

Pacific Journal of Mathematics

**A GENERALIZATION OF
“EXISTENCE AND BEHAVIOR OF THE RADIAL LIMITS OF
A BOUNDED CAPILLARY SURFACE AT A CORNER”**

JULIE N. CRENSHAW, ALEXANDRA K. ECHART AND KIRK E. LANCASTER

A GENERALIZATION OF “EXISTENCE AND BEHAVIOR OF THE RADIAL LIMITS OF A BOUNDED CAPILLARY SURFACE AT A CORNER”

JULIE N. CRENSHAW, ALEXANDRA K. ECHART AND KIRK E. LANCASTER

The principal existence theorem (i.e., Theorem 1) of “Existence and behavior of the radial limits of a bounded capillary surface at a corner” (*Pacific J. Math.* 176:1 (1996), 165–194) is extended to the case of a contact angle γ which is not bounded away from 0 and π (and depends on position in a bounded domain $\Omega \in \mathbb{R}^2$ with a convex corner at $\mathcal{O} = (0, 0)$). The lower bound on the size of “side fans” (i.e., Theorem 2 in the above paper) is extended to the case of such contact angles for convex and nonconvex corners.

1. Introduction and theorems

Consider the capillary problem

$$\begin{aligned} (1) \quad & Nf = \kappa f + \lambda \quad \text{in } \Omega, \\ (2) \quad & Tf \cdot \mathbf{v} = \cos \gamma \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a region in \mathbb{R}^2 with a corner at \mathcal{O} , $\mathcal{O} \in \partial\Omega$, $Nf = \nabla \cdot Tf$, $Tf = \nabla f / \sqrt{1 + |\nabla f|^2}$, κ and λ are constants, \mathbf{v} is the exterior unit normal on $\partial\Omega$, and $\gamma = \gamma(s)$ is a function of position on $\partial\Omega$, $0 \leq \gamma(s) \leq \pi$. The surface $z = f(x, y)$ describes the shape of the static liquid–gas interface in a vertical cylindrical tube of cross-section Ω ; see [Finn 1986; Lancaster and Siegel 1996] for background.

We are interested in the behavior of solutions to (1) and (2) in a neighborhood of a corner point of the boundary. We take the corner point to be $\mathcal{O} = (0, 0)$. Let $\Omega^* = \Omega \cap B_{\delta^*}(\mathcal{O})$, where $B_{\delta^*}(\mathcal{O})$ is the ball of radius δ^* about \mathcal{O} . Polar coordinates relative to \mathcal{O} will be denoted by r and θ . We assume that $\partial\Omega$ is piecewise smooth and that $\partial\Omega \cap B_{\delta^*}(\mathcal{O})$ consists of two arcs $\partial^+\Omega^*$ and $\partial^-\Omega^*$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point \mathcal{O} is approached. The points where $\partial B_{\delta^*}(\mathcal{O})$ intersect $\partial\Omega$ are labeled A and B ; also, $\Gamma^* = \partial B_{\delta^*}(\mathcal{O}) \cap \overline{\Omega}$. Set

$$\Omega_\infty = \{(r \cos(\theta), r \sin(\theta)) : r > 0, -\alpha < \theta < \alpha\}.$$

MSC2010: primary 35J93, 58E12; secondary 35J60, 53A10.

Keywords: radial limits, capillary surfaces, corners, discontinuities.

Let $(x^+(s), y^+(s))$ be an arclength parametrization of $\partial^+\Omega^*$ and $(x^-(s), y^-(s))$ be an arclength parametrization of $\partial^-\Omega^*$, each measured from the corner at \mathcal{O} , so that $(x^\pm(0), y^\pm(0)) = (0, 0)$. Let $(x_*^+(s), y_*^+(s))$ be an arclength parametrization of $\partial^+\Omega_\infty = \{(r \cos(\alpha), r \sin(\alpha)) : r \geq 0\}$ and $(x_*^-(s), y_*^-(s))$ be an arclength parametrization of $\partial^-\Omega_\infty = \{(r \cos(-\alpha), r \sin(-\alpha)) : r \geq 0\}$, each measured from the corner at \mathcal{O} . Define

$$\gamma^+(s) = \gamma(x^+(s), y^+(s)) \quad \text{and} \quad \gamma^-(s) = \gamma(x^-(s), y^-(s)).$$

For $0 \leq \alpha \leq \frac{\pi}{2}$, the corner will be said to be *convex* and for $\frac{\pi}{2} < \alpha \leq \pi$, the corner will be said to be *nonconvex*.

In [Lancaster and Siegel 1996], the existence of radial limits of a bounded solution f to (1) that satisfies (2) on the smooth portions of $\partial\Omega$ is proven provided that γ was bounded away from 0 and π , and for a convex corner an additional condition is satisfied coupling γ^+ and γ^- . In this paper, we eliminate the requirement that γ is bounded away from 0 and π ; an additional condition must still be satisfied at a convex corner. The radial limits of f will be denoted by

$$Rf(\theta) = \lim_{r \rightarrow 0^+} f(r \cos \theta, r \sin \theta), \quad -\alpha < \theta < \alpha,$$

and $Rf(\pm\alpha) = \lim_{\partial^\pm\Omega^* \ni \mathbf{x} \rightarrow \mathcal{O}} f(\mathbf{x})$, $\mathbf{x} = (x, y)$, which are the limits of the boundary values of f on the two sides of the corner if these exist.

Theorem 1. *Let f be a bounded solution to (1) satisfying (2) on $\partial^\pm\Omega^* \setminus \{\mathcal{O}\}$, which is discontinuous at \mathcal{O} .*

- (a) *If $\alpha > \frac{\pi}{2}$ then $Rf(\theta)$ exists for all $\theta \in (-\alpha, \alpha)$.*
- (b) *If $\alpha \leq \frac{\pi}{2}$ and there exist constants $\underline{\gamma}^\pm, \bar{\gamma}^\pm, 0 \leq \underline{\gamma}^\pm \leq \bar{\gamma}^\pm \leq \pi$ satisfying*

$$\pi - 2\alpha < \underline{\gamma}^+ + \underline{\gamma}^- \leq \bar{\gamma}^+ + \bar{\gamma}^- < \pi + 2\alpha$$

such that $\underline{\gamma}^\pm \leq \gamma^\pm(s) \leq \bar{\gamma}^\pm$ for all $s \in (0, s_0)$, for some $s_0 > 0$, then $Rf(\theta)$ exists for all $\theta \in (-\alpha, \alpha)$.

Furthermore, in either case, $Rf(\theta)$ is a continuous function on $(-\alpha, \alpha)$ which behaves in one of the following ways:

- (i) *$Rf(\theta)$ is a constant function of θ and f has a nontangential limit at \mathcal{O} .*
- (ii) *There exist α_1 and α_2 so that $-\alpha \leq \alpha_1 < \alpha_2 \leq \alpha$ and Rf is constant on $(-\alpha, \alpha_1]$ and $[\alpha_2, \alpha)$ and strictly increasing or strictly decreasing on $[\alpha_1, \alpha_2] \cap (-\alpha, \alpha)$. Label these case (I) and case (D), respectively.*
- (iii) *There exist $\alpha_1, \alpha_L, \alpha_R, \alpha_2$ so that $-\alpha \leq \alpha_1 < \alpha_L < \alpha_R < \alpha_2 \leq \alpha$, $\alpha_R = \alpha_L + \pi$, and Rf is constant on $(-\alpha, \alpha_1]$, $[\alpha_L, \alpha_R]$, and $[\alpha_2, \alpha)$ and either increasing on $[\alpha_1, \alpha_L] \cap (-\alpha, \alpha)$ and decreasing on $[\alpha_R, \alpha_2] \cap (-\alpha, \alpha)$ or decreasing on*

$[\alpha_1, \alpha_L] \cap (-\alpha, \alpha)$ and increasing on $[\alpha_R, \alpha_2] \cap (-\alpha, \alpha)$. Label these case (ID) and case (DI), respectively.

In Theorem 1 of [Lancaster and Siegel 1996] and Theorem 1 above, the existence of two intervals $(-\alpha, \alpha_1]$ and $[\alpha_2, \alpha)$ on which $Rf(\cdot)$ is constant (i.e., “side fans”) is established but the relationship between the sizes of these side fans and the contact angle is unclear. Theorem 2 of [Lancaster and Siegel 1996] establishes lower bounds on these sizes when the

$$\lim_{\partial^+ \Omega \ni (x, y) \rightarrow \mathcal{O}} \gamma(x, y) = \gamma_0^+ \quad \text{and} \quad \lim_{\partial^- \Omega \ni (x, y) \rightarrow \mathcal{O}} \gamma(x, y) = \gamma_0^-$$

are assumed to exist. (In [Lancaster 2010; 2012], these lower bounds were shown to be the actual sizes of the side fans.) What happens if the limits of γ at \mathcal{O} do not exist? Theorem 2 and Corollary 3 provide lower bounds in this situation.

For $0 < b < 1$, define

$$A_I^\pm(b) = \liminf_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^{b\epsilon} \cos(\gamma^\pm(t)) dt \quad \text{and} \quad A_S^\pm(b) = \limsup_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^{b\epsilon} \cos(\gamma^\pm(t)) dt.$$

Notice that $b \cos(\limsup_{t \downarrow 0} \gamma^\pm(t)) \leq A_I^\pm(b) \leq A_S^\pm(b) \leq b \cos(\liminf_{t \downarrow 0} \gamma^\pm(t))$.

Theorem 2. *Let f be a bounded solution to (1) satisfying (2) on $\partial^\pm \Omega^* \setminus \{\mathcal{O}\}$, which is discontinuous at \mathcal{O} . Assume $Rf(\theta)$ exists for all $\theta \in (-\alpha, \alpha)$. Then:*

- (a) *$Rf(\theta)$ is a continuous function on $(-\alpha, \alpha)$ which behaves as described in (i), (ii) or (iii) of Theorem 1.*
- (b) *There exist fans of constant radial limits adjacent to each tangent direction at \mathcal{O} and lower bounds on the sizes of these side fans exist.*

In terms of the cases labeled in Theorem 1, the sizes of the side fans $\beta^- = \alpha_1 + \alpha$ and $\beta^+ = \alpha - \alpha_2$ satisfy the following conditions:

- (1) $A_I^+ \left(\frac{\sin(\lambda - \beta^+)}{\sin(\lambda)} \right) + \frac{\sin(\beta^+)}{\sin(\lambda)} \geq 1$ for all $\lambda \in (\beta^+, \pi)$ for (I) and (DI).
- (2) $A_I^- \left(\frac{\sin(\lambda - \beta^-)}{\sin(\lambda)} \right) + \frac{\sin(\beta^-)}{\sin(\lambda)} \geq 1$ for all $\lambda \in (\beta^-, \pi)$ for (D) and (DI).
- (3) $1 + A_S^- \left(\frac{\sin(\lambda - \beta^-)}{\sin(\lambda)} \right) \leq \frac{\sin(\beta^-)}{\sin(\lambda)}$ for all $\lambda \in (\beta^-, \pi)$ for (I) and (ID).
- (4) $1 + A_S^+ \left(\frac{\sin(\lambda - \beta^+)}{\sin(\lambda)} \right) \leq \frac{\sin(\beta^+)}{\sin(\lambda)}$ for all $\lambda \in (\beta^+, \pi)$ for (D) and (ID).

2. Proofs of Theorems 1 and 2

The proof of Theorem 1 follows that established in [Lancaster 1985] and [Elcrat and Lancaster 1986] in which (i) the graph of the solution in $\Omega \times \mathbb{R}$ is represented

in isothermal coordinates, (ii) comparison arguments are used to prove that the component functions of the isothermal parametrization of the graph are uniformly continuous and so extend to be continuous on the closure of the parameter domain, (iii) boundary regularity theory (e.g., [Heinz 1970]) is used to prove that radial limits exist for almost every direction, (iv) cusp solutions are excluded (e.g., [Echart and Lancaster 2017]) and (v) the behavior of the radial limit function is determined. The only step which does not follow from previous work is (ii) and so the proof of Theorem 1 comes down to establishing (ii). The proof of Theorem 2 follows from standard “blow up” arguments.

2.1. Proof of Theorem 1. When $\alpha > \frac{\pi}{2}$, Theorem 1 is a consequence of [Entekhabi and Lancaster 2016]. Suppose now that $\alpha \leq \frac{\pi}{2}$. Since f is bounded and the prescribed mean curvature is $H(x, y, z) = \kappa z + \lambda$, there exist $M_1 \in (0, \infty)$ and $M_2 \in [0, \infty)$ such that

$$(3) \quad \sup_{(x,y) \in \Omega} |f(x, y)| \leq M_1 \quad \text{and} \quad \sup_{(x,y) \in \Omega} |H(x, y, f(x, y))| \leq M_2.$$

In §2.1 of [Entekhabi and Lancaster 2017], a specific torus is constructed which depends solely on M_2 and which is used as a comparison surface; one should compare this with, for example, [Lancaster and Siegel 1996], where several types of comparison surfaces are used, or [Entekhabi and Lancaster 2016], where an unduloid is used as a comparison surface. We shall use this torus as our comparison surface here. We will denote by q the modulus of continuity of the function h^- whose graph is the set \mathcal{T} which is the inner half of a torus with axis of symmetry $\{(2, y, 0) : y \in \mathbb{R}^2\}$, major radius $R_0 = 2$, and minor radius r_0 ; here

$$(4) \quad r_0 = \begin{cases} 1 & \text{if } M_2 = 0, \\ \frac{1}{M_2} + 1 - \sqrt{\left(\frac{1}{M_2}\right)^2 + 1} & \text{if } M_2 > 0. \end{cases}$$

Then q is also the modulus of continuity of functions (i.e., h^+ , h_β^- , h_β^+) whose graphs are obtained by rotations and translations in the horizontal plane of \mathcal{T} (see [Entekhabi and Lancaster 2017, p. 59]).

Let $\mathcal{S}_0 = \text{gra}(f) = \{(x, y, f(x, y)) : (x, y) \in \Omega^*\}$ and allow \mathcal{S} to be the closure of \mathcal{S}_0 in \mathbb{R}^3 . As in §2.2 of [Entekhabi and Lancaster 2017], there exists an isothermal parametrization $Y : E \rightarrow \mathbb{R}^3$ given by

$$Y(u, v) = (a(u, v), b(u, v), c(u, v))$$

such that $Y(\bar{E}) = \mathcal{S}$, $Y(E) = \mathcal{S}_0$, and (a_1) – (a_5) of [Entekhabi and Lancaster 2017] hold, where $E = B_1(\mathcal{O}) = \{(u, v) : u^2 + v^2 < 1\}$. By (a_2) of that paper, if we let $G(u, v) = (a(u, v), b(u, v))$ for $(u, v) \in E$, then $G \in C^0(\bar{E})$. From (a_3) of that paper, there exists a connected arc $\sigma \subset \partial E$ that Y maps strictly monotonically onto

$\{(x, y, f(x, y)) : (x, y) \in \partial\Omega^* \setminus \{\mathcal{O}\}\}$. Let the endpoints of σ be denoted \mathbf{o}_1 and \mathbf{o}_2 . There exists points $\mathbf{a}, \mathbf{b} \in \sigma$ such that $G(\mathbf{a}) = A$, $G(\mathbf{b}) = B$, G maps the arc $\mathbf{o}_2\mathbf{a}$ onto $\partial^-\Omega$ and G maps the arc $\mathbf{o}_1\mathbf{b}$ onto $\partial^+\Omega$. We must consider the two cases:

(A) $\mathbf{o}_1 = \mathbf{o}_2$,

(B) $\mathbf{o}_1 \neq \mathbf{o}_2$.

Assume first that (A) holds. Set $\mathbf{o} = \mathbf{o}_1 = \mathbf{o}_2$. We wish to prove that c is uniformly continuous on E and hence c extends to be continuous on \bar{E} . If so, then the existence and behavior of the radial limits of f follows as in [Entekhabi and Lancaster 2017; Lancaster and Siegel 1996]. There are three possible cases:

(i) $\underline{\gamma}^- > 0$ and $\bar{\gamma}^- < \pi$,

(ii) $\underline{\gamma}^+ > 0$ and $\bar{\gamma}^+ < \pi$,

(iii) $(\underline{\gamma}^- = 0 \text{ or } \bar{\gamma}^- = \pi)$ and $(\underline{\gamma}^+ = 0 \text{ or } \bar{\gamma}^+ = \pi)$.

Case (i). Let $\lambda_1 = \underline{\gamma}^+$, $\lambda_2 = \bar{\gamma}^+$, $\gamma_2 = \underline{\gamma}^-$. We observe that $\lambda_2 = \bar{\gamma}^+ < \pi + 2\alpha - \bar{\gamma}^-$, $\lambda_1 = \underline{\gamma}^+ > \pi - 2\alpha - \underline{\gamma}^-$, and so $\lambda_2 - \lambda_1 < 4\alpha$. We wish to use the argument in the proof of Theorem 2 of [Entekhabi and Lancaster 2017]. Since $\pi - 2\alpha - \lambda_1 < \gamma_2 < \pi + 2\alpha - \lambda_2$, we can choose $\tau_1, \tau_2 \in (0, \pi)$ such that $\tau_1 \in (\pi - 2\alpha - \lambda_1, \gamma_2)$ and $\tau_2 \in (\gamma_2, \pi + 2\alpha - \lambda_2)$. Set $\beta_1 = \frac{\pi}{2} - \tau_1$ and $\beta_2 = \tau_2 - \frac{\pi}{2}$. With these choices of β_1 and β_2 , notice that

$$T(h^- \circ T_{\beta_1})(x_1, 0) \cdot (0, -1) = \cos(\tau_1) > \cos(\gamma_2) \quad \text{for } 0 < x_1 < 2 - r_0$$

and

$$T(h^+ \circ T_{\beta_2})(x_1, 0) \cdot (0, -1) = \cos(\tau_2) < \cos(\gamma_2) \quad \text{for } 0 < x_1 < 2 - r_0$$

(see [Entekhabi and Lancaster 2017, p. 59]). This implies that for $\delta_1 = \delta_1(\beta_1, \beta_2) > 0$ small enough and $\mathbf{x} \in \partial^-\Omega$ with $|\mathbf{x}| < \delta_1$, we have

$$(5) \quad T(h_{\beta_1}^-)(\mathbf{x}) \cdot \vec{\nu}(\mathbf{x}) > \cos(\gamma(\mathbf{x})) \quad \text{and} \quad T(h_{\beta_2}^+)(\mathbf{x}) \cdot \vec{\nu}(\mathbf{x}) < \cos(\gamma(\mathbf{x})).$$

Since $\beta_1, \beta_2 \neq \pm\frac{\pi}{2}$, there exists $R = R(\beta_1, \beta_2) > 0$ such that $B_R(\mathcal{O}) \cap \Omega^* \subset \Delta_{\beta_1} \cap \Delta_{\beta_2}$, where Δ_β is as in §2.1 of [Entekhabi and Lancaster 2017]. For each $\delta \in (0, 1)$, allow

$$(6) \quad p(\delta) = \sqrt{\frac{8\pi M_0}{\ln(1/\delta)}}$$

where M_0 is the area of S_0 .

Let $\epsilon > 0$. Choose $\delta > 0$ such that

$$\begin{aligned} \sqrt{\delta} &< \min\{\|\mathbf{o} - \mathbf{a}\|, \|\mathbf{o} - \mathbf{b}\|\}, \\ p(\delta) &< \delta_1(\beta_1, \beta_2), \quad p(\delta) < R(\beta_1, \beta_2), \quad p(\delta) + q(p(\delta)) < \frac{\epsilon}{2}. \end{aligned}$$

Let $\mathbf{w}_0 = (u_0, v_0) \in E$. From the Courant–Lebesgue lemma, there exists a $\rho(\delta) \in (\delta, \sqrt{\delta})$ such that the arclength $l_{\rho(\delta)}$ of $C'_{\rho(\delta)}$ is less than $p(\delta)$, where $C_\delta = \{\mathbf{w} \in E : \|\mathbf{w} - \mathbf{w}_0\| = \delta\}$ and $C'_\delta = Y(C_\delta)$. Set $B_\delta = \{\mathbf{w} \in E : \|\mathbf{w} - \mathbf{w}_0\| < \delta\}$ and $B'_\delta = Y(B_\delta)$. Then, for $\mathbf{w} \in C'_{\rho(\delta)}$, there exist functions

$$(7) \quad b^+(x, y) = f(\mathbf{w}) + p(\delta) + h_{\beta_1}^-(x, y) \quad \text{for } (x, y) \in \Delta_{\beta_1},$$

$$(8) \quad b^-(x, y) = f(\mathbf{w}) - p(\delta) - h_{\beta_2}^+(x, y) \quad \text{for } (x, y) \in \Delta_{\beta_2}$$

where $\beta_1 = \frac{\pi}{2} - \tau_1$ and $\beta_2 = \tau_2 - \frac{\pi}{2}$. From (10) of [Entekhabi and Lancaster 2017], we have that $\operatorname{div}(b^+) \leq -M_2$ in Δ_{β_1} and $\operatorname{div}(b^-) \geq M_2$ in Δ_{β_2} . So in $\Omega \cap \Delta_{\beta_1}$, $\operatorname{div}(Tb^+) \leq \operatorname{div}(Tf)$. On $\partial^-\Omega \cap B_{\delta_1}(\mathcal{O})$, $Tb^+ \cdot \nu \geq Tf \cdot \nu$. As in the proof of Theorem 2 of that paper,

$$(9) \quad f(x, y) < b^+(x, y) \quad \text{for } (x, y) \in \Delta_{\beta_1} \cap B'_{\rho(\delta)},$$

where $B'_{\rho(\delta)} = Y(B_{\rho(\delta)})$. This follows since $Tb^+ \cdot \nu \geq Tf \cdot \nu$ on $\partial^+\Omega \cap B_{\delta_2}(\mathcal{O})$ by (15) of that paper if $\tau_1 + 2\alpha \leq \pi$ and no boundary condition on $\partial^+\Omega$ is required if $\tau_1 + 2\alpha > \pi$.

Repeat the same argument with $\lambda_1 = \underline{\gamma}^+$, $\lambda_2 = \bar{\gamma}^+$ and $\gamma_2 = \bar{\gamma}^-$. In the same way as above, there exist functions

$$(10) \quad b_*^+(x, y) = f(\mathbf{w}) + p(\delta) + h_{\beta_1}^-(x, y) \quad \text{for } (x, y) \in \Delta_{\beta_1},$$

$$(11) \quad b_*^-(x, y) = f(\mathbf{w}) - p(\delta) - h_{\beta_2}^+(x, y) \quad \text{for } (x, y) \in \Delta_{\beta_2}$$

such that

$$(12) \quad b_*^-(x, y) < f(x, y)$$

for $(x, y) \in \Delta_{\beta_2} \cap B'_{\rho(\delta)}$ where $B'_{\rho(\delta)} = Y(B_{\rho(\delta)})$. Then combining (9) and (12) we get

$$(13) \quad b_*^-(x, y) < f(x, y) < b^+(x, y)$$

for $(x, y) \in \Delta_{\beta_1} \cap \Delta_{\beta_2} \cap B'_{\rho(\delta)}$. As in [Entekhabi and Lancaster 2017], it follows that $c(u, v)$ is uniformly continuous on E .

Case (ii). Case (ii) is simply case (i) reflected about the xz -plane and the proof follows as above.

Case (iii). Notice that

$$0 \leq \pi - 2\alpha < \underline{\gamma}^+ + \underline{\gamma}^- \leq \bar{\gamma}^+ + \bar{\gamma}^- < \pi + 2\alpha \leq \pi$$

and so $\underline{\gamma}^- = 0$ and $\underline{\gamma}^+ = 0$ cannot both occur and $\bar{\gamma}^+ = \pi$ and $\bar{\gamma}^- = \pi$ cannot both occur. The result follows from this, using the arguments in cases 1 and 2. In particular, if $\underline{\gamma}^- > 0$, then we obtain a supersolution b^+ as in case (i) (see Figure 1)

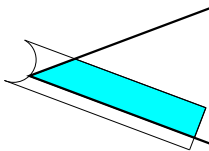


Figure 1. The domain of a supersolution in case (i).

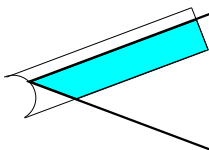


Figure 2. The domain of a supersolution in case (ii).

and if $\underline{\gamma}^- = 0$, we obtain a supersolution b^+ as in case (ii) (see Figure 2); if $\bar{\gamma}^- < \pi$, we obtain a subsolution b_*^- as in case (i) and if $\bar{\gamma}^- = \pi$, we obtain a subsolution as in case (ii).

Now assume (B) holds. Let $B = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1, y \geq 0\}$ and let \bar{B} be the closure of B in \mathbb{R}^2 . Let $g : \bar{B} \rightarrow \bar{E}$ be a conformal or anticonformal map taking $\{(u, 0) : -1 \leq u \leq 1\}$ onto $\partial E \setminus \sigma$ such that the map $X = Y \circ g : B \rightarrow \mathbb{R}^3$ has a downward orientation (i.e., the normal $X_u \times X_v$ to \mathcal{S}_0 gives a downward orientation). Writing $X(u, v) = (x(u, v), y(u, v), z(u, v))$ and $K(u, v) = (x(u, v), y(u, v))$, we have $K \in C^0(\bar{B})$ and $K(u, 0) = (0, 0)$ while $X(u, 0) = (0, 0, z(u, 0))$ for $u \in [-1, 1]$. Then the argument follows from [Lancaster and Siegel 1996] and the previous argument here, as explained in [Entekhabi and Lancaster 2017]. \square

2.2. Proof of Theorem 2. We first note that if $\delta_1, \delta_2 \in (-\alpha, \alpha)$ with $\delta_1 < \delta_2$ and $Rf(\delta_1)$ and $Rf(\delta_2)$ both exist, then it follows from [Elcrat and Lancaster 1986] that $Rf(\theta)$ exists for all $\theta \in [\delta_1, \delta_2]$ and $Rf(\theta)$ is a continuous function of θ on $[\delta_1, \delta_2]$ which behaves as described in (i), (ii) or (iii) of Theorem 1. The first part of Theorem 2 (i.e., (a)) follows from this.

Suppose $\{\epsilon_j\}$ is a decreasing sequence with $\lim_{j \rightarrow \infty} \epsilon_j = 0$. Let $I = (-1, 1)$ and set

$$\gamma_j(s) = \begin{cases} \gamma^+(\epsilon_j s) & \text{if } 0 < s < 1, \\ \gamma^-(-\epsilon_j s) & \text{if } -1 < s < 0 \end{cases}$$

for $j \in \mathbb{N}$; then $\{\cos(\gamma_j) : j \in \mathbb{N}\}$ is a subset of the unit ball in $L^\infty(I) = (L^1(I))^*$. By the Banach–Alaoglu theorem, there exist a subsequence $\{\epsilon_{j_k}\}$ of $\{\epsilon_j\}$ and a function $h = h_{\{\epsilon_{j_k}\}} \in L^\infty(I)$ such that $\cos(\gamma_{j_k})$ converges weak-star to h ; that is, for each $m \in L^1(I)$,

$$\lim_{k \rightarrow \infty} \int_{-1}^1 \cos(\gamma_{j_k}(s)) m(s) ds = \int_{-1}^1 h(s) m(s) ds.$$

Let us define $\gamma^* = \gamma_{\{\epsilon_{j_k}\}}^* = \cos^{-1}(h)$ (almost everywhere on $(-1, 1)$). For any $b \in (0, 1)$, by choosing m to be the characteristic function of the interval $(0, b)$ we see that

$$\int_0^b h(s) ds = \lim_{k \rightarrow \infty} \int_0^b \cos(\gamma_{j_k}(s)) ds = \lim_{k \rightarrow \infty} \frac{1}{\epsilon_{j_k}} \int_0^{b\epsilon_{j_k}} \cos(\gamma^+(t)) dt$$

and, by choosing m to be the characteristic function of the interval $(-b, 0)$,

$$\int_{-b}^0 h(s) ds = \lim_{k \rightarrow \infty} \int_{-b}^0 \cos(\gamma_{j_k}(s)) ds = \lim_{k \rightarrow \infty} \frac{1}{\epsilon_{j_k}} \int_0^{b\epsilon_{j_k}} \cos(\gamma^-(t)) dt;$$

hence

$$\int_0^b \cos(\gamma^*(s)) ds = \lim_{k \rightarrow \infty} \frac{1}{\epsilon_{j_k}} \int_0^{b\epsilon_{j_k}} \cos(\gamma^+(t)) dt \geq \liminf_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^{\epsilon b} \cos(\gamma^+(t)) dt$$

and

$$\int_{-b}^0 \cos(\gamma^*(s)) ds = \lim_{k \rightarrow \infty} \frac{1}{\epsilon_{j_k}} \int_0^{b\epsilon_{j_k}} \cos(\gamma^-(t)) dt \geq \liminf_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^{\epsilon b} \cos(\gamma^-(t)) dt.$$

Thus

$$(14) \quad \liminf_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^{\epsilon b} \cos(\gamma^\pm(t)) dt \leq \liminf_{j \rightarrow \infty} \int_0^b \cos(\gamma^\pm(\epsilon_j s)) ds.$$

Choose a sequence $\{\epsilon_j\}$ with $\lim_{j \rightarrow \infty} \epsilon_j = 0$ such that

$$\lim_{j \rightarrow \infty} \frac{1}{\epsilon_j} \int_0^{b\epsilon_j} \cos(\gamma^+(t)) dt = \liminf_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^{\epsilon b} \cos(\gamma^+(t)) dt;$$

as above, there exist a subsequence $\{\epsilon_{j_k}\}$ of $\{\epsilon_j\}$ and $\gamma_* \in L^\infty(I)$ such that $\cos(\gamma_{j_k})$ converges weak-star to $\cos(\gamma_*)$. Then

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^{\epsilon b} \cos(\gamma^+(t)) dt &= \lim_{k \rightarrow \infty} \frac{1}{\epsilon_{j_k}} \int_0^{b\epsilon_{j_k}} \cos(\gamma^+(t)) dt \\ &= \lim_{k \rightarrow \infty} \int_0^b \cos(\gamma_{j_k}(s)) ds = \int_0^b \cos(\gamma_*(s)) ds. \end{aligned}$$

Case 1. Suppose case (I) or (DI) of Theorem 1 holds and $\alpha_2 = \alpha - \beta^+$. Let us assume there exists $\lambda \in (\beta^+, \pi)$ such that

$$(15) \quad A_I^+ \left(\frac{\sin(\lambda - \beta^+)}{\sin(\lambda)} \right) + \frac{\sin(\beta^+)}{\sin(\lambda)} < 1;$$

we shall show that this leads to a contradiction. Set

$$b = \frac{\sin(\lambda - \beta^+)}{\sin(\lambda)}.$$

Choose a sequence $\{\epsilon_j\}$ with $\lim_{j \rightarrow \infty} \epsilon_j = 0$ such that

$$(16) \quad \lim_{j \rightarrow \infty} \frac{1}{\epsilon_j} \int_0^{b\epsilon_j} \cos(\gamma^+(t)) dt = \liminf_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^{\epsilon b} \cos(\gamma^+(t)) dt;$$

as before, there exist a subsequence $\{\epsilon_{j_k}\}$ of $\{\epsilon_j\}$ and $\gamma_* \in L^\infty(I)$ such that $\cos(\gamma_{j_k})$ converges weak-star to $\cos(\gamma_*)$. Then

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^{\epsilon b} \cos(\gamma^+(t)) dt &= \lim_{k \rightarrow \infty} \frac{1}{\epsilon_{j_k}} \int_0^{b\epsilon_{j_k}} \cos(\gamma^+(t)) dt \\ &= \lim_{k \rightarrow \infty} \int_0^b \cos(\gamma_{j_k}(s)) ds = \int_0^b \cos(\gamma_*(s)) ds. \end{aligned}$$

Let $\theta_0 \in (\sigma, \alpha_2)$, where $\sigma = \alpha_1$ if case (I) holds and $\sigma = \alpha_R$ if case (DI) holds, and $z_0 = Rf(\theta_0)$. Set $\Omega_k = \{(x, y) \in \mathbb{R}^2 : (\epsilon_{j_k}x, \epsilon_{j_k}y) \in \Omega\}$ and define $f_k \in C^\infty(\Omega_k)$ by

$$f_k(x, y) = \frac{1}{\epsilon_{j_k}} (f(\epsilon_{j_k}x, \epsilon_{j_k}y) - z_0)$$

for $(x, y) \in \Omega_k$. Let γ_k be defined on $\partial\Omega_k \setminus \{\mathcal{O}\}$ by

$$\gamma_k(x, y) = \gamma(\epsilon_{j_k}x, \epsilon_{j_k}y)$$

and let $v_k = v_k(x, y)$ denote the outward unit normal to $\partial\Omega_k$. Then f_k satisfies the capillary problem

$$\begin{aligned} Nf_k(x, y) &= \epsilon_{j_k} \kappa f(\epsilon_{j_k}x, \epsilon_{j_k}y) + \epsilon_{j_k} \lambda, & (x, y) \in \Omega_k, \\ Tf_k \cdot v_k &= \cos(\gamma_k) & \text{on } \partial\Omega_k \setminus \{\mathcal{O}\}. \end{aligned}$$

Since $Rf(\theta) < z_0$ if $\sigma < \theta < \theta_0$ and $Rf(\theta) > z_0$ if $\theta_0 < \theta < \alpha$, we see (e.g., [Lancaster 2010; 2012; Simon 1980]; also see [Tam 1984; 1986]) that $\{f_k\}$ converges locally to the generalized solution f_∞ (in the sense of Miranda [1977] and Giusti [1980; 1984]) of the functional

$$\mathcal{F}_\infty(g) = \iint_{\Omega_\infty} \sqrt{1 + |Dg|^2} dx - \int_{\partial\Omega_\infty} \cos(\gamma_*(s)) g ds,$$

where

$$f_\infty(r \cos(\theta), r \sin(\theta)) = \begin{cases} -\infty & \text{if } -\alpha < \theta < \theta_0, \\ \infty & \text{if } \theta_0 < \theta < \alpha \end{cases}$$

if case (I) holds or case (DI) holds and $z_0 > Rf(\theta)$ for all $\theta \in (-\alpha, \alpha_L)$ and

$$f_\infty(r \cos(\theta), r \sin(\theta)) = \begin{cases} \infty & \text{if } -\alpha < \theta < \theta_h, \\ -\infty & \text{if } \theta_h < \theta < \theta_0, \\ \infty & \text{if } \theta_0 < \theta < \alpha \end{cases}$$

with $Rf(\theta_h) = z_0$ and $\theta_h < \alpha_L$ otherwise.

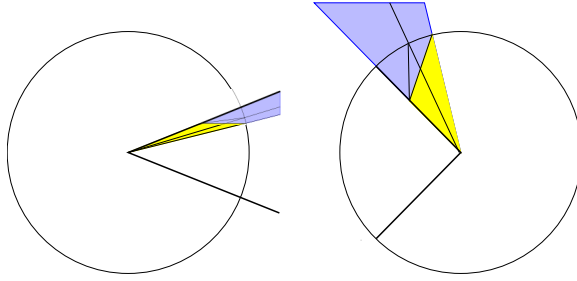


Figure 3. The yellow region represents Σ_{θ_0} .

Let us now define the sets

$$\mathcal{P} = \{(x, y) \in \Omega_\infty : f_\infty(x, y) = \infty\} \quad \text{and} \quad \mathcal{N} = \{(x, y) \in \Omega_\infty : f_\infty(x, y) = -\infty\}.$$

These sets have a special structure which follows from the fact that \mathcal{P} minimizes the functional

$$\Phi(A) = \iint_{\Omega_\infty} |D\chi_A| - \int_{\partial\Omega_\infty} \cos(\gamma_*) \chi_A dH^1$$

and \mathcal{N} minimizes the functional

$$\Psi(A) = \iint_{\Omega_\infty} |D\chi_A| + \int_{\partial\Omega_\infty} \cos(\gamma_*) \chi_A dH^1$$

in the appropriate sense (e.g., [Giusti 1980; Lancaster and Siegel 1996; Miranda 1977]). Let Σ_{θ_0} denote the (open) triangular region whose boundary is the triangle with vertices $(0, 0)$, $B = (b \cos(\alpha), b \sin(\alpha))$ and $C = (\cos(\theta_0), \sin(\theta_0))$ and set $A = \mathcal{P} \setminus \Sigma_{\theta_0}$ (see Figure 3). Simple trigonometric computations with $R > 2$ show that

$$(17) \quad \Phi(B(\mathcal{O}, R) \cap \mathcal{P}) - \Phi(B(\mathcal{O}, R) \cap \mathcal{P} \setminus \Sigma_{\theta_0}) = (1 - A_I^+(b)) - \left(\frac{\sin(\alpha - \theta_0)}{\sin(\omega)} \right),$$

where $\pi - \omega$ is the angle $\angle OBC$. This holds for all $\theta_0 < \alpha_2 = \alpha - \beta^+$; taking the limit as $\theta_0 \uparrow \alpha - \beta^+$ and noticing that $\omega \rightarrow \lambda$ as $\theta_0 \uparrow \alpha - \beta^+$, we see that

$$\Phi(B(\mathcal{O}, R) \cap \mathcal{P}) - \Phi(B(\mathcal{O}, R) \cap \mathcal{P} \setminus \Sigma_{\alpha_2}) = (1 - A_I^+(b)) - \left(\frac{\sin(\beta^+)}{\sin(\lambda)} \right) > 0$$

or

$$\Phi(B(\mathcal{O}, R) \cap \mathcal{P}) > \Phi(B(\mathcal{O}, R) \cap \mathcal{P} \setminus \Sigma_{\alpha_2});$$

this contradicts the fact that \mathcal{P} (locally) minimizes Φ . Therefore (15) is false. This completes case 1.

Case 2. Suppose case (D) or (DI) of Theorem 1 holds and $\alpha_1 = -\alpha + \beta^-$. Let us assume there exists $\lambda \in (\beta^-, \pi)$ such that

$$(18) \quad A_I^- \left(\frac{\sin(\lambda - \beta^-)}{\sin(\lambda)} \right) + \frac{\sin(\beta^-)}{\sin(\lambda)} < 1.$$

Using an argument similar to that in case 1, we reach a contradiction.

Case 3. Suppose case (I) or (ID) of Theorem 1 holds and $\alpha_1 = -\alpha + \beta^-$. Let us assume there exists $\lambda \in (\beta^-, \pi)$ such that

$$(19) \quad 1 + A_S^- \left(\frac{\sin(\lambda - \beta^-)}{\sin(\lambda)} \right) > \frac{\sin(\beta^-)}{\sin(\lambda)}.$$

Set

$$b = \frac{\sin(\lambda - \beta^-)}{\sin(\lambda)}.$$

Arguing as in case 1, we see that the set $\mathcal{N} = \{(x, y) \in \Omega_\infty : f_\infty(x, y) = -\infty\}$ minimizes the functional

$$\Psi(A) = \iint_{\Omega_\infty} |D\chi_A| + \int_{\partial\Omega_\infty} \cos(\gamma_*) \chi_A dH^1$$

in the appropriate sense (e.g., [Giusti 1980; Lancaster and Siegel 1996; Miranda 1977]). Let Σ_{θ_0} denote the (open) triangular region whose boundary is the triangle with vertices $(0, 0)$, $B = (b \cos(-\alpha), b \sin(-\alpha))$ and $C = (\cos(\theta_0), \sin(\theta_0))$ and set $A = \mathcal{N} \setminus \Sigma_{\theta_0}$ (see Figure 3). Simple trigonometric computations with $R > 2$ show that

$$\Psi(B(\mathcal{O}, R) \cap \mathcal{N}) - \Psi(B(\mathcal{O}, R) \cap \mathcal{N} \setminus \Sigma_{\theta_0}) = (1 + A_S^-(b)) - \left(\frac{\sin(\alpha + \theta_0)}{\sin(\omega)} \right),$$

where ω is the angle $\angle OBC$. This holds for all $\theta_0 > \alpha_1 = -\alpha + \beta^-$; taking the limit as $\theta_0 \downarrow -\alpha + \beta^-$ and noticing that $\omega \rightarrow \lambda$ as $\theta_0 \downarrow -\alpha + \beta^-$, we see that

$$\Psi(B(\mathcal{O}, R) \cap \mathcal{N}) - \Psi(B(\mathcal{O}, R) \cap \mathcal{N} \setminus \Sigma_{\alpha_1}) = (1 + A_S^-(b)) - \left(\frac{\sin(\beta^-)}{\sin(\lambda)} \right) > 0$$

or

$$\Psi(B(\mathcal{O}, R) \cap \mathcal{N}) > \Psi(B(\mathcal{O}, R) \cap \mathcal{N} \setminus \Sigma_{\alpha_1});$$

this contradicts the fact that \mathcal{N} (locally) minimizes Ψ . Therefore (19) is false. This completes case 3.

Case 4. Suppose case (D) or (ID) of Theorem 1 holds and $\alpha_2 = \alpha - \beta^+$. Let us assume there exists $\lambda \in (\beta^+, \pi)$ such that

$$(20) \quad 1 + A_S^+ \left(\frac{\sin(\lambda - \beta^+)}{\sin(\lambda)} \right) > \frac{\sin(\beta^+)}{\sin(\lambda)}.$$

Using an argument similar to that in case 3, we reach a contradiction. The proof of Theorem 2 is then complete. \square

3. Corollaries and examples

Corollary 3. Suppose $m \in [-1, 1]$; set $\sigma = \cos^{-1}(m) \in [0, \pi]$.

- (a) If $A_I^+(b) \leq mb$ and case (I) or (DI) holds, then $\beta^+ \geq \sigma$.
- (b) If $A_I^-(b) \leq mb$ and case (D) or (DI) holds, then $\beta^- \geq \sigma$.
- (c) If $A_S^-(b) \geq mb$ and case (I) or (ID) holds, then $\beta^- \geq \pi - \sigma$.
- (d) If $A_S^+(b) \geq mb$ and case (D) or (ID) holds, then $\beta^+ \geq \pi - \sigma$.

Proof. (a) Suppose case (I) or (DI) of Theorem 1 holds, $\sigma \in [0, \pi]$, $\cos(\sigma) = m$, and $\beta^+ < \sigma$. By Theorem 2(a), we know that

$$\sin(\sigma) \left(\frac{\sin(\lambda - \beta^+)}{\sin(\lambda)} \right) + \frac{\sin(\beta^+)}{\sin(\lambda)} \geq A_I^+ \left(\frac{\sin(\lambda - \beta^+)}{\sin(\lambda)} \right) + \frac{\sin(\beta^+)}{\sin(\lambda)} \geq 1$$

or

$$\frac{\cos(\sigma) \sin(\lambda - \beta^+) + \sin(\beta^+)}{\sin(\lambda)} \geq 1$$

for all $\lambda \in (\beta^+, \pi)$. Since $\sigma > \beta^+$, we may set $\lambda = \sigma$ and obtain

$$\cos(\sigma - \beta^+) = \frac{\cos(\sigma) \sin(\sigma - \beta^+) + \sin(\beta^+)}{\sin(\sigma)} \geq 1,$$

which is a contradiction since $\sigma - \beta^+ \neq 0$. Thus $\beta^+ \geq \sigma$.

(b) This is essentially the same as (a).

(c) Suppose case (I) or (ID) of Theorem 1 holds, $\sigma \in [0, \pi]$, $\cos(\sigma) = m$, and $\beta^- < \pi - \sigma$. By Theorem 2(c), we know that

$$1 + \sin(\sigma) \left(\frac{\sin(\lambda - \beta^-)}{\sin(\lambda)} \right) \leq 1 + A_S^- \left(\frac{\sin(\lambda - \beta^-)}{\sin(\lambda)} \right) \leq \frac{\sin(\beta^-)}{\sin(\lambda)}$$

or

$$\frac{\sin(\lambda) + \cos(\sigma) \sin(\lambda - \beta^-) - \sin(\beta^-)}{\sin(\lambda)} \leq 0$$

for all $\lambda \in (\beta^-, \pi)$. Since $\beta^- < \pi - \sigma$, we may set $\lambda = \pi - \sigma$ and obtain

$$1 + \cos(\sigma + \beta^-) = \frac{\sin(\sigma) + \cos(\sigma) \sin(\sigma + \beta^-) - \sin(\beta^-)}{\sin(\sigma)} \leq 0,$$

which is a contradiction since $\sigma + \beta^- < \pi$. Thus $\beta^- \geq \pi - \sigma$.

(d) This is essentially the same as (c). \square

Example 4. Let $\alpha \in (0, \pi]$ and $\gamma_1^\pm, \gamma_2^\pm \in [0, \pi]$ with $\gamma_1^+ \leq \gamma_2^+$ and $\gamma_1^- \leq \gamma_2^-$. Set

$$\Omega = \{(r \cos(\theta), r \sin(\theta)) : 0 < r < 1, -\alpha < \theta < \alpha\}.$$

For each $n \in \mathbb{N}$, let $A_n = (2^{-n^2}, 2^{-n(n-1)}]$ and $B_n = (2^{-n(n+1)}, 2^{-n^2}]$. Define

$$\gamma(s) = \sum_{n=1}^{\infty} (\gamma_1^+ I_{A_n}(s) + \gamma_2^+ I_{B_n}(s) + \gamma_1^- I_{A_n}(-s) + \gamma_2^- I_{B_n}(-s)),$$

so that γ is defined on $\partial\Omega \cap B(\mathcal{O}, 1)$ by

$$\gamma(r \cos(\theta), r \sin(\theta)) = \begin{cases} \gamma_1^+ & \text{if } \theta = \alpha, 2^{-n^2} < r \leq 2^{-n(n-1)} \text{ for some } n \in \mathbb{N}, \\ \gamma_2^+ & \text{if } \theta = \alpha, 2^{-n(n+1)} < r \leq 2^{-n^2} \text{ for some } n \in \mathbb{N}, \\ \gamma_1^- & \text{if } \theta = -\alpha, 2^{-n^2} < r \leq 2^{-n(n-1)} \text{ for some } n \in \mathbb{N}, \\ \gamma_2^- & \text{if } \theta = -\alpha, 2^{-n(n+1)} < r \leq 2^{-n^2} \text{ for some } n \in \mathbb{N}. \end{cases}$$

Set

$$c_j = \begin{cases} 2^{-\frac{j}{2}(\frac{j}{2}+1)} & \text{if } j \text{ is even,} \\ 2^{-\left(\frac{j+1}{2}\right)^2} & \text{if } j \text{ is odd.} \end{cases}$$

Let $b \in (0, 1)$ be fixed for now. Set $\epsilon_j = c_{2j}/b$ ($j \in \mathbb{N}$); notice that $c_{2j+1}/c_{2j} = 2^{-(j+1)}$. Then

$$\begin{aligned} b \cos(\gamma_1^\pm) &\geq A_S^\pm(b) \\ &\geq \lim_{j \rightarrow \infty} \frac{1}{\epsilon_j} \int_0^{\epsilon_j b} \cos(\gamma^\pm(t)) dt \\ &= \lim_{j \rightarrow \infty} b \int_0^1 \cos(\gamma_j^\pm(sb)) ds \\ &= \lim_{j \rightarrow \infty} b \int_0^1 \cos(\gamma^\pm(c_{2j}s)) ds \\ &= \lim_{j \rightarrow \infty} b \left(\int_{\frac{c_{2j+1}}{c_{2j}}}^1 \cos(\gamma^\pm(c_{2j}s)) ds + \int_0^{\frac{c_{2j+1}}{c_{2j}}} \cos(\gamma^\pm(c_{2j}s)) ds \right) \\ &= \lim_{j \rightarrow \infty} b \left(\cos(\gamma_1^\pm)(1 - 2^{-(j+1)}) + \int_0^{2^{-(j+1)}} \cos(\gamma^\pm(c_{2j}s)) ds \right) \\ &= b \cos(\gamma_1^\pm). \end{aligned}$$

Using a similar argument for $A_I^\pm(b)$ with $\epsilon_j = c_{2j+1}/b$, $j \in \mathbb{N}$, we see that

$$(21) \quad A_I^\pm(b) = b \cos(\gamma_2^\pm) \quad \text{and} \quad A_S^\pm(b) = b \cos(\gamma_1^\pm).$$

Example 5. Let $\alpha \in (0, \pi]$ and $\gamma_1^\pm, \gamma_2^\pm \in [0, \pi]$ with $\gamma_1^+ \leq \gamma_2^+$ and $\gamma_1^- \leq \gamma_2^-$. Set

$$\Omega = \{(r \cos(\theta), r \sin(\theta)) : 0 < r < 1, -\alpha < \theta < \alpha\}.$$

For each $n \in \mathbb{N}$, let $A_n = \left(\frac{2}{4^n}, \frac{4}{4^n}\right)$, $B_n = \left(\frac{1}{4^n}, \frac{2}{4^n}\right)$, and $C_n = \left\{\frac{4}{4^n}\right\}$. Define

$$\gamma(s) = \sum_{n=1}^{\infty} (\gamma_1^+ I_{A_n}(s) + \gamma_2^+ I_{B_n}(s) + \pi I_{C_n}(s) + \gamma_1^- I_{A_n}(-s) + \gamma_2^- I_{B_n}(-s) + \pi I_{C_n}(-s)),$$

so that γ is defined on $\partial\Omega \cap B(\mathcal{O}, 1)$ by

$$\gamma(r \cos(\theta), r \sin(\theta)) = \begin{cases} \gamma_1^+ & \text{if } \theta = \alpha, 2/4^n < r < 4/4^n \text{ for some } n \in \mathbb{N}, \\ \gamma_2^+ & \text{if } \theta = \alpha, 1/4^n < r < 2/4^n \text{ for some } n \in \mathbb{N}, \\ \pi & \text{if } \theta = \alpha, r = 4/4^n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } \theta = \alpha, r = 2/4^n \text{ for some } n \in \mathbb{N}, \\ \gamma_1^- & \text{if } \theta = -\alpha, 2/4^n < r < 4/4^n \text{ for some } n \in \mathbb{N}, \\ \gamma_2^- & \text{if } \theta = -\alpha, 1/4^n < r < 2/4^n \text{ for some } n \in \mathbb{N}, \\ \pi & \text{if } \theta = -\alpha, r = 4/4^n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } \theta = -\alpha, r = 2/4^n \text{ for some } n \in \mathbb{N}. \end{cases}$$

Then

$$\begin{aligned} \liminf_{r \rightarrow 0} \gamma(r \cos(\pm\alpha), r \sin(\pm\alpha)) &= 0, \\ \limsup_{r \rightarrow 0} \gamma(r \cos(\pm\alpha), r \sin(\pm\alpha)) &= \pi, \\ \text{ess } \liminf_{r \rightarrow 0} \gamma(r \cos(\pm\alpha), r \sin(\pm\alpha)) &= \gamma_1^\pm, \\ \text{ess } \limsup_{r \rightarrow 0} \gamma(r \cos(\pm\alpha), r \sin(\pm\alpha)) &= \gamma_2^\pm. \end{aligned}$$

Thus $A_I^\pm(b) \geq b \cos(\gamma_2^\pm)$ and $A_S^\pm(b) \leq b \cos(\gamma_1^\pm)$.

Let $b \in (0, 1)$ be fixed for now. If we set $\epsilon_j = 1/(b4^j)$ ($j \in \mathbb{N}$), then $\gamma_j(s) = \gamma(s/b)$ and so

$$A_S^+(b) \geq \lim_{j \rightarrow \infty} \int_0^b \cos(\gamma_j(s)) ds = b \int_0^1 \cos(\gamma(s)) ds = b\left(\frac{2}{3} \cos(\gamma_1^+) + \frac{1}{3} \cos(\gamma_2^+)\right),$$

and, if we set $\epsilon_j = 2/(b4^j)$ ($j \in \mathbb{N}$), then $\gamma_j(s) = \gamma(s/(2b))$ and so

$$A_I^+(b) \leq \lim_{j \rightarrow \infty} \int_0^b \cos(\gamma_j(s)) ds = 2b \int_0^{\frac{1}{2}} \cos(\gamma(s)) ds = b\left(\frac{1}{3} \cos(\gamma_1^+) + \frac{2}{3} \cos(\gamma_2^+)\right);$$

similar estimates hold on $\partial^-\Omega$. Now suppose (η_j) is any decreasing sequence in $(0, 1)$ converging to zero. For each $j \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $\frac{1}{4} \leq 4^{k-1} \eta_j b < 1$ and, since γ is piecewise constant, a direct calculation shows

that $\int_0^b \cos(\gamma^\pm(\eta_j s)) ds = b \int_0^1 \cos(\gamma^\pm(\eta_j b s)) ds$ equals

$$\begin{cases} b \left(\frac{1}{6 \cdot 4^{k-1} \eta_j b} \cos \gamma_1^\pm + \left(1 - \frac{1}{6 \cdot 4^{k-1} \eta_j b} \right) \cos \gamma_2^\pm \right) & \text{if } \frac{1}{4} \leq 4^{k-1} \eta_j b < \frac{1}{2}, \\ b \left(\left(1 - \frac{1}{3 \cdot 4^{k-1} \eta_j b} \right) \cos \gamma_1^\pm + \frac{1}{3 \cdot 4^{k-1} \eta_j b} \cos \gamma_2^\pm \right) & \text{if } \frac{1}{2} \leq 4^{k-1} \eta_j b < 1. \end{cases}$$

The minimum occurs when $4^{k-1} \eta_j b = \frac{1}{2}$ and the minimum of $\int_0^b \cos(\gamma^\pm(\eta_j s)) ds$ is $b(\frac{1}{3} \cos(\gamma_1^\pm) + \frac{2}{3} \cos(\gamma_2^\pm))$. The maximum occurs when $4^{k-1} \eta_j b = \frac{1}{4}$ and the maximum of $\int_0^b \cos(\gamma^\pm(\eta_j s)) ds$ is $b(\frac{2}{3} \cos(\gamma_1^\pm) + \frac{1}{3} \cos(\gamma_2^\pm))$. Thus

$$(22) \quad A_I^\pm(b) = b(\frac{1}{3} \cos(\gamma_1^\pm) + \frac{2}{3} \cos(\gamma_2^\pm))$$

and

$$(23) \quad A_S^\pm(b) = b(\frac{2}{3} \cos(\gamma_1^\pm) + \frac{1}{3} \cos(\gamma_2^\pm)).$$

In these examples, we have the same essential limits inferior and superior at \mathcal{O} and yet A_I^\pm and A_S^\pm behave differently. In Example 4, we have the “extreme values” (21); the “effective” contact angles in (a) and (b) of Corollary 3 are γ_2^\pm and in (c) and (d) of Corollary 3 are γ_1^\pm . On the other hand, in Example 5, we have the “intermediate values” (22) and (23). For Example 5, the “effective” contact angles in (a) and (b) of Corollary 3 are σ_2^\pm and in (c) and (d) of Corollary 3 are σ_1^\pm , where $\sigma_1^\pm, \sigma_2^\pm \in [0, \pi]$ satisfy

$$\cos \sigma_1^\pm = \frac{2}{3} \cos \gamma_1^\pm + \frac{1}{3} \cos \gamma_2^\pm \quad \text{and} \quad \cos \sigma_2^\pm = \frac{1}{3} \cos \gamma_1^\pm + \frac{2}{3} \cos \gamma_2^\pm.$$

If f is a bounded solution of (1) satisfying (2) on $\partial^\pm \Omega^* \setminus \{\mathcal{O}\}$ which is discontinuous at \mathcal{O} and $Rf(\theta)$ exists for all $\theta \in (-\alpha, \alpha)$, then bounds on the sizes β^+ and β^- of side fans can be computed using Corollary 3; the lower bounds on the sizes of these side fans differ between these two examples.

4. Comments and extensions

The last section of [Lancaster and Siegel 1996] dealt with extensions of (1) to equations of prescribed mean curvature. Consider the prescribed mean curvature contact angle problem

$$(24) \quad Nf = 2H(\cdot, f) \quad \text{in } \Omega,$$

$$(25) \quad Tf \cdot \nu = \cos \gamma \quad \text{a.e. on } \partial\Omega.$$

Suppose $f \in C^2(\Omega)$ satisfies (24) and (25) and also suppose the following conditions hold:

- (i) $\sup_{x \in \Omega} |f(x)| < \infty$ and $\sup_{x \in \Omega} |H(x, f(x))| < \infty$.
- (ii) $H(x, y, t)$ is weakly increasing in t for each $(x, y) \in \Omega$.

Using [Echart and Lancaster 2017], we see that Theorems 1 and 2 continue to hold for solutions f as above; the argument is the same as that in [Lancaster and Siegel 1996].

One might ask if the case considered in Theorem 2 is of “physical interest.” Is it possible for the contact angle to fail to have a limit at the corner \mathcal{O} ? In a sense this is a silly question since, at a small enough scale, the macroscopic description of a capillary surface becomes meaningless. On the other hand, one sometimes uses devices (e.g., homogenization) to obtain useful macroscopic information from knowledge of “small scale” properties. An experiment which might be of some interest would be to form a vertical wedge consisting of two planes of glass which have been coated in increasing narrow vertical strips with a nonwetting substance (e.g., paraffin) as the edge at which the two planes meet is approached; this would approximate the situation considered in Theorem 2 and one wonders if there is a “effective” contact angle at the corner which is larger than that for glass and smaller than that for paraffin.

References

- [Echart and Lancaster 2017] A. Echart and K. Lancaster, “On cusp solutions to a prescribed mean curvature equation”, *Pacific J. Math.* **288**:1 (2017), 47–54. MR Zbl
- [Elcrat and Lancaster 1986] A. R. Elcrat and K. E. Lancaster, “Boundary behavior of a nonparametric surface of prescribed mean curvature near a reentrant corner”, *Trans. Amer. Math. Soc.* **297**:2 (1986), 645–650. MR Zbl
- [Entekhabi and Lancaster 2016] M. Entekhabi and K. Lancaster, “Radial limits of bounded nonparametric prescribed mean curvature surfaces”, *Pacific J. Math.* **283**:2 (2016), 341–351. MR Zbl
- [Entekhabi and Lancaster 2017] M. Entekhabi and K. Lancaster, “Radial limits of capillary surfaces at corners”, *Pacific J. Math.* **288**:1 (2017), 55–67. MR Zbl
- [Finn 1986] R. Finn, *Equilibrium capillary surfaces*, Grundlehren Math. Wissenschaften **284**, Springer, New York, 1986. MR Zbl
- [Giusti 1980] E. Giusti, “Generalized solutions for the mean curvature equation”, *Pacific J. Math.* **88**:2 (1980), 297–321. MR Zbl
- [Giusti 1984] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics **80**, Birkhäuser, Boston, 1984. MR Zbl
- [Heinz 1970] E. Heinz, “Über das Randverhalten quasilinearer elliptischer Systeme mit isothermen Parametern”, *Math. Z.* **113** (1970), 99–105. MR Zbl
- [Lancaster 1985] K. E. Lancaster, “Boundary behavior of a nonparametric minimal surface in \mathbb{R}^3 at a nonconvex point”, *Analysis* **5**:1-2 (1985), 61–69. MR Zbl
- [Lancaster 2010] K. E. Lancaster, “A proof of the Concus–Finn conjecture”, *Pacific J. Math.* **247**:1 (2010), 75–108. MR Zbl
- [Lancaster 2012] K. E. Lancaster, “Remarks on the behavior of nonparametric capillary surfaces at corners”, *Pacific J. Math.* **258**:2 (2012), 369–392. MR Zbl

- [Lancaster and Siegel 1996] K. E. Lancaster and D. Siegel, “Existence and behavior of the radial limits of a bounded capillary surface at a corner”, *Pacific J. Math.* **176**:1 (1996), 165–194. Correction in **179**:2 (1997), 397–402. MR Zbl
- [Miranda 1977] M. Miranda, “Superficie minime illimitate”, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **4**:2 (1977), 313–322. MR Zbl
- [Simon 1980] L. Simon, “Regularity of capillary surfaces over domains with corners”, *Pacific J. Math.* **88**:2 (1980), 363–377. MR Zbl
- [Tam 1984] L.-F. Tam, *The behavior of capillary surfaces as gravity tends to zero*, Ph.D. thesis, Stanford University, 1984. Zbl
- [Tam 1986] L.-F. Tam, “Regularity of capillary surfaces over domains with corners: borderline case”, *Pacific J. Math.* **124**:2 (1986), 469–482. MR Zbl

Received January 17, 2017. Revised June 19, 2017.

JULIE N. CRENSHAW
 DEPARTMENT OF MATHEMATICS, STATISTICS AND PHYSICS
 WICHITA STATE UNIVERSITY
 WICHITA, KS
 UNITED STATES
 crenshaw@math.wichita.edu

ALEXANDRA K. ECHART
 DEPARTMENT OF MATHEMATICS, STATISTICS AND PHYSICS
 WICHITA STATE UNIVERSITY
 WICHITA, KS
 UNITED STATES
 echart@math.wichita.edu

KIRK E. LANCASTER
 WICHITA, KANSAS
 WICHITA, KS
 UNITED STATES
 redwoodsrunner@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.


See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2018 is US \$475/year for the electronic version, and \$640/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2018 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 292 No. 2 February 2018

Locally helical surfaces have bounded twisting	257
DAVID BACHMAN, RYAN DERBY-TALBOT and ERIC SEDGWICK	
Superconvergence to freely infinitely divisible distributions	273
HARI BERCOVICI, JIUN-CHAU WANG and PING ZHONG	
Norm constants in cases of the Caffarelli–Kohn–Nirenberg inequality	293
AKSHAY L. CHANILLO, SAGUN CHANILLO and ALI MAALAOUI	
Noncommutative geometry of homogenized quantum $\mathfrak{sl}(2, \mathbb{C})$	305
ALEX CHIRVASITU, S. PAUL SMITH and LIANG ZE WONG	
A generalization of “Existence and behavior of the radial limits of a bounded capillary surface at a corner”	355
JULIE N. CRENSHAW, ALEXANDRA K. ECHART and KIRK E. LANCASTER	
Norms in central simple algebras	373
DANIEL GOLDSTEIN and MURRAY SCHACHER	
Global existence and blowup of smooth solutions of 3-D potential equations with time-dependent damping	389
FEI HOU, INGO WITT and HUICHENG YIN	
Formal confluence of quantum differential operators	427
BERNARD LE STUM and ADOLFO QUIRÓS	
Rigidity of Hawking mass for surfaces in three manifolds	479
JIACHENG SUN	
Addendum to “A strong multiplicity one theorem for SL_2 ”	505
QING ZHANG	



0030-8730(201802)292:2;1-X