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Alexander S. Sivatski

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ON RATIONAL POINTS OF CERTAIN AFFINE HYPERSURFACES

ALEXANDER S. SIVATSKI

Let F be a field with char $F \neq 2$, let $a_1, \dots, a_n \in F^*$, and let $f \in F[y]$ be a monic polynomial of degree $2m$. Let further S be an affine hypersurface over F determined by the equation $f(y) = \sum_{i=1}^n a_i x_i^2$. In the first part of the paper we prove a certain version of Springer's theorem. Namely, we show that if the form $\psi \simeq \langle 1, -a_1, \dots, -a_n \rangle$ is anisotropic and S has an L-rational point for some odd-degree extension L/F, then S has an L-rational point for some odd-degree extension L/F with $[L : F] \leq m$, and the last inequality is strict in general.

In the second part we consider the case where the polynomial f is quartic. We show that the surface S has a rational point if and only if the quadratic form $\psi \perp \langle -x, g(x) \rangle$ is isotropic over $F(x)$, where $g(x) \in F[x]$ is a certain polynomial of degree at most 3, whose coefficients are expressed in a polynomial way via the coefficients of f.

In the third part we describe all Pfister forms that belong to the Witt kernel $W(F(C)/F)$, where C is the plane nonsingular curve determined by the equation $y^2 = a_4x^4 + a_2x^2 + a_1x + a_0$. In the case where the u-invariant of F is at most 10, we describe generators of the ideal $W(F(C)/F)$.

Introduction

Let F be a field of characteristic different from 2. We investigate some properties of the affine hypersurface S determined by the equation $f(y) = \sum_{i=1}^n a_i x_i^2$, where $a_i \in F^*$ and f is a monic polynomial of degree $2m$. In Section 1, we prove a version of Springer's theorem for S (Proposition 1.1). In particular, we show that if $m = 2$ (i.e., the polynomial f is quartic), and S has a K-rational point for some odd-degree extension K/F, then S has an F-rational point. Sections 2 and 3 can be considered as generalizations of some results in [Haile and Han 2007; Shick 1994]. Namely, for the affine hyperelliptic curve C with the equation $f(y) = ax^2$ over a field F, where $a \in F^* \setminus F^{*2}$ and $f(y)$ is a quartic polynomial, two questions have been investigated in [Haile and Han 2007]. First, it has been shown that existence

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of a rational point on C is equivalent to triviality of a certain quaternion algebra over a certain quadratic extension of the rational function field $F(x)$. It is easy to see that this is equivalent to isotropicity of some 4-dimensional quadratic form over $F(x)$. In Proposition 2.1 we obtain a similar criterion for the affine hypersurface $S : f(y) = \sum_{i=1}^{n} a_i x_i^2$, where $a_i \in F^*$, the form $\langle 1, -a_1, \dots, -a_n\rangle$ is anisotropic, and f is a monic quartic polynomial. This proves independently of Section 1 that existence of a rational point over any odd-degree field extension K/F implies existence of a rational point of S over the field F itself.

Another result in [Haile and Han 2007; Shick 1994] is a computation of the relative Brauer group $\mathrm{Br}(F(C)/F)$, where C is the affine hyperelliptic curve above. Obviously, this is equivalent to description of all 2-fold Pfister forms π over F such that $\pi_{F(C)} = 0$. Section 3 is devoted to investigation of the Witt kernel $W(F(C)/F)$. Applying an invertible change of variables, we may assume that the curve C is determined by the equation $y^2 = a_4x^4 + a_2x^2 + a_1x + a_0$, where $a_i \in F$, $a_4 \neq 0$. We will also assume that C is nonsingular, for the opposite case is trivial. Let $e \in F$. Set

$$d(e) = -\det \begin{pmatrix} a_4 & 0 & \frac{1}{2}(a_2 - e) \\ 0 & e & \frac{1}{2}a_1 \\ \frac{1}{2}(a_2 - e) & \frac{1}{2}a_1 & a_0 \end{pmatrix}.$$

In Proposition 3.1 we show that if $0 \neq Q \in \mathrm{Br}(F(C)/F)$, then either $Q = (a_4, e)$, where $e \neq 0$, $d(e) \in F^{*2} \cup \{0\}$, or $a_1 = 0$ and $Q = (a_4, a_2^2 - 4a_0a_4)$. Conversely, any quaternion algebra of the types above belongs to $\mathrm{Br}(F(C)/F)$.

Proposition 3.1 is not new, but we give it for the convenience of the reader, and because we need its proof a bit later in Proposition 3.2. In fact, the original proof of Proposition 3.1, which is very similar to ours, is given in [Shick 1994]. However, in Proposition 3.2 and Corollary 3.3 we describe *all* Pfister forms π (not necessarily 2-fold) over F such that $\pi_{F(C)} = 0$. More precisely, if $\pi_{F(C)} = 0$, then either π is divisible by a 2-fold Pfister form ρ such that $\rho_{F(C)} = 0$, or there exist $e, r \in F$, $e \neq 0$, $r^2 - d(e) \neq 0$ such that $\langle\langle a_4, e, r^2 - d(e)\rangle\rangle \subset \pi$. Conversely, $\langle\langle a_4, e, r^2 - d(e)\rangle\rangle \in W(F(C)/F)$ for any $e, r \in F$, $e \neq 0$, $r^2 - d(e) \neq 0$. If the u-invariant of F is at most 10, this is sufficient for the computation of the Witt kernel $W(F(C)/F)$.

A few words about the notation. Throughout all the fields have characteristic different from 2. By a form we always mean a quadratic form over a field. For $a_1, \dots a_n \in F^*$ we denote the Pfister form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ as $\langle\langle a_1, \dots, a_n\rangle\rangle$ (take notice of signs!), and $D(\varphi)$ is the set of all nonzero values of the form φ. If the form φ is considered as an element of the Witt ring $W(F)$, then $\dim \varphi$ denotes the dimension of the anisotropic part of φ.

If φ is a regular form over the field F, $\dim \varphi \geq 3$, then by $F(\varphi)$ we denote the function field of the corresponding projective quadric.

Slightly abusing notation, we often identify a form with its symmetric matrix.

1. A version of Springer's theorem

The well-known Springer's theorem claims that if K/F is an odd-degree field extension, and a projective quadric X has a rational point over K, then it has a rational point over F. Below we give an affine version of this theorem for certain hypersurfaces.

Proposition 1.1. *Let F be a field, let $a_1, \dots, a_n \in F^*$, and let $f \in F[y]$ be a monic polynomial of degree $2m$. Let $S = S(f, a_1, \dots, a_n)$ be the affine hypersurface over F determined by the equation $f(y) = \sum_{i=1}^n a_i x_i^2$. Suppose that S has a K-rational point for some odd-degree extension K/F.*

(1) *If the form $\langle a_1, \dots, a_n\rangle$ is anisotropic, then S has an L-rational point for some odd-degree extension L/F with $[L : F] \le 2m - 1$.*
(2) *If the form $\langle 1, -a_1, \dots, -a_n\rangle$ is anisotropic, then S has an L-rational point for some odd-degree extension L/F with $[L : F] \le m$, and the last inequality is strict in general. In particular, if $m = 2$, i.e., f is a quartic polynomial, then S has an F-rational point.*
(3) *If the form $\langle a_1, \dots, a_n\rangle$ is isotropic, then S has an F-rational point.*

Proof. (1)–(2) Assume the form $\langle a_1, \dots, a_n\rangle$ is anisotropic, and K/F is an odd-degree field extension. Suppose $[K : F] \ge s$, where $s = m + 1$ if the form $\langle 1, -a_1, \dots, -a_n\rangle$ is anisotropic, and $s = 2m+1$ otherwise. Let $f(\alpha) = \sum_{i=1}^n a_i \beta_i^2$ for some $\alpha, \beta_i \in K$. It suffices to find an odd-degree field extension L/F with $[L : F] < [K : F]$ such that S has a rational L-point. Since $[K : F(\alpha)]$ is odd, we get by Springer's theorem, applied to the extension $K/F(\alpha)$, that the form $\langle a_1, \dots, a_n, -f(\alpha)\rangle$ is isotropic over $F(\alpha)$. Hence we may assume that $\beta_i \in F(\alpha)$ for each i. We may assume also that $[F(\alpha) : F] \ge s$, for otherwise there is nothing to be proved. Let g be the minimal polynomial of α. In particular, $\deg g = [F(\alpha) : F] \ge s$. Let $\beta_i = p_i(\alpha)$, where $p_i \in F[x]$, $\deg p_i \le \deg g - 1$. Also $\deg f = 2m \le 2(s - 1) \le 2(\deg g - 1)$. We have

$$\sum_{i=1}^n a_i p_i^2 - f = gh \quad \text{for some } h \in F[x], \quad \text{and} \quad \deg\Bigl(\sum_{i=1}^n a_i p_i^2 - f\Bigr) \le 2(\deg g - 1).$$

If $\deg\bigl(\sum_{i=1}^n a_i p_i^2 - f\bigr)$ is even, then $\deg h$ is odd, and

$$\deg h \le 2(\deg g - 1) - \deg g = \deg g - 2 = [F(\alpha) : F] - 2 \le [K : F] - 2.$$

Hence S has an L-rational point, where $L = F[x]/p(x)$, and p is an arbitrary odd-degree prime divisor of h. Moreover, $[L : F] < [K : F]$.

If $\deg\bigl(\sum_{i=1}^n a_i p_i^2 - f\bigr)$ is odd, or $h = 0$, then, since f is monic of even degree, the form $\langle 1, -a_1, \dots, -a_n\rangle$ is isotropic. Hence $s = 2m + 1$, and $\deg\bigl(\sum_{i=1}^n a_i p_i^2\bigr) = \deg f = 2m$. Therefore, in this case $h = 0$, and so S has an F-rational point.

Now let us show that in the inequality $[L : F] \le m$ in the second part of Proposition 1.1, the number m cannot be replaced by a smaller number, provided we consider all fields F and all odd-degree extensions K/F. Consider two cases:

Case (a): m is odd. Let F be a field such that there exists an irreducible polynomial p of degree m over F. Consider the equation

$$p(y)^2 = \sum_{i=1}^{n} a_i x_i^2.$$

Clearly, it has a solution over the field $K = F[y]/p(y)$ with $x_1 = \cdots = x_n = 0$. Suppose that L/F is an odd-degree extension, $\alpha, \beta_i \in L$, and $p(\alpha)^2 = \sum_{i=1}^{n} a_i \beta_i^2$. Since the form $\langle 1, -a_1, \dots, -a_n\rangle$ is anisotropic, we get by Springer's theorem applied to the odd-degree extension K/F that $p(\alpha) = \beta_1 = \cdots = \beta_n = 0$. Hence $m = \deg p = [F(\alpha) : F] \le [L : F]$.

Case (b): m is even. Let k be a field, let $F = k((t))$ be the Laurent series field, and let the hypersurface S be determined by the equation $(y^{m-1}+t)(y^{m+1}+t) = \sum_{i=1}^{n} a_i x_i^2$. Let L/F be an odd-degree extension, $[L : F] \le m-3$. Obviously, the field L is complete with respect to a discrete valuation v such that $1 \le v(t) \le m-3$. It is easy to show that $(\alpha^{m-1}+t)(\alpha^{m+1}+t) \in L^{*2}$ for any $\alpha \in L$. Therefore, by Springer's theorem

$$(\alpha^{m-1}+t)(\alpha^{m+1}+t) \ne \sum_{i=1}^{n} a_i \beta_i^2 \quad \text{for any } \beta_i \in L.$$

(3) This is obvious, since any element of F is a value of the form $\langle a_1, \dots, a_n\rangle$. $\square$

Remark 1.2. The hypothesis that the form $\langle 1, -a_1, \dots, -a_n\rangle$ is anisotropic is essential in the second part of Proposition 1.1, at least for $m = 2$. Indeed, consider the equation $y^4 + 2 = x^2$ over $\mathbb{Q}$. Let $L = F(\delta)$, where δ is a root of the irreducible polynomial $p(u) = 2u^3 - u^2 + 2$. Obviously, $x = \delta^2 - \delta,\ y = \delta$ is a solution of the equation in question over L.

Let us prove now that this equation has no solution over $\mathbb{Q}$. It suffices to show that if $x, y, z \in \mathbb{Z}$, and $y^4 + 2z^4 = x^2$, then $z = 0$. Assume the contrary, so we may suppose that $y^4 + 2z^4 = x^2,\ z > 0$ and z is as small as possible. In particular, y and z are coprime; hence y is odd. Over $\mathbb{Q}(\sqrt{-2})$ we have $(y^2 + z^2\sqrt{-2})(y^2 - z^2\sqrt{-2}) = x^2$, and it is easy to see that the numbers $y^2 + z^2\sqrt{-2}$ and $y^2 - z^2\sqrt{-2}$ are coprime in the Euclidean ring $\mathbb{Z}[\sqrt{-2}]$. Since the group of units of the ring $\mathbb{Z}[\sqrt{-2}]$ consists of 1 and -1, we get that $y^2 + z^2\sqrt{-2} = \pm(u + v\sqrt{-2})^2$ for some $u, v \in \mathbb{Z},\ v > 0$. If $y^2 + z^2\sqrt{-2} = -(u + v\sqrt{-2})^2$, then $y^2 = 2v^2 - u^2,\ z^2 = -2uv$. The equality $y^2 = 2v^2 - u^2$ implies that u and v are odd. But then, clearly, the equality $z^2 = -2uv$ is impossible.

Thus $y^2 + z^2\sqrt{-2} = (u + v\sqrt{-2})^2$, which means that $y^2 = u^2 - 2v^2,\ z^2 = 2uv$. In particular, u is odd. Since $(u - y)(u + y) = 2v^2$, and the numbers $\frac{1}{2}(u - y)$,

$\frac{1}{2}(u+y)$ are, obviously, coprime, we may assume, changing if needed the sign of y, that $\frac{1}{2}(u-y)=t^2$, $\frac{1}{2}(u+y)=2s^2$ for some coprime $s, t>0$. Therefore, we have

$$\begin{cases} u=2s^2+t^2, \\ y=2s^2-t^2, \\ v=2st; \end{cases}$$

hence $z^2=2uv=4st(2s^2+t^2)$, and so $s=\alpha^2$, $t=\beta^2$, $2s^2+t^2=\gamma^2$, which implies $\beta^4+2\alpha^4=\gamma^2$ for some positive integers α, β, γ. Moreover, obviously,

$$0<\alpha=\sqrt{s}<\sqrt{v}<z,$$

a contradiction to the minimality of z.

In fact, there are similar counterexamples for any characteristic. Namely, let k be a field, t indeterminate, and $F=k(t)$. By an argument similar to the one for the equation $y^4+2=x^2$ over $\mathbb{Q}$, one can easily show that the equation $y^4-t=x^2$ has no solution in F. On the other hand, $x=\alpha^2-\alpha$, $y=\alpha$ is a solution of the same equation over the field $F(\alpha)$, where α is a root of the polynomial $p(u)=2u^3-u^2-t$.

However, we do not know if there exists a counterexample for each finite field, and for each number field.

Proposition 1.3. *Let F be a field, $a_1, \dots, a_n \in F^*$, and the form $\langle 1, -a_1, \dots, -a_n\rangle$ be isotropic. Let further $f \in F[y]$ be a monic polynomial of degree $2m$, where m is not divisible by* char F. *Then the hypersurface $S=S(f, a_1, \dots, a_n)$ has an L-rational point for some odd-degree field extension L/F with $[L:F]\le 2m-1$.*

Proof. Since $1 \in D(\langle a_1, \dots, a_n\rangle)$, we may assume that $n=1$ and $a_1=1$. Replacing if needed y by $y+c$, where $c\in F^*$, we may assume that the coefficient a at y^{2m-1} of the polynomial $f(y)$ is nonzero. Then setting $x=z+y^m$, one can see that the equation $f(y)=x^2$ is equivalent to the equation $ay^{2m-1}+\sum_{i=0}^{2m-2} p_i(z)y^i=0$, where $p_i(z)\in F[z]$. It is clear that the last equation has a required point. □

Remark 1.4. We do not know whether Proposition 1.3 remains valid if m is divisible by char F.

Another natural question is whether the inequality $[L:F]\le 2m-1$ in the first part of Proposition 1.1 is strict for each m. In view of Remark 1.2 it is strict for $m=2$.

2. A criterion for existence of rational points for certain affine hypersurfaces

We give a criterion in the language of quadratic forms for the existence of a rational point for the hypersurface S in the case where $m=2$ (the polynomial f is quartic) and the form $\langle 1, -a_1, \dots, -a_n\rangle$ is anisotropic. The main ingredient in the sequel is the strong form of the Cassels–Pfister theorem [Pfister 1995, Chapter 1, Generalization 2.3 of Theorem 2.2], which reads as follows:

Theorem. *Let $\varphi(x_1, \dots, x_n) = \sum_{1\le i,j\le n} l_{ij}(t)x_i x_j$ be an anisotropic form over $F(t)$, where $l_{ij}(t) \in F[t]$, and $\deg l_{ij}(t) \le 1$. Suppose $f \in F[t] \cap D(\varphi)$. Then there exist polynomials $p_i \in F[t]$ such that $f = \varphi(p_1, \dots, p_n)$.*

In the following statement, using the theorem above, we get a criterion for existence of rational points for the hypersurface S in the case of a quartic polynomial f.

Proposition 2.1. *Let F be a field, $a_1, \dots, a_n \in F^*$, and $u_1, u_2, u_3 \in F$. Suppose that the form $\psi \simeq \langle 1, -a_1, \dots, -a_n\rangle$ is anisotropic. Then the following two conditions are equivalent*:

(1) $-u_3^2x^3 + u_2x^2 + u_1x + 1 \in D(\psi \perp \langle -x\rangle)$, *i.e., the form*

$$\psi \perp \langle -x, u_3^2x^3 - u_2x^2 - u_1x - 1\rangle$$

is isotropic over $F(x)$.

(2) *The affine hypersurface S determined by the equation*

$$y^4 + 2u_1y^2 - 8u_3y + u_1^2 - 4u_2 = \sum_{i=1}^n a_i x_i^2$$

has a rational point.

Moreover, if, in contrast the form ψ is isotropic, and $u_3 \ne 0$, then both conditions necessarily hold. If the form ψ is isotropic, and $u_3 = 0$, then condition (1) *necessarily holds, but in general condition* (2) *does not.*

Proof. (1) $\Longrightarrow$ (2): Obviously, the form $\psi \perp \langle -x\rangle$ is anisotropic. By the strong form of the Cassels–Pfister theorem

$$-u_3^2x^3 + u_2x^2 + u_1x + 1 \in D(\psi \perp \langle -x\rangle)$$

if and only if

$$-u_3^2x^3 + u_2x^2 + u_1x + 1 = p_0^2 - a_1p_1^2 - \cdots - a_np_n^2 - xp_{n+1}^2$$

for some $p_i \in F[x]$. Since the form ψ is anisotropic, we get $p_i(x) = \alpha_i x + \beta_i$ for each i, where $\alpha_i, \beta_i \in F$. Moreover, $\alpha_{n+1}^2 = u_3^2$; hence we may assume that $\alpha_{n+1} = u_3$. Therefore, α_i, β_i satisfy the equations

$$(*) \qquad \begin{cases} \alpha_0^2 - a_1\alpha_1^2 - \cdots - a_n\alpha_n^2 - 2u_3\beta_{n+1} = u_2, \\ 2\alpha_0\beta_0 - 2a_1\alpha_1\beta_1 - \cdots - 2a_n\alpha_n\beta_n - \beta_{n+1}^2 = u_1, \\ \beta_0^2 - a_1\beta_1^2 - \cdots - a_n\beta_n^2 = 1. \end{cases}$$

Let $\boldsymbol{u} = (\alpha_0, \alpha_1, \dots, \alpha_n)$ and $\boldsymbol{v} = (\beta_0, \beta_1, \dots, \beta_n)$. Obviously, the system $(*)$ is equivalent to the system

$$(**) \qquad \begin{cases} \psi(\boldsymbol{u}) = u_2 + 2u_3\beta_{n+1}, \\ \psi(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{2}(u_1 + \beta_{n+1}^2), \\ \psi(\boldsymbol{v}) = 1. \end{cases}$$

If the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly dependent, then the system (∗∗) implies

$$\det\begin{pmatrix} u_2+2\alpha_{n+1}\beta_{n+1} & \frac{1}{2}(u_1+\beta_{n+1}^2) \\ \frac{1}{2}(u_1+\beta_{n+1}^2) & 1 \end{pmatrix} = u_2+2u_3\beta_{n+1}-\tfrac{1}{4}(u_1+\beta_{n+1}^2)^2=0.$$

Hence S has a rational point $x_i=0,\ y=\beta_{n+1}$.

If the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent, then the 2-dimensional form τ with the matrix

$$\begin{pmatrix} u_2+2u_3\beta_{n+1} & \frac{1}{2}(u_1+\beta_{n+1}^2) \\ \frac{1}{2}(u_1+\beta_{n+1}^2) & 1 \end{pmatrix}$$

is a subform of ψ with the underlying linear space generated by the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Obviously,

$$\tau \simeq \big\langle 1,\, u_2+2\alpha_{n+1}\beta_{n+1}-\tfrac{1}{4}(u_1+\beta_{n+1}^2)^2\big\rangle.$$

Therefore,

$$-u_2-2u_3\beta_{n+1}+\tfrac{1}{4}(u_1+\beta_{n+1}^2)^2 \in D(\langle a_1,\dots,a_n\rangle),$$

which means that $(u_1+\beta_{n+1}^2)^2-8u_3\beta_{n+1}-4u_2=\sum_{i=1}^n a_i x_i^2$ for some $x_i\in F$, and we are done.

(2) $\Longrightarrow$ (1): Assume that S has a rational point, say, $y=\beta_{n+1},\ x_i=c_i$. If $c_1=\dots=c_n=0$, then $u_2+2\alpha_{n+1}\beta_{n+1}-\frac{1}{4}(u_1+\beta_{n+1}^2)^2=0$. Put

$$\begin{cases} \alpha_1=\dots=\alpha_n=0, \\ \alpha_0=\frac{1}{2}(u_1+\beta_{n+1}^2), \\ \beta_0=1, \\ \beta_1=\dots=\beta_n=0. \end{cases}$$

Since the elements α_i, β_i satisfy the system (∗), we get $-u_3^2x^3+u_2x^2+u_1x+1\in D(\psi\perp\langle -x\rangle)$.

If at least one of c_i is not zero, then, since the form $\langle a_1,\dots,a_n\rangle$ is anisotropic,

$$-\det\begin{pmatrix} u_2+2u_3\beta_{n+1} & \frac{1}{2}(u_1+\beta_{n+1}^2) \\ \frac{1}{2}(u_1+\beta_{n+1}^2) & 1 \end{pmatrix} \in D(\langle a_1,\dots,a_n\rangle),$$

or, equivalently, the form with the matrix

$$\begin{pmatrix} u_2+2u_3\beta_{n+1} & \frac{1}{2}(u_1+\beta_{n+1}^2) \\ \frac{1}{2}(u_1+\beta_{n+1}^2) & 1 \end{pmatrix}$$

is a subform of the form ψ. In other words, there are linearly independent vectors $\boldsymbol{u}=(\alpha_0,\alpha_1,\dots,\alpha_n)$, $\boldsymbol{v}=(\beta_0,\beta_1,\dots,\beta_n)$ such that the system (∗∗) holds. Hence in this case we have $-u_3^2x^3+u_2x^2+u_1x+1\in D(\psi\perp\langle -x\rangle)$ as well.

If the form ψ is isotropic, then, obviously,

$$-u_3^2x^3+u_2x^2+u_1x+1\in D(\psi)\subset D(\psi\perp\langle -x\rangle).$$

Assume $u_3\neq 0$. We may suppose $a_1=1$, and put $y=-u_2/(2u_3)$, $x_1=u_2^2/(4u_3^2)+u_1$, $x_2=\cdots=x_n=0$.

Finally, if ψ is isotropic, and $u_3=0$, then Remark 1.2 shows that in general S does not always have a rational point. □

In the following example we show how Proposition 2.1 can be applied to construct elements from $\mathrm{Br}(F(S)/F)$ in the case $n=1$.

Example 2.2. Let $n=1$ and $a\in F^*\setminus F^{*2}$. Proposition 2.1 claims that

$$-u_3^2x^3+u_2x^2+u_1x+1\in D(\langle 1,-a,-x\rangle)$$

if and only if the equation

$$az^2=y^4+2u_1y^2-8u_3y+u_1^2-4u_2$$

has a solution over F, or, equivalently, multiplying by $4a$, and setting $t=2az$, if and only if the equation

$$t^2=4ay^4+8au_1y^2-32au_3y+4a(u_1^2-4u_2)$$

has a solution over F. Let us set

$$a_4=4a,\quad a_2=8au_1,\quad a_1=-32au_3,\quad a_0=4a(u_1^2-4u_2)$$

(here the meaning of the elements a_i is different from the previous one). Hence we get that the equation $t^2=a_4y^4+a_2y^2+a_1y+a_0$ has a solution over F if and only if

$$-\left(\frac{a_1}{32a}\right)^2x^3+\frac{4a(a_2/(8a))^2-a_0}{16a}x^2+\frac{a_2}{8a}x+1\in D(\langle 1,-a,-x\rangle).$$

A straightforward computation shows that the last condition is equivalent to

$$z^3-2a_2z^2+(a_2^2-4a_0a_4)z+a_1^2a_4\in D(\langle z,a_4,-a_4z\rangle),$$

where $z=-4a_4/x$, which means that $(a_4,z)_{F(z)(\sqrt{g(z)})}=0$, where

$$g(z)=z^3-2a_2z^2+(a_2^2-4a_0a_4)z+a_1^2a_4.$$

This is the result of [Haile and Han 2007, Propositions 5 and 17], originally obtained by means of algebraic geometry and cohomology groups.

Further, if $(a_4,z)_{F(z)(\sqrt{g(z)})}=0$, by the evaluating argument we get $(a,e)=0$ if $g(e)\in F^{*2}$ and $e\neq 0$. Therefore, $(a,e)\in \mathrm{Br}(F(S)/F)$ for each $e\in F^*$ such that $g(e)\in F^{*2}$.

Note also that

$$z^3-2a_2z^2+(a_2^2-4a_0a_4)z+a_1^2a_4=-4\det\begin{pmatrix} a_4 & 0 & \frac{1}{2}(a_2-z)\\ 0 & z & \frac{1}{2}a_1\\ \frac{1}{2}(a_2-z) & \frac{1}{2}a_1 & a_0\end{pmatrix}.$$

Later, in Proposition 3.1 we will see why this determinant is involved here.

Example 2.3. Suppose S has the equation $(y^2-b)^2=\sum_{i=1}^n a_ix_i^2$, where $b\in F^*$. Then, since the form $\psi\simeq\langle 1,-a_1,\dots,-a_n\rangle$ is anisotropic, it is easy to see that the surface S has a rational point if and only if $b\in F^{*2}$. On the other hand, in this case $u_1=-b$, $u_2=u_3=0$. Hence Proposition 2.1 claims that S has a rational point if and only if the form $\langle 1,-a_1,\dots,-a_n,-x,bx-1\rangle$ is isotropic. By Brumer's theorem [1978] this is the case if and only if the forms $\langle 1,-a_1,\dots,-a_n,0,-1\rangle$ and $\langle 0,0,\dots,0,-1,b\rangle$ have a common nontrivial zero. It is easy to verify independently that this is equivalent to $b\in F^{*2}$.

In the algebraic theory of quadratic forms over fields, there are many results concerning splitting of forms by the function field of a quadric. In the following statements (Corollaries 2.4–2.7) we consider the similar questions for the hypersurface S from Proposition 2.1. In particular, we assume that $a_1,\dots,a_n\in F^*$, and the form $\langle 1,-a_1,\dots,-a_n\rangle$ is anisotropic.

Let $W(k)$ be the Witt group of a field k. It is well known, see, for example, [Scharlau 1985], that the sequence of abelian groups

$$0\to W(k)\xrightarrow{\text{res}} W(k(t))\xrightarrow{\coprod\partial_p}\coprod_{p\in\mathbb{A}_k^1} W(k_p)\to 0$$

is split exact. We consider here a point $p\in\mathbb{A}_k^1$ as a monic irreducible polynomial over k. We denote by $k_p=k[t]/p$ the corresponding residue field and by $\partial_p: W(k(t))\to W(k_p)$ the residue homomorphism well defined by the rule

$$\partial_p(\langle f\rangle)=\begin{cases}0 & \text{if } v_p(f)=0,\\ \langle\overline{fp^{-1}}\rangle & \text{if } v_p(f)=1.\end{cases}$$

There is a splitting map $W(k(t))\to W(k)$ defined by the rule $\langle f\rangle\to\langle l(f)\rangle$, where $l(f)$ is the leading coefficient of the polynomial $f\in k[t]$.

Corollary 2.4. *In the notation of Proposition 2.1, assume that the hypersurface S has no F-rational point, $n=1$, and φ is a 3-dimensional form over F. Then S has no $F(\varphi)$-rational point.*

Proof. Let π be the 2-fold Pfister form corresponding to φ. We may assume that $\pi\neq 0$. Suppose that S determined by the equation

$$az^2=y^4+2u_1y^2-8u_3y+u_1^2-4u_2$$

has an $F(\varphi)$-rational point. In view of Example 2.2 we have $\langle\langle a, z\rangle\rangle_{F(\pi)(\sqrt{g(z)})} = 0$, where $g(z) = z^3 - 2a_2z^2 + (a_2^2 - 4a_0a_4)z + a_1^2a_4$. Then, since S has no rational point, i.e., $\langle\langle a, z\rangle\rangle_{F(\sqrt{g(z)})} \neq 0$, we get $\langle\langle a, z\rangle\rangle = \langle\langle g(z)\rangle\rangle\tau + \pi$ for some $\tau \in W(F(z))$. Therefore,

$$0 = l(\langle\langle a, z\rangle\rangle) - l(\langle\langle g(z)\rangle\rangle\tau) = l(\pi) = \pi,$$

a contradiction. □

Corollary 2.5. *Assume S has no F-rational point, $n = 2$, and φ is a 4-dimensional anisotropic form over F,* $\operatorname{disc}\varphi = d \neq 1$. *The following conditions are equivalent*:

(1) *S has an $F(\varphi)$-rational point.*

(2) $u_3^2x^3 - u_2x^2 - u_1x - 1 = h(x)q(x)$, *where* $h, q \in F[x]$, $\deg h \leq 1$, $\deg q = 2$, *q is monic and irreducible*, $-\overline{a_1a_2x} \in F_q^{*2}$, $\operatorname{disc} q = d$, *and φ is similar to the form* $\langle 1, -a_1, -a_2, a_1a_2d\rangle$.

Proof. (2) $\Longrightarrow$ (1): Consider first the case $u_3 \neq 0$. Since $-\overline{a_1a_2x} \in F_q^{*2}$, we have $N_{F_q/F}(\bar{x}) \in F^{*2}$; hence $q(x) = x^2 + cx + b^2$ for some $c, b \in F$, $b \neq 0$. Therefore, $h(x) = u_3^2x - b^{-2}$, in particular, $h \in D(\langle -1, x\rangle)$. Hence $u_3^2x^3 - u_2x^2 - u_1x - 1 = h(x)q(x) \in D(\langle -q, qx\rangle)$. It follows that

$$\text{(2-1)}\quad \langle 1, -a_1, -a_2, -x, hq\rangle \subset \langle 1, -a_1, -a_2, -x, -q, qx\rangle \subset \langle 1, -a_1, -a_2, -x\rangle\langle\langle q\rangle\rangle.$$

On the other hand,

$$\text{(2-2)}\quad \langle 1, -a_1, -a_2, -x\rangle\langle\langle q\rangle\rangle = \langle\langle a_1, a_2, q\rangle\rangle + \langle -a_1a_2, -x\rangle\langle\langle q\rangle\rangle = \langle\langle a_1, a_2, q\rangle\rangle,$$

as $-\overline{a_1a_2x} \in F_q^{*2}$.

Finally, $\varphi\langle\langle q\rangle\rangle \sim \langle 1, -a_1, -a_2, a_1a_2d\rangle\langle\langle q\rangle\rangle = \langle\langle a_1, a_2, q\rangle\rangle$, since $\langle\langle d, q\rangle\rangle = 0$. We conclude that $\langle\langle a_1, a_2, q\rangle\rangle_{F(x)(\varphi)} = 0$. In view of (2-1) and (2-2), the form

$$\langle 1, -a_1, -a_2, -x, hq\rangle_{F(x)(\varphi)} = \langle 1, -a_1, -a_2, -x, u_3^2x^3 - u_2x^2 - u_1x - 1\rangle_{F(x)(\varphi)}$$

is isotropic, which implies by Proposition 2.1 that S has an $F(\varphi)$-rational point.

The case $u_3 = 0$ is similar. In this case

$$-u_2x^2 - u_1x - 1 = -u_2q = -u_2(x^2 + cx + b^2);$$

hence $u_2 \in F^{*2}$, and obviously,

$$\langle 1, -a_1, -a_2, -x, -u_2q\rangle \subset \langle 1, -a_1, -a_2, -x\rangle\langle\langle q\rangle\rangle.$$

Now we can finish the proof as in the case $u_3 \neq 0$.

(1) $\Longrightarrow$ (2): Assume that S has an $F(\varphi)$-rational point. Then by Proposition 2.1 the form $\Phi \simeq \langle 1, -a_1, -a_2, -x, u_3^2x^3 - u_2x^2 - u_1x - 1\rangle$ is anisotropic over $F(x)$, but isotropic over $F(x)(\varphi)$. Consider two possible cases:

Case (a): $\operatorname{ind}(\Phi)=4$. Then by [Hoffmann 1995] there exists a squarefree $p \in F[x]$ such that $p\varphi \subset \Phi$. Comparing the determinants we get

$$\Phi \simeq p\varphi \perp \langle -a_1a_2 \operatorname{disc}(\varphi)x(u_3^2x^3 - u_2x^2 - u_1x - 1)\rangle. \tag{2-3}$$

Note that p is not divisible by x, for otherwise (2-3) would imply $\dim \partial_x(\Phi) \geq 3$, a contradiction. Comparing the residues at x of the left-hand and the right-hand parts of (2-3), we get $a_1a_2 \operatorname{disc}(\varphi) = -1$; hence

$$\Phi \simeq p\varphi \perp \langle x(u_3^2x^3 - u_2x^2 - u_1x - 1)\rangle. \tag{2-4}$$

Applying the "leading coefficient" homomorphism $l : W(F(x)) \to W(F)$ to both sides of (2-4), we get

$$\langle 1, -a_1, -a_2, -1, 1\rangle \simeq l(p)\varphi \perp \langle 1\rangle$$

if $u_3 \neq 0$, or

$$\langle 1, -a_1, -a_2, -1, -u_2\rangle \simeq l(p)\varphi \perp \langle -u_2\rangle$$

if $u_3 = 0$ (if $u_3 = 0$, then it easily follows that $u_2 \neq 0$). Hence in any case $l(p)\varphi \simeq \langle 1, -a_1, -a_2, -1\rangle$, so φ is isotropic, a contradiction.

Case (b): $\operatorname{ind}(\Phi)=2$. Then Φ is a Pfister neighbor of some anisotropic 3-fold Pfister form π over $F(x)$, say, $\pi \simeq \Phi \perp \sigma$. Since $\pi_{F(x)(\varphi)}$ is isotropic (or, equivalently, hyperbolic), $\pi \simeq \langle\langle a_1, a_2, P\rangle\rangle$ for some squarefree $P \in F[x]$. We claim that P does not have any odd-degree irreducible divisor p. Indeed, otherwise, taking into account that $\pi \simeq \varphi\langle\langle h(x)\rangle\rangle$ for some $h(x) \in F[x]$ [Wadsworth 1975], we get that $\langle\langle a_1, a_2\rangle\rangle_{F_p} = \partial_p(\langle\langle a_1, a_2, P\rangle\rangle)$ either equals 0 or is similar to φ_{F_p}. But since $\langle 1, -a_1, -a_2\rangle$ is anisotropic, $\operatorname{disc}(\varphi) \neq 1$, and $\deg p$ is odd, both cases are impossible.

Furthermore, if $s \neq x$ is a monic irreducible divisor of P, which is not a divisor of $u_3^2x^3 - u_2x^2 - u_1x - 1$, then

$$\langle\langle a_1, a_2\rangle\rangle \sim \partial_s(\pi) = \partial_s(\Phi) + \partial_s(\sigma) = \partial_s(\sigma).$$

Since $\dim \partial_s(\sigma) \leq 3$, we get $\langle\langle a_1, a_2\rangle\rangle_{F_s} = 0$; hence $\langle\langle a_1, a_2, s\rangle\rangle = 0$, and so we can replace P by P/s.

Thus, we may assume that P divides $u_3^2x^3 - u_2x^2 - u_1x - 1$, and P is an irreducible quadratic polynomial. Therefore, $u_3^2x^3 - u_2x^2 - u_1x - 1 = hq$, where $h, q \in F[x]$, $\deg h \leq 1$ ($\deg h = 0$ if and only if $u_3 = 0$), $\deg q = 2$, and q is monic irreducible. Obviously, $P = \lambda q$ for some $\lambda \in F^*$. We have $\dim l(\Phi) \leq 3$; hence $\dim l(\pi) \leq 3 + \dim l(\sigma) \leq 6$, which implies that $\dim l(\pi) = 0$. Therefore, we can replace P by q, so $\pi \simeq \langle\langle a_1, a_2, q\rangle\rangle$. In particular, $\langle\langle a_1, a_2\rangle\rangle_{F_q} \neq 0$. Since $\langle 1, -a_1, -a_2, -x\rangle$ is a subform of π, we have $\langle 1, -a_1, -a_2, -x\rangle\langle\langle R\rangle\rangle \simeq \langle\langle a_1, a_2, q\rangle\rangle$

for some squarefree $R \in F[x]$ [Wadsworth 1975]. In other words,

(2-5) $$\begin{cases} \langle\langle a_1, a_2, q\rangle\rangle = \langle\langle a_1, a_2, R\rangle\rangle, \\ \langle\langle -a_1a_2x, R\rangle\rangle = 0. \end{cases}$$

From the first equality of (2-5) we get that q divides R, since $\partial_q(\langle\langle a_1, a_2, q\rangle\rangle) = \langle\langle a_1, a_2\rangle\rangle_{F_q} \neq 0$. Therefore,

$$\bar{1} = \partial_q(\langle\langle -a_1a_2x, R\rangle\rangle) = -\overline{a_1a_2x} \in F_q^*/{F_q}^{*2}.$$

Hence $N_{F_q/F}(x) \in F^{*2}$, and $q(x) = x^2 + cx + b^2$ for some $c, b \in F$, $b \neq 0$. Further, since $\langle\langle a_1, a_2, q\rangle\rangle_{F(x)(\varphi)} = 0$, we have $\langle\langle a_1, a_2, q\rangle\rangle \sim \varphi\langle\langle T\rangle\rangle$ for some $T \in F[x]$. This implies that q divides T, and $\varphi_{F_q} \sim \langle\langle a_1, a_2\rangle\rangle_{F_q}$, i.e., $\varphi_{F(\sqrt{\operatorname{disc} q})} \sim \langle\langle a_1, a_2\rangle\rangle_{F(\sqrt{\operatorname{disc} q})}$. Therefore, $\operatorname{disc}\varphi = \operatorname{disc} q = d$. Finally, by [Wadsworth 1975] we get that $\varphi \sim \langle 1, -a_1, -a_2, a_1a_2d\rangle$. The verification of the implication (1) $\Longrightarrow$ (2) is done. □

Corollary 2.6. *Assume S has no F-rational point, $n = 2$, and φ is a 5-dimensional anisotropic form over F. Then S has no $F(\varphi)$-rational point.*

Proof. Let $\sigma \subset \varphi$ be a 4-dimensional subform of $\varphi_{F(t)}$, which does not satisfy condition (2) in Corollary 2.5 (with replacement of the ground field F by $F(t)$). Then S has no $F(t)(\sigma)$-rational points; hence S has no $F(t)(\varphi)$-rational points. □

Recall that u-invariant of the field k is the maximum of dimensions of anisotropic forms over k.

Corollary 2.7. *In the notation of Proposition 2.1, assume that the hypersurface S has no F-rational point*:

(1) *If $n = 1$, then there exists a field extension L/F such that S_L has no rational point, L does not have an odd-degree field extension, and $u(L) = 2$. In particular*, $\operatorname{cd}_2 L = 1$.

(2) *If $n = 2$, then there exists a field extension L/F such that S_L has no rational point, L does not have an odd-degree field extension, and $u(L) = 4$. In particular*, $\operatorname{cd}_2 L = 2$.

Proof. (1) By Proposition 1.1 and Corollary 2.4 the field L can be constructed by subsequent splitting of all 2-fold Pfister forms and passing to a maximal odd-degree extension; see, for instance, [Elman et al. 2008, Theorem 38.4]. Clearly, $u(L) = 2$.

(2) Similar to (1), the field L can be constructed by subsequent splitting of all 5-dimensional forms and passing to a maximal odd-degree extension. □

Corollary 2.8. *In the notation of Proposition 2.1, the following conditions are equivalent*:

(1) *The polynomial $f(y) = y^4 + 2u_1y^2 - 8u_3y + u_1^2 - 4u_2$ has a root in F.*

(2) *Let $p(x)$ be any monic polynomial divisor of $g(x) = -u_3^2x^3 + u_2x^2 + u_1x + 1$ such that $v_p(-u_3^2x^3 + u_2x^2 + u_1x + 1)$ is odd. Then $\bar{x}$ is a square in the field F_p.*

Proof. (1) $\Longrightarrow$ (2): Since the polynomial $f(y)$ has a root α in F, the affine curve $f(y)=tx^2$ has a rational point, namely $(0,\alpha)$, over the Laurent series field $F((t))$. Hence by Proposition 2.1 the form $\langle 1,-t,-x,-g(x)\rangle$ is isotropic, which implies that the form $\langle 1,-x,-g(x)\rangle$ is isotropic as well. This means that the Pfister form $\langle\langle x, g(x)\rangle\rangle$ is trivial. Then $\bar{x}$ is a square in the field F_p.

(2) $\Longrightarrow$ (1): In view of the exact sequence for $W(F(x))$ the Pfister form $\langle\langle x, g(x)\rangle\rangle$ is trivial; hence the form $\langle 1,-x,-g(x)\rangle$ is isotropic. By Proposition 2.1 the affine curve $f(y)=tx^2$ has a rational point over $F((t))$, say (x_0, y_0). Suppose f has no root in F. Let v be the discrete F-valuation on $F((t))$ such that $v(t)=1$. Obviously, $v(tx_0^2)$ is odd, but $v(y_0^4+2u_1y_0^2-8u_3y_0+u_1^2-4u_2)$ is even, a contradiction. □

3. On the Witt kernel $W(F(C)/F)$ for the plane curve C with the equation $y^2=a_4x^4+a_2x^2+a_1x+a_0$

If C is a nonsingular algebraic curve over the field F with a rational point $p\in C$ and the function field $F(C)$, then the composition of the restriction map $W(F)\to W(F(C))$ and the *first* residue map $\partial_p : W(F(C))\to W(F)$ is the identity; hence $W(F(C)/F)=0$. More generally, applying Springer's theorem, it is easy to see that $W(F(C)/F)=0$ if C has a point of odd degree. In the opposite case the computation of $W(F(C)/F)$ can hardly be done in general. In this section we describe all Pfister forms from the ideal $W(F(C)/F)$, with C being the affine plane curve determined by the equation $y^2=f(x)$, where $f(x)=a_4x^4+a_2x^2+a_1x+a_0\in F[x]$ is a squarefree quartic polynomial, $a_4\neq 0$. Obviously, the last equation is equivalent to the equation $y^4+2u_1y^2-8u_3y+u_1^2-4u_2=ax^2$, $a\neq 0$, under an invertible change of the coefficients. As a consequence, we compute $W(F(C)/F)$ if the u-invariant of the field F is at most 10.

The description of 2-fold Pfister forms in $W(F(C)/F)$, or, equivalently quaternion algebras in $\mathrm{Br}(F(C)/F)$, was made in [Shick 1994; Haile and Han 2007] correspondingly. The proof of Proposition 3.1 below, is, in fact, very similar to that in [Shick 1994, Theorem 9], but we give it here for the sake of completeness, and because we need it in Proposition 3.2.

Let $e\in F$. Set

$$M=\begin{pmatrix} a_4 & 0 & \frac{1}{2}(a_2-e)\\ 0 & e & \frac{1}{2}a_1\\ \frac{1}{2}(a_2-e) & \frac{1}{2}a_1 & a_0 \end{pmatrix},$$

and $d(e)=-\det(M)$.

Proposition 3.1. *Assume that* $0\neq Q\in \mathrm{Br}(F(C)/F)$. *Then either* $Q=(a_4,e)$, *where* $e\neq 0$, $d(e)\in F^{*2}\cup\{0\}$, *or* $a_1=0$ *and* $Q=(a_4, a_2^2-4a_0a_4)$. *Conversely, any quaternion algebra of the types above belongs to* $\mathrm{Br}(F(C)/F)$.

Proof. Let $0 \neq Q \in \mathrm{Br}(F(C)/F)$, let π be the 2-fold Pfister form corresponding to Q, and let $-\varphi$ be the pure subform of π, i.e., $\pi \simeq \langle 1\rangle \perp -\varphi$. Let V be the underlying vector space of φ. Assume that $Q_{F(C)} = 0$. Since φ is anisotropic, by the Cassels–Pfister theorem there exist $v_0, v_1, v_2 \in V$ such that $\varphi(x^2 v_2 + x v_1 + v_0) = f(x)$. Comparing the coefficients on the left-hand and the right-hand sides of the last equality, we get the system

$$(\star)\qquad \begin{cases} \varphi(v_2, v_2) = a_4, \\ \varphi(v_1, v_2) = 0, \\ \varphi(v_1, v_1) + 2\varphi(v_0, v_2) = a_2, \\ \varphi(v_0, v_1) = \frac{1}{2}a_1, \\ \varphi(v_0, v_0) = a_0. \end{cases}$$

If $d(e) \neq 0$, then M is the matrix of φ with respect to the basis (v_2, v_1, v_0), and so $d(e) \in F^{*2}$.

If $e \neq 0$, then $\langle a_4, e\rangle$ is a regular subform of the form $\varphi|_{\langle v_0, v_1, v_2\rangle}$. Since $\det\varphi = -1$, we get $\varphi \simeq \langle a_4, e, -a_4 e\rangle$, which implies $\pi \simeq \langle\langle a_4, e\rangle\rangle$ and $Q = (a_4, e)$. If $e = 0$, then

$$M = \begin{pmatrix} a_4 & 0 & \frac{1}{2}a_2 \\ 0 & 0 & \frac{1}{2}a_1 \\ \frac{1}{2}a_2 & \frac{1}{2}a_1 & a_0 \end{pmatrix}.$$

If additionally $a_1 \neq 0$, then

$$\mathbb{H} = \begin{pmatrix} 0 & \frac{1}{2}a_1 \\ \frac{1}{2}a_1 & a_0 \end{pmatrix}$$

is a regular subform of φ; hence $Q = 0$, a contradiction. If $e = a_1 = 0$, then, since f is squarefree, $a_2^2 - 4a_0a_4 \neq 0$. Hence

$$\begin{pmatrix} a_4 & \frac{1}{2}a_2 \\ \frac{1}{2}a_2 & a_0 \end{pmatrix}$$

is a regular subform of φ, so $\varphi \simeq \langle a_4, -a_4(a_2^2 - 4a_0a_4), a_2^2 - 4a_0a_4\rangle$, $\pi \simeq \langle\langle a_4, a_2^2 - 4a_0a_4\rangle\rangle$, and $Q = (a_4, a_2^2 - 4a_0a_4)$.

Conversely, assume that $d(e) \in F^{*2}$, $e \neq 0$. Consider the form φ with the matrix

$$M = \begin{pmatrix} a_4 & 0 & \frac{1}{2}(a_2 - e) \\ 0 & e & \frac{1}{2}a_1 \\ \frac{1}{2}(a_2 - e) & \frac{1}{2}a_1 & a_0 \end{pmatrix}$$

with respect to a certain basis v_2, v_1, v_0. Then $\varphi \simeq \langle a_4, e, -a_4 e\rangle$. Hence system $(\star)$ implies

$$f = \varphi(x^2 v_2 + x v_1 + v_0) \in D(\varphi) = D(\langle a_4, e, -a_4 e\rangle),$$

so $(a_4, e)_{F(C)} = 0$. Assume now that $d(e) = 0$, $e \neq 0$. Then φ is degenerate, and $\langle a_4, e\rangle$ is a regular subform of φ; hence $\varphi \simeq \langle a_4, e, 0\rangle$. Therefore, $f \in D(\langle a_4, e, 0\rangle) = D(\langle a_4, e\rangle)$, so again $(a_4, e)_{F(C)} = 0$.

Finally, if $a_1 = 0$, then

$$f(x) = a_4x^4 + a_2x^2 + a_0 = a_4\left(x^2 + \frac{a_2}{2a_4}\right)^2 + \left(a_0 - \frac{a_2^2}{4a_4}\right) \in D\left\langle a_4, a_0 - \frac{a_2^2}{4a_4}\right\rangle,$$

so $(a_4, a_2^2 - 4a_0a_4)_{F(C)} = (a_4, a_0 - a_2^2/(4a_4))_{F(C)} = 0$. □

Let $n \geq 3$. Let $P_n(f)$ be the set of n-fold Pfister forms π over F such that $\pi_{F(C)} = 0$, where f and C are as in Proposition 3.1. We say that $\pi \in P_n(f)$ is standard if $\rho \subset \pi$ for some $\rho \in P_2(f)$. Otherwise we say that $\pi \in P_n(f)$ is nonstandard.

Proposition 3.2. *Assume $n \geq 3$, $\pi \in P_n(f)$ is nonstandard, and $d(e)$ has the same meaning as in Proposition 3.1. Then there exist $e, r \in F$, $e \neq 0$, $r^2 - d(e) \neq 0$, such that $\langle\langle a_4, e, r^2 - d(e)\rangle\rangle \subset \pi$. Moreover, $\langle\langle a_4, e, r^2 - d(e)\rangle\rangle \in P_3(f)$ for any $e, r \in F$, $e \neq 0$, $r^2 - d(e) \neq 0$.*

Proof. Assume that $\pi \in P_n(f)$, or, equivalently, $f \in D(-\pi')$. Then the proof of Proposition 3.1 shows there is some $e \in F$ such that one of the following cases holds:

(1) $e \neq 0$, $d(e) \neq 0$,

$$\begin{pmatrix} a_4 & 0 & \frac{1}{2}(a_2 - e) \\ 0 & e & \frac{1}{2}a_1 \\ \frac{1}{2}(a_2 - e) & \frac{1}{2}a_1 & a_0 \end{pmatrix} \subset -\pi',$$

where π' is the pure subform of π.

(2) $e \neq 0$, $d(e) = 0$, $\langle a_4, e\rangle \subset -\pi'$.

(3) $a_1 = 0$,

$$\begin{pmatrix} a_4 & \frac{1}{2}a_2 \\ \frac{1}{2}a_2 & a_0 \end{pmatrix} \subset -\pi'.$$

In the second case, $\langle\langle a_4, e\rangle\rangle \subset \pi$ and $d(e) = 0$. In the third case $\langle\langle a_4, a_2^2 - 4a_0a_4\rangle\rangle \subset \pi$. In both cases, π is standard.

In the first case, $\langle a_4, e, -a_4ed(e)\rangle \subset -\pi'$. Set $\tau \simeq \langle 1, -a_4, -e, a_4ed(e)\rangle$. Hence $\tau \subset \pi$, which implies $\pi_{F(\tau)} = 0$. By [Fitzgerald 1983, Corollary 1.5] there is a 3-fold Pfister form ρ such that $\tau \subset \rho \subset \pi$. In particular, by [Wadsworth 1975] there is $s \in F^*$ such that $\rho \simeq \tau \otimes \langle\langle s\rangle\rangle$. Since $\rho \in I^3(F)$, we have $\langle\langle d(e), s\rangle\rangle = 0$; i.e., $\langle\langle s\rangle\rangle \simeq \langle\langle r^2 - d(e)\rangle\rangle$ for some $r \in F$. Therefore, $\rho \simeq \langle\langle a_4, e, r^2 - d(e)\rangle\rangle$.

Conversely, let $\delta \simeq \langle\langle a_4, e, r^2 - d(e)\rangle\rangle \neq 0$ for some $e, r \in F$, $e \neq 0$, $r^2 - d(e) \neq 0$. In particular, $d(e) \neq 0$. Then $\delta \simeq \tau \otimes \langle\langle r^2 - d(e)\rangle\rangle$, where $\tau \simeq \langle 1, -a_4, -e, a_4ed(e)\rangle$

as earlier. The form $\langle a_4, e, -a_4ed(e)\rangle \subset -\delta'$ is isomorphic to the form φ with the matrix

$$M_\varphi = \begin{pmatrix} a_4 & 0 & \frac{1}{2}(a_2-e) \\ 0 & e & \frac{1}{2}a_1 \\ \frac{1}{2}(a_2-e) & \frac{1}{2}a_1 & a_0 \end{pmatrix}$$

with respect to a certain basis v_2, v_1, v_0, which implies that $f = \varphi(x^2v_2 + xv_1 + v_0) \in D(-\delta')$. Therefore, $\delta_{F(C)} = 0$, and we are done. Certainly, δ is not necessarily nonstandard. □

Corollary 3.3. *Let $\pi \in P_n(f)$, $n \geq 3$. Then there are $s_1, \dots, s_{n-3} \in F^*$ and $\rho \in P_3(f)$ such that $\pi \simeq \rho \otimes \langle\langle s_1, \dots, s_{n-3}\rangle\rangle$.*

Proof. This follows at once from the definition of standard Pfister forms and Proposition 3.2. □

If the u-invariant of F is small enough, then one can give a complete description of the ideal $W(F(C)/F)$.

Proposition 3.4. *Let F be a field with $u(F) \leq 10$ (for instance, F is the function field of a 3-dimensional variety over an algebraically closed field). Then any element of $W(F(C)/F)$ is a sum of an element from $P_2(f)$ and an element from $P_3(f)$.*

Proof. Let $\varphi \in W(F(C)/F)$. Since $\operatorname{disc}(\varphi)_{F(x)(\sqrt{f(x)})} = 1$, $a_4 \neq 0$, and $f(x) = a_4x^4 + a_2x^2 + a_1x + a_0$ is squarefree, we have $\operatorname{disc}(\varphi) = 1$. Since $C(\varphi)_{F(x)(\sqrt{f(x)})} = 0$, we get that $C(\varphi)$ is a quaternion. Let $\pi \in P_2(f)$ be a 2-fold Pfister form associated with $C(\varphi)$. If $\pi = 0$, then $C(\varphi) = 0$. Since $\dim(\varphi) \leq 10$, a result of Pfister implies that $\varphi \in I^3(F)$ [Scharlau 1985, Chapter 2, Theorem 14.4] (also this follows from Merkurjev's theorem, but we do not need this profound result here). Since $u(F) \leq 10$, it follows that φ is a 3-fold Pfister form [Lam 2005, Chapter XII, Proposition 2.8].

If $\pi \neq 0$, then similarly $\varphi - \pi \in I^3(F)$; hence $\varphi = \pi + (\varphi - \pi)$ is a sum of a 2-fold Pfister form and a 3-fold one from $W(F(C)/F)$. □

Open Question. Is the ideal $W(F(C)/F)$ generated by 2-fold and 3-fold Pfister forms in general?

A natural question arises as to whether nonstandard Pfister forms exist. The following statement shows that this is really the case.

Proposition 3.5. *Let $f(x) = a_4x^4 + a_2x^2 + a_1x + a_0$ be a squarefree polynomial over a field k. Let C be the curve with the equation $y^2 = f(x)$. The following conditions are equivalent:*

(1) *The curve C has no rational point over k.*

(2) *There exists a field extension F/k with a nonstandard 3-fold Pfister form over F for the curve C_F.*

(3) *There exist a field extension* K/k *such that* $\operatorname{cd}_2 K = 1$, *and a nonstandard 3-fold Pfister form over the rational function field* $F = K(u, v)$ *for the curve* C_F. *Moreover, in this example* $\operatorname{Br}(F(C)/F) = 0$.

Proof. (2) $\Longrightarrow$ (1): This is obvious, since if C had a k-rational point, then $W(F(C)/F)$ would be trivial for any field extension F/k.

(3) $\Longrightarrow$ (2): This is also obvious.

(1) $\Longrightarrow$ (3): In view of Corollary 2.7, there is a field extension K/k such that $\operatorname{cd}_2 K = 1$ and C has no K-rational point. Set $F = K(u, v)$ and consider the Pfister form $\pi \simeq \langle\langle a_4, u, v^2 - d(u)\rangle\rangle \in W(F(C)/F)$. Since C has no K-rational point, we get by Example 2.2 that $\partial_{v^2-d(u)}(\pi) = \langle\langle a_4, u\rangle\rangle_{K(u)(\sqrt{d(u)})} \neq 0$. Therefore, $\pi \neq 0$. Now to check that π is nonstandard, it suffices to show that $\operatorname{Br}(F(C)/F) = 0$. Since ${}_2\operatorname{Br}(K) = 0$, this is a direct consequence of the following:

Lemma 3.6. *The restriction map* $\operatorname{Br}(L(C)/L) \to \operatorname{Br}(L(u)(C)/L(u))$ *is an isomorphism for any field extension* L/k.

Proof. Obviously, the map in question is injective. By Proposition 3.1 any element of $\operatorname{Br}(L(u)(C)/L(u))$ equals $(a_4, p(u))$ for some $p \in L[u]$. Let q be a prime divisor of p. We have

$$\bar{a}_4 = \partial_q(a_4, p) \in \ker(L_q^*/L_q^{*2} \to L_q(C)^*/L_q(C)^{*2}).$$

Since $f(x) = a_4x^4 + a_2x^2 + a_1x + a_0$ is squarefree, $a_4 \in L_q^{*2}$; that is, $\partial_q(a_4, p) = \bar{1}$. Therefore, $(a_4, p) \in \operatorname{Br}(L(C)/L)$, so the lemma is proven, which completes also the proof of the implication (1) $\Longrightarrow$ (3). □

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ALEXANDER S. SIVATSKI
DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DO RIO GRANDE DO NORTE
NATAL
BRAZIL
alexander.sivatski@gmail.com

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