NORM CONSTANTS IN CASES OF
THE CAFFARELLI–KOHN–NIRENBERG INEQUALITY

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Based on elementary linear algebra, we provide radically simplified proofs using quasiconformal changes of variables to obtain sharp constants and optimizers in cases of the Caffarelli–Kohn–Nirenberg inequality. Some of our results were obtained earlier by Lam and Lu.

1. Introduction

The aim of this elementary note is to obtain sharp constants and also the minimizer for special cases of the Caffarelli–Kohn–Nirenberg (CKN) inequality via change of variables employing a suitable quasiconformal map. We start by recalling the main inequality that we are interested in, proved first in [Caffarelli et al. 1984]:

$$(1\text{-}1) \quad \left(\int_{\mathbb{R}^n} |f(x)|^r |x|^{\alpha n} \, dx \right)^{1/r}$$
$$\leq C \left(\int_{\mathbb{R}^n} |f(x)|^s |x|^{\alpha n} \, dx \right)^{(1-t)/s} \left(\int_{R^n} |\nabla f(x)|^p |x|^{\alpha(n-p)} \, dx \right)^{t/p},$$

where $1 \leq p < n$ and $1/r = (1-t)/s + t(n-p)/(np)$ with $0 \leq t \leq 1$ and $a > -1$.

In our investigation of the best constant C for which inequality (1-1) holds, we will use the constant $M(s, r, p)$, for s, r and p as above, appearing in the inequality

$$(1\text{-}2) \qquad \qquad \|f\|_r \leq M(s, r, p) \|f\|_s^{1-t} \|\nabla f\|_p^t.$$

The constant and optimizers of this last inequality are the subjects of this result:

Theorem 1.1 [del Pino and Dolbeault 2013]. *Let $1 < p \leq n$ and $r > 1$ such that $r \leq p(n-1)/(n-p)$, and $s = p(r-1)/(p-1)$. There exists a positive constant $M(s, r, p)$ such that for every $f \in L^r(\mathbb{R}^n)$ with $\nabla f \in L^p(\mathbb{R}^n)$, we have*

$$\begin{cases} \|f\|_r \leq M(s, r, p) \|f\|_s^{1-t} \|\nabla f\|_p^t & \text{if } r > p, \\ \|f\|_s \leq M(s, r, p) \|f\|_r^{1-t} \|\nabla f\|_p^t & \text{if } r < p, \end{cases}$$

where t is taken as above. Moreover the best constant is achieved by the function:

(1-3) $$f_0(x) = A(1 + B|x - x_0|^{p/(p-1)})^{-(p-1)/(r-p)},$$

with A and B $\in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and B has the sign of a − p.

The proof of (1-2) follows by noticing that

$$\|f\|_r \leq \|f\|_s^{1-t} \|f\|_{np/(n-p)}^t.$$

We then apply the Sobolev inequality to the second term on the right

$$\|f\|_{np/(n-p)} \leq C_p \|\nabla f\|_p,$$

to conclude. In this paper, we show that the optimization problem for the best constant of (1-1) exhibit different behavior for the two cases $\alpha > 0$ and $0 > \alpha > -1$. In the case $0 > \alpha > -1$, we compute the best constant and we show that the optimizer is radial. In the case $\alpha > 0$, we also compute the best constant and we show that in this case there is a break in the symmetry of the optimizers, since the best constant cannot be obtained by radial functions anymore. More importantly we establish that there is no optimizer. We mainly rely on a study of the eigenvectors and eigenvalues of the differential of the quasiconformal change of variable that we will use. In the sequel we set $\phi(x) = x\,|x|^\alpha$. The results of this paper can be stated as follows:

Theorem 1.2. *The sharp constant in the CKN inequality* (1-1) *for $-1 < \alpha < 0$ and any function f, radial or otherwise, is given by*

$$(1 + \alpha)^{t/n-t} M(s, r, p),$$

where $M(s, r, p)$ is the constant in (1-2). *Moreover the optimizer of* (1-1) *is then a radial function and can be taken to be $f \circ \phi$, where f can be taken to be the radial optimizers in the cases investigated in* [del Pino and Dolbeault 2013], *namely* (1-3).

The result in Theorem 1.2 was established earlier by Nguyen Lam and Guozhen Lu [2017] using the quasiconformal map that we use in our work. Our proof in part is a radically simplified approach. A very comprehensive list of references on this topic is found in [Lam and Lu 2017]. In [Dolbeault et al. 2016], the authors also investigate the symmetry and symmetry breaking of the optimizers of the CKN, in the case $p = 2$ and $t = 1$ using a nonlinear flow approach. In the case $t \neq 1$ and $p = 2$, the best constants of the CKN inequality were investigated in [Dolbeault and Esteban 2012]. In addition to Theorem 1.2 we prove a symmetry breaking phenomenon for the case $\alpha > 0$:

Theorem 1.3. *The sharp constant for the CKN inequality for $\alpha > 0$ is given by*

$$(1 + \alpha)^{t/n} M(s, r, p).$$

Moreover, there is no optimizer for the inequality and this constant is strictly bigger than the one obtained for radial functions.

Our proofs also show the following theorem for radial functions for all $\alpha > -1$:

Theorem 1.4. *The sharp constant for inequality* (1-1) *restricted to radial functions for $\alpha > -1$ is given by*

$$(1 + \alpha)^{t/n - t} M(s, r, p).$$

2. Proofs of the theorems

Our aim is now to make quasiconformal changes of variables in (1-2) with explicit information on the Jacobian and eigenvalues of the quasiconformal changes of variables. To this end, the eigenvalues and Jacobians can be calculated explicitly using a linear algebra trick in [Chanillo and Torchinsky 1986, Lemma 2.23, p. 14]. This method was introduced there to calculate the Hessian that appears in stationary phase calculations, but the linear algebra idea goes back to the calculation of what are called permanents and bordered matrices and may be found in classical books on algebra.

If we set $y = \phi(x)$ in (1-2), then

$$Dy = D\phi(x)$$

and so if we set $A = D\phi$ and use the chain rule, (1-2) becomes

$$(2\text{-}1) \quad \left(\int_{\mathbb{R}^n} |f \circ \phi(x)|^r |J_\phi(x)| dx \right)^{1/r} \leq M(s, r, p) \left(\int_{\mathbb{R}^n} |f \circ \phi(x)|^s |J_\phi(x)| dx \right)^{(1-t)/s}$$
$$\times \left(\int_{\mathbb{R}^n} |(A^{-1})^\star \nabla (f \circ \phi)|^p |J_\phi(x)| dx \right)^{t/p}.$$

Here J_ϕ is the Jacobian of the map $y = \phi(x)$, that is $|J_\phi(x)| = |\det D\phi(x)|$ and $B^\star$ denotes the transpose of the matrix B. To obtain the Caffarelli–Kohn–Nirenberg inequalities in some cases we simply choose the explicit quasiconformal map (see also [Chanillo and Wheeden 1992]):

$$(2\text{-}2) \qquad\qquad \phi(x) = x|x|^\alpha, \quad \alpha > -1.$$

Thus the goal is to calculate explicitly the eigenvalues of the differential of (2-2) and thus we have full information of the matrix A above and in particular the Jacobian of (2-2).

Remark. We remark that using [del Pino and Dolbeault 2013] we can also consider the case of $p = n$ and the Onofri inequality and Moser–Trudinger type inequalities.

Lemma 2.1. *Given the map $\phi(x)$ as in* (2-2), *the differential $A = D\phi(x)$ is unitarily diagonalizable and the eigenvalues of A are given by*

$$\lambda_1 = (1 + \alpha)|x|^\alpha, \quad \lambda_2 = \cdots = \lambda_n = |x|^\alpha.$$

Thus as a corollary we obtain that the Jacobian $J_\phi(x)$ is

$$|J_\phi(x)| = (1+\alpha)|x|^{\alpha n}.$$

Proof. The proof of Lemma 2.1 involves implementing the elementary proof of Lemma 2.23, in [Chanillo and Torchinsky 1986] in this special situation. Since $D\phi$ is a symmetric matrix, $D\phi$ is unitarily diagonalizable, that is one can write $A = QRQ^t$, where Q is a rotation matrix and R a diagonal matrix. It is enough to compute the eigenvalues for $D\phi$; the Jacobian formula follows by multiplying the eigenvalues. First note

$$D\phi(x) = A = \begin{pmatrix} |x|^\alpha + \alpha x_1^2 |x|^{\alpha-2} & \cdots & \alpha x_1 x_j |x|^{\alpha-2} & \cdots & \alpha x_1 x_n |x|^{\alpha-2} \\ \vdots & & \vdots & & \vdots \\ \alpha x_1 x_n |x|^{\alpha-2} & \cdots & \alpha x_n x_j |x|^{\alpha-2} & \cdots & |x|^\alpha + \alpha x_n^2 |x|^{\alpha-2} \end{pmatrix}.$$

Next

$$A - \lambda I = \begin{pmatrix} |x|^\alpha + \alpha x_1^2 |x|^{\alpha-2} - \lambda & \cdots & \alpha x_1 x_j |x|^{\alpha-2} & \cdots & \alpha x_1 x_n |x|^{\alpha-2} \\ \vdots & & \vdots & & \vdots \\ \alpha x_1 x_n |x|^{\alpha-2} & \cdots & \alpha x_n x_j |x|^{\alpha-2} & \cdots & |x|^\alpha + \alpha x_n^2 |x|^{\alpha-2} - \lambda \end{pmatrix}.$$

It follows that

$$\det(A - \lambda I) = |x|^{n(\alpha-2)} \det C,$$

where

$$C = \begin{pmatrix} |x|^2 + \alpha x_1^2 - \lambda |x|^{2-\alpha} & \cdots & \alpha x_1 x_j & \cdots & \alpha x_1 x_n \\ \vdots & & \vdots & & \vdots \\ \alpha x_1 x_n & \cdots & \alpha x_n x_j & \cdots & |x|^2 + \alpha x_n^2 - \lambda |x|^{2-\alpha} \end{pmatrix}.$$

To compute the characteristic polynomial of $D\phi$ we simply compute $\det C$. It is now that we use the trick in [Chanillo and Torchinsky 1986]. We simply add an extra row and column to C such that the new matrix now with $n+1$ rows and $n+1$ columns has the same determinant as C. Thus we form the matrix D given by

$$D = \begin{pmatrix} 1 & x_1 & \cdots & x_j & \cdots & x_n \\ 0 & |x|^2 + \alpha x_1^2 - \lambda |x|^{2-\alpha} & \cdots & \alpha x_1 x_j & \cdots & \alpha x_1 x_n \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & \alpha x_1 x_n & \cdots & \alpha x_n x_j & \cdots & |x|^2 + \alpha x_n^2 - \lambda |x|^{2-\alpha} \end{pmatrix}.$$

Note $\det C = \det D$. Now we perform elementary row operations in D that preserve the determinant. We replace row R_j, $j \geq 2$ by $R_j - \alpha x_{j-1} R_1$ where R_1 is row 1.

The new matrix we get is

$$E = \begin{pmatrix} 1 & x_1 & \cdots & x_j & \cdots & x_n \\ -\alpha x_1 & |x|^2 - \lambda|x|^{2-\alpha} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ -\alpha x_j & 0 & \cdots & |x|^2 - \lambda|x|^{2-\alpha} & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ -\alpha x_n & 0 & \cdots & 0 & \cdots & |x|^2 - \lambda|x|^{2-\alpha} \end{pmatrix}.$$

Note $\det D = \det E$ and the matrix we get if we remove the first row and first column of E is a diagonal matrix with $|x|^2 - \lambda|x|^{2-\alpha}$ on the diagonal. To compute $\det E$ we simply expand for the determinant using the first row of E and then expanding by the j-th row of the subsequent cofactor matrices we get for the entry $a_{1,j+1}$. We get

$$\det E = (|x|^2 - \lambda|x|^{2-\alpha})^n + \alpha(|x|^2 - \alpha|x|^{2-\alpha})^{n-1} \sum_{j=1}^{n} x_j^2,$$

which is

$$(|x|^2 - \lambda|x|^{2-\alpha})^n + \alpha(|x|^2 - \alpha|x|^{2-\alpha})^{n-1} |x|^2.$$

The expression above obviously factors as

$$(2\text{-}3) \qquad (|x|^2 - \lambda|x|^{2-\alpha})^{n-1}((1+\alpha)|x|^2 - \lambda|x|^{2-\alpha}).$$

From (2-3) the conclusion of our Lemma follows because

$$\det(A - \lambda I) = 0 = (|x|^2 - \lambda|x|^{2-\alpha})^{n-1}((1+\alpha)|x|^2 - \lambda|x|^{2-\alpha}). \qquad \square$$

First note that $A = D\phi$ is a symmetric matrix and thus there exist rotation matrices Q such that,

$$A^{-1} = (A^{-1})^\star = Q\hat{D}Q^t,$$

where $\hat{D}$ is diagonal. Using the eigenvalues of A computed from Lemma 2.1 above we may write

$$\hat{D} = |x|^{-\alpha} D,$$

where

$$D = \mathrm{diag}((1+\alpha)^{-1}, 1, 1, \ldots, 1).$$

Thus, by Lemma 2.1,

$$\int_{\mathbb{R}^n} |(A^{-1})^\star \nabla f|^p |J_\phi(x)| \, dx = (1+\alpha) \int_{\mathbb{R}^n} |QDQ^t(\nabla f)|^p |x|^{\alpha(n-p)} \, dx.$$

We now apply Lemma 2.1 to (2-1) and we get with $M(s, r, p)$ the constant that occurs in (1-2),

$$(2\text{-}4) \quad \left(\int_{\mathbb{R}^n} |f(x)|^r |x|^{\alpha n} \, dx\right)^{1/r} \leq (1+\alpha)^{t/n} A_\alpha M(s, r, p) \left(\int_{\mathbb{R}^n} |f(x)|^s |x|^{\alpha n} \, dx\right)^{(1-t)/s}$$

$$\times \left(\int_{\mathbb{R}^n} |\nabla f(x)|^p |x|^{\alpha(n-p)} \, dx\right)^{t/p},$$

where we define

$$A_\alpha = \sup_f \left[\frac{\int_{\mathbb{R}^n} |QDQ^t(\nabla f)(x)|^p |x|^{\alpha(n-p)} \, dx}{\int_{\mathbb{R}^n} |\nabla f(x)|^p |x|^{\alpha(n-p)} \, dx}\right]^{t/p}.$$

The supremum is taken over those functions f where the denominator in the definition above is finite.

Lemma 2.2. *For $\alpha > -1$,*

$$B_\alpha \leq A_\alpha \leq C_\alpha = \begin{cases} 1, & \alpha \geq 0, \\ (1+\alpha)^{-t}, & -1 < \alpha < 0, \end{cases}$$

with

$$B_\alpha = \left[\frac{\int_{S^{n-1}} \left(\left[\frac{1}{(1+\alpha)^2} - 1\right] \cos^2 \psi(\sigma) + 1\right)^{p/2} d\sigma}{\int_{S^{n-1}} d\sigma}\right]^{t/p},$$

where

$$\cos \psi(\sigma) = \langle \sigma, v \rangle,$$

where v is a unit eigenvector for the eigenvalue $1/(1+\alpha)$ at $\sigma \in S^{n-1}$. Moreover, on radial functions f we may take $A_\alpha = B_\alpha$ for any $\alpha > -1$ and the ratio defining A_α is identically B_α for all radial functions without the supremum.

Proof. We now verify the assertions made about A_α. We note that pointwise

$$|QDQ^t(\nabla f)(x)| = |DQ^t \nabla f(x)| \leq C_\alpha^{1/t} |\nabla f(x)|.$$

This establishes $A_\alpha \leq C_\alpha$.

Next we establish the lower bound on A_α. Here we assume f is radial. Now note

$$(2\text{-}5) \qquad\qquad |x|^{-\alpha} A = QD^{-1}Q^t.$$

The coefficients of A are homogeneous of degree α and thus the coefficients of the left side of (2-5) are homogeneous of degree 0. Since D is a constant matrix, it follows that the coefficients of Q, Q^t are functions of $\sigma \in S^{n-1}$. We now wish to consider for f radial the expression

$$\frac{\int_{\mathbb{R}^n} |QDQ^t(\nabla f)|^p |x|^{\alpha(n-p)} \, dx}{\int_{\mathbb{R}^n} |\nabla f|^p |x|^{\alpha(n-p)} \, dx}.$$

Using the fact that the coefficients of Q depend only on σ we see when f is radial, the expression above when converted to polar coordinates is identical to

$$(2\text{-}6) \qquad \frac{\int_{S^{n-1}} |Q(\sigma)DQ^t(\sigma)\nabla r|^p \, d\sigma}{\int_{S^{n-1}} d\sigma}.$$

Now let $\{e_i(\sigma)\}_{i=1}^n$ be an orthonormal basis of eigenvectors for QDQ^t. Then since $\nabla r = \sigma$, we see that

$$|QDQ^t\nabla r|^2 = \frac{1}{(1+\alpha)^2}\langle e_1, \sigma\rangle^2 + \sum_{j=2}^n \langle \sigma, e_j\rangle^2.$$

The expression above can be rearranged as

$$\left(\frac{1}{(1+\alpha)^2} - 1\right)\cos^2\psi + 1.$$

Substituting this expression into (2-6) we readily establish that $B_\alpha \le A_\alpha$. Since we have equality at every step in the computation above, we also obtain that $A_\alpha = B_\alpha$ when f is radial. $\qquad\square$

Lemma 2.3. *For $\alpha > -1$, we have,*

$$B_\alpha = (1+\alpha)^{-t}.$$

Proof. From the formula for $A = D\phi$, the eigenvalue equation for $A - (1+\alpha)I$ is

$$(2\text{-}7) \qquad (\sigma_j^2 - 1)y_j + \sum_{k \ne j} \sigma_k\sigma_j y_k = 0, \quad j = 1, 2, \ldots, n,$$

where the eigenvector is $v = y = (y_1, y_2, \ldots, y_n)$. Now we set $y = \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and we get the left side of (2-7) is

$$\sigma_j(\sigma_j^2 - 1) + \sigma_j \sum_{k \ne j} \sigma_k^2 = \sigma_j(\sigma_j^2 - 1) + \sigma_j(1 - \sigma_j^2) = 0.$$

Thus σ is the unit eigenvector for the eigenvalue $(1+\alpha)$, which we already know has a 1-dimensional eigenspace. Thus,

$$\cos\psi = \langle \sigma, \sigma\rangle = 1,$$

and it follows from the expression for B_α in the statement of Lemma 2.2 that for any $\alpha > -1$,

$$B_\alpha = (1+\alpha)^{-t}. \qquad\square$$

This lemma shows in particular that if we restrict (1-1) to radial functions, then the sharp constant is $(1+\alpha)^{t/n-t}M(s, r, p)$, as stated in Theorem 1.4.

Corollary 2.4. *When* $-1 < \alpha < 0$, *then*

$$B_\alpha = A_\alpha = C_\alpha = (1 + \alpha)^{-t}.$$

Proof. The proof follows in an obvious manner by combining the conclusions of Lemmas 2.2 and 2.3, which yields $B_\alpha = C_\alpha$ when $-1 < \alpha < 0$.   □

Thus the sharp constant in (2-4) is established when $-1 < \alpha < 0$.

3. The case $\alpha > 0$

For the case $\alpha > 0$, so far we have the upper and lower bounds for the sharp constant that is $(1 + \alpha)^{-t} \le A_\alpha \le 1$, in (2-4). Also if we restrict to radial functions then

$$A_\alpha(\text{radial}) = (1 + \alpha)^{-t}.$$

Now if we check closely the computations in Lemma 2.3, we obtain

$$|QDQ^t(\nabla f)(x)|^2 = \left[\left(\frac{1}{(1+\alpha)^2} - 1 \right) \cos^2 \psi + 1 \right] |\nabla f(x)|^2,$$

where $\cos \psi = \langle v, w \rangle$, v is a unit vector in the direction of the eigenvector for $(1 + \alpha)$ and w the unit vector in the direction of ∇f. In particular v is radial at all points $x \in \mathbb{R}^n$. But now for $\alpha > 0$, $(1 + \alpha)^{-2} - 1 < 0$, and so it is advantageous to arrange $\cos \psi = 0$ as opposed to $\alpha < 0$ when $(1 + \alpha)^{-2} - 1 > 0$ and so there it is advantageous to have $\cos \psi = 1$ or functions to be radial. So the idea of proving Theorem 1.3 is to choose a function gradient having a big angular component that dominates the radial component. If one wants an optimizer for the case $\alpha > 0$, ∇f needs to be orthogonal to v, that is tangent to the sphere at all points. Notice the tangential directions to the sphere are eigenvectors to the eigenvalue 1 for $A = D\phi$. But in particular in 3D, $\nabla \times \nabla f = 0$; no such functions exist, or if f has some smoothness the vector field ∇f on S^2 will be smooth and tangential to S^2 which cannot happen by the hairy ball theorem. In fact, we show the following:

Lemma 3.1. *If* $\alpha > 0$, *then* $A_\alpha = 1$.

Proof. Indeed, based on the computations above, we have that the matrix $\tilde{A} = QDQ^t$ has two eigenvalues. The first one is $1/(1 + \alpha) < 1$, corresponding to the radial direction ∇r and the second eigenvalue is 1 with multiplicity $(n - 1)$ corresponding to the angular directions (tangential to S^{n-1}). We would like to estimate the quantity

$$F(f) = \frac{\int_{\mathbb{R}^n} |\tilde{A} \nabla f|^p |x|^{\alpha(n-p)} \, dx}{\int_{\mathbb{R}^n} |\nabla f|^p |x|^{\alpha(n-p)} \, dx}$$

for some choice of function f knowing that $A_\alpha = \sup_f F(f)$. We use spherical coordinates $(r, \phi_1, \ldots, \phi_{n-1})$, and we form

$$f_k(r, \phi_1, \ldots, \phi_{n-1}) = h(r) \sin(\phi_1) \cdots \sin(\phi_{n-2}) \cos(k\phi_{n-1}),$$

where $h : [0, \infty) \to \mathbb{R}$ is smooth and $h(t) = 0$ for $t < 1$ and $t > 4$ and $k \in \mathbb{N}$. For the sake of simplicity, we do the computation in $n = 3$; the higher dimensional case is similar.

So $f_k = h(r) \sin \phi \cos(k\theta)$, and thus

$$\nabla f_k = h'(r) \sin \phi \cos(k\theta) \boldsymbol{u}_r + \frac{h(r)}{r} \cos(\phi) \cos(k\theta) \boldsymbol{u}_\phi - \frac{h(r)}{r} k \sin(k\theta) \boldsymbol{u}_\theta,$$

where $(\boldsymbol{u}_r, \boldsymbol{u}_\phi, \boldsymbol{u}_\theta)$ is the standard orthonormal base defining the spherical coordinate system. Thus

$$\tilde{A} \nabla f_k = \frac{h'(r) \sin \phi \, \cos(k\theta)}{(1 + \alpha)} \boldsymbol{u}_r + \frac{h(r)}{r} \cos(\phi) \cos(k\theta) \boldsymbol{u}_\phi - \frac{h(r)}{r} k \sin(k\theta) \boldsymbol{u}_\theta.$$

We compute then

$$|\tilde{A} \nabla f_k|^p = \left[\frac{h'(r)^2 \sin^2 \phi \cos^2(k\theta)}{(1 + \alpha)^2} \right.$$
$$\left. + \left(\frac{h(r)}{r} \right)^2 \cos^2 \phi \cos^2(k\theta) + \left(\frac{h(r)}{r} \right)^2 k^2 \sin^2(k\theta) \right]^{\frac{p}{2}}$$
$$= \left[\cos^2(k\theta) \left(\frac{h'(r)^2 \sin^2 \phi}{(1 + \alpha)^2} + \left(\frac{h(r)}{r} \right)^2 \cos^2 \phi \right) + k^2 \left(\frac{h(r)}{r} \right)^2 \sin^2(k\theta) \right]^{\frac{p}{2}}.$$

Hence,

$$\int_{\mathbb{R}^3} |\tilde{A} \nabla f_k|^p |x|^{\alpha(n-p)} \, dx$$
$$= \int_1^4 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[\cos^2(k\theta) \left(\frac{h'(r)^2 \sin^2 \phi}{(1 + \alpha)^2} + \left(\frac{h(r)}{r} \right)^2 \cos^2 \phi \right) \right.$$
$$\left. + k^2 \left(\frac{h(r)}{r} \right)^2 \sin^2(k\theta) \right]^{\frac{p}{2}} r^{\alpha(n-p)+2} \sin \phi \, d\theta \, d\phi \, dr$$
$$= k^p \int_1^4 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[\frac{\cos^2(k\theta)}{k^2} \left(\frac{h'(r)^2 \sin^2 \phi}{(1 + \alpha)^2} + \left(\frac{h(r)}{r} \right)^2 \cos^2 \phi \right) \right.$$
$$\left. + \left(\frac{h(r)}{r} \right)^2 \sin^2(k\theta) \right]^{\frac{p}{2}} r^{\alpha(n-p)+2} \sin \phi \, d\theta \, d\phi \, dr$$
$$= k^p \int_1^4 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[\frac{\cos^2(u)}{k^2} \left(\frac{h'(r)^2 \sin^2 \phi}{(1 + \alpha)^2} + \left(\frac{h(r)}{r} \right)^2 \cos^2 \phi \right) \right.$$
$$\left. + \left(\frac{h(r)}{r} \right)^2 \sin^2(u) \right]^{\frac{p}{2}} r^{\alpha(n-p)+2} \sin \phi \, du \, d\theta \, dr.$$

Therefore we have

$$\int_{\mathbb{R}^3} |\tilde{A}\nabla f_k|^p |x|^{\alpha(n-p)} \, dx$$
$$= k^p \left[\int_1^4 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[\frac{h(r)}{r} \sin(u) \right]^p r^{\alpha(n-p)+2} \sin\phi \, du \, d\phi \, dr + o(1) \right].$$

A similar computation yields

$$\int_{\mathbb{R}^3} |\nabla f_k|^p |x|^{\alpha(n-p)} \, dx$$
$$= k^p \left[\int_1^4 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \left[\frac{h(r)}{r} \sin(u) \right]^p r^{\alpha(n-p)+2} \sin\phi \, du \, d\phi \, dr + o(1) \right].$$

Therefore

$$F(f_k) = 1 + o(1), \quad \text{as} \quad k \to \infty.$$

Combining this last estimate with Lemma 2.2, we get the conclusion. □

Notice that with this lemma, we have the proof of Theorem 1.3.

Remark. One can see that this sequence of functions f_k always satisfies

$$F(f_k) \to 1 \quad \text{as} \quad k \to \infty$$

for all $\alpha > -1$, but if $\alpha < 0$, we have that $A_\alpha = (1+\alpha)^{-t} > 1$, thus the sequence f_k in the case $\alpha < 0$ is not optimizing and as we saw earlier, the optimizer is radially symmetric. The sequence f_k gains importance in the case $\alpha > 0$ since $(\alpha + 1)^{-t} < 1$. Hence there is a symmetry breaking phenomenon and the radially symmetric functions cannot be optimizers anymore.

On the other hand, by the Riemann–Lebesgue lemma, $f_k \rightharpoonup 0$ as $k \to \infty$, hence we do not obtain an optimizer in this case.

Acknowledgement

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