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GLOBAL EXISTENCE AND BLOWUP OF SMOOTH SOLUTIONS OF 3-D POTENTIAL EQUATIONS WITH TIME-DEPENDENT DAMPING

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In this paper, we are concerned with the global existence and blowup of smooth solutions of the 3-D irrotational compressible Euler equation with time-dependent damping

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + p\mathrm{I}_3) = -\alpha(t)\rho u, \\ \rho(0, x) = \bar{\rho} + \varepsilon\rho_0(x), \quad u(0, x) = \varepsilon u_0(x), \end{cases}$$

where $x \in \mathbb{R}^3$, the frictional coefficient $\alpha(t) = \mu/(1+t)^\lambda$ with $\mu > 0$ and $\lambda \geq 0$, $\bar{\rho} > 0$ is a constant, $\rho_0, u_0 \in C_0^\infty(\mathbb{R}^3)$, $(\rho_0, u_0) \not\equiv 0$, $\rho(0, x) > 0$, $\operatorname{curl} u_0 \equiv 0$, and $\varepsilon > 0$ is sufficiently small. For $0 \leq \lambda \leq 1$, we show that there exists a global $C^\infty([0, \infty) \times \mathbb{R}^3)$-smooth solution (ρ, u) by introducing and establishing some uniform time-weighted energy estimates of (ρ, u), while for $\lambda > 1$, in general, the smooth solution (ρ, u) blows up in finite time. Therefore, $\lambda = 1$ appears to be the critical value for the global existence of small amplitude smooth solution (ρ, u).

1. Introduction

In this paper, we are concerned with the global existence and blowup of smooth solutions of the three-dimensional irrotational compressible Euler equations with time-dependent damping

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + p\mathrm{I}_3) = -\alpha(t)\rho u, \\ \rho(0, x) = \bar{\rho} + \varepsilon\rho_0(x), \quad u(0, x) = \varepsilon u_0(x), \end{cases} \tag{1-1}$$

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where $x=(x_1,x_2,x_3)$, ρ, $u=(u_1,u_2,u_3)$, and p stand for the density, velocity, and pressure, respectively, I_3 is the 3×3 identity matrix, the frictional coefficient $\alpha(t)=\mu/(1+t)^\lambda$ with $\mu>0$ and $\lambda\geq 0$, and $u_0=(u_{1,0},u_{2,0},u_{3,0})$,

$$\operatorname{curl} u_0=(\partial_2u_{3,0}-\partial_3u_{2,0},\partial_3u_{1,0}-\partial_1u_{3,0},\partial_1u_{2,0}-\partial_2u_{1,0})\equiv 0.$$

The equation of state of the gases is assumed to be $p(\rho)=A\rho^\gamma$, where $A>0$ and $\gamma>1$ are constants. Furthermore, $\bar\rho>0$ is a constant, $\rho_0,u_0\in C_0^\infty(\mathbb{R}^3)$, $(\rho_0,u_0)\not\equiv 0$, $\rho(0,x)>0$, and $\varepsilon>0$ is sufficiently small. With respect to the physical background of (1-1), it can be found in [Dafermos 1995].

For $\mu=0$ in $\alpha(t)$, (1-1) is the standard compressible Euler equation. It is well known that C^∞-smooth solution (ρ,u) of (1-1) will in general blow up in finite time. For the extensive literature on blowup results and the blowup mechanism for (ρ,u), see [Alinhac 1999a; 1999b; 1993; Christodoulou 2007; Christodoulou and Miao 2014; Christodoulou and Lisibach 2016; Ding et al. 2016; Hörmander 1997; Sideris 1997; 1985; Speck 2016; Yin and Qiu 1999; Yin 2004] and so on.

For $\lambda=0$ in $\alpha(t)$, it has been shown that (1-1) admits a global C^∞-smooth solution (ρ,u) and the large time behavior of (ρ,u) is governed by a parabolic equation derived by using Darcy's law; see [Dafermos 1995; Hsiao and Serre 1996; Hsiao and Liu 1992; Kawashima and Yong 2004; Nishihara 1997; Pan and Zhao 2009; Sideris et al. 2003; Tan and Guochun 2012; Wang and Yang 2001].

For $\mu>0$ and $\lambda>0$ in $\alpha(t)$, an interesting problem arises: does the C^∞-smooth solution (ρ,u) of (1-1) blow up in finite time or does it exist globally? In this paper, we will systematically study this problem under the assumption of $\operatorname{curl} u_0\equiv 0$. In this case it is not hard to see that $\operatorname{curl} u(t,\cdot)\equiv 0$ for all $t\geq 0$ as long as the smooth solution (ρ,u) of (1-1) exists. Then one can introduce a potential function $\varphi=\varphi(t,x)$ such that $u=\nabla\varphi$ (here and below, $\nabla=\nabla_x$), where the C^∞ scalar function φ has a compact support in x (as $u(t,\cdot)$ has a compact support for any fixed $t\geq 0$ in view of $u_0\in C_0^\infty(\mathbb{R}^3)$ and admits a finite propagation speed which holds for hyperbolic systems). Substituting $u=\nabla\varphi$ into the second equation of (1-1), we obtain

$$\partial_t\varphi+\tfrac12|\nabla\varphi|^2+h(\rho)+\frac{\mu}{(1+t)^\lambda}\varphi=0, \tag{1-2}$$

where $h'(\rho)=c^2(\rho)/\rho$ with $c(\rho)=\sqrt{p'(\rho)}$ and $h(\bar\rho)=0$.

From $h'(\rho)>0$ for $\rho>0$ we have that

$$\rho=h^{-1}\biggl(-\biggl(\partial_t\varphi+\tfrac12|\nabla\varphi|^2+\frac{\mu}{(1+t)^\lambda}\varphi\biggr)\biggr), \tag{1-3}$$

where $\bar\rho=h^{-1}(0)$ and h^{-1} is the inverse function of $h=h(\rho)$.

Substituting (1-3) into the first equation of (1-1) yields

$$\partial_t^2\varphi - c^2(\rho)\Delta\varphi + 2\sum_{k=1}^{3}(\partial_k\varphi)\partial_{tk}^2\varphi + \sum_{i,k=1}^{3}(\partial_i\varphi)(\partial_k\varphi)\partial_{ik}^2\varphi + \frac{\mu}{(1+t)^\lambda}|\nabla\varphi|^2 + \partial_t\Big(\frac{\mu}{(1+t)^\lambda}\varphi\Big) = 0. \tag{1-4}$$

As for the initial data $\varphi(0,x)$ and $\partial_t\varphi(0,x)$ for (1-4): Obviously, $\varphi(0,x) = \varepsilon\varphi_0(x)$, where

$$\varphi_0(x) = \int_{-\infty}^{x_1} u_{1,0}(s,x_2,x_3)\,ds.$$

Note that $\varphi_0 \in C_0^\infty(\mathbb{R}^3)$ in view of $\operatorname{curl} u_0 \equiv 0$ and $u_0 \in C_0^\infty(\mathbb{R}^3)$. Furthermore, from (1-2) we infer that $\partial_t\varphi(0,x) = \varepsilon\varphi_1(x) + \varepsilon^2 g(x,\varepsilon)$, where

$$\varphi_1 = -\Big(\mu\varphi_0 + \frac{c^2(\bar\rho)}{\bar\rho}\rho_0\Big)$$

and

$$g(x,\varepsilon) = -\rho_0^2(x)\int_0^1 \Big(\frac{c^2(\rho)}{\rho}\Big)'\Big|_{\rho=\bar\rho+\theta\varepsilon\rho_0(x)} d\theta - \frac{1}{2}\sum_{i=1}^{3} u_{i,0}^2(x).$$

Notice that $g(x,\varepsilon)$ is smooth in (x,ε) and has compact support in x. Consequently, studying problem (1-1) under the assumption $\operatorname{curl} u_0 \equiv 0$ is equivalent to investigating the problem

$$\begin{cases} \partial_t^2\varphi - c^2(\rho)\,\Delta\varphi + 2\sum_{k=1}^{3}(\partial_k\varphi)\partial_{tk}^2\varphi + \sum_{i,k=1}^{3}(\partial_i\varphi)(\partial_k\varphi)\partial_{ik}^2\varphi \\ \qquad + \frac{\mu}{(1+t)^\lambda}|\nabla\varphi|^2 + \partial_t\Big(\frac{\mu}{(1+t)^\lambda}\varphi\Big) = 0, \\ \varphi(0,x) = \varepsilon\varphi_0(x), \quad \partial_t\varphi(0,x) = \varepsilon\varphi_1(x) + \varepsilon^2 g(x,\varepsilon). \end{cases} \tag{1-5}$$

Here we mention that

$$c^2(\rho) = c^2(\bar\rho) - (\gamma-1)\Big(\partial_t\varphi + \tfrac{1}{2}|\nabla\varphi|^2 + \frac{\mu}{(1+t)^\lambda}\varphi\Big)$$

which follows by direct computation.

We now state the first main result of this paper.

Theorem 1.1 (global existence for $0 \le \lambda \le 1$). *Suppose that* $\operatorname{curl} u_0 \equiv 0$. *If* $\mu > 0$ *and* $0 \le \lambda \le 1$, *then, for* $\varepsilon > 0$ *small enough*, (1-5) *admits a global* C^∞*-smooth solution* φ. *As a consequence*, (1-1) *has a global* C^∞*-smooth solution* (ρ, u) *which fulfills* $\rho > 0$ *and which is uniformly bounded for* $t \ge 0$ *together with all its derivatives.*

Remark. The principal part of the linearization of the equation in (1-5) about $(\rho, \varphi) = (\bar{\rho}, 0)$ is

$$\mathcal{L}(\dot{\varphi}) \equiv \partial_t^2 \dot{\varphi} - c^2(\bar{\rho})\, \Delta\dot{\varphi} + \frac{\mu}{(1+t)^\lambda}\, \partial_t \dot{\varphi} - \frac{\mu\lambda}{(1+t)^{\lambda+1}}\, \dot{\varphi}. \tag{1-6}$$

For the linear operator $\mathcal{L}_0$ with

$$\mathcal{L}_0(\dot{\varphi}) \equiv \partial_t^2 \dot{\varphi} - c^2(\bar{\rho})\Delta\dot{\varphi} + \frac{\mu}{(1+t)^\lambda} \partial_t \dot{\varphi},$$

which appears as part of (1-6), it is shown in [Wirth 2006; 2007] that the large-term behavior of solutions $\dot{\varphi}$ of $\mathcal{L}_0(\dot{\varphi}) = 0$ depends on the value of λ. For $0 \le \lambda < 1$ it is the same as the large-term behavior of solutions of the linear heat equation $\partial_t \dot{\varphi} - c^2(\bar{\rho})\Delta\dot{\varphi} = 0$, while for $\lambda > 1$ it is the same as the large-term behavior of solutions of the linear wave equation $\partial_t^2 \dot{\varphi} - c^2(\bar{\rho})\Delta\dot{\varphi} = 0$. In addition, precise microlocal large-term decay properties of solutions $\dot{\varphi}$ of $\mathcal{L}(\dot{\varphi}) = 0$ have been established in [do Nascimento and Wirth 2015] for a special range of values of λ and μ. It seems to be difficult, however, to apply these microlocal estimates to attack the quasilinear problem (1-5). (In general, those microlocal estimates are useful when treating semilinear damped wave equations; see [D'Abbicco and Reissig 2014; D'Abbicco et al. 2015].)

Remark. For the 1-D Burgers equation with time-dependent damping term

$$\begin{cases} \partial_t w + w \partial_x w = -\dfrac{\mu}{(1+t)^\lambda}\, w, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ w(0, x) = \varepsilon w_0(x), \end{cases} \tag{1-7}$$

where $\mu > 0$ and $\lambda \ge 0$ are constants, $w_0 \in C_0^\infty(\mathbb{R})$, $w_0 \not\equiv 0$, and $\varepsilon > 0$ is sufficiently small, one concludes by the method of characteristics that

$$\begin{cases} T_\varepsilon = \infty & \text{if } 0 \le \lambda < 1 \text{ or } \lambda = 1,\ \mu > 1, \\ T_\varepsilon < \infty & \text{if } \lambda > 1 \text{ or } \lambda = 1,\ 0 \le \mu \le 1, \end{cases}$$

where T_ε is the lifespan of the C^∞-smooth solution w of (1-7). Therefore, $\lambda = 1$ again appears to be the critical value for the global existence of smooth solutions w of (1-7) in the presence of the damping term

$$\frac{\mu}{(1+t)^\lambda}\, w.$$

Remark. The smallness of $\varepsilon > 0$ in Theorem 1.1 is necessary in order to guarantee the global existence of smooth solution (ρ, u). Indeed, as in [Sideris et al. 2003], large amplitude smooth solution of (1-1) may blow up in finite time even for $0 \le \lambda \le 1$. See also Theorem 4.1.

Next we concentrate on the case of $\lambda > 1$. As in [Sideris 1985], introduce the two functions

$$q_0(l) = \int_{|x|>l} \frac{(|x|-l)^2}{|x|} (\rho(0,x) - \bar{\rho})\, dx,$$
$$q_1(l) = \int_{|x|>l} \frac{|x|^2 - l^2}{|x|^3} x \cdot (\rho u)(0,x)\, dx.$$

Before stating our blowup result for problem (1-1) with $\lambda > 1$, we require to introduce a special hypergeometric function $\Psi(a,b,c;z)$, where the constants a and b satisfy $a+b=1$ and $ab = \frac{1}{2}\mu\lambda$, $c \in \mathbb{R}^+$, the variable $z \in \mathbb{R}$, and

$$\Psi(a,b,c;z) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n$$

with $(a)_n = a(a+1)\cdots(a+n-1)$ and $(a)_0 = 1$. It is known from [Erdélyi et al. 1953] that $\Psi(a,b,c;z)$ is an analytic function of z in $(-1,1)$ and $\Psi(a,b,c;0) = \Psi(a+1,b+1,c;0) = 1$. Therefore, there exists a small constant $\delta_0 > 0$ depending on a and b (i.e., μ and λ) such that for $-\frac{1}{2}\delta_0 \le z \le 0$,

$$\tfrac{1}{2} \le \Psi(a,b,1;z),\ \Psi(a+1,b+1,2;z) \le \tfrac{3}{2}. \tag{1-8}$$

Theorem 1.2 (blowup for $\lambda > 1$). *Suppose* $\operatorname{supp}\rho_0, \operatorname{supp} u_0 \subseteq \{x : |x| \le M\}$ *and let*

$$q_0(l) > 0, \tag{1-9}$$
$$q_1(l) \ge 0 \tag{1-10}$$

hold for all $l \in (\widetilde{M}, M)$, *where* $\widetilde{M}$ *is some fixed constant satisfying* $0 \le \widetilde{M} < M$. *Moreover, we assume that there exist two constants* $M_0 \ge \widetilde{M}$ *and* $\Lambda \ge \frac{3}{2}\mu\lambda$ *such that*

$$q_1(l) \ge \Lambda q_0(l), \tag{1-11}$$

holds for all $l \in (M_0, M)$, *where* $M - M_0 < \delta_0$ *and* δ_0 *is given in* (1-8). *If* $\mu > 0$ *and* $\lambda > 1$, *then there exists an* $\varepsilon_0 > 0$ *such that, for* $0 < \varepsilon \le \varepsilon_0$, *the lifespan* T_ε *of* C^∞*-smooth solution* (ρ, u) *of* (1-1) *is finite.*

Remark. It is not hard to find a large number of initial data $(\rho, u)(0,x)$ such that (1-9)–(1-11) are satisfied. For instance, choosing $\rho_0(x) > 0$ and $u_0(x) = x\rho_0(x)\Lambda/\bar{\rho}$, then we get (1-9)–(1-11).

Remark. Sideris [1985] showed the formation of singularities in three-dimensional compressible equations under the assumptions of (1-9)–(1-10). However, in order to prove the blowup result of smooth solution (ρ, u) to problem (1.1) and overcome the difficulty arisen by the time-dependent frictional coefficient $\mu/(1+t)^\lambda$ with $\mu > 0$

and $\lambda > 1$, we pose an extra assumption (1-11) except (1-9)–(1-10), which leads to the nonnegativity of $P(t,l)$ in (3-7) so that an ordinary typed blowup inequalities (3-23)–(3-24) can be established. One can see more details in Section 3.

Let us indicate the proofs of Theorems 1.1 and 1.2. To prove Theorem 1.1, we first introduce the function $\psi = \varphi/(1+t)^\lambda$ which fulfills the second-order quasilinear wave equation

$$\partial_t^2\psi - \Delta\psi + \frac{\mu}{(1+t)^\lambda}\partial_t\psi + \frac{2\lambda}{1+t}\partial_t\psi - \frac{\lambda(1-\lambda)}{(1+t)^2}\psi = Q(\psi, \partial\psi, \partial^2\psi),$$

where $Q(\psi,\partial\psi,\partial^2\psi)$ stands for an error term which is of the second order in $(\psi,\partial\psi,\partial^2\psi)$; $\partial = (\partial_t,\nabla)$. Then, in order to establish the global existence of ψ, we introduce the time-weighted energy

$$E_N(\psi)(t) = \sum_{0\le|a|\le N}\int_{\mathbb{R}^3}((1+t)^{2\lambda}|\partial\Gamma^a\psi|^2 + |\Gamma^a\psi|^2)\,dx,$$

where $N \ge 8$ is a fixed number, $\Gamma = (\Gamma_0,\Gamma_1,\dots,\Gamma_7) = (\partial,\Omega,S)$ with $\Omega = (\Omega_1,\Omega_2,\Omega_3) = x\wedge\nabla$, $S = t\partial_t + \sum_{k=1}^3 x_k\partial_k$, and $\Gamma^a = \Gamma_0^{a_0}\Gamma_1^{a_1}\dots\Gamma_7^{a_7}$. Note that the vector fields Γ which appear in the definition of the energy $E_N(\psi)(t)$ only comprise part of the standard Klainerman vector fields $\{\partial,\Omega,S,H\}$, where $H = (H_1,H_2,H_3) = (x_1\partial_t + t\partial_1, x_2\partial_t + t\partial_2, x_3\partial_t + t\partial_3)$. This is due to the fact that the equation in (1-5) is not invariant under the Lorentz transformations H in view of the presence of the time-dependent damping. By a rather technical and involved analysis of the resulting equation for ψ, we eventually show that $E_N(\psi)(t) \le \frac{1}{2}K^2\varepsilon^2$ when $E_N(\psi)(t) \le K^2\varepsilon^2$ is assumed for some suitably large constant $K > 0$ and small $\varepsilon > 0$. Here we emphasize that the condition of $0 \le \lambda \le 1$ plays an essential role in the process of deriving the uniform boundedness of $E_N(\psi)(t)$ (see Lemmas 2.3–2.5). This, together with the continuous induction argument, yields the global existence of ψ and further completes the proof of Theorem 1.1 for $0 \le \lambda \le 1$. To prove the blowup result of Theorem 1.2 for $\lambda > 1$, as in [Sideris 1985], we derive a related second-order ordinary differential inequality. From this and assumptions (1-9)–(1-11), an upper bound of the lifespan T_ε is derived by making essential use of $\lambda > 1$. In this way the proof of Theorem 1.2 is completed. In Theorem 4.1, we show that for large data smooth solution (ρ,u) of (1-1), even in case $0 \le \lambda \le 1$, (ρ,u) will in general blow up in finite time. In addition, the proof on the nonnegativity of $P(t,l)$, which is introduced in (3-1), is given in the Appendix.

Throughout, we shall use the following notation and conventions:

- ∇ stands for ∇_x;
- $r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$;
- $\langle r-t\rangle = \bigl(1+(r-t)^2\bigr)^{1/2}$;

- $\|u(t,x)\| = \left(\int_{\mathbb{R}^3} |u(t,x)|^2\,dx\right)^{1/2}$ and $\|u(t,x)\|_{L^\infty} = \sup_{x\in\mathbb{R}^3} |u(t,x)|$;
- Γ denotes one of the vector fields $\{\partial, S, \Omega\}$ on $\mathbb{R}_+ \times \mathbb{R}^3$, where $\partial = (\partial_t, \nabla)$, $S = t\partial_t + \sum_{k=1}^3 x_k\partial_k$, $\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla$;
- β is the solution of $\beta'(t) = \frac{\mu}{(1+t)^\lambda}\beta(t)$ for $t \ge 0$, $\beta(0) = 1$, i.e.,

$$\beta(t) \equiv \begin{cases} e^{\frac{\mu}{1-\lambda}[(1+t)^{1-\lambda}-1]}, & \lambda \ge 0,\ \lambda \ne 1, \\ (1+t)^\mu, & \lambda = 1; \end{cases} \tag{1-12}$$

- $c(\bar\rho) = 1$ will be assumed throughout (introduce $X = x/c(\bar\rho)$ as a new space coordinate if necessary).

2. Global existence for small amplitude in case $0 \le \lambda \le 1$

Throughout this section, $C > 0$ stands for a generic constant which is independent of K, ε, and t.

We start by recalling a Sobolev-type inequality (see [Klainerman 1987]).

Lemma 2.1. *Let $u = u(t,x)$ be a smooth function of $(t,x) \in [0,\infty) \times \mathbb{R}^3$. Then*

$$|u(t,x)| \le C(1+r)^{-1} \sum_{|a|\le 2} \|\Gamma^a u(t,x)\|. \tag{2-1}$$

Moreover, we shall make use of the following inequalities (see [Klainerman and Sideris 1996, Lemma 3.1 and Theorem 5.1]).

Lemma 2.2. *For $u \in C^2([0,\infty)\times\mathbb{R}^3)$,*

$$\|\langle r-t\rangle \nabla\partial u(t,x)\| \le C\left(\sum_{|b|\le 1} \|\partial\Gamma^b u(t,x)\| + t\|\Box u(t,x)\|\right), \tag{2-2}$$

$$(1+r)\langle r-t\rangle |\nabla\partial u(t,x)| \le C\left(\sum_{|b|\le 3} \|\partial\Gamma^b u(t,x)\| + t\|\Box u(t,x)\|\right), \tag{2-3}$$

where $\Box = \partial_t^2 - \Delta = \partial_t^2 - \sum_{k=1}^3 \partial_k^2$.

We now reformulate problem (1-5). Let $\psi = \varphi/(1+t)^\lambda$. From (1-5) and $c(\bar\rho) = 1$ we then have

$$\Box\psi + \frac{\mu}{(1+t)^\lambda}\partial_t\psi + \frac{2\lambda}{1+t}\partial_t\psi - \frac{\lambda(1-\lambda)}{(1+t)^2}\psi = Q(\psi, \partial\psi, \partial^2\psi), \tag{2-4}$$

where

$$Q(\psi,\partial\psi,\partial^2\psi) = (c^2(\rho)-1)\Delta\psi - 2(1+t)^\lambda\partial_t\nabla\psi\cdot\nabla\psi - 2\lambda(1+t)^{\lambda-1}|\nabla\psi|^2 - \mu|\nabla\psi|^2 - (1+t)^{2\lambda}\sum_{1\le i,j\le 3}(\partial_i\psi)(\partial_j\psi)\partial_{ij}^2\psi.$$

We define a time-weighted energy for (2-4),

$$E_N(\psi(t)) = \sum_{0\le|a|\le N} \int_{\mathbb{R}^3} \big((1+t)^{2\lambda}|\partial\Gamma^a\psi|^2 + |\Gamma^a\psi|^2\big)\,dx,$$

where $N \ge 8$ is a fixed number. Moreover, we assume that for any $t \ge 0$,

$$E_N(\psi(t)) \le K^2\varepsilon^2, \tag{2-5}$$

where $K > 0$ is a suitably large constant. It follows from (2-1) and (2-5) that, for all $|a| \le N-2$,

$$\begin{aligned} |\partial\Gamma^a\psi| &\le C(1+r)^{-1}\sum_{|b|\le 2}\|\Gamma^b\partial\Gamma^a\psi(t,x)\| \\ &\le C(1+r)^{-1}\sum_{|b|\le N}\|\partial\Gamma^b\psi(t,x)\| \\ &\le C(1+r)^{-1}(1+t)^{-\lambda}\sqrt{E_N(\psi(t))} \\ &\le CK\varepsilon(1+r)^{-1}(1+t)^{-\lambda} \end{aligned} \tag{2-6}$$

and

$$|\Gamma^a\psi| \le C(1+r)^{-1}\sum_{|b|\le N}\|\Gamma^b\psi(t,x)\| \le CK\varepsilon(1+r)^{-1}. \tag{2-7}$$

In view of Lemma 2.2 and (2-5), we have

Lemma 2.3. *Let* ψ *be a solution of* (2-4). *Then, for all* $|a| \le N-3$ *and* $0 \le \lambda \le 1$, *we have the pointwise estimate*

$$\|\nabla\partial\Gamma^a\psi\|_{L^\infty} \le CK\varepsilon(1+t)^{-2\lambda}. \tag{2-8}$$

Moreover, for $0 \le l \le N-1$, *the weighted* L^2 *estimate*

$$\begin{aligned} \sum_{|b|\le l}\|\langle r-t\rangle\nabla\partial\Gamma^b\psi(t,x)\| &\le C\sum_{|c|\le l+1}\|\partial\Gamma^c\psi(t,x)\| + C(1+t)^{1-\lambda}\sum_{|c|\le l}\|\nabla\Gamma^c\psi(t,x)\| \\ &\quad + C(1-\lambda)(1+t)^{-1}\sum_{|c|\le l}\|\Gamma^c\psi(t,x)\| \end{aligned} \tag{2-9}$$

holds.

Proof. It follows from (2-3)–(2-4) and (2-6)–(2-7) that

$$
\begin{aligned}
(1+t)&\sum_{|a|\le N-3}|\nabla\partial\Gamma^a\psi|\\
&\le C\sum_{|a|\le N-3}(1+r)\langle r-t\rangle|\nabla\partial\Gamma^a\psi|\\
&\le C\sum_{|c|\le N}\|\partial\Gamma^c\psi\|+Ct\sum_{|a|\le N-3}\|\Box\Gamma^a\psi\|\\
&\le CK\varepsilon(1+t)^{-\lambda}+C(1+t)^{1-\lambda}\sum_{|a|\le N-3}\|\partial_t\Gamma^a\psi\|+C(1+t)^{-1}\sum_{|a|\le N-3}\|\Gamma^a\psi\|\\
&\quad+C(1+t)\sum_{|b|+|c|\le N-3}\|\nabla\partial\Gamma^b\psi\Gamma^c\psi\|+C(1+t)^{1+\lambda}\sum_{|a|\le N-3}\|\Gamma^a(\partial_t\nabla\psi\cdot\nabla\psi)\|\\
&\le CK\varepsilon(1+t)^{1-2\lambda}+CK\varepsilon(1+t)\sum_{|a|\le N-3}\|\nabla\partial\Gamma^a\psi\|_{L^\infty},
\end{aligned}
$$

which derives (2-7) in view of the smallness of $\varepsilon>0$.

By (2-2), (2-6)–(2-8) and (2-4), we have that, for $l\le N-1$,

$$
\begin{aligned}
&\sum_{|b|\le l}\|\langle r-t\rangle\nabla\partial\Gamma^b\psi\| \qquad (2\text{-}10)\\
&\quad\le C\sum_{|c|\le l+1}\|\partial\Gamma^c\psi\|+Ct\sum_{|b|\le l}\|\Gamma^b\Box\psi\|\\
&\quad\le C\sum_{|c|\le l+1}\|\partial\Gamma^c\psi\|+C(1+t)^{1-\lambda}\sum_{|c|\le l}\|\nabla\Gamma^c\psi\|+C(1-\lambda)(1+t)^{-1}\sum_{|c|\le l}\|\Gamma^c\psi\|\\
&\qquad+C(1+t)^{1+\lambda}\sum_{|b|\le l}\|\Gamma^b(\partial_t\nabla\psi\cdot\nabla\psi)\|\\
&\qquad+C(1+t)\sum_{\substack{|c|\le N-3,\\|b|\le l-|c|}}\|\langle r-t\rangle^{-1}\Gamma^c\psi\|_{L^\infty}\|\langle r-t\rangle\nabla\partial\Gamma^b\psi\|\\
&\qquad+C(1+t)\sum_{\substack{2-N\le|c|\le l,\\|b|\le l+2-N}}\|(1+r)\nabla\partial\Gamma^b\psi\|_{L^\infty}\|(1+r)^{-1}\Gamma^c\psi\|\\
&\quad\le C\sum_{|c|\le l+1}\|\partial\Gamma^c\psi\|+C(1+t)^{1-\lambda}\sum_{|c|\le l}\|\nabla\Gamma^c\psi\|+C(1-\lambda)(1+t)^{-1}\sum_{|c|\le l}\|\Gamma^c\psi\|\\
&\qquad+CK\varepsilon\sum_{|b|\le l}\|\langle r-t\rangle\nabla\partial\Gamma^b\psi\|+CK\varepsilon(1+t)^{1-\lambda}\sum_{2-N\le|c|\le l}\|(1+r)^{-1}\Gamma^c\psi\|.
\end{aligned}
$$

Note that $\Gamma^c\psi(t,x)$ is supported in $\{x : |x| \le t+M\}$. Then it follows from Hardy inequality that

$$\|(1+r)^{-1}\Gamma^c\psi\| \le C\|\nabla\Gamma^c\psi\|. \tag{2-11}$$

Substituting (2-11) into (2-10) and applying the smallness of ε, we derive (2-9). □

Next we derive the time-weighted energy estimate for the solution ψ of (2-4).

Lemma 2.4. *Let $\mu > 0$ and $\lambda \in (0,1]$. Under assumption (2-5), for all $t > 0$ and $N \ge 8$, it holds that*

$$\sum_{0\le|a|\le N}\int_{\mathbb{R}^3}\big((1+t)^{2\lambda}|\partial\partial^a\psi|^2+\psi^2\big)\,dx+C\sum_{0\le|a|\le N}\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}|\partial\partial^a\psi|^2\,dx\,d\tau \\ \le C\varepsilon^2+C(1+K\varepsilon)\int_0^t A(\tau)\sum_{0\le|a|\le N}\int_{\mathbb{R}^3}\big((1+\tau)^{2\lambda}|\partial\partial^a\psi|^2+\psi^2\big)\,dx\,d\tau, \tag{2-12}$$

where $A(\cdot)$ stands for a generic nonnegative function such that $A \in L^1((0,\infty))$, and $\|A\|_{L^1}$ is independent of K but dependent on μ and λ.

Proof. First we show (2-12) in case $|a| = 0$. Multiplying (2-4) by $m(1+t)^{2\lambda}\partial_t\psi + (1+t)^{2\lambda-1}\psi$ yields by a direct computation

$$\begin{aligned}&\tfrac12\partial_t\big[m(1+t)^{2\lambda}|\partial\psi|^2+2(1+t)^{2\lambda-1}\psi\partial_t\psi+(\mu(1+t)^{\lambda-1}+2\lambda(1+t)^{2\lambda-2})\psi^2\big]\\ &\quad+\operatorname{div}(\cdots)+\big(\mu m(1+t)^{\lambda}+(\lambda m-1)(1+t)^{2\lambda-1}\big)(\partial_t\psi)^2\\ &\quad+(1-\lambda m)(1+t)^{2\lambda-1}|\nabla\psi|^2+\frac{\mu}{2}(1-\lambda)(1+t)^{\lambda-2}\psi^2\\ &\quad+C_1(\lambda-1)(1+t)^{2\lambda-2}\psi\partial_t\psi+C_2(\lambda-1)(1+t)^{2\lambda-3}\psi^2\\ &\qquad=\big(m(1+t)^{2\lambda}\partial_t\psi+(1+t)^{2\lambda-1}\psi\big)Q(\psi,\partial\psi,\partial^2\psi),\end{aligned} \tag{2-13}$$

where the constant $m > 0$ will be determined later and C_i $(i = 1, 2)$ are suitable constants. Note that in the square bracket of the first line in (2-13),

$$\begin{aligned}&m(1+t)^{2\lambda}|\partial\psi|^2+2(1+t)^{2\lambda-1}\psi\partial_t\psi+(\mu(1+t)^{\lambda-1}+2\lambda(1+t)^{2\lambda-2})\psi^2\\ &\quad=m(1+t)^{2\lambda}\big(\tfrac13|\partial_t\psi|^2+|\nabla\psi|^2\big)+\Big(\mu(1+t)^{\lambda-1}+\Big(2\lambda-\frac{3}{2m}\Big)(1+t)^{2\lambda-2}\Big)\psi^2\\ &\qquad+\bigg(\sqrt{\frac{2m}{3}}(1+t)^{\lambda}\partial_t\psi+\sqrt{\frac{3}{2m}}(1+t)^{\lambda-1}\psi\bigg)^2.\end{aligned} \tag{2-14}$$

We choose $m > 0$ to fulfill

$$\lambda < \frac{1}{m} < \min\{\mu+\lambda, 2\lambda\};$$

together with $\lambda \le 1$ (i.e., $2\lambda-2 \le \lambda-1 \le 0$), this yields that (2-14) is equivalent to

$$(1+t)^{2\lambda}|\partial\psi|^2+(1+t)^{\lambda-1}\psi^2.$$

On the other hand, the coefficients

$$\mu m(1+t)^{\lambda}+(\lambda m-1)(1+t)^{2\lambda-1}$$

and

$$(1-\lambda m)(1+t)^{2\lambda-1}$$

of $(\partial_t\psi)^2$ and $|\nabla\psi|^2$ in the left-hand side of (2-13) are both positive.

Then integrating (2-13) over $[0,t]\times\mathbb{R}^3$ yields

$$\begin{aligned}(2\text{-}15)\quad &\int_{\mathbb{R}^3}\big((1+t)^{2\lambda}|\partial\psi|^2+(1+t)^{\lambda-1}\psi^2\big)\,dx\\ &+C\int_0^t\int_{\mathbb{R}^3}\big((1+\tau)^{\lambda}(\partial_t\psi)^2+(1+\tau)^{2\lambda-1}|\nabla\psi|^2+(1+\tau)^{\lambda-2}\psi^2\big)\,dx\,d\tau\\ &\le C\varepsilon^2+\int_0^t A(\tau)\int_{\mathbb{R}^3}(1+\tau)^{\lambda-1}\psi^2\,dx\,d\tau\\ &\quad+C\left|\int_0^t\int_{\mathbb{R}^3}\big(m(1+\tau)^{2\lambda}\partial_t\psi+(1+\tau)^{2\lambda-1}\psi\big)Q(\psi,\partial\psi,\partial^2\psi)\,dx\,d\tau\right|.\end{aligned}$$

Next we improve the time-weighted estimate of ψ in the left-hand side of (2-15). Multiplying both sides of (2-4) by $(1+t)^{\lambda}\psi$ yields by direct computation

$$\begin{aligned}&\partial_t\Big((1+t)^{\lambda}\psi\partial_t\psi+\frac{\mu}{2}\psi^2\Big)+\operatorname{div}(\cdots)-(1+t)^{\lambda}(\partial_t\psi)^2-\lambda(1+t)^{\lambda-1}\psi\partial_t\psi\\ &\quad+(1+t)^{\lambda}|\nabla\psi|^2+2\lambda(1+t)^{\lambda-1}\psi\partial_t\psi+\lambda(\lambda-1)(1+t)^{\lambda-2}\psi^2\\ &\qquad\qquad=(1+t)^{\lambda}\psi Q(\psi,\partial\psi,\partial^2\psi).\end{aligned}$$

From this and (2-15), we can choose the multiplier

$$m(1+t)^{2\lambda}\partial_t\psi+(1+t)^{2\lambda-1}\psi+\kappa(1+t)^{\lambda}\psi$$

for (2-4) with a small $\kappa>0$ and then obtain

$$\begin{aligned}(2\text{-}16)\quad &\int_{\mathbb{R}^3}\big((1+t)^{2\lambda}|\partial\psi|^2+\psi^2\big)\,dx+C\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}|\partial\psi|^2\,dx\,d\tau\\ &\le C\varepsilon^2+\int_0^t A(\tau)\int_{\mathbb{R}^3}\psi^2\,dx\,d\tau\\ &\quad+C\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{2\lambda}(\partial_t\psi)Q(\psi,\partial\psi,\partial^2\psi)\,dx\,d\tau\right|\\ &\quad+C\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}\psi Q(\psi,\partial\psi,\partial^2\psi)\,dx\,d\tau\right|.\end{aligned}$$

Next we derive the time-weighted estimates of $\partial^a\psi$ with $1\le|a|\le N$. Taking ∂^a on both sides of (2-4) yields

$$\begin{aligned}\Box\partial^a\psi+\frac{\mu}{(1+t)^\lambda}\partial_t\partial^a\psi+\frac{2\lambda}{1+t}\partial_t\partial^a\psi&\\=\partial^a Q(\psi,\partial\psi,\partial^2\psi)+\sum_{1\le|b|\le|a|}\frac{1}{(1+t)^\lambda}\big(1+O((1+t)^{\lambda-1})\big)\partial^b\psi&\\-\lambda(\lambda-1)\partial^a\Big(\frac{1}{(1+t)^2}\Big)\psi.&\end{aligned}$$

Exactly as for (2-16), multiplying this by

$$m(1+t)^{2\lambda}\partial_t\partial^a\psi+(1+t)^{2\lambda-1}\partial^a\psi+\kappa(1+t)^\lambda\partial^a\psi,$$

we obtain

$$\begin{aligned}\sum_{0\le|a|\le N}\int_{\mathbb{R}^3}\big((1+t)^{2\lambda}|\partial\partial^a\psi|^2+\psi^2\big)\,dx+C\sum_{0\le|a|\le N}\int_0^t\int_{\mathbb{R}^3}(1+\tau)^\lambda|\partial\partial^a\psi|^2\,dx\,d\tau&\\\le C\varepsilon^2+\int_0^t A(\tau)\int_{\mathbb{R}^3}\psi^2\,dx\,d\tau&\\+C\sum_{0\le|a|\le N}\Big|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{2\lambda}(\partial_t\partial^a\psi)\partial^a Q(\psi,\partial\psi,\partial^2\psi)\,dx\,d\tau\Big|&\\+C\sum_{0\le|a|\le N}\Big|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}(\partial^a\psi)\partial^a Q(\psi,\partial\psi,\partial^2\psi)\,dx\,d\tau\Big|.&\end{aligned}\tag{2-17}$$

We now deal with the last two terms in the right-hand side of (2-17). We first analyze the integrand $(1+t)^{2\lambda}(\partial_t\partial^a\psi)\partial^a Q(\psi,\partial\psi,\partial^2\psi)$ of the penultimate term. Direct computation yields

$$\begin{aligned}&\partial^a Q(\psi,\partial\psi,\partial^2\psi)\\&\quad=(c^2(\rho)-1)\Delta\partial^a\psi-2(1+t)^\lambda\nabla\partial_t\partial^a\psi\cdot\nabla\psi-(1+t)^{2\lambda}(\partial_i\psi)(\partial_j\psi)\partial^2_{ij}\partial^a\psi+\text{l.o.t.}\end{aligned}$$

and

$$\begin{aligned}&(1+t)^{2\lambda}(\partial_t\partial^a\psi)\partial^a Q(\psi,\partial\psi,\partial^2\psi)\\&\quad=\operatorname{div}\big((1+t)^{2\lambda}(c^2(\rho)-1)(\partial_t\partial^a\psi)\nabla\partial^a\psi\big)-\operatorname{div}\big((1+t)^{3\lambda}|\partial_t\partial^a\psi|^2\nabla\psi\big)\\&\qquad-\tfrac12\partial_t\big((1+t)^{2\lambda}(c^2(\rho)-1)|\nabla\partial^a\psi|^2\big)\\&\qquad+(1+t)^{3\lambda}|\partial_t\partial^a\psi|^2\Delta\psi+\lambda(1+t)^{2\lambda-1}(c^2(\rho)-1)|\nabla\partial^a\psi|^2\\&\qquad+\tfrac12(1+t)^{2\lambda}(c^2(\rho))'\partial_t\rho|\nabla\partial^a\psi|^2\\&\qquad-(1+t)^{4\lambda}(\partial_i\psi)(\partial_j\psi)(\partial^2_{ij}\partial^a\psi)\partial_t\partial^a\psi+\text{l.o.t.},\end{aligned}\tag{2-18}$$

where here and below l.o.t. designates lower-order terms which are of the form

$$(\partial^{b_1}\psi)(\partial^{b_2}\psi)\dots(\partial^{b_l}\psi)$$

(multiplied by $\partial\partial^a\psi$ or $\partial^a\psi$) with $l \geq 2$ and $1 \leq |b_1| + \cdots + |b_l| \leq |a| + 1$. Here we are concerned with the top-order derivatives only. Note that the term $(1+t)^{4\lambda}(\partial_i\psi)(\partial_j\psi)(\partial^2_{ij}\partial^a\psi)\partial_t\partial^a\psi$ in (2-18) can be expressed as

$$\begin{aligned}(1+t)^{4\lambda}&(\partial_i\psi)(\partial_j\psi)(\partial^2_{ij}\partial^a\psi)\partial_t\partial^a\psi\\ &= \tfrac{1}{2}\big\{\partial_i\big((1+t)^{4\lambda}(\partial_i\psi)(\partial_j\psi)(\partial_j\partial^a\psi)\partial_t\partial^a\psi\big)\\ &\quad + \partial_j\big((1+t)^{4\lambda}(\partial_i\psi)(\partial_j\psi)(\partial_i\partial^a\psi)\partial_t\partial^a\psi\big)\\ &\quad - \partial_t\big((1+t)^{4\lambda}(\partial_i\psi)(\partial_j\psi)(\partial_i\partial^a\psi)\partial_j\partial^a\psi\big)\\ &\quad + \partial_t\big((1+t)^{4\lambda}(\partial_i\psi)\partial_j\psi\big)(\partial_i\partial^a\psi)\partial_j\partial^a\psi + \text{l.o.t.}\big\}.\end{aligned} \tag{2-19}$$

Similarly, for the integrand of

$$\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}(\partial^a\psi)\partial^a Q(\psi,\partial\psi,\partial^2\psi)\,dx\,d\tau\right|,$$

one has

$$\begin{aligned}(1+t)^{\lambda}&\partial^a\psi\partial^a Q(\psi,\partial\psi,\partial^2\psi)\\ &= \operatorname{div}\big((1+t)^{\lambda}(c^2(\rho)-1)\nabla(\partial^a\psi)\partial^a\psi\big) - \tfrac{1}{2}\partial_i\big((1+t)^{3\lambda}(\partial_i\psi)\partial^a(|\nabla\psi|^2)\partial^a\psi\big)\\ &\quad - \partial_t\big((1+t)^{\lambda}\partial^a(|\nabla\psi|^2)\partial^a\psi\big) - (1+t)^{\lambda}(c^2(\rho)-1)|\nabla\partial^a\psi|^2\\ &\quad - (1+t)^{\lambda}(c^2(\rho))'\nabla\rho\cdot\nabla(\partial^a\psi)\partial^a\psi + \lambda(1+t)^{\lambda-1}\partial^a(|\nabla\psi|^2)\partial^a\psi\\ &\quad + (1+t)^{\lambda}\partial^a(|\nabla\psi|^2)\partial_t\partial^a\psi + \tfrac{1}{2}(1+t)^{3\lambda}(\Delta\psi)\partial^a(|\nabla\psi|^2)\partial^a\psi\\ &\quad + \tfrac{1}{2}(1+t)^{3\lambda}\nabla\psi\cdot\nabla(\partial^a\psi)\partial^a(|\nabla\psi|^2) + \text{l.o.t.}\end{aligned} \tag{2-20}$$

From the expression $(\partial^{b_1}\psi)(\partial^{b_2}\psi)\dots(\partial^{b_l}\psi)$ ($l \geq 2$, $1 \leq |b_1|+\cdots+|b_l| \leq N+1$) of the lower-order terms one readily obtains that there exists at most one b_j ($1 \leq j \leq l$) such that

$$\left[\frac{N+3}{2}\right] < |b_j| \leq N+1.$$

Moreover, $\left[\frac{N+3}{2}\right] \leq N-2$ by $N \geq 8$. Thus, applying (2-5)–(2-7) and subsequently substituting (2-18)–(2-20) into (2-17) completes the proof of Lemma 2.4. □

Next we focus on the general time-weighted energy estimate of $\partial\Gamma^a\psi$ with $0 \leq |a| \leq N$ and $N \geq 8$.

Lemma 2.5 (time-weighted energy estimate of $\partial\Gamma^a\psi$ for $|a|\le N$). *Let* $\mu>0$ *and* $\lambda\in(0,1]$. *Under assumption* (2-5), *we have that, for* $t>0$,

$$\begin{aligned}(2\text{-}21)\quad &\sum_{0\le|a|\le N}\int_{\mathbb{R}^3}\big((1+t)^{2\lambda}|\partial\Gamma^a\psi|^2+|\Gamma^a\psi|^2\big)\,dx\\ &+C\sum_{0\le|\alpha|\le N}\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}|\partial\Gamma^a\psi|^2\,dx\,d\tau\\ &\le C\varepsilon^2+C(1+K\varepsilon)\int_0^t A(\tau)\sum_{0\le|a|\le N}\int_{\mathbb{R}^3}\big((1+\tau)^{2\lambda}|\partial\Gamma^a\psi|^2+\psi^2\big)\,dx\,d\tau,\end{aligned}$$

where the function A has been defined in Lemma 2.4.

Proof. Writing $\Gamma^a=\tilde\Gamma^b\partial^c$ with $\tilde\Gamma\in\{\Omega,S\}$, we will use induction on $|b|$ to prove (2-21). In view of Lemma 2.4, it is enough to assume that $|c|=0$.

Suppose that (2-21) holds for $|b|\le l-1$, where $1\le l\le N$. We then intend to establish (2-21) for $|b|=l$.

Acting with $\tilde\Gamma^a$ (where $a=b$ and $|b|=l$) on both sides of (2-4) yields

$$\begin{aligned}(2\text{-}22)\quad &\Box\tilde\Gamma^a\psi+\frac{\mu}{(1+t)^{\lambda}}\partial_t\tilde\Gamma^a\psi+\frac{2\lambda}{1+t}\partial_t\tilde\Gamma^a\psi\\ &=\sum_{|b_1|<|b|}\tilde\Gamma^{b_1}\partial^c\Box\psi+\tilde\Gamma^a Q(\psi,\partial\psi,\partial^2\psi)\\ &\quad-\Big[\tilde\Gamma^a,\frac{\mu}{(1+t)^{\lambda}}\partial_t\Big]\psi-\Big[\tilde\Gamma^a,\frac{2\lambda}{1+t}\partial_t\Big]\psi+\tilde\Gamma^a\big((\lambda-1)(1+t)^{-2}\psi\big).\end{aligned}$$

Starting from (2-22), as in the proof of Lemma 2.4, we can choose the multiplier

$$m(1+t)^{2\lambda}\partial_t\tilde\Gamma^a\psi+(1+t)^{2\lambda-1}\tilde\Gamma^a\psi+\kappa(1+t)^{\lambda}\tilde\Gamma^a\psi$$

to derive (2-21). For the commutators, we see from (2-4) that

$$\begin{aligned}(2\text{-}23)\quad &\Big|\int_0^t\int_{\mathbb{R}^3}\Big[\tilde\Gamma^a,\frac{\mu}{(1+t)^{\lambda}}\partial_t\Big]\psi\,(1+t)^{\lambda}\tilde\Gamma^a\psi\,dx\,d\tau\Big|\\ &\le C\sum_{|a_1|<|a|}\Big|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}\Box\tilde\Gamma^{a_1}\psi\,\tilde\Gamma^a\psi\,dx\,d\tau\Big|\\ &\quad+C\sum_{|a_1|<|a|}\Big|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}\tilde\Gamma^{a_1}Q(\psi,\partial\psi,\partial^2\psi)\tilde\Gamma^a\psi\,dx\,d\tau\Big|\\ &\quad+C\sum_{|a_1|<|a|}\Big|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda-1}\tilde\Gamma^a\psi\big(\partial_t\tilde\Gamma^{a_1}\psi+(1-\lambda)(1+\tau)^{-1}\tilde\Gamma^{a_1}\psi\big)\,dx\,d\tau\Big|\end{aligned}$$

$$
\begin{aligned}
&\le C\varepsilon^2 + C\sum_{|a_1|<|a|}\left|\int_{\mathbb{R}^3}(1+t)^\lambda \partial_t \tilde{\Gamma}^{a_1}\psi\,\tilde{\Gamma}^a\psi\,dx\right| \\
&\quad + C\sum_{|a_1|<|a|}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^\lambda \tilde{\Gamma}^{a_1}Q(\psi,\partial\psi,\partial^2\psi)\tilde{\Gamma}^a\psi\,dx\,d\tau\right| \\
&\quad + C\sum_{|a_1|<|a|}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda-1}\tilde{\Gamma}^a\psi\big(\partial_t\tilde{\Gamma}^{a_1}\psi+(1-\lambda)(1+\tau)^{-1}\tilde{\Gamma}^{a_1}\psi\big)\,dx\,d\tau\right| \\
&\quad + C\sum_{|a_1|<|a|}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^\lambda \partial\tilde{\Gamma}^{a_1}\psi\,\partial\tilde{\Gamma}^a\psi\,dx\,d\tau\right|.
\end{aligned}
$$

By the finite propagation speed, we have for $a>0$

$$
|\tilde{\Gamma}^a\psi| \le C(1+t)\sum_{|a_1|<|a|}|\partial\tilde{\Gamma}^{a_1}\psi|. \tag{2-24}
$$

It follows from (2-23)–(2-24) and a direct computation that

$$
\begin{aligned}
&\sum_{\substack{|b|=l,\\ |c|\le N-l}}\int_{\mathbb{R}^3}\big((1+t)^{2\lambda}|\partial\tilde{\Gamma}^b\partial^c\psi|^2+|\tilde{\Gamma}^b\partial^c\psi|^2\big)\,dx \\
&\qquad\qquad + C\sum_{\substack{|b|=l,\\ |c|\le N-l}}\int_0^t\int_{\mathbb{R}^3}(1+\tau)^\lambda|\partial\tilde{\Gamma}^b\partial^c\psi|^2\,dx\,d\tau \\
&\le C\varepsilon^2 + CE_{l-1}(\psi(t)) + C\sum_{\substack{|b_1|<l,\\ |c_1|\le N-|b_1|}}\int_0^t\int_{\mathbb{R}^3}(1+\tau)^\lambda|\partial\tilde{\Gamma}^{b_1}\partial^{c_1}\psi|^2\,dx\,d\tau \\
&\quad + C(1+K\varepsilon)\int_0^t A(\tau)\sum_{\substack{|b_1|\le l,\\ |c_1|\le N-|b_1|}}\int_{\mathbb{R}^3}\big((1+\tau)^{2\lambda}|\partial\tilde{\Gamma}^{b_1}\partial^{c_1}\psi|^2+|\tilde{\Gamma}^{b_1}\partial^{c_1}\psi|^2\big)\,dx\,d\tau \\
&\quad + C\sum_{\substack{|b_1|\le l,\\ |c_1|\le N-|b_1|}}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{2\lambda}(\partial_t\tilde{\Gamma}^a\psi)\tilde{\Gamma}^{b_1}\partial^{c_1}Q(\psi,\partial\psi,\partial^2\psi)\,dx\,d\tau\right| \\
&\quad + C\sum_{\substack{|b_1|\le l,\\ |c_1|\le N-|b_1|}}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}(\tilde{\Gamma}^a\psi)\tilde{\Gamma}^{b_1}\partial^{c_1}Q(\psi,\partial\psi,\partial^2\psi)\,dx\,d\tau\right|.
\end{aligned} \tag{2-25}
$$

Next we deal with the last two terms in the right-hand side of (2-25). Note that

$$
c^2(\rho)-1 = -G(\psi,\partial\psi)\int_0^1 (c^2)'(-sG(\psi,\partial\psi))\,ds,
$$

where $G(\psi, \partial\psi) = (1+t)^\lambda \partial_t \psi + (1+t)^{\lambda-1}\psi + (1+t)^{2\lambda}|\nabla\psi|^2/2 + \mu\psi$. From this, it is readily seen that the typical terms in $Q(\psi, \partial\psi, \partial^2\psi)$ are of the form $\psi\,\Delta\psi$, $(1+t)^\lambda \partial_t \nabla\psi \cdot \nabla\psi$, and $(1+t)^{2\lambda}(\partial_i\psi)(\partial_j\psi)\partial_{ij}\psi$. We analyze them separately. Without loss of generality, we assume $|c_1| = 0$ in the last two terms of (2-25); the treatment of the other cases is easier.

Part A: *Estimates of*

$$\sum_{|b_1|\le N} \left| \int_0^t \int_{\mathbb{R}^3} (1+\tau)^{2\lambda} (\partial_t \tilde{\Gamma}^a \psi) \tilde{\Gamma}^{b_1}(\psi\,\Delta\psi)\,dx\,d\tau \right|$$

and

$$\sum_{|b_1|\le N} \left| \int_0^t \int_{\mathbb{R}^3} (1+\tau)^{\lambda} (\tilde{\Gamma}^a \psi) \tilde{\Gamma}^{b_1}(\psi\,\Delta\psi)\,dx\,d\tau \right|.$$

Note that

$$\tilde{\Gamma}^{b_1}(\psi\,\Delta\psi) = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,$$

where

$$\begin{aligned} \mathrm{I}_1 &= \psi\,\Delta\tilde{\Gamma}^{b_1}\psi, \\ \mathrm{I}_2 &= \sum_{\substack{|b_1|=|b_2|+|b_3|,\\ 1\le|b_2|\le N-2}} (\tilde{\Gamma}^{b_2}\psi)\Delta\tilde{\Gamma}^{b_3}\psi, \\ \mathrm{I}_3 &= \sum_{\substack{|b_1|=|b_2|+|b_3|,\\ N-1\le|b_2|\le l}} (\tilde{\Gamma}^{b_2}\psi)\Delta\tilde{\Gamma}^{b_3}\psi. \end{aligned}$$

In view of $b_1 = a$ and

$$\begin{aligned} &(1+t)^{2\lambda}(\partial_t\tilde{\Gamma}^a\psi)\psi\,\Delta\tilde{\Gamma}^a\psi \\ &= \operatorname{div}\big((1+t)^{2\lambda}(\partial_t\tilde{\Gamma}^a\psi)\psi\nabla\tilde{\Gamma}^a\psi\big) + \tfrac{1}{2}\partial_t\big((1+t)^{2\lambda}|\nabla\tilde{\Gamma}^a\psi|^2\psi\big) \\ &\quad -(1+t)^{2\lambda}(\partial_t\tilde{\Gamma}^a\psi)\nabla\psi\cdot\nabla\tilde{\Gamma}^a\psi - \lambda(1+t)^{\lambda-1}|\nabla\tilde{\Gamma}^a\psi|^2\psi - \tfrac{1}{2}(1+t)^{2\lambda}|\nabla\tilde{\Gamma}^a\psi|^2\partial_t\psi, \end{aligned}$$

we have by an integration by parts and (2-6)–(2-7)

$$\begin{aligned} &\left| \int_0^t \int_{\mathbb{R}^3} (1+\tau)^{2\lambda}(\partial_t\tilde{\Gamma}^a\psi)\mathrm{I}_1\,dx\,d\tau \right| \\ &\quad \le C\varepsilon^2 + CK\varepsilon \sum_{0\le|a|\le N} \int_{\mathbb{R}^3} (1+t)^{2\lambda}|\partial\tilde{\Gamma}^a\psi|^2\,dx \\ &\qquad + CK\varepsilon \sum_{0\le|a|\le N} \int_0^t \int_{\mathbb{R}^3} (1+\tau)^{\lambda}|\partial\tilde{\Gamma}^a\psi|^2\,dx\,d\tau. \end{aligned} \tag{2-26}$$

Moreover, it follows from (2-7) and (2-9) that

$$
\begin{aligned}
\text{(2-27)}\quad & \int_{\mathbb{R}^3} \big|(1+t)^{2\lambda}(\partial_t \tilde{\Gamma}^a \psi)\mathrm{I}_2\big|\,dx \\
&\le (1+t)^{2\lambda}\|\langle r-t\rangle^{-1}\tilde{\Gamma}^{b_2}\psi\|_{L^\infty}\cdot\|\partial_t\tilde{\Gamma}^a\psi\|\cdot\|\langle r-t\rangle\Delta\tilde{\Gamma}^{b_3}\psi\| \\
&\le CK\varepsilon(1+t)^{\lambda}\|\partial_t\tilde{\Gamma}^a\psi\| \sum_{|b_4|\le|b_3|+1}\big(\|\nabla\tilde{\Gamma}^{b_4}\psi\|+(1-\lambda)(1+t)^{-1}\|\tilde{\Gamma}^{b_4}\psi\|\big) \\
&\le CK\varepsilon(1+t)^{\lambda}\|\partial_t\tilde{\Gamma}^a\psi\| \sum_{|b_4|\le|b_3|+1}\|\nabla\tilde{\Gamma}^{b_4}\psi\| + CK\varepsilon(1+t)^{\lambda}\|\partial_t\tilde{\Gamma}^a\psi\|^2 \\
&\qquad\qquad + CK\varepsilon(1-\lambda)(1+t)^{\lambda-2}\sum_{|b_4|\le|b_3|+1}\|\tilde{\Gamma}^{b_4}\psi\|^2.
\end{aligned}
$$

On the other hand, we have that by (2-6) and Hardy's inequality

$$
\begin{aligned}
\text{(2-28)}\quad & \int_{\mathbb{R}^3} \big|(1+t)^{2\lambda}(\partial_t \tilde{\Gamma}^a \psi)\mathrm{I}_3\big|\,dx \\
&\le (1+t)^{2\lambda}\|(1+r)\Delta\tilde{\Gamma}^{b_3}\psi\|_{L^\infty}\cdot\|\partial_t\tilde{\Gamma}^a\psi\|\|(1+r)^{-1}\tilde{\Gamma}^{b_2}\psi\| \\
&\le CK\varepsilon(1+t)^{\lambda}\|\partial_t\tilde{\Gamma}^a\psi\| \sum_{|b_4|\le|b_2|}\|\nabla\tilde{\Gamma}^{b_4}\psi\|.
\end{aligned}
$$

Combining (2-26)–(2-28) together with $0\le\lambda\le 1$ (this means that the coefficient $CK\varepsilon(1-\lambda)(1+t)^{\lambda-2}$ of $\sum_{|b_4|\le|b_3|+1}\|\tilde{\Gamma}^{b_4}\psi\|^2$ in the last line of (2-27) is nonnegative and in $L^1(0,\infty)$) yields

$$
\begin{aligned}
\text{(2-29)}\quad & \sum_{|b_1|\le N}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{2\lambda}(\partial_t\Gamma^a\psi)\Gamma^{b_1}(\psi\Delta\psi)\,dx\,d\tau\right| \\
&\le C\varepsilon^2 + CK\varepsilon\sum_{0\le|a|\le N}\int_{\mathbb{R}^3}(1+t)^{2\lambda}|\partial\tilde{\Gamma}^a\psi|^2\,dx \\
&\qquad + CK\varepsilon\sum_{0\le|a|\le N}\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}|\partial\tilde{\Gamma}^a\psi|^2\,dx\,d\tau \\
&\qquad\qquad + CK\varepsilon\sum_{|b_4|\le N}\int_0^t A(\tau)\int_{\mathbb{R}^3}|\tilde{\Gamma}^{b_4}\psi|^2\,dx\,d\tau.
\end{aligned}
$$

Note that

$$
\begin{aligned}
(1+t)^{\lambda}(\tilde{\Gamma}^a\psi)\tilde{\Gamma}^{b_1}(\psi\Delta\psi) &= \sum_{|b_2|+|b_3|=|b_1|}(1+t)^{\lambda}(\tilde{\Gamma}^a\psi)(\tilde{\Gamma}^{b_2}\psi)\Delta\tilde{\Gamma}^{b_3}\psi \\
&= \operatorname{div}\Big(\sum_{|b_2|+|b_3|=|b_1|}(1+t)^{\lambda}(\tilde{\Gamma}^a\psi)(\tilde{\Gamma}^{b_2}\psi)\nabla\tilde{\Gamma}^{b_3}\psi\Big)+\sum_{i=4}^{5}\mathrm{I}_i,
\end{aligned}
$$

where

$$\mathrm{I}_4 = -\sum_{\substack{|b_2|\le N-2,\\ |b_2|+|b_3|=|b_1|}} (1+t)^\lambda (\widetilde{\Gamma}^{b_2}\psi)(\nabla\widetilde{\Gamma}^{a}\psi)\cdot(\nabla\widetilde{\Gamma}^{b_3}\psi),$$

$$\mathrm{I}_5 = -\sum_{\substack{N-1\le|b_2|\le l-1,\\ |b_2|+|b_3|=|b_1|}} (1+t)^\lambda (\widetilde{\Gamma}^{b_2}\psi)(\nabla\widetilde{\Gamma}^{a}\psi)\cdot(\nabla\widetilde{\Gamma}^{b_3}\psi) - \sum_{|b_2|+|b_3|=|b_1|} (1+t)^\lambda (\widetilde{\Gamma}^{a}\psi)(\nabla\widetilde{\Gamma}^{b_2}\psi)\cdot(\nabla\widetilde{\Gamma}^{b_3}\psi).$$

Therefore, by (2-7) and Hardy's inequality, we have

$$\int_{\mathbb{R}^3} |\mathrm{I}_4|\,dx \le CK\varepsilon(1+t)^\lambda \,\|\nabla\widetilde{\Gamma}^{a}\psi\| \sum_{|b_1|+3-N\le|b_3|\le N} \|\nabla\widetilde{\Gamma}^{b_3}\psi\|$$

and

$$\int_{\mathbb{R}^3} |\mathrm{I}_5|\,dx \le CK\varepsilon\|(1+r)^{-1}\widetilde{\Gamma}^{b_2}\psi\nabla\widetilde{\Gamma}^{a}\psi\|_{L^1} \le CK\varepsilon\|\nabla\widetilde{\Gamma}^{b_2}\psi\|\|\nabla\widetilde{\Gamma}^{a}\psi\|.$$

This yields

$$\sum_{|b_1|\le N}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^\lambda(\widetilde{\Gamma}^{a}\psi)\widetilde{\Gamma}^{b_1}(\psi\Delta\psi)\,dx\,d\tau\right| \le CK\varepsilon\sum_{0\le|a|\le N}\int_0^t\int_{\mathbb{R}^3}(1+\tau)^\lambda|\partial\widetilde{\Gamma}^{a}\psi|^2\,dx\,d\tau. \tag{2-30}$$

Part B: *Estimates of*

$$\sum_{|b_1|\le N}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{2\lambda}(\partial_t\widetilde{\Gamma}^{a}\psi)\widetilde{\Gamma}^{b_1}\big((1+\tau)^\lambda\partial_t\nabla\psi\cdot\nabla\psi\big)\,dx\,d\tau\right|$$

and

$$\sum_{|b_1|\le N}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}(\widetilde{\Gamma}^{a}\psi)\widetilde{\Gamma}^{b_1}\big((1+\tau)^\lambda\partial_t\nabla\psi\cdot\nabla\psi\big)\,dx\,d\tau\right|.$$

One has

$$\begin{aligned}\widetilde{\Gamma}^{b_1}\big((1+t)^\lambda\partial_t\nabla\psi\cdot\nabla\psi\big) &= (1+t)^\lambda\partial_t\nabla\widetilde{\Gamma}^{b_1}\psi\cdot\nabla\psi + \sum_{N-2\le|b_2|\le l-1}(1+t)^\lambda(\partial_t\nabla\widetilde{\Gamma}^{b_2}\psi)\nabla\widetilde{\Gamma}^{b_3}\psi \\ &\quad + \sum_{|b_2|\le N-3}(1+t)^\lambda(\partial_t\nabla\widetilde{\Gamma}^{b_2}\psi)\nabla\widetilde{\Gamma}^{b_3}\psi \\ &= \mathrm{II}_1+\mathrm{II}_2+\mathrm{II}_3.\end{aligned}$$

By (2-8), we have

$$\sum_{|b_1|\le N}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{2\lambda}(\partial_t\widetilde{\Gamma}^a\psi)\mathrm{II}_1\,dx\,d\tau\right| \le CK\varepsilon\sum_{0\le|a|\le N}\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}|\partial\widetilde{\Gamma}^a\psi|^2\,dx\,d\tau. \tag{2-31}$$

In addition, it follows from (2-6), (2-9) and a direct computation that

$$\begin{aligned}(1+t)^{2\lambda}&\|(\partial_t\Gamma^a\psi)\mathrm{II}_2\|_{L^1}\\ &\le(1+t)^{3\lambda}\sum_{|b_2|\le N-4}\|\langle r-t\rangle^{-1}\nabla\Gamma^{b_3}\psi\|_{L^\infty}\cdot\|\partial_t\Gamma^a\psi\|\cdot\|\langle r-t\rangle\,\partial_t\nabla\Gamma^{b_2}\psi\|\\ &\le CK\varepsilon(1+t)^{\lambda}\|\partial_t\Gamma^a\psi\|\sum_{|c|\le|b_2|+1}\big(\|\nabla\Gamma^c\psi\|+(1-\lambda)(1+t)^{-1}\|\Gamma^c\psi\|\big)\\ &\le CK\varepsilon(1+t)^{\lambda}\|\partial_t\widetilde{\Gamma}^a\psi\|\sum_{|b_4|\le|b_3|+1}\|\nabla\widetilde{\Gamma}^{b_4}\psi\|+CK\varepsilon(1+t)^{\lambda}\|\partial_t\widetilde{\Gamma}^a\psi\|^2\\ &\qquad+CK\varepsilon(1-\lambda)(1+t)^{\lambda-2}\sum_{|b_4|\le|b_3|+1}\|\widetilde{\Gamma}^{b_4}\psi\|^2.\end{aligned} \tag{2-32}$$

Treating II_3, we obtain by (2-8)

$$\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{2\lambda}(\partial_t\widetilde{\Gamma}^a\psi)\mathrm{II}_3\,dx\,d\tau\right|\le CK\varepsilon\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}|\partial\widetilde{\Gamma}^a\psi|^2dx\,d\tau. \tag{2-33}$$

Collecting (2-31)–(2-33) together with $0\le\lambda\le1$ (this means that the coefficient $CK\varepsilon(1-\lambda)(1+t)^{\lambda-2}$ of $\sum_{|b_4|\le|b_3|+1}\|\widetilde{\Gamma}^{b_4}\psi\|^2$ in the last line of (2-32) is nonnegative and in $L^1(0,\infty)$) yields

$$\begin{aligned}\sum_{|b_1|\le N}&\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{2\lambda}(\partial_t\Gamma^a\psi)\Gamma^{b_1}\big((1+t)^{\lambda}\partial_t\nabla\psi\cdot\nabla\psi\big)\,dx\,d\tau\right|\\ &\le CK\varepsilon\sum_{0\le|a|\le N}\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}|\partial\widetilde{\Gamma}^a\psi|^2\,dx\,d\tau\\ &\qquad+CK\varepsilon\sum_{|b_4|\le N}\int_0^t A(\tau)\int_{\mathbb{R}^3}|\widetilde{\Gamma}^{b_4}\psi|^2\,dx\,d\tau.\end{aligned} \tag{2-34}$$

In addition, one notes that

$$\begin{aligned}2(1+t)^{2\lambda}&(\widetilde{\Gamma}^a\psi)\widetilde{\Gamma}^a(\partial_t\nabla\psi\cdot\nabla\psi)\\ &=\sum_{|c|\le|a|}\partial_t\big((1+t)^{2\lambda}\widetilde{\Gamma}^a\psi\,\Gamma^c(|\nabla\psi|^2)\big)\\ &\qquad-2\lambda(1+t)^{2\lambda-1}(\widetilde{\Gamma}^a\psi)\widetilde{\Gamma}^c(|\nabla\psi|^2)-(1+t)^{2\lambda}(\partial_t\widetilde{\Gamma}^a\psi)\widetilde{\Gamma}^c(|\nabla\psi|^2).\end{aligned}$$

From this, (2-6) and Hardy's inequality, we have

$$\sum_{|b_1|\le N}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^\lambda(\tilde\Gamma^a\psi)\tilde\Gamma^{b_1}\big((1+\tau)^\lambda\partial_t\nabla\psi\cdot\nabla\psi\big)\,dx\,d\tau\right| \le C\varepsilon^2+CK\varepsilon\sum_{0\le|a|\le N}\int_{\mathbb{R}^3}(1+t)^{2\lambda}|\partial\tilde\Gamma^a\psi|^2\,dx + CK\varepsilon\sum_{0\le|a|\le N}\int_0^t\int_{\mathbb{R}^3}(1+\tau)^\lambda|\partial\tilde\Gamma^a\psi|^2\,dx\,d\tau. \tag{2-35}$$

Part C: *Estimates of*

$$\sum_{|b_1|\le N}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{2\lambda}(\partial_t\tilde\Gamma^a\psi)\tilde\Gamma^{b_1}\big((1+\tau)^{2\lambda}(\partial_i\psi)(\partial_j\psi)\partial_{ij}\psi\big)\,dx\,d\tau\right|$$

and

$$\sum_{|b_1|\le N}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}(\tilde\Gamma^a\psi)\tilde\Gamma^{b_1}\big((1+\tau)^{2\lambda}(\partial_i\psi)(\partial_j\psi)\partial_{ij}\psi\big)\,dx\,d\tau\right|.$$

A direct computation yields

$$\begin{aligned}\tilde\Gamma^{b_1}&((\partial_i\psi)(\partial_j\psi)\partial_{ij}\psi)\\ &=\partial_i\psi\,\partial_j\psi\,\partial_{ij}\tilde\Gamma^{b_1}\psi+\sum_{N-2\le|b_2|\le|b_1|-1}(\nabla^2\tilde\Gamma^{b_2}\psi)(\nabla\tilde\Gamma^{b_3}\psi)\nabla\tilde\Gamma^{b_4}\psi\\ &\qquad+\sum_{|b_2|\le N-3}(\nabla^2\tilde\Gamma^{b_2}\psi)(\nabla\tilde\Gamma^{b_3}\psi)\nabla\tilde\Gamma^{b_4}\psi\\ &=\mathrm{III}_1+\mathrm{III}_2+\mathrm{III}_3.\end{aligned}$$

As in the treatment of II_1 in Part B, we have

$$\sum_{|b_1|\le N}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{2\lambda}(\partial_t\tilde\Gamma^a\psi)\mathrm{III}_1\,dx\,d\tau\right| \le C\varepsilon^2+CK\varepsilon\sum_{0\le|a|\le N}\int_0^t\int_{\mathbb{R}^3}(1+\tau)^\lambda|\partial\tilde\Gamma^a\psi|^2\,dx\,d\tau. \tag{2-36}$$

By (2-6) and (2-9), for the term III_2, we have

$$\begin{aligned}(1+t)^{4\lambda}&\|(\partial_t\tilde\Gamma^a\psi)(\nabla^2\tilde\Gamma^{b_2}\psi)(\nabla\tilde\Gamma^{b_3}\psi)\nabla\tilde\Gamma^{b_4}\psi\|_{L^1}\\ &\le(1+t)^{4\lambda}\|\langle r-t\rangle^{-1}(\nabla\tilde\Gamma^{b_3}\psi)\nabla\tilde\Gamma^{b_4}\psi\|_{L^\infty}\cdot\|\partial_t\tilde\Gamma^a\psi\|\cdot\|\langle r-t\rangle\nabla^2\tilde\Gamma^{b_2}\psi\|\\ &\le CK\varepsilon(1+t)^\lambda\|\partial_t\tilde\Gamma^a\psi\|\sum_{|b_4|\le|b_3|+1}\|\nabla\tilde\Gamma^{b_4}\psi\|+CK\varepsilon(1+t)^\lambda\|\partial_t\tilde\Gamma^a\psi\|^2\\ &\qquad+CK\varepsilon(1-\lambda)(1+t)^{\lambda-2}\sum_{|b_4|\le|b_3|+1}\|\tilde\Gamma^{b_4}\psi\|^2.\end{aligned}\tag{2-37}$$

By (2-6) and (2-8), for the term III_3, one has

$$(1+t)^{4\lambda}\|(\partial_t\widetilde{\Gamma}^a\psi)(\nabla^2\widetilde{\Gamma}^{b_2}\psi)(\nabla\widetilde{\Gamma}^{b_3}\psi)\nabla\widetilde{\Gamma}^{b_4}\psi\|_{L^1} \le CK\varepsilon(1+t)^{\lambda}\|\partial_t\widetilde{\Gamma}^a\psi\|\sum_{|c|\le|b_1|}\|\nabla\widetilde{\Gamma}^c\psi\|. \tag{2-38}$$

Collecting (2-36)–(2-38) together with $0\le\lambda\le 1$ (this means that the coefficient $CK\varepsilon(1-\lambda)(1+t)^{\lambda-2}$ of $\sum_{|b_4|\le|b_3|+1}\|\widetilde{\Gamma}^{b_4}\psi\|^2$ in the last line of (2-37) is nonnegative and in $L^1(0,\infty)$) yields

$$\begin{aligned}\sum_{|b_1|\le N}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{2\lambda}(\partial_t\widetilde{\Gamma}^a\psi)\widetilde{\Gamma}^{b_1}\big((1+\tau)^{2\lambda}(\partial_i\psi)(\partial_j\psi)\partial_{ij}\psi\big)\,dx\,d\tau\right| \\ \le CK\varepsilon\sum_{0\le|a|\le N}\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}|\partial\widetilde{\Gamma}^a\psi|^2\,dx\,d\tau \\ +CK\varepsilon\sum_{|b_4|\le N}\int_0^t A(\tau)\int_{\mathbb{R}^3}|\widetilde{\Gamma}^{b_4}\psi|^2\,dx\,d\tau.\end{aligned} \tag{2-39}$$

In addition,

$$\begin{aligned}2(1+t)^{3\lambda}(\Gamma^a\psi)\Gamma^{b_1}\big((\partial_i\psi)(\partial_j\psi)\partial_{ij}\psi\big) &= \operatorname{div}\big((1+t)^{3\lambda}(\Gamma^a\psi)(\nabla\psi)\Gamma^{b_1}(|\nabla\psi|^2)\big)-(1+t)^{3\lambda}(\nabla\Gamma^a\psi)(\nabla\psi)\Gamma^{b_1}(|\nabla\psi|^2) \\ &\quad -(1+t)^{3\lambda}(\Gamma^a\psi)(\Delta\psi)\Gamma^{b_1}(|\nabla\psi|^2) \\ &\quad +\sum_{|b_2|\le|b_1|-1}(1+t)^{3\lambda}(\Gamma^a\psi)(\nabla^2\Gamma^{b_2}\psi)(\nabla\Gamma^{b_3}\psi)\nabla\Gamma^{b_4}(|\psi|^2).\end{aligned}$$

Together with (2-6) and Hardy's inequality this yields

$$\begin{aligned}\sum_{|b_1|\le N}\left|\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}(\Gamma^a\psi)\Gamma^{b_1}\big((1+\tau)^{2\lambda}(\partial_i\psi)(\partial_j\psi)\partial_{ij}\psi\big)\,dx\,d\tau\right| \\ \le CK\varepsilon\sum_{|a|\le N}\int_0^t\int_{\mathbb{R}^3}(1+\tau)^{\lambda}|\partial\Gamma^a\psi|^2\,dx\,d\tau.\end{aligned} \tag{2-40}$$

Therefore, substituting (2-29)–(2-30), (2-34)–(2-35), and (2-39)–(2-40) into (2-25) and utilizing the smallness of $\varepsilon>0$ gives (2-21). □

Based on Lemmas 2.4 and 2.5, we now prove Theorem 1.1.

Proof of Theorem 1.1. By Lemmas 2.4 and 2.5, one has that, for fixed $N\ge 8$,

$$E_N(t)\le C\varepsilon^2+C(1+K\varepsilon)\int_0^t A(t')E_N(t')\,dt'.$$

Choosing the constants $K > 0$ large and $\varepsilon > 0$ small, by Gronwall's inequality one gets that, for any $t \geq 0$,

$$E_N(t) \leq e^{C(1+K\varepsilon)\|A(t)\|_{L^1}} E_N(0) \leq \tfrac{1}{2} K^2 \varepsilon^2.$$

Thus, Theorem 1.1 is proved by the assumption that $E_N(t) \leq K^2\varepsilon^2$ and a continuous induction argument. □

3. Blowup for small data in case λ > 1

In this section, we shall prove the blowup result of Theorem 1.2 which is valid in case $\lambda > 1$.

Proof of Theorem 1.2. We divide the proof into two cases.

Case 1: $\gamma = 2$. Let (ρ, u) be a smooth solution of (1-1). For $l > 0$, we define

$$P(t,l) = \int_{|x|>l} \eta(x,l)(\rho(t,x) - \bar{\rho})\,dx, \tag{3-1}$$

where

$$\eta(x,l) = |x|^{-1}(|x| - l)^2.$$

Employing the first equation in (1-1) and an integration by parts, we see that

$$\begin{aligned} \partial_t P(t,l) &= \int_{|x|>l} \eta(x,l)\partial_t(\rho(t,x) - \bar{\rho})\,dx = -\int_{|x|>l} \eta(x,l)\,\mathrm{div}(\rho u)(t,x)\,dx \\ &= \int_{|x|>l} (\nabla_x \eta)(x,l) \cdot (\rho u)(t,x)\,dx, \end{aligned}$$

where we have used the fact that $\eta(x,l) = 0$ on $|x| = l$ and that $u(t,x) = 0$ for $|x| \geq t + M$.

By differentiating $\partial_t P(t,l)$ again and using the second equation in (1-1), we find that

$$\begin{aligned} \partial_t^2 P(t,l) &= \int_{|x|>l} (\nabla_x \eta)(x,l) \cdot \partial_t(\rho u)(t,x)\,dx \\ &= -\sum_{i,j} \int_{|x|>l} (\partial_{x_i}\eta)\,\partial_{x_j}(\rho u_i u_j)\,dx - \int_{|x|>l} (\nabla_x \eta)(x,l) \cdot \nabla(p - \bar{p})\,dx \\ &\quad - \frac{\mu}{(1+t)^\lambda} \int_{|x|>l} (\nabla_x \eta)(x,l) \cdot (\rho u)(t,x)\,dx, \end{aligned} \tag{3-2}$$

where $\nabla_x \eta(x,l) = |x|^{-3}(|x|^2 - l^2)x$ vanishes on $|x| = l$ and $\bar{p} = p(\bar{\rho})$. Integration by parts implies that

$$\begin{aligned}(3\text{-}3)\qquad &\partial_t^2 P(t,l) + \frac{\mu}{(1+t)^\lambda}\partial_t P(t,l)\\ &= \sum_{i,j}\int_{|x|>l}(\partial^2_{x_i x_j}\eta)\rho u_i u_j\,dx + \int_{|x|>l}(\Delta\eta)(p-\bar{p})\,dx\\ &\equiv J_1(t,l) + J_2(t,l),\end{aligned}$$

where we have used that $p - \bar{p}$ vanishes for $|x| \ge t + M$. A direct computation of $\partial^2_{x_i x_j}\eta$ shows that

$$\begin{aligned}(3\text{-}4)\quad J_1(t,l) = &\int_{|x|>l}\frac{2l^2}{|x|^3}\rho\Big(\frac{x}{|x|}\cdot u\Big)^2 dx\\ &-\int_{|x|>l}\frac{|x|^2-l^2}{|x|^3}\rho\Big(\frac{x}{|x|}\cdot u\Big)^2 dx + \int_{|x|>l}\frac{|x|^2-l^2}{|x|^3}\rho|u|^2\,dx \ge 0.\end{aligned}$$

On the other hand, notice that

$$(3\text{-}5)\qquad \partial_l^2\eta(x,l) = 2|x|^{-1} = \Delta_x\eta(x,l).$$

Then

$$(3\text{-}6)\quad J_2(t,l) = \int_{|x|>l}\partial_l^2\eta(x,l)(p(t,x)-\bar{p})\,dx = \partial_l^2\int_{|x|>l}\eta(x,l)(p(t,x)-\bar{p})\,dx,$$

where we have used the fact that η and $\partial_l\eta$ vanish on $|x| = l$. Combining (3-3)–(3-6), we arrive at

(3-7)

$$\partial_t^2 P(t,l) - \partial_l^2 P(t,l) + \frac{\mu}{(1+t)^\lambda}\partial_t P(t,l) = f(t,l) \equiv J_1(t,l) + G(t,l) \ge G(t,l),$$

where

(3-8)

$$G(t,l) = \partial_l^2\int_{|x|>l}\eta(x,l)(p-\bar{p}-(\rho-\bar{\rho}))\,dx = \int_{|x|>l}2|x|^{-1}(p-\bar{p}-(\rho-\bar{\rho}))\,dx.$$

Thanks to $\gamma = 2$ and the sound speed $\bar{c} = \sqrt{2A\bar{\rho}} = 1$, we have

$$(3\text{-}9)\qquad p-\bar{p}-(\rho-\bar{\rho}) = A(\rho^2-\bar{\rho}^2-2\bar{\rho}(\rho-\bar{\rho})) = A(\rho-\bar{\rho})^2.$$

Substituting (3-9) into (3-8) gives

$$G(t,l) \ge 0.$$

For M_0 satisfying the condition (1-11), let $\Sigma \equiv \{(t,l) : t \ge 0,\ t + M_0 \le l \le t + M\}$ be the strip domain. By applying Riemann's representation (see [Courant and Hilbert

1962, §5.5]) with the assumptions (1-9)–(1-11), we see that the solution $P(t,l)$ to (3-7) is nonnegative in Σ. We put its proof in the Appendix. Rewrite (3-7) as

$$\partial_t^2 P(t,l)-\partial_l^2 P(t,l)+\frac{\mu}{(1+t)^\lambda}\big(\partial_t P(t,l)-\partial_l P(t,l)\big)=f(t,l)-\frac{\mu}{(1+t)^\lambda}\partial_l P(t,l).$$

By the method of characteristics we have

$$\begin{aligned}P(t,l)&=\tfrac{1}{2}P(0,l+t)+\frac{1}{2\beta(t)}P(0,l-t)+\frac{1}{2}\int_0^t\frac{1}{\beta(\tau)}\frac{\mu}{(1+\tau)^\lambda}P(0,l+t-2\tau)\,d\tau\\&\quad+\int_0^t\frac{1}{\beta(\tau)}\partial_t P(0,l+t-2\tau)\,d\tau+\frac{1}{2}\int_0^t\int_{l-t+\tau}^{l+t-\tau}\frac{\beta(\tau)}{\beta\big(\frac{l+t+\tau-y}{2}\big)}f(\tau,y)\,dy\,d\tau\\&\quad+\frac{1}{2}\int_0^t\frac{\beta(\tau)}{\beta(t)}\frac{\mu}{(1+\tau)^\lambda}P(\tau,l-t+\tau)\,d\tau\\&\quad+\frac{1}{2}\int_0^t\int_\tau^t\frac{\beta(\tau)}{\beta(s)}\frac{\mu^2}{(1+\tau)^\lambda(1+s)^\lambda}P(\tau,l+t-2s+\tau)\,ds\,d\tau\\&\quad-\frac{1}{2}\int_0^t\frac{\mu}{(1+\tau)^\lambda}P(\tau,l+t-\tau)\,d\tau;\end{aligned}$$

see (1-12). Together with assumptions (1-9)–(1-10) and $P(t,l)\ge 0$ in Σ this yields, for $l\ge t+M_0$,

$$\text{(3-10)}\quad P(t,l)\ge\frac{1}{2\beta(t)}q_0(l-t)+\frac{1}{2}\int_0^t\int_{l-t+\tau}^{l+t-\tau}\frac{\beta(\tau)}{\beta\big(\frac{l+t+\tau-y}{2}\big)}G(\tau,y)\,dy\,d\tau-\frac{1}{2}\int_0^t\frac{\mu}{(1+\tau)^\lambda}P(\tau,l+t-\tau)\,d\tau.$$

Define the function

$$\text{(3-11)}\qquad F(t)\equiv\int_0^t(t-\tau)\int_{\tau+M_0}^{\tau+M}P(\tau,l)\frac{dl}{l}\,d\tau.$$

Then, by (3-10), we have that

(3-12)

$$\begin{aligned}F''(t)&=\int_{t+M_0}^{t+M}P(t,l)\frac{dl}{l}\\&\ge\frac{1}{2\beta(t)}\int_{t+M_0}^{t+M}q_0(l-t)\frac{dl}{l}+\frac{1}{2}\int_{t+M_0}^{t+M}\int_0^t\int_{l-t+\tau}^{l+t-\tau}\frac{\beta(\tau)}{\beta\big(\frac{l+t+\tau-y}{2}\big)}G(\tau,y)\,dy\,d\tau\frac{dl}{l}\\&\qquad-\frac{1}{2}\int_{t+M_0}^{t+M}\int_0^t\frac{\mu}{(1+\tau)^\lambda}P(\tau,l+t-\tau)\,d\tau\frac{dl}{l}\\&\equiv J_3+J_4-J_5.\end{aligned}$$

From $\lambda > 1$ and assumption (1-9), we see that

$$J_3 \geq \frac{c_1}{t+M}\int_{t+M_0}^{t+M} q_0(l-t)\,dl = \frac{c_1}{t+M}\int_{M_0}^{M} q_0(l)\,dl = \frac{c_2\varepsilon}{t+M}, \tag{3-13}$$

where $c_1, c_2 > 0$ are constants independent of ε. Note that $P(\tau, y)$ is supported in $\{y : y \leq \tau+M\}$ and nonnegative in Σ. Hence, there exists a constant $C_1 > 0$ such that

$$J_5 \leq \frac{C_1}{(1+t)^\lambda}\int_0^t \int_{\tau+M_0}^{\tau+M} P(\tau, y)\frac{dy}{y}d\tau = \frac{C_1}{(1+t)^\lambda}F'(t). \tag{3-14}$$

Substituting (3-14) into (3-12) gives

$$F''(t) + \frac{C_1}{(1+t)^\lambda}F'(t) \geq J_3 + J_4. \tag{3-15}$$

To bound J_4 from below, we write

$$\begin{aligned} J_4 &= \frac{1}{2}\int_0^{t-M_1}\int_{\tau+M_0}^{\tau+M} G(\tau, y)\int_{t+M_0}^{y+t-\tau}\frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)}\frac{dl}{l}\,dy\,d\tau \\ &\quad + \frac{1}{2}\int_{t-M_1}^{t}\int_{\tau+M_0}^{2t-\tau+M_0} G(\tau, y)\int_{t+M_0}^{y+t-\tau}\frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)}\frac{dl}{l}\,dy\,d\tau \\ &\quad + \frac{1}{2}\int_{t-M_1}^{t}\int_{2t-\tau+M_0}^{\tau+M} G(\tau, y)\int_{y-t+\tau}^{y+t-\tau}\frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)}\frac{dl}{l}\,dy\,d\tau \\ &\equiv J_{4,1} + J_{4,2} + J_{4,3}, \end{aligned} \tag{3-16}$$

where $M_1 = (M-M_0)/2$. For $t < M_1$, $t-M_1$ in the limits of integration is replaced by 0. By $\lambda > 1$, for the integrand in $J_{4,1}$ we have that

$$\int_{t+M_0}^{y+t-\tau}\frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)}\frac{dl}{l} \geq c\,\frac{y-\tau-M_0}{t+M} \geq c\,\frac{(t-\tau)(y-\tau-M_0)^2}{(t+M)^2}. \tag{3-17}$$

Analogously, for the integrands in $J_{4,2}$ and $J_{4,3}$ we have that

$$\int_{t+M_0}^{y+t-\tau}\frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)}\frac{dl}{l} \geq c\,\frac{(t-\tau)(y-\tau-M_0)^2}{(t+M)^2} \tag{3-18}$$

and

$$\int_{y-t+\tau}^{y+t-\tau}\frac{\beta(\tau)}{\beta\left(\frac{l+t+\tau-y}{2}\right)}\frac{dl}{l} \geq c\,\frac{t-\tau}{t+M} \geq c\,\frac{(t-\tau)(y-\tau-M_0)^2}{(t+M)^2}, \tag{3-19}$$

where $c > 0$ is a constant. Substituting (3-17)–(3-19) into (3-16) yields

$$J_4 \geq \frac{c}{(t+M)^2}\int_0^t (t-\tau)\int_{\tau+M_0}^{\tau+M}(y-\tau-M_0)^2\partial_y^2\widetilde{G}(\tau, y)\,dy\,d\tau,$$

where $\widetilde{G}(t,l)=\int_{|x|>l}\eta(x,l)(p-\bar{p}-(\rho-\bar{\rho}))\,dx$. Note that $\widetilde{G}(\tau,y)=\partial_y\widetilde{G}(\tau,y)=0$ for $y=\tau+M$. Thus, it follows from the integration by parts together with (3-8)–(3-9) that

$$J_4 \geq \frac{c}{(t+M)^2}\int_0^t(t-\tau)\int_{\tau+M_0}^{\tau+M}\widetilde{G}(\tau,y)\,dy\,d\tau \\ \geq \frac{c}{(t+M)^2}\int_0^t(t-\tau)\int_{\tau+M_0}^{\tau+M}\int_{|x|>y}\eta(x,y)(\rho(\tau,x)-\bar{\rho})^2\,dx\,dy\,d\tau \\ \equiv \frac{c}{(t+M)^2}J_6. \tag{3-20}$$

By applying the Cauchy–Schwartz inequality to $F(t)$ defined by (3-11), we arrive at

$$F^2(t)\leq J_6\int_0^t(t-\tau)\int_{\tau+M_0}^{\tau+M}\int_{y<|x|<\tau+M}\eta(x,y)\,dx\frac{dy}{y^2}d\tau\equiv J_6J_7. \tag{3-21}$$

We estimate J_7 as

$$J_7=\int_0^t(t-\tau)\int_{\tau+M_0}^{\tau+M}\int_{y<|x|<\tau+M}\frac{(|x|-y)^2}{|x|}\,dx\frac{dy}{y^2}d\tau \\ =\int_0^t(t-\tau)\int_{\tau+M_0}^{\tau+M}\int_y^{\tau+M}4\pi l(l-y)^2\,dl\frac{dy}{y^2}d\tau \\ \leq C\int_0^t(t-\tau)\int_{\tau+M_0}^{\tau+M}(\tau+M)(\tau+M-y)^3\frac{dy}{y^2}d\tau \\ \leq C\int_0^t(t-\tau)(\tau+M)\int_{\tau+M_0}^{\tau+M}\frac{dy}{y^2}d\tau \\ \leq C\int_0^t\frac{t-\tau}{\tau+M}\,d\tau\leq C(t+M)\log(t/M+1). \tag{3-22}$$

Combining (3-13), (3-15) and (3-20)–(3-22) gives the ordinary differential inequalities

$$F''(t)+\frac{C_1}{(1+t)^\lambda}F'(t)\geq\frac{c_2\varepsilon}{t+M}, \qquad t\geq 0, \tag{3-23}$$

$$F''(t)+\frac{C_1}{(1+t)^\lambda}F'(t)\geq C\big[(t+M)^3\log(t/M+1)\big]^{-1}F^2(t), \qquad t\geq 0. \tag{3-24}$$

Next, we apply (3-23)–(3-24) to prove that the lifespan T_ε of smooth solution $F(t)$ is finite for all $0<\varepsilon\leq\varepsilon_0$. The fact that $F(0)=F'(0)=0$, together with (3-23), yields

$$F'(t)\geq C\varepsilon\log(t/M+1), \qquad t\geq 0, \tag{3-25}$$

$$F(t)\geq C\varepsilon(t+M)\log(t/M+1), \qquad t\geq t_1\equiv Me^2, \tag{3-26}$$

where the constant $C > 0$ is independent of ε. Substituting (3-26) into (3-24) derives

$$F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \geq C\varepsilon^2 (t+M)^{-1} \log(t/M+1), \quad t \geq t_1,$$

which leads to the improvement

$$F(t) \geq C\varepsilon^2 (t+M) \log^2(t/M+1), \quad t \geq t_2 \equiv Me^3 > t_1. \tag{3-27}$$

Substituting this into (3-24) derives

$$F''(t) + \frac{C_1}{(1+t)^\lambda} F'(t) \geq C\varepsilon^2 (t+M)^{-2} \log(t/M+1) F(t), \quad t \geq t_2. \tag{3-28}$$

It follows from (3-25) that $F'(t) \geq 0$ for $t \geq 0$. Then multiplying (3-28) by $F'(t)$ and integrating from t_3 (which will be chosen later) to t yield

$$F'(t)^2 \geq C_2 F'(t_3)^2 + C_3 \varepsilon^2 \int_{t_3}^t (s+M)^{-2} \log(s/M+1) [F(s)^2]' \, ds.$$

Integrating by parts yields

(3-29)

$$\begin{aligned} F'(t)^2 \geq\ & C_2 F'(t_3)^2 \\ & + C_3 \varepsilon^2 \big((t+M)^{-2} \log(t/M+1) F(t)^2 - (t_3+M)^{-2} \log(t_3/M+1) F(t_3)^2 \big) \\ & - \int_{t_3}^t \left(\frac{\log(s/M+1)}{(s+M)^2} \right)' F(s)^2 \, ds, \quad t \geq t_3, \end{aligned}$$

where

$$\left(\frac{\log(s/M+1)}{(s+M)^2} \right)' \leq 0$$

for $t \geq t_3 \geq t_2$. On the other hand, (3-23) implies

$$\big(e^{-\frac{C_1}{\lambda-1}[(1+t)^{1-\lambda}-1]} F'(t) \big)' \geq 0, \quad t \geq 0,$$

which yields for $0 \leq t \leq \tau$

$$F'(t) \leq e^{\frac{C_1}{\lambda-1}[(1+t)^{1-\lambda} - (1+\tau)^{1-\lambda}]} F'(\tau). \tag{3-30}$$

Together with $F(0) = 0$, this yields

$$F(t) = \int_0^t F'(s) \, ds \leq C_4 t F'(t), \quad t > 0. \tag{3-31}$$

Choose

$$t_3 = M \big(e^{\frac{C_2}{2C_3 C_4 \varepsilon^2}} - 1 \big) \tag{3-32}$$

which satisfies $2C_3C_4\log(t_3/M+1)\varepsilon^2 = C_2$. Together with (3-29) and (3-31), this yields

$$F'(t) \ge \sqrt{C}_3\varepsilon(t+M)^{-1}\log^{\frac{1}{2}}(t/M+1)F(t), \quad t \ge t_3. \tag{3-33}$$

By integrating (3-33) from t_3 to t, we arrive at

$$\log\frac{F(t)}{F(t_3)} \ge C\varepsilon\log^{\frac{3}{2}}\left(\frac{t+M}{t_3+M}\right), \quad t \ge t_3.$$

If $t \ge t_4 \equiv Ct_3^2$, we then have

$$\log\frac{F(t)}{F(t_3)} \ge 8\log(t/M+1).$$

Together with (3-27) for $F(t_3)$, this yields

$$F(t) \ge C\varepsilon^2(t+M)^8, \quad t \ge t_4. \tag{3-34}$$

Substituting this into (3-24) derives

$$F''(t) + \frac{C_1}{(1+t)^\lambda}F'(t) \ge C\varepsilon F(t)^{\frac{3}{2}}, \quad t \ge t_4.$$

Multiplying this differential inequality by $F'(t)$ and integrating from t_4 to t yields

$$F'(t)^2 \ge C\varepsilon\big(F(t)^{\frac{5}{2}} - F(t_4)^{\frac{5}{2}}\big).$$

On the other hand, (3-30) and (3-31) imply that, for $t \ge t_4$,

$$F(t) = F'(\xi)(t-t_4) + F(t_4) \ge CF'(t_4)(t-t_4) \ge CF(t_4)\frac{t-t_4}{t_4},$$

where $t_4 \le \xi \le t$. If $t \ge t_5 \equiv Ct_4$, then we have

$$F(t)^{\frac{5}{2}} - F(t_4)^{\frac{5}{2}} \ge \tfrac{1}{2}F(t)^{\frac{5}{2}}.$$

Thus

$$F'(t) \ge C\sqrt{\varepsilon}F(t)^{\frac{5}{4}}, \quad t \ge t_5. \tag{3-35}$$

If $T_\varepsilon > 2t_5$, then integrating (3-35) from t_5 to T_ε derives

$$F(t_5)^{-\frac{1}{4}} - F(T_\varepsilon)^{-\frac{1}{4}} \ge C\sqrt{\varepsilon}T_\varepsilon.$$

We see from (3-34) and $t_5 = Ct_3^2$ that

$$F(t_5) \ge C\varepsilon^2 e^{C/\varepsilon^2},$$

which together with $F(T_\varepsilon) > 0$ is a contradiction. Thus, $T_\varepsilon \le 2t_5 = Ct_3^2$. From the choice of t_3 in (3-32), we see that $T_\varepsilon \le e^{C/\varepsilon^2}$.

Case 2: $\gamma > 1$ and $\gamma \neq 2$. Recall that the sound speed is $\bar{c} = \sqrt{\gamma A \bar{\rho}^{\gamma-1}} = 1$. Instead of (3-9) we have

$$p - \bar{p} - (\rho - \bar{\rho}) = A\big(\rho^\gamma - \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1}(\rho - \bar{\rho})\big) \equiv A\psi(\rho, \bar{\rho}).$$

The convexity of ρ^γ for $\gamma > 1$ implies that $\psi(\rho, \bar{\rho})$ is positive for $\rho \neq \bar{\rho}$. Applying Taylor's theorem, we have

$$\psi(\rho, \bar{\rho}) \geq C(\gamma, \bar{\rho})\,\Phi_\gamma(\rho, \bar{\rho}),$$

where $C(\gamma, \bar{\rho})$ is a positive constant and Φ_γ is given by

$$\Phi_\gamma(\rho, \bar{\rho}) = \begin{cases} (\bar{\rho} - \rho)^\gamma, & \rho < \frac{1}{2}\bar{\rho}, \\ (\rho - \bar{\rho})^2, & \frac{1}{2}\bar{\rho} \leq \rho \leq 2\bar{\rho}, \\ (\rho - \bar{\rho})^\gamma, & \rho > 2\bar{\rho}. \end{cases}$$

For $\gamma > 2$, we have that $(\bar{\rho} - \rho)^\gamma = (\bar{\rho} - \rho)^2(\bar{\rho} - \rho)^{\gamma-2} \geq C(\gamma, \bar{\rho})(\rho - \bar{\rho})^2$ for $2\rho < \bar{\rho}$ and $(\rho - \bar{\rho})^\gamma = (\rho - \bar{\rho})^2(\rho - \bar{\rho})^{\gamma-2} \geq C(\gamma, \bar{\rho})(\rho - \bar{\rho})^2$ for $\rho > 2\bar{\rho}$. Thus, $\psi(\rho, \bar{\rho}) \geq C(\gamma, \bar{\rho})(\rho - \bar{\rho})^2$. In this case, Theorem 1.2 can be shown completely analogously to Case 1.

Next we treat the case $1 < \gamma < 2$. We define $F(t)$ as in (3-11),

$$F(t) = \int_0^t \int_{\tau+M_0}^{\tau+M} \frac{1}{l} \int_{|x|>l} \frac{(|x| - l)^2}{|x|}(\rho(\tau, x) - \bar{\rho})\,dx\,dl\,d\tau.$$

Similarly to the case of $\gamma = 2$, we have

$$F''(t) \geq J_3 + J_4 - J_5, \tag{3-36}$$

where

$$J_3 \geq \frac{C\varepsilon}{t + M},$$
$$J_4 \geq C(t + M)^{-2}\tilde{J}6,$$
$$J_5 \leq \frac{C_1}{(1 + t)^\lambda} F'(t),$$

and

$$\tilde{J}_6 = \int_0^t (t - \tau) \int_{\tau+M_0}^{\tau+M} \int_{|x|>y} \frac{(|x| - y)^2}{|x|}\,\Phi_\gamma(\rho(\tau, x) - \bar{\rho})\,dx\,dy\,d\tau.$$

Denote $\Omega_1 = \{(\tau, x) : \bar{\rho} \leq \rho(\tau, x) \leq 2\bar{\rho}\}$, $\Omega_2 = \{(\tau, x) : \rho(\tau, x) > 2\bar{\rho}\}$, and $\Omega_3 = \{(\tau, x) : \rho(\tau, x) < \bar{\rho}\}$. Divide $F(t)$ into a sum of the three integrals over the domains Ω_i $(1 \leq i \leq 3)$

$$F(t) = F_1(t) + F_2(t) + F_3(t) \equiv \int_{\Omega_1} \cdots + \int_{\Omega_2} \cdots + \int_{\Omega_3} \cdots.$$

Corresponding to the three parts of $F(t)$, we define $\tilde{J}_6 \equiv \tilde{J}_{6,1} + \tilde{J}_{6,2} + \tilde{J}_{6,3}$. In view of $F(t) \ge 0$ and $F_3(t) \le 0$, we have

$$F(t) \le F_1(t) + F_2(t).$$

Applying Hölder's inequality for the domains Ω_1 and Ω_2, we obtain that

$$\begin{aligned} F(t) \le \tilde{J}_{6,1}^{\frac{1}{2}} &\Big(\int_0^t (t-\tau) \int_{\tau+M_0}^{\tau+M} \frac{1}{y^2} \int_{y<|x|\le\tau+M} \frac{(|x|-y)^2}{|x|} \, dx\, dy\, d\tau \Big)^{\frac{1}{2}} \\ &+ \tilde{J}_{6,2}^{\frac{1}{\gamma}} \Big(\int_0^t (t-\tau) \int_{\tau+M_0}^{\tau+M} \frac{1}{y^{\frac{\gamma}{\gamma-1}}} \int_{y<|x|\le\tau+M} \frac{(|x|-y)^2}{|x|} \, dx\, dy\, d\tau \Big)^{\frac{\gamma-1}{\gamma}} \\ &\le \tilde{J}_6^{\frac{1}{2}} (t+M)^{\frac{1}{2}} \log^{\frac{1}{2}}(t/M+1) + \tilde{J}_6^{\frac{1}{\gamma}} (t+M)^{\frac{\gamma-1}{\gamma}} \\ &= \big(\tilde{J}_6 (t+M)^{-1}\big)^{\frac{1}{2}} (t+M) \log^{\frac{1}{2}}(t/M+1) + \big(\tilde{J}_6 (t+M)^{-1}\big)^{\frac{1}{\gamma}} (t+M). \end{aligned}$$

In view of $1<\gamma<2$, we have $\frac{1}{2\gamma} < \frac{1}{2} < \frac{1}{\gamma}$. Applying Young's inequality yields

$$F(t) \le \Big(\big(\tilde{J}_6 (t+M)^{-1}\big)^{\frac{1}{2\gamma}} + \big(\tilde{J}_6 (t+M)^{-1}\big)^{\frac{1}{\gamma}} \Big) (t+M) \log^{\frac{1}{2}}(t/M+1), \quad t \ge \tilde{t}_1 \equiv Me.$$

Together with the fact that $F(t) \ge C\varepsilon (t+M)\log(t/M+1)$, this yields

$$\tilde{J}_6 \ge C F(t)^{\gamma} (t+M)^{1-\gamma} \log^{-\frac{\gamma}{2}}(t/M+1), \quad t \ge \tilde{t}_1.$$

Substituting this into (3-36) yields

$$F''(t) + \frac{C_1}{(1+t)^{\lambda}} F'(t) \ge \frac{C\varepsilon}{t+M}, \quad t \ge 0, \tag{3-37}$$

$$F''(t) + \frac{C_1}{(1+t)^{\lambda}} F'(t) \ge C F(t)^{\gamma} (t+M)^{-1-\gamma} \log^{-\frac{\gamma}{2}}(t/M+1), \quad t \ge \tilde{t}_1. \tag{3-38}$$

Substituting $F(t) \ge C\varepsilon (t+M)\log(t/M+1)$ into (3-38) yields

$$F''(t) + \frac{C_1}{(1+t)^{\lambda}} F'(t) \ge C\varepsilon^{\gamma} (t+M)^{-1} \log^{\frac{\gamma}{2}}(t/M+1).$$

Integrating this yields

$$F(t) \ge C\varepsilon^{\gamma} (t+M) \log^{\frac{\gamma+2}{2}}(t/M+1).$$

Substituting this into (3-38) again gives

$$\begin{aligned} F''(t) &+ \frac{C_1}{(1+t)^{\lambda}} F'(t) \\ &\ge C\varepsilon^{\gamma^2} (t+M)^{-1} \log^{\frac{\gamma(\gamma+1)}{2}}(t/M+1) = C\varepsilon^{\gamma^2} (t+M)^{-1} \log^{\frac{\gamma(\gamma^2-1)}{2(\gamma-1)}}(t/M+1). \end{aligned}$$

Repeating this process n times, we see that

$$F''(t)+\frac{C_1}{(1+t)^\lambda}F'(t)\ge C\varepsilon^{\gamma^n}(t+M)^{-1}\log^{\frac{\gamma(\gamma^n-1)}{2(\gamma-1)}}(t/M+1), \tag{3-39}$$

where $n=[\log_\gamma 2]$. Solving (3-39) yields

$$F(t)\ge C\varepsilon^{\gamma^n}(t+M)\log^{\frac{\gamma(\gamma^n-1)}{2(\gamma-1)}+1}(t/M+1),\quad t\ge\tilde t_2,$$

where $\tilde t_2>0$ is a constant only depending on γ. Substituting this into (3-38) derives

$$\begin{aligned}F''(t)+\frac{C_1}{(1+t)^\lambda}&F'(t)\\ &\ge CF(t)\varepsilon^{\gamma^n(\gamma-1)}(t+M)^{-2}\log^{\frac{\gamma^{n+1}-2}{2}}(t/M+1),\quad t\ge\tilde t_2,\end{aligned}\tag{3-40}$$

where $\frac{1}{2}(\gamma^{n+1}-2)>0$ by the choice of $n=[\log_\gamma 2]$. Since (3-40) is analogous to (3-28), as in Case 1, we can choose

$$\tilde t_3=O\Big(e^{C\varepsilon^{-\frac{2\gamma^n(\gamma-1)}{\gamma^{n+1}-2}}}\Big)$$

such that

$$F'(t)\ge C\varepsilon^{\frac{\gamma^n(\gamma-1)}{2}}(t+M)^{-1}\log^{\frac{\gamma^{n+1}-2}{4}}(t/M+1)F(t),\quad t\ge\tilde t_3,$$

which is similar to (3-33) and yields

$$F(t)\ge C\varepsilon^{C_\gamma}(t+M)^{\frac{2(\gamma+2)}{\gamma-1}},\quad t\ge\tilde t_4\equiv C\tilde t_3^2, \tag{3-41}$$

where $C_\gamma>0$ is a constant depending on γ. Substituting (3-41) into (3-38) yields

$$F''(t)+\frac{C_1}{(1+t)^\lambda}F'(t)\ge C\varepsilon^{C_\gamma}F(t)^{\frac{\gamma+1}{2}},\quad t\ge\tilde t_4. \tag{3-42}$$

Multiplying (3-42) by $F'(t)$ and integrating over the variable t as in Case 1, we have

$$F'(t)\ge C\varepsilon^{C_\gamma}F(t)^{\frac{\gamma+3}{4}},\quad t\ge\tilde t_5\equiv C\tilde t_4.$$

Together with $\gamma>1$ and the choice of $\tilde t_3$, this yields $T_\varepsilon<\infty$.

Both Case 1 and Case 2 complete the proof of Theorem 1.2. □

4. Blowup for large data

In this section, we establish a blowup result for large amplitude smooth solutions of (1-1) which is valid for all $\lambda\ge 0$. More precisely, instead of (1-1) we consider the Cauchy problem

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + p I_3) = -\dfrac{\mu}{(1+t)^\lambda}\rho u, \\ \rho(0,x) = \bar\rho + \tilde\rho_0(x), \quad u(0,x) = \tilde u_0(x), \end{cases} \tag{4-1}$$

where $\tilde\rho_0, \tilde u_0 \in C_0^\infty(\mathbb{R}^3)$, supp $\tilde\rho_0$, supp $\tilde\rho_0 \subseteq B(0,M) \equiv \{x : |x| \le M\}$, and $\rho(0,\cdot) > 0$. Motivated by the treatment of the special case of $\lambda = 0$ in [Sideris et al. 2003], we introduce the functions

$$H(t) \equiv \int_{\mathbb{R}^3} x \cdot (\rho u)(t,x)\,dx, \qquad L(t) \equiv \int_{\mathbb{R}^3} (\rho(t,x) - \bar\rho)\,dx,$$
$$\gamma(t) \equiv (t+M)^2\Big(L(0) + \frac{4\pi^2\bar\rho}{3}(t+M)^3\Big),$$

and also remind the reader of the definition of the function β in (1-12).

Then we have the following result:

Theorem 4.1. *Suppose that $L(0) \ge 0$ and*

$$H(0)\int_0^{T^*} \frac{d\tau}{\gamma(\tau)\beta(\tau)} > 1. \tag{4-2}$$

for some $T^* > 0$. Then $T < T^*$ holds for any solution $(\rho, u) \in C^1([0,T]\times\mathbb{R}^3)$ of (4-1).

Proof. From the first equation of (4-1), we see that

$$L'(t) = -\int_{\mathbb{R}^3} \operatorname{div}(\rho u)\,dx = 0,$$

which implies $L(t) = L(0)$. Applying the second equation of (4-1), we find that

$$H'(t) = \int_{\mathbb{R}^3} x \cdot \partial_t(\rho u)(t,x)\,dx = \int_{\mathbb{R}^3} x \cdot \Big[-\operatorname{div}(\rho u \otimes u) - \nabla p - \frac{\mu}{(1+t)^\lambda}\rho u\Big]\,dx.$$

An integration by parts gives

$$H'(t) + \frac{\mu}{(1+t)^\lambda} H(t) = \int_{\mathbb{R}^3} \big(\rho|u|^2 + 3\big(p(\rho) - p(\bar\rho)\big)\big)\,dx. \tag{4-3}$$

Note that the convexity of $p = A\rho^\gamma$ for $\gamma > 1$ and $c(\bar\rho) = 1$ imply that

$$\int_{\mathbb{R}^3} \big(p(\rho) - p(\bar\rho)\big)\,dx \ge \int_{\mathbb{R}^3} A\gamma\bar\rho^{\gamma-1}(\rho - \bar\rho)\,dx = L(0). \tag{4-4}$$

Furthermore, by applying the Cauchy–Schwartz inequality to $H(t)$ and taking into account $\operatorname{supp} u(t,\cdot) \subseteq B(0, M+t)$ for any fixed $t \geq 0$, we have

$$H(t)^2 \leq \left(\int_{\mathbb{R}^3} \rho|u|^2\,dx\right)\left(\int_{|x|\leq t+M} \rho|x|^2\,dx\right) \tag{4-5}$$
$$\leq (t+M)^2\left(L(0)+\frac{4\pi^2\bar{\rho}}{3}(t+M)^3\right)\int_{\mathbb{R}^3}\rho|u|^2\,dx$$
$$=\gamma(t)\int_{\mathbb{R}^3}\rho|u|^2\,dx.$$

Substituting (4-4)–(4-5) into (4-3) yields

$$H'(t)+\frac{\mu}{(1+t)^\lambda}H(t) \geq \frac{H(t)^2}{\gamma(t)}+3L(0). \tag{4-6}$$

Together with $L(0) \geq 0$ and $H(0) > 0$ due to (4-2), this shows that $H(t) > 0$ for all $t \in [0, T]$. Denoting $G(t) \equiv \beta(t)H(t)$, from (1-12) and (4-6) we then get that

$$G'(t) \geq \frac{G^2(t)}{\gamma(t)\beta(t)}. \tag{4-7}$$

Now suppose that $T \geq T^*$. Then integrating (4-7) from 0 to T yields

$$\frac{1}{H(0)}-\frac{1}{G(T)} \geq \int_0^T \frac{d\tau}{\gamma(\tau)\beta(\tau)} \geq \int_0^{T^*} \frac{d\tau}{\gamma(\tau)\beta(\tau)},$$

which is a contradiction in view of $G(T) > 0$ and (4-2). □

Appendix: Proof of the nonnegativity of $P(t,l)$ in $\Sigma \equiv \{(t,l) : t \geq 0,\ t+M_0 \leq l \leq t+M\}$

We fixed a point $A = (t_A, l_A) \in \Sigma$. In the characteristic coordinates $\xi = 1+t-l$ and $\zeta = 1+t+l$, (3-7) can be written as

$$\mathscr{L}\bar{P} \equiv \partial^2_{\xi\zeta}\bar{P}+\frac{2^{\lambda-2}\mu}{(\xi+\zeta)^\lambda}(\partial_\xi\bar{P}+\partial_\zeta\bar{P}) = \frac{\bar{f}}{4}, \tag{A-1}$$

where $\bar{P}(\xi,\zeta) \equiv P\left(\frac{\zeta+\xi}{2}-1, \frac{\zeta-\xi}{2}\right)$. The adjoint operator $\mathscr{L}^*$ of $\mathscr{L}$ has the form

$$\mathscr{L}^*\mathcal{R} \equiv \partial^2_{\xi\zeta}\mathcal{R}-\frac{2^{\lambda-2}\mu}{(\xi+\zeta)^\lambda}(\partial_\xi\mathcal{R}+\partial_\zeta\mathcal{R})+\frac{2^{\lambda-1}\mu\lambda}{(\xi+\zeta)^{\lambda+1}}\mathcal{R}. \tag{A-2}$$

For the point $A = (\xi_A, \zeta_A)$ with $\xi_A+\zeta_A = 2(1+t_A) \geq 2$, write $B = (2-\zeta_A, \zeta_A)$ and $C = (\xi_A, 2-\xi_A)$, and let $\mathscr{D}$ the domain surrounded by the triangle ABC (see Figure 1 below).

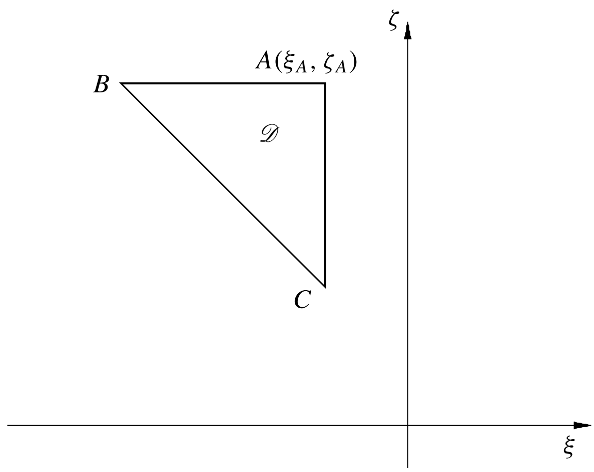

Figure 1. (ξ, ζ)-plane.

Let the numbers a and b satisfy $a+b=1$ and $ab=\frac{1}{2}\mu\lambda$. We define

$$z \equiv -\frac{(\xi_A-\xi)(\zeta_A-\zeta)}{(\xi_A+\zeta_A)(\xi+\zeta)} \tag{A-3}$$

and

$$\mathcal{R}(\xi,\zeta;\xi_A,\zeta_A) \equiv \left[\frac{\beta(\xi+\zeta-1)}{\beta(\xi_A+\zeta_A-1)}\right]^{2^{\lambda-2}} \Psi(a,b,1;z); \tag{A-4}$$

here the definition of function β is given in (1-12) and Ψ is the hypergeometric function. From this and direct calculation, we infer

$$\mathscr{L}^*\mathcal{R} = \left[\frac{2^{\lambda-2}\mu\lambda}{(\xi+\zeta)^{\lambda+1}} - \frac{\mu\lambda}{2(\xi+\zeta)^2} - \frac{4^{\lambda-2}\mu^2}{(\xi+\zeta)^{2\lambda}}\right]\mathcal{R}. \tag{A-5}$$

On the other hand, from (A-1)–(A-2) we arrive at

$$\mathcal{R}\mathscr{L}\bar{P} - \bar{P}\mathscr{L}^*\mathcal{R} = \partial_\zeta\left(\mathcal{R}\partial_\xi\bar{P} + \frac{2^{\lambda-2}\mu}{(\xi+\zeta)^\lambda}\mathcal{R}\bar{P}\right) - \partial_\xi\left(\bar{P}\partial_\zeta\mathcal{R} - \frac{2^{\lambda-2}\mu}{(\xi+\zeta)^\lambda}\mathcal{R}\bar{P}\right).$$

Integrating this over $\mathscr{D}$ yields

$$\begin{aligned}\bar{P}(A) = {} & \tfrac{1}{2}\mathcal{R}(C;A)\bar{P}(C) + \tfrac{1}{2}\mathcal{R}(B;A)\bar{P}(B) \\ & + \iint_{\mathscr{D}}(\mathcal{R}\mathscr{L}\bar{P} - \bar{P}\mathscr{L}^*\mathcal{R})\,d\xi\,d\zeta + \int_{BC}\left(\tfrac{1}{2}\mathcal{R}\partial_\xi\bar{P} - \tfrac{1}{2}\bar{P}\partial_\xi\mathcal{R} + \frac{\mu}{4}\mathcal{R}\bar{P}\right)d\xi \\ & + \left(\tfrac{1}{2}\bar{P}\partial_\zeta\mathcal{R} - \tfrac{1}{2}\mathcal{R}\partial_\zeta\bar{P} - \frac{\mu}{4}\mathcal{R}\bar{P}\right)d\zeta.\end{aligned} \tag{A-6}$$

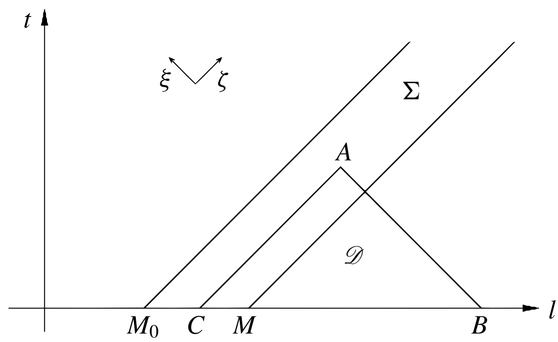

Figure 2. (t, l)-plane.

Returning to the variable (t, l) (see Figure 2), we find in the second line of (A-6) that

$$\text{(A-7)}\quad \begin{aligned}\int_{BC}\cdots &= \int_B^C \Big[\tfrac{1}{4}\mathcal{R}(\partial_t-\partial_l)P - \tfrac{1}{4}P(\partial_t-\partial_l)\mathcal{R} + \frac{\mu}{4}\mathcal{R}P\Big](-dl)\\ &\qquad + \Big[\tfrac{1}{4}P(\partial_t+\partial_l)\mathcal{R} - \tfrac{1}{4}\mathcal{R}(\partial_t+\partial_l)P - \frac{\mu}{4}\mathcal{R}P\Big]\,dl\\ &= \int_{l_A-t_A}^{l_A+t_A} \Big[\frac{\mu}{2}\mathcal{R}P + \tfrac{1}{2}\mathcal{R}\partial_t P - \tfrac{1}{2}P\partial_t\mathcal{R}\Big]\Big|_{t=0}\,dl\\ &= \int_{l_A-t_A}^{l_A+t_A} \beta(t_A)^{-\frac{1}{2}}\Big[\Psi(a,b,1;z|_{t=0})\Big(\frac{\mu}{4}q_0(l)+\tfrac{1}{2}q_1(l)\Big)\\ &\qquad\qquad - \frac{\mu\lambda}{4}\Psi(a+1,b+1,2;z|_{t=0})q_0(l)z_t|_{t=0}\Big]\,dl,\end{aligned}$$

where we have used the formula $\Psi'(a,b,c;z) = \frac{ab}{c}\Psi(a+1,b+1,c+1;z)$ (see [Erdélyi et al. 1953, page 58]). From the definition (A-3), we arrive at

$$z = -\frac{(t_A - l_A - t + l)(t_A + l_A - t - l)}{4(1+t_A)(1+t)}$$

and

$$\text{(A-8)}\qquad z_t|_{t=0} = \frac{t_A}{2(1+t_A)} - z|_{t=0}.$$

If $(t, l) \in \Sigma \cap \overline{\mathscr{D}}$, we infer

$$\text{(A-9)}\qquad 0 \ge z \ge -\tfrac{1}{2}(M - M_0) \ge -\tfrac{1}{2}\delta_0,$$

which implies that (1-8) holds. This, together with (A-7)–(A-9) and the assumption (1-11) of $\Lambda \ge \frac{3}{2}\mu\lambda$, yields that the integral in the second line of (A-6) is nonnegative.

Next we prove that $P(t,l) \geq 0$ for all $(t,l) \in \Sigma$. Define

$$\bar{t} \equiv \inf\{t : \exists\, l \in (t+M_0, t+M) \text{ such that } P(t,l) < 0\}.$$

From assumption (1-9), we get $\bar{t} > 0$. If $\bar{t} < +\infty$, we see that there exists $\bar{l} \in (\bar{t}+M_0, \bar{t}+M)$ such that $P(\bar{t},\bar{l}) = 0$. Moreover, we have $P(t,l) \geq 0$ for $t < \bar{t}$. Choose $A = (t_A, l_A) = (\bar{t}, \bar{l})$ in (A-6). From (A-4)–(A-5) and (1-8) we infer $\mathscr{L}^*\mathcal{R} \leq 0$ for $\lambda > 1$ and $(t,l) \in \Sigma \cap \mathscr{D}$. It follows from $f(t,l) \geq 0$ in (3-7), (1-8), (1-9), and (A-6) that

$$P(\bar{t},\bar{l}) \geq \tfrac{1}{2}\mathcal{R}(C;A)P(0,\bar{l}-\bar{t}) + \iint_{\Sigma\cap\mathscr{D}} (\mathcal{R}\mathscr{L}\bar{P} - \bar{P}\mathscr{L}^*\mathcal{R})\, d\xi\, d\zeta \geq \tfrac{1}{4}q_0(\bar{l}-\bar{t}) > 0,$$

which is a contradiction with $P(\bar{t},\bar{l}) = 0$. Consequently, we conclude that $\bar{t} = +\infty$ and $P(t,l) \geq 0$ for all $(t,l) \in \Sigma$.

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