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We prove superconvergence results for all freely infinitely divisible distributions. Given a nondegenerate freely infinitely divisible distribution ν, let μ_n be a sequence of probability measures and let k_n be a sequence of integers tending to infinity such that $\mu_n^{\boxplus k_n}$ converges weakly to ν. We show that the density $d\mu_n^{\boxplus k_n}/dx$ converges uniformly, as well as in all L^p-norms for $p > 1$, to the density of ν except possibly in the neighborhood of one point. Applications include the global superconvergence to freely stable laws and that to free compound Poisson laws over the whole real line.

1. Introduction

Consider a sequence $\{X_i\}_{i=1}^{\infty}$ of independent identically distributed random variables with zero mean and unit variance. The classical central limit theorem states that variables

$$S_n = \frac{X_1+X_2+\cdots+X_n}{\sqrt{n}}$$

converge in distribution to the standard normal law. Note that the variables S_n might always be discrete, even though their limit is absolutely continuous. This means that the convergence of S_n to a normal law must be expressed in terms of distribution functions, rather than densities.

Assume now that, instead of being independent, the variables $\{X_i\}_{i=1}^{\infty}$ are *freely* independent in the sense of [Voiculescu et al. 1992]. We still assume them identically distributed with zero mean and unit variance. Under the additional condition that the variables are bounded, it was shown in [Bercovici and Voiculescu 1995] that the distribution of S_n is absolutely continuous for sufficiently large n, and these densities converge uniformly, along with all of their derivatives, to the density of the semicircle law

$$\frac{1}{2\pi}\sqrt{4-t^2}$$

on any interval $[a, b] \subset (-2, 2)$. This phenomenon was called *superconvergence* in that paper. In [Wang 2010], the assumption that X_i be bounded was removed. Even

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when the variables X_i are not identically distributed, but are uniformly bounded, the support of S_n was shown by Kargin [2007] to converge to the interval $[-2, 2]$ as $n \to \infty$. See also [Anshelevich et al. 2014] for multiplicative superconvergence results.

The purpose of this paper is to demonstrate that the phenomenon of superconvergence is not limited to convergence to the semicircle law. Consider a nondegenerate probability measure ν on $\mathbb{R}$, which is infinitely divisible in the free sense (that is, $\boxplus$-infinitely divisible). It is known that its Cauchy transform,

$$G_\nu(z) = \int_{-\infty}^{+\infty} \frac{1}{z-t}\, d\nu(t), \tag{1-1}$$

defined for $\Im z > 0$, extends continuously to all points $z \in \mathbb{R}$ with at most one exception t_ν. The measure ν is absolutely continuous on $\mathbb{R} \setminus \{t_\nu\}$ and its density is locally analytic when strictly positive. To formulate our result, assume that for every positive integer n, we are given k_n freely independent, identically distributed random variables $X_{n1}, X_{n2}, \dots, X_{nk_n}$ such that $\lim_{n\to\infty} k_n = \infty$ and the sums

$$S_n = X_{n1} + X_{n2} + \cdots + X_{nk_n}$$

converge in distribution to the measure ν. (Necessary and sufficient conditions for such a convergence to take place are found in [Bercovici and Pata 1999].) Our main result, Theorem 4.1, implies the following statement. For convenience, we denote by D_ν the singleton $\{t_\nu\}$ if this point exists. Otherwise, $D_\nu = \varnothing$.

Theorem 1.1. *Given any open set $U \supset D_\nu$, the distribution ν_n of S_n is absolutely continuous on $\mathbb{R} \setminus U$ for sufficiently large n, and the density of ν_n converges to the density of ν uniformly and in L^p-norms for $p > 1$ on $\mathbb{R} \setminus U$.*

Note that U can be taken to be empty if $D_\nu = \varnothing$.

In Proposition 5.1, we provide the necessary and sufficient conditions for the existence of the singularity t_ν, as well as a formula to compute it when this point exists. These conditions and the formula are further used to investigate the quality of convergence to freely stable and free compound Poisson densities.

To prove this result, we first approximate ν_n by a closely related $\boxplus$-infinitely divisible measure ρ_n and we use the fact that G_{ρ_n} is a conformal map. Related considerations appear in the work of Chistyakov and Götze [2013].

The remainder of this paper is organized as follows. In Section 2, we review some relevant preliminaries on free convolution and freely infinitely divisible distributions. Section 3 is devoted to describing the subordination function appearing in free convolution powers. Section 4 contains the proof of our main result, and some examples and applications are given in Section 5.

2. Free convolution and freely infinitely divisible distributions

Let $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ be the complex upper half-plane, and let ν be a probability measure on $\mathbb{R}$. Recall that the Cauchy transform $G_\nu(z)$ of ν is defined by (1-1) for $z \in \mathbb{C}^+$. The measure ν can be recovered as the weak limit of the measures

$$d\nu_y(x) = -\frac{1}{\pi}\Im G_\nu(x+iy)\,dx, \quad x \in \mathbb{R},\ y > 0,$$

as $y \to 0$, and the atoms of ν can be calculated as follows:

$$\lim_{y\to 0} iyG_\nu(\alpha+iy) = \nu(\{\alpha\}), \quad \alpha \in \mathbb{R}. \tag{2-1}$$

The reciprocal $F_\nu = 1/G_\nu$ is an analytic self-map of $\mathbb{C}^+$ and plays a role in the calculation of free convolution. More precisely, for any $\eta > 0$ there exists a positive constant $M = M(\eta, \nu)$ such that the function F_ν has an analytic right inverse F_ν^{-1} (relative to the composition) defined in the truncated cone

$$\Gamma_{\eta,M} = \{x+iy : y > M \text{ and } |x| < \eta y\}.$$

The *Voiculescu transform* φ_ν of ν is then defined as $\varphi_\nu(z) = F_\nu^{-1}(z) - z$, and for any probability law μ on $\mathbb{R}$, we have

$$\varphi_{\mu\boxplus\nu}(z) = \varphi_\mu(z) + \varphi_\nu(z)$$

for all z in a region of the form $\Gamma_{\eta,M}$ where all three transforms are defined (see [Bercovici and Voiculescu 1993] for the proof). In this sense, the Voiculescu transform linearizes the free convolution $\boxplus$.

The set of all finite Borel measures on $\mathbb{R}$ is equipped with the topology of weak convergence from duality with continuous bounded functions. Denoting by $\mathcal{M}$ the class of all Borel probability measures on $\mathbb{R}$, we can translate weak convergence of measures in $\mathcal{M}$ into convergence properties of the corresponding Voiculescu transforms. We recall the following result from [Bercovici and Pata 1999].

Proposition 2.1. *Let $\mu, \mu_1, \mu_2, \ldots$ be measures in $\mathcal{M}$. Then the sequence μ_n converges weakly to the law μ if and only if there exist $\eta, M > 0$ such that the functions φ_{μ_n} are defined on $\Gamma_{\eta,M}$ for every n, $\lim_{n\to\infty} \varphi_{\mu_n}(iy) = \varphi_\mu(iy)$ for every $y > M$, and $\varphi_{\mu_n}(iy) = o(y)$ uniformly in n as $y \to \infty$.*

A measure $\nu \in \mathcal{M}$ is said to be $\boxplus$-*infinitely divisible* if for every positive integer n, there exists a measure $\nu_n \in \mathcal{M}$ such that

$$\nu = \underbrace{\nu_n \boxplus \nu_n \boxplus \cdots \boxplus \nu_n}_{n \text{ times}}.$$

We denote by $\mathcal{ID}(\boxplus)$ the set of all $\boxplus$-infinitely divisible measures in $\mathcal{M}$. It was shown in [Bercovici and Voiculescu 1993] that $\nu \in \mathcal{ID}(\boxplus)$ if and only if the function

φ_ν extends analytically to a map from $\mathbb{C}^+$ into $\mathbb{C}^- \cup \mathbb{R}$, in which case there exist a real constant γ and a finite Borel measure σ on $\mathbb{R}$ such that φ_ν has the following *free Lévy–Khintchine representation*:

$$\varphi_\nu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t}\, d\sigma(t).$$

The pair (γ, σ) is uniquely determined. Conversely, given such a pair (γ, σ), there exists a unique probability law $\nu = \nu_{\boxplus}^{\gamma,\sigma} \in \mathcal{ID}(\boxplus)$ satisfying the above integral formula. We shall call the pair (γ, σ) the *free generating pair* for $\nu_{\boxplus}^{\gamma,\sigma}$. Weak convergence of $\boxplus$-infinitely divisible laws can be characterized in terms of their free generating pairs; namely, $\nu_{\boxplus}^{\gamma_n,\sigma_n} \to \nu_{\boxplus}^{\gamma,\sigma}$ weakly if and only if $\gamma_n \to \gamma$ and $\sigma_n \to \sigma$ weakly [Barndorff-Nielsen et al. 2006, Theorem 5.13].

We review some useful results related to the F-transforms of freely infinitely divisible distributions, which were proved in [Belinschi and Bercovici 2005; Huang 2015], and are closely related to Biane's work [1997]. Given $\nu = \nu_{\boxplus}^{\gamma,\sigma}$ in $\mathcal{ID}(\boxplus)$, the function F_ν is a conformal map, and its inverse is the function

$$H_\nu(z) = z + \varphi_\nu(z) = z + \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t}\, d\sigma(t), \quad z \in \mathbb{C}^+.$$

This means that $H_\nu(F_\nu(z)) = z$ for all $z \in \mathbb{C}^+$. Note that $H_\nu : \mathbb{C}^+ \to \mathbb{C}$ is an analytic function satisfying $\Im H_\nu(z) \le \Im z$ for all $z \in \mathbb{C}^+$. The following result is a consequence of [Belinschi and Bercovici 2005, Theorem 4.6].

Proposition 2.2. *The function F_ν has a one-to-one continuous extension to $\mathbb{C}^+ \cup \mathbb{R}$, and it satisfies*

$$|F_\nu(z_1) - F_\nu(z_2)| \ge \tfrac{1}{2}|z_1 - z_2|, \quad z_1, z_2 \in \mathbb{C}^+ \cup \mathbb{R}. \tag{2-2}$$

If $\alpha \in \mathbb{R}$ is a point such that $\Im F_\nu(\alpha) > 0$, then F_ν can be continued analytically to a neighborhood of α.

The inequality (2-2) implies that

$$|H_\nu(z_1) - H_\nu(z_2)| \le 2|z_1 - z_2|, \quad z_1, z_2 \in \Omega_\nu,$$

where $\Omega_\nu = F_\nu(\mathbb{C}^+)$. The function H_ν has a one-to-one continuous extension to the closure $\overline{\Omega}_\nu$. This extension is still denoted H_ν. Thus, we have the following inversion relationships:

$$H_\nu(F_\nu(z)) = z, \quad z \in \mathbb{C}^+ \cup \mathbb{R}, \quad \text{and} \quad F_\nu(H_\nu(z)) = z, \quad z \in \overline{\Omega}_\nu.$$

We describe now the boundary set $\partial\Omega_\nu$. Given $x \in \mathbb{R}$ and $y > 0$, observe

$$\Im H_\nu(x+iy) = y\left(1 - \int_{\mathbb{R}} \frac{1+t^2}{(t-x)^2 + y^2}\, d\sigma(t)\right).$$

It follows that

$$\Im H_\nu(x+iy)=0$$

if and only if

$$\int_{\mathbb{R}} \frac{1+t^2}{(t-x)^2+y^2}\, d\sigma(t)=1. \tag{2-3}$$

On the other hand, note that for any $x\in\mathbb{R}$, the positive function

$$y\mapsto \int_{\mathbb{R}} \frac{1+t^2}{(t-x)^2+y^2}\, d\sigma(t)$$

is continuous and strictly decreasing in y, provided that $\sigma\neq 0$; the case $\sigma=0$ corresponds to a measure ν which is a point mass. Thus, for any $x\in\mathbb{R}$, there exists at most one value $y>0$ satisfying (2-3). It is natural to introduce two sets

$$A_\nu=\{x\in\mathbb{R}: g(x)>1\}$$

and

$$B_\nu=\mathbb{R}\setminus A_\nu=\{x\in\mathbb{R}: g(x)\leq 1\},$$

where the function

$$g(x)=\int_{\mathbb{R}} \frac{1+t^2}{(t-x)^2}\, d\sigma(t)=\sup_{y>0}\int_{\mathbb{R}} \frac{1+t^2}{(t-x)^2+y^2}\, d\sigma(t), \qquad x\in\mathbb{R},$$

is a lower semicontinuous function of x, so that A_ν is an open set. For $x\in A_\nu$, define $u_\nu(x)$ to be the unique y in $(0,\infty)$ satisfying (2-3); for $x\in B_\nu$, set $u_\nu(x)=0$.

Proposition 2.3 [Huang 2015]. *The function F_ν maps $\mathbb{R}$ bicontinuously to the graph γ_ν of the function u_ν, that is,*

$$F_\nu(\mathbb{R})=\gamma_\nu=\{x+iu_\nu(x): x\in\mathbb{R}\}.$$

In particular, the function u_ν is continuous on $\mathbb{R}$.

We note for further reference that the set A_ν is merely the collection of all $x\in\mathbb{R}$ such that $u_\nu(x)>0$. Moreover, for any $t\in\mathbb{R}$, we have $\Im F_\nu(t)>0$ if and only if $\Re F_\nu(t)\in A_\nu$. The graph γ_ν is precisely the boundary set $\partial\Omega_\nu$, and one has $\Omega_\nu=\{z\in\mathbb{C}^+: H_\nu(z)\in\mathbb{C}^+\}$. The following result now follows easily from these facts; see also [Biane 1997; Huang 2015].

Proposition 2.4. *The function $t\mapsto \Re F_\nu(t)$ is a strictly increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$.*

As shown in [Bercovici and Voiculescu 1993], the measure ν has at most one atom. From (2-1), we see that α is an atom of ν if and only if $F_\nu(\alpha)=0$ (which gives us the uniqueness of the atom by Proposition 2.2) and the Julia–Carathéodory derivative $F_\nu'(\alpha)$ is finite. (See [Shapiro 1993] for the definition, existence, and properties of the Julia–Carathéodory derivative.) The value of this derivative is given by

$$F_\nu'(\alpha) = \frac{1}{\nu(\{\alpha\})}.$$

By the Stieltjes inversion formula, the density of ν (relative to Lebesgue measure) is given by

$$\frac{d\nu}{dx}(t) = -\frac{1}{\pi}\Im G_\nu(t) = \frac{1}{\pi}\frac{\Im F_\nu(t)}{|F_\nu(t)|^2},$$

at points other than the possible atom α. (This uses the continuous extension of F_ν to $\mathbb{R}$.)

Lemma 2.5. *Consider a measure $\nu \in \mathcal{ID}(\boxplus)$, and denote by s_ν the density of the absolutely continuous part of ν. We have $\lim_{|t|\to\infty} s_\nu(t) = 0$.*

Proof. Inequality (2-2) implies that

$$|F_\nu(t) - F_\nu(i)| \geq \tfrac{1}{2}|t-i| > \tfrac{1}{2}|t|, \quad t \in \mathbb{R},$$

so that $|F_\nu(t)| > \frac{1}{3}|t|$ for $|t| > 6|F_\nu(i)|$. Then the value of density s_ν at such t can be estimated as follows:

$$s_\nu(t) = \frac{1}{\pi}\frac{\Im F_\nu(t)}{|F_\nu(t)|^2} \leq \frac{1}{\pi}\frac{1}{|F_\nu(t)|} < \frac{1}{\pi}\frac{3}{|t|}, \quad |t| > 6|F_\nu(i)|. \tag{2-4}$$

The conclusion follows. □

The preceding result shows that if $F_\nu(t_\nu) = 0$, then we must have $|t_\nu| \leq 6|F_\nu(i)|$. Moreover, for any $p > 1$ and any neighborhood U of the point t_ν, the estimate (2-4) implies that the p-th power $|s_\nu|^p$ is continuous and integrable over $\mathbb{R} \setminus U$. If such a zero t_ν does not exist, then the density s_ν is a continuous function which belongs to the space $L^p(\mathbb{R}, dx)$ for all $p > 1$.

The next result follows from the proof of Theorem 4.6 in [Belinschi and Bercovici 2005]. Here we offer a more direct argument.

Lemma 2.6. *The derivative of H_ν is nonzero at $z = x + iu_\nu(x)$, for any $x \in A_\nu$.*

Proof. We have

$$H_\nu'(z) = 1 - \int_{\mathbb{R}} \frac{1+t^2}{(z-t)^2}\,d\sigma(t), \quad z \in \mathbb{C}^+.$$

When $x \in A_\nu$ and $z = x + iu_\nu(x)$, a straightforward calculation and the definition of u_ν lead to

$$\left|\int_{\mathbb{R}} \frac{1+t^2}{(z-t)^2}\,d\sigma(t)\right| < \int_{\mathbb{R}} \frac{1+t^2}{|z-t|^2}\,d\sigma(t) = \int_{\mathbb{R}} \frac{1+t^2}{(t-x)^2 + u_\nu(x)^2}\,d\sigma(t) = 1,$$

which implies the desired conclusion. □

Lemma 2.7. *Consider measures ν, $\nu_n \in \mathcal{ID}(\boxplus)$, $n \in \mathbb{N}$, such that $\nu_n \to \nu$ weakly as $n \to \infty$, and let $I \subset \mathbb{R}$ be a compact interval such that the limiting density $d\nu/dx$ is bounded away from zero on I. Then the density $d\nu_n/dx$ converges uniformly on I to $d\nu/dx$ as $n \to \infty$.*

Proof. Let $(\gamma, \sigma), (\gamma_n, \sigma_n)$ be the free generating pairs of ν and ν_n, respectively. As seen earlier, $\gamma_n \to \gamma$ and $\sigma_n \to \sigma$ weakly as $n \to \infty$. Thus, the sequence H_{ν_n} converges to the function H_ν uniformly on compact subsets of $\mathbb{C}^+$.

It is clear that $\Re F_\nu(I) \subset A_\nu$. Thus, by Lemma 2.6, $H_\nu'(z) \neq 0$ for $z \in F_\nu(I)$, and its inverse function F_ν has a conformal continuation to a neighborhood of I. Expressing inverse functions using the Cauchy integral formula, we conclude that, for large n, F_{ν_n} also has a conformal continuation to a neighborhood of I. Moreover, these continuations converge uniformly on I to the continuation of F_ν. Since $0 \notin F_\nu(I)$, the lemma follows from the Stieltjes inversion formula. □

3. Free convolution powers and subordination functions

Given two probability measures μ_1 and μ_2 on $\mathbb{R}$, there exist two unique analytic functions $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ such that $F_{\mu_1 \boxplus \mu_2}(z) = F_{\mu_1}(\omega_1(z)) = F_{\mu_2}(\omega_2(z))$ and

$$F_{\mu_1 \boxplus \mu_2}(z) = \omega_1(z) + \omega_2(z) - z$$

for all $z \in \mathbb{C}^+$ (see [Voiculescu 1993; Biane 1998; Bercovici and Voiculescu 1998]).

Consider now a sequence $\{\mu_n\}_{n=1}^\infty$ in $\mathcal{M}$ and positive integers $k_n \geq 2$, and denote by $\mu_n^{\boxplus k_n}$ the k_n-fold free convolution power of μ_n. Belinschi and Bercovici [2005] showed that $\mu_n^{\boxplus k_n}$ has at most one atom and otherwise $\mu_n^{\boxplus k_n}$ is absolutely continuous, and they studied the analytic subordination for these free convolution powers. Thus, let $\omega_n : \mathbb{C}^+ \to \mathbb{C}^+$ be the subordination function of $F_{\mu_n^{\boxplus k_n}}$ with respect to F_{μ_n}, that is,

$$F_{\mu_n^{\boxplus k_n}}(z) = F_{\mu_n}(\omega_n(z)).$$

Then we have

$$F_{\mu_n^{\boxplus k_n}}(z) = F_{\mu_n}(\omega_n(z)) = \omega_n(z) + \frac{1}{k_n - 1}(\omega_n(z) - z), \quad z \in \mathbb{C}^+. \tag{3-1}$$

Equation (3-1) implies that the inverse function

$$\omega_n^{-1}(z) = z + (k_n - 1)(z - F_{\mu_n}(z))$$

for $z \in \Gamma_{\eta,M}$, where η, M are positive constants. On the other hand, the function ω_n can be regarded as the F-transform of a unique probability measure on $\mathbb{R}$ by the characterization of F-transforms (see [Bercovici and Voiculescu 1993, Proposition 5.2]). Let ρ_n be the probability measure on $\mathbb{R}$ such that $\omega_n(z) = F_{\rho_n}(z)$, so

$$\varphi_{\rho_n}(z) = (k_n - 1)(z - F_{\mu_n}(z)). \tag{3-2}$$

This implies that the measure ρ_n is $\boxplus$-infinitely divisible. In particular, the function ω_n extends continuously to $\mathbb{C}^+ \cup \mathbb{R}$ and so, too, does the function $F_{\mu_n^{\boxplus k_n}}$ by (3-1).

Denote by $E_\mu(z) = z - F_\mu(z)$ the self-energy of μ. Given two measures $\mu_1, \mu_2 \in \mathcal{M}$, their Boolean convolution $\mu_1 \uplus \mu_2$, introduced in [Speicher and Woroudi 1997], is the unique probability measure on $\mathbb{R}$ satisfying

$$E_{\mu_1 \uplus \mu_2}(z) = E_{\mu_1}(z) + E_{\mu_2}(z), \quad z \in \mathbb{C}^+.$$

Every probability measure on $\mathbb{R}$ is $\uplus$-infinitely divisible. Given a measure $\nu \in \mathcal{M}$, the function E_ν is a map from $\mathbb{C}^+$ to $\mathbb{C}^- \cup \mathbb{R}$ and satisfies $E_\nu(iy)/iy \to 0$ as $y \to \infty$. (The latter limit actually holds uniformly for ν in any tight family of probability measures [Bercovici and Voiculescu 1993].) Thus, E_ν admits a unique *Nevanlinna representation*:

$$E_\nu(z) = \gamma + \int_{\mathbb{R}} \frac{1+tz}{z-t}\, d\sigma(t), \quad z \in \mathbb{C}^+.$$

Conversely, every such formula defines an analytic function which is of the form E_ν for a unique probability measure ν. We will write $\nu = \nu_\uplus^{\gamma,\sigma}$ to indicate this correspondence. Note $E_{\nu_\uplus^{\gamma,\sigma}}(z) = \varphi_{\nu_\boxplus^{\gamma,\sigma}}(z)$, and that the map $\nu_\boxplus^{\gamma,\sigma} \to \nu_\uplus^{\gamma,\sigma}$ is a bijective map from the set $\mathcal{ID}(\boxplus)$ into the set $\mathcal{M}$. Finally, it is easy to verify that if a sequence ν_n converges weakly to a law ν in $\mathcal{M}$, then the limit $\lim_{n\to\infty} E_{\nu_n}(z) = E_\nu(z)$ holds for $z \in \mathbb{C}^+$.

We record for further use the following result from [Bercovici and Pata 1999, Theorem 6.3].

Theorem 3.1. *Fix a free generating pair (γ, σ), a sequence $\{\mu_n\}_{n=1}^\infty$ in $\mathcal{M}$, and a sequence $\{k_n\}_{n=1}^\infty$ of unbounded positive integers. Then the sequence $\mu_n^{\boxplus k_n}$ converges weakly to $\nu_\boxplus^{\gamma,\sigma}$ if and only if the sequence $\mu_n^{\uplus k_n}$ converges weakly to $\nu_\uplus^{\gamma,\sigma}$.*

Boolean limit theorems are used in the proof of the following result.

Proposition 3.2. *Let $\{\mu_n\}_{n=1}^\infty \subset \mathcal{M}$ and let $\{k_n\}_{n=1}^\infty \subset \mathbb{N}$ such that* $\lim_{n\to\infty} k_n = \infty$. *Suppose $\mu_n^{\boxplus k_n}$ converges weakly to a law $\nu \in \mathcal{ID}(\boxplus)$. For each n, choose $\rho_n \in \mathcal{ID}(\boxplus)$, such that*

$$F_{\mu_n^{\boxplus k_n}}(z) = F_{\mu_n}(F_{\rho_n}(z)), \quad z \in \mathbb{C}^+.$$

Then $\rho_n \to \nu$ weakly.

Proof. Assume that (γ, σ) is the free generating pair of ν. By Proposition 2.1, the weak convergence $\mu_n^{\boxplus k_n} \to \nu_\boxplus^{\gamma,\sigma}$ implies the existence of $M > 0$ such that

$$\lim_{n\to\infty} k_n \varphi_{\mu_n}(iy) = \varphi_{\nu_\boxplus^{\gamma,\sigma}}(iy)$$

for all $y > M$, and $k_n \varphi_{\mu_n}(iy) = o(y)$ uniformly in n as $y \to \infty$. In particular, it follows that the sequence μ_n converges weakly to the unit point mass at 0. On the other hand, Theorem 3.1 shows that $\mu_n^{\uplus k_n} \to \nu_\uplus^{\gamma,\sigma}$ weakly.

By (3-2), we have

$$\varphi_{\rho_n}(z) = E_{\mu_n^{\uplus k_n}}(z) - E_{\mu_n}(z), \quad z \in \mathbb{C}^+.$$

Since the two sequences $\{\mu_n^{\uplus k_n}\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ are both tight, the last formula implies that $\varphi_{\rho_n}(iy) = o(y)$ uniformly in n as $y \to \infty$. To determine the limit of $\{\rho_n\}_{n=1}^{\infty}$, we calculate

$$\lim_{n\to\infty} \varphi_{\rho_n}(iy) = \lim_{n\to\infty} [E_{\mu_n^{\uplus k_n}}(iy) - E_{\mu_n}(iy)] = E_{\nu_{\uplus}^{\gamma,\sigma}}(iy) = \varphi_{\nu_{\boxplus}^{\gamma,\sigma}}(iy)$$

for every $y > M$. The desired conclusion follows from Proposition 2.1. □

4. The main result

In the following statement, F_ν is viewed as a continuous function defined on $\mathbb{C}^+ \cup \mathbb{R}$.

Theorem 4.1. *Consider a nondegenerate $\boxplus$-infinitely divisible distribution ν on $\mathbb{R}$, a sequence $\{\mu_n\}_{n=1}^{\infty}$ of probability measures on $\mathbb{R}$, and a sequence $\{k_n\}_{n=1}^{\infty}$ of positive integers tending to infinity such that the measures $\mu_n^{\boxplus k_n}$ converge weakly to ν.*

(1) *If $0 \notin F_\nu(\mathbb{R})$, then the measure ν has no atom and there exists $N > 0$ such that the measure $\mu_n^{\boxplus k_n}$ is Lebesgue absolutely continuous with a continuous density on $\mathbb{R}$ for every $n \geq N$. Moreover, the density of the measure $\mu_n^{\boxplus k_n}$ converges uniformly on $\mathbb{R}$ to the density of the measure ν.*

(2) *If $0 \in F_\nu(\mathbb{R})$, and $U \subset \mathbb{R}$ is an open interval containing the singleton $F_\nu^{-1}(\{0\})$, then there exists $N > 0$ such that the restriction of the measure $\mu_n^{\boxplus k_n}$ to $\mathbb{R} \setminus U$ is absolutely continuous with a continuous density on $\mathbb{R} \setminus U$ for $n \geq N$. Moreover, the density of the measure $\mu_n^{\boxplus k_n}$ converges uniformly on $\mathbb{R} \setminus U$ to the density of the measure ν.*

(3) *In all cases, the limit*

$$\lim_{n\to\infty} \left\| \frac{d\mu_n^{\boxplus k_n}}{dx} - \frac{d\nu}{dx} \right\|_{L^p(\mathbb{R}\setminus U)} = 0$$

holds for $p > 1$, with $U = \varnothing$ in case (1).

Remark. The condition that $0 \in F_\nu(\mathbb{R})$ is necessary for ν to have an atom, but it is not sufficient (see Proposition 5.1). If $F_\nu(t_\nu) = 0$, then the function G_ν extends continuously to all points $t \in \mathbb{R} \setminus \{t_\nu\}$. Theorem 1.1 follows from Theorem 4.1 and this observation.

Proof. As seen earlier, there exist measures $\rho_n \in \mathcal{ID}(\boxplus)$ satisfying

$$F_{\mu_n^{\boxplus k_n}}(z) = F_{\mu_n}(F_{\rho_n}(z)), \quad z \in \mathbb{C}^+.$$

To each n, denote by s_n and s the density of the absolutely continuous part of $\mu_n^{\boxplus k_n}$ and that of ν, respectively. Relation (3-1) shows that $|F_{\mu_n^{\boxplus k_n}} - F_{\rho_n}|$ is small relative to $|F_{\rho_n}|$. Thus, it suffices to focus on the asymptotic behavior of F_{ρ_n}.

Given $\varepsilon > 0$, we first prove that there exists $M > 0$ such that $|s_n(t) - s(t)| < \varepsilon$ for $|t| > M$ and for sufficiently large n. Since the measures ρ_n converge weakly to ν by Proposition 3.2, we have $|F_{\rho_n}(i)| \to |F_\nu(i)|$ as $n \to \infty$. In the sequel, we shall only consider the integers n which satisfy the following two conditions:

$$k_n > 13 \quad \text{and} \quad 9|F_\nu(i)| > 6|F_{\rho_n}(i)|.$$

Applying the estimate (2-4) to ρ_n, we get $|F_{\rho_n}(t)| > \frac{1}{3}|t|$ for all such n and for $|t| > 9|F_\nu(i)|$. It follows from (3-1) that $|F_{\mu_n^{\boxplus k_n}}(t)| > \frac{1}{4}|t|$ for the same n and t. Combining this with another application of (2-4) to the density s, we get

$$|s_n(t) - s(t)| < \frac{7}{\pi}\frac{1}{|t|}, \quad |t| > 9|F_\nu(i)|, \tag{4-1}$$

for these large n. Therefore, the desired cutoff constant M can be chosen as

$$M = \max\left\{9|F_\nu(i)|, \frac{7}{\varepsilon\pi}\right\}.$$

We conclude that it suffices to prove the uniform convergence of s_n to s on a set of the form $I \setminus U$, where $I = [-M, M]$. To this purpose, fix $I = [-M, M]$ with $M > 0$, and let $\delta > 0$ be arbitrary but fixed. Recall that the map

$$t \mapsto \Re F_\nu(t)$$

is an increasing homeomorphism of $\mathbb{R}$. Thus, the set

$$J = \Re F_\nu(I) = \{x \in \mathbb{R} : \Re F_\nu(-M) \le x \le \Re F_\nu(M)\}$$

is a compact interval. Set

$$\Gamma = \{x \in J : u_\nu(x) \ge \delta\}$$

and

$$\Delta = \left\{x \in J : u_\nu(x) > \frac{\delta}{2}\right\}.$$

We have $\Gamma \subset \Delta \subset J$, where Γ is closed, and Δ is relatively open in J. We conclude that Γ is contained in the union of finitely many connected components of Δ. Taking the closure of those components, we find a finite family $J_1, J_2, \dots, J_K$ of pairwise disjoint, closed intervals such that

$$\Gamma \subset \bigcup_{1 \le \ell \le K} J_\ell \subset \overline{\Delta}.$$

We have $u_\nu \ge \delta/2$ on the union $\bigcup_{1 \le \ell \le K} J_\ell$ and $u_\nu \le \delta$ on the complement $J' = J \setminus \left(\bigcup_{1 \le \ell \le K} J_\ell\right)$.

Denote $I_\ell = \{t \in I : \Re F_\nu(t) \in J_\ell\}$ for each $1 \le \ell \le K$. Note that

$$\Im F_\nu(t) \ge \delta/2$$

for each $t \in \bigcup_{1 \le \ell \le K} I_\ell$. Thus, the density of ν is bounded away from zero on $\bigcup_{1 \le \ell \le K} I_\ell$. From Lemma 2.7, we see that the functions F_ν and F_{ρ_n} both extend analytically to a neighborhood of the set $\bigcup_{1\le\ell\le K} I_\ell$ for sufficiently large n. These extensions are injective. Moreover, the convergence $F_{\rho_n} \to F_\nu$ holds uniformly in that neighborhood. By virtue of (3-1), we conclude that the functions $F_{\mu_n^{\boxplus k_n}}$ will have the same behavior on the set $\bigcup_{1\le\ell\le K} I_\ell$ as $n \to \infty$. It follows that the measure $\mu_n^{\boxplus k_n}$ has no atom in the union $\bigcup_{1\le\ell\le K} I_\ell$ for large n and $s_n \to s$ uniformly on this set by the Stieltjes inversion formula.

We prove next the uniform convergence on the set I' (or on $I' \setminus U$), where

$$I' = \{t \in I : \Re F_\nu(t) \in J'\} = I \setminus \Big(\bigcup\nolimits_{1\le\ell\le K} I_\ell\Big). \tag{4-2}$$

We claim that

$$\sup\nolimits_{x\in J'} u_{\rho_n}(x) \le 2\delta \tag{4-3}$$

for sufficiently large n. Assume, to get a contradiction, that there exist positive integers $n_1 < n_2 < \cdots \to \infty$ and points $x_1, x_2, \ldots \in J'$ such that $u_{\rho_{n_k}}(x_k) > 2\delta$. By the definition of u_{ρ_n} given in Section 2, we have

$$\int_{\mathbb{R}} \frac{1+t^2}{(t-x_k)^2 + u_{\rho_{n_k}}(x_k)^2}\, d\sigma_{n_k}(t) = 1, \quad k \ge 1, \tag{4-4}$$

where σ_{n_k} is the free generating measure of ρ_{n_k}. By passing to a subsequence if necessary, we assume that $x_k \to x_0 \in \overline{J'}$ as $k \to \infty$. Then, denoting $\nu_{\boxplus}^{\gamma,\sigma}$ by ν, the identity (4-4) and the fact that $\sigma_n \to \sigma$ weakly imply

$$1 \le \int_{\mathbb{R}} \frac{1+t^2}{(t-x_k)^2 + (2\delta)^2}\, d\sigma_{n_k}(t) \to \int_{\mathbb{R}} \frac{1+t^2}{(t-x_0)^2 + (2\delta)^2}\, d\sigma(t)$$

as $k \to \infty$. We conclude that $u_\nu(x_0) \ge 2\delta$, which is in contradiction to the fact that $x_0 \in \overline{J'}$. Thus, the estimate (4-3) is proved.

The rest of the proof is divided into two cases according to whether $U = \varnothing$ or $U \ne \varnothing$. By translating the measure ν if necessary, we may assume that $\Re F_\nu(0) = 0$.

Case (1): $0 \notin F_\nu(\mathbb{R})$ and $U = \varnothing$. In this case, $u_\nu(0) > 0$ and thus $0 \in A_\nu$. Since the set A_ν is open, there exists a small number $a > 0$ such that the interval $[-4a, 4a]$ is contained in A_ν. By considering a smaller δ if necessary, we may assume further that

$$[-4a, 4a] \subset \bigcup_{1\le\ell\le K} J_\ell. \tag{4-5}$$

Since the map $t \mapsto \Re F_\nu(t)$ is an increasing homeomorphism of $\mathbb{R}$, the uniform convergence of $F_{\rho_n} \to F_\nu$ on $\bigcup_{1\le\ell\le K} I_\ell$ implies that there exists some integer $N > 0$ such that

$$[-2a, 2a] \subset \Big\{\Re F_{\rho_n}(t) : t \in \bigcup_{1\le\ell\le K} I_\ell\Big\}, \quad n \ge N.$$

Since the map $t \mapsto \Re F_{\rho_n}(t)$ is also a homeomorphism of the same nature, we have

$$\inf_{t \in I'} |\Re F_{\rho_n}(t)| \geq 2a, \quad n \geq N,$$

by recalling the definition (4-2) of the complement I'. Using (3-1) and enlarging N if necessary, we conclude that

$$\inf_{t \in I'} |\Re F_{\mu_n^{\boxplus k_n}}(t)| \geq a, \quad n \geq N. \tag{4-6}$$

Further enlarging N, the inequality (4-3) and the relation (3-1) imply that

$$\Im F_{\mu_n^{\boxplus k_n}}(t) \leq 3\delta, \quad t \in I',\ n \geq N. \tag{4-7}$$

From (4-6) and (4-7), we see that

$$0 \leq s_n(t) \leq \frac{3\delta}{a^2 \pi}$$

for $t \in I'$ and $n \geq N$. On the other hand, the relation (4-5) and the fact that $u_\nu \leq \delta$ on J' yield

$$0 \leq s(t) \leq \frac{\delta}{16a^2 \pi}$$

for $t \in I'$. As the parameter δ can be arbitrarily small, we have proved the uniform convergence of $s_n \to s$ on I'. This finishes the proof of Theorem 4.1(1).

Case (2): $0 \in F_\nu(\mathbb{R})$. In this case, $u_\nu(0) = 0$ and $F_\nu(0) = 0 = H_\nu(0)$ by our normalization. Let a_n be the unique real number such that $\Re F_{\rho_n}(a_n) = 0$ (and hence $F_{\rho_n}(a_n) = i u_{\rho_n}(0)$). We first show that a_n is small for large n. To this end, we write $U = (-2b, 2b)$ where $b > 0$ and set $c = b/5$. Observe that $H_\nu(ic) \in \mathbb{C}^+$, and that the Lipschitz property of H_ν yields

$$|H_\nu(ic)| = |H_\nu(ic) - H_\nu(0)| \leq 2c.$$

Since $\lim_{n\to\infty} H_{\rho_n}(ic) = H_\nu(ic)$, there exists an integer $N > 0$ such that $H_{\rho_n}(ic) \in \mathbb{C}^+$ for all $n \geq N$. Consequently, we have $u_{\rho_n}(0) < c$ for such n; for if $u_{\rho_n}(0) \geq c > 0$, we will get

$$\begin{aligned} 1 = \int_{\mathbb{R}} \frac{1+t^2}{t^2 + u_{\rho_n}(0)^2}\, d\sigma_n(t) &\leq \int_{\mathbb{R}} \frac{1+t^2}{t^2+c^2}\, d\sigma_n(t) \\ &= 1 - \frac{1}{c} \Im H_{\rho_n}(ic) < 1, \end{aligned}$$

a contradiction. Note further that

$$|H_{\rho_n}(ic) - a_n| = |H_{\rho_n}(ic) - H_{\rho_n}(i u_{\rho_n}(0))| \leq 2(c - u_{\rho_n}(0)) \leq 2c$$

for all $n \geq N$. (We have used the inversion relationship $a_n = H_{\rho_n}(F_{\rho_n}(a_n))$ here.) Therefore, by enlarging N if necessary, we conclude that $|a_n| < 5c = b$ for $n \geq N$.

Now, (2-2) shows that for any $t \in I' \setminus U$ and $n \geq N$, we have

$$|F_{\rho_n}(t) - F_{\rho_n}(a_n)| \geq \frac{1}{2}|t - a_n| > \frac{b}{2}.$$

This implies further that

$$|F_{\rho_n}(t)| > \frac{b}{2} - |F_{\rho_n}(a_n)| = \frac{b}{2} - |u_{\rho_n}(0)| > \frac{b}{4}, \quad t \in I' \setminus U,\ n \geq N.$$

In other words, for such values of t and n, $|F_{\rho_n}(t)|$ is always bounded away from zero. Then an argument similar to the proof of Case (1) yields the absolute continuity of the free convolution $\mu_n^{\boxplus k_n}$ and the uniform convergence $s_n \to s$ on $I' \setminus U$, finishing the proof of Theorem 4.1(2).

Finally, the L^p-convergence result in Theorem 4.1(3) follows from the estimate (4-1) and the dominated convergence theorem. □

Remark (Local analyticity and approximation). An important feature of superconvergence are the analyticity properties of the distributions in the limiting process. Indeed, under the weak convergence assumption of Theorem 4.1, if I is a finite interval on which the limit density $d\nu/dx$ is bounded away from zero (and hence it admits an analytic continuation to a neighborhood of I), then the restriction of the free convolution $\mu_n^{\boxplus k_n}$ on I becomes absolutely continuous in finite time and its density continues analytically to a neighborhood of I. Moreover, these extensions can be approximated uniformly by the analytic continuation of $d\nu/dx$ on I, thanks to Lemma 2.7 and the identity (3-1).

5. Applications

In this section, we apply our main result to some of the most important limit theorems in free probability. We begin by examining the geometric condition: $0 \in F_\nu(\mathbb{R})$. Note that the singular integral in the following result takes values in $(0, \infty]$.

Proposition 5.1. *Let $\nu = \nu_{\boxplus}^{\gamma,\sigma}$ be a nondegenerate law in $\mathcal{ID}(\boxplus)$. We have*:

(1) $0 \in F_\nu(\mathbb{R})$ *if and only if*

$$L = \sup_{\varepsilon>0} \frac{-\Im\varphi_\nu(i\varepsilon)}{\varepsilon} = \int_{\mathbb{R}} \frac{1+t^2}{t^2}\, d\sigma(t) \leq 1. \tag{5-1}$$

In this case, the value of the unique zero t_ν of F_ν is given by

$$t_\nu = \gamma - \int_{\mathbb{R}} \frac{1}{t}\, d\sigma(t).$$

(2) $\nu(\{t_\nu\}) > 0$ *if and only if $L < 1$, and we have $\nu(\{t_\nu\}) = 1 - L$ in this case.*

Proof. The identity

$$\sup_{\varepsilon>0}(-\Im\varphi_\nu(i\varepsilon))/\varepsilon = \int_{\mathbb{R}} \frac{1+t^2}{t^2}\, d\sigma(t)$$

follows from the free Lévy–Khintchine formula

$$-\Im\varphi_\nu(i\varepsilon) = \varepsilon \int_{\mathbb{R}} \frac{1+t^2}{\varepsilon^2+t^2}\, d\sigma(t)$$

and the monotone convergence theorem, and we see that the supremum here is in fact a genuine limit:

$$\sup_{\varepsilon>0}(-\Im\varphi_\nu(i\varepsilon))/\varepsilon = \lim_{\varepsilon\to 0^+}(-\Im\varphi_\nu(i\varepsilon))/\varepsilon.$$

Next, recall from [Belinschi and Bercovici 2005, Proposition 4.7] that $0 \in F_\nu(\mathbb{R})$ if and only if the limit

$$t_\nu = H_\nu(0) = \lim_{\varepsilon\to 0^+} H_\nu(i\varepsilon)$$

exists, $t_\nu \in \mathbb{R}$, and the Julia–Carathéodory derivative $H'_\nu(0) \geq 0$. Note that if the limit t_ν exists and is real, then the derivative

$$H'_\nu(0) = \lim_{\varepsilon\to 0^+} \frac{\Im H_\nu(i\varepsilon)}{\varepsilon} \tag{5-2}$$

always exists and belongs to the interval $[-\infty, 1)$. Moreover, if $0 \in F_\nu(\mathbb{R})$ and $H'_\nu(0) > 0$ then we have the Julia–Carathéodory derivative $F'_\nu(t_\nu) = 1/H'_\nu(0)$.

Now, if $0 \in F_\nu(\mathbb{R})$, then we know the limit $t_\nu \in \mathbb{R}$. Hence, (5-2) implies $H'_\nu(0) = 1 - L$. Since $H'_\nu(0) \geq 0$ in this case, we conclude that $1 \geq L$. On the other hand, since $F_\nu(\mathbb{R}) = \partial\Omega_\nu$, the inversion formula shows that

$$F_\nu(t_\nu) = F_\nu(H_\nu(0)) = 0.$$

Conversely, if the singular integral L converges and $1 \geq L$, then we have $\Im H_\nu(i\varepsilon) \to 0 \cdot (1-L) = 0$ as $\varepsilon \to 0^+$. On the other hand, the estimate

$$\frac{|t|}{\varepsilon^2+t^2} \leq \frac{1+t^2}{\varepsilon^2+t^2} \leq \frac{1+t^2}{t^2} \in L^1(\sigma), \qquad t \in \mathbb{R},\ \varepsilon > 0,$$

and the dominated convergence theorem imply that the function $t \mapsto 1/t$ belongs to $L^1(\sigma)$ and

$$\Re H_\nu(i\varepsilon) = \gamma + (\varepsilon^2 - 1)\int_{\mathbb{R}} \frac{t}{\varepsilon^2+t^2}\, d\sigma(t) \to \gamma - \int_{\mathbb{R}} \frac{1}{t}\, d\sigma(t)$$

as $\varepsilon \to 0^+$. It follows that the vertical limit t_ν is equal to

$$\gamma - \int_{\mathbb{R}} \frac{1}{t}\, d\sigma(t) \in \mathbb{R}.$$

As seen earlier, this fact and the formula (5-2) imply that $H'_\nu(0) = 1 - L$. Therefore, we have $H'_\nu(0) \geq 0$, and the proof of (1) is finished.

The statement (2) follows from the fact that the derivative $F'_\nu(t_\nu) = 1/\nu(\{t_\nu\})$. □

We remark that the results in [Belinschi and Bercovici 2005] were proved using Denjoy–Wolff analysis for boundary fixed points of analytic self-maps on $\mathbb{C}^+$. A different approach to the same results has been used in [Huang and Wang 2015], which yields a more general description for the points on the boundary set $\partial\Omega_\nu$.

Stable approximation. Recall that two measures $\mu, \nu \in \mathcal{M}$ are said to have the same *type* (and we write $\mu \sim \nu$) if there exist constants $a > 0$ and $b \in \mathbb{R}$ such that $\mu(E) = \nu(aE + b)$ for all Borel sets $E \subset \mathbb{R}$. The relation $\sim$ is an equivalence relationship among all probability laws, and hence the set $\mathcal{M}$ is partitioned into a union of distributions with inequivalent types. A nondegenerate distribution $\nu \in \mathcal{M}$ is said to be $\boxplus$-*stable* if $\nu \sim \nu_1 \boxplus \nu_2$ whenever $\nu_1 \sim \nu \sim \nu_2$. Clearly, within one type either all distributions are stable or else none of them is stable.

Each $\boxplus$-stable law ν is associated with a unique *stability index* $\alpha \in (0, 2]$, so that if X and Y are free random variables drawn from the same law ν and $a, b > 0$, then the distribution of the sum $aX + bY$ is a translate of the distribution of the scaled variable $(a^\alpha + b^\alpha)^{1/\alpha} X$. Stable laws of the same type share the same index.

Freely stable laws are $\boxplus$-infinitely divisible and absolutely continuous, and they can be classified using the stability index α. Following [Bercovici and Voiculescu 1993], every $\boxplus$-stable law has the same type as a unique distribution whose Voiculescu transform falls into the following list:

(1) $\varphi(z) = 1/z$ for $\alpha = 2$.

(2) $\varphi(z) = bz^{1-\alpha}$ for $1 < \alpha < 2$, where $|b| = 1$ and $\arg b \in [(\alpha - 2)\pi, 0]$.

(3) $\varphi(z) = bz^{1-\alpha}$ for $0 < \alpha < 1$, where $|b| = 1$ and $\arg b \in [\pi, (1 + \alpha)\pi]$.

(4) $\varphi(z) = -2bi + [2(2b - 1)/\pi] \log z$ for $\alpha = 1$, where $b \in [0, 1]$.

Here, the complex power and logarithmic functions are given by their principal value in $\mathbb{C}^+$. One can also find a formula for the density of the $\boxplus$-stable laws in [Bercovici and Pata 1999]. Above all, we mention that the case $\alpha = 2$ corresponds to the stable type of the standard semicircular law.

The interest in the class of freely stable laws arises from the fact that a measure ν is $\boxplus$-stable if and only if there exist a sequence $\{X_i\}_{i=1}^\infty$ of identically distributed free random variables and constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that the distribution of the normalized sum $S_n = \sum_{i=1}^n (X_i - b_n)/a_n$ converges weakly to the law ν. In this case, the common distribution of the sequence X_i is said to belong to the *free domain of attraction* of the stable law ν. Thus, up to a change of scale and location, the distributional behavior of a large free convolution $\mu^{\boxplus n}$ for a measure μ in a free domain of attraction can be estimated using the corresponding freely stable law.

Free domains of attraction for $\boxplus$-stable laws are determined in [Bercovici and Pata 1999], showing that these domains of attraction coincide with their classical counterparts relative to the classical convolution. In the semicircular case, the free domain of attraction consists of all nondegenerate measures $\mu \in \mathcal{M}$ such that the truncated variance function

$$H_\mu(x) = \int_{-x}^{x} t^2\, d\mu(t), \quad x > 0,$$

satisfies $\lim_{x\to\infty} H_\mu(cx)/H_\mu(x) = 1$ for any given $c > 0$. This is in parallel to the classical theory of central limit theorems, that is, convergence to a Gaussian law.

With that being said, the following result shows that the quality of freely stable approximation is in fact much better than its classical counterpart. This result is stated in the general framework of triangular arrays with identical rows.

Proposition 5.2. *Let ν be a $\boxplus$-stable law for which the weak approximation $\mu_n^{\boxplus k_n} \to \nu$ holds. Then the measure $\mu_n^{\boxplus k_n}$ superconverges to the law ν on $\mathbb{R}$.*

Proof. This is a direct consequence of Theorem 4.1 and the criterion (5-1). Indeed, one has $L = \infty$ in all cases of the index α, which implies that $0 \notin F_\nu(\mathbb{R})$. □

In particular, the preceding result generalizes the superconvergence for measures with finite variance in [Wang 2010] to the entire free domain of attraction of the semicircular law.

Notice that stable approximation to the free sum S_n could fail for any choice of constants a_n and b_n if the common distribution μ of the summands X_i does not belong to any free domain of attraction, but even in this case one may still have weak convergence along some subsequence S_{k_n}. The limit ν in this situation is necessarily $\boxplus$-infinitely divisible, and hence Theorem 4.1 still applies to this case. The law μ in this case is said to belong to the *free domain of partial attraction* of the law ν. In fact, a probability distribution is $\boxplus$-infinitely divisible if and only if its free domain of partial attraction is nonempty. It is also well known that the domain of partial attraction of a stable law is strictly larger than its domain of attraction in both free and classical theories. We refer to [Bercovici and Pata 1999] for the details of these results.

Poisson approximation. Here we study an example of freely infinitely divisible approximation relative to Poisson type limit theorems. Let μ be an arbitrary probability measure on $\mathbb{R}$, $\mu \neq \delta_0$, and let $\lambda > 0$ be a given parameter. Recall that the *compound free Poisson distribution* $\nu_{\lambda,\mu}$ with rate λ and jump distribution μ is the weak limit of

$$[(1-\lambda/n)\delta_0 + (\lambda/n)\mu]^{\boxplus n}$$

as $n \to \infty$ [Voiculescu et al. 1992]. The law $\nu_{\lambda,\mu}$ is $\boxplus$-infinitely divisible, and its free generating pair is given by

$$\gamma = \lambda \int_{\mathbb{R}} \frac{t}{1+t^2}\, d\mu(t), \qquad d\sigma(t) = \lambda \frac{t^2}{1+t^2}\, d\mu(t).$$

Thus, we see immediately that $L = \lambda$ and $t_{\nu_{\lambda,\mu}} = 0$ in this case, which leads further to the following result:

Proposition 5.3. *The origin is an atom of mass* $1-\lambda$ *for the law* $\nu_{\lambda,\mu}$ *if and only if the parameter* λ *is less than* 1. *If* $\lambda > 1$, *then the superconvergence phenomenon in any weak approximation* $\mu_n^{\boxplus k_n} \to \nu_{\lambda,\mu}$ *holds globally on* $\mathbb{R}$.

Note the case $\mu = \delta_1$ corresponds to the approximation by Marčenko–Pastur law:

$$d\nu_{\lambda,\delta_1}(t) = \begin{cases} \frac{1}{2\pi t}\sqrt{4\lambda - (t-1-\lambda)^2}\,\chi(t)\,dt & \text{if } \lambda \geq 1; \\ (1-\lambda)\delta_0 + \frac{1}{2\pi t}\sqrt{4\lambda - (t-1-\lambda)^2}\,\chi(t)\,dt & \text{if } 0 < \lambda < 1, \end{cases}$$

where χ stands for the indicator function of the open interval $((1-\sqrt{\lambda})^2, (1+\sqrt{\lambda})^2)$. Clearly, the measure ν_{1,δ_1} has no atom and yet $F_{\nu_{1,\delta_1}}(0) = 0$.

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