{ij}(t) \in F[t] \), and \( \deg l_{ij}(t) \leq 1 \). Suppose \( f \in F[t] \cap D(\varphi) \). Then there exist polynomials \( p_i \in F[t] \) such that \( f = \varphi(p_1, \ldots, p_n) \).

In the following statement, using the theorem above, we get a criterion for existence of rational points for the hypersurface \( S \) in the case of a quartic polynomial \( f \).

Proposition 2.1. Let \( F \) be a field, \( a_1, \ldots, a_n \in F^* \), and \( u_1, u_2, u_3 \in F \). Suppose that the form \( \psi \simeq \langle 1, -a_1, \ldots, -a_n \rangle \) is anisotropic. Then the following two conditions are equivalent:

1. \( -u_3^2 x^3 + u_2 x^2 + u_1 x + 1 \in D(\psi \perp \langle -x \rangle) \), i.e., the form \( \psi \perp \langle -x, u_2 x^3 - u_2 x^2 - u_1 x - 1 \rangle \) is isotropic over \( F(x) \).

2. The affine hypersurface \( S \) determined by the equation

\[
y^4 + 2u_1 y^2 - 8u_3 y + u_1^2 - 4u_2 = \sum_{i=1}^{n} a_i x_i^2
\]

has a rational point.

Moreover, if, in contrast the form \( \psi \) is isotropic, and \( u_3 \neq 0 \), then both conditions necessarily hold. If the form \( \psi \) is isotropic, and \( u_3 = 0 \), then condition (1) necessarily holds, but in general condition (2) does not.

Proof. (1) \( \implies \) (2): Obviously, the form \( \psi \perp \langle -x \rangle \) is anisotropic. By the strong form of the Cassels–Pfister theorem

\[
-u_3^2 x^3 + u_2 x^2 + u_1 x + 1 \in D(\psi \perp \langle -x \rangle)
\]

if and only if

\[
-u_3^2 x^3 + u_2 x^2 + u_1 x + 1 = p_0^2 - a_1 p_1^2 - \cdots - a_n p_n^2 - x p_{n+1}^2
\]

for some \( p_i \in F[x] \). Since the form \( \psi \) is anisotropic, we get \( p_i(x) = \alpha_i x + \beta_i \) for each \( i \), where \( \alpha_i, \beta_i \in F \). Moreover, \( \alpha_{n+1}^2 = u_3^2 \); hence we may assume that \( \alpha_{n+1} = u_3 \). Therefore, \( \alpha_i, \beta_i \) satisfy the equations

\[
\begin{pmatrix}
\alpha_0^2 - a_1 \alpha_1^2 - \cdots - a_n \alpha_n^2 - 2u_3 \beta_{n+1} &= u_2, \\
2\alpha_0 \beta_0 - 2a_1 \alpha_1 \beta_1 - \cdots - 2a_n \alpha_n \beta_n - \beta_{n+1}^2 &= u_1, \\
\beta_0^2 - a_1 \beta_1^2 - \cdots - a_n \beta_n^2 &= 1.
\end{pmatrix}
\]

Let \( \mathbf{u} = (\alpha_0, \alpha_1, \ldots, \alpha_n) \) and \( \mathbf{v} = (\beta_0, \beta_1, \ldots, \beta_n) \). Obviously, the system (*) is equivalent to the system

\[
\begin{pmatrix}
\psi(\mathbf{u}) &= u_2 + 2u_3 \beta_{n+1}, \\
\psi(\mathbf{u}, \mathbf{v}) &= \frac{1}{2}(u_1 + \beta_{n+1}^2), \\
\psi(\mathbf{v}) &= 1.
\end{pmatrix}
\]
If the vectors \( u \) and \( v \) are linearly dependent, then the system (***) implies

\[
\det \left( \begin{array}{cc}
u_2 + 2\alpha_{n+1}\beta_{n+1} & \frac{1}{2}(u_1 + \beta_{n+1}^2) \\
\frac{1}{2}(u_1 + \beta_{n+1}^2) & 1
\end{array} \right) = u_2 + 2u_3\beta_{n+1} - \frac{1}{4}(u_1 + \beta_{n+1}^2)^2 = 0.
\]

Hence \( S \) has a rational point \( x_i = 0, \ y = \beta_{n+1} \).

If the vectors \( u \) and \( v \) are linearly independent, then the 2-dimensional form \( \tau \) with the matrix

\[
\begin{pmatrix}
u_2 + 2u_3\beta_{n+1} & \frac{1}{2}(u_1 + \beta_{n+1}^2) \\
\frac{1}{2}(u_1 + \beta_{n+1}^2) & 1
\end{pmatrix}
\]

is a subform of \( \psi \) with the underlying linear space generated by the vectors \( u \) and \( v \).

Obviously,

\[
\tau \simeq \{1, u_2 + 2\alpha_{n+1}\beta_{n+1} - \frac{1}{4}(u_1 + \beta_{n+1}^2)^2\}.
\]

Therefore,

\[
-u_2 - 2u_3\beta_{n+1} + \frac{1}{4}(u_1 + \beta_{n+1}^2)^2 \in D(\langle a_1, \ldots, a_n \rangle),
\]

which means that \((u_1 + \beta_{n+1}^2)^2 - 8u_3\beta_{n+1} - 4u_2 = \sum_{i=1}^n a_ix_i^2 \) for some \( x_i \in F \), and we are done.

(2) \( \Rightarrow \) (1): Assume that \( S \) has a rational point, say, \( y = \beta_{n+1}, \ x_i = c_i \). If \( c_1 = \cdots = c_n = 0, \) then \( u_2 + 2\alpha_{n+1}\beta_{n+1} - \frac{1}{4}(u_1 + \beta_{n+1}^2)^2 = 0. \) Put

\[
\begin{cases}
\alpha_1 = \cdots = \alpha_n = 0, \\
\alpha_0 = \frac{1}{2}(u_1 + \beta_{n+1}^2), \\
\beta_0 = 1, \\
\beta_1 = \cdots = \beta_n = 0.
\end{cases}
\]

Since the elements \( \alpha_i, \beta_i \) satisfy the system (**), we get \( -u_2 \beta_{n+1} + \frac{1}{4}(u_1 + \beta_{n+1}^2)^2 \in D(\langle a_1, \ldots, a_n \rangle), \)

If at least one of \( c_i \) is not zero, then, since the form \( \langle a_1, \ldots, a_n \rangle \) is anisotropic,

\[
-\det \left( \begin{array}{cc}
u_2 + 2u_3\beta_{n+1} & \frac{1}{2}(u_1 + \beta_{n+1}^2) \\
\frac{1}{2}(u_1 + \beta_{n+1}^2) & 1
\end{array} \right) \in D(\langle a_1, \ldots, a_n \rangle),
\]

or, equivalently, the form with the matrix

\[
\begin{pmatrix}
u_2 + 2u_3\beta_{n+1} & \frac{1}{2}(u_1 + \beta_{n+1}^2) \\
\frac{1}{2}(u_1 + \beta_{n+1}^2) & 1
\end{pmatrix}
\]

is a subform of the form \( \psi \). In other words, there are linearly independent vectors \( u = (\alpha_0, \alpha_1, \ldots, \alpha_n), \ v = (\beta_0, \beta_1, \ldots, \beta_n) \) such that the system (**) holds. Hence in this case we have \( -u_2 \beta_{n+1} + \frac{1}{4}(u_1 + \beta_{n+1}^2)^2 \in D(\psi \perp \langle -x \rangle) \) as well.
If the form \( \psi \) is isotropic, then, obviously,
\[ -u_3^2x^3 + u_2x^2 + u_1x + 1 \in D(\psi) \subset D(\psi \perp \langle -x \rangle). \]
Assume \( u_3 \neq 0 \). We may suppose \( a_1 = 1 \), and put \( y = -u_2/(2u_3) \), \( x_1 = u_2^2/(4u_3^2) + u_1 \), \( x_2 = \cdots = x_n = 0 \).

Finally, if \( \psi \) is isotropic, and \( u_3 = 0 \), then Remark 1.2 shows that in general \( S \) does not always have a rational point. \( \square \)

In the following example we show how Proposition 2.1 can be applied to construct elements from \( \text{Br}(F(S)/F) \) in the case \( n = 1 \).

**Example 2.2.** Let \( n = 1 \) and \( a \in F^* \setminus F^{*2} \). Proposition 2.1 claims that
\[ -u_3^2x^3 + u_2x^2 + u_1x + 1 \in D(\langle 1, -a, -x \rangle) \]
if and only if the equation
\[ az^2 = y^4 + 2u_1y^2 - 8u_3y + u_1^2 - 4u_2 \]
has a solution over \( F \), or, equivalently, multiplying by \( 4a \), and setting \( t = 2az \), if and only if the equation
\[ t^2 = 4ay^4 + 8au_1y^2 - 32au_3y + 4a(u_1^2 - 4u_2) \]
has a solution over \( F \). Let us set
\[ a_4 = 4a, \quad a_2 = 8au_1, \quad a_1 = -32au_3, \quad a_0 = 4a(u_1^2 - 4u_2) \]
(here the meaning of the elements \( a_i \) is different from the previous one). Hence we get that the equation \( t^2 = a_4y^4 + a_2y^2 + a_1y + a_0 \) has a solution over \( F \) if and only if
\[ -\left( \frac{a_1}{32a} \right)^2 x^3 + \frac{4a(a_2/(8a))^2 - a_0}{16a} x^2 + \frac{a_2}{8a} x + 1 \in D(\langle 1, -a, -x \rangle). \]

A straightforward computation shows that the last condition is equivalent to
\[ z^3 - 2a_2z^2 + (a_2^2 - 4a_0a_4)z + a_1^2a_4 \in D(\langle z, a_4, -a_4z \rangle), \]
where \( z = -4a_4/x \), which means that \( (a_4, z)_{F(\sqrt{g(z)})} = 0 \), where
\[ g(z) = z^3 - 2a_2z^2 + (a_2^2 - 4a_0a_4)z + a_1^2a_4. \]

This is the result of [Haile and Han 2007, Propositions 5 and 17], originally obtained by means of algebraic geometry and cohomology groups.

Further, if \( (a_4, z)_{F(\sqrt{g(z)})} = 0 \), by the evaluating argument we get \( (a, e) = 0 \) if \( g(e) \in F^{*2} \) and \( e \neq 0 \). Therefore, \( (a, e) \in \text{Br}(F(S)/F) \) for each \( e \in F^* \) such that \( g(e) \in F^{*2} \).
Note also that
\[ z^3 - 2a_2z^2 + (a_2^2 - 4a_0a_4)z + a_1^2a_4 = -4 \det \begin{pmatrix}
    a_4 & 0 & \frac{1}{2}(a_2 - z) \\
    0 & z & \frac{1}{2}a_1 \\
    \frac{1}{2}(a_2 - z) & \frac{1}{2}a_1 & a_0
\end{pmatrix}. \]

Later, in Proposition 3.1 we will see why this determinant is involved here.

**Example 2.3.** Suppose \( S \) has the equation \((y^2 - b)^2 = \sum_{i=1}^n a_i x_i^2\), where \( b \in F^* \). Then, since the form \( \psi \simeq \langle 1, -a_1, \ldots, -a_n \rangle \) is anisotropic, it is easy to see that the surface \( S \) has a rational point if and only if \( b \in F^{*2} \). On the other hand, in this case \( u_1 = -b, u_2 = u_3 = 0 \). Hence Proposition 2.1 claims that \( S \) has a rational point if and only if the form \( \langle 1, -a_1, \ldots, -a_n, -x, bx - 1 \rangle \) is isotropic. By Brumer’s theorem [1978] this is the case if and only if the forms \( \langle 1, -a_1, \ldots, -a_n, 0, -1 \rangle \) and \( \langle 0, 0, \ldots, 0, -1, b \rangle \) have a common nontrivial zero. It is easy to verify independently that this is equivalent to \( b \in F^{*2} \).

In the algebraic theory of quadratic forms over fields, there are many results concerning splitting of forms by the function field of a quadric. In the following statements (Corollaries 2.4–2.7) we consider the similar questions for the hypersurface \( S \) from Proposition 2.1. In particular, we assume that \( a_1, \ldots, a_n \in F^* \), and the form \( \langle 1, -a_1, \ldots, -a_n \rangle \) is anisotropic.

Let \( W(k) \) be the Witt group of a field \( k \). It is well known, see, for example, [Scharlau 1985], that the sequence of abelian groups
\[ 0 \to W(k) \xrightarrow{p \to k_p} W(k(t)) \xrightarrow{\prod_p} \bigoplus_{p \in \mathbb{A}_k^1} W(k_p) \to 0 \]
is split exact. We consider here a point \( p \in \mathbb{A}_k^1 \) as a monic irreducible polynomial over \( k \). We denote by \( k_p = k[t]/p \) the corresponding residue field and by \( \partial_p : W(k(t)) \to W(k_p) \) the residue homomorphism well defined by the rule
\[ \partial_p((f)) = \begin{cases} 0 & \text{if } v_p(f) = 0, \\ (fp^{-1}) & \text{if } v_p(f) = 1. \end{cases} \]

There is a splitting map \( W(k(t)) \to W(k) \) defined by the rule \( \langle f \rangle \to \langle l(f) \rangle \), where \( l(f) \) is the leading coefficient of the polynomial \( f \in k[t] \).

**Corollary 2.4.** In the notation of Proposition 2.1, assume that the hypersurface \( S \) has no \( F \)-rational point, \( n = 1 \), and \( \varphi \) is a 3-dimensional form over \( F \). Then \( S \) has no \( F(\varphi) \)-rational point.

**Proof.** Let \( \pi \) be the 2-fold Pfister form corresponding to \( \varphi \). We may assume that \( \pi \neq 0 \). Suppose that \( S \) determined by the equation
\[ az^2 = y^4 + 2u_1y^2 - 8u_3y + u_1^2 - 4u_2 \]
has an \( F(\varphi) \)-rational point. In view of Example 2.2 we have \( \langle a, z \rangle_{F(\sqrt{g(z)})} = 0 \), where \( g(z) = z^3 - 2a_2z^2 + (a_2^2 - 4a_0a_4)z + a_1^2a_4 \). Then, since \( S \) has no rational point, i.e., \( \langle a, z \rangle_{F(\sqrt{g(z)})} \neq 0 \), we get \( \langle a, z \rangle = \langle g(z) \rangle + \tau \) for some \( \tau \in W(F(z)) \). Therefore,

\[
0 = l(\langle a, z \rangle) - l(\langle g(z) \rangle) + l(\tau) = l(\tau) = \pi,
\]
a contradiction. \( \square \)

**Corollary 2.5.** Assume \( S \) has no \( F \)-rational point, \( n = 2 \), and \( \varphi \) is a 4-dimensional anisotropic form over \( F \), disc \( \varphi = d \neq 1 \). The following conditions are equivalent:

1. \( S \) has an \( F(\varphi) \)-rational point.
2. \( u_3^2x^3 - u_2x^2 - u_1x - 1 = h(x)q(x) \), where \( h, q \in F[x] \), \( \deg h \leq 1 \), \( \deg q = 2 \), \( q \) is monic and irreducible, \(-a_1a_2\bar{x} \in F_q^{*2} \), disc \( q = d \), and \( \varphi \) is similar to the form \( \langle 1, -a_1, -a_2, a_1a_2d \rangle \).

**Proof.** (2) \( \Rightarrow \) (1): Consider first the case \( u_3 \neq 0 \). Since \(-a_1a_2\bar{x} \in F_q^{*2} \), we have \( N_{F_q/F}(\bar{x}) = F_q^{*2} \), hence \( q(x) = x^2 + cx + b^2 \) for some \( c, b \in F, b \neq 0 \). Therefore, \( h(x) = u_3^2x - b^{-2} \), in particular, \( h \in D(\langle -1, x \rangle) \). Hence \( u_3^2x^3 - u_2x^2 - u_1x - 1 = h(x)q(x) \in D(\langle -q, qx \rangle) \). It follows that

\[
\langle 1, -a_1, -a_2, -x, h q \rangle \subset \{ 1, -a_1, -a_2, -x, -q, qx \} \subset \{ 1, -a_1, -a_2, -x \} \langle q \rangle.
\]

On the other hand,

\[
\langle 1, -a_1, -a_2, -x \rangle \langle q \rangle = \langle a_1, a_2, q \rangle + \langle -a_1a_2, -x \rangle \langle q \rangle = \langle a_1, a_2, q \rangle.
\]

as \(-a_1a_2\bar{x} \in F_q^{*2} \).

Finally, \( \varphi \langle q \rangle \sim \langle 1, -a_1, -a_2, a_1a_2d \rangle \langle q \rangle = \langle a_1, a_2, q \rangle \), since \( \langle d, q \rangle = 0 \). We conclude that \( \langle a_1, a_2, q \rangle_{F(x)(\varphi)} = 0 \). In view of (2-1) and (2-2), the form 

\[
\langle 1, -a_1, -a_2, -x, h q \rangle_{F(x)(\varphi)} = \langle 1, -a_1, -a_2, -x, u_3^2x^3 - u_2x^2 - u_1x - 1 \rangle_{F(x)(\varphi)}
\]

is isotropic, which implies by Proposition 2.1 that \( S \) has an \( F(\varphi) \)-rational point.

The case \( u_3 = 0 \) is similar. In this case

\[
-u_2x^2 - u_1x - 1 = -u_2 q = -u_2(x^2 + cx + b^2);
\]

hence \( u_2 \in F_q^{*2} \), and obviously,

\[
\langle 1, -a_1, -a_2, -x, -u_2 q \rangle \subset \{ 1, -a_1, -a_2, -x \} \langle q \rangle.
\]

Now we can finish the proof as in the case \( u_3 \neq 0 \).

(1) \( \Rightarrow \) (2): Assume that \( S \) has an \( F(\varphi) \)-rational point. Then by Proposition 2.1 the form \( \Phi \simeq \langle 1, -a_1, -a_2, -x, u_3^2x^3 - u_2x^2 - u_1x - 1 \rangle \) is anisotropic over \( F(x) \), but isotropic over \( F(x)(\varphi) \). Consider two possible cases:
Case (a): \( \text{ind}(\Phi) = 4 \). Then by [Hoffmann 1995] there exists a squarefree \( p \in F[x] \) such that \( p \varphi \subset \Phi \). Comparing the determinants we get

\[
(2-3) \quad \Phi \simeq p \varphi \perp (-a_1 a_2 \text{disc}(\varphi)x(u_3^2 x^3 - u_2 x^2 - u_1 x - 1)).
\]

Note that \( p \) is not divisible by \( x \), for otherwise (2-3) would imply \( \dim \partial_s(\Phi) \geq 3 \), a contradiction. Comparing the residues at \( x \) of the left-hand and the right-hand parts of (2-3), we get \( a_1 a_2 \text{disc}(\varphi) = -1 \); hence

\[
(2-4) \quad \Phi \simeq p \varphi \perp (x(u_3^2 x^3 - u_2 x^2 - u_1 x - 1)).
\]

Applying the “leading coefficient” homomorphism \( l : W(F(x)) \to W(F) \) to both sides of (2-4), we get

\[
\langle 1, -a_1, -a_2, -1, 1 \rangle \simeq l(p) \varphi \perp \langle 1 \rangle
\]

if \( u_3 \neq 0 \), or

\[
\langle 1, -a_1, -a_2, -1, -u_2 \rangle \simeq l(p) \varphi \perp (-u_2)
\]

if \( u_3 = 0 \) (if \( u_3 = 0 \), then it easily follows that \( u_2 \neq 0 \)). Hence in any case \( l(p) \varphi \simeq \langle 1, -a_1, -a_2, -1 \rangle \), so \( \varphi \) is isotropic, a contradiction.

Case (b): \( \text{ind}(\Phi) = 2 \). Then \( \Phi \) is a Pfister neighbor of some anisotropic 3-fold Pfister form \( \pi \) over \( F(x) \), say, \( \pi \simeq \Phi \perp \sigma \). Since \( \pi \perp \sigma \) is isotropic (or, equivalently, hyperbolic), \( \pi \simeq \langle a_1, a_2, P \rangle \) for some squarefree \( P \in F[x] \). We claim that \( P \) does not have any odd-degree irreducible divisor \( p \). Indeed, otherwise, taking into account that \( \pi \simeq \varphi \langle h(x) \rangle \) for some \( h(x) \in F[x] \) [Wadsworth 1975], we get that \( \langle a_1, a_2 \rangle_{F_p} = \partial_p(\langle a_1, a_2, P \rangle) \) either equals 0 or is similar to \( \varphi_{F_p} \). But since \( \langle 1, -a_1, -a_2 \rangle \) is anisotropic, \( \text{disc}(\varphi) \neq 1 \), and \( \deg p \) is odd, both cases are impossible.

Furthermore, if \( s \neq x \) is a monic irreducible divisor of \( P \), which is not a divisor of \( u_3^2 x^3 - u_2 x^2 - u_1 x - 1 \), then

\[
\langle a_1, a_2 \rangle \sim \partial_s(\pi) = \partial_s(\Phi) + \partial_s(\sigma) = \partial_s(\sigma).
\]

Since \( \dim \partial_s(\sigma) \leq 3 \), we get \( \langle a_1, a_2 \rangle_{F_s} = 0 \); hence \( \langle a_1, a_2, s \rangle = 0 \), and so we can replace \( P \) by \( P/s \).

Thus, we may assume that \( P \) divides \( u_3^2 x^3 - u_2 x^2 - u_1 x - 1 \), and \( P \) is an irreducible quadratic polynomial. Therefore, \( u_3^2 x^3 - u_2 x^2 - u_1 x - 1 = hq \), where \( h, q \in F[x] \), \( \deg h \leq 1 \) (deg \( h = 0 \) if and only if \( u_3 = 0 \)), \( \deg q = 2 \), and \( q \) is monic irreducible. Obviously, \( P = \lambda q \) for some \( \lambda \in F^* \). We have \( \dim(l(\Phi)) \leq 3 \); hence \( \dim(l(\pi)) \leq 3 + \dim(l(\sigma)) \leq 6 \), which implies that \( \dim(l(\pi)) = 0 \). Therefore, we can replace \( P \) by \( q \), so \( \pi \simeq \langle a_1, a_2 \rangle \). In particular, \( \langle a_1, a_2 \rangle_{F_q} \neq 0 \). Since \( \langle 1, -a_1, -a_2, -x \rangle \) is a subform of \( \pi \), we have \( \langle 1, -a_1, -a_2, -x \rangle \langle R \rangle \simeq \langle a_1, a_2, q \rangle \).
for some squarefree $R \in F[x]$ [Wadsworth 1975]. In other words,

\begin{equation}
\langle\langle a_1, a_2, q \rangle\rangle = \langle\langle a_1, a_2, R \rangle\rangle,
\langle\langle -a_1a_2x, R \rangle\rangle = 0.
\end{equation}

From the first equality of (2-5) we get that $q$ divides $R$, since $\partial_q(\langle\langle a_1, a_2, q \rangle\rangle) = \langle\langle a_1, a_2 \rangle\rangle_{F_q} \neq 0$. Therefore,

$$\bar{1} = \partial_q(\langle\langle -a_1a_2x, R \rangle\rangle) = -\overline{a_1a_2x} \in F_q^*/F_q^{*2}.$$  

Hence $N_{F_q/F}(x) \in F^*^2$, and $q(x) = x^2 + cx + b^2$ for some $c, b \in F$, $b \neq 0$. Further, since $\langle\langle a_1, a_2, q \rangle\rangle_{F(x)(\varphi)} = 0$, we have $\langle\langle a_1, a_2, q \rangle\rangle \sim \varphi \langle\langle T \rangle\rangle$ for some $T \in F[x]$. This implies that $q$ divides $T$, and $\varphi_{F_q} \sim \langle\langle a_1, a_2 \rangle\rangle_{F_q}$, i.e., $\varphi_{F(\sqrt{\text{disc}q})} \sim \langle\langle a_1, a_2 \rangle\rangle_{F(\sqrt{\text{disc}q})}$. Therefore, $\text{disc} \varphi = \text{disc} q = d$. Finally, by [Wadsworth 1975] we get that $\varphi \sim \langle 1, -a_1, -a_2, a_1a_2d \rangle$. The verification of the implication (1) $\implies$ (2) is done. \qed

**Corollary 2.6.** Assume $S$ has no $F$-rational point, $n = 2$, and $\varphi$ is a 5-dimensional anisotropic form over $F$. Then $S$ has no $F(\varphi)$-rational point.

**Proof.** Let $\sigma \subset \varphi$ be a 4-dimensional subform of $\varphi_{F(t)}$, which does not satisfy condition (2) in Corollary 2.5 (with replacement of the ground field $F$ by $F(t)$). Then $S$ has no $F(t)(\sigma)$-rational points; hence $S$ has no $F(t)(\varphi)$-rational points. \qed

Recall that $u$-invariant of the field $k$ is the maximum of dimensions of anisotropic forms over $k$.

**Corollary 2.7.** In the notation of Proposition 2.1, assume that the hypersurface $S$ has no $F$-rational point:

1. If $n = 1$, then there exists a field extension $L/F$ such that $S_L$ has no rational point, $L$ does not have an odd-degree field extension, and $u(L) = 2$. In particular, $\text{cd}_L 2 = 1$.

2. If $n = 2$, then there exists a field extension $L/F$ such that $S_L$ has no rational point, $L$ does not have an odd-degree field extension, and $u(L) = 4$. In particular, $\text{cd}_L 2 = 2$.

**Proof.** (1) By Proposition 1.1 and Corollary 2.4 the field $L$ can be constructed by subsequent splitting of all 2-fold Pfister forms and passing to a maximal odd-degree extension; see, for instance, [Elman et al. 2008, Theorem 38.4]. Clearly, $u(L) = 2$.

(2) Similar to (1), the field $L$ can be constructed by subsequent splitting of all 5-dimensional forms and passing to a maximal odd-degree extension. \qed

**Corollary 2.8.** In the notation of Proposition 2.1, the following conditions are equivalent:

1. The polynomial $f(y) = y^4 + 2u_1y^2 - 8u_3y + u_1^2 - 4u_2$ has a root in $F$.

2. Let $p(x)$ be any monic polynomial divisor of $g(x) = -u_3^2x^3 + u_2x^2 + u_1x + 1$ such that $v_p(-u_3^2x^3 + u_2x^2 + u_1x + 1)$ is odd. Then $\bar{x}$ is a square in the field $F_p$. 

Proof. (1) $\implies$ (2): Since the polynomial $f(y)$ has a root $\alpha$ in $F$, the affine curve $f(y) = tx^2$ has a rational point, namely $(0, \alpha)$, over the Laurent series field $F((t))$. Hence by Proposition 2.1 the form $(1, -t, -x, -g(x))$ is isotropic, which implies that the form $(1, -x, -g(x))$ is isotropic as well. This means that the Pfister form $\langle x, g(x) \rangle$ is trivial. Then $\bar{x}$ is a square in the field $F_p$.

(2) $\implies$ (1): In view of the exact sequence for $W(F(x))$ the Pfister form $\langle x, g(x) \rangle$ is trivial; hence the form $(1, -x, -g(x))$ is isotropic. By Proposition 2.1 the affine curve $f(y) = tx^2$ has a rational point over $F((t))$, say $(x_0, y_0)$. Suppose $f$ has no root in $F$. Let $v$ be the discrete $F$-valuation on $F((t))$ such that $v(t) = 1$. Obviously, $v(tx_0^2)$ is odd, but $v(y_0^4 + 2u_1y_0^2 - 8u_3y_0 + u_1^2 - 4u_2)$ is even, a contradiction. \hfill $\Box$

3. On the Witt kernel $W(F(C)/F)$ for the plane curve $C$ with the equation $y^2 = a_4x^4 + a_2x^2 + a_1x + a_0$

If $C$ is a nonsingular algebraic curve over the field $F$ with a rational point $p \in C$ and the function field $F(C)$, then the composition of the restriction map $W(F) \to W(F(C))$ and the first residue map $\partial_p : W(F(C)) \to W(F)$ is the identity; hence $W(F(C)/F) = 0$. More generally, applying Springer’s theorem, it is easy to see that $W(F(C)/F) = 0$ if $C$ has a point of odd degree. In the opposite case the computation of $W(F(C)/F)$ can hardly be done in general. In this section we describe all Pfister forms from the ideal $W(F(C)/F)$, with $C$ being the affine plane curve determined by the equation $y^2 = f(x)$, where $f(x) = a_4x^4 + a_2x^2 + a_1x + a_0 \in F[x]$ is a squarefree quartic polynomial, $a_4 \neq 0$. Obviously, the last equation is equivalent to the equation $y^4 + 2u_1y^2 - 8u_3y + u_1^2 - 4u_2 = ax^2$, $a \neq 0$, under an invertible change of the coefficients. As a consequence, we compute $W(F(C)/F)$ if the $u$-invariant of the field $F$ is at most 10.

The description of 2-fold Pfister forms in $W(F(C)/F)$, or, equivalently quaternion algebras in $Br(F(C)/F)$, was made in [Shick 1994; Haile and Han 2007] correspondingly. The proof of Proposition 3.1 below is, in fact, very similar to that in [Shick 1994, Theorem 9], but we give it here for the sake of completeness, and because we need it in Proposition 3.2.

Let $e \in F$. Set

$$M = \begin{pmatrix} a_4 & 0 & \frac{1}{2}(a_2 - e) \\ 0 & e & \frac{1}{2}a_1 \\ \frac{1}{2}(a_2 - e) & \frac{1}{2}a_1 & a_0 \end{pmatrix},$$

and $d(e) = -\det(M)$.

Proposition 3.1. Assume that $0 \neq Q \in Br(F(C)/F)$. Then either $Q = (a_4, e)$, where $e \neq 0$, $d(e) \in F^{\times 2} \cup \{0\}$, or $a_1 = 0$ and $Q = (a_4, a_2^2 - 4a_0a_4)$. Conversely, any quaternion algebra of the types above belongs to $Br(F(C)/F)$. 

Proof. Let \( 0 \neq Q \in \Br(F(C)/F) \), let \( \pi \) be the 2-fold Pfister form corresponding to \( Q \), and let \( -\varphi \) be the pure subform of \( \pi \), i.e., \( \pi \simeq (1) \perp -\varphi \). Let \( V \) be the underlying vector space of \( \varphi \). Assume that \( Q_{F(C)} = 0 \). Since \( \varphi \) is anisotropic, by the Cassels–Pfister theorem there exist \( v_0, v_1, v_2 \in V \) such that \( \varphi(x^2v_2 + xv_1 + v_0) = f(x) \).

Comparing the coefficients on the left-hand and the right-hand sides of the last equality, we get the system

\[
\begin{align*}
\varphi(v_2, v_2) &= a_4, \\
\varphi(v_1, v_2) &= 0, \\
\varphi(v_1, v_1) + 2\varphi(v_0, v_2) &= a_2, \\
\varphi(v_0, v_1) &= \frac{1}{2}a_1, \\
\varphi(v_0, v_0) &= a_0.
\end{align*}
\]

(\( \ast \))

If \( d(e) \neq 0 \), then \( M \) is the matrix of \( \varphi \) with respect to the basis \( (v_2, v_1, v_0) \), and so \( d(e) \in F^* \).

If \( e \neq 0 \), then \( \langle a_4, e \rangle \) is a regular subform of the form \( \varphi_{(v_0, v_1, v_2)} \). Since \( \det \varphi = -1 \), we get \( \varphi \simeq \langle a_4, e, -a_4e \rangle \), which implies \( \pi \simeq \langle \langle a_4 \rangle \rangle \) and \( Q = (a_4, e) \). If \( e = 0 \), then

\[
M = \begin{pmatrix}
a_4 & 0 & \frac{1}{2}a_2 \\
0 & 0 & \frac{1}{2}a_1 \\
\frac{1}{2}a_2 & \frac{1}{2}a_1 & a_0
\end{pmatrix}.
\]

If additionally \( a_1 \neq 0 \), then

\[
H = \begin{pmatrix}
0 & \frac{1}{2}a_1 \\
\frac{1}{2}a_1 & a_0
\end{pmatrix}
\]

is a regular subform of \( \varphi \); hence \( Q = 0 \), a contradiction. If \( e = a_1 = 0 \), then, since \( f \) is squarefree, \( a_2^2 - 4a_0a_4 \neq 0 \). Hence

\[
\begin{pmatrix}
a_4 & \frac{1}{2}a_2 \\
\frac{1}{2}a_2 & a_0
\end{pmatrix}
\]

is a regular subform of \( \varphi \), so \( \varphi \simeq \langle a_4, -a_4(a_2^2 - 4a_0a_4), a_2^2 - 4a_0a_4 \rangle, \pi \simeq \langle \langle a_4, a_2^2 - 4a_0a_4 \rangle \rangle, \) and \( Q = (a_4, a_2^2 - 4a_0a_4) \).

Conversely, assume that \( d(e) \in F^* \), \( e \neq 0 \). Consider the form \( \varphi \) with the matrix

\[
M = \begin{pmatrix}
a_4 & 0 & \frac{1}{2}(a_2 - e) \\
0 & e & \frac{1}{2}a_1 \\
\frac{1}{2}(a_2 - e) & \frac{1}{2}a_1 & a_0
\end{pmatrix}
\]

with respect to a certain basis \( v_2, v_1, v_0 \). Then \( \varphi \simeq \langle a_4, e, -a_4e \rangle \). Hence system (\( \ast \)) implies

\[
f = \varphi(x^2v_2 + xv_1 + v_0) \in D(\varphi) = D(\langle a_4, e, -a_4e \rangle),
\]
so \((a_4, e)_{F(C)} = 0\). Assume now that \(d(e) = 0, e \neq 0\). Then \(\varphi\) is degenerate, and \(\langle a_4, e \rangle\) is a regular subform of \(\varphi\); hence \(\varphi \simeq \langle a_4, e \rangle\). Therefore, \(f \in D(\langle a_4, e, 0 \rangle) = D(\langle a_4, e \rangle)\), so again \((a_4, e)_{F(C)} = 0\).

Finally, if \(a_1 = 0\), then

\[
\begin{align*}
f(x) = a_4x^4 &+ a_2x^2 + a_0 = a_4\left(x^2 + \frac{a_2}{2a_4}\right)^2 + \left(a_0 - \frac{a_2^2}{4a_4}\right) \\
&\in D\left(\langle a_4, a_0 - \frac{a_2^2}{4a_4}\rangle\right),
\end{align*}
\]

so \((a_4, a_2^2 - 4a_0a_4)_{F(C)} = (a_4, a_0 - \frac{a_2^2}{4a_4})_{F(C)} = 0.\]

Let \(n \geq 3\). Let \(P_n(f)\) be the set of \(n\)-fold Pfister forms \(\pi\) over \(F\) such that \(\pi_{F(C)} = 0\), where \(f\) and \(C\) are as in Proposition 3.1. We say that \(\pi \in P_n(f)\) is standard if \(\rho \subset \pi\) for some \(\rho \in P_2(f)\). Otherwise we say that \(\pi \in P_n(f)\) is nonstandard.

**Proposition 3.2.** Assume \(n \geq 3\), \(\pi \in P_n(f)\) is nonstandard, and \(d(e)\) has the same meaning as in Proposition 3.1. Then there exist \(e, r \in F, e \neq 0, r^2 - d(e) \neq 0\), such that \(\langle a_4, e, r^2 - d(e) \rangle \subset \pi\). Moreover, \(\langle a_4, e, r^2 - d(e) \rangle \in P_3(f)\) for any \(e, r \in F, e \neq 0, r^2 - d(e) \neq 0\).

**Proof.** Assume that \(\pi \in P_n(f)\), or, equivalently, \(f \in D(-\pi')\). Then the proof of Proposition 3.1 shows there is some \(e \in F\) such that one of the following cases holds:

1. \(e \neq 0, d(e) \neq 0\),

\[
\begin{pmatrix}
a_4 & 0 & \frac{1}{2}(a_2 - e) \\
0 & e & \frac{1}{2}a_1 \\
\frac{1}{2}(a_2 - e) & \frac{1}{2}a_1 & a_0
\end{pmatrix} \subset -\pi',
\]

where \(\pi'\) is the pure subform of \(\pi\).

2. \(e \neq 0, d(e) = 0, \langle a_4, e \rangle \subset -\pi'\).

3. \(a_1 = 0, \)

\[
\begin{pmatrix}
a_4 & \frac{1}{2}a_2 \\
\frac{1}{2}a_2 & a_0
\end{pmatrix} \subset -\pi'.
\]

In the second case, \(\langle a_4, e, e \rangle \subset \pi\) and \(d(e) = 0\). In the third case \(\langle a_4, a_2^2, 4a_0a_4 \rangle \subset \pi\). In both cases, \(\pi\) is standard.

In the first case, \(\langle a_4, e, -a_4ed(e) \rangle \subset -\pi'\). Set \(\tau : (1, -a_4, -e, a_4ed(e))\). Hence \(\tau \subset \pi\), which implies \(\pi_{F(\tau)} = 0\). By [Fitzgerald 1983, Corollary 1.5] there is a 3-fold Pfister form \(\rho\) such that \(\tau \subset \rho \subset \pi\). In particular, by [Wadsworth 1975] there is \(s \in F^\ast\) such that \(\rho \simeq \tau \otimes \langle s \rangle\). Since \(\rho \in \text{I}^3(F)\), we have \(\langle d(e), s \rangle = 0\); i.e., \(\langle s \rangle \simeq \langle r^2 - d(e) \rangle\) for some \(r \in F\). Therefore, \(\rho \simeq \langle a_4, e, r^2 - d(e) \rangle\).

Conversely, let \(\delta \simeq \langle a_4, e, r^2 - d(e) \rangle \neq 0\) for some \(e, r \in F, e \neq 0, r^2 - d(e) \neq 0\). In particular, \(d(e) \neq 0\). Then \(\delta \simeq \tau \otimes \langle r^2 - d(e) \rangle\), where \(\tau \simeq \langle 1, -a_4, -e, a_4ed(e) \rangle\).
as earlier. The form \( \langle a_4, e, -a_4ed(e) \rangle \subset -\delta' \) is isomorphic to the form \( \varphi \) with the matrix

\[
M_{\varphi} = \begin{pmatrix}
a_4 & 0 & \frac{1}{2}(a_2 - e) \\
0 & e & \frac{1}{2}a_1 \\
\frac{1}{2}(a_2 - e) & \frac{1}{2}a_1 & a_0
\end{pmatrix}
\]

with respect to a certain basis \( v_2, v_1, v_0 \), which implies that \( f = \varphi(x^2v_2 + xv_1 + v_0) \in D(-\delta') \). Therefore, \( \delta_{F(C)} = 0 \), and we are done. Certainly, \( \delta \) is not necessarily nonstandard. \( \square \)

**Corollary 3.3.** Let \( \pi \in P_n(f) \), \( n \geq 3 \). Then there are \( s_1, \ldots, s_{n-3} \in F^* \) and \( \rho \in P_3(f) \) such that \( \pi \simeq \rho \otimes \langle s_1, \ldots, s_{n-3} \rangle \).

**Proof.** This follows at once from the definition of standard Pfister forms and Proposition 3.2. \( \square \)

If the \( u \)-invariant of \( F \) is small enough, then one can give a complete description of the ideal \( W(F(C)/F) \).

**Proposition 3.4.** Let \( F \) be a field with \( u(F) \leq 10 \) (for instance, \( F \) is the function field of a \( 3 \)-dimensional variety over an algebraically closed field). Then any element of \( W(F(C)/F) \) is a sum of an element from \( P_2(f) \) and an element from \( P_3(f) \).

**Proof.** Let \( \varphi \in W(F(C)/F) \). Since \( \text{disc}(\varphi)_{F(x)(\sqrt{f(x)})} = 1 \), \( a_4 \neq 0 \), and \( f(x) = a_4x^4 + a_2x^2 + a_1x + a_0 \) is squarefree, we have \( \text{disc}(\varphi) = 1 \). Since \( C(\varphi)_{F(x)(\sqrt{f(x)})} = 0 \), we get that \( C(\varphi) \) is a quaternion. Let \( \pi \in P_2(f) \) be a 2-fold Pfister form associated with \( C(\varphi) \). If \( \pi = 0 \), then \( C(\varphi) = 0 \). Since \( \text{dim}(\varphi) \leq 10 \), a result of Pfister implies that \( \varphi \in I^3(F) \) [Scharlau 1985, Chapter 2, Theorem 14.4] (also this follows from Merkurjev’s theorem, but we do not need this profound result here). Since \( u(F) \leq 10 \), it follows that \( \varphi \) is a 3-fold Pfister form [Lam 2005, Chapter XII, Proposition 2.8].

If \( \pi \neq 0 \), then similarly \( \varphi - \pi \in I^3(F) \); hence \( \varphi = \pi + (\varphi - \pi) \) is a sum of a 2-fold Pfister form and a 3-fold one from \( W(F(C)/F) \). \( \square \)

**Open Question.** Is the ideal \( W(F(C)/F) \) generated by 2-fold and 3-fold Pfister forms in general?

A natural question arises as to whether nonstandard Pfister forms exist. The following statement shows that this is really the case.

**Proposition 3.5.** Let \( f(x) = a_4x^4 + a_2x^2 + a_1x + a_0 \) be a squarefree polynomial over a field \( k \). Let \( C \) be the curve with the equation \( y^2 = f(x) \). The following conditions are equivalent:

1. The curve \( C \) has no rational point over \( k \).
2. There exists a field extension \( F/k \) with a nonstandard 3-fold Pfister form over \( F \) for the curve \( C_F \).
(3) There exist a field extension $K/k$ such that $\mathcal{d}_2K = 1$, and a nonstandard 3-fold Pfister form over the rational function field $F = K(u, v)$ for the curve $C_F$. Moreover, in this example $\text{Br}(F(C)/F) = 0$.

**Proof.** (2) $\Rightarrow$ (1): This is obvious, since if $C$ had a $k$-rational point, then $W(F(C)/F)$ would be trivial for any field extension $F/k$.

(3) $\Rightarrow$ (2): This is also obvious.

(1) $\Rightarrow$ (3): In view of Corollary 2.7, there is a field extension $K/k$ such that $\mathcal{d}_2K = 1$ and $C$ has no $K$-rational point. Set $F = K(u, v)$ and consider the Pfister form $\pi \simeq \langle a_4, u, v^2 - d(u) \rangle \in W(F(C)/F)$. Since $C$ has no $K$-rational point, we get by Example 2.2 that $\partial_{a_4^2 - d(u)}(\pi) = \langle a_4, u \rangle \neq 0$. Therefore, $\pi \neq 0$. Now to check that $\pi$ is nonstandard, it suffices to show that $\text{Br}(F(C)/F) = 0$. Since $2\text{Br}(K) = 0$, this is a direct consequence of the following:

**Lemma 3.6.** The restriction map $\text{Br}(L(C)/L) \rightarrow \text{Br}(L(u)(C)/L(u))$ is an isomorphism for any field extension $L/k$.

**Proof.** Obviously, the map in question is injective. By Proposition 3.1 any element of $\text{Br}(L(u)(C)/L(u))$ equals $(a_4, p(u))$ for some $p \in L[u]$. Let $q$ be a prime divisor of $p$. We have

$$\bar{a}_4 = \partial_q(a_4, p) \in \ker(L_q^*/L_q^{*2}) \rightarrow L_q^*(C)/L_q^*(C)^{*2}.$$  

Since $f(x) = a_4x^4 + a_2x^2 + a_1x + a_0$ is squarefree, $a_4 \in L_q^{*2}$; that is, $\partial_q(a_4, p) = \bar{1}$. Therefore, $(a_4, p) \in \text{Br}(L(C)/L)$, so the lemma is proven, which completes also the proof of the implication (1) $\Rightarrow$ (3). \qed

**References**


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