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ADDENDUM TO A STRONG MULTIPLICITY ONE THEOREM FOR SL_2

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We remove the restrictions on the residue characteristic in the strong multiplicity one theorem proved by Chai and Zhang (2016). Similar considerations can remove the restrictions on the residue characteristic in the stability and local converse theorem for unitary groups considered by Zhang (2017).

Introduction

This is an addendum to the paper [Chai and Zhang 2016]. In this addendum, we remove the residue characteristic 2 restriction in the strong multiplicity one theorem for SL_2 [Chai and Zhang 2016, Theorem 4.8]. Following the proof of [Chai and Zhang 2016, Theorem 4.8], it suffices to remove the residue characteristic 2 restriction in the local converse theorem [Chai and Zhang 2016, Theorem 3.10]. The idea is to lift an irreducible representation of $\mathrm{SL}_2(F)$ to $\mathrm{GL}_2(F)$. Similar considerations can remove the residue characteristic 2 restrictions in the stability and local converse theorem for unitary groups considered by Zhang [2017a; 2017b; 2017c].

1. Strong multiplicity one theorem for SL_2

We start from the following stability and local converse theorem for SL_2.

Theorem 1.1. *Let F be a p-adic field and ψ be a nontrivial additive character of F, which is also viewed as a character of $N(F)$. Let π and π' be two irreducible ψ-generic representations of $\mathrm{SL}_2(F)$ with the same central character*:

(1) *There exists an integer $l = l(\pi, \pi')$ such that if η is a quasicharacter of $F^\times$ with $\mathrm{cond}(\eta) > l$, then*

$$\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi).$$

(2) *If $\gamma(s, \pi \times \eta, \psi) = \gamma(s, \pi' \times \eta, \psi)$ for all quasicharacters η, then $\pi \cong \pi'$.*

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Here the gamma factor $\gamma(s, \pi, \eta, \psi)$ is defined in [Chai and Zhang 2016, §2]. If the residue characteristic of F is not 2, Theorem 1.1 is [Chai and Zhang 2016, Theorem 3.10]. As noted in that paper, the difficulty in the proof when the residue characteristic is 2 comes from the fact that [Chai and Zhang 2016, Lemma 3.3] does not hold in that case. To remedy this, in the following, we lift a representation of $\mathrm{SL}_2(F)$ to $\mathrm{GL}_2(F)$ and give a proof of Theorem 1.1 uniformly. In the following F is p-adic field, $\mathcal{O}$ is the ring of integers of F and $\mathcal{P}$ is the maximal ideal of $\mathcal{O}$.

Before the proof, we recall some facts related to representations of SL_2 and GL_2. All of these facts can be found in [Gelbart and Knapp 1982]. Let $\tilde{\pi}$ be an irreducible smooth representation of $\mathrm{GL}_2(F)$, then $\tilde{\pi}|_{\mathrm{SL}_2(F)}$ is a finite direct sum of irreducible representations of $\mathrm{SL}_2(F)$ with multiplicity one. Here the multiplicity one statement can be found in [Adler and Prasad 2006]. Moreover, for any irreducible smooth representation π of $\mathrm{SL}_2(F)$, there exists an irreducible smooth representation $\tilde{\pi}$ of $\mathrm{GL}_2(F)$ such that π is a direct summand of $\tilde{\pi}|_{\mathrm{SL}_2(F)}$.

For a representation π of $\mathrm{SL}_2(F)$ (or $\mathrm{GL}_2(F)$), denote its central character by ω_π.

Lemma 1.2. *Let π and π' be two irreducible representations of $\mathrm{SL}_2(F)$ with the same central character. Then there exist irreducible smooth representations $\tilde{\pi}$ and $\tilde{\pi}'$ of $\mathrm{GL}_2(F)$ with the same central character such that π and π' are direct summands of $\tilde{\pi}|_{\mathrm{SL}_2(F)}$ and $\tilde{\pi}'|_{\mathrm{SL}_2(F)}$, respectively.*

Proof. Let $\tilde{\pi}$ and $\tilde{\pi}'$ be two representations of $\mathrm{GL}_2(F)$ such that π and π' are direct summands of $\tilde{\pi}|_{\mathrm{SL}_2(F)}$ and $\tilde{\pi}'|_{\mathrm{SL}_2(F)}$, respectively. For any quasicharacter χ of $F^\times$, we have $\chi \otimes \tilde{\pi}|_{\mathrm{SL}_2(F)} = \tilde{\pi}|_{\mathrm{SL}_2(F)}$. Thus π is also a direct summand of $\chi \otimes \tilde{\pi}$. Since the central character of $\chi \otimes \tilde{\pi}$ is $\chi^2 \omega_\pi$, it suffices to find a quasicharacter χ of $F^\times$ such that $\chi^2 = \omega_{\tilde{\pi}'} \omega_{\tilde{\pi}}^{-1}$. Denote $\eta = \omega_{\tilde{\pi}'} \omega_{\tilde{\pi}}^{-1}$. Since $\omega_\pi = \omega_{\pi'}$ by assumption, we get $\eta(\pm 1) = 1$. Thus there exists a character χ_1 of $F^{\times 2}$ such that $\eta = \chi_1^2$. Any extension χ of χ_1 to $F^\times$ satisfies the desired property. □

We now fix $\tilde{\pi}$ and $\tilde{\pi}'$ as in Lemma 1.2. We repeat part of the notations from [Chai and Zhang 2016, §3]. Let $\lambda \in \mathrm{Hom}_{N(F)}(\pi, \psi)$ be a nonzero ψ-Whittaker functional of π. Since π is a direct summand of $\tilde{\pi}$, λ can also be viewed as a ψ-Whittaker functional of $\tilde{\pi}$. Let v be a vector in the space V_π. We consider the Whittaker function

$$W_v(g) = \lambda(\tilde{\pi}(g)v), \quad g \in \mathrm{GL}_2(F).$$

We fix a vector $v \in V_\pi$ such that $W_v(1) = 1$. Similarly, we consider a Whittaker functional λ' of π' and fix a vector $v' \in V_{\pi'}$ such that $W_{v'}(1) = 1$. Let $C = C(v, v')$ be a positive integer such that v and v' are fixed by K_C under the action of $\tilde{\pi}$ and $\tilde{\pi}'$, respectively, where $K_C = I_2 + \mathrm{Mat}_{2\times 2}(\mathcal{P}^C)$ is the standard congruence subgroup of $\mathrm{GL}_2(F)$ with level C. Recall that we have defined Howe vectors $v_m, v'_m, m \geq C$ associated with v and v' [Chai and Zhang 2016, (3-1)].

Denote by B the upper triangular Borel subgroup of $\mathrm{GL}_2(F)$.

Lemma 1.3. *For $m \geq C$, we have*

$$W_{v_m}(b) = W_{v'_m}(b), \quad \forall b \in B.$$

Proof. This is proved in [Baruch 1995, Corollary 6.2.11]. We give a sketch here. Since $\tilde{\pi}$ and $\tilde{\pi}'$ are both ψ-generic and have the same central character by Lemma 1.2, it suffices to show that $W_{v_m}(\mathrm{diag}(a, 1)) = W_{v'_m}(\mathrm{diag}(a, 1))$ for all $a \in F^\times$. As in the proof of [Chai and Zhang 2016, Corollary 3.4], we can check that $W_{v_m}(\mathrm{diag}(a, 1)) = 1$ if $a \in 1 + \mathcal{P}^m$ and $W_{v_m}(\mathrm{diag}(a, 1)) = 0$ if $a \notin 1 + \mathcal{P}^m$. The same is true for $W_{v'_m}$. The assertion follows. □

After Lemma 1.3, all of the proof in [Chai and Zhang 2016, §3C] goes through. Part (1) of Theorem 1.1 is proved in this way. From the proof of [Chai and Zhang 2016, Theorem 3.10], if we assume the condition $\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi)$ for all quasicharacter η of $F^\times$, we can obtain that

$$W_{v_m}(h) = W_{v'_m}(h), \quad \forall h \in \mathrm{SL}_2(F).$$

Since $\tilde{\pi}(h)v_m = \pi(h)v_m$ and $\tilde{\pi}'(h)v'_m = \tilde{\pi}'(h)v'_m$, we get

$$\lambda(\pi(h)v_m) = \lambda'(\pi'(h)v'_m), \quad \forall h \in \mathrm{SL}_2(F).$$

Then by the uniqueness of the Whittaker models, we get $\pi \cong \pi'$. This proves Theorem 1.1.

From the proof of the strong multiplicity one theorem given in [Chai and Zhang 2016], one can see that the restriction on the residue characteristic 2 in [Chai and Zhang 2016, Theorem 4.8] can be removed. We record the theorem for completeness.

Theorem 1.4. *Let F be a number field and $\mathbb{A}$ be its ring of adeles. Let N be the upper triangular unipotent subgroup of SL_2. Let ψ be a nontrivial additive character of $F \backslash \mathbb{A}_F$ which is also viewed as a character of $N(F) \backslash N(\mathbb{A})$. Let S be a finite set of finite places of F. Let $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ be two irreducible cuspidal automorphic representations of $\mathrm{SL}_2(\mathbb{A}_F)$ with the same central character. If π and π' are both generic with respect to ψ and $\pi_v \cong \pi'_v$ for all $v \notin S$, then $\pi = \pi'$.*

2. Stability and local converse theorem for unitary groups

Using the same trick as the SL(2) case in §1, one could remove the restrictions of the residue characteristic in the local converse theorem for $\mathrm{U}(1, 1)$ and $\mathrm{U}(2, 2)$ in [Zhang 2017a; 2017b] and the stability results for $\mathrm{U}(n, n)$ in [Zhang 2017c]. To be more precise, we introduce the following notations. Let F be a p-adic field and E/F be a quadratic field extension. Let $\mathrm{U}_{E/F}(n, n)$ be the unitary group defined by the skew-Hermitian form $J_n = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$, where I_n is the $n \times n$ identity matrix. Let π be an irreducible smooth generic representation of $\mathrm{U}_{E/F}(n,n)$ and τ be an irreducible smooth generic representation of $\mathrm{GL}_k(E)$ with $k \le n$. Then one can define a local gamma factor $\gamma(s, \pi \times \tau, \psi)$, where ψ is a fixed additive character of F. These local gamma factors come from the local functional equations of the local zeta integrals considered in [Ben-Artzi and Soudry 2009]. See [Zhang 2017a; 2017b] for some details when $n = 1, 2$ and [Zhang 2017c] for some details for the gamma factors for $\mathrm{U}_{E/F} \times \mathrm{GL}_1(E)$.

Theorem 2.1. (1) *Suppose that $n = 1, 2$. Let π and π' be two irreducible smooth generic representations of $\mathrm{U}_{E/F}(n,n)$ with the same central character and*

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \pi' \times \tau, \psi)$$

for all irreducible smooth generic representations of $\mathrm{GL}_k(E)$ for all $k \le n$. Then $\pi \cong \pi'$.

(2) *Suppose that n is an arbitrary positive integer. Let π and π' be two irreducible smooth generic representations of $\mathrm{U}_{E/F}(n,n)$ with the same central character. Then there exists a positive integer $l := l(\pi, \pi')$ such that for all quasicharacters η of $E^\times$ with $\mathrm{cond}(\eta) > l$, one has*

$$\gamma(s, \pi \times \eta, \psi) = \gamma(s, \pi' \times \eta, \psi).$$

If E/F is unramified, or E/F is ramified but the residue characteristic of F is not 2, part (1) of the above theorem is proved in [Zhang 2017a; 2017b] and part (2) is proved in [Zhang 2017c]. To prove the general case, one can embed the representation π of $\mathrm{U}_{E/F}(n,n)$ into the similitude unitary group $\mathrm{GU}_{E/F}(n,n)$, where

$$\mathrm{GU}_{E/F}(n,n) = \{g \in \mathrm{GL}_{2n}(E) : {}^t\bar{g} J_n g = \lambda J_n,\ \lambda \in F^\times\}.$$

Here $x \mapsto \bar{x}$ is the nontrivial element in the Galois group of E/F. Note that the center of $\mathrm{GU}_{E/F}$ is $E^\times$ and the center of $\mathrm{U}_{E/F}(n,n)$ is E^1, where E^1 is the norm one element in $E^\times$.

The theory in [Gelbart and Knapp 1982] also works for the pair $\mathrm{GU}_{E/F}(n,n)$, $\mathrm{U}_{E/F}(n,n)$. In particular, the restriction of an irreducible smooth representation $\tilde{\pi}$ of $\mathrm{GU}_{E/F}(n,n)$ to $\mathrm{U}_{E/F}(n,n)$ is semisimple; on the other hand, for any irreducible smooth representation π of $\mathrm{U}_{E/F}(n,n)$, there exists an irreducible smooth representation $\tilde{\pi}$ of $\mathrm{GU}_{E/F}(n,n)$, such that π is a constituent of $\tilde{\pi}|_{\mathrm{U}_{E/F}(n,n)}$. As in Lemma 1.2, one has:

Lemma 2.2. *Let π and π' be two irreducible smooth representations of $\mathrm{U}_{E/F}(n,n)$ with the same central character. Then there exist irreducible smooth representations $\tilde{\pi}$ and $\tilde{\pi}'$ of $\mathrm{GU}_{E/F}(n,n)$ with the same central character such that π and π' are constituents of $\tilde{\pi}|_{\mathrm{U}_{E/F}(n,n)}$ and $\tilde{\pi}|_{\mathrm{U}_{E/F}(n,n)}$, respectively.*

Proof. Let $\mathrm{Sim} : \mathrm{GU}_{E/F}(n,n) \to F^\times$ be the similitude character of the group $\mathrm{GU}_{E/F}(n,n)$. Via the composition $\chi \circ \mathrm{Sim}$, a quasicharacter χ of $F^\times$ can be viewed as a character of $\mathrm{GU}_{E/F}(n,n)$. Let $\tilde{\pi}$ be an irreducible smooth representation of $\mathrm{GU}_{E/F}(n,n)$ with central character $\omega_{\tilde{\pi}}$. Then we can consider the representation $\chi \otimes \tilde{\pi}$. Note that $\chi \otimes \tilde{\pi}|_{\mathrm{U}(n,n)} = \tilde{\pi}|_{\mathrm{U}(n,n)}$ and the central character of $\chi \otimes \tilde{\pi}$ is $(\chi \circ \mathrm{Nm}_{E/F})\omega_{\tilde{\pi}}$. Denote $\eta = \omega_{\tilde{\pi}'}\omega_{\tilde{\pi}}^{-1}$. As in the proof of Lemma 1.2, it suffices to show that there is a character of χ of $F^\times$ such that $\chi \circ \mathrm{Nm}_{E/F} = \eta$. This follows from the fact that $\eta|_{E^1} = 1$. □

To proceed, we need the following multiplicity one property.

Proposition 2.3. *Let π be an irreducible generic representation of $\mathrm{U}_{E/F}(n,n)$ and let $\tilde{\pi}$ be an irreducible representation of $\mathrm{GU}_{E/F}(n,n)$. Then*

$$\dim \mathrm{Hom}_{\mathrm{U}_{E/F}(n,n)}(\tilde{\pi}, \pi) \leq 1.$$

Remark. For the pairs (GL, SL), (GSp, Sp) and (GO, O) (similitude orthogonal group and orthogonal group), similar results were proved in [Adler and Prasad 2006] without the genericity assumption.

Proof. Let U be the maximal unipotent subgroup of a fixed Borel subgroup of $\mathrm{U}_{E/F}(n,n)$ and let ψ be a generic character of U. We assume that π is ψ-generic. Since U is also the maximal unipotent subgroup of a Borel subgroup of $\mathrm{GU}_{E/F}(n,n)$, we have $\dim \mathrm{Hom}_U(\tilde{\pi}, \psi) \leq 1$ by the uniqueness of Whittaker model for $\tilde{\pi}$. If π appears as a constituent of $\tilde{\pi}|_{\mathrm{U}_{E/F}(n,n)}$, a Whittaker functional of π gives a Whittaker functional of $\tilde{\pi}$. Now the assertion follows from the uniqueness of Whittaker model of $\tilde{\pi}$. □

After Lemma 2.2 and Proposition 2.3, the proof of Theorem 2.1 reduces the cases considered in [Zhang 2017a; 2017b; 2017c] as in the SL_2 case sketched in §1.

Finally, we remark that Theorem 2.1 also holds for symplectic groups with the same proof.

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References

[Adler and Prasad 2006] J. D. Adler and D. Prasad, "On certain multiplicity one theorems", *Israel J. Math.* **153** (2006), 221–245. MR Zbl

[Baruch 1995] E. M. Baruch, *Local factors attached to representations of p-adic groups and strong multiplicity one*, Ph.D. thesis, Yale University, 1995, http://tinyurl.com/baruchthesis.

[Ben-Artzi and Soudry 2009] A. Ben-Artzi and D. Soudry, "L-functions for $\mathrm{U}_m \times R_{E/F}\mathrm{GL}_n$ $\left(n \leq \left[\frac{m}{2}\right]\right)$", pp. 13–59 in *Automorphic forms and L-functions, I: Global aspects*, edited by D. Ginzburg et al., Contemp. Math. **488**, American Mathematical Society, Providence, RI, 2009. MR Zbl

[Chai and Zhang 2016] J. Chai and Q. Zhang, "A strong multiplicity one theorem for SL_2", *Pacific J. Math.* **285**:2 (2016), 345–374. MR Zbl

[Gelbart and Knapp 1982] S. S. Gelbart and A. W. Knapp, "L-indistinguishability and R groups for the special linear group", *Adv. in Math.* **43**:2 (1982), 101–121. MR Zbl

[Zhang 2017a] Q. Zhang, "A local converse theorem for $\mathrm{U}(1, 1)$", *Int. J. Number Theory* **13**:8 (2017), 1931–1981. MR Zbl

[Zhang 2017b] Q. Zhang, "A local converse theorem for $\mathrm{U}(2, 2)$", *Forum Math.* (online publication January 2017).

[Zhang 2017c] Q. Zhang, "Stability of Rankin–Selberg gamma factors for $\mathrm{Sp}(2n)$, $\widetilde{\mathrm{Sp}}(2\mathrm{n})$ and $\mathrm{U}(n, n)$", *Int. J. Number Theory* (2017), 1–40.

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