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To Robert Steinberg, a cherished teacher, colleague, and friend

Let A be a central simple algebra over a number field K . We study the question of which integers of K are reduced norms of integers of A . We prove that if K contains an integer that is the reduced norm of an element of A but not the reduced norm of an integer of A , then A is a totally definite quaternion algebra over a totally real field (i.e., A fails the *Eichler condition*).

1. Introduction

Let A be a central simple algebra over a number field K . Write $\text{Norm}(\cdot)$ for the reduced norm from A to K . If x is an integer in A , then clearly $\text{Norm}(x)$ lies in R , the ring of integers of K . It is also clear that x must be positive at the real primes of K at which A is ramified. Suppose that $m \in R$ satisfies this property, and so m is a norm from A (see Theorem 2). If m is not the reduced norm of an integer of A , we call m an *outlier for A* (this terminology is not standard).

The main result of this paper (combining Theorem A and Lemmas 6 and 7) is that if K contains an outlier for A , then K is totally real, A is a quaternion algebra over K , and A is totally definite. (One says in this case that A fails the Eichler condition).

We also prove a theorem of Deligne in Section 8 (because we couldn't find a proof in the literature), which states that if $n \geq 2$ is an integer, and E_1, \dots, E_n and F_1, \dots, F_n are supersingular elliptic curves defined over an algebraic closure of the finite field $GF(p)$, the field of p elements, then

$$E_1 \times \cdots \times E_n \cong F_1 \times \cdots \times F_n.$$

The main ingredient is Eichler's theorem on the uniqueness of a maximal order in a central simple algebra in which Eichler's condition holds. We also exploit the known fact that the endomorphism algebra of such an E_i is a maximal order in the quaternion algebra A_p over the rational field \mathbb{Q} ramified at p and ∞ and unramified everywhere else (and every maximal order arises in this context). Using this connection also allows one to interpret outliers in \mathbb{Q} for A_p as positive integers

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m for which no supersingular elliptic curve defined over the algebraic closure of $GF(p)$ has an endomorphism of degree m .

2. Notation and terminology

Throughout this paper, K is a number field, R its ring of integers, and A is a *central simple algebra* over K . By definition, A is a finite-dimensional algebra over K , the center of A is equal to K , and A has no nonzero 2-sided ideals. Equivalently, $A \otimes_K \bar{K}$ is isomorphic to the matrix algebra $M_n(\bar{K})$, where \bar{K} denotes an algebraic closure of K . For basic facts about central simple algebras, see [Pierce 1982].

The positive integer n is the *degree* of A . A central division algebra D is a central simple algebra, as is $M_k(D)$ for any k , and conversely every central simple algebra over K is of this form by Wedderburn's Theorem [Weil 1967, Chapter IX, §1, Proposition 2].

A division algebra of degree $n = 2$ is called a *quaternion algebra*.

If L is a field extension of K , then $A \otimes_K L$ is a central simple algebra over L . If $A \otimes_K L$ is isomorphic to $M_n(L)$ then L is said to *split* A .

Let M denote the set of places of the number field K . For each place $v \in M$, $A_v := A \otimes_K K_v$ is a central simple algebra over the completion K_v . By Wedderburn's theorem, it is a ring of matrices over a local division ring D_v central over K_v . We set $n_v = \text{degree}(D_v)$; n_v is called the *local degree*. A is said to be *split* at v if K_v splits A ($n_v = 1$); otherwise it is *ramified* at v ($n_v > 1$). A key fact is that a central simple algebra over K splits at all but finitely many places v of K .

We have the following splitting criterion:

Lemma 1 [Reiner 1975]. *Let A be a central simple algebra over the number field K . A finite extension L of K splits A if and only if, for each place v of K and for each extension w of v to L , the local dimension $[L_w : K_v]$ is a multiple of the local degree n_v .*

Note that to determine, using Lemma 1, whether a given finite extension L over K splits A , it is enough to check the stated condition at the finite set of places v of K where A is ramified.

The notion of reduced norms in a central simple algebra A is bound up with the two notions of subfields and splitting fields. A field extension L of K is a subfield of A if L embeds in A ; a maximal subfield of A is a maximal such. All maximal subfields of A have dimension $n = \text{degree}(A)$ over K . A maximal subfield of A is a splitting field for A , and conversely every n -dimensional splitting field for A embeds in A as a maximal subfield [Reiner 1975, Chapter 1, Section 7]. When A is a quaternion algebra, this translates as: maximal subfields of A are quadratic over K , and quadratic splitting fields of A embed in A . We will use this association later.

If A is a central simple algebra over K , L a maximal subfield of A , and $x \in L$, then the *reduced norm* $\text{Norm}(x)$ is the ordinary field norm from L to K . This notion is independent of the choice of L , or of the embedding of L into A . By norm we will always mean reduced norm, and the notation will be $\text{Norm}(x)$. In particular, for $a \in K$, $\text{Norm}(a) = a^n$. The usual property holds: $\text{Norm}(xy) = \text{Norm}(x)\text{Norm}(y)$ for $x, y \in A$, whether or not x and y commute. It follows that $\text{Norm}(ax) = a^n \text{Norm}(x)$ for $a \in K$.

An element $a \in A$ is an *integer* if the monic irreducible polynomial of a over K has coefficients in R . Sums and products of commuting integers are integers, but, as we shall see later, products of integers need not be integers.

Suppose A is central simple over K of degree n . Which elements of K are reduced norms of elements of A ? The answer is given by the theorem of Hasse, Maass and Schilling; see [Reiner 1975, p. 289]:

Theorem 2 (Hasse–Maass–Schilling). *An element m of K is a reduced norm of an element of A if and only if m is positive at every real place of K at which A is ramified.*

For convenience we will call this the HMS theorem. Note that there is no condition at the complex places of K , at the finite places of K , or at the real places of K where A does not ramify.

Suppose $m \in R$ and m is a norm in A . It need not happen that m is the norm of an integer of A . We will call $m \in R$ an *outlier* if m is a norm in A but not the norm of an integer. Equivalently, m is not the norm of an element of any maximal order. We will be concerned with the existence of, and properties of, outliers.

If K is a number field, we say K is *totally real* if K_v is real at all the infinite places v of K . If K is totally real, m in K is *totally positive* if the real number m_v is > 0 at all the infinite places v of K . The m in K for which $m_v > 0$ at the real places of A that ramify are, by Theorem 2, the reduced norms of elements of A , and conversely.

We recast the identification of outliers in terms of Lemma 1. Suppose A is central simple over K of degree n , R the ring of integers of K .

Lemma 3. *Suppose $m \in R$ is a norm in A . Then m is **not** an outlier if and only if there is a monic irreducible polynomial $f(t) \in R[t]$ such that*

- (1) *We have $f(0) = (-1)^n m$.*
- (2) *For each place v of K , let $f(t) = \prod f_i(t)$ be the factorization of $f(t)$ into irreducible monic factors in $K_v[t]$. Then each $d_i = \text{degree}(f_i)$ is a multiple of the local degree $n_v(A)$.*

Proof. Let $L = K(\alpha)$ be the root field of f . Then $[L : K] = n$ since f is irreducible, and α is an integer since $f \in R[t]$ is monic. The first condition says that the norm of

α is m . The second condition, by Lemma 1, says that L splits A , and so L embeds in A since its dimension is n . Then the reduced norm of α is m . The other direction of Lemma 3 is clear. \square

Corollary 4. *Suppose A and B are central simple algebras over K of the same degree n , and that the local degree $n_v(A)$ divides the local degree $n_v(B)$ for all places v of K . Then any outlier m of A is a priori also an outlier of B . In particular, if B has no outliers then A has no outliers.*

Proof. The polynomial requirements of Lemma 3 for B are more restrictive than those for A . \square

In Section 3 we review maximal orders in the central simple algebra A and recall Eichler's condition. In Section 4 we prove that when Eichler's condition is satisfied, then there are no outliers. In other words, if A has outliers then A is a quaternion algebra over a totally real number field K , and all real places of K are ramified in A . However, this condition is sufficient but not necessary: there are definite quaternion algebras over totally real number fields that have no outliers. We remark that there is no logical relation between having outliers and having a unique (up to conjugacy) maximal order; neither condition implies the other. In Section 5 we study quaternion algebras over the field of rational numbers \mathbb{Q} . We particularly study definite quaternion algebras ramified at a single finite prime. We write A_r for the definite quaternion algebra over \mathbb{Q} unramified away from the places ∞ and r . We show, for example, that if A_r has an outlier then it has an outlier less than an explicit bound $(r^2/16)$.

We give heuristic evidence that for infinitely many r , A_r has no outliers, as well as examples when chosen square-free integers are outliers.

We are grateful to Joel Rosenberg for many discussions about the contents of this paper, and for posing the questions which started us on this research.

3. Maximal orders

Let K be a number field, R its ring of integers, and A a central simple algebra over K . A subring O of R that contains 1, is finitely generated as an R module, and that contains a basis of A over K is called an *order* of A . Any order O of A is a projective R -module of rank equal to n , the degree of A over K . A *maximal order* of A is an order which is maximal with respect to containment. Maximal orders are isomorphic if and only if they are conjugate, so we will speak of conjugacy classes of maximal orders. All elements of a maximal order are *integral* over R , and every integral element of A is contained in some maximal order.

It is known that the number of maximal orders of A , up to conjugacy by an element of A , is finite. Let $\{O_1, \dots, O_t\}$ be a set of representatives.

For any given maximal order O of A , let $I(O)$ be the group of two-sided fractional ideals of O modulo principal two-sided fractional ideals. Set $i(O) = |O(I)|$. It is known that each $i(O)$ is finite although the cardinalities $i(O_1), \dots, i(O_t)$ may be distinct. Their sum $c := i(O_1) + \dots + i(O_t)$ turns out to be equal to the number of fractional left ideals of O modulo principal fractional left ideals for any maximal order O .

The terminology is that t is called the type number and c is called the class number. We have just seen that the type number is at most the class number.

Let A be a CSA over a number field K . Consider the following three conditions:

- (1) A is a quaternion algebra.
- (2) The field K is totally real.
- (3) A is ramified at every infinite place of K .

It is customary to say that A *fails* the Eichler condition when all three conditions hold. For example, the quaternion algebra A_p over \mathbb{Q} ramified at p and ∞ , and unramified away from those places, fails the Eichler condition.

This description can be refined if the Eichler condition holds. Assume now that A satisfies the Eichler condition. Then the $i(O_1), \dots, i(O_t)$ are all equal. In fact each $I(O_i)$ can be identified with an abelian group $I = I(A)$, as do the types T and the classes C . These three abelian groups fit into an exact sequence

$$0 \rightarrow T \rightarrow C \rightarrow I \rightarrow 0.$$

These three groups are related, via the reduced norm map to certain generalizations of the class group of the center K of A , by results of Eichler.

The group C is isomorphic to the group C' of fractional ideals of K modulo principal fractional ideals that can be generated by an invertible element $a \in K$ that is positive at all infinite places of K that ramify in A .

Let n be as usual the square root of $\dim_K(A)$. If \mathfrak{p} is a prime ideal of K that is ramified in A , then at the corresponding finite place v of K , $A \otimes K_v = M_r D'$ for some division algebra D' over K_v and for some r dividing n . The group T is isomorphic to the subgroup T' of C generated by nC and the class of \mathfrak{p}^r for each finite prime \mathfrak{p} (and note that this gives nothing new for the unramified primes since $r = n$).

I is isomorphic to the (abelian, finite) quotient group C'/T' .

4. Higher degree central simple algebras

Let A be a central simple algebra of degree n over the number field K . The main result of this section is:

Theorem A. *If $n > 2$ then A has no outliers.*

We need first a review of the proof of the HMS theorem in order to build a variant that works for integers. A first ingredient is:

Krasner's Lemma. *Let v be a place of K , and $f(t) = t^n + a_1t^{n-1} + \cdots + a_n$ a separable irreducible polynomial in $K_v[t]$. If $g(t) \in K_v[t]$ is close enough to $f(t)$, then g is separable irreducible and $K_v[a] = K_v[b]$, where a is a root of $f(t)$ and b is a root of $g(t)$.*

Eichler's proof of the HMS theorem goes as follows. Let R be the integers of K , and $m \in R$ satisfy the required condition: m is positive at all places v of K which are real and ramified in A . Let S be the set of infinite places of K at which A ramifies. Let S' be a finite set of finite primes of K , including those that ramify in A . We insist that S' be nonempty; if necessary, we include an irrelevant extra prime where A is unramified but where the polynomial constructed below is irreducible. We construct a polynomial

$$(5) \quad f(t) = t^n + c_1t^{n-1} + \cdots + (-1)^n m \in K[t]$$

so that:

- For each $v \in S'$, c_i is close enough to an irreducible polynomial $f_v(t) = t^n + a_1t^{n-1} + \cdots + (-1)^n m \in R_v[t]$ to guarantee f is irreducible in $K_v[t]$. There is such a polynomial [Weil 1967, XI, §3, Lemma 2] but we don't show that here.
- For each $v \in S$, f is close to $f_v(t) = t^n + (-1)^n m$, i.e., each c_i is positive and close to 0. (Note that if any such v exists, then n is necessarily even). This guarantees f_v has no real roots. If A is not ramified at any infinite place of K , then this condition is vacuous.

Since S' is nonempty, f is irreducible in $K[t]$. Let $L = K(\alpha)$ where α is a root of f ; $[L : K] = n$. The first condition on f says that L splits A at the finite primes, and the second condition guarantees that L splits A at the ramified infinite places, since the root field of f must be complex. The sign $(-1)^n$ guarantees that the norm from L to K of α is m . Finally, since L is a splitting field of degree n , then L embeds in A as a maximal subfield, and the reduced norm of α is m .

This is the proof rendered by Eichler, and is the one presented in [Reiner 1975], [Vignéras 1980], and [Weil 1967]. Note that it made crucial use of the weak approximation theorem.

To go further, we use the strong approximation theorem [Weil 1967, Corollary 2, page 70], which better suits our purposes. Let w be a place of K at which A is unramified. Then we can insist that the c_i are in R_v for all $v \neq w$. We call this the strong proof of the HMS theorem. We conclude any $m \in R$ which is positive at all real places of K that ramify in A is the reduced norm of an element α of A that is

integral at all places v of K not equal to w . So if K has a complex place, or a real place that is not ramified in A , then, taking this for w shows that A has no outliers.

Lemma 6. *If A has an outlier then, K is totally real, A is totally definite, i.e., A is ramified at all the real infinite places of K .*

Proof. Let $m \in R$ be a norm in A . If the conditions are not satisfied, then A must have an infinite place w at which A is unramified. We use this extra place in the strong proof of the HMS theorem. Then the polynomial f is in $R[t]$, α is an integer, and m is the norm of an integer. □

Lemma 7. *If A has an outlier then there is a finite place of K that ramifies in A .*

Proof. Suppose A is unramified at all finite places. By Lemma 6, we may assume n is even. Let $m \in R$ be totally positive. The polynomial $t^n + m$ does the trick in the strong proof of HMS. □

We finish the proof of Theorem A. By Lemma 6 we may assume that K is totally real, A is ramified at all real places, $n > 2$ is even, and m is positive at all infinite places. First we treat the finite places. By [Weil 1967, Ch. XI, §3, Lemma 2], for each finite place v , for any n , and for any nonzero m in K there exists a monic degree- n irreducible polynomial $f(t) \in K_v[t]$ with coefficients in R_v such that $f(0) = (-1)^n m$.

Let M_f denote the set of finite places of K that ramify in A . Note that M_f is finite, and for each v in M_f we have $f_v(t)$ as required by Lemma 3, but we have not yet treated the infinite places.

For each $1 \leq k \leq n$, apply the Chinese remainder theorem to the coefficient of t^k in f_v to get a monic polynomial $g(t) \in K[t]$ with $g(0) = m$ and integral coefficients so that each localization $g_v(t)$ at each K_v is close enough to f_v to be irreducible by Krasner's Lemma. We have lifted the required polynomials at the finite primes, but the infinite places are still at bay; there is yet no reason why $g(t)$ has only complex embeddings.

Each $v \in M_f$ lies over some rational prime p_v . Let $N = \prod_{v \in M_f} p_v$ be their product.

Let M_{inf} be the set of real places of K that ramify in A . For any $v \in M_{\text{inf}}$ we have a real polynomial g_v which is positive at $-\infty$, ∞ and 0 by construction. Therefore, there is some integer multiple M_v of N so that $g_v(t) + M_v t^2$ is positive everywhere. Let M be the largest of the M_v . Furthermore by replacing M by $N^k M$ for a sufficiently large k , we can insure by Krasner's Lemma again that $g_v(t) + N^k M t^2$ is irreducible at each $v \in M_f$.

The polynomial $f(t) = g(t) + N^k M t^2 \in K[t]$ does the trick: it is monic of degree n , has no real roots, and for each place of K that ramifies in A , each irreducible factor of f_v has degree a multiple of $n_v(A)$. This finishes the proof of Theorem A. □

Note that the coefficient of t^2 was available for modification only because $n > 2$. For quaternion algebras, the coefficient of t^2 is constant equal to 1. We get to that case next.

5. Quaternion algebras

We write \mathbb{Q} for the field of rational numbers and \mathbb{Z} for the ring of integers.

We consider definite quaternion algebras over \mathbb{Q} with special attention to $A_r =$ the definite quaternion algebra ramified at the prime r and unramified at all other finite primes. Of course A_r is also ramified at ∞ , and so at all infinite places. The simplification here is that the integers which are norms in A_r are exactly the set of positive integers, and so the only issue is whether they are norms of integers. We now investigate how this could happen.

Let m be a positive integer. Let $f(t) = t^2 + bt + m$ with $b \in \mathbb{Z}$. Let $L = \mathbb{Q}(\alpha)$ with $f(\alpha) = 0$. Then L splits A_r if and only if:

- f is r -adically irreducible,
- f has degree 2 at ∞ , i.e., $d = b^2 - 4m < 0$.

When either of the conditions above hold, then f is irreducible and $[L : \mathbb{Q}] = 2$. When they both hold, L embeds in A_r by Lemma 1, $\text{Norm}(\alpha) = m$, and so m is the norm of an integer in A_r . Moreover, m is the norm of an integer if and only if this search succeeds for some $b \in \mathbb{Z}$. There are a finite number of eligible b by the last condition; $|b| < \sqrt{4m}$. Furthermore, b can be assumed to be positive; if α is a root of $t^2 + bt + m$ then $-\alpha$ is a root of $t^2 - bt + m$. Of course $b = 0$ is legitimate as a possibility. We record this in:

Lemma 8. *The positive integer m is the norm of an integer in A_r if and only if there is a polynomial $f(t) = t^2 + bt + m$ satisfying the two conditions above for some $b \in \mathbb{Z}$. It is sufficient to search only in the range $0 \leq b < \sqrt{4m}$.*

For polynomials of the right shape, they are irreducible r -adically if and only if they are irreducible mod r . So when is $m = 2$ an outlier in A_r ? We illustrate the search below, where we assume $r > 2$:

$$\begin{aligned}
 & b = 0 \quad d = -8 \\
 (9) \quad & b = 1 \quad d = -7 \\
 & b = 2 \quad d = -4
 \end{aligned}$$

Of course -8 is an r -adic square if and only if -2 is, and this happens if and only if the Legendre symbol $\left(\frac{-2}{r}\right) = 1$. Similarly, -4 is a square if and only if -1 is. For each of the three conditions in (9), a random prime r satisfies it with probability $\frac{1}{2}$. We conclude:

Theorem 10. *The integer 2 is an outlier in A_r if and only if*

$$(11) \quad \left(\frac{-2}{r}\right) = \left(\frac{-7}{r}\right) = \left(\frac{-1}{r}\right) = 1.$$

By considering the value of $r \pmod{56}$, it follows from the Dirichlet density theorem [Serre 1973, Chapter VI, §4, Theorem 2] that the set of primes r for which this holds has density $\frac{1}{8}$. In particular it is infinite.

We do this once more to determine when 3 is an outlier. The data gives the following list:

$$(12) \quad \begin{array}{ll} b = 0 & d = -12 \\ b = 1 & d = -11 \\ b = 2 & d = -8 \\ b = 3 & d = -3 \end{array}$$

There is a redundancy; -12 is a square if and only if -3 is. We conclude, for $r > 3$:

Theorem 13. *The integer 3 is an outlier in A_r if and only if*

$$(14) \quad \left(\frac{-3}{r}\right) = \left(\frac{-11}{r}\right) = \left(\frac{-2}{r}\right) = 1.$$

The set of primes r for which this holds is infinite and has density $\frac{1}{8}$.

By similar analysis we get, for $r > 6$:

Theorem 15. *The integer 6 is an outlier in A_r if and only if*

$$(16) \quad \left(\frac{-2}{r}\right) = \left(\frac{-3}{r}\right) = \left(\frac{-5}{r}\right) = \left(\frac{-23}{r}\right) = 1$$

The set of primes r for which this holds is infinite and has density $\frac{1}{16}$

Suppose 6 is an outlier for A_r . It does not follow that 2 and 3 are outliers. There may be integral α and β with $\text{Norm}(\alpha) = 2$ and $\text{Norm}(\beta) = 3$, and then $\text{Norm}(\alpha \cdot \beta) = 6$. It might happen that for all such occurrences α and β are in different maximal orders, and $\alpha \cdot \beta$ is not integral. When 6 is minimal as an outlier, this is what had to happen. This can be quantified; we state without proof:

Theorem 17. *A_r has the property that 2 and 3 are not outliers and 6 is an outlier if and only if $-2, -3, -5, -23$ are squares mod r and either*

$$(18) \quad \left(\frac{-1}{r}\right) = -1 \quad \text{or} \quad -1 \text{ is a square mod } r \text{ and } 11, 7 \text{ are nonsquares mod } r$$

The set of primes r for which this holds is infinite and has density $\frac{5}{128} = \left(\frac{1}{16}\right)\left(\frac{1}{2} + \frac{1}{8}\right)$.

We have not yet determined all outliers in A_r , nor have we answered whether they are infinite when nonempty. We need two results to prepare for this. We take on the second issue first. Since the next result holds more generally than for the A_r , we state it in full generality. In all of the following, the symbol (a, b) stands for the quaternion algebra over some ground field generated by i and j where $i^2 = a$, $j^2 = b$, $ij = -ji$.

Theorem 19. *Let A be a definite quaternion algebra over \mathbb{Q} ramified at the finite prime r . If m is a positive integer, then m is an outlier for A if and only if mr^2 is also.*

Proof. For the easy direction: if $\text{Norm}(\alpha) = m$ with α an integral element of A , then $\text{Norm}(r \cdot \alpha) = mr^2$. We need to show conversely that when mr^2 is the norm of an integer, so is m .

Let O be any maximal order of A . It is enough to show that whenever mr^2 is a norm of an element α of O , then $\alpha/r \in O$. The completion of O at r is the norm form of the unique quaternion algebra D over \mathbb{Q}_r . By [Serre 1979], D has the form (a, r) where a is an appropriate nonresidue mod r . When r is odd, any nonresidue will do, whereas when $r = 2$, $a = -3$ will do (in all cases, \sqrt{a} determines the unique unramified quadratic extension). The norm form for this algebra is:

$$(20) \quad F = x^2 - ay^2 - r(z^2 - aw^2)$$

Assume that $F(x, y, z, w) = mr^2$. It follows that $x^2 - ay^2 \equiv 0 \pmod{r}$. As a is a nonresidue, this forces x and y to be $\equiv 0 \pmod{r}$. But then $x^2 - ay^2 \equiv 0 \pmod{r^2}$, and so $r(z^2 - aw^2)$ is $0 \pmod{r^2}$. It follows that $z^2 - aw^2 \equiv 0 \pmod{r}$, so that z and w are $0 \pmod{r}$. Now all four coefficients x, y, z, w of α are divisible by r . Thus α/r is in O and has norm m . We conclude that whenever $\text{Norm}(\alpha)$ is mr^2 with α in O , then α/r is in O and has norm m . Since this holds for all maximal orders, the lemma is established. \square

Corollary 21. *With A as in Theorem 19, if the set of outliers for A is nonempty, then it is infinite; if m is an outlier for A , then so is mr^{2n} for any positive integer n .*

Remark 22. Corollary 21 allows division by r^2 , but not by r . In fact, if m is an outlier for A_r and relatively prime to r , then mr is not an outlier. The polynomial $t^2 + mr$ is irreducible at r by Eisenstein's criterion, and also irreducible at infinity; it satisfies the requirements of Lemma 3.

We need a bound up to which we can check for outliers not governed by Theorem 19. We do this for A_r ; the generalizations to definite quaternion algebras will be clear. One more preliminary is necessary.

Lemma 23. *Let $p > 2$ be a prime and m in $GF(p)$ nonzero. Then there exists b in $GF(p)$ such that $b^2 - 4m$ is a nonsquare mod p .*

Proof. Suppose not. Then, for every b , $b^2 - 4m$ is a square. But then $(b^2 - 4m) - 4m$ is a square and by induction $b^2 - 4mj$ is a square for all j . By our hypotheses $4m$ is invertible in $GF(p)$ so all elements of $GF(p)$ are squares, contradiction. \square

We can now establish a bound for A_r .

Theorem 24. *Suppose $r > 2$ is prime and m is a positive integer coprime to r . Set $C(r) = r^2/16$. If $m > C(r)$, then m is not an outlier for A_r .*

Proof. By Lemma 23, we choose an integer b such that $b^2 - 4m$ is a nonsquare mod r . We are free to assume of course that $b < r/2$. Set $f = t^2 + bt + m$, and $d = b^2 - 4m$. One checks that the bounds on b and m say that $d < 0$, so f is irreducible at infinity. Since d is a nonresidue at r , f is also irreducible in $\mathbb{Q}_r[t]$. Then f satisfies the requirements of Lemma 3, and so m is the norm of an integer. \square

Remark 25. Theorem 24 gives an effective strategy for finding all outliers in A_r . One checks all m in the interval $[0, C(r)]$ using Lemma 3. For $m > C(r)$: m is not an outlier if m is not divisible by r . If m is divisible by r to the first power, then m is not an outlier by Remark 22. If m is divisible by higher powers of r , then successive uses of Theorem 19 gets us to the case of first power or the range $[0, C(r)]$.

Here is one case where all outliers can be determined:

Corollary 26. *If $r = 67$, then the only outliers for A_r are of form $3 \cdot r^{2n}$, $n = 1, 2, 3, \dots$*

Proof. One checks in the range $[0, C(r)]$ that the only outlier is 3 using Lemma 3 for each possible m . Then Remark 25 does the rest. \square

Remark 27. Note that this corollary says that division by r^2 is not always possible when r is not a ramified prime. In A_{67} , $12 = 3 \cdot 2^2$ is the norm of an integer, but 3 is not, so division by the square of the unramified prime 2 is not possible.

An effective bound for more general definite quaternion algebras is not difficult. Suppose A is a quaternion algebra central over \mathbb{Q} ramified at infinity and the finite primes comprising a set S . Let C be the product of the finite ramified primes of A , and $M = C^2/16$. Then

Theorem 28. *M is an effective bound for determining all the outliers for A .*

Proof. The proof is exactly as in Theorem 24 and Remark 25. \square

The symbol $B = (-58, -17)$ over \mathbb{Q} is ramified at infinity and the finite primes $S = \{2, 17, 29\}$. Using Theorem 28 one can show:

Corollary 29. *The outliers for B are the set*

$$\{10r^{2n} : n = 1, 2, 3, \dots; r \text{ a product of elements of } S\}.$$

The minimal outlier of B is 10. Therefore, there are integers α and β in B with $\text{Norm}(\alpha) = 2$ and $\text{Norm}(\beta) = 5$. Whenever this happens, the product $\alpha\beta$ is not integral.

The appearance of 6 and 10 in this context is general, as seen in the next theorem.

Theorem 30. *Let m be a positive integer that is not a square. Then there are infinitely many primes r such that $\{m \cdot r^{2n} : n \geq 1\}$ are outliers for A_r .*

Proof. For b in the range $[0, C]$, $C = \sqrt{4m}$, and $d = b^2 - 4m$, we must have the Legendre symbol $\left(\frac{d}{r}\right)$ equal to 1; the Chebotarev density theorem says there are infinitely many such primes r . In fact their density is $1/2^s$ for some appropriate integer s . \square

6. Open questions

We begin with:

Are there infinitely many rational primes r such that A_r has no outliers?

Heuristically, the answer is yes. Computer searches for small bounds show that A_r has no outliers a little more than half the time.

We have seen that the set of primes r for which $m = 2$ is an outlier for A_r has density $\frac{1}{8}$. Similarly, the set of primes for which $m = 3$ is an outlier for A_r has density $\frac{1}{8}$. Adding together these probabilities for small m appears to give something like density 0.7; this is roughly the probability that neither 2 nor 3 is an outlier. However for large m , the density of primes r for which m is an outlier in A_r should be something like $2^{-c\sqrt{4m}}$, since we are asking that the floor of $\sqrt{4m} + 1$ numbers are all squares r -adically and some constant c is required because these numbers may not be linearly independent in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$. However the sum $\sum_{m>0} 2^{-c\sqrt{4m}}$ converges. Therefore we cannot distinguish whether our set is finite or infinite.

Another interesting question concerns totally definite quaternion algebras over totally real number fields. Do they have outliers? Sometimes? Often?

We have worked out only one example. Let $B = (-1, -7)_K$ where K is the real subfield of seventh roots of unity. Then B has no outliers. The argument is technical, so we will not reproduce it here; it requires a detailed study of units, totally positive units, class number, and the establishment of a bound as in Theorem 24; the bound is 1792. However, when $K = \mathbb{Q}(\sqrt{2})$, the same algebra tensored up to K does have outliers. Thus, restriction maps may or may not preserve the property of having no outliers.

On the other hand, let A be the algebra $(-1, -67)$ over \mathbb{Q} ; by Corollary 26, 3 is an outlier for A . If $K = \mathbb{Q}(\sqrt{67})$, then $A \otimes_{\mathbb{Q}} K$ is ramified at only the infinite places of K , and so by Lemma 7 has no outliers. Thus the restriction map may also fail to preserve the property of having outliers.

The last remark can be generalized. From Lemma 7, if A is a quaternion algebra over \mathbb{Q} , then there is a real quadratic field K such that:

- $A \otimes_{\mathbb{Q}} K$ is a division ring,
- $A \otimes_{\mathbb{Q}} K$ has no outliers.

7. Application to supersingular elliptic curves and surfaces

We review the connection between supersingular elliptic curves in characteristic r and maximal orders in A_r , where A_r is the definite quaternion algebra ramified at ∞ and r and unramified away from these places.

Let E be a supersingular elliptic curve defined over an algebraic closure Ω of $GF(r)$, and write $\text{End}(E)$ for its endomorphism ring. Then $\text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to A_r . Under this isomorphism, $\text{End}(E)$ is a maximal order in A_r , and, conversely, any maximal order M of A_r is isomorphic to $\text{End}(E)$ for some E .

Furthermore the norm of an endomorphism $\phi : E \rightarrow E$ is, under this isomorphism, equal to the reduced norm of the corresponding $m \in M$.

The statement “ m is an outlier for A_r ,” translates to: no supersingular elliptic curve defined over Ω has an endomorphism of degree m .

So we see, for example, that for every integer $m > 1$ there are infinitely many primes p such that no supersingular elliptic curve defined over Ω has an endomorphism of degree m .

Next we turn to products of supersingular elliptic curves.

Corollary 31. *Let E be a supersingular elliptic curve defined over an algebraic closure of $GF(p)$, and set $A = E^g$ for $g \geq 2$ an integer. Then the abelian variety A has an endomorphism of degree m for every positive integer m .*

Remark 32. Here we are considering all endomorphisms of A , not just those that preserve the obvious principal polarization.

Proof 1. $\text{End}(A)$ (which happens to equal $S = \text{Mat}_g(M)$) is a maximal order in the central simple algebra $\text{Mat}_g(A_r)$ of dimension $4g^2$ over \mathbb{Q} . Eichler’s methods, as outlined in Section 3, imply that S is the unique maximal order up to conjugacy since $g > 1$. Therefore, by Theorem A, m is a reduced norm of an element $\alpha \in \text{End}(S)$. However, reduced norm is in this case equal to the degree of the map α . □

Our second proof of Corollary 31 uses a well-known theorem of Deligne, which we state below. As we have not found an adequate proof in the literature, for the reader’s convenience we include one in the next section.

Theorem 33 (Deligne). *Let p be a prime and let $n \geq 2$ be an integer. If E_1, \dots, E_n and F_1, \dots, F_n are supersingular elliptic curves defined over an algebraic closure of $GF(p)$, then*

$$E_1 \times \dots \times E_n \cong F_1 \times \dots \times F_n$$

Proof 2 of Corollary 31. It turns out it is enough, using Deligne’s result, to show that for each rational prime ℓ that $A = E^n$ has an endomorphism of degree ℓ .

There does exist for each ℓ an isogeny of supersingular elliptic curves $\phi : E \rightarrow E'$ of degree ℓ (for $\ell = p$, the Frobenius has degree p and for $\ell \neq p$ mod out by any subgroup H of order ℓ). However, Deligne’s theorem gives an isomorphism

$$\psi : E^n \cong E' \times E^{n-1}.$$

Thus the composite $\psi^{-1} \circ (\phi \times \text{id}^{n-1})$ furnishes the desired endomorphism of degree ℓ . □

We are grateful to Bruce Jordan for suggesting the second proof of Corollary 31.

8. Proof of Deligne’s theorem

In this section we prove Theorem 33. For p a prime, let Ω denote an algebraic closure of $GF(p)$.

Remark 34. (1) It would suffice by induction to prove the theorem for $n = 2$ (although we will not use this remark).

(2) It will suffice to show (by transitivity of isomorphism) that $F_1 \times \cdots \times F_n \cong E^n$ for some particular supersingular elliptic curve E defined over Ω .

The remainder of this section is devoted to the proof of Theorem 33.

Note that $\Delta = \text{End}(E)$ is a maximal order in the quaternion algebra $A_p = \Delta \otimes_{\mathbb{Z}} \mathbb{Q}$. The left Δ -module $\text{Hom}(F_1 \times \cdots \times F_n, E)$ being a projective module of rank $n \geq 2$ is free by [Reiner 1975][Corollary 35.11 (iv)] (by the results of Section 3 since the Eichler condition holds for $M_n(\Delta)$, and since visibly any ray class field over \mathbb{Q} is trivial). This is the key point in the proof.

Let ϕ_1, \dots, ϕ_n be a basis. The freeness means that any homomorphism ψ from $F_1 \times \cdots \times F_n$ to E is uniquely a sum

$$\psi = \delta_1 \circ \phi_1 + \cdots + \delta_n \circ \phi_n$$

for some $\delta_1, \dots, \delta_n$ in Δ , noting that the Δ -action on $\text{Hom}(A, E)$, is composition of functions. Setting $\Phi = (\phi_1, \dots, \phi_n)$, we have constructed a homomorphism

$$\Phi : F_1 \times \cdots \times F_n \rightarrow E^n,$$

and to finish the proof of the theorem it will suffice to prove that Φ is an isomorphism.

Let K be the kernel of Φ . If Φ is not an isomorphism, then K is nontrivial, and therefore some projection $\pi_i(K)$ is nontrivial in F_i . Let $\rho : F_i \rightarrow E$ be a homomorphism, and set $\psi : F_1 \times \cdots \times F_n \rightarrow E$ to be $\psi(x_1, \dots, x_n) = \rho(x_i)$. It follows that ρ and therefore any homomorphism from F_i to E must kill $\pi_i(K)$.

Lemma 35. *There is a supersingular elliptic curve E_0 defined over $GF(p)$.*

Proof. Let E be a supersingular elliptic curve defined over Ω . Then there is only one isogeny of order p from E to another elliptic curve, namely the Frobenius isogeny $Fr : E \rightarrow E^{(p)}$. It follows that E has an endomorphism of degree p if and only if E is defined over $GF(p)$.

Consider now the element $\sqrt{-p}$ in the quadratic number field $L = \mathbb{Q}(\sqrt{-p})$. It has norm p and is integral. As L splits A_p , it embeds in A_p . So $\mathbb{Z}[\sqrt{-p}]$ a fortiori embeds and thus is contained in a maximal order O of A_p . The usual norm on $\mathbb{Z}[\sqrt{-p}]$ is equal to the restriction of the reduced norm under the embedding. In the correspondence between maximal orders of A_p and supersingular elliptic curves over Ω , the elliptic curve corresponding to O is thus defined over $GF(p)$. \square

The proof of Deligne’s theorem will be completed by the following lemma.

Lemma 36. *Let E and F be supersingular elliptic curves over Ω . Then the intersection, as subgroup schemes of $\ker(\phi)$ as ϕ ranges over $\text{Hom}(E, F)$, is trivial.*

Remark 37. (1) It will suffice to find a collection of isogenies from E to F whose degrees are coprime.

(2) It will suffice to prove the lemma for a fixed elliptic curve E_0 (and F varying), then precomposing with the dual isogenies from E_0 to E coming from (1).

Proof.

First of all we know that $\text{Hom}(E, F)$ is nonzero. If $O = \text{End}(E)$ is the maximal order of $\text{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$ corresponding to E , then O has an ideal whose right order is equal to the maximal order corresponding to F and this furnishes a nonzero isogeny. So $\text{Hom}(E, F)$ is a finitely generated projective left module over O (and not the zero module). Let K denote the intersection (as subgroup schemes) of $\ker(\phi)$ as ϕ ranges over $\text{Hom}(E, F)$. We have just showed that K is finite. Among all isogenies from E to F , let ϕ be one of least degree. Let ℓ be a prime dividing $\text{deg}(\phi)$, hence also the order of K . We first treat the somewhat easier case $\ell \neq p$. If $\phi(E[\ell]) = 0$ then $(1/\ell)\phi$ is a nonzero isogeny from E to F of smaller degree, a contradiction. Thus $W = \ker(\phi) \cap E[\ell]$ is one-dimensional. However, $\text{End}(E)$ acts transitively on the one-dimensional subspaces of $E[\ell]$. Thus there is a σ in $\text{End}(E)$ that does not fix W . Then $\phi + \phi \circ \sigma$ is an isogeny from E to F of order prime to ℓ .

We finish the proof of the lemma in the case $\ell = p$.

It is enough by transitivity to assume (by Lemma 35) that $E = E_0$ is defined over $GF(p)$. Assume that every isogeny from E_0 to E has degree divisible by p . Let $\phi : E_0 \rightarrow E$ be the nonzero isogeny of least degree. If ϕ has degree divisible by p then ϕ factors through the Frobenius isogeny: $\phi = \psi \circ Fr$ for some $\psi : E_0^{(p)} \rightarrow E$. But since $E_0^{(p)} = E_0$, $\psi : E_0 \rightarrow E$ is an isogeny of degree smaller than $\text{deg}(\phi)$. \square

This finishes the proof of Lemma 36 and of Theorem 33.

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
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