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NONCOMMUTATIVE GEOMETRY OF HOMOGENIZED QUANTUM $\mathfrak{sl}(2, \mathbb{C})$

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We examine the relationship between certain noncommutative analogues of projective 3-space, $\mathbb{P}^3$, and the quantized enveloping algebras $U_q(\mathfrak{sl}_2)$. The relationship is mediated by certain noncommutative graded algebras S, one for each $q \in \mathbb{C}^\times$, having a degree-two central element c such that $S[c^{-1}]_0 \cong U_q(\mathfrak{sl}_2)$. The noncommutative analogues of $\mathbb{P}^3$ are the spaces $\mathrm{Proj}_{\mathrm{nc}}(S)$. We show how the points, fat points, lines, and quadrics, in $\mathrm{Proj}_{\mathrm{nc}}(S)$, and their incidence relations, correspond to finite-dimensional irreducible representations of $U_q(\mathfrak{sl}_2)$, Verma modules, annihilators of Verma modules, and homomorphisms between them.

1. Introduction

This paper concerns the interplay between the geometry of some noncommutative analogues of $\mathbb{P}^3$ and the representation theory of the quantized enveloping algebras, $U_q(\mathfrak{sl}_2)$, of $\mathfrak{sl}(2, \mathbb{C})$. We always assume that q is not a root of unity.

1A. $\mathrm{Proj}_{\mathrm{nc}}(S)$ *and* $U_q(\mathfrak{sl}_2)$. In Section 2D, we define a family of noncommutative graded algebras $S = \mathbb{C}[E, F, K, K']$ depending on a parameter $q \in \mathbb{C} - \{0, \pm 1, \pm i\}$ that have the same Hilbert series and the "same" homological properties as the polynomial ring in four variables. For these reasons the noncommutative spaces $\mathrm{Proj}_{\mathrm{nc}}(S)$ have much in common with $\mathbb{P}^3$. The element KK' belongs to the center of S and $S[(KK')^{-1}]_0 \cong U_q(\mathfrak{sl}_2)$. Thus, $U_q(\mathfrak{sl}_2)$ is the coordinate ring of the "open

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complement” to the union of the “hyperplanes” $\{K=0\}$ and $\{K'=0\}$ in $\mathrm{Proj}_{\mathrm{nc}}(S)$. This analogy can be formalized: there is an abelian category $\mathrm{QCOH}(\,\cdot\,)$, defined below, that plays the role of the category of quasicoherent sheaves and an adjoint pair of functors

$$\mathrm{QCOH}(\mathrm{Proj}_{\mathrm{nc}}(S)) \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathsf{Mod}(U_q(\mathfrak{sl}_2)) \tag{1-1}$$

that behave like the inverse and direct image functors for an open immersion $j:\mathbb{P}^3 - \{\text{two planes}\} \to \mathbb{P}^3$.

1A1. By definition, $\mathrm{QCOH}(\mathrm{Proj}_{\mathrm{nc}}(S))$ is the quotient category

$$\mathsf{QGr}(S) := \frac{\mathsf{Gr}(S)}{\mathsf{Fdim}(S)},$$

where $\mathsf{Gr}(S)$ denotes the category of $\mathbb{Z}$-graded left S-modules and $\mathsf{Fdim}(S)$ denotes the full subcategory of $\mathsf{Gr}(S)$ consisting of those modules that are the sum of their finite-dimensional submodules. If S were the polynomial ring on four variables, then the category $\mathsf{QGr}(S)$ would be equivalent to $\mathrm{QCOH}(\mathbb{P}^3)$, the category of quasi-coherent sheaves on $\mathbb{P}^3$, and this equivalence would send a graded module M to the $\mathcal{O}_{\mathbb{P}^3}$-module that Hartshorne denotes by $\widetilde{M}$.

1A2. There is an exact functor $\pi^*:\mathsf{Gr}(S)\to\mathsf{QGr}(S)$ that sends a graded S-module M to M viewed as an object in $\mathsf{QGr}(S)$. The composition

$$\mathsf{Gr}(S) \xrightarrow{\pi^*} \mathsf{QGr}(S) = \mathrm{QCOH}(\mathrm{Proj}_{\mathrm{nc}}(S)) \xrightarrow{j^*} \mathsf{Mod}(U_q(\mathfrak{sl}_2)) \tag{1-2}$$

sends a graded S-module M to $M[(KK')^{-1}]_0$.

1A3. The main theme of this paper is the interaction between noncommutative geometry (where $\mathsf{QGr}(S)$ belongs) and representation theory (where $\mathsf{Mod}(U_q(\mathfrak{sl}_2))$ belongs). We show how the points, fat points, lines, and quadrics, in $\mathrm{Proj}_{\mathrm{nc}}(S)$, and their incidence relations, correspond to finite-dimensional irreducible representations of $U_q(\mathfrak{sl}_2)$, Verma modules, annihilators of Verma modules, and homomorphisms between them.

Just as passing from affine to projective geometry provides a more elegant picture that unifies seemingly different objects (affine vs. projective conic sections, for example), passing from the “affine” category $\mathsf{Mod}(U_q(\mathfrak{sl}_2))$ to the “projective” category $\mathrm{QCOH}(\mathrm{Proj}_{\mathrm{nc}}(S))$ results in a more complete picture of $\mathsf{Mod}(U_q(\mathfrak{sl}_2))$.

1B. ***Lines and Verma modules, fat points and finite-dimensional irreducible representations.*** The most important $U_q(\mathfrak{sl}_2)$-modules are its finite-dimensional irreducible representations and its Verma modules. In Section 5, we show that for each Verma module $V\in\mathsf{Mod}(U_q(\mathfrak{sl}_2))$ there is a graded S-module M such that

(1) $V \cong j^*\pi^* M$;

(2) $M \cong S/S\ell^\perp$, where $\ell^\perp \subseteq S_1$ is a codimension-two subspace;

(3) $\dim(M_i) = i + 1$ for all $i \ge 0$, i.e., M has the same Hilbert series as the polynomial ring on two variables;

(4) M is a line module for S;

(5) M is a Cohen–Macaulay S-module.

In Section 5, we also show that for each finite-dimensional irreducible representation L of $U_q(\mathfrak{sl}_2)$ there is a graded S-module F such that

(1) $L \cong j^*\pi^* F$;

(2) $\dim(F_i) = \dim(L)$ for all $i \ge 0$ and $\dim(F_i) = 0$ for all $i < 0$;

(3) every proper quotient of F is finite-dimensional, whence F is a simple object in QCOH($\mathrm{Proj}_{\mathrm{nc}}(S)$);

(4) F is a fat point module for S;

(5) F is a Cohen–Macaulay S-module.

Items (4) are, essentially, definitions; see Section 2B.

1B1. *Point modules and line modules.* Let R be the polynomial ring on four variables with its standard grading. The points in $\mathbb{P}^3 = \mathrm{Proj}(R)$ are in bijection with the graded modules R/I such that $\dim(R_i/I_i) = 1$ for all $i \ge 0$. The lines in $\mathbb{P}^3$ are in bijection with the modules R/I such that $\dim(R_i/I_i) = i+1$ for all $i \ge 0$.

If S is one of the algebras in Section 2D, a graded S-module M is called a *point module*, resp. a *line module*, if it is isomorphic to S/I for some left ideal I such that $\dim(S_i/I_i) = 1$, resp. $\dim(S_i/I_i) = i+1$, for all $i \ge 0$.

There are fine moduli spaces that parametrize the point modules and line modules for S. These fine moduli spaces are called the *point scheme* and *line scheme* respectively. The point scheme for S is a closed subscheme of $\mathbb{P}^3 = \mathbb{P}(S_1^*)$ and the line scheme for S is a closed subscheme of the Grassmannian $\mathbb{G}(1, 3)$ consisting of the lines in $\mathbb{P}^3$.

In Section 4, we determine the line modules and the point modules for S.

1B2. *The point modules for S.* The point scheme, $\mathcal{P}_S$, for S is $C \cup C' \cup L \cup \{p_1, p_2\} \subseteq \mathbb{P}(S_1^*) = \mathbb{P}^3$, the union of two plane conics, C and C', meeting at two points, the line L through those two points, and two additional points (Theorem 4.2). If $M_p = S/Sp^\perp$ is the point module corresponding to $p \in \mathcal{P}_S$, then $(M_p)_{\ge 1}$ is a shifted point module; i.e., $(M_p)_{\ge 1}(1)$ is a point module and therefore isomorphic to $M_{p'}$ for some point $p' \in \mathcal{P}_S$. General results show there is an automorphism $\sigma : \mathcal{P}_S \to \mathcal{P}_S$ such that $p' = \sigma^{-1} p$. Thus, $(M_p)_{\ge 1} \cong M_{\sigma^{-1}p}(-1)$. We determine $\mathcal{P}_S$ and σ in Section 4.

1B3. *The line modules for S.* Theorem 4.5 says that the lines $\ell \subseteq \mathbb{P}^3 = \mathbb{P}(S_1^*)$ for which $S/S\ell^{\perp}$ is a line module are precisely those lines that meet $C \cup C'$ with multiplicity two; i.e., the secant lines to $C \cup C'$. These are exactly the lines lying on a certain pencil of quadrics $Q(\lambda) \subseteq \mathbb{P}^3$, $\lambda \in \mathbb{P}^1$. This should remind the reader of the analogous result for the 4-dimensional Sklyanin algebras in which the lines in $\mathbb{P}^3$ that correspond to line modules are exactly the secant lines to the quartic elliptic curve E.

The labeling of the line modules is such that the Verma module $M(\lambda)$ is isomorphic to $j^*\pi^*(S/S\ell^{\perp})$ for a unique line $\ell \subseteq Q(\lambda)$.

1B4. *Incidence relations.* If $(p)+(p')$ is a degree-two divisor on $C \cup C'$, we write $M_{p,p'}$ for the line module $S/S\ell^{\perp}$, where $\ell^{\perp}$ is the subspace of S_1 that vanishes on the line $\ell \subseteq \mathbb{P}^3 = \mathbb{P}(S_1^*)$ whose scheme-theoretic intersection with $C \cup C'$ is $(p)+(p')$. Proposition 4.8 shows there is an exact sequence

$$0 \to M_{\sigma p,\sigma^{-1}p'}(-1) \to M_{p,p'} \to M_p \to 0.$$

Proposition 4.9 shows that if the line ℓ just referred to meets the line $\{K = K' = 0\} \subseteq \mathcal{P}_S$ at a point p'', there is an exact sequence

$$0 \to M_{\sigma^{-1}p,\sigma^{-1}p'}(-1) \to M_{p,p'} \to M_{p''} \to 0.$$

1B5. *Finite-dimensional simple modules.* Let $n \in \mathbb{N}$. If q is not a root of unity there are two simple $U_q(\mathfrak{sl}_2)$-modules of dimension $n+1$. We label them $L(n,\pm)$ in such a way that there are exact sequences

$$0 \to M(\pm q^{-n-2}) \to M(\pm q) \to L(n,\pm) \to 0 \tag{1-3}$$

in which $M(\lambda)$ denotes the Verma module of highest weight λ.

In Section 5 we show there are S-modules $V(n,\pm)$ that are also $S[(KK')^{-1}]$-modules, and hence modules over $S[(KK')^{-1}]_0 \cong U_q(\mathfrak{sl}_2)$ and, as such, $V(n,\pm) \cong L(n,\pm)$. We define graded S-modules $F(n,\pm)$ such that $F(n,\pm)[(KK')^{-1}]_0 \cong L(n,\pm)$; i.e., if we view $F(n,\pm)$ as an object in $\mathsf{QGr}(S)$, then

$$j^*F(n,\pm) \cong L(n,\pm).$$

Furthermore, we show there are exact sequences

$$0 \to M_{\ell'_{\pm}}(-n-1) \to M_{\ell_{\pm}} \to F(n,\pm) \to 0 \tag{1-4}$$

in $\mathrm{QCOH}(\mathrm{Proj}_{\mathrm{nc}}(S))$ and that (1-3) is obtained from (1-4) by applying the functor j^*, i.e., by restricting the exact sequence (1-4) in $\mathrm{QCOH}(\mathrm{Proj}_{\mathrm{nc}}(S))$ to the "open affine subscheme" $\{KK' \neq 0\}$. Here $M_{\ell_{\pm}}$ denotes the line module $S/S\ell_{\pm}^{\perp}$ corresponding to a line $\ell_{\pm} \subseteq \mathbb{P}(S_1^*) = \mathbb{P}^3$.

1B6. *Heretical Verma modules.* The connections we establish between Verma modules and line modules highlights one way in which the q-deformation $U_q(\mathfrak{sl}_2)$ is "more rigid" or "less symmetric" than the enveloping algebra $U(\mathfrak{sl}_2)$: there is a $\mathbb{P}^1$-family of Borel subalgebras of $\mathfrak{sl}_2$, but there are only two reasonable candidates for the role of the quantized enveloping algebra of a "Borel subalgebra" of quantum $\mathfrak{sl}_2$.

Fix one of the two "Borel subalgebras", $U_q(\mathfrak{b}) \subseteq U_q(\mathfrak{sl}_2)$. It gives rise by induction to Verma modules $M_{\mathfrak{b}}(\lambda) = U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{b})} \mathbb{C}_\lambda$, $\lambda \in \mathbb{C}^\times$. Thus, one obtains two 1-parameter families of Verma modules for $U_q(\mathfrak{sl}_2)$. In sharp contrast, by varying both the Borel subalgebra and the highest weight one obtains a 2-parameter family of Verma modules for $U(\mathfrak{sl}_2)$. Our perspective on $U_q(\mathfrak{sl}_2)$ as a noncommutative open subscheme of a noncommutative $\mathbb{P}^3$ allows us to fit the two 1-parameter families of Verma modules for $U_q(\mathfrak{sl}_2)$ into a single 2-parameter family of modules, thus undoing the rigidification phenomenon alluded to in the previous paragraph. It is these additional Verma-like modules that we call "heretical" in the title of this subsection.

For simplicity of discussion, fix a finite-dimensional simple module $L(n,+)$ and the corresponding fat point module $F(n,+)$ for which $j^*F(n,+) \cong L(n,+)$. The module $L(n,+)$ appears in exactly two sequences of the form (1-3), one for each "Borel subalgebra" of $U_q(\mathfrak{sl}_2)$; in contrast, $F(n,+)$ appears in a 1-parameter family of sequences of the form (1-4), one for each line in one of the rulings on the quadric $Q(q^n)$. Likewise, a fixed finite-dimensional simple $U(\mathfrak{sl}_2)$-module fits into a 1-parameter family of sequences of the form (1-3). If we broadened the definition of a Verma module for $U_q(\mathfrak{sl}_2)$ so as to include j^*M_ℓ for all line modules M_ℓ one would then obtain a 1-parameter family of sequences of the form (1-3).

1B7. *Annihilators of Verma modules and quadrics in* $\operatorname{Proj}_{nc}(S)$. When q is not a root of unity, the center of $U_q(\mathfrak{sl}_2)$ is generated by a single central element C called the Casimir element. A Verma module is annihilated by $C-\nu$ for a unique $\nu \in \mathbb{C}$ and given ν there are, usually, four Verma modules annihilated by $C-\nu$.

There is a nonzero central element $\Omega \in S_2$ such that $C = \Omega(KK')^{-1}$ under the isomorphism $U_q(\mathfrak{sl}_2) \cong S[(KK')^{-1}]_0$. A line module for S is annihilated by $\Omega - \nu KK'$ for a unique $\nu \in \mathbb{C} \cup \{\infty\}$ and given ν there are, usually, two 1-parameter families of line modules annihilated by $\Omega - \nu KK'$. There is an isomorphism

$$\frac{S}{(\Omega-\nu KK')}[(KK')^{-1}]_0 \cong \frac{U_q(\mathfrak{sl}_2)}{(C-\nu)}$$

and an adjoint pair of functors

$$\operatorname{QCOH}\left(\operatorname{Proj}_{nc}\left(\frac{S}{(\Omega-\nu KK')}\right)\right) \underset{j_*}{\overset{j^*}{\rightleftarrows}} \operatorname{Mod}\left(\frac{U_q(\mathfrak{sl}_2)}{(C-\nu)}\right). \tag{1-5}$$

We think of $\mathrm{Proj}_{\mathrm{nc}}(S/(\Omega-\nu KK'))$ as a noncommutative quadric hypersurface in $\mathrm{Proj}_{\mathrm{nc}}(S)$ and think of $U_q(\mathfrak{sl}_2)/(C-\nu)$ as the coordinate ring of a noncommutative affine quadric. Noncommutative quadrics in noncommutative analogues of $\mathbb{P}^3$ were examined in [Smith and Van den Bergh 2013]. The results there apply to the present situation. The line modules for S that are annihilated by $\Omega-\nu KK'$ provide rulings on the noncommutative quadric and the noncommutative quadric is smooth if and only if it has two rulings. We note that $\mathrm{Proj}_{\mathrm{nc}}(S/(\Omega-\nu KK'))$ is smooth if and only if $U_q(\mathfrak{sl}_2)/(C-\nu)$ has finite global dimension.

In Section 1B2, we mentioned the pencil of quadrics $Q(\lambda)\subseteq\mathbb{P}^3$, $\lambda\in\mathbb{P}^1$, that contain $C\cup C'$. The $Q(\lambda)$'s are commutative quadrics and should not be confused with the noncommutative ones in the previous paragraph. If ℓ is a line on $Q(\lambda)$, then $M_\ell=S/S\ell^\perp$ is a line module so is annihilated by $\Omega-\nu KK'$ for some $\nu\in\mathbb{C}\cup\infty$.

1B8. *What happens for $U(\mathfrak{sl}_2)$?* Le Bruyn and Smith [1993] considered a graded algebra $H(\mathfrak{sl}_2)$ that has a central element t in H_1 such that $H[t^{-1}]_0$ is isomorphic to the enveloping algebra $U(\mathfrak{sl}_2)$. They call $H(\mathfrak{sl}_2)$ a *homogenization* of $U(\mathfrak{sl}_2)$,

Since the Hilbert series of H equals that of the polynomial ring in four variables with its standard grading, and since H has "all" the good homological properties the polynomial ring does, they view H as a homogeneous coordinate ring of a noncommutative analogue of $\mathbb{P}^3$, denoted $\mathrm{Proj}_{\mathrm{nc}}(H)$. Because $H[t^{-1}]_0\cong U(\mathfrak{sl}_2)$, there is an adjoint pair of functors j^* and j_* fitting into diagrams like those in (1-1) and (1-2). Because t has degree one, j^* and j_* behave like the inverse and direct image functors associated to the open complement to the hyperplane at infinity in $\mathbb{P}^3$. Le Bruyn and Smith examined the point and line modules for H and showed that these modules are related to the finite-dimensional irreducible representations and Verma modules for $\mathfrak{sl}_2$. The situation for $U(\mathfrak{sl}_2)$ is simpler than that for $U_q(\mathfrak{sl}_2)$.

1B9. *Richard Chandler's results.* We are not the first to compute the point modules and line modules for S. Richard Chandler did this in his Ph.D. thesis [Chandler 2016]. His approach differs from ours. Following a method introduced by Shelton and Vancliff [2002b], he used *Mathematica* to compute a system of 45 quadratic polynomials in the Plücker coordinates on the Grassmannian $\mathbb{G}(1,3)$, the common zero locus of which is the line scheme for S. In contrast, we use the results on central extensions in [Le Bruyn et al. 1996] to determine which lines in $\mathbb{P}^3$ correspond to line modules. The two approaches are complementary.

1C. ***The structure of the paper.*** In Section 2, we define the algebra S, the central focus of our paper, and discuss its position as a degenerate version of the 4-dimensional Sklyanin algebra and a homogenization of $U_q(\mathfrak{sl}_2)$. We introduce the category $\mathsf{QGr}(S)$ and its noncommutative geometry. We focus on point, line, and fat point modules.

$\mathsf{Gr}(S)$	$\mathsf{Proj}_{\mathrm{nc}}(S)$	$\mathsf{Mod}(U_q(\mathfrak{sl}_2))$
Point modules	Points	Finite-dimensional irreducible modules
Line modules	Lines	Verma modules

Table 1. Relation to $U_q(\mathfrak{sl}_2)$-modules.

In Section 3, we examine a Zhang twist D of S. It has the property that $\mathsf{Gr}(D) \equiv \mathsf{Gr}(S)$. Moreover, D has a central element $z \in D_1$ such that $A = D/(z)$ is a 3-dimensional Artin–Schelter regular algebra, thereby making D a central extension of A. This allows us to use the results in [Le Bruyn et al. 1996] to determine the point and line modules of D in terms of those for A.

In Section 4, we use the equivalence $\mathsf{Gr}(D) \equiv \mathsf{Gr}(S)$ to transfer the results about D back to S.

In Section 5, we relate our results about point and line modules for S to results about the finite-dimensional irreducible representations and Verma modules of $U_q(\mathfrak{sl}_2)$. Table 1 summarizes some of these relations.

In Section 6, we show that some of our results can be obtained as "degenerations" of results in [Smith and Stafford 1992; Chirvasitu and Smith 2017; Smith and Staniszkis 1993] about the 4-dimensional Sklyanin algebra.

Figure 1 summarizes the algebras in this paper and their relationships to S.

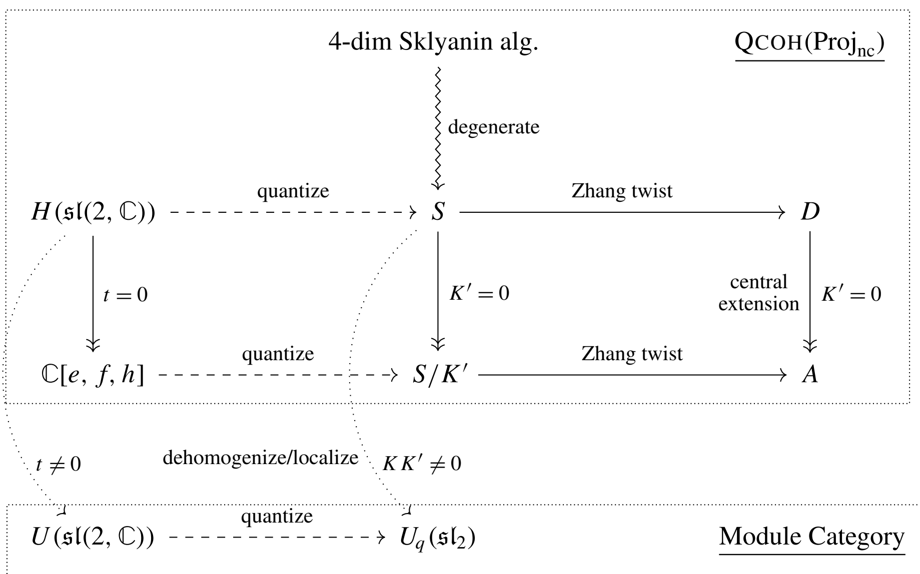

Figure 1. Algebras in this paper, their relationship to S, and their associated categories.

2. Preliminary notions

2A. ***The category* QGr.** Let $\Bbbk$ be a field and R a $\mathbb{Z}$-graded $\Bbbk$-algebra. The category $\mathsf{QGr}(R)$ is defined to be the quotient category

$$\mathsf{QGr}(R) := \frac{\mathsf{Gr}(R)}{\mathsf{Fdim}(R)},$$

where $\mathsf{Gr}(R)$ denotes the category of $\mathbb{Z}$-graded left R-modules with degree-preserving homomorphisms and $\mathsf{Fdim}(R)$ denotes the full subcategory of $\mathsf{Gr}(R)$ consisting of those modules that are the sum of their finite-dimensional submodules.

The categories $\mathsf{QGr}(R)$ and $\mathsf{Gr}(R)$ have the same objects but different morphisms. There is an exact functor $\pi^* : \mathsf{Gr}(R) \to \mathsf{QGr}(R)$ that is the identity on objects. In the situations considered in this paper π^* has a right adjoint π_*. A morphism $f : M \to M'$ becomes an isomorphism in $\mathsf{QGr}(R)$, i.e., $\pi^* f$ is an isomorphism, if and only if $\ker(f)$ and $\operatorname{coker}(f)$ are in $\mathsf{Fdim}(R)$. In particular, a graded R-module is isomorphic to 0 in $\mathsf{QGr}(R)$ if and only if it is the sum of its finite-dimensional modules. Two modules in $\mathsf{Gr}(R)$ are *equivalent* if they are isomorphic in $\mathsf{QGr}(R)$.

If $M \in \mathsf{Gr}(R)$ and $n \in \mathbb{Z}$ we write $M(n)$ for the graded R-module that is M as a left R-module but with new homogeneous components, $M(n)_i = M_{n+i}$. The rule $M \rightsquigarrow M(n)$ extends to an autoequivalence of $\mathsf{Gr}(R)$. Because it sends finite-dimensional modules to finite-dimensional modules, it induces an autoequivalence of $\mathsf{QGr}(R)$ that we denote by $\mathcal{M} \rightsquigarrow \mathcal{M}(n)$.

If $M \in \mathsf{Gr}(R)$ we define $M_{\geq n} := M_n + M_{n+1} + \cdots$. If $R = R_{\geq 0}$, then $M_{\geq n}$ is a submodule of M.

2B. ***Linear modules.*** The importance of linear modules for noncommutative analogues of $\mathbb{P}^n$ was first recognized by Artin, Tate, and Van den Bergh. We recall a few notions from their papers [Artin et al. 1990; 1991]. Let $M \in \mathsf{Gr}(R)$. If $M_n = 0$ for $n \ll 0$ and $\dim M_n < \infty$ for all n, the *Hilbert series* of M is the formal Laurent series

$$H_M(t) = \sum_n (\dim M_n) t^n.$$

We are particularly interested in cyclic modules M with Hilbert series having the form

$$H_M(t) = \frac{n}{(1-t)^d}$$

for some $n, d \in \mathbb{N}$. The *Gelfand–Kirillov* (*GK*) *dimension* of such a module is $d(M) = d$ and its *multiplicity* is n. If $d(M) = d$ and $d(M/N) < d$ for all nonzero submodules N, then M is called *d-critical*. Equivalent modules (in the sense of Section 2A) have the same GK dimension, and also have the same multiplicity if they are not equivalent to 0, so the notions of GK dimension and multiplicity carry over to $\mathsf{QGr}(R)$ as well.

We call M a *linear* module if it is cyclic and its Hilbert series is $(1-t)^{-d}$. The cases $d = 1$ and $d = 2$ play a key role: we call a linear module M a

- *point module* if it is cyclic, 1-critical, and $H_M(t) = (1-t)^{-1}$;
- *line module* if it is cyclic, 2-critical, and $H_M(t) = (1-t)^{-2}$.

We are also interested in modules of higher multiplicity: we call M a

- *fat point module* if it is 1-critical, generated by M_0, and $H_M(t) = n(1-t)^{-1}$ for some $n > 1$.

Point modules and fat point modules are important because, as objects in $\mathsf{QGr}(R)$, they are simple (or irreducible): all proper quotient modules of a 1-critical module are finite-dimensional and therefore zero in $\mathsf{QGr}(R)$. The following result illustrates the relationship between finite-dimensional simple modules and fat point modules.

Lemma 2.1. *Let V be a simple left R-module of dimension $n < \infty$. Let $\mathbb{C}[z]$ be the polynomial ring generated by a degree-one indeterminate, z. Let $V \otimes \mathbb{C}[z]$ be the graded left R-module whose degree-j component is $V \otimes z^j$ with $a \in R_i$ acting as $a(v \otimes z^j) := (av) \otimes z^{i+j}$. Let $\pi : V \otimes \mathbb{C}[z] \to V$ be the R-module homomorphism $v \otimes z^j \mapsto v$.*

(1) *$V \otimes \mathbb{C}[z]$ is a fat point module of multiplicity n.*

(2) *If M is a graded left R-module such that $M = M_{\geq 0}$ and $\psi : M \to V$ is a homomorphism in $\mathsf{Mod}(R)$, then there is a unique homomorphism $\widetilde{\psi} : M \to V \otimes \mathbb{C}[z]$ in $\mathsf{Gr}(R)$ such that $\psi = \pi\psi$, namely $\widetilde{\psi}(m) = \psi(m) \otimes z^n$ for $m \in M_n$.*

2C. *Geometry in* $\mathbf{Proj_{nc}}(R)$. The "noncommutative scheme" $\mathrm{Proj_{nc}}(R)$ is defined implicitly by declaring that the category of "quasicoherent sheaves" on it is $\mathsf{QGr}(R)$,

$$\mathrm{QCOH}(\mathrm{Proj_{nc}}(R)) := \mathsf{QGr}(R).$$

The isomorphism class of a (fat) point module in $\mathsf{QGr}(R)$ is called a (fat) point of $\mathrm{Proj_{nc}}(R)$. Likewise, the isomorphism class of a line module in $\mathsf{QGr}(R)$ is called a line in $\mathrm{Proj_{nc}}(R)$.

2C1. *Origin of the terminology.* Let $\Bbbk$ be an algebraically closed field, and let $R = \Bbbk[x_0, \dots, x_n]$ be the commutative polynomial ring with its standard grading, $\deg(x_j) = 1$ for all j. Then $\mathrm{Proj}(R)$ is $\mathbb{P}^n$, projective n-space, and there is a bijection between closed points in $\mathbb{P}^n$ and isomorphism classes of point modules for R: a point module is isomorphic to R/I for a unique ideal I, and I is generated by a codimension-1 subspace of $\mathbb{C}x_0 + \cdots + \mathbb{C}x_n$; conversely, if I is such an ideal, then R/I is a point module. Under the equivalence $\mathsf{QGr}(R) \xrightarrow{\sim} \mathrm{QCOH}(\mathbb{P}^n)$, $M \mapsto \widetilde{M}$, the point module R/I corresponds to the skyscraper sheaf $\mathcal{O}_p$ at the point $p \in \mathbb{P}^n$ where I vanishes. Similarly, if M is a line module for R, then $M \cong R/I$ for an ideal I that is generated by a codimension-2 subspace of $\mathbb{C}x_0 + \cdots + \mathbb{C}x_n$ and the zero locus of I is a line in $\mathbb{P}^n$, and this sets up a bijection between the lines in $\mathbb{P}^n$ and the isomorphism classes of line modules. Indeed, there is a bijection between linear subspaces of $\mathbb{P}^n$ and isomorphism classes of linear modules over the polynomial ring R.

Theorem 2.2 [Levasseur and Smith 1993, Theorem 1.13]. *Let* $R = \Bbbk[x_0, \dots, x_n]$ *be a polynomial ring in* $n+1$ *variables, graded by setting* $\deg(x_j) = 1$ *for all* j. *Let* M *be a finitely generated graded* R*-module. The following conditions on a graded* R*-module* M *are equivalent*:

(1) M *is cyclic with Hilbert series* $(1-t)^{-d}$;

(2) $M \cong R/R\ell^{\perp}$ *for some codimension-*d *subspace* $\ell \subseteq R_1$ *or, equivalently, for some* $(d-1)$*-dimensional linear subspace* $\ell \subseteq \mathbb{P}(R_1^*)$;

(3) M *is a Cohen–Macaulay* R*-module having GK dimension* d *and multiplicity* 1.

Thus, linear modules of GK dimension d correspond to linear subspaces of $\mathbb{P}^n$ having dimension $d-1$.

2C2. *Points, fat points, and lines in* $\mathrm{Proj}_{\mathrm{nc}}(R)$. For the noncommutative graded algebras R in this paper, the points and lines in $\mathrm{Proj}_{\mathrm{nc}}(R)$ are parametrized by genuine (commutative) varieties [Artin et al. 1990; 1991].

Let R be any $\mathbb{N}$-graded $\Bbbk$-algebra such that $R_0 = \Bbbk$ and R is generated by R_1 as a $\Bbbk$-algebra. Let $\mathbb{P}(R_1^*)$ denote the projective space whose points are the 1-dimensional subspaces of $R_1^* = \mathrm{Hom}_{\Bbbk}(R_1, \Bbbk)$.

For V a linear subspace of R_1^*, define $V^{\perp} := \{x \in R_1 \mid \xi(x) = 0 \text{ for all } \xi \in V\}$. Let

$$\mathcal{P}_R := \{p \in \mathbb{P}(R_1^*) \mid R/Rp^{\perp} \text{ is a point module}\},$$
$$\mathcal{L}_R := \{\text{lines } \ell \text{ in } \mathbb{P}(R_1^*) \mid R/R\ell^{\perp} \text{ is a line module}\}.$$

For the algebras in this paper there are moduli problems for which $\mathcal{P}_R$ and $\mathcal{L}_R$ are fine moduli spaces; see [Artin et al. 1990, Corollary 3.13] and [Shelton and Vancliff 2002a, Corollary 1.5]. We call $\mathcal{P}_R$ and $\mathcal{L}_R$ the *point scheme* and *line scheme* for R.

Clearly, a line module $R/R\ell^{\perp}$ surjects onto a point module $R/Rp^{\perp}$ if and only if p lies on the line ℓ. Thus, the incidence relations between points and lines in $\mathrm{Proj}_{\mathrm{nc}}(R)$ coincides with the incidence relations between *certain* points and lines in $\mathbb{P}(R_1^*)$. In such a situation the phrase *"p lies on ℓ"* is a statement about points and lines in $\mathbb{P}(R_1^*)$ and *also* a statement about points and lines in $\mathrm{Proj}_{\mathrm{nc}}(R)$. If a line module $R/R\ell^{\perp}$ surjects onto a fat point module F in $\mathsf{QGr}(R)$ we say that the corresponding fat point lies on the line ℓ and understand this as a statement about the geometry of $\mathrm{Proj}_{\mathrm{nc}}(R)$.

Proposition 2.3 [Levasseur and Smith 1993]. *The kernel of a surjective homomorphism $\psi : M_\ell \to M_p$ in $\mathsf{Gr}(S)$ from a line module to a point module is isomorphic to a shifted line module $M_{\ell'}(-1)$.*

Proof. There are elements $u, v, w \in S_1$ for which there is a commutative diagram

$$\begin{array}{ccc} M_\ell & \longrightarrow & S/Su+Sv \\ \psi\downarrow & & \downarrow\psi' \\ M_p & \longrightarrow & S/Su+Sv+Sw \end{array}$$

in which the horizontal arrows are isomorphisms and ψ' is the obvious map. The kernel of ψ' is isomorphic to the submodule $S\bar{w} = Su+Sv+Sw/Su+Sv$. Because M_ℓ is a critical Cohen–Macaulay module of GK dimension 2 and multiplicity 1, and M_p has GK dimension 1, the kernel is a Cohen–Macaulay module of GK dimension 2 and multiplicity 1. By [Levasseur and Smith 1993, Proposition 2.12], the kernel of ψ' is isomorphic to a shifted line module. □

The associated exact sequence $0 \to M_{\ell'}(-1) \to M_\ell \to M_p \to 0$ is the analogue of an exact sequence $0 \to M(\lambda') \to M(\lambda) \to L \to 0$ in which $M(\lambda')$ and $M(\lambda)$ are Verma modules.

2C3. *Noncommutative analogues of quadrics and* $\mathbb{P}^3$. Let S be one of the algebras in Section 2D. The Hilbert series of S is $(1-t)^{-4}$, the same as that of the polynomial ring on four variables. Furthermore, S has the "same" homological properties as that polynomial ring and, as a consequence, it is a domain [Artin et al. 1991, Theorem 3.9]. For these reasons we think of $\mathrm{Proj}_{nc}(S)$ as a noncommutative analogue of $\mathbb{P}^3$.

If Ω is a homogeneous, degree-two, central element in S we call $\mathrm{Proj}_{nc}(S/(\Omega))$ a *quadric hypersurface* in $\mathrm{Proj}_{nc}(S)$ and sometimes denote it by the symbols $\{\Omega = 0\}$. A line module $S/S\ell^\perp$ is annihilated by Ω if and only if there is a surjective map $S/(\Omega) \to S/S\ell^\perp$. If so we say that *"the line ℓ lies on the quadric* $\{\Omega = 0\}$*"* and interpret this as a statement about the geometry of $\mathrm{Proj}_{nc}(S)$.

2D. ***The algebras S.*** The algebras of interest to us are the noncommutative $\mathbb{C}$-algebras S with generators x_0, x_1, x_2, x_3 subject to the relations

$$\begin{aligned} &[x_0,x_1]=0, && \{x_0,x_1\}=2x_0x_1=[x_2,x_3], \\ &[x_0,x_2]=b^2\{x_1,x_3\}, && \{x_0,x_2\}=[x_3,x_1], \\ &[x_0,x_3]=-b^2\{x_1,x_2\}, && \{x_0,x_3\}=[x_1,x_2], \end{aligned} \tag{2-1}$$

where $\{x,x'\} = xx' + x'x$, $[x,x'] = xx' - x'x$, and $b \in \mathbb{C} - \{0, \pm i\}$.

The algebras S occupy an interesting position between the nondegenerate 4-dimensional Sklyanin algebras and the quantized enveloping algebras $U_q(\mathfrak{sl}_2)$. We now introduce these algebras and, in Proposition 2.4 below, describe their relation to S.

2D1. *S is a degenerate Sklyanin algebra.* A *nondegenerate Sklyanin algebra* is a $\mathbb{C}$-algebra $S(\alpha, \beta, \gamma)$ with generators x_0, x_1, x_2, x_3 subject to the relations

$$\begin{aligned} [x_0, x_1] &= \alpha\{x_2, x_3\}, & \{x_0, x_1\} &= [x_2, x_3], \\ [x_0, x_2] &= \beta\{x_1, x_3\}, & \{x_0, x_2\} &= [x_3, x_1], \\ [x_0, x_3] &= \gamma\{x_1, x_2\}, & \{x_0, x_3\} &= [x_1, x_2], \end{aligned} \tag{2-2}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ are such that $\alpha + \beta + \gamma + \alpha\beta\gamma = 0$, and further satisfy the nondegeneracy condition

$$\{\alpha, \beta, \gamma\} \cap \{0, 1, -1\} = \varnothing. \tag{2-3}$$

With this notation, $S = S(0, b^2, -b^2)$, and is degenerate.

For the rest of the paper, S will denote $S(0, b^2, -b^2)$ and $S(\alpha, \beta, \gamma)$ will denote a nondegenerate Sklyanin algebra.

The noncommutative space $\operatorname{Proj}_{\mathrm{nc}}(S(\alpha, \beta, \gamma))$ is well understood. Its point scheme was computed in [Smith and Stafford 1992], its lines and the incidence relations between its points and lines were determined in [Levasseur and Smith 1993], and its fat points and the incidence relations between fat points and lines were determined in [Smith and Staniszkis 1993]. A short account of these and related results can be found in the survey article [Smith 1994]. In this paper we carry out the same computations for S and compare them to what has been obtained for nondegenerate $S(\alpha, \beta, \gamma)$. This is the subject of Section 6.

2D2. *S as a homogenization of $U_q(\mathfrak{sl}(2, \mathbb{C}))$.* The quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ is the $\mathbb{C}$-algebra with generators $e, f, k^{\pm}$ subject to the relations

$$ke = q^2 ek, \quad kf = q^{-2} fk, \quad kk^{-1} = k^{-1}k = 1, \quad \text{and} \quad [e, f] = \frac{k - k^{-1}}{q - q^{-1}}, \tag{2-4}$$

where $q \neq 0, \pm 1, \pm i$.

The representation theory of $U_q(\mathfrak{sl}_2)$ is the subject of books by Brown and Goodearl [2002], Jantzen [1996], Kassel [1995], Klimyk and Schmüdgen [1997], and others.[1]

[1] A slightly different algebra was studied by Jimbo [1985] and by Lusztig [1988]: they replace the last of the above relations by $[e, f] = (k - k^{-1})/(q^2 - q^{-2})$. Lusztig [1990] replaced that relation by the one in (2-4) and that seems to have become the "official" quantized enveloping algebra of $\mathfrak{sl}(2, \mathbb{C})$ used by subsequent authors. We call the algebra studied in [Jimbo 1985] and [Lusztig 1988] the "unofficial" quantized enveloping algebra of $\mathfrak{sl}(2, \mathbb{C})$. That unofficial version is a quotient of the algebra S in Proposition 2.4.

Before showing that S is a homogenization of $U_q(\mathfrak{sl}_2)$, we introduce notation that will be used throughout the paper:

$$\begin{aligned} q &= \frac{1-ib}{1+ib}, & E &= \frac{i}{2}(1-ib)(x_2+ix_3), & K &= x_0+bx_1, \\ \kappa &= \frac{1}{q^{-1}-q}, & F &= \frac{i}{2}(1+ib)(x_2-ix_3), & K' &= x_0-bx_1. \end{aligned} \tag{2-5}$$

Proposition 2.4. *The algebra S is the $\mathbb{C}$-algebra generated by E, F, K, K' modulo the relations*

$$\begin{aligned} KE &= qEK, & KF &= q^{-1}FK, & KK' &= K'K, \\ K'E &= q^{-1}EK', & K'F &= qFK', & [E, F] &= \frac{K^2-K'^2}{q-q^{-1}}. \end{aligned} \tag{2-6}$$

Further, $S[(KK')^{-1}]_0$ is isomorphic to $U_q(\mathfrak{sl}_2)$ via

$$EK^{-1} \mapsto \sqrt{q}e, \quad F(K')^{-1} \mapsto \sqrt{q}f, \quad K(K')^{-1} \mapsto k, \tag{2-7}$$

where $\sqrt{q}$ is a fixed square root of q.

Proof. A few tedious but straightforward calculations show that E, F, K, K' satisfy the relations in (2-6). For example, $KE = qEK$ because

$$\begin{aligned} &(1+ib)KE-(1-ib)EK \\ &\qquad = [K, E]+ib\{K, E\} \\ &\qquad = \frac{i}{2}(1-ib)\big([x_0, x_2]+i[x_0, x_3]+b[x_1, x_2]+ib[x_1, x_3] \\ &\qquad\qquad +ib\{x_0, x_2\}-b\{x_0, x_3\}+ib^2\{x_1, x_2\}-b^2\{x_1, x_3\}\big) \\ &\qquad = \frac{i}{2}(1-ib)\big([x_0, x_2]-b^2\{x_3, x_1\}+i[x_0, x_3]+ib^2\{x_1, x_2\} \\ &\qquad\qquad +ib\{x_0, x_2\}-ib[x_3, x_1]-b\{x_0, x_3\}+b[x_1, x_2]\big) \\ &\qquad = 0. \end{aligned}$$

Similar calculations show $KF = q^{-1}FK$, $K'E = q^{-1}EK'$ and $K'F = qFK'$.

Since $K^2-K'^2 = 4bx_0x_1 = 2b\{x_0, x_1\}$ and $\frac{i}{2}(1-ib)\cdot\frac{i}{2}(1+ib) = -\frac{1}{4}(1+b^2)$, we have

$$-\frac{4}{1+b^2}[E, F]+ib^{-1}(K^2-K'^2) = 2i[x_3, x_2]+2i\{x_0, x_1\} = 0.$$

However,

$$q-q^{-1} = \frac{1-ib}{1+ib}-\frac{1+ib}{1-ib} = -\frac{4ib}{1+b^2},$$

so

$$[E, F] = -\frac{1+b^2}{4ib}(K^2-K'^2) = \frac{K^2-K'^2}{q-q^{-1}}.$$

For the second part of the proposition, it is clear that $S[(KK')^{-1}]_0$ is generated by

$$e := \frac{1}{\sqrt{q}} EK^{-1}, \quad f := \frac{1}{\sqrt{q}} F(K')^{-1}, \quad k := K(K')^{-1}, \quad \text{and} \quad k^{-1}.$$

Similar straightforward calculations then show that these elements satisfy the relations in (2-4). Hence $S[(KK')^{-1}]_0 \cong U_q(\mathfrak{sl}_2)$. □

Since S is an Artin–Schelter regular algebra of global dimension 4 and has Hilbert series $(1-t)^{-4}$ we think of it as a homogeneous coordinate ring of a noncommutative analogue of $\mathbb{P}^3$. Since $SK = KS$ and $SK' = K'S$ we think of $S/(K)$ and $S/(K')$ as homogeneous coordinate rings of noncommutative analogues of $\mathbb{P}^2$.

Further, we think of $S[(KK')^{-1}]_0$, i.e., $U_q(\mathfrak{sl}_2)$, as the coordinate ring of the noncommutative affine scheme that is the "open complement" of the "union" of the "hyperplanes" $\{K = 0\}$ and $\{K' = 0\}$. These "hyperplanes" are effective divisors in the sense of Van den Bergh [2001, §3.6]. From this perspective, $U_q(\mathfrak{sl}_2)$ can be considered an "affine piece" of S. As explained in Section 1A2, this point of view can be formalized in terms of an adjoint pair of functors $j^* \dashv j_*$.

The left adjoint $j^* : \mathsf{QGr}(S) \to \mathsf{Mod}(S[(KK')^{-1}]_0)$ sends a graded S-module, X, viewed as an object in $\mathsf{QGr}(S)$ to $X[(KK')^{-1}]_0 \in \mathsf{Mod}(S[(KK')^{-1}]_0)$. The action of j^* on a morphism $f : M \to N$ in $\mathsf{QGr}(S)$ is defined by choosing a lift of f to a morphism ϕ in $\mathsf{Gr}(S)$, then applying the localization functor $X \mapsto X[(KK')^{-1}]_0$ to ϕ.

3. Point and line modules for D, a Zhang twist of S

In this section, we replace S by a Zhang twist of itself [Zhang 1996]. The appropriate Zhang twist is an algebra D that has a central element $z \in D_1$ such that $D/(z)$ a 3-dimensional Artin–Schelter regular algebra. In the terminology of [Le Bruyn et al. 1996], this makes D a *central extension of* $D/(z)$. We use the results in that paper to determine the point and line modules for D. The point and line modules for $D/(z)$ are already understood due to [Artin et al. 1990; 1991].

In Section 4 we use Zhang's fundamental equivalence $\mathsf{Gr}(D) \equiv \mathsf{Gr}(S)$ [Zhang 1996] to transfer the results about the point and line modules for D to S.

3A. ***The Zhang twist.*** Let S be a graded $\Bbbk$-algebra and ϕ a degree-preserving $\Bbbk$-algebra automorphism of S. Define D to be the $\Bbbk$-algebra that is equal to S as a graded $\Bbbk$-vector space, but endowed with the new multiplication

$$c * d := \phi^n(c)d$$

for $c \in D = S$ and $d \in D_n = S_n$. We call D a *Zhang twist* of S. Zhang [1996] showed that there is an equivalence of categories $\Phi : \mathsf{Gr}(S) \to \mathsf{Gr}(D)$ defined as follows: if M is a graded left S-module, then ΦM is M as a graded $\Bbbk$-vector space, but endowed with the D-action

$$c * m := \phi^n(c)m$$

for $c \in D = S$ and $m \in (\Phi M)_n = M_n$.

On morphisms Φ is the "identity": if $f \in \mathrm{Hom}_{\mathsf{Gr}(S)}(M, M')$, then $\Phi(f) = f$ considered now as a morphism $\Phi M \to \Phi M'$. Note that f *is* a morphism of graded D-modules because if $c \in D$ and $m \in M_n$, then $f(c * m) = f(\phi^n(c)m) = \phi^n(c) f(m) = c * f(m)$.

We use the graded algebra automorphism $\phi : S \to S$ defined by

$$\phi(s) := K' s (K')^{-1}. \tag{3-1}$$

This is a homomorphism because $K'S = SK'$, and is an automorphism because S is a 4-dimensional Artin–Schelter regular algebra and therefore a domain [Artin et al. 1991, Theorem 3.9].

Proposition 3.1. *Let D be the Zhang twist of S with respect to ϕ (3-1). Then D is isomorphic to $\mathbb{C}\langle E, F, K, K'\rangle$ modulo the relations*

$$[K', E] = [K', F] = [K', K] = 0,$$
$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad qEF - q^{-1} FE = \frac{K^2 - K'^2}{q - q^{-1}}.$$

In particular, K' belongs to the center of D.

Proof. Since $\phi(E) = q^{-1}E$, $\phi(F) = qF$, $\phi(K) = K$, and $\phi(K') = K'$,

$$K' * E = \phi(K')E = K'E = q^{-1}EK' = \phi(E)K' = E * K',$$
$$K' * F = \phi(K')F = K'F = qFK' = \phi(F)K' = F * K',$$
$$K * E = \phi(K)E = KE = qEK = q^2\phi(E)K = q^2 E * K,$$
$$K * F = \phi(K)F = KF = q^{-1}FK = q^{-2}\phi(F)K = q^{-2} F * K,$$

and

$$qE * F - q^{-1} F * E = EF - FE = \frac{K^2 - K'^2}{q - q^{-1}}.$$

By the very definition of ϕ, K' belongs to the center of D. □

Corollary 3.2. *Let A be $\mathbb{C}\langle E, F, K\rangle$ modulo the relations*

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad qEF - q^{-1} FE = \frac{K^2}{q - q^{-1}}.$$

Then $A \cong D/(K')$ and D is a central extension of A in the sense of [Le Bruyn et al. 1996, Definition 3.1.1].

3B. ***Applying the results in [Le Bruyn et al. 1996].*** In the notation of [Le Bruyn et al. 1996], our (E, F, K, K') is their (x_1, x_2, x_3, z). Following the notation of equation (3.1) and Section 4.2 in that paper, if $A = \mathbb{C}\langle x_1, x_2, x_3\rangle/(f_1, f_2, f_3)$, then the defining relations for a central extension D of A can be written as[2]

$$\begin{aligned} zx_i - x_i z &= 0, \quad j = 1, 2, 3, \\ g_j := f_j + zl_j + \alpha_j z^2 &= 0, \quad j = 1, 2, 3, \end{aligned}$$

for some $l_j \in A_1$ and $\alpha_j \in \mathbb{C}$. For our D,

$$\begin{aligned} \boldsymbol{f} := \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} -q^3KF + qFK \\ q^{-3}KE - q^{-1}EK \\ qEF - q^{-1}FE + \kappa K^2 \end{pmatrix}, \quad \boldsymbol{l} := \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ \boldsymbol{\alpha} := \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\kappa \end{pmatrix}. \end{aligned} \tag{3-2}$$

Thus

$$\boldsymbol{g} := \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} -q^3KF + qFK \\ q^{-3}KE - q^{-1}EK \\ qEF - q^{-1}FE + \kappa K^2 - \kappa K'^2 \end{pmatrix}. \tag{3-3}$$

The relations of A are said to be in *standard form* [Artin et al. 1990, p. 34] if, in the notation of [Le Bruyn et al. 1996, p. 181], there is a 3×3 matrix M, and a matrix $Q \in \mathsf{GL}(3)$, such that $\boldsymbol{f} = M\boldsymbol{x}$ and $\boldsymbol{x}^\mathsf{T} M = (Q\boldsymbol{f})^\mathsf{T}$, where $\boldsymbol{f}^\mathsf{T} = (f_1, f_2, f_3)$ and A is generated as an algebra by the entries of the column vector $\boldsymbol{x}$.

Proposition 3.3. *The relations $\boldsymbol{f}$ for A in (3-2) are in standard form, where*

$$\begin{aligned} \boldsymbol{x} &= (E, F, K)^\mathsf{T}, \\ Q &= \mathsf{diag}(q^{-4}, q^4, 1), \end{aligned} \tag{3-4}$$

and

$$M = \begin{pmatrix} 0 & -q^3K & qF \\ q^{-3}K & 0 & -q^{-1}E \\ -q^{-1}F & qE & \kappa K \end{pmatrix}. \tag{3-5}$$

Proof. It is easy to check that $\boldsymbol{f} = M\boldsymbol{x}$. On the other hand,

$$\boldsymbol{x}^\mathsf{T} M = (E, F, K)M = (q^{-3}FK - q^{-1}KF, -q^3EK + qKE, qEF - q^{-1}FE + \kappa K^2),$$

[2] In the notation of [Le Bruyn et al. 1996, Theorem 3.1.3], $\gamma_j = 0$ for all j.

so

$$(\boldsymbol{x}^{\mathsf{T}} M)^{\mathsf{T}} = \begin{pmatrix} q^{-3}FK - q^{-1}KF \\ -q^3 EK + qKE \\ qEF - q^{-1}FE + \kappa K^2 \end{pmatrix} = \begin{pmatrix} q^{-4} & 0 & 0 \\ 0 & q^4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{f}.$$

Thus $\boldsymbol{x}^{\mathsf{T}} M = (Q\boldsymbol{f})^{\mathsf{T}}$ as claimed. □

We use (E, F, K) as homogeneous coordinates on the plane $\mathbb{P}(A_1^*) \cong \mathbb{P}^2$ and identify this plane with the hyperplane $\{K' = 0\}$ in $\mathbb{P}(D_1^*)$.

Proposition 3.4. *The point scheme* $(\mathcal{P}_A, \sigma_A)$ *for* A *is the cubic divisor on* $\{K' = 0\} = \mathbb{P}(A_1^*)$ *consisting of the line* $K = 0$ *and the conic* $\kappa^2 K^2 + EF = 0$. *The line meets the conic at the points* $(1, 0, 0)$ *and* $(0, 1, 0)$.

(1) *If* (ξ_1, ξ_2, ξ_3) *lies on the conic* $\kappa^2 K^2 + EF = 0$, *then*

$$\sigma_A(\xi_1, \xi_2, \xi_3) = (q^2 \xi_1, q^{-2} \xi_2, \xi_3).$$

(2) *If* (ξ_1, ξ_2, ξ_3) *lies on the line* $K = 0$; *i.e.*, $\xi_3 = 0$, *then*

$$\sigma_A(\xi_1, \xi_2, 0) = (q\xi_1, q^{-1}\xi_2, 0).$$

Proof. By [Artin et al. 1990], the subscheme of $\mathbb{P}(A_1^*)$ parametrizing the left point modules for A is given by the equation

$$\det \begin{pmatrix} 0 & -q^2 K & F \\ q^{-2} K & 0 & -E \\ -q^{-1} F & qE & \kappa K \end{pmatrix} = 0.$$

The vanishing locus of this determinant is the union of the line $K = 0$ and the smooth conic $\kappa^2 K^2 + EF = 0$. The line meets the conic at the points $(1, 0, 0)$ and $(0, 1, 0)$.

We denote this cubic curve by $\mathcal{P}_A$. The point module corresponding to a point $p \in \mathcal{P}_A$ is $M_p := A/Ap^{\perp}$, where $p^{\perp}$ is the subspace of A_1 consisting of the linear forms that vanish at p.

If M_p is a point module for A so is $(M_p)_{\geq 1}(1)$. In keeping with the notation in [Le Bruyn et al. 1996], we write σ_A (in this proof we will use σ for brevity) for the automorphism of $\mathcal{P}_A$ such that

$$M_{\sigma^{-1}(p)} \cong (M_p)_{\geq 1}(1). \tag{3-6}$$

To determine σ explicitly, let $p \in \mathcal{P}_A$ and suppose that $p = (\xi'_1, \xi'_2, \xi'_3)$ and $\sigma^{-1}(p) = (\xi_1, \xi_2, \xi_3)$ with respect to the homogeneous coordinates (E, F, K). Then M_p has a homogeneous basis $e_0, e_1, \ldots$, where $\deg(e_n) = n$, and

$$Ee_0 = \xi'_1 e_1, \quad Fe_0 = \xi'_2 e_1, \quad Ke_0 = \xi'_3 e_1,$$

and

$$Ee_1 = \xi_1 e_2, \quad Fe_1 = \xi_2 e_2, \quad Ke_1 = \xi_3 e_2.$$

Since $KE - q^2 EK = 0$ in A, $(KE - q^2 EK)e_0 = 0$; i.e., $\xi_3\xi_1' - q^2\xi_1\xi_3' = 0$. The other two relations for A in Corollary 3.2 imply $\xi_3\xi_2' - q^{-2}\xi_2\xi_3' = 0$ and $q\xi_1\xi_2' - q^{-1}\xi_2\xi_1' + \kappa\xi_3\xi_3' = 0$. These equalities can be expressed as the single equality

$$\begin{pmatrix} 0 & \xi_3 & -q^{-2}\xi_2 \\ \xi_3 & 0 & -q^2\xi_1 \\ -q^{-1}\xi_2 & q\xi_1 & \kappa\xi_3 \end{pmatrix}\begin{pmatrix} \xi_1' \\ \xi_2' \\ \xi_3' \end{pmatrix} = 0.$$

Since $\sigma(\xi_1, \xi_2, \xi_3) = (\xi_1', \xi_2', \xi_3')$, we can now determine σ.

If (ξ_1, ξ_2, ξ_3) lies on the line $K = 0$; i.e., $\xi_3 = 0$, then

$$\begin{pmatrix} 0 & 0 & -q^{-2}\xi_2 \\ 0 & 0 & -q^2\xi_1 \\ -q^{-1}\xi_2 & q\xi_1 & 0 \end{pmatrix}\begin{pmatrix} \xi_1' \\ \xi_2' \\ \xi_3' \end{pmatrix} = 0,$$

so $\sigma(\xi_1, \xi_2, 0) = (q\xi_1, q^{-1}\xi_2, 0)$. If (ξ_1, ξ_2, ξ_3) lies on the conic $\kappa^2 K^2 + EF = 0$, then

$$\begin{pmatrix} 0 & \xi_3 & -q^{-2}\xi_2 \\ \xi_3 & 0 & -q^2\xi_1 \\ -q^{-1}\xi_2 & q\xi_1 & \kappa\xi_3 \end{pmatrix}\begin{pmatrix} q^2\xi_1 \\ q^{-2}\xi_2 \\ \xi_3 \end{pmatrix} = 0,$$

so $\sigma(\xi_1, \xi_2, \xi_3) = (q^2\xi_1, q^{-2}\xi_2, \xi_3)$. □

The algebra A is of Type S_1' in the terminology of [Artin et al. 1990, Proposition 4.13, p. 54]. See also [Le Bruyn et al. 1996, p. 187], where it is stated that D is the unique central extension of A that is not a polynomial extension, up to the notion of equivalence at [Le Bruyn et al. 1996, §3.1, p. 180].

In the next result, which is similar to Proposition 3.4, we use (E, F, K') as homogeneous coordinate functions on the plane $K = 0$ in $\mathbb{P}(D_1^*)$.

Proposition 3.5. *Let $A' = D/(K)$. The point scheme $(\mathcal{P}_{A'}, \sigma_{A'})$ for A' is the cubic divisor on the plane $K = 0$ consisting of the line $K' = 0$ and the smooth conic $EF + \kappa^2 K'^2 = 0$. The line meets the conic at the points* $(1, 0, 0)$ *and* $(0, 1, 0)$.

(1) *If* (ξ_1, ξ_2, ξ_4) *lies on the conic* $\kappa^2 K'^2 + EF = 0$, *then*

$$\sigma_{A'}(\xi_1, \xi_2, \xi_4) = (\xi_1, \xi_2, \xi_4).$$

(2) *If* (ξ_1, ξ_2, ξ_4) *lies on the line* $K = 0$; *i.e.*, $\xi_4 = 0$, *then*

$$\sigma_{A'}(\xi_1, \xi_2, 0) = (q\xi_2, q^{-1}\xi_1, 0).$$

Proof. Since $[E, K'] = [F, K'] = qEF - q^{-1}FE - \kappa K'^2 = 0$ are defining relations for $A' = \mathbb{C}[E, F, K']$, the left point modules for A' are naturally parametrized by the scheme-theoretic zero locus of

$$\det\begin{pmatrix} 0 & K' & -F \\ K' & 0 & -E \\ -q^{-1}F & qE & -\kappa K' \end{pmatrix}$$

in $\mathbb{P}(A_1'^*)$, namely the union of the line $K' = 0$ and the smooth conic $\kappa^2 K'^2 + EF = 0$.

We denote this cubic curve by $\mathcal{P}_{A'}$ and define $\sigma_{A'} : \mathcal{P}_{A'} \to \mathcal{P}_{A'}$ (in this proof we will use σ for brevity) by the requirement that $M_{\sigma^{-1}(p)} \cong (M_p)_{\geq 1}(1)$ for all $p \in \mathcal{P}_{A'}$. Calculations like those in Proposition 3.4 show that σ is the identity on the conic and is given by $\sigma(\xi_1, \xi_2, 0) = (q\xi_2, q^{-1}\xi_1, 0)$ on the line $K' = 0$. □

3C. *The point scheme for D.* By [Le Bruyn et al. 1996, Theorem 4.2.2] the point scheme $(\mathcal{P}_D, \sigma_D)$ for D exists. That result also gives an explicit description of $\mathcal{P}_D$. It is also pointed out there that the restriction of σ_D to $\mathcal{P}_D - \mathcal{P}_A$ is the identity.

Warning: The g_1, g_2, g_3 in (3-3) belong to the tensor algebra $T(D_1)$. The g_1, g_2, g_3 in the next result are the images of the g_1, g_2, g_3 in (3-3) in the polynomial ring generated by the indeterminates E, F, K, K'.

Proposition 3.6 [Le Bruyn et al. 1996, Lemma 4.2.1 and Theorem 4.2.2]. *Let $x_1 = E$, $x_2 = F$, and $x_3 = K$. The equations for $\mathcal{P}_D$ are*

(1) $g_1 = g_2 = g_3 = 0$ *on* $\mathcal{P}_D \cap \{K' \neq 0\}$, *where*

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} (q - q^3)FK \\ (q^{-3} - q^{-1})EK \\ -\kappa^{-1}EF + \kappa K^2 - \kappa K'^2 \end{pmatrix} = \kappa^{-1}\begin{pmatrix} q^2FK \\ q^{-2}EK \\ -EF + \kappa^2(K^2 - K'^2) \end{pmatrix},$$

(2) $K'g_1 = K'g_2 = K'g_3 = h_i = 0$ *on* $\mathcal{P}_D \cap \{x_i \neq 0\}$, *where*

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \kappa^{-1}K\begin{pmatrix} E(EF + \kappa^2K^2 - \kappa^2q^2K'^2) \\ F(EF + \kappa^2K^2 - \kappa^2q^{-2}K'^2) \\ K(EF + \kappa^2K^2 - \kappa^2K'^2) \end{pmatrix}.$$

Proof. The polynomials h_1, h_2, and h_3 are defined in [Le Bruyn et al. 1996, Lemma 4.2.1].

Denote the columns of M by M_1, M_2, M_3, so that $M = [M_1\, M_2\, M_3]$, and note that

$$\det(M) = (\kappa K^2 + \kappa^{-1}EF)K.$$

In this case, since $\boldsymbol{l} = 0$, the definitions of h_1, h_2, and h_3, in [Le Bruyn et al. 1996, Lemma 4.2.1] become

$$h_1 = E\det(M) + z^2 \det[\boldsymbol{\alpha}\, M_2\, M_3],$$
$$h_2 = F\det(M) + z^2 \det[M_1\, \boldsymbol{\alpha}\, M_3],$$
$$h_3 = K\det(M) + z^2 \det[M_1\, M_2\, \boldsymbol{\alpha}].$$

Since $\boldsymbol{\alpha} = (0, 0, -\kappa)^{\mathsf{T}}$,

$$h_1 = EK(\kappa K^2 + \kappa^{-1} EF) - \kappa q^2 K'^2 EK,$$
$$h_2 = FK(\kappa K^2 + \kappa^{-1} EF) - \kappa q^{-2} K'^2 FK,$$
$$h_3 = K^2(\kappa K^2 + \kappa^{-1} EF) - \kappa K'^2 K^2.$$

Hence the result. □

Theorem 3.7. *The point scheme $\mathcal{P}_D$ is reduced and is the union of*

(1) *the conics $EF + \kappa^2 K^2 = K' = 0$ and $EF + \kappa^2 K'^2 = K = 0$,*

(2) *the line $K = K' = 0$, and*

(3) *the points $(0, 0, 1, \pm 1)$.*

Let $p \in \mathcal{P}_D$.

(4) *If $p = (\xi_1, \xi_2, \xi_3, 0)$ is on the conic $EF + \kappa^2 K^2 = K' = 0$, then $\sigma_D(p) = (q^2\xi_1, q^{-2}\xi_2, \xi_3, 0)$.*

(5) *If $p = (\xi_1, \xi_2, 0, \xi_4)$ is on the conic $EF + \kappa^2 K'^2 = K = 0$, then $\sigma_D(p) = p$.*

(6) *If $p = (\xi_1, \xi_2, 0, 0)$ is on the line $K = K' = 0$, then $\sigma_D(p) = (q\xi_1, q^{-1}\xi_2, 0, 0)$.*

(7) *If $p = (0, 0, 1, \pm 1)$, then $\sigma_D(p) = p$.*

Proof. By [Le Bruyn et al. 1996, Theorem 4.2.2], $(\mathcal{P}_D)_{\text{red}} = (\mathcal{P}_A)_{\text{red}} \cup V(g_1, g_2, g_3)_{\text{red}}$, where $V(g_1, g_2, g_3)$ is the scheme-theoretic zero locus of the ideal $(EK, FK, EF - \kappa^2(K'^2 - K^2))$. Certainly $\mathcal{P}_A$ is reduced. Straightforward computations on the open affine pieces $E \neq 0$, $F \neq 0$, $K \neq 0$, and $K' \neq 0$, show that $V(g_1, g_2, g_3)$ is reduced. Hence $\mathcal{P}_D$ is reduced.

If $p = (0, 0, 1, \pm 1)$, then $M_p = D/Dp^{\perp} = D/DE + DF + D(K \mp K')$. But $DE + DF + D(K \mp K')$ is a two-sided ideal and the quotient by it is the polynomial ring in one variable. Hence $\sigma_D(p) = p$. □

3D. ***The line modules for D.*** We now use the results in [Le Bruyn et al. 1996, §5] to characterize the line modules for D. Recall from Section 2C2 that

$$\mathcal{L}_D = \{\text{lines } \ell \text{ in } \mathbb{P}(D_1^*) \mid D/D\ell^{\perp} \text{ is a line module}\}.$$

For each point $p \in \mathcal{P}_A$, let

$$\mathcal{L}_p := \{\ell \in \mathcal{L}_D \mid p \in \ell\}.$$

Then

$$\mathcal{L}_D = \{\text{lines on the plane } K' = 0\} \cup \bigcup_{p \in \mathcal{P}_A} \mathcal{L}_p.$$

Proposition 3.8. *Let M be a line module for D.*

(1) *There is a unique line ℓ in $\mathbb{P}(D_1^*)$ such that $M \cong D/D\ell^\perp$.*
(2) *If $K'M = 0$, then $\ell \subseteq \{K' = 0\}$ and M is a line module for $A = D/(K')$.*
(3) *The line modules for A are, up to isomorphism, $A/A\ell^\perp$, where $\ell \subseteq \{K' = 0\}$.*
(4) *If $K'M \neq 0$, then $M/K'M$ is a point module for A and isomorphic to $A/Ap^\perp$, where $\{p\} = \ell \cap \{K' = 0\}$.*

Proof. (1) This is a consequence of [Levasseur and Smith 1993, Proposition 2.8], which says that if A is a noetherian, Auslander regular, graded $\Bbbk$-algebra having Hilbert series $(1-t)^{-4}$, and is generated by A_1, and satisfies the Cohen–Macaulay property, then there is a bijection

$$\{\text{lines } u = v = 0 \text{ in } \mathbb{P}(A_1^*) \mid \text{there is a rank 2 relation } a \otimes u - b \otimes v\} \longleftrightarrow \frac{A}{Au + Av}$$

between certain lines in $\mathbb{P}(A_1^*)$ and the set of isomorphism classes of line modules for A.

(2) This is obvious.

(3) Since A is a 3-dimensional Artin–Schelter regular algebra, by [Artin et al. 1990], the isoclasses of the line modules for A are the modules $A/A\ell^\perp$, where ℓ ranges over all lines in $\mathbb{P}(A_1^*) = \{K' = 0\}$.

(4) See the discussion at [Le Bruyn et al. 1996, p. 204]. □

3D1. *The quadrics Q_p.* By [Le Bruyn et al. 1996, Theorem 5.1.6], if $p \in \mathcal{P}_A$ there is a quadric Q_p containing p such that

$$\mathcal{L}_p = \left\{\text{lines } \ell \subseteq \{K' = 0\} \text{ such that } p \in \ell\right\} \cup \left\{\text{lines } \ell \subseteq Q_p \text{ such that } p \in \ell\right\}.$$

By [Le Bruyn et al. 1996, Proposition 5.1.7], if $\sigma_D(p) = (\zeta_1, \zeta_2, \zeta_3, 0)$, then Q_p is given by the equation $\boldsymbol{\zeta}^\mathsf{T} Q \boldsymbol{g} = 0$, where $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)^\mathsf{T}$, Q is the matrix in Proposition 3.3, $\boldsymbol{g} = (g_1, g_2, g_3)^\mathsf{T} = \boldsymbol{f} + \boldsymbol{\alpha} K'^2$, $\boldsymbol{f}$ is the image in the polynomial ring $\mathbb{C}[E, F, K]$ of the column vector $\boldsymbol{f}$ introduced in the proof of Proposition 3.3, and $\boldsymbol{\alpha} = (0, 0, -\kappa)^\mathsf{T}$. Thus, Q_p is given by the equation

$$(\zeta_1, \zeta_2, \zeta_3) \begin{pmatrix} q^{-4} & 0 & 0 \\ 0 & q^4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \kappa^{-1} \begin{pmatrix} q^2 FK \\ q^{-2} EK \\ -EF + \kappa^2 K^2 - \kappa^2 K'^2 \end{pmatrix} = 0$$

or, equivalently, by

$$\kappa^{-1}(\zeta_1, \zeta_2, \zeta_3)\begin{pmatrix} q^{-2}FK \\ q^2EK \\ -EF+\kappa^2K^2-\kappa^2K'^2 \end{pmatrix} = 0.$$

The next result determines $\mathcal{L}_p$ for each $p \in \mathcal{P}_A$. We take coordinates with respect to the coordinate functions (E, F, K, K').

Proposition 3.9. *Suppose* $p = (\xi_1, \xi_2, \xi_3, 0) \in \mathcal{P}_A$.

(1) *If* $p = (1, 0, 0, 0)$, *then*

$$\mathcal{L}_p = \big\{\textit{lines } \ell \subseteq \{F = 0\} \cup \{K = 0\} \cup \{K' = 0\} \textit{ such that } p \in \ell\big\}.$$

(2) *If* $p = (0, 1, 0, 0)$, *then*

$$\mathcal{L}_p = \big\{\textit{lines } \ell \subseteq \{E = 0\} \cup \{K = 0\} \cup \{K' = 0\} \textit{ such that } p \in \ell\big\}.$$

(3) *If* $\xi_3 = 0$ *and* $\xi_1\xi_2 \neq 0$, *then*

$$\mathcal{L}_p = \big\{\textit{lines } \ell \subseteq \{K = 0\} \cup \{K' = 0\} \textit{ such that } p \in \ell\big\}.$$

(4) *If* $\xi_3 \neq 0$, *then* Q_p *is the cone with vertex* p *given by the equation*

$$\xi_1 FK + \xi_2 EK + \xi_3(-EF + \kappa^2K^2 - \kappa^2K'^2) = 0,$$

and

$$\mathcal{L}_p = \big\{\textit{lines } \ell \subseteq Q_p \cup \{K' = 0\} \textit{ such that } p \in \ell\big\}.$$

Proof. (1)–(3) Suppose $\xi_3 = 0$. Then p is on the line $\{K = K' = 0\}$ whence $\sigma_D(p) = (\xi_1, \xi_2, 0, 0)$ and Q_p is given by the equation

$$\kappa^{-1}(\xi_1 q^{-2}F + \xi_2 q^2 E)K = 0.$$

Thus, ℓ either lies on the pair of planes $\{KK' = 0\}$ or the plane $\{\xi_1 q^{-2}F + \xi_2 q^2 E = 0\}$. Suppose $\ell \not\subseteq \{KK' = 0\}$. We are assuming that $q^4 + 1 \neq 0$ so, if p is neither $(1, 0, 0, 0)$ nor $(0, 1, 0, 0)$, then $\xi_1 q^{-2}F + \xi_2 q^2 E$ does not vanish at p whence there are no lines through p that lie on the plane $\{\xi_1 q^{-2}F + \xi_2 q^2 E = 0\}$.

Suppose $p = (1, 0, 0, 0)$. Then $Q_p = \{FK = 0\}$ and $\mathcal{L}_p$ consists of the lines through p that are contained in $\{F = 0\} \cup \{K = 0\}$. Similarly, if $p = (0, 1, 0, 0)$, then $\mathcal{L}_p$ consists of the lines through p that are contained in $\{E = 0\} \cup \{K = 0\}$.

(4) Suppose $\xi_3 \neq 0$. Then p lies on the conic $\kappa^2K^2 + EF = K' = 0$ so $\sigma_A(p) = (q^2\xi_1, q^{-2}\xi_2, \xi_3, 0)$. Thus, Q_p is given by the equation

$$\kappa^{-1}(q^2\xi_1, q^{-2}\xi_2, \xi_3)\begin{pmatrix} q^{-2}FK \\ q^2EK \\ -EF+\kappa^2K^2-\kappa^2K'^2 \end{pmatrix} = 0;$$

i.e., by the equation $\xi_1 FK + \xi_2 EK + \xi_3(-EF + \kappa^2 K^2 - \kappa^2 K'^2) = 0$. The quadric Q_p is singular at p (and is therefore a cone) because the partial derivatives at the point $p = (\xi_1, \xi_2, \xi_3, 0)$ are

$$\begin{aligned}
\partial_E &: (\xi_2 K - \xi_3 F)|_p = 0,\\
\partial_F &: (\xi_1 K - \xi_3 E)|_p = 0,\\
\partial_K &: (\xi_1 F + \xi_2 E + 2\xi_3 \kappa^2 K)|_p = 0,\\
\partial_{K'} &: 2\xi_3 \kappa^2 K'|_p = 0.
\end{aligned}$$

It follows that every line contained in Q_p passes through p, and hence all such lines correspond to line modules by [Le Bruyn et al. 1996, Theorem 5.1.6], as recalled above in Section 3D1. □

Let C denote the conic $K' = \kappa^2 K^2 + EF = 0$ and C' the conic $K = \kappa^2 K'^2 + EF = 0$. The two isolated points in $\mathcal{P}_D$, namely (0, 0, 1, ±1), lie on Q_p for every $p \in \{K' = \kappa^2 K^2 + EF = 0\}$ so there is a pencil of lines (or two pencils) through each of these points that correspond to line modules.

Since Q_p is singular at p, [Le Bruyn et al. 1996, Lemma 5.1.10] implies that $p = p^\vee$ in the notation defined on page 204 of *loc. cit.* Thus, according to [Le Bruyn et al. 1996, Definition 5.1.9], p is of the third kind. Thus, we are in the last case of [Le Bruyn et al. 1996, Table 1].

4. Points, lines, and quadrics in $\mathbf{Proj}_{\mathbf{nc}}(S)$

We now transfer the results in Section 3 from D to S. Recall that the automorphism $\phi : S \to S$ defined in (3-1) induces an equivalence of categories $\Phi : \mathsf{Gr}(S) \to \mathsf{Gr}(D)$. We first note how ϕ and Φ act on linear modules.

4A. ***Left ideals and linear modules over S and D.*** If W is a graded subspace of $S = D$, then $D_m * W_j = \phi^j(S_m)W_j = S_m W_j$ so, dropping the $*$, $DW = SW$, i.e., the left ideal of D generated by W is equal to the left ideal of S generated by W. In particular, if I is a graded left ideal of S, then $I = SI = DI$ so I is also a left ideal of D. Likewise, if J is graded left ideal of D, then $J = DJ = SJ$ so J is also a left ideal of S.

In summary, D and S have exactly the same left ideals.

Let I be a graded left ideal of S. The equality $S/I = D/I$ is an equality in the category of graded vector spaces. In fact, more is true: $\Phi(S/I) = D/I$ in the category $\mathsf{Gr}(D)$. To see this, observe, first, that the result of applying Φ to the exact sequence $0 \to I \to S \to S/I \to 0$ in $\mathsf{Gr}(S)$ is the exact sequence $0 \to I \to D \to \Phi(S/I) \to 0$ in $\mathsf{Gr}(D)$, where $I \to S$ and $I \to D$ are the inclusion maps, then use the fact that $\Phi(S/I) = S/I = D/I$ as graded vector spaces.

Now let M be a d-linear S-module. Then $M \cong S/I$ for a unique graded left ideal I in S. Hence $\Phi M \cong \Phi(S/I) = D/I$. In particular, $\Phi(M)$ is a d-linear D-module. Hence

$$\{\text{left ideals } I \text{ in } S \mid S/I \text{ is } d\text{-linear}\} = \{\text{left ideals } I \text{ in } D \mid D/I \text{ is } d\text{-linear}\}.$$

Similarly, if $S_0 = k$, then

$$\{\text{subspaces } W \subset S_1 \mid S/SW \text{ is } d\text{-linear}\} = \{\text{subspaces } W \subset D_1 \mid D/DW \text{ is } d\text{-linear}\}.$$

Lemma 4.1. (1) *$\mathcal{P}_S = \mathcal{P}_D$ and $\mathcal{L}_S = \mathcal{L}_D$.*

(2) *If $\sigma_S : \mathcal{P}_S \to \mathcal{P}_S$ is a bijection such that $(M_p)_{\geq 1}(1) \cong M_{\sigma_S^{-1} p}$ for all $p \in \mathcal{P}_S$, then there is a bijection $\sigma_D : \mathcal{P}_D \to \mathcal{P}_D$ such that $(M_p)_{\geq 1}(1) \cong M_{\sigma_D^{-1} p}$ for all $p \in \mathcal{P}_D$, namely $\sigma_D = \sigma_S \phi$.*

Proof. (1) This follows from the discussion prior to the lemma.

(2) Let $p \in \mathcal{P}_S$. Suppose that $p = (\xi_0', \ldots, \xi_n')$ and $\sigma_S^{-1}(p) = (\xi_0, \ldots, \xi_n)$ with respect to an ordered basis $x_1, \ldots, x_n$ for S_1. There is a homogeneous basis $e_0, e_1, \ldots,$ where $\deg(e_n) = n$, for the S-module M_p such that $x_i e_0 = \xi_i' e_1$ and $x_i e_1 = \xi_i e_2$ for all i. Hence $(\xi_i x_j - \xi_j x_i) e_1 = 0$ for all $1 \leq i, j \leq n$.

Since

$$\begin{aligned} \sigma_D^{-1}(p)^{\perp} &= \{x \in D_1 \mid x * e_1 = 0\} \\ &= \{x \in D_1 \mid \phi(x) e_1 = 0\} \\ &= \{\phi^{-1}(x) \in D_1 \mid x e_1 = 0\} \\ &= \phi^{-1}(\{x \in D_1 = S_1 \mid x e_1 = 0\}) \\ &= \phi^{-1}(\sigma_S^{-1}(p)^{\perp}) \\ &= \phi^{-1}(\sigma_S^{-1}(p))^{\perp}, \end{aligned}$$

$\sigma_D^{-1} = \phi^{-1} \sigma_S^{-1}$ and $\sigma_D = \sigma_S \phi$. □

4B. ***Points in* $\mathbf{Proj_{nc}}(S)$*, the point scheme of S, and point modules.*** We restate Lemma 4.1(1) explicitly in the following theorem.

Theorem 4.2. *The point scheme $\mathcal{P}_S$ is reduced. It is the union of*

(1) *the conics $EF + \kappa^2 K^2 = K' = 0$ and $EF + \kappa^2 K'^2 = K = 0$,*

(2) *the line $K = K' = 0$, and*

(3) *the points $(0, 0, 1, \pm 1)$.*

Furthermore,

(4) *if $p = (\xi_1, \xi_2, \xi_3, 0)$ is on the conic $EF + \kappa^2 K^2 = K' = 0$, then $\sigma_S(p) = (q\xi_1, q^{-1}\xi_2, \xi_3, 0)$;*

(5) *if* $p=(\xi_1,\xi_2,0,\xi_4)$ *is on the conic* $EF+\kappa^2K'^2=K=0$, *then* $\sigma_S(p)=(q^{-1}\xi_1,q\xi_2,0,\xi_4)$;

(6) *if* $p=(\xi_1,\xi_2,0,0)$ *is on the line* $K=K'=0$, *then* $\sigma_S(p)=p$;

(7) *if* $p=(0,0,1,\pm 1)$, *then* $\sigma_S(p)=p$.

Proof. By Lemma 4.1, $\mathcal{P}_S=\mathcal{P}_D$. However, $\mathcal{P}_D$ is reduced so $\mathcal{P}_S$ is reduced. By Theorem 3.7, the irreducible components of $\mathcal{P}_D$ are the varieties in parts (1), (2) and (3) of this theorem. Hence the same is true of $\mathcal{P}_S$.

Now suppose there is an ordered basis $x_1,\dots,x_n$ for S_1 and scalars $\lambda_1,\dots,\lambda_n$ such that $\phi(x_i)=\lambda_i x_i$ for all i. Let $p=(\xi_1,\dots,\xi_n)\in\mathbb{P}(S_1^*)$, where the coordinates are written with respect to the ordered basis $x_1,\dots,x_n$. Let ξ be the point in S_1^* with coordinates $(\xi_1,\dots,\xi_n)$; i.e., $x_i(\xi)=\xi_i$ or, equivalently, $\xi(x_i)=\xi_i$. Since $\phi(\xi)(x_i)=\xi(\phi^{-1}x_i)=\xi(\lambda_i^{-1}x_i)$, $\phi(\xi)=(\lambda_1^{-1}\xi_1,\dots,\lambda_n^{-1}\xi_n)$. Hence

$$\phi(p)=(\lambda_1^{-1}\xi_1,\dots,\lambda_n^{-1}\xi_n).$$

Note that E, F, K, and K' in S are eigenvectors for ϕ with eigenvalues q^{-1}, q, 1, and 1, respectively. Thus if $p=(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbb{P}(S_1^*)$ with respect to the ordered basis E,F,K,K', then $\phi(p)=(q\xi_1,q^{-1}\xi_2,\xi_3,\xi_4)$ and $\phi^{-1}(p)=(q^{-1}\xi_1,q\xi_2,\xi_3,\xi_4)$.

We now use the description of σ_D in Theorem 3.7 to obtain:

(1) If $p=(\xi_1,\xi_2,\xi_3,0)\in\{EF+\kappa^2K^2=K'=0\}$, then $\sigma_D(p)=(q^2\xi_1,q^{-2}\xi_2,\xi_3,0)$ so $\sigma_S(p)=\sigma_D\phi^{-1}(p)=(q\xi_1,q^{-1}\xi_2,\xi_3,0)$.

(2) If $p=(\xi_1,\xi_2,0,\xi_4)\in\{EF+\kappa^2K'^2=K=0\}$, then $\sigma_D(p)=p$ so $\sigma_S(p)=\sigma_D\phi^{-1}(p)=(q^{-1}\xi_1,q\xi_2,\xi_3,0)$.

(3) If $p=(\xi_1,\xi_2,0,0)\in\{K=K'=0\}$, then $\sigma_D(p)=(q\xi_1,q^{-1}\xi_2,0,0)$ so $\sigma_S(p)=\phi^{-1}(p)=(\xi_1,\xi_2,0,0)$.

(4) If $p=(0,0,1,\pm 1)$, then $\sigma_D(p)=p$ so $\sigma_S(p)=\phi^{-1}(p)=p$. □

The algebra D is less symmetric than S: the fact that σ_D is the identity on one of the conics but not on the other indicates a certain asymmetry about D. The asymmetry is a result of the fact that we favored K' over K when we formed the Zhang twist of S which made K', but not K, a central element. Theorem 4.2 shows that the symmetry is restored when the results for $\mathcal{P}_D$ are transferred to $\mathcal{P}_S$.

4C. ***Lines and quadrics in*** $\mathbf{Proj_{nc}}(S)$**.** Proposition 3.9 classified the line modules for D, and therefore the line modules for S. Theorem 4.5 below gives a new description of the line modules for S: it says that the line modules correspond to the lines lying on a certain pencil of quadrics. This is analogous to the description in [Le Bruyn and Smith 1993, Theorem 2] of the line modules for the homogenized enveloping algebra of $\mathfrak{sl}_2$ and the description in [Levasseur and Smith

1993, Theorem 4.5] of the line modules for the 4-dimensional Sklyanin algebra $S(\alpha,\beta,\gamma)$.

The new description provides a unifying picture. The pencil of quadrics becomes more degenerate as one passes from the $S(\alpha,\beta,\gamma)$'s to the homogenizations of the various $U_q(\mathfrak{sl}_2)$ and more degenerate still for $H(\mathfrak{sl}_2)$. The pencil for $H(\mathfrak{sl}_2)$ contains a double plane $t^2=0$, that for S contains the pair of planes $\{KK'=0\}$, and that for $S(\alpha,\beta,\gamma)$ contains 4 cones and the other quadrics in the pencil are smooth.

The vertices of the cones in each pencil play a special role: for $H(\mathfrak{sl}_2)$ there is only one cone and its vertex corresponds to the trivial representation of $U(\mathfrak{sl}_2)$; for S there are two cones and their vertices correspond to the two 1-dimensional representations of $U_q(\mathfrak{sl}_2)$; for $S(\alpha,\beta,\gamma)$ the vertices of the four cones "correspond" to the four special 1-dimensional representations.

4C1. The line through two points $(\xi_1,\xi_2,\xi_3,0)$ and $(\eta_1,\eta_2,0,\eta_4)$ in $\mathbb{P}(S_1^*)$ is given by the equations

$$\xi_3\eta_4E-\xi_1\eta_4K-\xi_3\eta_1K'=\xi_3\eta_4F-\xi_2\eta_4K-\xi_3\eta_2K'=0. \tag{4-1}$$

4C2. *The pencil of quadrics $Q(\lambda)\subseteq\mathbb{P}(S_1^*)$.* For each $\lambda\in\mathbb{P}^1$, let $Q(\lambda)\subseteq\mathbb{P}(D_1^*)$ be the quadric where

$$g_\lambda:=\kappa^{-2}EF+K^2-(\lambda+\lambda^{-1})KK'+K'^2$$

vanishes. The points on the conics

$$C':\kappa^2K'^2+EF=K=0,$$
$$C:\kappa^2K^2+EF=K'=0$$

correspond to point modules for S. If $\lambda\neq0,\infty$, then

$$C'=Q(\lambda)\cap\{K=0\}\quad\text{and}\quad C=Q(\lambda)\cap\{K'=0\}.$$

Proposition 4.3. (1) *The base locus of the pencil $Q(\lambda)$ is $C\cup C'$.*

(2) *The $Q(\lambda)$'s are the only quadrics that contain $C\cup C'$.*

(3) *The singular quadrics in the pencil are the cones $Q(\pm1)$ with vertices at $(0,0,1,\pm1)$ respectively, and $Q(0)=Q(\infty)=\{KK'=0\}$.*

(4) *The lines on $Q(1)$ are $\kappa^{-1}E-s(K-K')=s\kappa^{-1}F+(K-K')=0$, $s\in\mathbb{P}^1$, and the lines on $Q(-1)$ are $\kappa^{-1}E-s(K+K')=s\kappa^{-1}F+(K+K')=0$, $s\in\mathbb{P}^1$.*

(5) *Suppose* $\lambda \notin \{0, \pm 1, \infty\}$. *The two rulings on* $Q(\lambda)$ *are*

$$\kappa^{-1}E - s(K - \lambda K') = s\kappa^{-1}F + (K - \lambda^{-1}K') = 0, \qquad s \in \mathbb{P}^1, \tag{4-2}$$

$$\kappa^{-1}E - s(K - \lambda^{-1}K') = s\kappa^{-1}F + (K - \lambda K') = 0, \qquad s \in \mathbb{P}^1. \tag{4-3}$$

Proof. (1) The base locus is, by definition, the intersection of all $Q(\mu)$ so is given by the equations $KK' = \kappa^{-2}EF + K^2 + K'^2 = 0$ so is $\{K' = \kappa^{-2}EF + K^2 = 0\} \cup \{K = \kappa^{-2}EF + K'^2 = 0\}$.

(2), (3) The proofs are straightforward. To prove (3) observe that the determinant of the symmetric matrix representing the g_λ has zeroes at $\lambda = \pm 1$ and a zero at $\lambda = 0, \infty$.

(4), (5) Let ℓ be the line defined by (4-2). Suppose $s \notin \{0, \infty\}$. Then $\kappa^{-1}E = s(K - \lambda K')$ and $s\kappa^{-1}F = -(K - \lambda^{-1}K')$ on ℓ, so $s\kappa^{-2}EF = -s(K - \lambda K')(K - \lambda^{-1}K')$ on ℓ. Canceling s, this says that $\kappa^{-2}EF + (K - \lambda K')(K - \lambda^{-1}K')$ vanishes on ℓ. Since the equation for $Q(\lambda)$ can be written as $\kappa^{-2}EF + (K - \lambda K')(K - \lambda^{-1}K') = 0$, $\ell \subseteq Q(\lambda)$. If $s = 0$, then ℓ is the line $E = K - \lambda^{-1}K' = 0$ which is on $Q(\lambda)$. If $s = \infty$, then ℓ is the line $K + \lambda^{-1}K' = F = 0$ which is on $Q(\lambda)$. The other case, (4-3), is similar. □

4C3. There are exactly four singular quadrics in a generic pencil of quadrics in $\mathbb{P}^3$. The point modules for the 4-dimensional Sklyanin algebras $S(\alpha, \beta, \gamma)$ are parametrized by a quartic elliptic curve $E \subseteq \mathbb{P}^3$ and 4 isolated points that are the vertices of the singular quadrics that contain E. The point modules corresponding to those isolated points correspond to the four 1-parameter families of 1-dimensional representations of a 4-dimensional Sklyanin algebra.

4C4. The vertices of the cones $Q(\pm 1)$ are the points $(0, 0, 1, \pm 1)$. These are the isolated points in the point scheme $\mathcal{P}_S$ (see Theorem 4.2). Later, we will see that the points $(0, 0, 1, \pm 1)$ correspond to the two 1-dimensional $U_q(\mathfrak{sl}_2)$-modules. More precisely, if p is one of those points, then $M_p[(KK')^{-1}]_0$ is a 1-dimensional $U_q(\mathfrak{sl}_2)$-module.

4C5. The lines on $Q(1)$ meet C' and C at points of the form $(\xi_1, \xi_2, \xi_3, 0)$ and $(\xi_1, \xi_2, 0, -\xi_3)$ respectively. The lines on $Q(-1)$ meet C' and C at points of the form $(\xi_1, \xi_2, \xi_3, 0)$ and $(\xi_1, \xi_2, 0, \xi_3)$ respectively.

Theorem 4.4. *Let* ℓ *be a line in* $\mathbb{P}(S_1^*)$. *Then* $S/S\ell^\perp$ *is a line module if and only if* $\ell \subseteq Q(\lambda)$ *for some* $\lambda \in \mathbb{P}^1$.

Proof. ($\Rightarrow$) Suppose $S/S\ell^\perp$ is a line module. By Section 4A, $D/D\ell^\perp$ is a line module for D.

The result is true if $\ell \subseteq \{KK' = 0\} = Q(\infty)$ so, from now on, suppose $\ell \not\subseteq \{KK' = 0\}$.

Let $p=(\xi_1,\xi_2,\xi_3,0)$ be the point where ℓ meets $\{K'=0\}$. By the discussion at the beginning of Section 3D, $A/Ap^{\perp}$ is a point module for $A=D/(K')$ so $p\in C\cup\{K=K'=0\}$.

Suppose $p=(1,0,0,0)$. By Proposition 3.9(1), ℓ is on the plane $\{F=0\}$. Since $p\in\ell$ it follows that $\ell=\{F=K-\lambda K'=0\}$ for some $\lambda\in\mathbb{P}^1$. Since $\ell\not\subseteq\{KK'=0\}$, $\lambda\neq 0,\infty$. Thus ℓ is the line in (4-2) corresponding to the point $s=\infty\in\mathbb{P}^1$. Hence $\ell\subseteq Q(\lambda)$.

If $p=(0,1,0,0)$, a similar argument shows that ℓ lies on some $Q(\lambda)$.

Now suppose that $p\notin\{(1,0,0,0),(0,1,0,0)\}$. Since $\ell\not\subseteq\{KK'=0\}$, it follows from Proposition 3.9(3) that $\xi_3\neq 0$. Hence by Proposition 3.9(4), ℓ lies on the quadric

$$\xi_1FK+\xi_2EK+\xi_3(-EF+\kappa^2K-\kappa^2K'^2)=0.$$

The conic $C'=\{K=EF+\kappa^2K'^2=0\}$ also lies on this quadric so $C'\cap\ell\neq\varnothing$. by a result analogous to Proposition 3.9(3), $\eta_4\neq 0$. Let $(\eta_1,\eta_2,0,\eta_4)\in C'\cap\ell$. Then ℓ is given by the equations in (4-1) so lies on the surface cut out by the equation

$$\begin{aligned}\xi_3^2\eta_4^2EF&=(\xi_1\eta_4K+\xi_3\eta_1K')(\xi_2\eta_4K+\xi_3\eta_2K')\\&=\xi_1\xi_2\eta_4^2K^2+\xi_3\eta_4(\xi_1\eta_2+\xi_2\eta_1)KK'+\xi_3^2\eta_1\eta_2K'^2\\&=-\kappa^2\xi_3^2\eta_4^2K^2+\xi_3\eta_4(\xi_1\eta_2+\xi_2\eta_1)KK'-\kappa^2\xi_3^2\eta_4^2K'^2,\end{aligned}$$

which can be rewritten as $\xi_3^2\eta_4^2(\kappa^{-2}EF+K^2+K'^2)-\xi_3\eta_4(\xi_1\eta_2+\xi_2\eta_1)KK'=0$. Thus, ℓ lies on some $Q(\lambda)$.

($\Leftarrow$) Let ℓ be a line on $Q(\lambda)$. If $\ell\subseteq\{K'=0\}$, then $D/D\ell^{\perp}=A/A\ell^{\perp}$ *is* a line module. From now on suppose that $\ell\not\subseteq\{K'=0\}$.

To show that $D/D\ell^{\perp}$, and hence $S/S\ell^{\perp}$, is a line module we must show there is a point, p say, in $\ell\cap C$ such that $\ell\subseteq Q_p$, where Q_p is the quadric in Section 3D1.

Suppose $\ell\subseteq Q(1)$. By Proposition 4.3(4), ℓ is given by

$$\kappa^{-1}E-s(K-K')=s\kappa^{-1}F+(K-K')=0$$

for some $s\in\mathbb{P}^1$. Since the point $p=(-s^2,1,-s\kappa^{-1},0)$ belongs to $\ell\cap C$, $S/S\ell^{\perp}$ is a line module if and only if $\ell\subseteq Q_p$. Since Q_p is given by the equation

$$-s^2FK+EK-s\kappa^{-1}(-EF+\kappa^2K^2-\kappa^2K'^2)=0,$$

the point $(-s^2,1,0,s\kappa^{-1})$ is in $\ell\cap Q_p$. Thus, ℓ passes through the vertex of the cone Q_p and through a second point on Q_p, whence $\ell\subseteq Q_p$. Therefore $S/S\ell^{\perp}$ is a line module.

The case $\ell\subseteq Q(-1)$ is similar.

Suppose $\lambda\notin\{0,\pm1,\infty\}$. Since $\ell\subseteq Q(\lambda)$ we suppose, without loss of generality, that ℓ belongs to the ruling (4-3) on $Q(\lambda)$. Thus $\ell=\{\kappa^{-1}E-s(K-\lambda K')=$

$s\kappa^{-1}F + (K - \lambda^{-1}K') = 0\}$ for some $s \in \mathbb{P}^1$. The point $p = (-\kappa s^2, \kappa, -s, 0)$, which is the vertex of the cone Q_p, is in $\ell \cap C$. Thus ℓ passes through the vertex of Q_p and the point $(-\kappa s^2 \nu^2, \kappa, 0, s\nu)$ which is also on Q_p, $\ell \subseteq Q_p$. Hence $S/S\ell^\perp$ is a line module. □

Theorem 4.5. *Let ℓ be a line in $\mathbb{P}(S_1^*)$. Then $S/S\ell^\perp$ is a line module if and only if ℓ meets $C \cup C'$ with multiplicity* 2; *i.e., if and only if ℓ is a secant line to $C \cup C'$.*

Proof. (⇒) Since $S/S\ell^\perp$ is a line module for S, $D/D\ell^\perp$ is a line module for D. Let λ be such that $\ell \subseteq Q(\lambda)$.

Suppose $Q(\lambda)$ is smooth. The Picard group of $Q(\lambda)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and equal to $\mathbb{Z}[L] \oplus \mathbb{Z}[L']$, where $[L]$ is the class of the line in (4-2) corresponding to $s = 0$ and $[L']$ is the class of the line in (4-3) corresponding to $s = 0$. Since $L = \{E = K - \lambda^{-1}K' = 0\}$, the scheme-theoretic intersection $L \cap (C \cup C')$ is the zero locus of the ideal

$$\begin{aligned}(E, K - \lambda^{-1}K') &+ (KK', g_\lambda)\\ &= (E, K - \lambda^{-1}K', KK', \kappa^{-2}EF + (K - \lambda K')(K - \lambda^{-1}K'))\\ &= (E, K - \lambda^{-1}K', KK').\end{aligned}$$

Hence $L \cap (C \cup C')$ is a finite scheme of length 2. Therefore $[L] \cdot [C \cup C'] = 2$. A similar calculation shows that $[L'] \cdot [C \cup C'] = 2$. Hence $[C \cup C'] = 2[L] + 2[L']$. It follows that $[\ell] \cdot [C \cup C'] = 2$.

Suppose ℓ lies on the cone $Q(1)$. Then ℓ is the line $\kappa^{-1}E - s(K + K') = s\kappa^{-1}F + (K + K') = 0$ for some $s \in \mathbb{P}^1$. Therefore the scheme-theoretic intersection $\ell \cap (C \cup C')$ is the zero locus of the ideal

$$(\kappa^{-1}E - s(K - K'), s\kappa^{-1}F + (K - K')) + (KK', g_1). \tag{4-4}$$

Since $\ell \subset Q(1)$, g_1 belongs to the ideal vanishing on ℓ. The ideal in (4-4) is therefore equal to $(\kappa^{-1}E - s(K - K'), s\kappa^{-1}F + (K - K'), KK')$. Thus $\ell \cap (C \cup C')$ is a finite scheme of length 2.

If $\ell \subseteq Q(-1)$, a similar argument shows that $\ell \cap (C \cup C')$ is a finite scheme of length 2.

Suppose $\ell \subseteq Q(\infty) = \{KK' = 0\}$. Without loss of generality we can, and do, assume that $\ell \subseteq \{K' = 0\}$. By Bézout's theorem, ℓ meets C with multiplicity two. Thus, if $\ell \cap C' = \varnothing$, then ℓ meets $C \cup C'$ with multiplicity two. Now suppose that ℓ meets C with multiplicity two and C' with multiplicity ≥ 1. If ℓ meets C at two distinct points, then ℓ is transversal to some $Q(\lambda')$ so meets $Q(\lambda')$, and hence $C \cup C'$, with multiplicity two. It remains to deal with the case where ℓ is tangent to C and meets C'. We now assume that is the case. Since $C \cap C' = \{(1, 0, 0, 0), (0, 1, 0, 0)$ it follows that ℓ is tangent to C at $(1, 0, 0, 0)$ or at $(0, 1, 0, 0)$. Since the two cases are similar we assume that ℓ is tangent to C at $(1, 0, 0, 0)$. It follows that

$\ell=\{K'=F=0\}$. The scheme-theoretic intersection $\ell\cap(C\cup C')$ is the zero locus of the ideal

$$(K,F)+(KK',g_1)=(K,F,KK',\kappa^{-2}EF+K^2+K'^2)=(K,F,K'^2).$$

Hence $\ell\cap(C\cup C')$ is a finite scheme of length 2.

($\Leftarrow$) Suppose ℓ is a line that meets $C\cup C'$ with multiplicity 2.

If ℓ lies on the plane $\{K'=0\}$, then $K'\in\ell^\perp$ so the ideal (K') of D is contained in $D\ell^\perp$ and $D/D\ell^\perp$ is a module over $A=D/(K')$. However, the dual of the map $D_1\to A_1$ embeds $\mathbb{P}(A_1^*)$ in $\mathbb{P}(D_1^*)$ and the image of this embedding is $\{K'=0\}$. In short, ℓ is a line in $\mathbb{P}(A_1^*)$. This implies that $A/A\ell^\perp$ is a line module for A and hence a line module for D. But $A/A\ell^\perp=D/D\ell^\perp$ so $D/D\ell^\perp$ is a line module for D. Therefore $S/S\ell^\perp$ is a line module for S. A similar argument shows that if ℓ lies on the plane $\{K=0\}$, then $S/S\ell^\perp$ is a line module.

For the remainder of the proof we assume that $\ell\not\subseteq\{KK'=0\}$.

A line that meets C with multiplicity two lies in the plane $K'=0$, so ℓ meets C and C' with multiplicity one. Let $p=(\xi_1,\xi_2,\xi_3,0)$ be the point where ℓ meets C and $p=(\eta_1,\eta_2,0,\eta_4)$ be the point where ℓ meets C'. Since p is the vertex of Q_p and $C'\subseteq Q_p$, ℓ passes through the vertex of Q_p and another point on Q_p. Hence $\ell\subseteq Q_p$. □

4C6. Lemmas 4.4 and 4.5 are analogous to results for the 4-dimensional Sklyanin algebras: the line modules correspond to the lines in $\mathbb{P}^3$ that lie on the quadrics that contain the quartic elliptic curve E, and those are exactly the lines in $\mathbb{P}^3$ that meet E with multiplicity two, i.e., the secant lines to E. Similar results hold for the homogenization of $\mathfrak{sl}_2$ [Le Bruyn and Smith 1993].

4C7. *Notation for line modules.* By Theorem 4.5, the lines that correspond to line modules for S are the secant lines to $C\cup C'$. If $(p)+(p')$ is a degree-two divisor on $C\cup C'$ we write $M_{p,p'}$ for the line module $M_\ell=S/S\ell^\perp$, where ℓ is the unique line that meets $C\cup C'$ at $(p)+(p')$. Thus, up to isomorphism, the line modules for S are

$$\{M_{p,p'}\mid p,p'\in C\cup C'\}.$$

4D. ***Incidence relations between lines and points in*** $\mathbf{Proj_{nc}}(S)$**.** Let $(p)+(p')$ be a degree-two divisor on $C\cup C'$. There is a surjective map $M_{p,p'}\twoheadrightarrow M_p$ in $\mathsf{Gr}(S)$ and, by [Levasseur and Smith 1993, Lemma 5.3], the kernel of that homomorphism is isomorphic to $M_{\ell'}(-1)$ for some ℓ'. Our next goal is to determine ℓ'. We do that in Proposition 4.8 below.

First we need the rather nice observation in the next lemma.

We call a degree-three divisor on a plane cubic curve *linear* if it is the scheme-theoretic intersection of that curve and a line.

Lemma 4.6. *Let C be a nondegenerate conic in $\mathbb{P}^2$, σ an automorphism of C that fixes two points, and L the line through those two points. Let $p, p' \in C$ and $p'' \in L$. The divisor $(p)+(p')+(p'') \in \mathrm{Div}(C \cup L)$ is linear if and only if $(\sigma p)+(\sigma^{-1}p')+(p'')$ is.*

Proof. By symmetry, it suffices to show that if $(p)+(p')+(p'')$ is linear so is $(\sigma p)+(\sigma^{-1}p')+(p'')$. That is what we will prove. So, assume $(p)+(p')+(p'')$ is linear.

If τ is an automorphism of $\mathbb{P}^1$ that fixes two points, there are nonzero scalars λ and μ and a choice of coordinates such that $\tau(s,t)=(\lambda s, \mu t)$ for all $(s,t) \in \mathbb{P}^1$. We assume, without loss of generality, that (C,σ) is the image of $(\mathbb{P}^1,\tau)$ under the 2-Veronese embedding. Thus, we can assume that C is the curve $xy-z^2=0$ and $\sigma(\alpha,\beta,\gamma)=(\lambda^2\alpha, \mu^2\beta, \lambda\mu\gamma)$. The line L is the line through $(1,0,0)$ and $(0,1,0)$.

Let $p=(\alpha,\beta,\gamma)$, $p'=(\alpha',\beta',\gamma')$, and $p''=(a,b,0)$. By hypothesis, these three points are collinear. Therefore

$$\det\begin{pmatrix} a & b & 0 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix} = 0.$$

I.e., $a(\beta\gamma'-\beta'\gamma)-b(\alpha\gamma'-\alpha'\gamma)=0$.

To show that $(\sigma p)+(\sigma^{-1}p')+(p'')$ is linear we must show that the points $\sigma p=(\lambda^2\alpha, \mu^2\beta, \lambda\mu\gamma)$, $\sigma^{-1}p'=(\lambda^{-2}\alpha', \mu^{-2}\beta', \lambda^{-1}\mu^{-1}\gamma')$, and p'', are collinear. This is the case if and only if

$$\det\begin{pmatrix} a & b & 0 \\ \lambda^2\alpha & \mu^2\beta & \lambda\mu\gamma \\ \lambda^{-2}\alpha' & \mu^{-2}\beta' & \lambda^{-1}\mu^{-1}\gamma' \end{pmatrix} = 0.$$

This determinant is $a(\lambda^{-1}\mu\beta\gamma'-\lambda\mu^{-1}\beta'\gamma)-b(\lambda\mu^{-1}\alpha\gamma'-\lambda^{-1}\mu\alpha'\gamma)$. It is zero if and only if

$$(\alpha\gamma'-\alpha'\gamma)(\lambda^{-1}\mu\beta\gamma'-\lambda\mu^{-1}\beta'\gamma)-(\beta\gamma'-\beta'\gamma)(\lambda\mu^{-1}\alpha\gamma'-\lambda^{-1}\mu\alpha'\gamma)=0.$$

This expression is equal to

$$\lambda^{-1}\mu(\alpha\beta\gamma'^2-\alpha'\beta'\gamma^2)+\lambda\mu^{-1}(\alpha'\beta'\gamma^2-\alpha\beta\gamma^2).$$

But $\alpha\beta-\gamma^2=\alpha'\beta'-\gamma'^2=0$ so $\alpha\beta\gamma'^2-\alpha'\beta'\gamma^2=0$. Thus, the determinant is zero and we conclude that $\sigma^{-1}p'$, σp, and p'', are collinear. □

Remark 4.7. For an alternative approach to Lemma 4.6, note first that the statement can be recast as the claim that if η is the involution of C obtained by “reflection across p''" meaning that

$$\eta(p) = \text{the second intersection of the line } pp'' \text{ with } C,$$

then $\pi = \eta \circ \sigma$ is an involution.

In turn, the involutivity of π follows from the fact that it interchanges the points p and p', and any automorphism of $\mathbb{P}^1$ that interchanges two points is, after a coordinate change identifying said points with 0, ∞, of the form $z \mapsto tz^{-1}$ for some constant t. ♦

The next result is analogous to [Levasseur and Smith 1993, Theorem 5.5], which shows for the 4-dimensional Sklyanin algebras that if $(p)+(p')$ is a degree-two divisor on the quartic elliptic curve E, then there is an exact sequence

$$0 \to M_{p+\tau,p'-\tau}(-1) \to M_{p,p'} \to M_p \to 0,$$

where τ is the point on E such that $\sigma p = p+\tau$ for all $p \in E$.

Proposition 4.8. *If $(p)+(p')$ is a degree-two divisor on $C \cup C'$, there is an exact sequence*

$$0 \to M_{\sigma p,\sigma^{-1}p'}(-1) \to M_{p,p'} \to M_p \to 0.$$

Proof. Let ℓ be the unique line in $\mathbb{P}^3 = \mathbb{P}(S_1^*)$ such that $\ell \cap C = (p)+(p')$; i.e., $M_{p,p'} = M_\ell$. By [Levasseur and Smith 1993, Lemma 5.3], there is an exact sequence $0 \to M_{\ell'}(-1) \to M_\ell \to M_p \to 0$ for some line module $M_{\ell'}$. We complete the proof by showing that $\ell' \cap C = (\sigma p)+(\sigma^{-1}p')$; i.e., $M_{\ell'} = M_{\sigma p,\sigma^{-1}p'}$. There are several cases depending on the location of p and p'.

<u>Case 0.</u> Suppose $\ell = L$. Then $KM_\ell = K'M_\ell = 0$ so M_ℓ, and consequently $M_{\ell'}$, is a module over $S/(K, K')$. Since $S/(K, K')$ is a commutative polynomial ring on two indeterminates it has a unique line module up to isomorphism, itself. In particular, $\ell' = \ell$. Hence there is an exact sequence $0 \to M_{p,p'}(-1) \to M_{p,p'} \to M_p \to 0$. But σ is the identity on L by Theorem 4.2, so $M_{p,p'} = M_{\sigma p,\sigma^{-1}p'}$. Thus, the previous exact sequence is exactly the sequence in the statement of this proposition.

<u>Case 1.</u> Suppose $p, p' \in C$. Then ℓ meets C with multiplicity two and therefore the plane $\{K' = 0\}$ with multiplicity ≥ 2. Hence $\ell \subseteq \{K' = 0\}$. It follows that M_ℓ and $M_{\ell'}$ are modules over $S/(K')$. Given Case 0 treated above, for the remainder of Case 1 we can, and do, assume that $\ell \neq L$.

Since $\ell \neq L$, $\ell \cap (C+L) = (p)+(p')+(p'')$, where p'' is the point where ℓ and L meet. Since $S/(K')$ is a 3-dimensional Artin–Schelter regular algebra, [Artin et al. 1991, Proposition 6.24] tells us that ℓ' is the unique line such that $\ell' \cap (C+L)$ contains the divisor $(\sigma^{-1}p')+(\sigma^{-1}p'')$.[3] By Theorem 4.2, $\sigma p'' = p''$ so ℓ' is the unique

[3]Since [Artin et al. 1991, Proposition 6.24] is for right modules and we are working with left modules we replaced σ by σ^{-1} in the conclusion of that result.

line in $\{K'=0\}$ such that $\ell'\cap(C+L)$ contains $(\sigma^{-1}p')+(p'')$. By Lemma 4.6, σp, $\sigma^{-1}p'$, and p'', are collinear. Therefore $\ell'\cap(C+L)=(\sigma p)+(\sigma^{-1}p')+(p'')$.

Case 2. If $p, p'\in C'$, the "same" argument as in Case 1 proves the proposition.

Case 3. Suppose $p\in C-C'$ and $p'\in C'-C$. Let $p=(\xi_1,\xi_2,\xi_3,0)$ and $p'=(\eta_1,\eta_2,0,\eta_4)$. Since $p\notin C'$, $\xi_3\neq 0$. Since $p'\notin C$, $\eta_4\neq 0$.

By (4-1), ℓ is given by the equations

$$X:=\xi_3\eta_4E-\xi_1\eta_4K-\xi_3\eta_1K'=0,$$
$$Y:=\xi_3\eta_4F-\xi_2\eta_4K-\xi_3\eta_2K'=0.$$

The corresponding linear modules are

$$M_\ell=M_{p,p'}=\frac{S}{SX+SY},$$
$$M_p=\frac{S}{SK'+SX+SY},$$
$$M_{p'}=\frac{S}{SK+SX+SY}.$$

By (4-1), the line through the points $\sigma p=(q\xi_1,q^{-1}\xi_2,\xi_3,0)$ and $\sigma^{-1}p'=(q\eta_1,q^{-1}\eta_2,0,\eta_4)$ is $\{X'=Y'=0\}$, where

$$X':=\xi_3\eta_4E-q\xi_1\eta_4K-q\xi_3\eta_1K',$$
$$Y':=\xi_3\eta_4F-q^{-1}\xi_2\eta_4K-q^{-1}\xi_3\eta_2K'.$$

The corresponding line module is $M_{\sigma p,\sigma^{-1}p'}=S/SX'+SY'$.

The image of K' in M_ℓ generates the kernel of $M_\ell\to M_p$. Since $X'K'=qK'X$ and $Y'K'=q^{-1}K'Y$, X' and Y' annihilate the image of K' in M_ℓ. It follows that there is a map from $M_{\sigma p,\sigma^{-1}p'}(-1)$ onto the kernel of $M_\ell\to M_p$. Thus, the kernel of $M_\ell\to M_p$ is isomorphic to a quotient of $M_{\sigma p,\sigma^{-1}p'}$. But every nonzero submodule of a line modules has GK dimension 2, and every proper quotient of a line module has GK dimension 1, so every nonzero homomorphism map $M_{\sigma p,\sigma^{-1}p'}(-1)\to M_\ell$ is injective. This shows that the kernel of $M_\ell\to M_p$ is isomorphic to $M_{\sigma p,\sigma^{-1}p'}(-1)$.

Case 4. If $p'\in C-C'$ and $p\in C'-C$, the "same" argument as in Case 3 proves the proposition. □

We continue to write L for the line $\{K=K'=0\}$.

Proposition 4.9. *Let ℓ be a line in* $\operatorname{Proj}_{\mathrm{nc}}(S)$[4] *and suppose $p''\in\ell\cap L$.*

(1) *There are points $p, p'\in C\cup C'$ such that the scheme-theoretic intersection $\ell\cap(C\cup L)$ contains the divisor $(p)+(p')+(p'')$.*

[4]This means that $M_\ell:=S/S\ell^\perp$ is a line module.

(2) *There is an exact sequence*

$$0 \to M_{\sigma^{-1}p,\sigma^{-1}p'}(-1) \to M_{p,p'} \to M_{p''} \to 0.$$

Proof. Since M_ℓ is a line module, ℓ meets $C \cup C'$ with multiplicity two. It therefore meets either $C \cup L$ or $C' \cup L$ with multiplicity ≥ 2. We can, and do, assume without loss of generality that ℓ meets $C \cup L$ with multiplicity ≥ 2. Hence ℓ meets the plane $\{K' = 0\}$ with multiplicity ≥ 2. Therefore $\ell \subseteq \{K' = 0\}$. By Bézout's theorem, ℓ is either equal to L or meets $C \cup L$ with multiplicity 3.

Suppose $\ell = L$. Then M_ℓ is a module over the commutative polynomial ring $S/(K, K')$ and there is an exact sequence $0 \to M_\ell(-1) \to M_\ell \to M_{p''} \to 0$. Let p and p' be the points where L meets $C \cup C'$. Then $M_\ell = M_L = M_{p,p'}$ and, since σ is the identity on L, $M_\ell = M_{\sigma p,\sigma^{-1}p'}$. Thus, (1) and (2) hold when $\ell = L$.

Suppose $\ell \neq L$. Let p and p' be the points in $\ell \cap C$; i.e., $\ell \cap C = (p) + (p')$. By [Artin et al. 1991, Proposition 6.24], there is an exact sequence $0 \to M_{\ell'}(-1) \to M_\ell = M_{p,p'} \to M_{p''} \to 0$, where ℓ' is the unique line whose scheme-theoretic intersection with $C \cup L$ is $\geq (\sigma^{-1}p) + (\sigma^{-1}p')$. Hence $M_{\ell'} = M_{\sigma^{-1}p,\sigma^{-1}p'}$. □

5. Relation to $U_q(\mathfrak{sl}_2)$-modules

In this section, we relate our results about fat point and line modules for S to classical results about the finite-dimensional irreducible representations and Verma modules of $U_q(\mathfrak{sl}_2)$. Briefly, fat points in $\mathrm{Proj}_{\mathrm{nc}}(S)$ correspond to finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules and lines in $\mathrm{Proj}_{\mathrm{nc}}(S)$ correspond to Verma modules.

5A. *Facts about $U_q(\mathfrak{sl}_2)$.* First, we recall a few facts about $U_q(\mathfrak{sl}_2)$ that can be found in [Jantzen 1996, Chapter 2].

5A1. *Verma modules.* For each $\lambda \in \mathbb{C}$, we call

$$M(\lambda) := \frac{U_q(\mathfrak{sl}_2)}{U_q(\mathfrak{sl}_2)e + U_q(\mathfrak{sl}_2)(k-\lambda)}$$

a *Verma module* for $U_q(\mathfrak{sl}_2)$, and λ its *highest weight*.

5A2. *Casimir element.* The *Casimir element*

(5-1) $$C := ef + \frac{q^{-1}k + qk^{-1}}{(q-q^{-1})^2} = fe + \frac{qk + q^{-1}k^{-1}}{(q-q^{-1})^2}$$

is in the center of $U_q(\mathfrak{sl}_2)$ and acts on $M(\lambda)$ as multiplication by

$$\frac{q\lambda + q^{-1}\lambda^{-1}}{(q-q^{-1})^2}.$$

5A3. *Finite-dimensional simple modules.* For each $n \geq 1$, there are exactly two simple $U_q(\mathfrak{sl}_2)$-modules of dimension $n+1$. They can be labeled $L(n,+)$ and $L(n,-)$ in such a way that there are exact sequences

$$0 \to M(\pm q^{-n-2}) \to M(\pm q^n) \to L(n,\pm) \to 0. \tag{5-2}$$

The module $L(n,\pm)$ has basis $m_0,\dots,m_n$ with action

$$km_i = \pm q^{n-2i} m_i,$$

$$fm_i = \begin{cases} m_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n, \end{cases} \qquad em_i = \begin{cases} \pm[i][n+1-i]m_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \end{cases} \tag{5-3}$$

where we have made use of the *quantum integers*

$$[m] := \frac{q^m - q^{-m}}{q - q^{-1}}.$$

5B. ***Lines in*** $\mathbf{Proj_{nc}}(S) \longleftrightarrow$ ***Verma modules for*** $U_q(\mathfrak{sl}_2)$**.** First, we show that Verma modules are "affine pieces" of line modules.

Proposition 5.1. *Let* $\lambda \in \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$ *and let* ℓ *be the line* $E = K - \lambda K' = 0$.

(1) ℓ *lies on the quadric* $Q(\lambda)$.

(2) $S/S\ell^{\perp}$ *is a line module.*

(3) *If* $\lambda \notin \{0,\infty\}$, *then* $(S/S\ell^{\perp})[(KK')^{-1}]_0 \cong M(\lambda)$.

Proof. A simple calculation proves (1), and then (2) follows from Theorem 4.4.

(3) The functor $j^*\pi^* : \mathsf{Gr}(S) \to \mathsf{Mod}(U_q(\mathfrak{sl}_2)$ defined by $j^*\pi^*M = M[(KK')^{-1}]_0$ is exact, so $(S/S\ell^{\perp})[(KK')^{-1}]_0$ is isomorphic to $S(KK')^{-1}]_0/(S\ell^{\perp})[(KK')^{-1}]_0$. Using the isomorphism given by (2-7), it is clear that $(S\ell^{\perp})[(KK')^{-1}]_0$ is the left ideal of $U_q(\mathfrak{sl}_2)$ generated by e and $k-\lambda$. □

5B1. *"Heretical" Verma modules.* Proposition 5.1 illustrates the importance of line modules for Artin–Schelter regular algebras with Hilbert series $(1-t)^{-4}$. Line modules are just like Verma modules. Indeed, Verma modules for $U(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2)$ are "affine pieces" of line modules.

From the point of view of noncommutative projective algebraic geometry, the line modules that correspond to Verma modules are no more special than other line modules. One is tempted to declare that if ℓ is any line on any $Q(\lambda)$, $\lambda \neq 0, \infty$, then $(S/S\ell^{\perp})[(KK')^{-1}]_0$ should be considered as a Verma module.

Doing that would place $U_q(\mathfrak{sl}_2)$ on a more equal footing with $U(\mathfrak{sl}_2)$: if one varies both the Borel subalgebra and the highest weight, then $U(\mathfrak{sl}_2)$ has a 2-parameter family of Verma modules; if were to define Verma modules for $U_q(\mathfrak{sl}_2)$ as "affine pieces" of line modules, then $U_q(\mathfrak{sl}_2)$ would also have a 2-parameter family of Verma modules.

5B2. *Central (Casimir) elements.* We define $\Omega(0) = \Omega(\infty) = KK'$ and, for each $\lambda \in \mathbb{C} - \{0, \infty\}$, we define

$$\begin{aligned}\Omega(\lambda) :=& EF + \frac{q^{-1}K^2 + qK'^2}{(q-q^{-1})^2} - \frac{q\lambda + q^{-1}\lambda^{-1}}{(q-q^{-1})^2}KK' \\ =& EF + \kappa^2(q^{-1}K - q\lambda K')(K - \lambda^{-1}K') \\ =& FE + \kappa^2(qK - q^{-1}\lambda^{-1}K')(K - \lambda K').\end{aligned}$$

The elements $\Omega(\lambda)$, $\lambda \in \mathbb{P}^1$, belong to the center of S and span a 2-dimensional subspace of S_2.

We take note that $\Omega(\lambda) = \Omega(q^{-2}\lambda^{-1})$ and $\Omega(\mu) \neq \Omega(\lambda)$ if $\mu \notin \{\lambda, q^{-2}\lambda^{-1}\}$.

Under the isomorphism $S[(KK')^{-1}]_0 \cong U_q(\mathfrak{sl}_2)$ given in (2-7), we have

$$\Omega(\lambda)(KK')^{-1} = C - \frac{q\lambda + q^{-1}\lambda^{-1}}{(q-q^{-1})^2},$$

where C is the Casimir element defined in (5-1).

The reader will notice similarities between the pencil of central subspaces $\mathbb{C}\Omega(\lambda) \subseteq S_2$ and the pencil of quadrics $Q(\lambda) \subseteq \mathbb{P}(S_1^*)$. For example, exactly one $\Omega(\lambda)$ is a product of two degree-1 elements, namely $\Omega(0) = \Omega(\infty) = KK'$, and exactly one $Q(\lambda)$ that is a union of two planes, namely $Q(0) = Q(\infty) = \{KK' = 0\}$. In a similar vein, we expect that $S/(\Omega(\lambda))$ is a prime ring if and only if λ is not 0 or ∞. The precise relation between the $\Omega(\lambda)$'s and the $Q(\lambda)$'s is established in Proposition 5.3.

Lemma 5.2. *Let $\lambda \in \mathbb{C}^\times$. The central element $\Omega(\lambda)$ annihilates M_ℓ for all lines ℓ of the form*

$$E - \kappa s(K - \lambda K') = sF + \kappa(K - \lambda^{-1}K') = 0, \quad s \in \mathbb{P}^1.$$

Proof. Let s be any point on $\mathbb{P}^1$. Since $\Omega(\lambda)$ equals

$$\begin{aligned}&FE + \frac{1}{(q-q^{-1})^2}(qK - q^{-1}\lambda^{-1}K')(K - \lambda K') \\ &\qquad = F(E - \kappa s(K - \lambda^{-1}K')) + \kappa(qK - q^{-1}\lambda^{-1}K')(sF + \kappa(K - \lambda K')),\end{aligned}$$

it belongs to the left ideal generated by $E - \kappa s(K - \lambda^{-1}K')$ and $sF + \kappa(K - \lambda K')$. That left ideal is $S\ell^\perp$ so, since $\Omega(\lambda)$ is in the center of S, it annihilates $S/S\ell^\perp$. □

Proposition 5.3. *Let M_ℓ be a line module. If $\lambda \in \mathbb{C}^\times$, then $\Omega(\lambda)$ annihilates M_ℓ if and only if either*

(1) *$\ell \subseteq Q(\lambda)$ and is in the same ruling as the line $E = K - \lambda K' = 0$, or*

(2) *$\ell \subseteq Q(q^{-2}\lambda^{-1})$ and is in the same ruling as the line $E = K - q^{-2}\lambda^{-1}K' = 0$.*

Furthermore, $\Omega(0)=\Omega(\infty)$ *annihilates* M_ℓ *if and only if* $\ell \subseteq Q(0)=Q(\infty)=\{KK'=0\}$.

Proof. It is easy to see that the last sentence in the statement of the proposition is true so we will assume that $\ell \not\subseteq \{KK'=0\}$. Since M_ℓ is a line module, ℓ lies on $Q(\mu)=Q(\mu^{-1})$ for some $\mu \in \mathbb{C}^\times$. We fix such a λ. Since $\ell \not\subseteq \{KK'=0\}$, $\mu \neq 0, \infty$.

($\Rightarrow$) Fix $\lambda \in \mathbb{C}^\times$ and suppose that $\Omega(\lambda)$ annihilates M_ℓ. The lines on $Q(\mu)$ are given by (4-2) and (4-3). Replacing μ by μ^{-1} if necessary, we can assume that ℓ belongs to the same ruling on $Q(\mu)$ as $E=K-\mu K'=0$. Hence M_ℓ is annihilated by $\Omega(\mu)$. If $\Omega(\mu)\neq\Omega(\lambda)$, then M_ℓ is annihilated by KK'. That is not the case, so $\Omega(\mu)=\Omega(\lambda)$. Hence $\mu \in \{\lambda, q^{-2}\lambda^{-1}\}$. Hence either (1) or (2) holds.

($\Leftarrow$) This implication follows from Lemma 5.2. If $\ell \subseteq Q(\lambda)$ and is in the same ruling as the line $E=K-\lambda K'=0$, then M_ℓ is annihilated by $\Omega(\lambda)$. If $\ell \subseteq Q(q^{-2}\lambda^{-1})$ and is in the same ruling as the line $E=K-q^{-2}\lambda^{-1}K'=0$, then M_ℓ is annihilated by $\Omega(q^{-2}\lambda^{-1})=\Omega(\lambda)$. □

We only care about the ideal generated by $\Omega(\lambda)$ and the matter of which modules are annihilated by which $\Omega(\lambda)$'s. Thus, we only care about $\Omega(\lambda)$ up to nonzero scalar multiples. For this reason it is often better to think of $\Omega(\lambda)$ as an element in $\mathbb{P}^1$.

5C. ***Fat points in*** $\mathbf{Proj_{nc}}(S)$ $\longleftrightarrow$ ***finite-dimensional simple*** $U_q(\mathfrak{sl}_2)$***-modules.*** As the title suggests, this subsection establishes a connection between the finite-dimensional simple $U_q(\mathfrak{sl}_2)$-modules $L(n,\pm)$ discussed in Section 5A3 and certain fat points $F(n,\pm)$ of the noncommutative scheme $\mathrm{Proj}_{\mathrm{nc}}(S)$ that are defined below. Proposition 5.6 makes this connection explicit. We have not addressed the question of whether the $F(n,\pm)$'s are all the fat points.

5C1. *Some finite-dimensional simple S-modules.* We fix a square root, $\sqrt{q}$, of q and adopt the convention that $q^{n/2-i}=(\sqrt{q})^{n-2i}$ and $q^{i-n/2}=(\sqrt{q})^{2i-n}$. Let $V(n,\pm)$ be the vector space with basis $v_0,\dots,v_n$ and define

$$Kv_i=\sqrt{\pm 1}\,q^{n/2-i}v_i, \qquad K'v_i=\pm\sqrt{\pm 1}\,q^{i-n/2}v_i,$$

$$Fv_i=\begin{cases}[n-i]v_{i+1} & \text{if } i<n,\\ 0 & \text{if } i=n,\end{cases} \qquad Ev_i=\begin{cases}\pm[i]v_{i-1} & \text{if } i>0,\\ 0 & \text{if } i=0.\end{cases}$$

5C2. *Automorphisms of S and autoequivalences of* $\mathsf{Gr}(S)$. Let $\theta : S\to S$ be the algebra automorphism defined by $\theta(K)=-K$, $\theta(K')=K'$, $\theta(E)=E$, $\theta(F)=F$.

If $\varepsilon\in\mathbb{C}^\times$ let $\phi_\varepsilon : S\to S$ be the algebra automorphism $\phi_\varepsilon(a)=\varepsilon^n a$ for all $a\in S_n$.

Let ϕ be a degree-preserving algebra automorphism of S. The functor $\phi^* : \mathsf{Gr}(S)\to\mathsf{Gr}(S)$ is defined as follows: if $M\in\mathsf{Gr}(S)$, then $\phi^*(M)$ is M as a graded vector space and if $a\in S$ and $m\in M^*$, then $a\cdot m=\phi(a)m$. The functor ϕ^* is an autoequivalence.

Proposition 5.4. *Let* $\varepsilon = -\sqrt{-1}$.

(1) $V(n, \pm)$ *is a simple* S*-module of dimension* $n+1$.

(2) $V(n, \pm)$ *is a* $S[(KK')^{-1}]$*-module with* $(KK')^{-1}$ *acting as the identity.*

(3) *Identifying* $U_q(\mathfrak{sl}_2)$ *with* $S[(KK')^{-1}]_0$ *as in Proposition 2.4,* $V(n, \pm) \cong L(n, \pm)$ *as a* $U_q(\mathfrak{sl}_2)$*-module.*

(4) $\Omega(\pm q^n)$ *annihilates* $V(n, \pm)$.

(5) $V(n, -) \cong \phi_\varepsilon^* \theta^* V(n, +)$.

Proof. (1) First we check that the action makes $V(n, \pm)$ a left S-module. If $v_{-1} = 0$, then $EKv_i = \pm\sqrt{\pm 1}\, q^{n/2-i}[i]v_{i-1}$ and $KEv_i = \pm\sqrt{\pm 1}\, q^{n/2-i+1}[i]v_{i-1} = qEKv_i$. Hence $KE - qEK$ acts on $V(n, \pm)$ as 0. With the understanding that $v_{n+1} = 0$, $FKv_i = \sqrt{\pm 1}\, q^{n/2-i}[n-i]v_{i-1}$ and $KFv_i = \sqrt{\pm 1}\, q^{n/2-i-1}[n-i]v_{i+1} = q^{-1}FKv_i$, so $KF - q^{-1}FK$ acts on $V(n, \pm)$ as 0. Similar calculations show that $K'E - q^{-1}EK'$ and $K'F - qFK'$ act on $V(n, \pm)$ as 0 also. Furthermore,

$$\begin{aligned}
[E, F]v_i &= \pm\big([n-i]Ev_{i+1} - [i]Fv_{i-1}\big) \\
&= \pm\big([n-i][i+1][i][n-i+1\big)v_i \\
&= \pm[n-2i]v_i \\
&= \frac{K^2 - K'^2}{q - q^{-1}} v_i,
\end{aligned}$$

so $V(n, \pm)$ really is a left S-module.

To see it is simple, first observe that the v_i are eigenvectors for K with pairwise distinct eigenvalues. It follows that if $V(n, \pm)$ is not simple, then there it has a proper submodule that contains some v_i. However, looking at the actions of E and F on the v_j, a submodule that contains one v_i contains all v_i. Hence $V(n, \pm)$ is simple.

(2) Since KK' acts on $V(n, \pm)$ as multiplication by 1, the module-action of S on $V(n, \pm)$ extends to a module-action of $S[(KK')^{-1}]$.

(3) Since $e = \dfrac{1}{\sqrt{q}} EK^{-1}$, $f = \dfrac{1}{\sqrt{q}} F(K')^{-1}$, and $k = K(K')^{-1}$,

$$kv_i = -q^{n-2i} v_i,$$

$$fv_i = \begin{cases} \dfrac{1}{\sqrt{\pm q}} q^{n/2-i}[n-i]v_{i+1} & \text{if } i < n, \\ 0 & \text{if } i = n, \end{cases} \qquad ev_i = \begin{cases} \dfrac{\pm 1}{\sqrt{\pm q}} q^{i-n/2}[i]v_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0. \end{cases}$$

Choose nonzero scalars $\lambda_0, \dots, \lambda_n$ such that

$$\lambda_{i-1}/\lambda_i = \sqrt{\pm q}\, q^{n/2-i}[n+1-i].$$

The linear isomorphism $\phi : V(n,\pm) \to L(n,\pm)$ defined by $\phi(v_i) = \lambda_i m_i$ is a $U_q(\mathfrak{sl}_2)$-module isomorphism because $\phi(kv_i) = k\phi(v_i)$,

$$\phi(ev_i) = \frac{\pm 1}{\sqrt{\pm q}} q^{i-n/2}[i]\lambda_{i-1}m_{i-1} = \pm[i][n+1-i]\lambda_i m_{i-1} = e\phi(v_i),$$

and

$$\phi(fv_i) = \frac{1}{\sqrt{\pm q}} q^{n/2-i}[n-i]\lambda_{i+1}v_{i+1} = \lambda_i m_{i+1} = f\phi(v_i).$$

Hence $V(n,\pm) \cong L(n,\pm)$ as claimed.

(4) By Schur's lemma, $\Omega(\lambda)$ acts on $V(n,\pm)$ as multiplication by a scalar. Therefore, if $\Omega(\lambda)$ annihilates v_0 it annihilates $V(n,\pm)$. Since $Ev_0 = 0$, $\Omega(\lambda)v_0 = \kappa^2(qK - q^{-1}\lambda^{-1}K')(K - \lambda K')v_0$. The result follows from $(K \mp q^n K')v_0 = 0$.

(5) Let $v_0, \dots, v_n$ be the basis for $V(n,+)$ in Section 5C1 and, to avoid confusion, write v_i' for the basis element v_i in $V(n,-)$. Thus, $Kv_i' = -\varepsilon q^{n/2-i}v_i'$.

Define $\psi : \phi_\varepsilon^*\theta^*V(n,+) \to V(n,-)$ by $\psi(v_i) := (-1)^i\varepsilon^i v_i'$. To show ψ is an S-module isomorphism it suffices to show it is an S-module homomorphism. To this end, consider v_i as an element in $\phi_\varepsilon^*\theta^*V(n,+)$. Because $\theta\phi_\varepsilon(K) = -\varepsilon K$, $Kv_i = -\varepsilon q^{n/2-i}v_i$. Hence

$$\psi(Kv_i) = \psi(-\varepsilon q^{n/2-i}v_i) = -\varepsilon q^{n/2-i}(-1)^i\varepsilon^i v_i' = (-1)^i\varepsilon^i Kv_i' = K\psi(v_i).$$

Similarly, because $\theta\phi_\varepsilon(K') = \varepsilon K'$ and $K'v_i' = \varepsilon q^{i-n/2}v_i'$,

$$\psi(K'v_i) = \psi(\varepsilon q^{i-n/2}v_i) = \varepsilon q^{i-n/2}(-1)^i\varepsilon^i v_i' = (-1)^i\varepsilon^i K'v_i' = K'\psi(v_i).$$

We also have

$$\psi(Fv_i) = \psi(\varepsilon[n-i]v_{i+1}) = \varepsilon[n-i](-1)^{i+1}\varepsilon^{i+1}v_{i+1}' = (-1)^i\varepsilon^i Fv_i' = F\psi(v_i)$$

and

$$\psi(Ev_i) = \psi(\varepsilon[i]v_{i-1}) = \varepsilon[i](-1)^{i-1}\varepsilon^{i-1}v_{i-1}' = (-1)^i\varepsilon^i Ev_i' = E\psi(v_i). \quad \square$$

5C3. *Fat points and fat point modules.* For each $n \in \mathbb{N}$ we define

$$F(n,\pm) := V(n,\pm) \otimes \mathbb{C}[z]$$

and make this a graded left S-module according to the recipe in Lemma 2.1. It is a fat point module. Proposition 5.6 makes the statement that the fat point (module) $F(n,\pm)$ corresponds to the finite-dimensional simple $U_q(\mathfrak{sl}_2)$-module $L(n,\pm)$ precise.

Lemma 5.5. *If θ is the automorphism in Section 5C2, then $\theta^*F(n,\pm) \cong F(n,\mp)$.*

Proof. If V is any left S-module and ϕ_ε the automorphism in Section 5C2 associated to $\varepsilon \in \Bbbk^\times$, then the map $\Phi : V \otimes \Bbbk[z] \to (\phi_\varepsilon^* V) \otimes \Bbbk[z]$, $\Phi(v \otimes z^i) = v \otimes (\varepsilon z)^i$, is an isomorphism in $\mathsf{Gr}(S)$. Hence

$$F(n,-) = \phi_\varepsilon^* \theta^* V(n,+) \otimes \Bbbk[z] \cong \theta^* V(n,+) \otimes \Bbbk[z] \cong \theta^*(V(n,+) \otimes \Bbbk[z]) = \theta^* F(n,+). \qquad \square$$

Proposition 5.6. *If* $\pi^* : \mathsf{Gr}(S) \to \mathsf{QGr}(S)$ *and* $j^* : \mathsf{QGr}(S) \to U_q(\mathfrak{sl}_2)$ *are the functors in Section 1A2, then* $j^*\pi^* F(n,\pm) \cong L(n,\pm)$; *i.e., there is an isomorphism of* $U_q(\mathfrak{sl}_2)$*-modules*

$$F(n,\pm)[(KK')^{-1}]_0 \cong L(n,\pm).$$

Proof. The functor $j^*\pi^*$ sends $M \in \mathsf{Gr}(S)$ to $M[(KK')^{-1}]_0$, where the latter is made into a $U_q(\mathfrak{sl}_2)$-module via the isomorphism $U_q(\mathfrak{sl}_2) \to S[(KK')^{-1}]_0$ in Proposition 2.4.

Since KK' acts on $V(n,\pm)$ as the identity, it acts on $F(n,\pm) = V(n,\pm) \otimes k[z]$ as multiplication by z^2. Hence, $F(n,\pm)[(KK')^{-1}]_0 = V(n,\pm) \otimes k[z,z^{-2}]_0 = V(n,\pm) \otimes 1$.

Let $\widehat{S} = S[(KK')^{-1}]$. Applying the functor $\widehat{S} \otimes_S -$ to the surjective S-module homomorphism $F(n,\pm) \to V(n,\pm)$, $v \otimes z^i \mapsto v$, produces a surjective homomorphism

$$\psi : F[(KK')^{-1}] = \widehat{S} \otimes_S F(n,\pm) \to \widehat{S} \otimes_S V(n,\pm)$$

of $\widehat{S}$-modules. Of course, ψ is a homomorphism of $\widehat{S}_0$-modules. Every homogeneous component of $F(n,\pm)[(KK')^{-1}]$ is an $\widehat{S}_0$-submodule of $F(n,\pm)[(KK')^{-1}]$ so ψ restricts to a homomorphism $F(n,\pm)[(KK')^{-1}]_0 \to \widehat{S} \otimes_S V(n,\pm)$ of $\widehat{S}_0$-modules. But $\widehat{S} \otimes_S V(n,\pm)$ is isomorphic to $L(n,\pm)$ as an $\widehat{S}_0$-module by Proposition 5.4(3) and, by the previous paragraph, $\dim(F(n,\pm)[(KK')^{-1}]_0) = \dim(V(n,\pm)) = n+1 = \dim(L(n,\pm))$ so the restriction of ψ to $F(n,\pm)[(KK')^{-1}]_0$ is an isomorphism of $\widehat{S}_0$-modules. $\square$

Proposition 5.7. *Let* $n \geq 0$. *Let* $\ell_\pm$ *be any line on* $Q(\pm q^n)$ *that is in the same ruling as the line* $E = K \mp q^n K' = 0$.

(1) *There is a surjective* S*-module homomorphism* $M_{\ell_\pm} \to V(n,\pm)$.

(2) *There is a homomorphism* $M_{\ell_\pm} \to F(n,\pm)$ *in* $\mathsf{Gr}(S)$ *that becomes an epimorphism in* $\mathsf{QGr}(S)$.

(3) *In* $\operatorname{Proj}_{nc}(S)$, *the fat point* $F(n,\pm)$ *lies on the line* $\ell_\pm$.

Proof. Let $s \in \mathbb{P}^1$ be such that $\ell_\pm$ is the line

$$\kappa(K \mp q^n K') - s^{-1}E = \kappa(K \mp q^{-n} K') + sF = 0.$$

Thus,

$$M_{\ell_\pm} \cong \frac{S}{SX_\pm + SY_\pm},$$

where $X_\pm = \kappa(K \mp q^{-n}K') - s^{-1}E$ and $Y_\pm = \kappa(K \mp q^n K') + sF$.

(1) Since $V(n,\pm)$ is a simple S-module it suffices to show there is a nonzero homomorphism $M_{\ell_\pm} \to V(n,\pm)$. For this, it suffices to show there is a nonzero element in $V(n,\pm)$ annihilated by both $X_\pm$ and $Y_\pm$.

If $s=0$, and $v_\pm = v_0 \in V(n,\pm)$, then $X_\pm v_\pm = Ev_0 = 0$ and $Y_\pm v_\pm = (K \mp q^n K')v_\pm = 0$. If $s=\infty$ and $v_\pm = v_n \in V(n,\pm)$, then $X_\pm v_\pm = (K \mp q^{-n}K')v_n = 0$ and $Y_\pm v_\pm = Fv_n = 0$. Thus, (1) is true if s equals 0 or ∞.

From now on, assume that $s \neq 0, \infty$. Let $\lambda_0, \dots, \lambda_n \in \Bbbk^\times$ be such that

$$\lambda_{i+1}/\lambda_i = \pm\sqrt{\pm 1}\,\frac{[n-i]}{[i+1]} s q^{-n/2}$$

for all i. If

$$v_\pm = \sum_{i=0}^{n} \lambda_i v_i \in V(n,\pm),$$

then

$$\begin{aligned} X_\pm v_\pm &= \sum_{i=0}^{n} \big(\kappa\sqrt{\pm 1}\,(q^{n/2-i} \mp q^{-n}q^{i-n/2})\lambda_i v_i - s^{-1}(\pm 1)[i]\lambda_i v_{i-1}\big) \\ &= \sum_{i=0}^{n} \big(-q^{-n/2}\sqrt{\pm 1}\,[n-i]\lambda_i \mp s^{-1}[i+1]\lambda_{i+1}\big) v_i \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} Y_\pm v_\pm &= \sum_{i=0}^{n} \big(\kappa\sqrt{\pm 1}\,(q^{n/2-i} \mp q^{n}q^{i-n/2})\lambda_i v_i + s[n-i]\lambda_i v_{i+1}\big) \\ &= \sum_{i=0}^{n} \big(-q^{n/2}\sqrt{\pm 1}\,[i]\lambda_i + s[n-i+1]\lambda_{i-1}\big) v_i \\ &= 0. \end{aligned}$$

(2) By Lemma 2.1, the existence of a nonzero homomorphism $M_{\ell_\pm} \to V(n,\pm)$ implies the existence of a nonzero homomorphism $M_{\ell_\pm} \to F(n,\pm)$ in $\mathsf{Gr}(S)$. However, as an object in $\mathsf{QGr}(S)$, $F(n,\pm)$ is irreducible so (2) follows.

(3) This is just terminology. □

If one of the lines $\ell_\pm = \{X_\pm = Y_\pm = 0\}$ in Proposition 5.7 meets C at $(\xi_1, \xi_2, \xi_3, 0)$, then it meets C' at $(q^{-n}\xi_1, q^n\xi_2, 0, \pm\xi_3)$. Combining this with Theorem 4.2(5) gives the following result.

Corollary 5.8. *Let* $p = (\xi_1, \xi_2, \xi_3, 0) \in C$ *and define* $p_\pm = (\xi_1, \xi_2, 0, \pm\xi_3)$. *Let* $\ell_\pm$ *be the secant line to* $C \cup C'$ *passing through* p *and* $\sigma_S^n(p_\pm)$. *There is a surjective homomorphism* $M_{\ell_\pm} \to V(n, \pm)$ *in* $\mathsf{Mod}(S)$ *and an epimorphism* $M_{\ell_\pm} \to F(n, \pm)$ *in* $\mathsf{QGr}(S)$.

The analogue of (5-2) requires results from the next section, and can be found in Theorem 6.2.

6. Relation to the nondegenerate Sklyanin algebras

We remind the reader that $S(\alpha, \beta, \gamma)$ denotes one of the nondegenerate Sklyanin algebras defined in (2-2).

In this section, we show that some of our results about S can be obtained as "degenerations" of results in [Smith and Stafford 1992; Chirvasitu and Smith 2017; Smith and Staniszkis 1993] about $S(\alpha, \beta, \gamma)$. We also complete the characterization of those line modules that surject onto fat point modules that we alluded to in the last section.

6A. ***The point scheme of a nondegenerate Sklyanin algebra.*** We follow [Smith and Stafford 1992]. The point scheme of $S(\alpha, \beta, \gamma)$ embedded in $\mathbb{P}^3$ with coordinates x_0, x_1, x_2, x_3 is

$$E' = E \cup \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}, \tag{6-1}$$

where E is the elliptic curve defined by

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0 = x_3^2 + \frac{1-\gamma}{1+\alpha} x_1^2 + \frac{1+\gamma}{1-\beta} x_2^2. \tag{6-2}$$

Equivalently, E is the intersection of any two of the following quadrics:

$$\begin{aligned} x_0^2 + x_1^2 + x_2^2 + x_3^2 &= 0, \\ x_0^2 - \beta\gamma x_1^2 - \gamma x_2^2 + \beta x_3^2 &= 0, \\ x_0^2 + \gamma x_1^2 - \alpha\gamma x_2^2 - \alpha x_3^2 &= 0, \\ x_0^2 - \beta x_1^2 + \alpha x_2^2 - \alpha x_3^2 &= 0. \end{aligned} \tag{6-3}$$

There is an automorphism σ of E' that fixes the four isolated points and on E is given by the formula

$$\sigma : \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} -2\alpha\beta\gamma & x_1x_2x_3 & - & x_0(-x_0^2 + \beta\gamma x_1^2 + \alpha\gamma x_2^2 + \alpha\beta x_3^2) \\ 2\alpha & x_0x_2x_3 & + & x_1(x_0^2 - \beta\gamma x_1^2 + \alpha\gamma x_2^2 + \alpha\beta x_3^2) \\ 2\beta & x_0x_1x_3 & + & x_2(x_0^2 + \beta\gamma x_1^2 - \alpha\gamma x_2^2 + \alpha\beta x_3^2) \\ 2\gamma & x_0x_1x_2 & + & x_3(x_0^2 + \beta\gamma x_1^2 + \alpha\gamma x_2^2 - \alpha\beta x_3^2) \end{pmatrix}. \tag{6-4}$$

6B. ***Degenerate point scheme.*** In the degenerate case, substituting $(\alpha, \beta, \gamma) = (0, b^2, -b^2)$ into equations (6-1) through (6-4) yields the following results.

We will compare the point scheme of $S = S(0, b^2, -b^2)$ to

$$E'_{\text{deg}} := E_{\text{deg}} \cup \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}, \tag{6-5}$$

where the curve E_{deg} is defined by

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0 = x_3^2 + (1 + b^2)x_1^2 + x_2^2, \tag{6-6}$$

or as the intersection of any two of the quadrics

$$\begin{aligned} x_0^2 + x_1^2 + x_2^2 + x_3^2 &= 0, \\ x_0^2 + b^4 x_1^2 + b^2 x_2^2 + b^2 x_3^2 &= 0, \\ x_0^2 - b^2 x_1^2 &= 0, \\ x_0^2 - b^2 x_1^2 &= 0. \end{aligned} \tag{6-7}$$

The automorphism on E'_{deg} fixes the four isolated points and is defined on E_{deg} by

$$\sigma_{\text{deg}} : \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_0(x_0^2 + b^4 x_1^2) \\ x_1(x_0^2 + b^4 x_1^2) \\ 2b^2 x_0 x_1 x_3 + x_2(x_0^2 - b^4 x_1^2) \\ -2b^2 x_0 x_1 x_2 + x_3(x_0^2 - b^4 x_1^2) \end{pmatrix} \tag{6-8}$$

6C. ***Comparison with our results.*** We now compare $(E'_{\text{deg}}, \sigma_{\text{deg}})$ with $(\mathcal{P}_S, \sigma_S)$ from Theorem 4.2. Recall our definitions of E, F, K, K' from (2-5):

$$\begin{gathered} E = \frac{i}{2}(1 - ib)(x_2 + ix_3), \quad F = \frac{i}{2}(1 + ib)(x_2 - ix_3), \\ K = x_0 + bx_1, \quad K' = x_0 - bx_1. \end{gathered} \tag{6-9}$$

With respect to the homogeneous coordinates E, F, K, and K',

$$\mathcal{P}_S = C \cup C' \cup L \cup \{(0, 0, 1, \pm 1)\},$$

where C, C' and L are given by

$$\begin{aligned} C' &: EF + \kappa^2 K'^2 = K = 0, \\ C &: EF + \kappa^2 K^2 = K' = 0, \\ L &: K = K' = 0. \end{aligned}$$

The conics C and C' lie on the planes $K' = 0$ and $K = 0$, respectively, and the line L is the intersection of those two planes. With respect to the homogeneous coordinates E, F, K, and K', (6-5) becomes

$$E'_{\text{deg}} = E_{\text{deg}} \cup \{(0, 0, 1, 1), (0, 0, 1, -1), (q, 1, 0, 0), (-q, 1, 0, 0)\}.$$

The isolated points $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ in (6-5) remain isolated after degeneration, but the points $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ in (6-5), which are $(q, 1, 0, 0)$ and $(-q, 1, 0, 0)$ in the E, F, K, K' coordinates, become points on the line L in $\mathcal{P}_S$ after degeneration.

Next, we compare E_{deg} with $C \cup C' \cup L$. The equation (6-6) yields

$$x_0^2 - b^2 x_1^2 = (x_0 - bx_1)(x_0 + bx_1) = KK' = 0.$$

Hence $E_{\text{deg}} \subseteq \{K = 0\} \cup \{K' = 0\}$.

On the plane $K' = 0$, $x_0 = bx_1$ so both sides of (6-6) for E_{deg} become

$$(1 + b^2)x_1^2 + x_2^2 + x_3^2 = 0.$$

On the other hand, C' is given by

$$\begin{aligned} 0 = EF + \kappa^2 K'^2 &= -\tfrac{1}{4}(1 + b^2)(x_2^2 + x_3^2) + 4\kappa^2 b^2 x_1^2 \\ &= -\tfrac{1}{4}(1 + b^2)(x_2^2 + x_3^2) - \tfrac{1}{4}(1 + b^2)^2 x_1^2 \\ &= -\tfrac{1}{4}(1 + b^2)(x_2^2 + x_3^2 + (1 + b^2)x_1^2). \end{aligned}$$

Hence $E_{\text{deg}} \cap \{K' = 0\} = C'$. A similar calculation yields the analogous result for the plane $K = 0$. We thus conclude that

$$E_{\text{deg}} = C \cup C'.$$

Finally, we compare σ_{deg} and σ_S. On the plane $K = 0$,

$$\sigma_{\text{deg}} : \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} (b^2 + b^4)x_1^2 x_0 \\ (b^2 + b^4)x_1^3 \\ 2b^4 x_1^2 x_3 + (b^2 - b^4)x_1^2 x_2 \\ -2b^4 x_1^2 x_2 + (b^2 - b^4)x_1^2 x_3 \end{pmatrix} = \begin{pmatrix} (1 + b^2)x_0 \\ (1 + b^2)x_1 \\ (1 - b^2)x_2 + 2b^2 x_3 \\ (1 - b^2)x_3 - 2b^2 x_2 \end{pmatrix}.$$

Changing coordinates,

$$\sigma_{\text{deg}} : \begin{pmatrix} x_2 + ix_3 \\ x_2 - ix_3 \\ x_0 + bx_1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} (1 - ib)^2(x_2 + ix_3) \\ (1 + ib)^2(x_2 - ix_3) \\ (1 + b^2)(x_0 + bx_1) \\ 0 \end{pmatrix} = \begin{pmatrix} q(x_2 + ix_3) \\ q^{-1}(x_2 - ix_3) \\ (x_0 + bx_1) \\ 0 \end{pmatrix}.$$

Therefore, in the E, F, K, K' coordinates, $\sigma_{\text{deg}}(\xi_1, \xi_2, \xi_3, 0) = (q\xi_1, q^{-1}\xi_2, \xi_3, 0) = \sigma_S(\xi_1, \xi_2, \xi_3, 0)$. Similar calculations on the plane $K' = 0$ and on the isolated points yield $\sigma_{\text{deg}} = \sigma_S$.

6D. *Degenerations of Heisenberg automorphisms.* Recall (e.g., from [Chirvasitu and Smith 2017, Proposition 2.6]) that the Heisenberg group of order 4^3 acts on the Sklyanin algebra $S(\alpha, \beta, \gamma)$ as follows.

	x_0	x_1	x_2	x_3
ϕ_1	bcx_1	$-ix_0$	$-ibx_3$	$-cx_2$
ϕ_2	acx_2	$-ax_3$	$-ix_0$	$-icx_1$
ϕ_3	abx_3	$-iax_2$	$-bx_1$	$-ix_0$

Table 2. Automorphisms of $S(\alpha, \beta, \gamma)$.

	E	F	K	K'
ϕ_1	bqF	$-bq^{-1}E$	$-ibK'$	ibK
ϕ_2	$\frac{1}{2}(1-ib)K'$	$\frac{1}{2}(1+ib)K$	0	0
ϕ_3	$\frac{i}{2}(1-ib)K'$	$-\frac{i}{2}(1+ib)K$	0	0

Table 3. Endomorphisms of $S(0, \beta, -\beta)$.

First, fix square roots a, b and c of α, β and γ respectively. We define automorphisms ϕ_i of $S(\alpha, \beta, \gamma)$ via Table 2.

Now fix $\nu_1, \nu_2, \nu_3 \in k^\times$ such that $a\nu_1^2 = b\nu_2^2 = c\nu_3^2 = -iabc$, and define $\varepsilon_1 = \nu_1^{-1}\phi_1,\ \varepsilon_2 = \nu_2^{-1}\phi_2,\ \varepsilon_3 = \nu_3^{-1}\phi_3$, and $\delta = i$. The subgroup $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \delta \rangle \subseteq \mathrm{Aut}(S)$ is isomorphic to the Heisenberg group of order 4^3, defined by generators and relations as

$$H_4 := \langle \varepsilon_1, \varepsilon_2, \delta \mid \varepsilon_1^4 = \varepsilon_2^4 = \delta^4 = 1,\ \delta\varepsilon_1 = \varepsilon_1\delta,\ \varepsilon_2\delta = \delta\varepsilon_2,\ \varepsilon_1\varepsilon_2 = \delta\varepsilon_2\varepsilon_1 \rangle.$$

The algebras we are considering are of the form $S(0, \beta, -\beta)$. We define $c = ib$. The map ϕ_1 still extends to an algebra automorphism but ϕ_2 and ϕ_3 degenerate to *endo*morphisms. In terms of E, F, K, and K', the endomorphisms ϕ_i act as in Table 3.

Although ϕ_2 and ϕ_3 are not isomorphisms, there are associated endofunctors ϕ_2^* and ϕ_3^* of $\mathsf{Gr}(S)$. The application of ϕ_2^* and ϕ_3^* to the point modules $M_{(0,0,1,\pm 1)} \in \mathsf{Gr}(S)$ produces point modules. Indeed,

$$\phi_2(S) = \phi_3(S) = \mathbb{C}[K, K'] \subseteq S,$$

and the two point modules referred to above are cyclic $\mathbb{C}[K, K']$-modules. With this in hand, the next result describes how the ϕ_i act on the four $S(0, \beta, -\beta)$-points obtained by degeneration from $S(\alpha, \beta, \gamma)$. The proof is a direct application of the formulas in Table 2 above.

Proposition 6.1. *The endomorphisms ϕ_i of S move the four special point modules of S as follows.*

(1) ϕ_1^* *interchanges* $M_{(0,0,1,\pm 1)}$ *and interchanges* $M_{(\pm q,1,0,0)}$;

(2) $\phi_2^* M_{(0,0,1,\pm 1)} \cong M_{(\pm q,1,0,0)}$;

(3) $\phi_3^* M_{(0,0,1,\pm 1)} \cong M_{(\mp q,1,0,0)}$.

6E. ***Degenerations of fat point–line incidences.*** In this section we describe resolutions of fat points by line modules by degenerating the analogous statements in [Smith and Staniszkis 1993] for the algebras $S(\alpha, \beta, \gamma)$.

If $\ell \subset \mathbb{P}^3$ is the line passing through $p, p' \in C \cup C'$, we will sometimes denote the line module M_ℓ by $M_{p,p'}$ for clarity. We will also use the following notation: if $p = (\xi_1, \xi_2, \xi_3, 0) \in C$, then $p_\pm = (\xi_1, \xi_2, 0, \pm\xi_3) \in C'$ is the point for which $M_{p,p_\pm}$ surjects onto $F(0, \pm)$. Similarly, in order to keep the notation symmetric, if $p \in C'$ then $p_\pm$ is the point on C for which $M_{p,p_\pm}$ surjects onto $F(0, \pm)$.

Finally, we denote by $\sigma = \sigma_S : (C \cup C')^2 \to (C \cup C')^2$ the diagonal action of $\sigma = \sigma_S$ on $(C \cup C')^2$, and by ψ the automorphism

$$\psi := (\mathrm{id}, \sigma) : (C \cup C')^2 \to (C \cup C')^2.$$

By a slight abuse of notation, we use the same symbols to refer to the induced automorphisms on the variety of lines through pairs of points on $C \cup C'$.

Theorem 6.2. *Let n be a nonnegative integer, $\ell_\pm$ a line through $p, p_\pm \in C \cup C'$, and $\ell_{\pm n}$ the line $\psi^n(\ell_\pm)$. In $\mathsf{QGr}(S)$, there is an exact sequence*

$$0 \to M_{\sigma^{-(n+1)}(\ell_{\pm n})}(-n-1) \to M_{\ell_\pm} \to F(n, \pm) \to 0. \tag{6-10}$$

Proof. We will prove this for ℓ_+. To that end, let ℓ be the line through p and p_+.

The relation $\{(p, p_\pm)\}$ on $C \cup C'$ is the fiber over $(0, b^2, -b^2)$ of a family of relations over the space of parameters (α, β, γ) for the Sklyanin algebras. Specifically, let us write

$$(x_0, x_1, x_2, x_3) \mapsto (-x_0, x_1, x_2, x_3)$$

for the $-$ maps on the elliptic curves $E = E(\alpha, \beta, \gamma)$ and let

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, -x_2, -x_3)$$

be addition by the 2-torsion point $\omega \in E$.

Claim. $\{(p, p_+)\}$ *is the limit of the graphs of the maps $p \mapsto \omega - p$.*

Proof of claim. In terms of the x_i coordinates, the map $p \mapsto \omega - p$ amounts to changing the sign of x_0. On the other hand, the discussion at the beginning of Section 6E shows that in (E, F, K, K')-coordinates the map $p \mapsto p_+$ simply interchanges K and K'. Since $C \cup C'$ is the degeneration of the family (6-2) of elliptic curves, the truth of the claim follows from the coordinate change formulas (6-9). □

The claim implies that the resolutions

$$0 \to M_{\sigma(p),\sigma(\omega-p)}(-1) \to M_{p,\omega-p} \to \bullet \to 0$$

of the point modules associated to $(x_0, x_1, x_2, x_3) = (1, 0, 0, 0)$ (e.g., from [Levasseur and Smith 1993, Theorem 5.7]) degenerate to (6-10) for $n = 0$ in the $+$ case.

Similarly, for larger n we have, in the nondegenerate case, resolutions

$$0 \to M_{\sigma^{-(n+1)}(p),\sigma^{-1}(\omega-p)}(-n-1) \to M_{p,\sigma^n(\omega-p)} \to \bullet \to 0$$

of 1-critical fat points of multiplicity $n+1$ as explained in [Smith and Staniszkis 1993, Proposition 4.4(b)]. These degenerate to a resolution of the form (6-10) of a certain fat S-point module of multiplicity $n+1$ (denoted momentarily by the same symbol $\bullet$):

$$0 \to M_{\sigma^{-(n+1)}(\ell_n)}(-n-1) \to M_{\ell_n} \to \bullet \to 0, \tag{6-11}$$

where ℓ is the line through p and p_+ and $\ell_n = \psi^n(\ell)$; note that $\bullet$ is the same fat point (up to isomorphism in $\mathsf{QGr}(S)$) for all choices of p.

Finally, to argue that $\bullet \cong F(n, +)$ in the present case, simply specialize to the line ℓ for which (6-11) is the homogenized version of the standard BGG resolution (5-2) of the simple $U_q(\mathfrak{sl}_2)$-module $L(n, +)$.

There is a similar argument for $F(n, -)$, or one can use the observation in Lemma 5.5 that $F(n, -) \cong \theta^* F(n, +)$. □

Remark 6.3. Incidentally, one can give a proof of Proposition 4.8 in the same spirit as that of Theorem 6.2 by degenerating the exact sequences

$$0 \to M_{\sigma p,\sigma^{-1}p'}(-1) \to M_{p,p'} \to M_p \to 0$$

from [Levasseur and Smith 1993, Theorem 5.5] for the Sklyanin algebras $S(\alpha, \beta, \gamma)$, where p, p' belong to the elliptic curve component $E = E(\alpha, \beta, \gamma)$ of the point scheme of $S(\alpha, \beta, \gamma)$ and σ is the translation automorphism of E. The result then follows from the observation made above that $E(\alpha, \beta, \gamma)$, together with its translation automorphism, degenerates to $C \cup C'$ equipped with our automorphism (also denoted by σ throughout) when $\alpha \to 0$. ♦

The next result completes the description of the fat point–line incidences.

Proposition 6.4. *For $n \geq 0$ the line modules M_{ℓ_n} from Theorem 6.2 are the only ones having $F(n, \pm)$ as a quotient in* $\mathsf{QGr}(S)$.

Proof. The only central element $\Omega(\lambda)$ annihilating $F(n, \pm)$ is $\Omega(\pm q^n)$. In turn, Proposition 5.3 tells us that the only line modules annihilated by $\Omega(\pm q^n)$ are the lines M_{ℓ_n} in question and the lines $M_{\sigma^{-(n+1)}(\ell_n)}$ appearing as the leftmost terms in (6-10). In conclusion, it suffices to show that there are no surjections

$$M_{\sigma^{-(n+1)(p)},\sigma^{-1}(p_\pm)} \to F(n, \pm) \tag{6-12}$$

in $\mathsf{QGr}(S)$.

Let us specialize to $F(n,+)$, to fix notation. Upon localizing to $S[(KK')^{-1}]_0 \cong U = U_q(\mathfrak{sl}(2))$, (6-12) becomes a surjection

$$\frac{U}{UX+UY} \to L(n,+), \tag{6-13}$$

where $X=\kappa(1-q^{n+2}k^{-1})-s^{-1}q^{-1/2}e$ and $Y=\kappa(k-q^{-(n+2)})+sq^{-1/2}f$ for some $s\in\mathbb{P}^1$. If $s=0$ or ∞ then the left-hand side of (6-13) is the simple Verma module of highest weight $q^{-(n+2)}$ (respectively lowest weight q^{n+2}), thus contradicting the existence of such a surjection. On the other hand, if $s\in\mathbb{C}^\times$, then we obtain surjections (6-13) for *all* $s\in\mathbb{C}^\times$ by applying the $\mathbb{G}_m$-action on U given by

$$k\mapsto k,\quad e\mapsto s^{-1}e,\quad f\mapsto sf\quad \text{for } s\in\mathbb{C}^\times.$$

By continuity in $s\in\mathbb{P}^1$, we then get such surjections for $s=0,\infty$ as well, and the previous argument applies. □

We end with the following remark on certain modules over $U=U_q(\mathfrak{sl}_2)$. In the proof of Proposition 6.4 we showed that the modules (6-13) of the form $U/(UX_\pm+UY_\pm)$ do not surject onto the simple modules $L(n,\pm)$ for

$$\begin{aligned} X_\pm &= \kappa(1\mp q^{n+2}k^{-1})-s^{-1}q^{-1/2}e,\\ Y_\pm &= \kappa(k\mp q^{-(n+2)})+sq^{-1/2}f, \end{aligned} \tag{6-14}$$

where $s\in\mathbb{P}^1$. In fact, we can do somewhat more:

Proposition 6.5. *For $X_\pm$ and $Y_\pm$ as in (6-14) the module $U/(UX_\pm+UY_\pm)$ is simple.*

Proof. As in the proof of Proposition 6.4, we focus on $X=X_+$ and $Y=Y_+$ to fix notation.

Assume otherwise. Then, using the equivalence between the category of modules over $U\cong S[(KK')^{-1}]_0$ and a full subcategory of $\mathsf{QGr}(S)$, this assumption implies that the line module

$$M=M_{\sigma^{-(n+1)(p)},\sigma^{-1}(p_\pm)}$$

from (6-12) has a nonobvious subobject in $\mathsf{QGr}(S)$. The criticality of line modules then implies that such a subobject would be a shifted line module, and hence there would be a surjection from M to a nonzero fat point. Localizing back to U this would give a surjection of $U/(UX+UY)$ onto a nonzero finite-dimensional U-module, which would be a contradiction as in the proof of Proposition 6.4. □

The significance of Proposition 6.5 is that it fits the simple Verma modules of highest and lowest weights $q^{-(n+2)}$ and respectively q^{n+2} into "continuous" $\mathbb{P}^1$-families of simple modules.

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