Pacific Journal of Mathematics

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Volume 292 No. 2 February 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 292, No. 2, 2018
dx.doi.org/10.2140/pjm.2018.292.355

A GENERALIZATION OF "EXISTENCE AND BEHAVIOR OF THE RADIAL LIMITS OF A BOUNDED CAPILLARY SURFACE AT A CORNER"

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The principal existence theorem (i.e., Theorem 1) of "Existence and behavior of the radial limits of a bounded capillary surface at a corner" (*Pacific J. Math.* 176:1 (1996), 165–194) is extended to the case of a contact angle γ which is not bounded away from 0 and π (and depends on position in a bounded domain $\Omega \in \mathbb{R}^2$ with a convex corner at $\mathcal{O} = (0, 0)$). The lower bound on the size of "side fans" (i.e., Theorem 2 in the above paper) is extended to the case of such contact angles for convex and nonconvex corners.

1. Introduction and theorems

Consider the capillary problem

$$Nf = \kappa f + \lambda \quad \text{in } \Omega, \tag{1}$$

$$Tf \cdot \boldsymbol{\nu} = \cos\gamma \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a region in $\mathbb{R}^2$ with a corner at $\mathcal{O}$, $\mathcal{O} \in \partial\Omega$, $Nf = \nabla \cdot Tf$, $Tf = \nabla f/\sqrt{1+|\nabla f|^2}$, κ and λ are constants, $\boldsymbol{\nu}$ is the exterior unit normal on $\partial\Omega$, and $\gamma = \gamma(s)$ is a function of position on $\partial\Omega$, $0 \le \gamma(s) \le \pi$. The surface $z = f(x, y)$ describes the shape of the static liquid–gas interface in a vertical cylindrical tube of cross-section Ω; see [Finn 1986; Lancaster and Siegel 1996] for background.

We are interested in the behavior of solutions to (1) and (2) in a neighborhood of a corner point of the boundary. We take the corner point to be $\mathcal{O} = (0, 0)$. Let $\Omega^* = \Omega \cap B_{\delta^*}(\mathcal{O})$, where $B_{\delta^*}(\mathcal{O})$ is the ball of radius δ^* about $\mathcal{O}$. Polar coordinates relative to $\mathcal{O}$ will be denoted by r and θ. We assume that $\partial\Omega$ is piecewise smooth and that $\partial\Omega \cap B_{\delta^*}(\mathcal{O})$ consists of two arcs $\partial^+\Omega^*$ and $\partial^-\Omega^*$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point $\mathcal{O}$ is approached. The points where $\partial B_{\delta^*}(\mathcal{O})$ intersect $\partial\Omega$ are labeled A and B; also, $\Gamma^* = \partial B_{\delta^*}(\mathcal{O}) \cap \overline{\Omega}$. Set

$$\Omega_\infty = \{(r\cos(\theta), r\sin(\theta)) : r > 0, -\alpha < \theta < \alpha\}.$$

MSC2010: primary 35J93, 58E12; secondary 35J60, 53A10.
Keywords: radial limits, capillary surfaces, corners, discontinuities.

Let $(x^+(s), y^+(s))$ be an arclength parametrization of $\partial^+\Omega^*$ and $(x^-(s), y^-(s))$ be an arclength parametrization of $\partial^-\Omega^*$, each measured from the corner at $\mathcal{O}$, so that $(x^\pm(0), y^\pm(0)) = (0, 0)$. Let $(x_*^+(s), y_*^+(s))$ be an arclength parametrization of $\partial^+\Omega_\infty = \{(r\cos(\alpha), r\sin(\alpha)) : r \ge 0\}$ and $(x_*^-(s), y_*^-(s))$ be an arclength parametrization of $\partial^-\Omega_\infty = \{(r\cos(-\alpha), r\sin(-\alpha)) : r \ge 0\}$, each measured from the corner at $\mathcal{O}$. Define

$$\gamma^+(s) = \gamma(x^+(s), y^+(s)) \quad \text{and} \quad \gamma^-(s) = \gamma(x^-(s), y^-(s)).$$

For $0 \le \alpha \le \frac{\pi}{2}$, the corner will be said to be *convex* and for $\frac{\pi}{2} < \alpha \le \pi$, the corner will be said to be *nonconvex*.

In [Lancaster and Siegel 1996], the existence of radial limits of a bounded solution f to (1) that satisfies (2) on the smooth portions of $\partial\Omega$ is proven provided that γ was bounded away from 0 and π, and for a convex corner an additional condition is satisfied coupling γ^+ and γ^-. In this paper, we eliminate the requirement that γ is bounded away from 0 and π; an additional condition must still be satisfied at a convex corner. The radial limits of f will be denoted by

$$Rf(\theta) = \lim_{r\to 0^+} f(r\cos\theta, r\sin\theta), \qquad -\alpha < \theta < \alpha,$$

and $Rf(\pm\alpha) = \lim_{\partial^\pm\Omega^* \ni \boldsymbol{x} \to \mathcal{O}} f(\boldsymbol{x})$, $\boldsymbol{x} = (x, y)$, which are the limits of the boundary values of f on the two sides of the corner if these exist.

Theorem 1. *Let f be a bounded solution to (1) satisfying (2) on $\partial^\pm\Omega^* \setminus \{\mathcal{O}\}$, which is discontinuous at $\mathcal{O}$.*

(a) *If $\alpha > \frac{\pi}{2}$ then $Rf(\theta)$ exists for all $\theta \in (-\alpha, \alpha)$.*

(b) *If $\alpha \le \frac{\pi}{2}$ and there exist constants $\underline{\gamma}^\pm, \overline{\gamma}^\pm$, $0 \le \underline{\gamma}^\pm \le \overline{\gamma}^\pm \le \pi$ satisfying*

$$\pi - 2\alpha < \underline{\gamma}^+ + \underline{\gamma}^- \le \overline{\gamma}^+ + \overline{\gamma}^- < \pi + 2\alpha$$

such that $\underline{\gamma}^\pm \le \gamma^\pm(s) \le \overline{\gamma}^\pm$ for all $s \in (0, s_0)$, for some $s_0 > 0$, then $Rf(\theta)$ exists for all $\theta \in (-\alpha, \alpha)$.

Furthermore, in either case, $Rf(\theta)$ is a continuous function on $(-\alpha, \alpha)$ which behaves in one of the following ways:

(i) *$Rf(\theta)$ is a constant function of θ and f has a nontangential limit at $\mathcal{O}$.*

(ii) *There exist α_1 and α_2 so that $-\alpha \le \alpha_1 < \alpha_2 \le \alpha$ and Rf is constant on $(-\alpha, \alpha_1]$ and $[\alpha_2, \alpha)$ and strictly increasing or strictly decreasing on $[\alpha_1, \alpha_2] \cap (-\alpha, \alpha)$. Label these case (I) and case (D), respectively.*

(iii) *There exist $\alpha_1, \alpha_L, \alpha_R, \alpha_2$ so that $-\alpha \le \alpha_1 < \alpha_L < \alpha_R < \alpha_2 \le \alpha$, $\alpha_R = \alpha_L + \pi$, and Rf is constant on $(-\alpha, \alpha_1]$, $[\alpha_L, \alpha_R]$, and $[\alpha_2, \alpha)$ and either increasing on $[\alpha_1, \alpha_L] \cap (-\alpha, \alpha)$ and decreasing on $[\alpha_R, \alpha_2] \cap (-\alpha, \alpha)$ or decreasing on*

$[\alpha_1, \alpha_L] \cap (-\alpha, \alpha)$ *and increasing on* $[\alpha_R, \alpha_2] \cap (-\alpha, \alpha)$. *Label these case* (*ID*) *and case* (*DI*), *respectively.*

In Theorem 1 of [Lancaster and Siegel 1996] and Theorem 1 above, the existence of two intervals $(-\alpha, \alpha_1]$ and $[\alpha_2, \alpha)$ on which $Rf(\cdot)$ is constant (i.e., “side fans”) is established but the relationship between the sizes of these side fans and the contact angle is unclear. Theorem 2 of [Lancaster and Siegel 1996] establishes lower bounds on these sizes when the

$$\lim_{\partial^+\Omega \ni (x,y)\to \mathcal{O}} \gamma(x,y) = \gamma_0^+ \quad \text{and} \quad \lim_{\partial^-\Omega \ni (x,y)\to \mathcal{O}} \gamma(x,y) = \gamma_0^-$$

are assumed to exist. (In [Lancaster 2010; 2012], these lower bounds were shown to be the actual sizes of the side fans.) What happens if the limits of γ at $\mathcal{O}$ do not exist? Theorem 2 and Corollary 3 provide lower bounds in this situation.

For $0 < b < 1$, define

$$A_I^{\pm}(b) = \liminf_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^{b\epsilon} \cos(\gamma^{\pm}(t))\, dt \quad \text{and} \quad A_S^{\pm}(b) = \limsup_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^{b\epsilon} \cos(\gamma^{\pm}(t))\, dt.$$

Notice that $b\cos(\limsup_{t\downarrow 0} \gamma^{\pm}(t)) \le A_I^{\pm}(b) \le A_S^{\pm}(b) \le b\cos(\liminf_{t\downarrow 0} \gamma^{\pm}(t))$.

Theorem 2. *Let f be a bounded solution to (1) satisfying (2) on $\partial^{\pm}\Omega^* \setminus \{\mathcal{O}\}$, which is discontinuous at $\mathcal{O}$. Assume $Rf(\theta)$ exists for all $\theta \in (-\alpha, \alpha)$. Then*:

(a) *$Rf(\theta)$ is a continuous function on $(-\alpha, \alpha)$ which behaves as described in* (*i*), (*ii*) *or* (*iii*) *of Theorem 1.*

(b) *There exist fans of constant radial limits adjacent to each tangent direction at $\mathcal{O}$ and lower bounds on the sizes of these side fans exist.*

In terms of the cases labeled in Theorem 1, the sizes of the side fans $\beta^- = \alpha_1 + \alpha$ and $\beta^+ = \alpha - \alpha_2$ satisfy the following conditions:

(1) $A_I^+\left(\dfrac{\sin(\lambda - \beta^+)}{\sin(\lambda)}\right) + \dfrac{\sin(\beta^+)}{\sin(\lambda)} \ge 1$ *for all* $\lambda \in (\beta^+, \pi)$ *for* (*I*) *and* (*DI*).

(2) $A_I^-\left(\dfrac{\sin(\lambda - \beta^-)}{\sin(\lambda)}\right) + \dfrac{\sin(\beta^-)}{\sin(\lambda)} \ge 1$ *for all* $\lambda \in (\beta^-, \pi)$ *for* (*D*) *and* (*DI*).

(3) $1 + A_S^-\left(\dfrac{\sin(\lambda - \beta^-)}{\sin(\lambda)}\right) \le \dfrac{\sin(\beta^-)}{\sin(\lambda)}$ *for all* $\lambda \in (\beta^-, \pi)$ *for* (*I*) *and* (*ID*).

(4) $1 + A_S^+\left(\dfrac{\sin(\lambda - \beta^+)}{\sin(\lambda)}\right) \le \dfrac{\sin(\beta^+)}{\sin(\lambda)}$ *for all* $\lambda \in (\beta^+, \pi)$ *for* (*D*) *and* (*ID*).

2. Proofs of Theorems 1 and 2

The proof of Theorem 1 follows that established in [Lancaster 1985] and [Elcrat and Lancaster 1986] in which (i) the graph of the solution in $\Omega \times \mathbb{R}$ is represented in isothermal coordinates, (ii) comparison arguments are used to prove that the component functions of the isothermal parametrization of the graph are uniformly continuous and so extend to be continuous on the closure of the parameter domain, (iii) boundary regularity theory (e.g., [Heinz 1970]) is used to prove that radial limits exist for almost every direction, (iv) cusp solutions are excluded (e.g., [Echart and Lancaster 2017]) and (v) the behavior of the radial limit function is determined. The only step which does not follow from previous work is (ii) and so the proof of Theorem 1 comes down to establishing (ii). The proof of Theorem 2 follows from standard "blow up" arguments.

2.1. ***Proof of Theorem 1.*** When $\alpha > \frac{\pi}{2}$, Theorem 1 is a consequence of [Entekhabi and Lancaster 2016]. Suppose now that $\alpha \le \frac{\pi}{2}$. Since f is bounded and the prescribed mean curvature is $H(x, y, z) = \kappa z + \lambda$, there exist $M_1 \in (0, \infty)$ and $M_2 \in [0, \infty)$ such that

$$\sup_{(x,y)\in\Omega} |f(x, y)| \le M_1 \quad \text{and} \quad \sup_{(x,y)\in\Omega} |H(x, y, f(x, y))| \le M_2. \tag{3}$$

In §2.1 of [Entekhabi and Lancaster 2017], a specific torus is constructed which depends solely on M_2 and which is used as a comparison surface; one should compare this with, for example, [Lancaster and Siegel 1996], where several types of comparison surfaces are used, or [Entekhabi and Lancaster 2016], where an unduloid is used as a comparison surface. We shall use this torus as our comparison surface here. We will denote by q the modulus of continuity of the function h^- whose graph is the set $\mathcal{T}$ which is the inner half of a torus with axis of symmetry $\{(2, y, 0) : y \in \mathbb{R}^2\}$, major radius $R_0 = 2$, and minor radius r_0; here

$$r_0 = \begin{cases} 1 & \text{if } M_2 = 0, \\ \dfrac{1}{M_2} + 1 - \sqrt{\left(\dfrac{1}{M_2}\right)^2 + 1} & \text{if } M_2 > 0. \end{cases} \tag{4}$$

Then q is also the modulus of continuity of functions (i.e., h^+, h^-_β, h^+_β) whose graphs are obtained by rotations and translations in the horizontal plane of $\mathcal{T}$ (see [Entekhabi and Lancaster 2017, p. 59]).

Let $\mathscr{S}_0 = \text{gra}(f) = \{(x, y, f(x, y)) : (x, y) \in \Omega^*\}$ and allow $\mathscr{S}$ to be the closure of $\mathscr{S}_0$ in $\mathbb{R}^3$. As in §2.2 of [Entekhabi and Lancaster 2017], there exists an isothermal parametrization $Y : E \to \mathbb{R}^3$ given by

$$Y(u, v) = (a(u, v), b(u, v), c(u, v))$$

such that $Y(\bar{E}) = \mathscr{S}$, $Y(E) = \mathscr{S}_0$, and (a_1)–(a_5) of [Entekhabi and Lancaster 2017] hold, where $E = B_1(\mathcal{O}) = \{(u, v) : u^2 + v^2 < 1\}$. By (a_2) of that paper, if we let $G(u, v) = (a(u, v), b(u, v))$ for $(u, v) \in E$, then $G \in C^0(\bar{E})$. From (a_3) of that paper, there exists a connected arc $\sigma \subset \partial E$ that Y maps strictly monotonically onto

$\{(x, y, f(x, y)) : (x, y) \in \partial\Omega^* \setminus \{\mathcal{O}\}\}$. Let the endpoints of σ be denoted $\mathbf{o}_1$ and $\mathbf{o}_2$. There exists points $\mathbf{a}, \mathbf{b} \in \sigma$ such that $G(\mathbf{a}) = A$, $G(\mathbf{b}) = B$, G maps the arc $\mathbf{o}_2\mathbf{a}$ onto $\partial^-\Omega$ and G maps the arc $\mathbf{o}_1\mathbf{b}$ onto $\partial^+\Omega$. We must consider the two cases:

(A) $\mathbf{o}_1 = \mathbf{o}_2$,

(B) $\mathbf{o}_1 \neq \mathbf{o}_2$.

Assume first that (A) holds. Set $\mathbf{o} = \mathbf{o}_1 = \mathbf{o}_2$. We wish to prove that c is uniformly continuous on E and hence c extends to be continuous on $\bar{E}$. If so, then the existence and behavior of the radial limits of f follows as in [Entekhabi and Lancaster 2017; Lancaster and Siegel 1996]. There are three possible cases:

(i) $\underline{\gamma}^- > 0$ and $\overline{\gamma}^- < \pi$,

(ii) $\underline{\gamma}^+ > 0$ and $\overline{\gamma}^+ < \pi$,

(iii) ($\underline{\gamma}^- = 0$ or $\overline{\gamma}^- = \pi$) and ($\underline{\gamma}^+ = 0$ or $\overline{\gamma}^+ = \pi$).

Case (i). Let $\lambda_1 = \underline{\gamma}^+$, $\lambda_2 = \overline{\gamma}^+$, $\gamma_2 = \underline{\gamma}^-$. We observe that $\lambda_2 = \overline{\gamma}^+ < \pi + 2\alpha - \overline{\gamma}^-$, $\lambda_1 = \underline{\gamma}^+ > \pi - 2\alpha - \underline{\gamma}^-$, and so $\lambda_2 - \lambda_1 < 4\alpha$. We wish to use the argument in the proof of Theorem 2 of [Entekhabi and Lancaster 2017]. Since $\pi - 2\alpha - \lambda_1 < \gamma_2 < \pi + 2\alpha - \lambda_2$, we can choose $\tau_1, \tau_2 \in (0, \pi)$ such that $\tau_1 \in (\pi - 2\alpha - \lambda_1, \gamma_2)$ and $\tau_2 \in (\gamma_2, \pi + 2\alpha - \lambda_2)$. Set $\beta_1 = \frac{\pi}{2} - \tau_1$ and $\beta_2 = \tau_2 - \frac{\pi}{2}$. With these choices of β_1 and β_2, notice that

$$T(h^- \circ T_{\beta_1})(x_1, 0) \cdot (0, -1) = \cos(\tau_1) > \cos(\gamma_2) \quad \text{for } 0 < x_1 < 2 - r_0$$

and

$$T(h^+ \circ T_{\beta_2})(x_1, 0) \cdot (0, -1) = \cos(\tau_2) < \cos(\gamma_2) \quad \text{for } 0 < x_1 < 2 - r_0$$

(see [Entekhabi and Lancaster 2017, p. 59]). This implies that for $\delta_1 = \delta_1(\beta_1, \beta_2) > 0$ small enough and $\boldsymbol{x} \in \partial^-\Omega$ with $|\boldsymbol{x}| < \delta_1$, we have

$$T(h^-_{\beta_1})(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) > \cos(\gamma(\boldsymbol{x})) \quad \text{and} \quad T(h^+_{\beta_2})(\boldsymbol{x}) \cdot \vec{\nu}(\boldsymbol{x}) < \cos(\gamma(\boldsymbol{x})). \tag{5}$$

Since $\beta_1, \beta_2 \neq \pm\frac{\pi}{2}$, there exists $R = R(\beta_1, \beta_2) > 0$ such that $B_R(\mathcal{O}) \cap \Omega^* \subset \Delta_{\beta_1} \cap \Delta_{\beta_2}$, where Δ_β is as in §2.1 of [Entekhabi and Lancaster 2017]. For each $\delta \in (0, 1)$, allow

$$p(\delta) = \sqrt{\frac{8\pi M_0}{\ln(1/\delta)}} \tag{6}$$

where M_0 is the area of S_0.

Let $\epsilon > 0$. Choose $\delta > 0$ such that

$$\sqrt{\delta} < \min\{\|\mathbf{o} - \mathbf{a}\|, \|\mathbf{o} - \mathbf{b}\|\},$$
$$p(\delta) < \delta_1(\beta_1, \beta_2), \quad p(\delta) < R(\beta_1, \beta_2), \quad p(\delta) + q(p(\delta)) < \frac{\epsilon}{2}.$$

Let $\boldsymbol{w}_0 = (u_0, v_0) \in E$. From the Courant–Lebesgue lemma, there exists a $\rho(\delta) \in (\delta, \sqrt{\delta})$ such that the arclength $l_{\rho(\delta)}$ of $C'_{\rho(\delta)}$ is less than $p(\delta)$, where $C_\delta = \{\boldsymbol{w} \in E : \|\boldsymbol{w} - \boldsymbol{w}_0\| = \delta\}$ and $C'_\delta = Y(C_\delta)$. Set $B_\delta = \{\boldsymbol{w} \in E : \|\boldsymbol{w} - \boldsymbol{w}_0\| < \delta\}$ and $B'_\delta = Y(B_\delta)$. Then, for $\boldsymbol{w} \in C'_{\rho(\delta)}$, there exist functions

$$b^+(x, y) = f(\boldsymbol{w}) + p(\delta) + h^-_{\beta_1}(x, y) \quad \text{for } (x, y) \in \Delta_{\beta_1}, \tag{7}$$

$$b^-(x, y) = f(\boldsymbol{w}) - p(\delta) - h^+_{\beta_2}(x, y) \quad \text{for } (x, y) \in \Delta_{\beta_2} \tag{8}$$

where $\beta_1 = \frac{\pi}{2} - \tau_1$ and $\beta_2 = \tau_2 - \frac{\pi}{2}$. From (10) of [Entekhabi and Lancaster 2017], we have that $\operatorname{div}(b^+) \le -M_2$ in Δ_{β_1} and $\operatorname{div}(b^-) \ge M_2$ in Δ_{β_2}. So in $\Omega \cap \Delta_{\beta_1}$, $\operatorname{div}(Tb^+) \le \operatorname{div}(Tf)$. On $\partial^-\Omega \cap B_{\delta_1}(\mathcal{O})$, $Tb^+ \cdot \nu \ge Tf \cdot \nu$. As in the proof of Theorem 2 of that paper,

$$f(x, y) < b^+(x, y) \quad \text{for } (x, y) \in \Delta_{\beta_1} \cap B'_{\rho(\delta)}, \tag{9}$$

where $B'_{\rho(\delta)} = Y(B_{\rho(\delta)})$. This follows since $Tb^+ \cdot \nu \ge Tf \cdot \nu$ on $\partial^+\Omega \cap B_{\delta_2}(\mathcal{O})$ by (15) of that paper if $\tau_1 + 2\alpha \le \pi$ and no boundary condition on $\partial^+\Omega$ is required if $\tau_1 + 2\alpha > \pi$.

Repeat the same argument with $\lambda_1 = \underline{\gamma}^+$, $\lambda_2 = \overline{\gamma}^+$ and $\gamma_2 = \overline{\gamma}^-$. In the same way as above, there exist functions

$$b^+_*(x, y) = f(\boldsymbol{w}) + p(\delta) + h^-_{\beta_1}(x, y) \quad \text{for } (x, y) \in \Delta_{\beta_1}, \tag{10}$$

$$b^-_*(x, y) = f(\boldsymbol{w}) - p(\delta) - h^+_{\beta_2}(x, y) \quad \text{for } (x, y) \in \Delta_{\beta_2} \tag{11}$$

such that

$$b^-_*(x, y) < f(x, y) \tag{12}$$

for $(x, y) \in \Delta_{\beta_2} \cap B'_{\rho(\delta)}$ where $B'_{\rho(\delta)} = Y(B_{\rho(\delta)})$. Then combining (9) and (12) we get

$$b^-_*(x, y) < f(x, y) < b^+(x, y) \tag{13}$$

for $(x, y) \in \Delta_{\beta_1} \cap \Delta_{\beta_2} \cap B'_{\rho(\delta)}$. As in [Entekhabi and Lancaster 2017], it follows that $c(u, v)$ is uniformly continuous on E.

Case (ii). Case (ii) is simply case (i) reflected about the xz-plane and the proof follows as above.

Case (iii). Notice that

$$0 \le \pi - 2\alpha < \underline{\gamma}^+ + \underline{\gamma}^- \le \overline{\gamma}^+ + \overline{\gamma}^- < \pi + 2\alpha \le \pi$$

and so $\underline{\gamma}^- = 0$ and $\underline{\gamma}^+ = 0$ cannot both occur and $\overline{\gamma}^+ = \pi$ and $\overline{\gamma}^- = \pi$ cannot both occur. The result follows from this, using the arguments in cases 1 and 2. In particular, if $\underline{\gamma}^- > 0$, then we obtain a supersolution b^+ as in case (i) (see Figure 1)

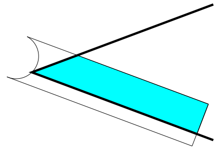

Figure 1. The domain of a supersolution in case (i).

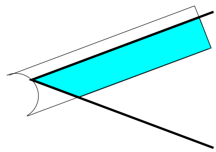

Figure 2. The domain of a supersolution in case (ii).

and if $\underline{\gamma}^- = 0$, we obtain a supersolution b^+ as in case (ii) (see Figure 2); if $\overline{\gamma}^- < \pi$, we obtain a subsolution b_*^- as in case (i) and if $\overline{\gamma}^- = \pi$, we obtain a subsolution as in case (ii).

Now assume (B) holds. Let $B = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1, y \geq 0\}$ and let $\bar{B}$ be the closure of B in $\mathbb{R}^2$. Let $g : \bar{B} \to \bar{E}$ be a conformal or anticonformal map taking $\{(u, 0) : -1 \leq u \leq 1\}$ onto $\partial E \setminus \sigma$ such that the map $X = Y \circ g : B \to \mathbb{R}^3$ has a downward orientation (i.e., the normal $X_u \times X_v$ to $\mathscr{S}_0$ gives a downward orientation). Writing $X(u, v) = (x(u, v), y(u, v), z(u, v))$ and $K(u, v) = (x(u, v), y(u, v))$, we have $K \in C^0(\bar{B})$ and $K(u, 0) = (0, 0)$ while $X(u, 0) = (0, 0, z(u, 0))$ for $u \in [-1, 1]$. Then the argument follows from [Lancaster and Siegel 1996] and the previous argument here, as explained in [Entekhabi and Lancaster 2017]. □

2.2. ***Proof of Theorem 2.*** We first note that if $\delta_1, \delta_2 \in (-\alpha, \alpha)$ with $\delta_1 < \delta_2$ and $Rf(\delta_1)$ and $Rf(\delta_2)$ both exist, then it follows from [Elcrat and Lancaster 1986] that $Rf(\theta)$ exists for all $\theta \in [\delta_1, \delta_2]$ and $Rf(\theta)$ is a continuous function of θ on $[\delta_1, \delta_2]$ which behaves as described in (i), (ii) or (iii) of Theorem 1. The first part of Theorem 2 (i.e., (a)) follows from this.

Suppose $\{\epsilon_j\}$ is a decreasing sequence with $\lim_{j\to\infty} \epsilon_j = 0$. Let $I = (-1, 1)$ and set

$$\gamma_j(s) = \begin{cases} \gamma^+(\epsilon_j s) & \text{if } 0 < s < 1, \\ \gamma^-(-\epsilon_j s) & \text{if } -1 < s < 0 \end{cases}$$

for $j \in \mathbb{N}$; then $\{\cos(\gamma_j) : j \in \mathbb{N}\}$ is a subset of the unit ball in $L^\infty(I) = (L^1(I))^*$. By the Banach–Alaoglu theorem, there exist a subsequence $\{\epsilon_{j_k}\}$ of $\{\epsilon_j\}$ and a function $h = h_{\{\epsilon_{j_k}\}} \in L^\infty(I)$ such that $\cos(\gamma_{j_k})$ converges weak-star to h; that is, for each $m \in L^1(I)$,

$$\lim_{k\to\infty} \int_{-1}^{1} \cos(\gamma_{j_k}(s)) m(s)\, ds = \int_{-1}^{1} h(s) m(s)\, ds.$$

Let us define $\gamma^* = \gamma^*_{\{\epsilon_{j_k}\}} = \cos^{-1}(h)$ (almost everywhere on $(-1, 1)$). For any $b \in (0, 1)$, by choosing m to be the characteristic function of the interval $(0, b)$ we see that

$$\int_0^b h(s)\,ds = \lim_{k\to\infty}\int_0^b \cos(\gamma_{j_k}(s))\,ds = \lim_{k\to\infty}\frac{1}{\epsilon_{j_k}}\int_0^{b\epsilon_{j_k}}\cos(\gamma^+(t))\,dt$$

and, by choosing m to be the characteristic function of the interval $(-b, 0)$,

$$\int_{-b}^0 h(s)\,ds = \lim_{k\to\infty}\int_{-b}^0 \cos(\gamma_{j_k}(s))\,ds = \lim_{k\to\infty}\frac{1}{\epsilon_{j_k}}\int_0^{b\epsilon_{j_k}}\cos(\gamma^-(t))\,dt;$$

hence

$$\int_0^b \cos(\gamma^*(s))\,ds = \lim_{k\to\infty}\frac{1}{\epsilon_{j_k}}\int_0^{b\epsilon_{j_k}}\cos(\gamma^+(t))\,dt \geq \liminf_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{\epsilon b}\cos(\gamma^+(t))\,dt$$

and

$$\int_{-b}^0 \cos(\gamma^*(s))\,ds = \lim_{k\to\infty}\frac{1}{\epsilon_{j_k}}\int_0^{b\epsilon_{j_k}}\cos(\gamma^-(t))\,dt \geq \liminf_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{\epsilon b}\cos(\gamma^-(t))\,dt.$$

Thus

$$\liminf_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{\epsilon b}\cos(\gamma^\pm(t))\,dt \leq \liminf_{j\to\infty}\int_0^b \cos(\gamma^\pm(\epsilon_j s))\,ds. \tag{14}$$

Choose a sequence $\{\epsilon_j\}$ with $\lim_{j\to\infty}\epsilon_j = 0$ such that

$$\lim_{j\to\infty}\frac{1}{\epsilon_j}\int_0^{b\epsilon_j}\cos(\gamma^+(t))\,dt = \liminf_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{\epsilon b}\cos(\gamma^+(t))\,dt;$$

as above, there exist a subsequence $\{\epsilon_{j_k}\}$ of $\{\epsilon_j\}$ and $\gamma_* \in L^\infty(I)$ such that $\cos(\gamma_{j_k})$ converges weak-star to $\cos(\gamma_*)$. Then

$$\begin{aligned}\liminf_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{\epsilon b}\cos(\gamma^+(t))\,dt &= \lim_{k\to\infty}\frac{1}{\epsilon_{j_k}}\int_0^{b\epsilon_{j_k}}\cos(\gamma^+(t))\,dt\\ &= \lim_{k\to\infty}\int_0^b \cos(\gamma_{j_k}(s))\,ds = \int_0^b \cos(\gamma_*(s))\,ds.\end{aligned}$$

Case 1. Suppose case (I) or (DI) of Theorem 1 holds and $\alpha_2 = \alpha - \beta^+$. Let us assume there exists $\lambda \in (\beta^+, \pi)$ such that

$$A_I^+\left(\frac{\sin(\lambda-\beta^+)}{\sin(\lambda)}\right) + \frac{\sin(\beta^+)}{\sin(\lambda)} < 1; \tag{15}$$

we shall show that this leads to a contradiction. Set

$$b = \frac{\sin(\lambda-\beta^+)}{\sin(\lambda)}.$$

Choose a sequence $\{\epsilon_j\}$ with $\lim_{j\to\infty}\epsilon_j = 0$ such that

$$\lim_{j\to\infty}\frac{1}{\epsilon_j}\int_0^{b\epsilon_j}\cos(\gamma^+(t))\,dt = \liminf_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{\epsilon b}\cos(\gamma^+(t))\,dt; \tag{16}$$

as before, there exist a subsequence $\{\epsilon_{j_k}\}$ of $\{\epsilon_j\}$ and $\gamma_* \in L^\infty(I)$ such that $\cos(\gamma_{j_k})$ converges weak-star to $\cos(\gamma_*)$. Then

$$\begin{aligned}\liminf_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{\epsilon b}\cos(\gamma^+(t))\,dt &= \lim_{k\to\infty}\frac{1}{\epsilon_{j_k}}\int_0^{b\epsilon_{j_k}}\cos(\gamma^+(t))\,dt\\ &= \lim_{k\to\infty}\int_0^b\cos(\gamma_{j_k}(s))\,ds = \int_0^b\cos(\gamma_*(s))\,ds.\end{aligned}$$

Let $\theta_0 \in (\sigma, \alpha_2)$, where $\sigma = \alpha_1$ if case (I) holds and $\sigma = \alpha_R$ if case (DI) holds, and $z_0 = Rf(\theta_0)$. Set $\Omega_k = \{(x, y) \in \mathbb{R}^2 : (\epsilon_{j_k}x, \epsilon_{j_k}y) \in \Omega\}$ and define $f_k \in C^\infty(\Omega_k)$ by

$$f_k(x, y) = \frac{1}{\epsilon_{j_k}}(f(\epsilon_{j_k}x, \epsilon_{j_k}y) - z_0)$$

for $(x, y) \in \Omega_k$. Let γ_k be defined on $\partial\Omega_k \setminus \{\mathcal{O}\}$ by

$$\gamma_k(x, y) = \gamma(\epsilon_{j_k}x, \epsilon_{j_k}y)$$

and let $\nu_k = \nu_k(x, y)$ denote the outward unit normal to $\partial\Omega_k$. Then f_k satisfies the capillary problem

$$\begin{aligned} Nf_k(x, y) &= \epsilon_{j_k}\kappa f(\epsilon_{j_k}x, \epsilon_{j_k}y) + \epsilon_{j_k}\lambda, && (x, y) \in \Omega_k,\\ Tf_k\cdot\nu_k &= \cos(\gamma_k) && \text{on } \partial\Omega_k \setminus \{\mathcal{O}\}.\end{aligned}$$

Since $Rf(\theta) < z_0$ if $\sigma < \theta < \theta_0$ and $Rf(\theta) > z_0$ if $\theta_0 < \theta < \alpha$, we see (e.g., [Lancaster 2010; 2012; Simon 1980]; also see [Tam 1984; 1986]) that $\{f_k\}$ converges locally to the generalized solution f_∞ (in the sense of Miranda [1977] and Giusti [1980; 1984]) of the functional

$$\mathcal{F}_\infty(g) = \iint_{\Omega_\infty}\sqrt{1 + |Dg|^2}\,dx - \int_{\partial\Omega_\infty}\cos(\gamma_*(s))g\,ds,$$

where

$$f_\infty(r\cos(\theta), r\sin(\theta)) = \begin{cases} -\infty & \text{if } -\alpha < \theta < \theta_0,\\ \infty & \text{if } \theta_0 < \theta < \alpha\end{cases}$$

if case (I) holds or case (DI) holds and $z_0 > Rf(\theta)$ for all $\theta \in (-\alpha, \alpha_L)$ and

$$f_\infty(r\cos(\theta), r\sin(\theta)) = \begin{cases} \infty & \text{if } -\alpha < \theta < \theta_h,\\ -\infty & \text{if } \theta_h < \theta < \theta_0,\\ \infty & \text{if } \theta_0 < \theta < \alpha\end{cases}$$

with $Rf(\theta_h) = z_0$ and $\theta_h < \alpha_L$ otherwise.

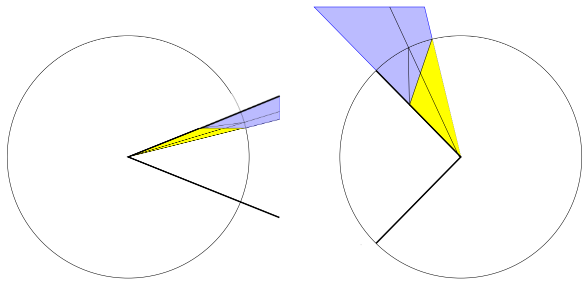

Figure 3. The yellow region represents Σ_{θ_0}.

Let us now define the sets

$$\mathcal{P} = \{(x, y) \in \Omega_\infty : f_\infty(x, y) = \infty\} \quad \text{and} \quad \mathcal{N} = \{(x, y) \in \Omega_\infty : f_\infty(x, y) = -\infty\}.$$

These sets have a special structure which follows from the fact that $\mathcal{P}$ minimizes the functional

$$\Phi(A) = \iint_{\Omega_\infty} |D\chi_A| - \int_{\partial\Omega_\infty} \cos(\gamma_*)\chi_A \, dH^1$$

and $\mathcal{N}$ minimizes the functional

$$\Psi(A) = \iint_{\Omega_\infty} |D\chi_A| + \int_{\partial\Omega_\infty} \cos(\gamma_*)\chi_A \, dH^1$$

in the appropriate sense (e.g., [Giusti 1980; Lancaster and Siegel 1996; Miranda 1977]). Let Σ_{θ_0} denote the (open) triangular region whose boundary is the triangle with vertices $(0, 0)$, $B = (b\cos(\alpha), b\sin(\alpha))$ and $C = (\cos(\theta_0), \sin(\theta_0))$ and set $A = \mathcal{P} \setminus \Sigma_{\theta_0}$ (see Figure 3). Simple trigonometric computations with $R > 2$ show that

$$\Phi\big(B(\mathcal{O}, R) \cap \mathcal{P}\big) - \Phi\big(B(\mathcal{O}, R) \cap \mathcal{P} \setminus \Sigma_{\theta_0}\big) = \big(1 - A_I^+(b)\big) - \left(\frac{\sin(\alpha - \theta_0)}{\sin(\omega)}\right), \tag{17}$$

where $\pi - \omega$ is the angle $\angle\mathcal{O}BC$. This holds for all $\theta_0 < \alpha_2 = \alpha - \beta^+$; taking the limit as $\theta_0 \uparrow \alpha - \beta^+$ and noticing that $\omega \to \lambda$ as $\theta_0 \uparrow \alpha - \beta^+$, we see that

$$\Phi\big(B(\mathcal{O}, R) \cap \mathcal{P}\big) - \Phi\big(B(\mathcal{O}, R) \cap \mathcal{P} \setminus \Sigma_{\alpha_2}\big) = \big(1 - A_I^+(b)\big) - \left(\frac{\sin(\beta^+)}{\sin(\lambda)}\right) > 0$$

or

$$\Phi\big(B(\mathcal{O}, R) \cap \mathcal{P}\big) > \Phi\big(B(\mathcal{O}, R) \cap \mathcal{P} \setminus \Sigma_{\alpha_2}\big);$$

this contradicts the fact that $\mathcal{P}$ (locally) minimizes Φ. Therefore (15) is false. This completes case 1.

Case 2. Suppose case (D) or (DI) of Theorem 1 holds and $\alpha_1 = -\alpha + \beta^-$. Let us assume there exists $\lambda \in (\beta^-, \pi)$ such that

$$A_I^- \left(\frac{\sin(\lambda - \beta^-)}{\sin(\lambda)} \right) + \frac{\sin(\beta^-)}{\sin(\lambda)} < 1. \tag{18}$$

Using an argument similar to that in case 1, we reach a contradiction.

Case 3. Suppose case (I) or (ID) of Theorem 1 holds and $\alpha_1 = -\alpha + \beta^-$. Let us assume there exists $\lambda \in (\beta^-, \pi)$ such that

$$1 + A_S^- \left(\frac{\sin(\lambda - \beta^-)}{\sin(\lambda)} \right) > \frac{\sin(\beta^-)}{\sin(\lambda)}. \tag{19}$$

Set

$$b = \frac{\sin(\lambda - \beta^-)}{\sin(\lambda)}.$$

Arguing as in case 1, we see that the set $\mathcal{N} = \{(x, y) \in \Omega_\infty : f_\infty(x, y) = -\infty\}$ minimizes the functional

$$\Psi(A) = \iint_{\Omega_\infty} |D\chi_A| + \int_{\partial\Omega_\infty} \cos(\gamma_*)\chi_A \, dH^1$$

in the appropriate sense (e.g., [Giusti 1980; Lancaster and Siegel 1996; Miranda 1977]). Let Σ_{θ_0} denote the (open) triangular region whose boundary is the triangle with vertices $(0, 0)$, $B = (b\cos(-\alpha), b\sin(-\alpha))$ and $C = (\cos(\theta_0), \sin(\theta_0))$ and set $A = \mathcal{N} \setminus \Sigma_{\theta_0}$ (see Figure 3). Simple trigonometric computations with $R > 2$ show that

$$\Psi\big(B(\mathcal{O}, R) \cap \mathcal{N}\big) - \Psi\big(B(\mathcal{O}, R) \cap \mathcal{N} \setminus \Sigma_{\theta_0}\big) = \big(1 + A_S^-(b)\big) - \left(\frac{\sin(\alpha + \theta_0)}{\sin(\omega)} \right),$$

where ω is the angle $\angle \mathcal{O}BC$. This holds for all $\theta_0 > \alpha_1 = -\alpha + \beta^-$; taking the limit as $\theta_0 \downarrow -\alpha + \beta^-$ and noticing that $\omega \to \lambda$ as $\theta_0 \downarrow -\alpha + \beta^-$, we see that

$$\Psi\big(B(\mathcal{O}, R) \cap \mathcal{N}\big) - \Psi\big(B(\mathcal{O}, R) \cap \mathcal{N} \setminus \Sigma_{\alpha_1}\big) = \big(1 + A_S^-(b)\big) - \left(\frac{\sin(\beta^-)}{\sin(\lambda)} \right) > 0$$

or

$$\Psi\big(B(\mathcal{O}, R) \cap \mathcal{N}\big) > \Psi\big(B(\mathcal{O}, R) \cap \mathcal{N} \setminus \Sigma_{\alpha_1}\big);$$

this contradicts the fact that $\mathcal{N}$ (locally) minimizes Ψ. Therefore (19) is false. This completes case 3.

Case 4. Suppose case (D) or (ID) of Theorem 1 holds and $\alpha_2 = \alpha - \beta^+$. Let us assume there exists $\lambda \in (\beta^+, \pi)$ such that

$$1 + A_S^+ \left(\frac{\sin(\lambda - \beta^+)}{\sin(\lambda)} \right) > \frac{\sin(\beta^+)}{\sin(\lambda)}. \tag{20}$$

Using an argument similar to that in case 3, we reach a contradiction. The proof of Theorem 2 is then complete. □

3. Corollaries and examples

Corollary 3. *Suppose $m \in [-1, 1]$; set $\sigma = \cos^{-1}(m) \in [0, \pi]$.*

(a) *If $A_I^+(b) \le mb$ and case (I) or (DI) holds, then $\beta^+ \ge \sigma$.*

(b) *If $A_I^-(b) \le mb$ and case (D) or (DI) holds, then $\beta^- \ge \sigma$.*

(c) *If $A_S^-(b) \ge mb$ and case (I) or (ID) holds, then $\beta^- \ge \pi - \sigma$.*

(d) *If $A_S^+(b) \ge mb$ and case (D) or (ID) holds, then $\beta^+ \ge \pi - \sigma$.*

Proof. (a) Suppose case (I) or (DI) of Theorem 1 holds, $\sigma \in [0, \pi]$, $\cos(\sigma) = m$, and $\beta^+ < \sigma$. By Theorem 2(a), we know that

$$\sin(\sigma) \left(\frac{\sin(\lambda - \beta^+)}{\sin(\lambda)} \right) + \frac{\sin(\beta^+)}{\sin(\lambda)} \ge A_I^+ \left(\frac{\sin(\lambda - \beta^+)}{\sin(\lambda)} \right) + \frac{\sin(\beta^+)}{\sin(\lambda)} \ge 1$$

or

$$\frac{\cos(\sigma)\sin(\lambda - \beta^+) + \sin(\beta^+)}{\sin(\lambda)} \ge 1$$

for all $\lambda \in (\beta^+, \pi)$. Since $\sigma > \beta^+$, we may set $\lambda = \sigma$ and obtain

$$\cos(\sigma - \beta^+) = \frac{\cos(\sigma)\sin(\sigma - \beta^+) + \sin(\beta^+)}{\sin(\sigma)} \ge 1,$$

which is a contradiction since $\sigma - \beta^+ \ne 0$. Thus $\beta^+ \ge \sigma$.

(b) This is essentially the same as (a).

(c) Suppose case (I) or (ID) of Theorem 1 holds, $\sigma \in [0, \pi]$, $\cos(\sigma) = m$, and $\beta^- < \pi - \sigma$. By Theorem 2(c), we know that

$$1 + \sin(\sigma) \left(\frac{\sin(\lambda - \beta^-)}{\sin(\lambda)} \right) \le 1 + A_S^- \left(\frac{\sin(\lambda - \beta^-)}{\sin(\lambda)} \right) \le \frac{\sin(\beta^-)}{\sin(\lambda)}$$

or

$$\frac{\sin(\lambda) + \cos(\sigma)\sin(\lambda - \beta^-) - \sin(\beta^-)}{\sin(\lambda)} \le 0$$

for all $\lambda \in (\beta^-, \pi)$. Since $\beta^- < \pi - \sigma$, we may set $\lambda = \pi - \sigma$ and obtain

$$1 + \cos(\sigma + \beta^-) = \frac{\sin(\sigma) + \cos(\sigma)\sin(\sigma + \beta^-) - \sin(\beta^-)}{\sin(\sigma)} \le 0,$$

which is a contradiction since $\sigma + \beta^- < \pi$. Thus $\beta^- \geq \pi - \sigma$.

(d) This is essentially the same as (c). $\square$

Example 4. Let $\alpha \in (0, \pi]$ and $\gamma_1^\pm, \gamma_2^\pm \in [0, \pi]$ with $\gamma_1^+ \leq \gamma_2^+$ and $\gamma_1^- \leq \gamma_2^-$. Set

$$\Omega = \{(r\cos(\theta), r\sin(\theta)) : 0 < r < 1, -\alpha < \theta < \alpha\}.$$

For each $n \in \mathbb{N}$, let $A_n = (2^{-n^2}, 2^{-n(n-1)}]$ and $B_n = (2^{-n(n+1)}, 2^{-n^2}]$. Define

$$\gamma(s) = \sum_{n=1}^{\infty} \bigl(\gamma_1^+ I_{A_n}(s) + \gamma_2^+ I_{B_n}(s) + \gamma_1^- I_{A_n}(-s) + \gamma_2^- I_{B_n}(-s)\bigr),$$

so that γ is defined on $\partial\Omega \cap B(\mathcal{O}, 1)$ by

$$\gamma(r\cos(\theta), r\sin(\theta)) = \begin{cases} \gamma_1^+ & \text{if } \theta = \alpha,\ 2^{-n^2} < r \leq 2^{-n(n-1)} \text{ for some } n \in \mathbb{N}, \\ \gamma_2^+ & \text{if } \theta = \alpha,\ 2^{-n(n+1)} < r \leq 2^{-n^2} \text{ for some } n \in \mathbb{N}, \\ \gamma_1^- & \text{if } \theta = -\alpha,\ 2^{-n^2} < r \leq 2^{-n(n-1)} \text{ for some } n \in \mathbb{N}, \\ \gamma_2^- & \text{if } \theta = -\alpha,\ 2^{-n(n+1)} < r \leq 2^{-n^2} \text{ for some } n \in \mathbb{N}. \end{cases}$$

Set

$$c_j = \begin{cases} 2^{-\frac{j}{2}\left(\frac{j}{2}+1\right)} & \text{if } j \text{ is even}, \\ 2^{-\left(\frac{j+1}{2}\right)^2} & \text{if } j \text{ is odd}. \end{cases}$$

Let $b \in (0, 1)$ be fixed for now. Set $\epsilon_j = c_{2j}/b$ ($j \in \mathbb{N}$); notice that $c_{2j+1}/c_{2j} = 2^{-(j+1)}$. Then

$$\begin{aligned} b\cos(\gamma_1^\pm) &\geq A_S^\pm(b) \\ &\geq \lim_{j\to\infty} \frac{1}{\epsilon_j} \int_0^{\epsilon_j b} \cos(\gamma^\pm(t))\,dt \\ &= \lim_{j\to\infty} b \int_0^1 \cos(\gamma_j^\pm(sb))\,ds \\ &= \lim_{j\to\infty} b \int_0^1 \cos(\gamma^\pm(c_{2j}s))\,ds \\ &= \lim_{j\to\infty} b \left(\int_{\frac{c_{2j+1}}{c_{2j}}}^1 \cos(\gamma^\pm(c_{2j}s))\,ds + \int_0^{\frac{c_{2j+1}}{c_{2j}}} \cos(\gamma^\pm(c_{2j}s))\,ds \right) \\ &= \lim_{j\to\infty} b \left(\cos(\gamma_1^\pm)(1 - 2^{-(j+1)}) + \int_0^{2^{-(j+1)}} \cos(\gamma^\pm(c_{2j}s))\,ds \right) \\ &= b\cos(\gamma_1^\pm). \end{aligned}$$

Using a similar argument for $A_I^\pm(b)$ with $\epsilon_j = c_{2j+1}/b$, $j \in \mathbb{N}$, we see that

$$A_I^\pm(b) = b\cos(\gamma_2^\pm) \quad \text{and} \quad A_S^\pm(b) = b\cos(\gamma_1^\pm). \tag{21}$$

Example 5. Let $\alpha \in (0, \pi]$ and $\gamma_1^{\pm}, \gamma_2^{\pm} \in [0, \pi]$ with $\gamma_1^+ \le \gamma_2^+$ and $\gamma_1^- \le \gamma_2^-$. Set

$$\Omega = \{(r\cos(\theta), r\sin(\theta)) : 0 < r < 1, -\alpha < \theta < \alpha\}.$$

For each $n \in \mathbb{N}$, let $A_n = \left(\frac{2}{4^n}, \frac{4}{4^n}\right)$, $B_n = \left(\frac{1}{4^n}, \frac{2}{4^n}\right)$, and $C_n = \left\{\frac{4}{4^n}\right\}$. Define

$$\gamma(s) = \sum_{n=1}^{\infty}\left(\gamma_1^+ I_{A_n}(s) + \gamma_2^+ I_{B_n}(s) + \pi I_{C_n}(s) + \gamma_1^- I_{A_n}(-s) + \gamma_2^- I_{B_n}(-s) + \pi I_{C_n}(-s)\right),$$

so that γ is defined on $\partial\Omega \cap B(\mathcal{O}, 1)$ by

$$\gamma(r\cos(\theta), r\sin(\theta)) = \begin{cases} \gamma_1^+ & \text{if } \theta = \alpha,\ 2/4^n < r < 4/4^n \text{ for some } n \in \mathbb{N}, \\ \gamma_2^+ & \text{if } \theta = \alpha,\ 1/4^n < r < 2/4^n \text{ for some } n \in \mathbb{N}, \\ \pi & \text{if } \theta = \alpha,\ r = 4/4^n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } \theta = \alpha,\ r = 2/4^n \text{ for some } n \in \mathbb{N}, \\ \gamma_1^- & \text{if } \theta = -\alpha,\ 2/4^n < r < 4/4^n \text{ for some } n \in \mathbb{N}, \\ \gamma_2^- & \text{if } \theta = -\alpha,\ 1/4^n < r < 2/4^n \text{ for some } n \in \mathbb{N}, \\ \pi & \text{if } \theta = -\alpha,\ r = 4/4^n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } \theta = -\alpha,\ r = 2/4^n \text{ for some } n \in \mathbb{N}. \end{cases}$$

Then

$$\liminf_{r\to 0} \gamma\left(r\cos(\pm\alpha), r\sin(\pm\alpha)\right) = 0,$$
$$\limsup_{r\to 0} \gamma\left(r\cos(\pm\alpha), r\sin(\pm\alpha)\right) = \pi,$$
$$\operatorname{ess}\liminf_{r\to 0} \gamma\left(r\cos(\pm\alpha), r\sin(\pm\alpha)\right) = \gamma_1^{\pm},$$
$$\operatorname{ess}\limsup_{r\to 0} \gamma\left(r\cos(\pm\alpha), r\sin(\pm\alpha)\right) = \gamma_2^{\pm}.$$

Thus $A_I^{\pm}(b) \ge b\cos(\gamma_2^{\pm})$ and $A_S^{\pm}(b) \le b\cos(\gamma_1^{\pm})$.

Let $b \in (0, 1)$ be fixed for now. If we set $\epsilon_j = 1/(b4^j)$ $(j \in \mathbb{N})$, then $\gamma_j(s) = \gamma(s/b)$ and so

$$A_S^+(b) \ge \lim_{j\to\infty} \int_0^b \cos(\gamma_j(s))\,ds = b\int_0^1 \cos(\gamma(s))\,ds = b\left(\tfrac{2}{3}\cos(\gamma_1^+) + \tfrac{1}{3}\cos(\gamma_2^+)\right),$$

and, if we set $\epsilon_j = 2/(b4^j)$ $(j \in \mathbb{N})$, then $\gamma_j(s) = \gamma(s/(2b))$ and so

$$A_I^+(b) \le \lim_{j\to\infty} \int_0^b \cos(\gamma_j(s))\,ds = 2b\int_0^{\frac{1}{2}} \cos(\gamma(s))\,ds = b\left(\tfrac{1}{3}\cos(\gamma_1^+) + \tfrac{2}{3}\cos(\gamma_2^+)\right);$$

similar estimates hold on $\partial^-\Omega$. Now suppose (η_j) is any decreasing sequence in $(0, 1)$ converging to zero. For each $j \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $\frac{1}{4} \le 4^{k-1}\eta_j b < 1$ and, since γ is piecewise constant, a direct calculation shows that $\int_0^b \cos(\gamma^{\pm}(\eta_j s))\,ds = b\int_0^1 \cos(\gamma^{\pm}(\eta_j bs))\,ds$ equals

$$\begin{cases} b\left(\dfrac{1}{6\cdot 4^{k-1}\eta_j b}\cos\gamma_1^{\pm} + \left(1-\dfrac{1}{6\cdot 4^{k-1}\eta_j b}\right)\cos\gamma_2^{\pm}\right) & \text{if } \frac14 \le 4^{k-1}\eta_j b < \frac12, \\ b\left(\left(1-\dfrac{1}{3\cdot 4^{k-1}\eta_j b}\right)\cos\gamma_1^{\pm} + \dfrac{1}{3\cdot 4^{k-1}\eta_j b}\cos\gamma_2^{\pm}\right) & \text{if } \frac12 \le 4^{k-1}\eta_j b < 1. \end{cases}$$

The minimum occurs when $4^{k-1}\eta_j b = \frac12$ and the minimum of $\int_0^b \cos(\gamma^{\pm}(\eta_j s))\,ds$ is $b\big(\frac13\cos(\gamma_1^{\pm}) + \frac23\cos(\gamma_2^{\pm})\big)$. The maximum occurs when $4^{k-1}\eta_j b = \frac14$ and the maximum of $\int_0^b \cos(\gamma^{\pm}(\eta_j s))\,ds$ is $b\big(\frac23\cos(\gamma_1^{\pm}) + \frac13\cos(\gamma_2^{\pm})\big)$. Thus

$$A_I^{\pm}(b) = b\big(\tfrac13\cos(\gamma_1^{\pm}) + \tfrac23\cos(\gamma_2^{\pm})\big) \tag{22}$$

and

$$A_S^{\pm}(b) = b\big(\tfrac23\cos(\gamma_1^{\pm}) + \tfrac13\cos(\gamma_2^{\pm})\big). \tag{23}$$

In these examples, we have the same essential limits inferior and superior at $\mathcal{O}$ and yet $A_I^{\pm}$ and $A_S^{\pm}$ behave differently. In Example 4, we have the "extreme values" (21); the "effective" contact angles in (a) and (b) of Corollary 3 are $\gamma_2^{\pm}$ and in (c) and (d) of Corollary 3 are $\gamma_1^{\pm}$. On the other hand, in Example 5, we have the "intermediate values" (22) and (23). For Example 5, the "effective" contact angles in (a) and (b) of Corollary 3 are $\sigma_2^{\pm}$ and in (c) and (d) of Corollary 3 are $\sigma_1^{\pm}$, where $\sigma_1^{\pm}, \sigma_2^{\pm} \in [0,\pi]$ satisfy

$$\cos\sigma_1^{\pm} = \tfrac23\cos\gamma_1^{\pm} + \tfrac13\cos\gamma_2^{\pm} \quad\text{and}\quad \cos\sigma_2^{\pm} = \tfrac13\cos\gamma_1^{\pm} + \tfrac23\cos\gamma_2^{\pm}.$$

If f is a bounded solution of (1) satisfying (2) on $\partial^{\pm}\Omega^*\setminus\{\mathcal{O}\}$ which is discontinuous at $\mathcal{O}$ and $Rf(\theta)$ exists for all $\theta \in (-\alpha,\alpha)$, then bounds on the sizes β^+ and β^- of side fans can be computed using Corollary 3; the lower bounds on the sizes of these side fans differ between these two examples.

4. Comments and extensions

The last section of [Lancaster and Siegel 1996] dealt with extensions of (1) to equations of prescribed mean curvature. Consider the prescribed mean curvature contact angle problem

$$Nf = 2H(\cdot, f) \quad \text{in } \Omega, \tag{24}$$

$$Tf\cdot\boldsymbol{\nu} = \cos\gamma \quad \text{a.e. on } \partial\Omega. \tag{25}$$

Suppose $f \in C^2(\Omega)$ satisfies (24) and (25) and also suppose the following conditions hold:

(i) $\sup_{x\in\Omega}|f(x)| < \infty$ and $\sup_{x\in\Omega}|H(x, f(x))| < \infty$.

(ii) $H(x, y, t)$ is weakly increasing in t for each $(x, y) \in \Omega$.

Using [Echart and Lancaster 2017], we see that Theorems 1 and 2 continue to hold for solutions f as above; the argument is the same as that in [Lancaster and Siegel 1996].

One might ask if the case considered in Theorem 2 is of "physical interest." Is it possible for the contact angle to fail to have a limit at the corner $\mathcal{O}$? In a sense this is a silly question since, at a small enough scale, the macroscopic description of a capillary surface becomes meaningless. On the other hand, one sometimes uses devices (e.g., homogenization) to obtain useful macroscopic information from knowledge of "small scale" properties. An experiment which might be of some interest would be to form a vertical wedge consisting of two planes of glass which have been coated in increasing narrow vertical strips with a nonwetting substance (e.g., paraffin) as the edge at which the two planes meet is approached; this would approximate the situation considered in Theorem 2 and one wonders if there is a "effective" contact angle at the corner which is larger than that for glass and smaller than that for paraffin.

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Received January 17, 2017. Revised June 19, 2017.

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PACIFIC JOURNAL OF MATHEMATICS

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 292 No. 2 February 2018