**FORMAL CONFLUENCE
OF QUANTUM DIFFERENTIAL OPERATORS**

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**We prove that a differential operator in the usual sense is formally the limit of quantum differential operators. For this purpose, we introduce the notion of a twisted differential operator of infinite level and prove that, formally, such an object is independent of the choice of the twist. Our method provides explicit formulas.**

## Introduction

Classically, the finite difference operator and the $q$-difference operator are given by the formulas

$$\Delta_h(f)(x) = \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad \delta_q(f)(x) = \frac{f(qx) - f(x)}{qx - x},$$

and we can obtain the differentiation operator by passing to the limit

$$\partial(f) = \lim_{h \to 0} \Delta_h(f) = \lim_{q \to 1} \delta_q(f).$$

By using any of these operators, we may consider the notions of finite difference systems, $q$-difference systems and differential systems. The confluence process

consists in using the solutions of a difference (finite difference or $q$-difference) system in order to approximate the solutions of a differential system.

In the late 19th century, C. Guichard [1887] solved the problem of integration along a finite difference operator. In the early 20th century, in a couple of articles in the Transactions of the AMS, Georges D. Birkhoff [1911] and R. D. Carmichael [1911] gave reasonable conditions for solving linear finite difference systems. Shortly after, Carmichael [1912] applied the same methods to linear $q$-difference systems. This was later improved by C. Raymond Adams [1928/29]. There exists an abundant literature on these questions and we would like to jump to the present and mention the recent paper by Anne Duval and Julien Roques [2008] where they study Fuchsian difference systems and their confluence. More generally, the school of Jean-Pierre Ramis produced a lot of interesting material during the last two decades (see [Di Vizio et al. 2003] for a survey). Among them, we mention in particular the work of Jacques Sauloy on resolution, classification and confluence of Fuchsian differential equations (see [Sauloy 2000] for example) and on the Galois theory of these equations [Sauloy 2003].

While these works look at equations over the complex numbers, difference systems have also been studied over $p$-adic fields by Lucia di Vizio [2004], for example, and Andrea Pulita [2008; 2017]. Pulita showed that, over a $p$-adic field, confluence is a very rigid process in the sense that we can approximate a linear differential system by a $q$-difference system that has exactly the same (formal) solutions. Finally, we want to mention the work of Charlotte Hardouin [2010] who studies $q$-difference Galois theory over a field of positive characteristic in the "exotic" case in which $q$ is a root of unity (which is also taken into account by Pulita).

Our approach here is very general in the sense that we do not make any assumption about the base field or the value of the parameter $q$ for example. Actually, differential systems and difference systems may be seen as particular instances of what we called *twisted* differential systems in [Le Stum and Quirós 2015b] (see also [André 2001] for a similar approach). Let us be more precise. If $R$ is a commutative ring, then a twisted $R$-algebra (or a difference $R$-algebra, if we name them in the spirit of [Cohn 1965]) is a commutative $R$-algebra $A$ endowed with an $R$-linear ring endomorphism $\sigma$. Actually, in [Le Stum and Quirós 2015b] we allowed more generally families of endomorphisms satisfying some relations but we will concentrate here on the one-dimensional case and we will use either one endomorphism or a system of roots of this endomorphism. One may then consider the $\sigma$-derivations $D$ of $A$, or more generally, the $\sigma$-derivations $D$ of an $A$-module $M$: $R$-linear maps that satisfy the twisted Leibniz rule

$$D(xs) = D(x)s + \sigma(x)D(s).$$

A $\sigma$-differential module is then defined as an $A$-module $M$ endowed with an action by $\sigma$-derivations of all the $\sigma$-derivations of $A$. In the classical situation above, we had $\sigma(x) = x$, $\sigma(x) = qx$ or $\sigma(x) = x + h$, giving rise to an action of $\partial$, $\delta_q$ or $\Delta_h$ respectively. In our more general setting, the confluence question amounts to comparing twisted differential systems for different twists.

Our method is original in the sense that it relies on a generalization of the classical notion of a ring of differential operators. In order to do that, we may first fix what we call a $\sigma$-coordinate $x \in A$, so that there exists a basis $\partial_{\sigma,A}$ for the $\sigma$-derivations of $A$ such that $\partial_{\sigma,A}(x) = 1$. We may then define the twisted Weyl algebra $D_\sigma$ as the Ore extension of $A$ by $\sigma$ and $\partial_{\sigma,A}$ (as in [Bourbaki 2012], Proposition 1.4). This is the free $A$-module on the generators $\partial_\sigma^k$ for $k \in \mathbb{N}$ that satisfies

$$\partial_\sigma \circ f = \partial_{\sigma,A}(f) + \sigma(f)\partial_\sigma$$

when $f \in A$. A $\sigma$-differential module is then the same thing as a $D_\sigma$-module. So now, what we want to understand is the relation between $D_\sigma$ and $D_\tau$ when $\tau$ is another $R$-linear endomorphism of $A$ with the same twisted coordinate $x$. In the end, we will be essentially interested in $\tau(x) = x$ (the usual case where we simply write $D$ instead of $D_\tau$) and $\sigma(x) = qx + h$ with $q, h \in R$ (the quantum case). Note that the condition of $q$ being a root of unity in the quantum case below is analogous to having a positive characteristic in the usual case and will require some care.

One may usually see a ring of differential operators as some dual of a ring of formal functions (see the first section of [Grothendieck 1967] for example). Doing this directly for $D_\sigma$ would require understanding the notion of $\sigma$-divided powers on the function side. This is a very interesting question that we postpone to a forthcoming article. Here, we will actually replace the twisted Weyl algebra $D_\sigma$ with a Grothendieck ring of differential operators $D_\sigma^{(\infty)}$ (so that the $\sigma$-divided powers live naturally on the differential operator side). It so happens that the classical construction works incredibly well for this particular generalization. It is actually sufficient to replace the usual powers of an ideal with the twisted powers introduced in [Le Stum and Quirós 2015a]. But now we are faced with two questions: the comparison between $D_\sigma^{(\infty)}$ and $D_\tau^{(\infty)}$ on one hand and the comparison of $D_\sigma$ with $D_\sigma^{(\infty)}$ on the other.

We will show that, if we denote by $\widehat{D}_\sigma^{(\infty)}$ the completion along the divided powers of $\partial_\sigma$, then there always exists a natural isomorphism $\widehat{D}_\sigma^{(\infty)} \simeq \widehat{D}_\tau^{(\infty)}$ (the formal deformation of Proposition 7.4) and we will be able to give very explicit formulas. For example, in the case $\tau(x) = x$ and $\sigma(x) = qx$, we can write (over a field of characteristic zero when $q$ is not a root of unity):

$$\partial_\sigma = \sum_{k=1}^{\infty} \frac{1}{k!}(q-1)^{k-1}x^{k-1}\partial^k \quad \text{and} \quad \partial = \sum_{k=1}^{\infty} \frac{(1-q)^k}{1-q^k}x^{k-1}\partial_\sigma^k.$$

In the quantum situation, there exists a canonical map $D_\sigma \to D_\sigma^{(\infty)}$ whose image is exactly the ring $\overline{D}_\sigma$ of small (or naive) differential operators. Under some conditions on $q$ ($R$ $q$-divisible and $q$ -char($R$) $= 0$, see [Le Stum and Quirós 2015b]), then all three rings $D_\sigma$, $\overline{D}_\sigma$ and $D_\sigma^{(\infty)}$ are actually equal. This happens for example in the usual case when $R$ is a $\mathbb{Q}$-algebra and in the quantum case when $q$ belongs to a subfield $K$ of $R$ and $q$ is not a root of unity. When all these conditions are satisfied, we easily derive from the formal deformation our first confluence Theorem 8.3: there exists a map $D_\sigma \to \widehat{D}$ with dense image.

We will also consider the case where $q$ is a primitive $p$-th root of unity (but still $R$ a $\mathbb{Q}$-algebra). It is then necessary to use a complete family $\underline{q}$ of $p^n$-th roots of $q$. We can define what we call a rooted Weyl algebra $D_{\underline{\sigma}}$ by taking the limit on all $D_{\sigma_n}$ and build a map $D_{\underline{\sigma}} \to \widehat{D}$ with dense image. This is the second confluence Theorem 9.13.

It is interesting to notice that this last theorem puts together the ring $D$ that supports the conjecture of Jacques Dixmier [1968] (whose higher version is equivalent to the Jacobian conjecture [Belov-Kanel and Kontsevich 2007]) and an Azumaya algebra where the Morita equivalence could be applied. Recall that the Jacobian conjecture (or global inversion theorem) states that an endomorphism of the complex affine space whose Jacobian is invertible is necessarily an isomorphism. Recall also that Dixmier's conjecture states that any endomorphism of a complex Weyl algebra such as $D$ is always an automorphism. This conjecture is not valid for $D_{\underline{\sigma}}$. However, this last ring is a lot easier to study because it is essentially a matrix algebra.

In a forthcoming paper, we will prove an ultrametric version of the results presented here. More precisely, we will introduce the notion of twisted differential operator of given radius on an affinoid algebra and show that this notion is essentially independent of the choice of the twist. This will lead to an explicit equivalence between differential systems and difference systems, generalizing a theorem of Pulita [2008]. Recall that the first results in this direction were already obtained by Yves André and Lucia Di Vizio [2004].

In a forthcoming joint paper with Michel Gros, we will introduce the notion of quantum divided power and apply our methods in order to obtain a ring of quantum differential operators of level zero. It happens to be isomorphic to the quantum Weyl algebra. However, this new approach also provides the notion of quantum $p$-curvature. We will then use some Frobenius action and obtain an Azumaya splitting of the quantum Weyl algebra as well as a quantum Simpson's correspondence much as in [Gros and Le Stum 2014].

Both authors thank Michel Gros for all the fruitful conversations that we had all together.

For us, a *ring* has an associative multiplication (not commutative in general) and a two-sided unit. Morphisms of rings are always assumed to preserve the unit. A module always means a *left* module. We will essentially consider 1-twisted

rings from [Le Stum and Quirós 2015b] and simply call them twisted rings. More precisely, throughout the paper, $R$ will denote a *commutative ring* and $A$ a *twisted commutative R-algebra:* a commutative $R$-algebra endowed with an $R$-linear ring endomorphism $\sigma_A$.

## 1. Twisted principal parts

We will introduce here the notion of twisted principal part (functions on twisted infinitesimal neighborhoods of the diagonal).

We will begin by ignoring the endomorphism $\sigma_A$ and concentrating on the commutative $R$-algebra $A$. The tensor product $P_{A/R} := A \otimes_R A$ has two $A$-algebra structures, one coming from the action on the left and the other one coming from the action on the right. Unless otherwise specified, we will use the *left* structure when we see $P_{A/R}$ as an $A$-module. However, when $M$ is an $A$-module, the notation $P \otimes_A M$ will implicitly mean that we use the action of $A$ on the *right* to build the tensor product and that the resulting object will be seen as an $A$-module using the action of $A$ on the *left*.

In practice, we will write $x := x \otimes 1 \in P_{A/R}$ and $\tilde{x} := 1 \otimes x \in P_{A/R}$. In other words, with these notations, the action on the left is multiplication by $x$ and the action on the right is multiplication by $\tilde{x}$. Any element of $P_{A/R}$ can be written as a finite sum $\sum x_i \tilde{y}_i$. At some point, we will call the embedding on the right the *Taylor map* and denote it by

$$\theta_{A/R} : A \to P_{A/R}, \quad x \mapsto \tilde{x}.$$

We will then call $\tilde{x} = \theta_{A/R}(x)$ the *Taylor expansion* of $x \in A$ (more on this vocabulary later on).

We will also consider the canonical map corresponding to the projection that forgets the middle term:

$$A \otimes_R A \xrightarrow{\delta_{A/R}} A \otimes_R A \otimes_R A, \quad x \otimes y \longmapsto x \otimes 1 \otimes y.$$

It is a morphism of $R$-algebras that may also be seen as a map

$$P_{A/R} \xrightarrow{\delta_{A/R}} P_{A/R} \otimes_A P_{A/R} \quad x\tilde{y} \longmapsto x \otimes \tilde{y},$$

where $A$ acts on the *right* on the first factor and on the *left* on the second one in the tensor product.

Let $I_{A/R}$ be the kernel of the multiplication morphism

$$P_{A/R} \to A, \quad x\tilde{y} \mapsto xy$$

(that corresponds to the diagonal embedding). The ideal $I_{A/R}$ is generated by the image of the linear map

$$A \xrightarrow{d_{A/R}} P_{A/R}, \quad x \longmapsto \xi := \tilde{x} - x$$

(that corresponds to the difference between the projections). In practice, we will usually drop the index $A/R$ and simply write $P$, $I$, $\theta$, $\delta$ and $d$.

Since we will make regular use of linearization, we introduce it formally now:

**Definition 1.1.** Let $M$, $N$ be two $A$-modules and $u : M \to N$ an $R$-linear map. Then the *A-linearization* of $u$ is the $A$-linear map

$$P \otimes_A M \xrightarrow[\cong]{\overset{\tilde{u}}{\overbrace{\qquad}}} A \otimes_R M \longrightarrow N$$
$$x \otimes s \longmapsto x u(s).$$

Recall that, in the tensor product, we use the action on the *right* on $P$. Therefore, we have for all $x$, $y \in A$, for all $s \in M$,

$$\tilde{u}(x\tilde{y} \otimes s) = xu(ys).$$

As an example, we see that the multiplication map $P \to A$ is the $A$-linearization of the identity of $A$, seen as an $R$-linear map.

For further use, we also mention the following result that follows immediately from the definitions:

**Lemma 1.2.** *If we are given two $R$-linear maps $\varphi : M \to N$ and $\psi : L \to M$, then the linearization of $\varphi \circ \psi$ factors as*

$$P \otimes_A L \xrightarrow{\delta \otimes \mathrm{Id}} P \otimes_A P \otimes_A L \xrightarrow{\mathrm{Id} \otimes_A \tilde{\psi}} P \otimes_A M \xrightarrow{\tilde{\varphi}} N.$$

Now we make the endomorphism $\sigma_A$ enter the game. We will consider $P := A \otimes_R A$ as a twisted $R$-algebra by using $\sigma_A$ on the left and the identity the right: in other words, we set $\sigma_P := \sigma_A \otimes_R \mathrm{Id}_A$. Alternatively, this is the unique structure of twisted $R$-algebra on $P$ such that for all $x \in A$,

$$\sigma_P(x) = \sigma_A(x) \quad \text{and} \quad \sigma_P(\tilde{x}) = \tilde{x}.$$

In particular, we may also consider $P$ as a twisted $A$-algebra, in the sense that $P$ is endowed with a $\sigma_A$-linear ring endomorphism $\sigma_P$. We will often drop the indexes $A$ and $P$ and simply write $\sigma$ for both maps (so that $\sigma(\tilde{x}) = \tilde{x}$) when there is no ambiguity.

Before we do anything else, let us prove the following result, which is quite elementary, but very useful:

**Lemma 1.3.** *If $x \in A$ and $\xi := \tilde{x} - x \in P_{A/R}$, then*

$$\sigma_P(\xi) = \xi + y \quad \text{with} \quad y := x - \sigma_A(x).$$

*Proof.* We have

$$\sigma(\xi) = \sigma(\tilde{x} - x) = \sigma(\tilde{x}) - \sigma(x) = \tilde{x} - \sigma(x) = \tilde{x} - x + x - \sigma(x) = \xi + y. \quad \square$$

**Lemma 1.4.** *The kernel of the A-linearization*

$$P_{A/R} \xrightarrow{\tilde{\sigma}_A} A, \quad x\tilde{y} \longmapsto x\sigma_A(y)$$

*of $\sigma_A$ is $\sigma_P(I_{A/R})$.*

*Proof.* Recall that $\sigma(I)$ denotes the *ideal* generated by the image of $I$. As a consequence, since $I$ is generated by the image of $d$, we see that $\ker \tilde{\sigma}$ will be generated by the image of

$$A \xrightarrow{\sigma \circ d} P, \quad x \longmapsto \tilde{x} - \sigma(x).$$

Now, we have for all $x \in A$, $\tilde{\sigma}(\tilde{x} - \sigma(x)) = \sigma(x) - \sigma(x) = 0$, and it follows that $\sigma(I) \subset \ker \tilde{\sigma}$. Conversely, by definition, we have for all $x \in A$,

$$\sigma(x) \equiv \tilde{x} \mod \sigma(I).$$

Therefore, if $f := \sum x_i \tilde{y}_i \in \ker \tilde{\sigma}$, we have

$$f \equiv \sum x_i \sigma(y_i) = \tilde{\sigma}(f) = 0 \mod \sigma(I),$$

and we see that $\ker \tilde{\sigma} \subset \sigma(I)$. $\qquad \square$

**Remarks.** (1) It is sometimes convenient to use the bimodule language. An *A-sesquimodule $M$* is an $A$-bimodule such that for all $x \in A$, $s \in M$,

$$\sigma_A(x) \cdot s = s \cdot x.$$

Note that we are using the *reverse* convention from André [2001] so that forgetting the right action induces an equivalence (an isomorphism) between $A$-sesquimodules and left $A$-modules (we will use this identification).

(2) The $R$-algebra $P$ has a canonical $A$-bimodule structure which is completely independent of the choice of $\sigma_A$. If we endow $A$ with its sesquimodule structure, then the linearization $\tilde{\sigma}$ of $\sigma_A$ is a morphism of $A$-bimodules: we always have for all $x, y \in A$, $f \in P$,

$$\tilde{\sigma}(x \cdot f \cdot y) = \tilde{\sigma}(xf\tilde{y}) = x\tilde{\sigma}(f)\sigma(y) = x \cdot \tilde{\sigma}(f) \cdot y.$$

It follows that $\ker \tilde{\sigma}$ has a natural $A$-bimodule structure. Actually, since $\tilde{\sigma}$ is a ring homomorphism, then $\ker \tilde{\sigma}$ is an ideal and therefore automatically an $A$-bimodule.

Recall from [Le Stum and Quirós 2015a] that the *twisted powers* of $I$ are

$$I^{(0)} = P, \quad I^{(1)} = I, \quad I^{(2)} := I\sigma(I), \quad \dots, \quad I^{(n)} := I\sigma(I)\cdots\sigma^{n-1}(I),$$

where images and products of ideals are meant as *ideals*. We will write $I_{A/R}^{(n)_\sigma}$ when we want to make clear the dependence on $\sigma$ and $A/R$.

**Definition 1.5.** The $A$-module of *twisted principal parts of order $n \in \mathbb{N}$ (and infinite level)* of $A$ is

$$P_{A/R,(n)_\sigma} := P_{A/R}/I_{A/R}^{(n+1)_\sigma}.$$

The $A$-module of *twisted principal parts of infinite order (and infinite level)* of $A$ is the *twisted completion*:

$$\widehat{P}_{A/R,\sigma} := \varprojlim P_{A/R,(n)_\sigma}.$$

Note that these $A$-modules all have natural $R$-algebra structures and that the definition also makes sense for $n = -1$ so that $P_{A/R,(-1)_\sigma} = 0$. Again, we will often drop the indexes $A/R$ when we believe that there is no risk of confusion and simply write $P_{(n)_\sigma}$ and $\widehat{P}_\sigma$. We will also drop the index $\sigma$ when $\sigma_A = \mathrm{Id}_A$.

**Examples.**  (1) When $A$ is trivially twisted, which means that $\sigma_A := \mathrm{Id}_A$, this notion of principal parts coincides with the usual one (definition 16.3.1 of [Grothendieck 1967]), and therefore many of the basic objects we will construct are twisted versions of those in that work.

(2) When $A = R[x]$, we have $P = R[x, \xi]$ with $\xi = \tilde{x} - x$ and $I = (\xi)$. Moreover, $\sigma(\xi) = \xi + y$ with $y = x - \sigma(x)$. It follows that

$$\sigma^n(\xi) = \xi + (n)_\sigma(y) = \xi + x - \sigma^n(x),$$

with $(n)_\sigma := 1 + \cdots + \sigma^{n-1}$. Therefore, we have

$$P_{(n)} = R[x, \xi]/\prod_{i=0}^{n}(\xi + (i)_\sigma(y)) = R[x, \xi]/\prod_{i=0}^{n}(\xi + x - \sigma^n(x)).$$

(a) In the case $\sigma(x) = x$, we get $P_{(n)} = R[x, \xi]/\xi^{n+1}$ as expected.

(b) More generally, if we assume that $\sigma(x) = x + h$ with $h \in R$, we obtain

$$P_{(n)} = R[x, \xi]/\prod_{i=0}^{n}(\xi - ih).$$

(c) On the other hand, if we let $\sigma(x) = qx$ with $q \in R$, we find

$$P_{(n)} = R[x, \xi]/\prod_{i=0}^{n}(\xi + (1 - q^i)x).$$

When $R \to R'$ is a homomorphism of commutative rings, we endow $A' := R' \otimes_R A$ with $\sigma_{A'} := \mathrm{Id}_{R'} \otimes_R \sigma_A$.

**Proposition 1.6.** *Let $R \to R'$ be a homomorphism of commutative rings and $A' := R' \otimes_R A$. Then we have for all $n \in \mathbb{N}$,*

$$A' \otimes_A P_{A/R,(n)_\sigma} \simeq P_{A'/R',(n)_\sigma}.$$

*Proof.* If we let $P' := P_{A'/R'}$, then there exists a canonical isomorphism $A' \otimes_A P \simeq R' \otimes_R P \simeq P'$. Moreover, if we denote by $I'$ the kernel of the multiplication morphism on $P'$, we have $A' \otimes_A I \simeq I'$. And finally, $\mathrm{Id}_A \otimes_A \sigma_P$ corresponds to $\sigma_{P'}$ under this isomorphism. Our assertion is therefore an immediate consequence of right exactness of tensor product. $\square$

Recall from [Le Stum and Quirós 2015a] that a twisted $A$-algebra is an $A$-algebra $B$ endowed with a $\sigma_A$-linear ring endomorphism $\sigma_B$ (which simply means that for all $f \in A$, $\sigma_B(f 1_B) = \sigma_A(f)$).

**Proposition 1.7.** *If $B$ is a twisted commutative $A$-algebra, then there exists a canonical morphism of $B$-algebras*

$$B \otimes_A P_{A/R,(n)_\sigma} \to P_{B/R,(n)_\sigma}.$$

*When $B$ is a quotient (resp. a localization) of $A$, this map is surjective (resp. bijective).*

Recall from Definition 1.7 of [Le Stum and Quirós 2015b] that we call such a $B$ a *twisted quotient* (resp. *twisted localization*) of $A$.

*Proof.* The morphism of twisted $R$-algebras $A \to B$ extends naturally to a morphism of twisted $R$-algebras $P_A \to P_B$. Since $I_A$ is sent into $I_B$, we see that, for all $n \in \mathbb{N}$, $\sigma^n(I_A)$ is sent into $\sigma^n(I_B)$ and the first assertion formally follows. In the case of a quotient map, all the maps involved are surjective.

Now, if $B := S^{-1}A$ is a twisted localization of $A$, then $P_B$ is the localization of $P_A$ with respect to the monoid $S'$ generated by $S$ and $\tilde{S}$, and we have $I_B = P_B \otimes_{P_A} I_A$. It immediately follows that for all $n \in \mathbb{N}$, we have $\sigma^n(I_B) = P_B \otimes_{P_A} \sigma^n(I_A)$, and therefore $I_B^{(n)} = P_B \otimes_{P_A} I_A^{(n)}$. Thus we see that

$$P_{B,(n)_\sigma} = P_B/(P_B \otimes_{P_A} I_A^{(n+1)}) = P_B \otimes_{P_A} P_{A,(n)} = B \otimes_A P_{A,(n)} \otimes_A B.$$

We need to remove the $B$ on the right-hand side and it is sufficient to show that $\tilde{x}$ is invertible in $B \otimes_A P_{A,(n)}$ whenever $x \in S$. But we have

$$\prod_{i=0}^{n}(\tilde{x} - \sigma^i(x)) = \prod_{i=0}^{n} \sigma^i(\tilde{x} - x) \in I_A^{(n+1)},$$

from which we derive that there exists $f \in P_A$ such that

$$f\tilde{x} \equiv \prod_{i=0}^{n} \sigma^i(x) \quad \mod I_A^{(n+1)}.$$

Since $B = S^{-1}A$, we must have $\sigma(S) \subset B^{\times}$ and it follows that $\prod_{i=0}^{n} \sigma^i(x) \in B^{\times}$. Thus, we see $f\tilde{x}$ is invertible in $B \otimes_A P_{A,(n)}$ and it follows $\tilde{x}$ is invertible too. $\square$

As an illustration, we can give explicit formulas in the quantum situation. Recall that we introduced in [Le Stum and Quirós 2015a] the notion of *twisted powers* of an element in a twisted ring. In particular, for $f \in P$, we will have

$$f^{(0)} := 1, \quad f^{(1)} := f, \quad f^{(2)} = f\sigma(f), \quad \ldots, \quad f^{(n+1)} = f\sigma(f)\cdots\sigma^n(f).$$

Recall also that the *quantum binomial coefficients* are defined by induction (see [Le Stum and Quirós 2015a] for example) as

$$\binom{n}{k}_q := \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q.$$

**Proposition 1.8.** *Assume $\sigma(x) = qx$ with $q \in R$ and let $\xi = \tilde{x} - x$. Then we have for all $n \in \mathbb{N}$,*

$$(1) \qquad \xi^{(n)} = \sum_{j=0}^{n} (-1)^j \binom{n}{j}_q q^{\frac{j(j-1)}{2}} x^j \tilde{x}^{n-j}$$

*and for all $n \in \mathbb{N}$,*

$$\tilde{x}^n = \sum_{i=0}^{n} \binom{n}{i}_q x^i \xi^{(n-i)}.$$

*Proof.* The first equality is essentially the quantum binomial formula (see proposition 2.14 in [Le Stum and Quirós 2015a]):

$$(\tilde{x} - x)^{(n)} = \sum_{j=0}^{n} \binom{n}{j}_q (-x)^{(j)} \tilde{x}^{(n-j)}.$$

For the second one, we compute the right-hand side with the help of formula (1):

$$\sum_{i=0}^{n} \binom{n}{i}_q x^i \xi^{(n-i)} = \sum_{i=0}^{n} \binom{n}{i}_q x^i \left( \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j}_q q^{\frac{j(j-1)}{2}} x^j \tilde{x}^{n-i-j} \right)$$

$$= \sum_{k=0}^{n} \left( \sum_{i=0}^{k} \binom{n}{i}_q \binom{n-i}{k-i}_q (-1)^{k-i} q^{\frac{(k-i)(k-i-1)}{2}} \right) x^k \tilde{x}^{n-k}$$

after rewriting $i + j = k$. Now, we have (using corollaries 2.7 and 2.8 in [Le Stum and Quirós 2015a], for example)

$$\sum_{i=0}^{k} \binom{n}{i}_q \binom{n-i}{k-i}_q (-1)^{k-i} q^{\frac{(k-i)(k-i-1)}{2}} = \binom{n}{k}_q \sum_{i=0}^{k} \binom{k}{i}_q (-1)^{k-i} q^{\frac{(k-i)(k-i-1)}{2}}.$$

But the quantum binomial formula again implies that for $k > 0$, we have

$$\sum_{i=0}^{k} \binom{k}{i}_q (-1)^{k-i} q^{\frac{(k-i)(k-i-1)}{2}} = \prod_{i=0}^{k-1} (1 - q^i) = 0.$$

And it follows that

$$\sum_{i=0}^{n} \binom{n}{i}_q x^i \xi^{(n-i)} = \tilde{x}^n,$$

as asserted. □

We will need a slightly stronger notion of coordinate than the one used in [Le Stum and Quirós 2015b]:

**Definition 1.9.** Let $x \in A$ and $\xi := \tilde{x} - x \in P_{A/R}$.

(1) Then $x$ is a *twisted coordinate* for the twisted $R$-algebra $A$ or a $\sigma$-*coordinate* for the $R$-algebra $A$ if for all $n \in \mathbb{N}$, $P_{A/R,(n)_\sigma}$ is freely generated as an $A$-module by the images of $1, \xi, \xi^2, \ldots, \xi^n$.

(2) If $x$ is a twisted coordinate such that $\sigma(x) = qx + h$ with $q, h \in R$, then we will call it a *quantum coordinate* or *$q$-coordinate* and call $A$ a *quantum $R$-algebra* or *$q$-$R$-algebra*.

**Examples.** (1) When $\sigma_A = \mathrm{Id}_A$, a twisted coordinate will be called a *usual coordinate*. In the case $A/R$ is smooth (of pure relative dimension one), then a usual coordinate is nothing but an *étale coordinate*: it means that the map

$$R[T] \to A, \quad T \mapsto x$$

is étale.

(2) If $A = R[x]$, then $x$ is always a twisted coordinate, whatever $\sigma$ is.

**Proposition 1.10.** (1) *If $R \to R'$ is a homomorphism of commutative rings and $x$ is a twisted coordinate on $A$, then $x$ becomes a twisted coordinate on $A' := R' \otimes_R A$ (relatively to $R'$).*

(2) *If $B$ is a twisted localization of $A$ and $x$ is a twisted coordinate on $A$, then $x$ becomes a twisted coordinate on $B$.*

*Proof.* Follows from propositions 1.6 and 1.7. □

**Remark.** Assume $A$ is a quantum $R$-algebra so that there exists a twisted coordinate $x$ on $A$ and $q, h \in R$ such that $\sigma(x) = qx + h$. Let us still denote by the same letter $x$ an indeterminate over $R$ and by $\sigma$ again the endomorphism of $R[x]$ given by the same formula. Then $A$ becomes an $R[x]$-twisted algebra and we have a canonical isomorphism (free $A$-modules of the same rank with the same generators):

$$A \otimes_{R[x]} P_{R[x]/R,(n)_\sigma} \simeq P_{A/R,(n)_\sigma}.$$

In the next statement, we use the letter $\xi$ as an indeterminate over $A$ so that $A[\xi]$ denotes the polynomial ring and $A[\xi]_{\leq n}$ the submodule of polynomial of degree at most $n$. Ultimately, this should not create any confusion due to Corollary 1.12 below.

**Proposition 1.11.** *Let $x \in A$ and $y := x - \sigma(x)$. We endow $A[\xi]$ with the unique $\sigma_A$-linear endomorphism such that $\sigma(\xi) = \xi + y$. Then $x$ is a twisted coordinate on $A$ if and only if the morphism of twisted algebras*

$$\phi : A[\xi] \to P_{A/R}, \quad \xi \mapsto \tilde{x} - x$$

*induces for all $n \in \mathbb{N}$ an isomorphism of $A$-algebras $A[\xi]/\xi^{(n+1)} \simeq P_{A/R,(n)_\sigma}$.*

*Proof.* First of all, it follows from Lemma 1.3 that there exists such a morphism for all $n \in \mathbb{N}$. On the other hand,

$$(2) \qquad\qquad \xi^{(n+1)} = \prod_0^n (\xi + (x - \sigma^i(x)))$$

is a monic polynomial of degree $n + 1$. Then euclidean division tells us that the composite map

$$A[\xi]_{\leq n} \to A[\xi] \to A[\xi]/\xi^{(n+1)}$$

is an isomorphism of $A$-modules. Therefore the condition on $\phi$ is equivalent to the fact that the map

$$A[\xi]_{\leq n} \to P_{(n)_\sigma}, \quad \xi \mapsto \overline{\tilde{x} - x}$$

is bijective. And this exactly means that $P_{(n)_\sigma}$ is freely generated by the $n + 1$ first powers of the images of $\tilde{x} - x$. $\qquad\square$

When the polynomial ring $A[\xi]$ is endowed with the structure of a $\sigma_A$-algebra, we will set

$$A[\![\xi]\!]_\sigma := \varprojlim A[\xi]/\xi^{(n+1)}.$$

**Corollary 1.12.** *With the same hypothesis, $x$ is a twisted coordinate on $A$ if and only if there exists an isomorphism of $A$-algebras*

$$A[\![\xi]\!]_\sigma \xrightarrow{\sim} \widehat{P}_{A/R,\sigma}, \quad \xi \longmapsto \tilde{x} - x.$$

**Corollary 1.13.** *Let $x \in A$ and $\xi := \tilde{x} - x \in P$. Then the following conditions are equivalent*:

(1) *$x$ is a twisted coordinate on $A$*;

(2) *for all $n \in \mathbb{N}$, the $A$-module $P_{(n)_\sigma}$ is freely generated by the images of $1$, $\xi$, $\xi^{(2)}, \ldots, \xi^{(n)}$*;

(3) *for all $n \in \mathbb{N}$, the $A$-module $I^{(n)}/I^{(n+1)}$ is free of rank one on the image of $\xi^{(n)}$.*

**Corollary 1.14.** *If $A$ is a twisted localization of $R[x]$, then $x$ is a twisted coordinate on $A$.*

*Proof.* Using the second part of Proposition 1.10, we may assume that $A = R[x]$, in which case this is a trivial consequence of Proposition 1.11. $\qquad\square$

## 2. Twisted differential forms

In this section, we study the module of twisted differential forms (of degree one) and make the link with twisted derivations. We use the same notations as before.

**Definition 2.1.** The $A$-module of *twisted differential forms* on $A/R$ is

$$\Omega^1_{A/R,\sigma} := I_{A/R}/I^{(2)_\sigma}_{A/R}.$$

Again, we will often drop the index $A/R$. Since we implicitly endow $P$ with the action of $A$ on the left, we will also always see $\Omega^1_\sigma$ as an $A$-module through the action on the left.

**Examples.** (1) When $A$ is trivially twisted, then $\Omega^1_\sigma = I/I^2$ is the usual module of differential forms of $A$ over $R$.

(2) If $A = R[x]$ is endowed with any $R$-algebra endomorphism $\sigma$, then $\Omega^1_\sigma$ is free of rank 1: with the notations of Lemma 1.3 (so that $y = x - \sigma(x)$), we have

$$\Omega^1_\sigma \simeq \xi R[x, \xi]/\xi(\xi + y) \simeq R[x, \xi]/(\xi + y) \simeq R[x].$$

**Remarks.** (1) Clearly, $\Omega^1_\sigma$ has a natural $A$-bimodule structure as a quotient of two ideals of $P$. It happens that this is identical to its $A$-sesquimodule structure: by definition, if $x \in A$, then $\tilde{x} \equiv \sigma(x) \mod \sigma(I)$ and it follows that for all $f \in I$,

(3) $$f\tilde{x} \equiv f\sigma(x) \mod I\sigma(I).$$

(2) The $\Omega^1_\sigma$ that appears in proposition 1.4.2.1 of [André 2001] is exactly the same as ours (André calls $k$ what we call $R$).

(3) Formula (3) is exactly the first step of the braiding described by Max Karoubi and Mariano Suárez-Álvarez [2003].

(4) One can define more generally the *twisted de Rham complex* $\Omega_\sigma^\bullet$ of $A$ as the quotient of the noncommutative tensor algebra of $I$ by the graded differential ideal generated by $I\sigma(I)$. We will not consider this complex here.

**Proposition 2.2.** *There exists a split exact sequence*

$$0 \to \Omega_{A/R,\sigma}^1 \to P_{A/R,(1)_\sigma} \xrightarrow{\tilde{\sigma}_A} A \to 0.$$

*Proof.* There exists such an exact sequence by definition of $P_{(1)_\sigma}$ and $\Omega_\sigma^1$. The $A$-module structure of $P$ provides a section of $\tilde{\sigma}$. $\qquad\square$

**Proposition 2.3.** (1) *If $R \to R'$ is a homomorphism of commutative rings and $A' := R' \otimes_R A$, then there exists an isomorphism*

$$A' \otimes_A \Omega_{A/R,\sigma}^1 \simeq \Omega_{A'/R',\sigma}^1.$$

(2) *If $B$ is a twisted commutative $A$-algebra, then there exists a canonical $B$-linear map*

(4) $$B \otimes_A \Omega_{A/R,\sigma}^1 \to \Omega_{B/R,\sigma}^1.$$

*When $B$ is a quotient (resp. a localization) of $A$ this map is surjective (resp. bijective).*

*Proof.* Using Proposition 2.2, this follows from propositions 1.6 and 1.7. $\qquad\square$

**Remark.** This last result does *not* hold however if we only require $A \to B$ to be an étale map (and not a localization map) as the following example shows. Let $R$ be any field of characteristic different from 2, $A := R[x]$ with $\sigma_A := \mathrm{Id}_A$ and $B := R[x, x^{-1}]$ with $\sigma_B(x) := -x$. Then the morphism $x \mapsto x^2$ is an étale twisted morphism but the morphism (4) is the zero map. More precisely, if $\xi = \tilde{x} - x$, we have

$$B \otimes_A \Omega_{A,\sigma}^1 = (\xi)/(\xi^2) \quad \text{and} \quad \Omega_{B,\sigma}^1 = (\xi)/(\xi^2 + 2x\xi),$$

where the ideals are taken inside $R[x, x^{-1}, \xi]$, and the morphism (4) is induced by $\xi \mapsto \xi^2 + 2x\xi$.

Recall from [Le Stum and Quirós 2015b] that a *twisted derivation* of $A$ is an $R$-linear map into an $A$-module $M$ that satisfies the *twisted Leibniz rule*: for all $x, y \in A$,

$$D(xy) = x D(y) + \sigma(y) D(x).$$

They form an $A$-module $\mathrm{Der}_{R,\sigma}(A, M)$.

**Proposition 2.4.** *The canonical map $A \to \Omega_{A/R,\sigma}^1$ induced by $d$ is a twisted derivation. It provides us with a natural isomorphism*

(5) $$\mathrm{Hom}_A(\Omega_{A/R,\sigma}^1, M) \xrightarrow{\sim} \mathrm{Der}_{R,\sigma}(A, M), \quad u \mapsto D := u \circ d$$

*whenever $M$ is an $A$-module.*

In the future, we will also denote by $d : A \to \Omega^1_\sigma$ this universal twisted derivation when there is no risk of confusion.

*Proof.* Using formula (3), we see that, inside $P$, we have

$$
\begin{aligned}
yd(x) + \sigma(x)d(y) &= y(\tilde{x} - x) + \sigma(x)(\tilde{y} - y) \\
&\equiv y(\tilde{x} - x) + \tilde{x}(\tilde{y} - y) \\
&= \widetilde{xy} - xy \\
&= d(xy) \quad \mathrm{mod}\ I\sigma(I).
\end{aligned}
$$

It follows that the induced map $d : A \to \Omega^1_\sigma$ is indeed a twisted derivation. This also implies that the map in (5) is well defined. And it is clearly injective because $I$ is generated by the image of $d$.

We now show that it is surjective. If $D$ is a twisted derivation of $M$, we can consider its linearization

$$
P \xrightarrow{\tilde{D}} M, \quad x\tilde{y} \longmapsto xD(y).
$$

By definition, the ideal $I\sigma(I)$ is generated by elements of the form

$$
f = (\tilde{x} - x)(\tilde{y} - \sigma(y)) = \tilde{x}\tilde{y} - x\tilde{y} - \sigma(y)\tilde{x} + x\sigma(y),
$$

and we have

$$
\tilde{D}(f) = D(xy) - xD(y) - \sigma(y)D(x) + x\sigma(y)D(1) = 0
$$

because $D$ is a twisted derivation (and in particular $D(1) = 0$). It follows that $\tilde{D}$ factors through $P/I\sigma(I)$ and we may consider the induced map $u : \Omega^1_\sigma \to M$. It only remains to notice that we have for all $x \in A$,

$$
u(d(x)) = \tilde{D}(d(x)) = \tilde{D}(\tilde{x} - x) = D(x) - xD(1) = D(x). \qquad \square
$$

**Remark.** There exists a very elegant proof of this last result through the theory of bimodules. It is based on the fact (see proposition 17 in [Bourbaki 1970], Chapter III, section 10) that $I$ is universal for bimodule derivations: there exists a natural isomorphism

$$
\mathrm{Hom}_{A\text{-Bim}}(I, M) \xrightarrow{\sim} \mathrm{Der}_R(A, M), \quad u \longmapsto D := u \circ d,
$$

where the right-hand side stands for bimodule derivations (see proposition 1.4.2.1 of [André 2001]).

As an immediate consequence of the proposition, writing $T_{A/R,\sigma} := \mathrm{Der}_{R,\sigma}(A, A)$, which we will often abbreviate to $T_\sigma$, we obtain the following:

**Corollary 2.5.** *The $A$-module $T_{A/R,\sigma}$ is the dual of $\Omega^1_{A/R,\sigma}$.*

**Proposition 2.6.** *Assume that $\Omega^1_{A/R,\sigma}$ is projective of finite rank. Then if $M$ is an A-module we have the following*:

(1) *If $R \to R'$ is a base extension and $A' = R' \otimes_R A$, then there exists a canonical isomorphism*

$$R' \otimes_R \mathrm{Der}_{R,\sigma}(A, M) \simeq \mathrm{Der}_{R',\sigma}(A', A' \otimes_A M).$$

(2) *If $B$ is a twisted A-algebra, there exists a canonical map*

$$B \otimes_A \mathrm{Der}_{R,\sigma}(A, M) \leftarrow \mathrm{Der}_{R,\sigma}(B, B \otimes_A M).$$

*It is injective* (*resp. bijective*) *when $B$ is a quotient* (*resp. a localization*) *of $A$.*

*Proof.* Both assertions follow from Proposition 2.3. More precisely, in the first case,

$$R' \otimes_R \mathrm{Der}_{R,\sigma}(A, M) \simeq R' \otimes_R \mathrm{Hom}_A(\Omega^1_{A/R,\sigma}, M) \simeq \mathrm{Hom}_A(\Omega^1_{A/R,\sigma}, A' \otimes_A M)$$

because $\Omega^1_{A/R,\sigma}$ is projective of finite rank, and then

$$\mathrm{Hom}_A(\Omega^1_{A/R,\sigma}, A' \otimes_A M) \simeq \mathrm{Hom}_{A'}(A' \otimes_A \Omega^1_{A/R,\sigma}, A' \otimes_A M)$$
$$\simeq \mathrm{Hom}_{A'}(\Omega^1_{A'/R',\sigma}, A' \otimes_A M) \simeq \mathrm{Der}_{R',\sigma}(A', A' \otimes_A M).$$

The proof of the second assertion follows exactly the same lines with the same arguments. We have

$$B \otimes_A \mathrm{Der}_{R,\sigma}(A, M) \simeq B \otimes_A \mathrm{Hom}_A(\Omega^1_{A/R,\sigma}, M)$$
$$\simeq \mathrm{Hom}_A(\Omega^1_{A/R,\sigma}, B \otimes_A M)$$
$$\simeq \mathrm{Hom}_B(B \otimes_A \Omega^1_{A/R,\sigma}, B \otimes_A M) \leftarrow \mathrm{Hom}_A(\Omega^1_{B/R,\sigma}, B \otimes_A M)$$
$$\simeq \mathrm{Der}_{R,\sigma}(B', B \otimes_A M),$$

and the only map which is not always an isomorphism will be injective (resp. bijective) when $B$ is a quotient (resp. a localization) of $A$. $\square$

The following immediate consequence is worth stating:

**Corollary 2.7.** *If $\Omega^1_{A/R,\sigma}$ is projective of finite rank, then $\mathrm{T}_{A/R,\sigma}$ and $\Omega^1_{A/R,\sigma}$ are dual to each other and we have*

(1) *If $R \to R'$ is a base extension and $A' = R' \otimes_R A$, then there exists a canonical isomorphism*

$$R' \otimes_R \mathrm{T}_{A/R,\sigma} \simeq \mathrm{T}_{A'/R',\sigma'}.$$

(2) *If $B$ is a twisted localization of $A$, then there exists a canonical isomorphism*

$$B \otimes_A \mathrm{T}_{A/R,\sigma} \simeq \mathrm{T}_{B/R,\sigma}.$$

**Definition 2.8.** A *twisted connection* on an $A$-module $M$ is an $R$-linear map

$$\nabla : M \mapsto M \otimes_A \Omega^1_{A/R,\sigma}$$

such that for all $s \in M$, for all $x \in A$,

$$\nabla(xs) = s \otimes \mathrm{d}(x) + \sigma(x)\nabla(s).$$

An $A$-linear map between two $A$-modules with twisted connections is said to be *horizontal* if it is compatible with the connections.

Clearly, $A$-modules endowed with a connection and horizontal maps form a category $\nabla_{\sigma_A}$-Mod.

**Remark.** This definition is compatible with definition 2.2.1 by André [2001]. In particular, all the tannakian formalism applies but this is not what we are interested in.

Recall from [Le Stum and Quirós 2015b] that if $D$ is a twisted derivation of $A$, then a twisted $D$-derivation of an $A$-module $M$ is an $R$-linear endomorphism $D_M$ of $M$ that satisfies the twisted Leibnitz rule: for all $x \in A$, for all $s \in M$,

$$D_M(xs) = D(x)s + \sigma_A(x)D_M(s).$$

One may then consider the notion of *action by twisted derivations* of $T_{A/R,\sigma}$ on $M$: it is an $R$-linear action such that whenever $D \in T_{A/R,\sigma}$, the map $D_M : s \mapsto D.s$ is a $D$-derivation.

**Proposition 2.9.** *There exists a functor from the category of $A$-modules endowed with a twisted connection to the category of $A$-modules endowed with a linear action of $T_{A/R,\sigma}$ by twisted derivations. It is an equivalence (an isomorphism) when $\Omega^1_{A/R,\sigma}$ is free of finite rank.*

*Proof.* If $M$ is endowed with a twisted connection $\nabla : M \mapsto M \otimes_A \Omega^1_\sigma$ and $D$ is a twisted derivation of $A$, we may write uniquely $D = u \circ d$ with $u : \Omega^1_\sigma \to A$ and consider the composite map $D_M := (\mathrm{Id}_M \otimes u) \circ \nabla : M \to M$. Then we will have

$$\begin{aligned}
D_M(xs) &= (\mathrm{Id}_M \otimes u)(\nabla(xs)) \\
&= (\mathrm{Id}_M \otimes u)(s \otimes \mathrm{d}(x) + \sigma(x)\nabla(s)) \\
&= (\mathrm{u} \circ d)(x)s + \sigma(x)(\mathrm{Id}_M \otimes u)(\nabla(s)) = D(x)s + \sigma(x)D_M(s).
\end{aligned}$$

Conversely, assume that $M$ is endowed with an action of $T_\sigma$ by twisted derivations. Let $D_1, \ldots, D_n$ be a basis of $T_\sigma$ and $\omega_1, \ldots, \omega_n$ be the dual basis in $\Omega^1_\sigma$. Then we

can define $\nabla(s) = \sum D_{i,M}(s) \otimes \omega_i$ and check that

$$
\begin{aligned}
\nabla(xs) &= \sum D_{i,M}(xs) \otimes \omega_i \\
&= \left( \sum D_i(x)s + \sigma(x)D_{i,M}(s) \right) \otimes \omega_i \\
&= s \otimes \sum D_i(x)\omega_i + \sigma(x) \sum D_{i,M}(s) \otimes \omega_i = s \otimes \mathrm{d}(x) + \sigma(x)\nabla(s).
\end{aligned}
$$

Clearly, this is an inverse to the previous functor.                    $\square$

**Proposition 2.10.** *If $x \in A$ is a **twisted coordinate** on $A$, then $\Omega^1_{A/R,\sigma}$ is free of rank 1, generated by $\mathrm{d}x$. Moreover, there exists a unique twisted derivation $\partial_{x,\sigma}$ of $A$ such that $\partial_{x,\sigma}(x) = 1$ and we have for all $D \in T_{A/R,\sigma}$,*

$$
D = D(x)\partial_{x,\sigma}.
$$

*Proof.* The first assertion is a particular case of Corollary 1.13. The second one then follows from Corollary 2.5.                    $\square$

In particular, we see that a twisted coordinate is also a coordinate in the sense of [Le Stum and Quirós 2015b]. In order to lighten the notations, we will usually drop the index $x$ but we must not forget that $\partial_\sigma$ depends on the choice of $x$. Also, we would rather write $\partial_{A,\sigma}$ than $\partial_{\sigma_A}$ when we want to make clear the dependence on $A$.

If $x$ is a twisted coordinate on $A$, one may consider the *twisted Weyl algebra* $D_{A/R,\sigma,\partial}$ (see [Le Stum and Quirós 2015b] for example), that we will usually denote by $D_{A/R,\sigma}$ and sometimes simply by $D_\sigma$. This is the noncommutative polynomial ring in one variable $\partial_\sigma$ over $A$ with the commutation rule for all $z \in A$,

$$
\partial_\sigma z = \partial_{A,\sigma}(z) + \sigma_A(z)\partial_\sigma.
$$

Moreover, there exists an equivalence (an isomorphism) of categories

$$
D_{A/R,\sigma}\text{-Mod} \simeq \partial_{A,\sigma}\text{-Mod},
$$

where the latter denotes the category of $A$-modules $M$ endowed with a $\partial_{A,\sigma}$-derivation.

**Proposition 2.11.** *Assume that $x$ is a twisted coordinate on $A$. Then there exists an equivalence (an isomorphism) of categories*

$$
\nabla_{\sigma_A}\text{-Mod} \simeq D_{A/R,\sigma}\text{-Mod}.
$$

*Proof.* Follows from Proposition 2.9.                    $\square$

## 3. Twisted binomial coefficient theorem for principal parts

We prove here the main theorem that will allow us to recover twisted differential operators from principal parts. We use the same notation as before.

In Section 1 we introduced the canonical map (it is a morphism of $R$-algebras)

$$P_{A/R} \xrightarrow{\delta_{A/R}} P_{A/R} \otimes_A P_{A/R}, \quad x\tilde{y} \longmapsto x \otimes \tilde{y}.$$

We want to investigate the interaction between $\sigma_A$ and $\delta_{A/R}$.

Recall that we also considered in Section 1 the maps

$$P_{A/R} \xrightarrow{\sigma_P} P_{A/R}, \quad x\tilde{y} \longmapsto \sigma_A(x)\tilde{y},$$

which is an $R$-linear ring homomorphism, and

$$A \xrightarrow{d_{A/R}} P_{A/R}, \quad x \longmapsto \tilde{x} - x,$$

which is only $R$-linear. As usual, we will drop the subscripts in order to lighten the notation, hoping that the meaning will always be clear from the context.

**Lemma 3.1.** *For all $i = 0, \dots, n$, we have in $P \otimes_A P$ for all $x \in A$,*

$$\delta(\sigma^n(d(x))) = 1 \otimes \sigma^i(d(x)) + \sigma^{n-i}(d(\sigma^i(x))) \otimes 1.$$

*Proof.* We do the computations in $A \otimes_R A \otimes_R A$. The right-hand side is

$$1 \otimes (1 \otimes x - \sigma^i(x) \otimes 1) + \sigma^{n-i}(1 \otimes \sigma^i(x) - \sigma^i(x) \otimes 1) \otimes 1$$
$$= 1 \otimes 1 \otimes x - 1 \otimes \sigma^i(x) \otimes 1 + 1 \otimes \sigma^i(x) \otimes 1 - \sigma^n(x) \otimes 1 \otimes 1$$
$$= 1 \otimes 1 \otimes x - \sigma^n(x) \otimes 1 \otimes 1.$$

If we develop the left-hand side, we obtain exactly the same thing:

$$\delta(\sigma^n(d(x))) = \delta(1 \otimes x - \sigma^n(x) \otimes 1) = 1 \otimes 1 \otimes x - \sigma^n(x) \otimes 1 \otimes 1. \qquad \square$$

We endow $P \otimes_A P$ with the endomorphism $\sigma_P \otimes_A \mathrm{Id}_P$ (which is the same thing as $\sigma_A \otimes_R \mathrm{Id}_A \otimes_R \mathrm{Id}_A$ on $A \otimes_R A \otimes_R A$).

**Proposition 3.2.** *The map $\delta : P \to P \otimes_A P$ is a morphism of twisted $R$-algebras.*

*Proof.* This is the case $n = 1$ and $i = 0$ of Lemma 3.1. More precisely, if $x \in A$ and $\xi = \tilde{x} - x$, we have

$$\delta(\sigma(\xi)) = 1 \otimes \xi + \sigma(\xi) \otimes 1 = \sigma(\delta(\xi)). \qquad \square$$

**Proposition 3.3.** *We have in $P \otimes_A P$ for all $n \in \mathbb{N}$,*

$$\delta(I^{(n)}) \subset \sum_{i=0}^{n} I^{(i)} \otimes I^{(n-i)}.$$

*Proof.* First of all, since $I$ is generated by the image of $d$, it follows from Lemma 3.1 that for all $i = 0, \ldots, n$, we have

$$\delta(\sigma^n(I)) \subset P \otimes \sigma^i(I) + \sigma^{n-i}(I) \otimes P.$$

Using induction, we obtain

$$\delta(I^{(n+1)}) = \delta(I^{(n)})\delta(\sigma^n(I))) \subset \sum_{i=0}^{n}(I^{(i)} \otimes I^{(n-i)})(P \otimes \sigma^{n-i}(I) + \sigma^i(I) \otimes P)$$

$$\subset \sum_{i=0}^{n}(I^{(i)} \otimes I^{(n-i+1)}) + \sum_{i=0}^{n}(I^{(i+1)} \otimes I^{(n-i)})$$

$$\subset \sum_{i=0}^{n+1} I^{(i)} \otimes I^{(n+1-i)}. \qquad \square$$

**Corollary 3.4.** *For all $m, n \in \mathbb{N}$, we have in $P \otimes_A P$:*

$$\delta(I^{(n+m+1)}) \subset P \otimes_A I^{(m+1)} + I^{(n+1)} \otimes_A P.$$

*In other words, $\delta$ induces a map*

$$P_{(n+m)_\sigma} \xrightarrow{\delta_{n,m}} P_{(n)_\sigma} \otimes_A P_{(m)_\sigma}.$$

*Proof.* If $0 \leq i \leq m+n+1$, we have either $i > n$ and then $I^{(i)} \subset I^{(n+1)}$ or else $i \leq n$ so that $m+n+1-i > m$ and then $I^{(m+n+1-i)} \subset I^{(m+1)}$.

Going to the limit, we obtain a canonical homomorphism of $R$-algebras

$$\widehat{P}_\sigma \xrightarrow{\widehat{\delta}} \widehat{P}_\sigma \widehat{\otimes}_A \widehat{P}_\sigma,$$

where the right-hand side is, *by definition*, the inverse limit of all the $P_{(n)_\sigma} \otimes_A P_{(m)_\sigma}$. In other words, we obtain a comultiplication on $\widehat{P}_\sigma$ that will allow us to turn its "dual" into a ring (more on this later). $\qquad \square$

We finish this section with the *quantum binomial theorem for principal parts*:

**Theorem 3.5.** *let $A$ be a twisted commutative $R$-algebra and $x \in A$ such that $\sigma(x) = qx + h$ with $q, h \in R$. If we set $\xi = \tilde{x} - x$, then we have*

$$\delta(\xi^{(n)}) := \sum_{i=0}^{n} \binom{n}{i}_q \xi^{(n-i)} \otimes \xi^{(i)}.$$

*Proof.* The formula is proved to be correct by induction on $n$. First of all, since $\delta$ is a ring homomorphism, we have

$$\delta(\xi^{(n+1)}) = \delta(\xi^{(n)}\sigma^n(\xi)) = \delta(\xi^{(n)})\delta(\sigma^n(\xi))).$$

Using induction and Lemma 3.6 below, we get

$$\delta(\xi^{(n+1)}) = \sum_{i=0}^{n} \binom{n}{i}_q (\xi^{(n-i)} \otimes \xi^{(i)})(1 \otimes \sigma^i(\xi) + q^i \sigma^{n-i}(\xi) \otimes 1)$$

$$= \sum_{i=0}^{n} \binom{n}{i}_q \xi^{(n-i)} \otimes \xi^{(i)} \sigma^i(\xi) + \sum_{i=0}^{n} \binom{n}{i}_q q^i \xi^{(n-i)} \sigma^{n-i}(\xi) \otimes \xi^{(i)}$$

$$= \sum_{i=0}^{n} \binom{n}{i}_q \xi^{(n-i)} \otimes \xi^{(i+1)} + \sum_{i=0}^{n} \binom{n}{i}_q q^i \xi^{(n-i+1)} \otimes \xi^{(i)}$$

$$= \sum_{i=1}^{n+1} \binom{n}{i-1}_q \xi^{(n-i+1)} \otimes \xi^{(i)} + \sum_{i=0}^{n} \binom{n}{i}_q q^i \xi^{(n-i+1)} \otimes \xi^{(i)}$$

$$= \sum_{i=0}^{n+1} \left( \binom{n}{i-1}_q + q^i \binom{n}{i}_q \right) \xi^{(n+1-i)} \otimes \xi^{(i)}$$

$$= \sum_{i=0}^{n+1} \binom{n+1}{i}_q \xi^{(n+1-i)} \otimes \xi^{(i)}. \qquad \square$$

**Lemma 3.6.** *Under the hypothesis of the proposition, we have for all $i = 0, \ldots, n$,*

$$\delta(\sigma^n(\xi)) = 1 \otimes \sigma^i(\xi) + q^i \sigma^{n-i}(\xi) \otimes 1.$$

*Proof.* We have

$$d(\sigma(x)) = \widetilde{\sigma(x)} - \sigma(x) = \widetilde{qx+h} - (qx+h) = q(\tilde{x} - x) = q\xi.$$

The analogous result holds for the endomorphism $\sigma^i$. It follows that $d(\sigma^i(x)) = q^i \xi$ and we finish with Lemma 3.1. $\qquad \square$

## 4. Twisted differential operators of infinite level

We are now able to define the ring of twisted differential operators (of infinite level). We keep the previous notation.

**Definition 4.1.** If $M$ and $N$ are two $A$-modules, then a *twisted differential operator* $\varphi : M \to N$ *of order at most $n$ (and infinite level)* is an $R$-linear map whose $A$-linearization

$$P \otimes_A M \xrightarrow[\;\simeq\;]{\tilde{\varphi}} A \otimes_R M \longrightarrow N$$
$$x \otimes s \longmapsto x\varphi(s)$$

factors through $P_{(n)_\sigma} \otimes_A M$.

Note that the condition means that the restriction of $\tilde{\varphi}$ to $I^{(n+1)} \otimes_A M$ is zero. We might still write $\tilde{\varphi}$ for the map induced on $P_{(n)_\sigma} \otimes_A M$ when there is no risk of confusion.

We denote by $\mathrm{Diff}_{n,\sigma}(M, N)$ the set of all twisted differential operators of order at most $n$. Thus, we have a canonical isomorphism

$$\mathrm{Diff}_{n,\sigma}(M, N) \simeq \mathrm{Hom}_A(P_{(n)_\sigma} \otimes_A M, N),$$

where $P_{(n)_\sigma}$ is seen as an $A$-module for the action on the *right* with respect to $\otimes_A$ and for the action on the *left* with respect to $\mathrm{Hom}_A$. In particular, $\mathrm{Diff}_{n,\sigma}(M, N)$ has the natural structure of a $P_{(n)_\sigma}$-module given by $x \tilde{y} \cdot \varphi := x \circ \varphi \circ y$, where multiplication by $y$ takes place in $M$ while we multiply by $x$ in $N$.

We will also denote by $\mathrm{Diff}_\sigma(M, N)$ the set of all twisted differential operators of any order so that

$$\mathrm{Diff}_\sigma(M, N) \simeq \varinjlim \mathrm{Hom}_A(P_{(n)_\sigma} \otimes_A M, N).$$

In particular, we see that $\mathrm{Diff}_\sigma(M, N)$ has the natural structure of a $\widehat{P}_\sigma$-module.

In the case $N = M$, we will write $\mathrm{Diff}_{n,\sigma}(M)$ and $\mathrm{Diff}_\sigma(M)$. We also set $\mathrm{D}_{A/R,\sigma}^{(\infty)} := \mathrm{Diff}_\sigma(A)$ and we will often drop the index $A/R$ and simply write $\mathrm{D}_\sigma^{(\infty)}$.

**Definition 4.2.** Let $x$ be a twisted coordinate on $A$ and $\xi = \tilde{x} - x$. Then the *standard basis* of $\mathrm{Diff}_{n,\sigma}(A)$ is the basis $\partial_\sigma^{[k]}$ dual to the images of the $\xi^{(k)}$ in $P_{(n)_\sigma}$. We call $\partial_\sigma^{[k]}$ the *standard twisted divided differential operator of order $k$ associated to $x$*.

In other words (we will come back to this in Section 7), the canonical basis is characterized by the property that for all $k, n \in \mathbb{N}$,

$$\widetilde{\partial_\sigma^{[k]}}(\xi^{(n)}) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\widetilde{\partial_\sigma^{[k]}}$ denotes the $A$-linearization of the $R$-linear endomorphism $\partial_\sigma^{[k]}$ of $A$.

Thus, when $x$ is a twisted coordinate, any $\varphi \in \mathrm{D}_\sigma^{(\infty)}$ can be uniquely written as a finite sum $\sum z_k \partial_\sigma^{[k]}$ with $z_k \in A$ (and conversely, any such sum is in $\mathrm{D}_\sigma^{(\infty)}$).

The next proposition shows that the $A$-module of twisted differential operators could have also been defined by induction on the order $n$ (this is sometimes more convenient and does not require one to work out the theory of principal parts). Instead, it uses the notion of twisted bracket for all $\varphi \in \mathrm{Hom}_R(M, N)$, for all $x \in A$,

$$[\varphi, x]_\sigma = \varphi \circ x - \sigma(x) \circ \varphi,$$

already used in [Le Stum and Quirós 2015b]. We will need this intermediate result:

**Lemma 4.3.** *Let $M$, $N$ be two $A$-modules, $\varphi \in \operatorname{Hom}_R(M, N)$ and $x \in A$. If we set $\varphi_x := [\varphi, x]_\sigma$, then*

$$\tilde{\varphi}_x = \tilde{\varphi} \circ (\sigma(\xi) \otimes_A \operatorname{Id}_M) : P \otimes_A M \to N.$$

*Proof.* We do the computations in $A \otimes_R M$. Let $y \in A$ and $s \in M$. Then

$$\begin{aligned}
\tilde{\varphi}(\sigma(\xi)(y \otimes s)) &= \tilde{\varphi}((1 \otimes x - \sigma(x) \otimes 1)(y \otimes s)) \\
&= \tilde{\varphi}(y \otimes xs) - \tilde{\varphi}(\sigma(x)y \otimes s) \\
&= y\varphi(xs) - \sigma(x)y\varphi(s) \\
&= y\varphi_x(s) = \tilde{\varphi}_x(y \otimes s). \qquad \square
\end{aligned}$$

**Proposition 4.4.** *Let $M$, $N$ be two $A$-modules and $\varphi \in \operatorname{Hom}_R(M, N)$. Then for all $n \in \mathbb{N}$, we have*

$$\varphi \in \operatorname{Diff}_{n,\sigma}(M, N) \Leftrightarrow \text{for all } x \in A, [\varphi, x]_{\sigma^n} \in \operatorname{Diff}_{n-1,\sigma}(M, N).$$

Note that we can start the induction process with $\operatorname{Diff}_{0,\sigma}(M, N) = \operatorname{Hom}_A(M, N)$ or with $\operatorname{Diff}_{-1,\sigma}(M, N) = 0$ if we prefer.

*Proof.* For $x \in A$, we set $\varphi_x := [\varphi, x]_{\sigma^n}$. Then we consider the linearizations

$$\tilde{\varphi}, \tilde{\varphi}_x : P \otimes_A M \to N$$

of $\varphi$ and $\varphi_x$, respectively, and apply Lemma 4.3 to $\sigma^n$ so that

$$\tilde{\varphi}_x = \tilde{\varphi} \circ (\sigma^n(\xi) \otimes_A \operatorname{Id}_M).$$

Thus we see that $\tilde{\varphi} = 0$ on $I^{(n+1)} \otimes_A M = I^{(n)} \sigma^n(I) \otimes_A M$ if and only if $\tilde{\varphi}_x = 0$ on $I^{(n)} \otimes_A M$ for all $x \in A$. In other words, $\tilde{\varphi}$ factors through $P_{(n)_\sigma} \otimes_A M$ if and only if all $\tilde{\varphi}_x$ factor through $P_{(n-1)_\sigma} \otimes_A M$. $\qquad \square$

**Corollary 4.5.** *A twisted differential operator of order at most $n$ (and infinite level) from $M$ to $N$ is an $R$-linear map $\varphi : M \to N$ such that for all $x_0, \ldots, x_n \in A$,*

$$[[\cdots [[\varphi, x_n]_{\sigma^n}, x_{n-1}]_{\sigma^{n-1}} \cdots ]_\sigma, x_0] = 0.$$

**Remarks.** (1) Be careful that, with the notation of Valery Lunts and Alexander L. Rosenberg [1997], our $\operatorname{Diff}_\sigma(M, N)$ is different from their $\operatorname{Diff}(M, N^\sigma)$ which is defined by the condition for all $x_0, \ldots, x_n \in A$,

$$[[\cdots [[\varphi, x_n]_\sigma, x_{n-1}]_\sigma \cdots ]_\sigma, x_0] = 0.$$

They only coincide when $n = 0, 1$.

(2) Our $\operatorname{Diff}_\sigma(M, N)$ should however coincide with some flavor of the $\operatorname{Diff}_\beta(M, N)$ of Lunts and Rosenberg. More precisely, in order to define this module, they need a $G$-grading on $A$ and a bilinear map $\beta : G \times G \to R^\times$ (they use $k$ and $R$ for our $R$

and $A$). In the simplest nontrivial case $A = R[x]$ and $\beta(m, n) = q^{-mn}$, we believe that their $D_\beta(A)$ coincides with our $D_{A/R,\sigma}^{(\infty)}$ but their $D_q(A)$ is bigger (see [Iyer and McCune 2002] for example).

(3) Charlotte Hardouin [2010, Definition 2.4] introduces what she calls an *iterative q-difference ring* or $\mathrm{ID}_q$-*ring* for short. She chooses some nonzero $q \in K$ where $K$ is a fixed algebraically closed field and endows the field $A := K(x)$ of rational functions on $K$ with the automorphism $\sigma(x) = qx$. Then if we look carefully at her conditions, we see that an $\mathrm{ID}_q$-ring is a finitely generated $A$-algebra $B$ with a structure of $D_{A/K,\sigma}^{(\infty)}$-module, denoted by $(\varphi, y) \mapsto \varphi(y)$, such that the map $y \mapsto \sigma(y)$ is an automorphism of the ring $B$ and for all $k \in \mathbb{N}$, for all $y, z \in B$,

$$(6) \qquad \partial_\sigma^{[k]}(yz) = \sum_{i+j=k} (\sigma^i \partial_\sigma^{[j]})(y) \partial_\sigma^{[i]}(z).$$

Note that $B$ becomes a twisted $A$-algebra, that is in fact inversive (which simply means that $\sigma$ is bijective on $B$, a condition that is built into Hardouin's definition), and that condition (6) is automatic if the $q$-characteristic of $B$ is zero.

**Proposition 4.6.** *Composition of twisted differential operators gives a twisted differential operator. Moreover, its order is at most the sum of the order of the components.*

*Proof.* We let $\varphi : M \to N$ be a twisted differential operator of order $n$ and $\psi : L \to M$ a twisted differential operator of order $m$ and consider the factorization

$$P \otimes_A L \xrightarrow{\delta \otimes \mathrm{Id}} P \otimes_A P \otimes_A L \xrightarrow{\mathrm{Id} \otimes_A \tilde{\psi}} P \otimes_A M \xrightarrow{\tilde{\varphi}} N$$

of Lemma 1.2. The map $\tilde{\varphi}$ factors through $P_{(n)_\sigma} \otimes_A M$ and $\mathrm{Id} \otimes \tilde{\psi}$ factors through $P \otimes_A P_{(m)_\sigma} \otimes_A L$. Therefore, their composite factors through $P_{(n)_\sigma} \otimes_A P_{(m)_\sigma} \otimes_A L$ and it follows from Corollary 3.4 that the whole thing will factor through $P_{(n+m)_\sigma} \otimes L$. Thus, $\varphi \circ \psi$ is a twisted differential operator of order at most $m + n$. $\square$

It is sometimes useful to have a general formula for the commutation of twisted differential operators with the twisted coordinate $x$:

**Proposition 4.7.** *We have for all $k \in \mathbb{N} \setminus \{0\}$,*

$$\partial_\sigma^{[k]} \circ x = \sigma^k(x) \partial_\sigma^{[k]} + \partial_\sigma^{[k-1]}.$$

*Proof.* In order to make the proof easier to understand, we will still write $\tilde{x} = 1 \otimes x \in P$, but we will denote the multiplication maps by

$$A \xrightarrow{m_x} A, \quad y \longmapsto xy \quad \text{and} \quad P \xrightarrow{m_{\tilde{x}}} P \quad \varphi \longmapsto \tilde{x}\varphi.$$

Then it is easy to see that $m_{\tilde{x}}$ splits as

$$P \xrightarrow{\delta} P \otimes_A P \xrightarrow{\mathrm{Id}_P \otimes_A \widetilde{m_x}} P,$$

where $\widetilde{m_x}$ denotes the linearization of $m_x$. It then follows from Lemma 1.2 that

$$\widetilde{\partial_\sigma^{[k]} \circ m_x} = \widetilde{\partial_\sigma^{[k]}} \circ m_{\tilde{x}}.$$

On the other hand, we have

$$\xi^{(n+1)} = \xi^{(n)}\sigma^n(\xi) = \xi^{(n)}(\tilde{x} - \sigma^n(x)) = \tilde{x}\xi^{(n)} - \sigma^n(x)\xi^{(n)},$$

from which we derive $\tilde{x}\xi^{(n)} = \xi^{(n+1)} + \sigma^n(x)\xi^{(n)}$. Putting all these together, we get

$$\begin{aligned}
\left(\widetilde{\partial_\sigma^{[k]} \circ m_x}\right)(\xi^{(n)}) &= \left(\widetilde{\partial_\sigma^{[k]}} \circ m_{\tilde{x}}\right)(\xi^{(n)}) \\
&= \widetilde{\partial_\sigma^{[k]}}(\tilde{x}\xi^{(n)}) \\
&= \widetilde{\partial_\sigma^{[k]}}(\xi^{(n+1)} + \sigma^n(x)\xi^{(n)}) = \begin{cases} 1 & \text{if } n = k-1, \\ \sigma^k(x) & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

In other words, we have

$$\partial_\sigma^{[k]} \circ m_x = \sigma^k(x)\partial_\sigma^{[k]} + \partial_\sigma^{[k-1]}. \qquad \square$$

**Corollary 4.8.** *If $x \in A^\times$, then for all $k \in \mathbb{N} \setminus \{0\}$,*

$$\partial_\sigma^{[k]} \circ x^{-1} = \sum_{i=0}^{k} \frac{(-1)^i}{\prod_{j=0}^{i} \sigma^{k-j}(x)} \partial_\sigma^{[k-i]}.$$

*Proof.* Using for a moment the convention $\partial_\sigma^{[-1]} = 0$, which makes Proposition 4.7 formally valid for $k = 0$, we compute

$$\begin{aligned}
\sum_{i=0}^{k} \frac{(-1)^i}{\prod_{j=0}^{i} \sigma^{k-j}(x)} &\partial_\sigma^{[k-i]} \circ x \\
&= \sum_{i=0}^{k} \frac{(-1)^i}{\prod_{j=0}^{i} \sigma^{k-j}(x)}(\sigma^{k-i}(x)\partial_\sigma^{[k-i]} + \partial_\sigma^{[k-i-1]}) \\
&= \sum_{i=0}^{k} \frac{(-1)^i}{\prod_{j=0}^{i-1} \sigma^{k-j}(x)}\partial_\sigma^{[k-i]} + \sum_{i=0}^{k-1} \frac{(-1)^i}{\prod_{j=0}^{i} \sigma^{k-j}(x)}\partial_\sigma^{[k-i-1]} = \partial_\sigma^{[k]}. \qquad \square
\end{aligned}$$

**Proposition 4.9.** *Let $M, N$ be two $A$-modules.*

(1) *Let $R \to R'$ be any morphism of commutative rings and $A' := R' \otimes_R A$, endowed with $\mathrm{Id}_{R'} \otimes_R \sigma_A$. Then we have*

$$\mathrm{Diff}_{\sigma_{A'}}(A' \otimes_A M, A' \otimes_A N) \simeq A' \otimes_A \mathrm{Diff}_{\sigma_A}(M, N).$$

(2) *If $A \to B$ is a twisted localization and $M$ is finitely presented, then we have*

$$\mathrm{Diff}_{\sigma_B}(B \otimes_A M, B \otimes_A N) \simeq B \otimes_A \mathrm{Diff}_{\sigma_A}(M, N).$$

*Proof.* Follows from propositions 1.6 and 1.7 and the fact that direct limits commute with tensor product (and localization is flat). ☐

**Remark.** (1) As a particular case, we will have

$$D_{A'/R',\sigma'}^{(\infty)} \simeq A' \otimes_A D_{A/R,\sigma}^{(\infty)} \quad \text{and} \quad D_{B/R,\sigma}^{(\infty)} \simeq B \otimes_A D_{A/R,\sigma}^{(\infty)}.$$

(2) When $A$ is a quantum $R$-algebra, we will always have (see the remark following Proposition 1.10) a natural isomorphism of $A$-modules

$$D_{A/R,\sigma}^{(\infty)} \simeq A \otimes_{R[x]} D_{R[x]/R,\sigma}^{(\infty)}.$$

Note however that the ring structure (or equivalently the action on $A$) plays a fundamental role.

**Proposition 4.10.** *We have* $\mathrm{Diff}_{0,\sigma}(A) = A$ *and* $\mathrm{Diff}_{1,\sigma}(A) = A \oplus \mathrm{T}_{A/R,\sigma}$.

*Proof.* The first assertion follows from the fact that $P_{(0)_\sigma} = A$ and the second one from Proposition 2.2. ☐

Recall that we introduced in [Le Stum and Quirós 2015b] the ring $\overline{D}_{A/R,\sigma}$ of *small* twisted differential operators of $A/R$ as the smallest subring of $\mathrm{End}_R(A)$ containing both $A$ and $\mathrm{T}_{A/R,\sigma}$. Again, we will simply write $\overline{D}_\sigma$ when we believe that there is no risk of confusion. Then we have the following:

**Corollary 4.11.** *The ring of small twisted differential operators is contained inside the ring of twisted differential operators of infinite level: we have*

$$\overline{D}_{A/R,\sigma} \subset D_{A/R,\sigma}^{(\infty)}.$$

When there exists a twisted coordinate, we can make the twisted Weyl algebra enter the picture and we have:

**Corollary 4.12.** *If $x$ is a twisted coordinate on $A$, there exists an epi-mono factorization*

$$D_{A/R,\sigma} \twoheadrightarrow \overline{D}_{A/R,\sigma} \hookrightarrow D_{A/R,\sigma}^{(\infty)}.$$

*Proof.* There exists a natural map $D_\sigma \to \overline{D}_\sigma$ that sends the parameter $\partial_\sigma$ of $D_{\sigma,\partial}$ to the corresponding endomorphism $\partial_{A,\sigma}$ of $A$. And it is surjective since $\partial_{A,\sigma}$ is a generator of $\mathrm{T}_\sigma$. ☐

None of the maps are in general bijective, even in characteristic zero. However, as we will see in Theorem 6.3 below, there are some important cases where both maps are bijective.

At some point, we will need to be able to compare twisted differential operators with respect to $\sigma$ and twisted differential operators with respect to the powers (or roots) of $\sigma$.

**Proposition 4.13.** *For all $m > 0$, if $M$, $N$ are two $A$-modules, we have*

$$\mathrm{Diff}_{\sigma^m}(M, N) \subset \mathrm{Diff}_\sigma(M, N).$$

*Proof.* Since

$$I^{(mn)_\sigma} = I\sigma(I)\cdots\sigma^{mn}(I) \subset I\sigma^m(I)\cdots\sigma^{mn}(I) = I^{(n)_{\sigma^m}},$$

there exists an natural surjective map

$$P_{(nm)_\sigma} \to P_{(n)_{\sigma^m}},$$

from which we derive an inclusion $\mathrm{Diff}_{n,\sigma^m}(M, N) \subset \mathrm{Diff}_{mn,\sigma}(M, N)$.    □

Recall from [Le Stum and Quirós 2015a] that a system of roots of $\sigma$ is a family $\underline{\sigma} := \{\sigma_n\}_{n\in S}$, with $S \subset \mathbb{N}$, of $R$-linear ring endomorphisms of $A$ such that $\sigma_n^m = \sigma_{n'}^{m'}$ whenever $m/n = m'/n'$ and $\sigma_n^n = \sigma$. We will always assume that $S$ is filtering for division in the sense that if $m, m' \in S$, then there always exists $m'' \in S$ such that $m \mid m''$ and $m' \mid m''$. We will call the pair $(A, \underline{\sigma})$ an *S-twisted R-algebra*.

**Definition 4.14.** Let $\underline{\sigma} := \{\sigma_n\}_{n\in S}$ be a system of roots of $\sigma$. Then the *ring of twisted differential operators (of infinite level)* $\mathrm{D}^{(\infty)}_{A/R,\underline{\sigma}}$ is the $R$-subalgebra of $\mathrm{End}_R(A)$ generated by all $\sigma_n$-differential operators (of infinite level) for all $n \in S$.

Proposition 4.13 then has the following consequence:

**Corollary 4.15.** *If $\underline{\sigma}$ is a system of roots of $\sigma$, we have*

$$\mathrm{D}^{(\infty)}_{A/R,\underline{\sigma}} = \bigcup \mathrm{D}^{(\infty)}_{A/R,\sigma_n}.$$

Note that in section 3 of [Le Stum and Quirós 2015b] we defined in exactly the same way the ring of small twisted differential operators $\overline{\mathrm{D}}_{A/R,\underline{\sigma}}$ for any family $\underline{\sigma}$ (not necessarily a root system), but we showed that the analogous statement is *not* true in general.

## 5. Twisted Taylor series

We will develop here the formalism of twisted Taylor maps which describes the formal solutions of twisted differential modules. Notations are as before.

**Lemma 5.1.** *If $M$ is a $\mathrm{D}^{(\infty)}_{A/R,\sigma}$-module, then the canonical map $\mathrm{D}^{(\infty)}_{A/R,\sigma} \to \mathrm{End}_R(M)$ induces, for all $n \in \mathbb{N}$, a $P_{A/R,(n)_\sigma}$-linear map*

$$\mathrm{Diff}_{n,\sigma}(A) \to \mathrm{Diff}_{n,\sigma}(M).$$

*Hence, there exists a canonical $\widehat{P}_\sigma$-linear map*

$$D^{(\infty)}_{A/R} \to \mathrm{Diff}_\sigma(M).$$

*Proof.* Since the canonical map $\lambda : D_\sigma^{(\infty)} \to \mathrm{End}_R(M)$ is a morphism of $A$-algebras, it will commute with the action of $P$. More precisely, for all $x$, $y \in A$ and $\varphi \in D_\sigma^{(\infty)}$, we have (see [Section 4](#) for the definition of the action of $P$ on $D_\sigma^{(\infty)}$)

$$\lambda(x\tilde{y} \cdot \varphi) = \lambda(x \circ \varphi \circ y) = x \circ \lambda(\varphi) \circ y = x\tilde{y} \cdot \lambda(\varphi).$$

In particular, if $\tilde{\varphi}$ is zero on $I^{(n+1)}$, then $\widetilde{\lambda(\varphi)}$ will be zero on $I^{(n+1)} \otimes_A M$. It means that the image of $\mathrm{Diff}_{n,\sigma}(A)$ falls inside $\mathrm{Diff}_{n,\sigma}(M)$. $\qquad\square$

We will usually denote by $\varphi_M \in \mathrm{Diff}_\sigma(M)$ the image of $\varphi \in D_\sigma^{(\infty)}$. In other words, for $\varphi \in D_\sigma^{(\infty)}$ and $s \in M$, we will have $\varphi_M(s) = \varphi s$.

**Definition 5.2.** A *twisted Taylor structure (of infinite level)* on an $A$-module $M$ is a compatible family of $A$-linear maps $\theta_n : M \to M \otimes_A P_{(n)_\sigma}$ (called *twisted Taylor maps*) with $\theta_0 = \mathrm{Id}$, making commutative all the diagrams

$$
\begin{array}{ccc}
M & \xrightarrow{\quad\theta_m\quad} & M \otimes_A P_{(m)_\sigma} \\
{\scriptstyle\theta_{n+m}}\downarrow & & \downarrow{\scriptstyle\theta_n\otimes\mathrm{Id}} \\
M \otimes_A P_{(n+m)_\sigma} & \xrightarrow{\mathrm{Id}\,\otimes\delta_{n,m}} & M \otimes_A P_{(n)_\sigma} \otimes_A P_{(m)_\sigma}
\end{array}
$$

For example, the canonical twisted Taylor structure on $A$ is defined by the family of composite maps

$$
\begin{array}{ccc}
A & \xrightarrow{\quad\theta\quad} & P \\
& {\scriptstyle\theta_n}\searrow & \downarrow \\
& & P_{(n)_\sigma}
\end{array}
$$

where the upper map is the Taylor map $x \mapsto \tilde{x}$ given by the action on the right (see [Section 1](#)).

There exists an obvious notion of morphism of $A$-modules endowed with a twisted Taylor structure and they form a category.

**Proposition 5.3.** *Let $M$ be an $A$-module endowed with a twisted Taylor structure $(\theta_n)_{n\in\mathbb{N}}$. Then there exists a unique structure of a $\mathrm{D}_{A/R,\sigma}^{(\infty)}$-module on $M$ such that, for all $n \in \mathbb{N}$, we have*

$$(7) \qquad \theta_n(s) = \sum s_k \otimes f_k \Rightarrow \text{for all } \varphi \in \mathrm{Diff}_{n,\sigma}(A), \varphi_M(s) = \sum \tilde{\varphi}(f_k)s_k.$$

*This is functorial in $M$ in the sense that any morphism of $A$-modules $M \to N$ which is compatible with some twisted Taylor structures on $M$ and $N$ will automatically be $\mathrm{D}_{A/R,\sigma}^{(\infty)}$-linear. Moreover, this is an equivalence (an isomorphism) of categories if all $P_{(n)_\sigma}$ are finite projective (for the left $A$-module structure).*

Note that the last condition is satisfied if there exists a twisted coordinate on $A$.

*Proof.* First of all, there exists for all $n \in \mathbb{N}$ a canonical morphism of $A$-modules

(8) $$M \otimes_A P_{(n)_\sigma} \to \mathrm{Hom}_A(\mathrm{Hom}_A(P_{(n)_\sigma}, A), M),$$

which is automatically $P_{(n)_\sigma}$-linear. Now, $A$-linear maps

(9) $$\theta_n : M \to M \otimes_A P_{(n)_\sigma}$$

correspond bijectively to $P_{(n)_\sigma}$-linear maps

$$\epsilon_n : P_{(n)_\sigma} \otimes_A M \to M \otimes_A P_{(n)_\sigma},$$

and we can compose with the map (8) in order to get

$$P_{(n)_\sigma} \otimes_A M \to \mathrm{Hom}_A(\mathrm{Hom}_A(P_{(n)_\sigma}, A), M),$$

or equivalently,

$$\mathrm{Hom}_A(P_{(n)_\sigma}, A) \to \mathrm{Hom}_A(P_{(n)_\sigma} \otimes_A M, M).$$

In other words, we obtain $P_{(n)_\sigma}$-linear maps

(10) $$\mathrm{Diff}_{n,\sigma}(A) \to \mathrm{Diff}_{n,\sigma}(M), \quad \varphi \mapsto \varphi_M.$$

Formula (7) follows directly from the construction. Compatibility for various $n$ in (10) follows from compatibility for various $n$ in (9). We need to show that the corresponding map $\mathrm{D}_\sigma^{(\infty)} \to \mathrm{End}(M)$ is a morphism of rings. To do that, one can use the description of composition of twisted differential operators given in Proposition 4.6: we need to verify that the maps (9) are compatible with $\delta$ which is exactly the condition in the definition of Taylor structure.

This construction is clearly functorial. Moreover, if $P_{(n)_\sigma}$ is finite projective (for the left $A$-module structure), then the map (8) is actually an isomorphism. And it follows from Lemma 5.1 that a $\mathrm{D}_\sigma^{(\infty)}$-module structure on $M$ will provide us with a compatible family of maps as in (10). $\qquad\square$

**Definition 5.4.** Let $M$ be an $A$-module endowed with a twisted Taylor structure. Then the *twisted Taylor map* of $M$ is the map

$$\hat{\theta} = \varprojlim \theta_n : M \mapsto M \widehat{\otimes}_A \widehat{P}_\sigma := \varprojlim M \otimes_A P_{(n)_\sigma}.$$

The *twisted Taylor series* of $s \in M$ is $\hat{\theta}(s) \in M \widehat{\otimes}_A \widehat{P}_\sigma$.

There exists a commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\hat{\theta}} & M \widehat{\otimes}_A \widehat{P}_\sigma \\
\downarrow{\scriptstyle\hat{\theta}} & & \downarrow{\scriptstyle\hat{\theta} \widehat{\otimes}\, \mathrm{Id}} \\
M \widehat{\otimes}_A \widehat{P}_\sigma & \xrightarrow{\mathrm{Id}\,\widehat{\otimes}\hat{\delta}} & M \widehat{\otimes}_A \widehat{P}_\sigma \widehat{\otimes}_A \widehat{P}_\sigma
\end{array}
$$

and the action of $A$ on $\widehat{P}_\sigma$ on the right is given by the Taylor map of $A$.

In practice, we will only consider the case of finitely presented $A$-modules $M$, and then the completed tensor product is the usual tensor product $M \otimes_A \widehat{P}_\sigma$.

We just showed in Proposition 5.3 that any $D_\sigma^{(\infty)}$-module comes with a canonical twisted Taylor structure. When there exists a twisted coordinate on $A$, we can describe it explicitly as follows:

**Proposition 5.5** (twisted Taylor formula). *Assume that $x$ is a twisted coordinate on $A$ and let $\xi = \tilde{x} - x$. If $M$ is a $D_\sigma^{(\infty)}$-module, we have for all $n \in \mathbb{N}$,*

$$\theta_n(s) = \sum_{k=0}^n \partial_\sigma^{[k]}(s) \otimes \xi^{(k)} \in M \otimes_A P_{(n)_\sigma}$$

*and*

$$\hat{\theta}(s) = \sum_{k=0}^\infty \partial_\sigma^{[k]}(s) \otimes \xi^{(k)} \in M \widehat{\otimes}_A \widehat{P}_\sigma.$$

*Proof.* This follows from equation (7): if we write

$$\theta_n(s) = \sum_{k=0}^n s_k \otimes \xi^{(k)},$$

we will have for all $l \in \mathbb{N}$,

$$\partial_\sigma^{[l]}(s) = \sum_{k=0}^n \tilde{\partial}_\sigma^{[l]}(\xi^{(k)}) s_k = s_l. \qquad \square$$

In particular, we see that, if $z \in A$, then the image in $\widehat{P}_\sigma$ of $\tilde{z} = 1 \otimes z \in P$ is the twisted Taylor series

$$\hat{\theta}(z) = \sum_k \partial_\sigma^{[k]}(z) \xi^{(k)}.$$

This explains why it is legitimate to call the map $z \mapsto \tilde{z}$ the Taylor map.

**Examples.** (1) If $x$ is a twisted coordinate on $A$, we have $\hat{\theta}(x) = x + \xi$ and

$$\hat{\theta}(x^2) = x^2 + (x + \sigma(x))\xi + \xi^2.$$

(2) Assume that $x \in A^\times$ is an invertible twisted coordinate on $A$ and that $\sigma(x) = qx$ with $q \in R^\times$. Then we have

$$\hat{\theta}\left(\frac{1}{x}\right) = \sum_{k=0}^\infty (-1)^k \frac{\xi^{(k)}}{q^{\frac{k(k+1)}{2}} x^{k+1}} = \frac{1}{x} - \frac{\xi}{qx^2} + \frac{\xi^{(2)}}{q^3 x^3} - \cdots.$$

**Remark.** If $A$ is a twisted localization of $R[x]$, then there exists at most one $R$-algebra homomorphism $\hat{\theta} : A \to \widehat{P}_\sigma$ such that $\hat{\theta}(x) = \tilde{x}$. It means that the twisted

Taylor map

$$A \xrightarrow{\hat{\theta}} \widehat{P}_\sigma, \quad z \longmapsto \sum_{k=0}^{\infty} \partial_\sigma^{[k]}(z) \xi^{(k)}$$

is the *unique* such map.

## 6. Quantum differential operators

In the quantum situation, we can be a lot more explicit as we shall see now. Thus, we assume in this section that $A$ is a quantum $R$-algebra: we are given a twisted coordinate $x$ such that $\sigma(x) = qx + h$ with $q, h \in R$.

**Proposition 6.1.** *We have for all $k, l \in \mathbb{N}$,*

$$\partial_\sigma^{[k]} \circ \partial_\sigma^{[l]} = \binom{k+l}{l}_q \partial_\sigma^{[k+l]}.$$

*Proof.* It follows from Lemma 1.2 that

$$\widetilde{\partial_\sigma^{[k]} \circ \partial_\sigma^{[l]}} = \widetilde{\partial_\sigma^{[k]}} \circ (\mathrm{Id} \otimes \widetilde{\partial_\sigma^{[l]}}) \circ \delta.$$

Thus, using Theorem 3.5, we see that

$$(\widetilde{\partial_\sigma^{[k]} \circ \partial_\sigma^{[l]}})(\xi^{(n)}) = \widetilde{\partial_\sigma^{[k]}}\left( (\mathrm{Id} \otimes \widetilde{\partial_\sigma^{[l]}})\left( \sum_{i=0}^{n} \binom{n}{i}_q \xi^{(n-i)} \otimes \xi^{(i)} \right) \right)$$

$$= \sum_{i=0}^{n} \binom{n}{i}_q \widetilde{\partial_\sigma^{[k]}}\left( \xi^{(n-i)} \otimes \widetilde{\partial_\sigma^{[l]}}(\xi^{(i)}) \right)$$

$$= \binom{n}{l}_q \widetilde{\partial_\sigma^{[k]}}(\xi^{(n-l)}) = \begin{cases} \binom{k+l}{l}_q & \text{if } n = k+l, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Recall from [Le Stum and Quirós 2015a] that if $m \in \mathbb{N}$, we write $(m)_q := \binom{m}{1}_q = 1 + \cdots + q^{m-1}$ and we define by induction $(m)_q! := (m)_q(m-1)_q!$.

**Corollary 6.2.** *We have for all $k \in \mathbb{N}$, for all $z \in A$,*

$$\partial_\sigma^k(z) = (k)_q! \partial_\sigma^{[k]}(z).$$

*Proof.* We proceed by induction on $k$ and obtain

$$\partial_\sigma^{k+1}(z) = \partial_\sigma^k(\partial_\sigma(z)) = (k)_q! \partial_\sigma^{[k]}(\partial_\sigma(z))$$

$$= (k)_q! (\partial_\sigma^{[k]} \partial_\sigma)(z)$$

$$= (k)_q! \binom{k+1}{1}_q \partial_\sigma^{[k+1]}(z) = (k+1)_q! \partial_\sigma^{[k+1]}(z). \qquad \square$$

The next result is important because it describes explicitly the relations between the different rings of twisted differential operators introduced so far. Recall from Corollary 4.12 that there exists an epi-mono factorization

$$\mathrm{D}_{A/R,\sigma} \twoheadrightarrow \bar{\mathrm{D}}_{A/R,\sigma} \hookrightarrow \mathrm{D}_{A/R,\sigma}^{(\infty)}.$$

Recall from [Le Stum and Quirós 2015a] that the ring $R$ is said to be *q-flat* (resp. *q-divisible*) if $(m)_q$ is never a zero-divisor (resp. is always invertible) unless $(m)_q = 0$ and also that the *q-characteristic* of $R$ is the smallest positive integer $p$ such that $(p)_q = 0$, if it exists, and 0 otherwise.

**Theorem 6.3.** *Assume $R$ is $q$-divisible and let $A$ be a $q$-$R$-algebra. Then:*

(1) *If $q$ -char$(A) = 0$, we have*

$$\mathrm{D}_{A/R,\sigma} = \bar{\mathrm{D}}_{A/R,\sigma} = \mathrm{D}_{A/R,\sigma}^{(\infty)}.$$

(2) *If $q$ -char$(A) = p > 0$, we have*

$$\mathrm{D}_{A/R,\sigma}/\partial_\sigma^p \simeq \bar{\mathrm{D}}_{A/R,\sigma} \simeq \mathrm{D}_{A/R,\sigma}^{(\infty)}/K_\sigma^{[p]},$$

*where $K_\sigma^{[p]}$ is the free $A$-module generated by all $\partial_\sigma^{[k]}$ for $k \geq p$.*

Note that with some extra conventions, assertions (1) and (2) could be combined.

*Proof.* We use the formula from Corollary 6.2.

In situation (1), all $q$-integers are invertible in $A$ and the composite map sends the canonical basis $\{\partial_\sigma^k\}_{k\in\mathbb{N}}$ of $\mathrm{D}_\sigma$ bijectively onto a basis of $\mathrm{D}_\sigma^{(\infty)}$. The first assertion follows.

In situation (2), we have $(p)_q = 0$ in $A$ and the element $\partial_\sigma^p \in \mathrm{D}_\sigma$ is therefore sent to 0. But since $R$ is $q$-divisible, we have $(m)_q \in R^\times$ for $m < p$, and the composite map sends the family $\{\partial_\sigma^k\}_{k<p}$ of $\mathrm{D}_\sigma$ bijectively onto a basis of $\mathrm{D}_\sigma^{(\infty)}/K_\sigma^{[p]}$. $\quad\square$

**Remarks.** (1) The hypothesis in (1) is satisfied in the classical cases of differential equations, finite difference equations and $q$-difference equations.

More precisely, it is satisfied for example when

(a) $q = 1$ and $R$ is a $\mathbb{Q}$-algebra, or

(b) $q$ is not equal to zero, not a root of 1 and belongs to a subfield $K$ of $R$.

The hypothesis in (2) is satisfied for differential equations and finite difference equations in positive characteristic as well as in the classical quantum case. More precisely, they are satisfied for example when

(a) $q = 1$ and $R$ is an $\mathbb{F}_p$-algebra, or

(b) $q$ is a nontrivial $p$-th root of 1 ($p$ not necessarily prime) and belongs to a subfield $K$ of $R$, or

(c) $q$ is a nontrivial $p$-th root of 1 with $p$ prime (but $q$ does not necessarily belong to a subfield of $R$).

(2) Since both $D_{A/R,\sigma}$ and $D_{A/R,\sigma}^{(\infty)}$ commute with extensions of $R$ (although $\bar{D}_{A/R,\sigma}$ does not), and $D_{A/R,\sigma}$ always commutes with extensions of $A$, we can sometimes (see the remark following Proposition 1.10) reduce questions to the generic case

(11) $$R = \mathbb{Q}(t)[s], \quad A = R[x], \quad q = t \quad \text{and} \quad h = s,$$

and work as well over the latter. In this case, thanks to the theorem, we can identify $D_{A/R,\sigma}$ with $D_{A/R,\sigma}^{(\infty)}$.

(3) The same proof shows that if $A$ is a twisted $R$-algebra which is only $q$-flat (but not necessarily $q$-divisible), then

(a) if $q$-char$(A) = 0$, then $D_{A/R,\sigma} = \bar{D}_{A/R,\sigma}$;

(b) if $q$-char$(A) = p > 0$, then $D_{A/R,\sigma}/\partial_\sigma^p \simeq \bar{D}_{A/R,\sigma}$.

The end of this section will be devoted to giving explicit formulas. They are usually quite formal to prove in the ring of twisted differential operators of infinite level and their analog in the twisted Weyl algebra is then easily obtained thanks to Theorem 6.3.

**Proposition 6.4.** *In* $D_{A/R,\sigma}^{(\infty)}$*, for all* $k > 0$,

$$\partial_\sigma^{[k]} \circ x = q^k x \partial_\sigma^{[k]} + (k)_q h \partial_\sigma^{[k]} + \partial_\sigma^{[k-1]}.$$

*In* $D_{A/R,\sigma}$*, for all* $k > 0$,

$$\partial_\sigma^k \circ x = q^k x \partial_\sigma^k + (k)_q (h \partial_\sigma^k + \partial_\sigma^{k-1}).$$

*Proof.* The first assertion is simply a reformulation of Proposition 4.7. For the second one, after a base change, we may reduce to the generic case (11) and thus assume that $R$ is $q$-divisible and $q$-char$(R) = 0$. And we may then replace $D_\sigma$ with $D_\sigma^{(\infty)}$ in which case we fall back onto the first equality. $\qquad\square$

Note that in order to prove the second formula, we cannot use Corollary 6.2 directly: our equality takes place in the Weyl algebra and is not just an assertion about endomorphisms of $A$.

We concentrate now on the case $\sigma(x) = qx$.

**Proposition 6.5.** *Assume* $h = 0$, $q \in R^\times$ *and* $x \in A^\times$. *Then in* $D_{A/R,\sigma}^{(\infty)}$*, for all* $k \in \mathbb{N}$,

$$\partial_\sigma^{[k]} \circ x^{-1} = \sum_{i=0}^{k} (-1)^i q^{-\frac{(2k-i)(i+1)}{2}} x^{-i-1} \partial_\sigma^{[k-i]}.$$

*In* $D_{A/R,\sigma}$*, for all* $k \in \mathbb{N}$,

$$\partial_\sigma^k \circ x^{-1} = \sum_{i=0}^{k} (-1)^i q^{-\frac{(2k-i)(i+1)}{2}} (k)_q \cdots (k-i+1)_q x^{-i-1} \partial_\sigma^{k-i}.$$

*Proof.* The first assertion is a particular case of Corollary 4.8 and the second one follows by the standard generic argument. □

**Proposition 6.6.** *Assume that* $h = 0$. *Then in* $\mathrm{D}_{A/R,\sigma}^{(\infty)}$, *for all* $k, n \in \mathbb{N}$,

$$\partial_\sigma^{[k]} \circ x^n = \sum_{i=0}^k q^{(n-i)(k-i)} \binom{n}{i}_q x^{n-i} \partial_\sigma^{[k-i]}.$$

*In* $\mathrm{D}_{A/R,\sigma}$, *for all* $k, n \in \mathbb{N}$,

$$\partial_\sigma^k \circ x^n = \sum_{i=0}^k q^{(n-i)(k-i)} (i)_q! \binom{k}{i}_q \binom{n}{i}_q x^{n-i} \partial_\sigma^{k-i}.$$

*Proof.* As usual, the second formula will follow from the first, which we prove directly by induction on $n$. We have

$$\partial_\sigma^{[k]} \circ x^n = \sum_{i=0}^k q^{(n-1-i)(k-i)} \binom{n-1}{i}_q x^{n-1-i} \partial_\sigma^{[k-i]} \circ x$$

$$= \sum_{i=0}^k q^{(n-1-i)(k-i)} \binom{n-1}{i}_q x^{n-1-i} (q^{k-i} x \partial_\sigma^{[k-i]} + \partial_\sigma^{[k-i-1]})$$

$$= \sum_{i=0}^k q^{(n-1-i)(k-i)} \binom{n-1}{i}_q x^{n-1-i} q^{k-i} x \partial_\sigma^{[k-i]}$$

$$\qquad + \sum_{i=0}^k q^{(n-1-i)(k-i)} \binom{n-1}{i}_q x^{n-1-i} \partial_\sigma^{[k-i-1]}$$

$$= \sum_{i=0}^k q^{(n-i)(k-i)} \binom{n-1}{i}_q x^{n-i} \partial_\sigma^{[k-i]}$$

$$\qquad + \sum_{i=0}^{k-1} q^{(n-i)(k-i+1)} \binom{n-1}{i-1}_q x^{n-i} \partial_\sigma^{[k-i]}$$

$$= \sum_{i=0}^k q^{(n-i)(k-i)} \left( \binom{n-1}{i}_q + q^{n-i} \binom{n-1}{i-1}_q \right) x^{n-i} \partial_\sigma^{[k-i]}$$

$$= \sum_{i=0}^k q^{(n-i)(k-i)} \binom{n}{i}_q x^{n-i} \partial_\sigma^{[k-i]}. \qquad \square$$

**Corollary 6.7.** *When* $h = 0$, *we have for all* $n \in \mathbb{N}$,

$$\partial_\sigma^{[k]}(x^n) = \begin{cases} \binom{n}{k}_q x^{n-k} & \text{if } n \geq k, \\ 0 & \text{otherwise,} \end{cases}$$

*and*

$$\partial_\sigma^k(x^n) = \begin{cases} (n)_q \cdots (n-k+1)_q x^{n-k} & \text{if } n \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The first assertion follows from the proposition and the second one uses the inclusion $\bar{\mathrm{D}}_{A/R,\sigma} \subset \mathrm{D}_{A/R,\sigma}^{(\infty)}$. □

**Corollary 6.8.** *Assume that $h = 0$. Then we have for all $n \in \mathbb{N} \setminus \{0\}$,*

$$\partial_\sigma(x^n) = (n)_q x^{n-1}.$$

*If $x \in A^\times$ and $q \in R^\times$, then the formula actually holds for all $n \in \mathbb{Z}$.*

*Proof.* Only the second part needs a proof: if $x$ is invertible, we have for $n > 0$

$$\begin{aligned} 0 = \partial_\sigma(1) &= \partial_\sigma(x^n x^{-n}) \\ &= \partial_\sigma(x^n)x^{-n} + \sigma(x^n)\partial_\sigma(x^{-n}) \\ &= (n)_q x^{n-1}x^{-n} + q^n x^n \partial_\sigma(x^{-n}) = (n)_q x^{-1} + q^n x^n \partial_\sigma(x^{-n}), \end{aligned}$$

from which we derive, when $q \in R^\times$,

$$\partial_\sigma(x^{-n}) = -q^{-n}(n)_q x^{-n-1} = (-n)_q x^{-n-1}. \qquad \Box$$

## 7. Formal deformations of twisted differential operators

In this section, we study the relation between twisted differential operators relative to various endomorphisms of $A$. We are particularly interested in the comparison of our twisted differential operators with usual differential operators. We assume that there exists a twisted coordinate $x$, that we fix for the rest of the section, and we write $y_\sigma := x - \sigma(x)$.

Recall that the ring $\mathrm{D}_\sigma^{(\infty)}$ of differential operators (of infinite level) comes with a natural increasing filtration by $A$-submodules

$$\mathrm{D}_\sigma^{(\infty)} = \cup_{m=0}^\infty \mathrm{Diff}_{m,\sigma}(A),$$

which is called the *order filtration*. The choice of the twisted coordinate determines a splitting of this filtration. To see this, we let for all $m \in \mathbb{N}$, $K_\sigma^{[m]} \subset \mathrm{D}_\sigma^{(\infty)}$ be the free $A$-module generated by all $\partial_\sigma^{[k]}$ for $k \geq m$. Note that this is actually a filtration by *left* ideals.

**Definition 7.1.** The decreasing filtration by the $K_\sigma^{[m]}$ is called the *ideal filtration* on $\mathrm{D}_\sigma^{(\infty)}$. The *module of twisted differential operators of infinite level and infinite order on $A$* is

$$\widehat{\mathrm{D}}_{A/R,\sigma}^{(\infty)} = \varprojlim \mathrm{D}_\sigma^{(\infty)}/K_\sigma^{[m+1]}.$$

We might again drop the index $A/R$ and write $\widehat{\mathrm{D}}_\sigma^{(\infty)}$. The decreasing filtration of $\widehat{\mathrm{D}}_\sigma^{(\infty)}$ by the closures $\widehat{K}_\sigma^{[m]}$ of the $K_\sigma^{[m]}$ will also be called the *ideal filtration*.

**Remarks.**   (1) The ideal filtration is *separated*, which means that $\cap_m K_\sigma^{[m+1]} = \{0\}$ and it follows that $D_\sigma^{(\infty)} \subset \widehat{D}_\sigma^{(\infty)}$. Actually, any $\varphi \in \widehat{D}_\sigma^{(\infty)}$ can be uniquely written as an infinite sum $\sum_0^\infty z_k \partial_\sigma^{[k]}$ with $z_k \in A$ (and conversely). In other words, we have the isomorphisms of $A$-modules

$$D_\sigma^{(\infty)} = \bigoplus_{k \in \mathbb{N}} A \partial_\sigma^{[k]} \quad \text{and} \quad \widehat{D}_\sigma^{(\infty)} = \prod_{k \in \mathbb{N}} A \partial_\sigma^{[k]}.$$

(2) The ideal filtration defines a splitting of the order filtration in the sense that

$$D_\sigma^{(\infty)} = \text{Diff}_{m,\sigma}(A) \oplus K_\sigma^{[m+1]} \quad \text{and} \quad \widehat{D}_\sigma^{(\infty)} = \text{Diff}_{m,\sigma}(A) \oplus \widehat{K}_\sigma^{[m+1]}$$

as $A$-modules.

(3) $\widehat{D}_\sigma^{(\infty)}$ is *not* a ring in general. More precisely, multiplication on $D_\sigma^{(\infty)}$ is not continuous for the ideal filtration: we always have $\partial_\sigma^{[m]} \to 0$ when $m \to \infty$ but, if $\sigma(x) = qx$ and $x \in A^\times$ for example, we can see that for all $m \in \mathbb{N}$,

$$\partial_\sigma^{[m]} \circ x^{-1} \equiv \partial_\sigma^{[m]}(x^{-1}) = x^{-m-1} \neq 0 \mod K_\sigma^{[1]}.$$

If $A$ is any ring and $A[\xi]$ denotes the polynomial ring in one variable $\xi$, then the *natural filtration* on $\text{Hom}_A(A[\xi], A)$ is the decreasing filtration by the kernels of the surjections

$$\text{Hom}_A(A[\xi], A) \to \text{Hom}_A(A[\xi]_{\leq m}, A),$$

where $A[\xi]_{\leq m}$ denotes as before the $A$-module of polynomials of degree at most $m$. The corresponding topology will be called the *natural topology* of $\text{Hom}_A(A[\xi], A)$. Note that $\text{Hom}_A(A[\xi], A)$ is separated and complete for the natural topology:

$$\varprojlim \text{Hom}_A(A[\xi]_{\leq m}, A) = \text{Hom}_A(\varinjlim A[\xi]_{\leq m}, A) = \text{Hom}_A(A[\xi], A).$$

**Lemma 7.2** (formal density). *The map*

$$A[\xi] \to P_{A/R}, \quad \xi \mapsto \tilde{x} - x$$

*induces by duality an isomorphism of topological $A$-modules*

(12) $$\widehat{D}_{A/R,\sigma}^{(\infty)} \xrightarrow{\sim} \text{Hom}_A(A[\xi], A).$$

*More precisely, the ideal filtration corresponds to the natural filtration.*

*Proof.* It follows from Corollary 1.13 that the cokernel of the map

$$A[\xi]_{\leq m} \hookrightarrow A[\xi] \to P \twoheadrightarrow P_{(n)_\sigma}$$

is generated by $\xi^{(k)}$ for $m < k \leq n$. Moreover, this map is injective when $m \leq n$. Dually, it means that the corresponding map

$$\text{Diff}_{n,\sigma}(A) \simeq \text{Hom}_A(P_{(n)_\sigma}, A) \to \text{Hom}_A(A[\xi]_{\leq m}, A)$$

is surjective for $m \leq n$ and that its kernel is exactly $K_\sigma^{[m+1]} \cap \mathrm{Diff}_{n,\sigma}(A)$ (it is generated by $\partial_\sigma^{[k]}$ for $n \geq k > m$). Taking direct limits on the left, we see that the map

$$\mathrm{D}_\sigma^{(\infty)} \to \mathrm{Hom}_A(A[\xi]_{\leq m}, A)$$

is surjective with kernel exactly $K_\sigma^{[m+1]}$. In other words, we get a canonical isomorphism of $A$-modules

$$\mathrm{D}_\sigma^{(\infty)}/K_\sigma^{[m+1]} \xrightarrow{\sim} \mathrm{Hom}_A(A[\xi]_{\leq m}, A).$$

Thus, taking inverse limits on both sides gives the result.                                    □

**Remarks.** (1) By construction, there exists a commutative diagram



(2) We will usually denote by $\tilde{\varphi} \in \mathrm{Hom}_A(A[\xi], A)$ the image of $\varphi \in \widehat{\mathrm{D}}_{A/R,\sigma}^{(\infty)}$. This is compatible with the previous notation for linearization. In particular, we have for all $k, n \in \mathbb{N}$,

$$\widetilde{\partial_\sigma^{[k]}}(\xi^{(n)}) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

(3) When $A = R[x]$, we actually get

$$\mathrm{D}_{A/R,\sigma}^{(\infty)} \hookrightarrow \widehat{\mathrm{D}}_{A/R,\sigma}^{(\infty)} \xrightarrow{\sim} \mathrm{Hom}_A(A[\xi], A) \xrightarrow{\sim} \mathrm{Hom}_A(P, A) \xrightarrow{\sim} \mathrm{End}_R(A),$$

and $\widehat{\mathrm{D}}_{A/R,\sigma}^{(\infty)}$ is a ring in this very special case.

(4) There exists a complex analytic analog of the density lemma (for usual differential operators) as explained by Zoghman Mebkhout and Luis Narváez [1998]: the ring of algebraic differential operators is dense in the ring of continuous endomorphisms of the structural sheaf.

Although it is difficult to give an explicit description of the isomorphism (12), we can at least show the following:

**Lemma 7.3.** *We have for all $n \in \mathbb{N} \setminus \{0\}$,*

$$\tilde{\partial}_\sigma(\xi^n) = (\sigma(x) - x)^{n-1}.$$

*More generally*, *if $\tau$ is another R-algebra endomorphism of A, we have for all*
$n \in \mathbb{N} \setminus \{0\}$,

$$\tilde{\partial}_\sigma(\xi^{(n)_\tau}) = \prod_{i=1}^{n-1}(\sigma(x) - \tau^i(x)),$$

*where $\xi^{(n)_\tau} := \xi\tau(\xi)\cdots\tau^{n-1}(\xi)$ denotes the twisted power with respect to the*
*endomorphism $\tau$.*

*Proof.* By definition, $\tilde{\partial}_\sigma$ is the unique $A$-linear function on $A[\xi]$ such that $\tilde{\partial}_\sigma(\xi^{(n)}) = 1$ for $n = 1$ and 0 otherwise. We consider now the unique $A$-linear function $u$ on $A[\xi]$ such that $u(\xi^n) = (\sigma(x) - x)^{n-1}$ for $n > 0$ and $u(1) = 0$. We want to show that $u = \tilde{\partial}_\sigma$. The map $u$ may be seen as the composition of division by $\xi$ on $A[\xi]$ (after removing the constant term) and evaluation at $\sigma(x) - x$. But we have

$$\xi^{(n)} := \xi\sigma(\xi)\cdots\sigma^{n-1}(\xi),$$

and we know that $\sigma(\xi) = \xi + x - \sigma(x)$. Therefore, it is clear that for $n \geq 2$, we will get $u(\xi^{(n)}) = 0$. Of course, for $n = 0$ we have $\xi^{(n)} = 1$ and we also obtain 0. Finally, for $n = 1$, we have $\xi^{(n)} = \xi$ and we get 1.

The proof of the second formula follows the same lines. From the first part, we can interpret $\tilde{\partial}_\sigma$ as the composition of division by $\xi$ on $A[\xi]$ and evaluation at $\sigma(x) - x$. We want to apply this to $\xi^{(n)_\tau} = \prod_{i=0}^{n-1}\tau^i(\xi)$ and we know that $\tau^i(\xi) = \xi + x - \tau^i(x)$. Thus, we see that

$$\tilde{\partial}_\sigma(\xi^{(n)_\tau}) = \prod_{i=1}^{n-1}(\sigma(x) - x + x - \tau^i(x)) = \prod_{i=1}^{n-1}(\sigma(x) - \tau^i(x)). \qquad \square$$

We can actually derive a remarkable consequence of the density lemma (recall that when $\tau$ is an endomorphism of the $R$-algebra $A$, we call any twisted coordinate relative to $\tau$ a $\tau$-coordinate):

**Proposition 7.4** (formal deformation). *If $x$ is a also a $\tau$-coordinate for another R-endomorphism $\tau$ of A, then there exists a canonical isomorphism of topological A-modules (that depends on $x$)*

$$\widehat{D}_{A/R,\sigma}^{(\infty)} \xrightarrow{\sim} \widehat{D}_{A/R,\tau}^{(\infty)}.$$

*More precisely, it is compatible with the ideal filtrations.*

Recall that the hypothesis is always satisfied when $A$ is a $\tau$-twisted localization of $R[x]$.

*Proof.* We may just compose the isomorphism of the formal density Lemma 7.2 with the inverse of the analogous isomorphism for $\tau$. $\qquad \square$

When the coordinate $x$ is fixed, we may (and will) *identify* these two topological (or even, filtered) $A$-modules. It is worth mentioning a particular case of this corollary that is of great interest (this is the case $\tau = \mathrm{Id}_A$, in which case we say *usual coordinate* instead of $\tau$-coordinate):

**Corollary 7.5.** *Assume that $x$ is not only a $\sigma$-coordinate, but also a usual coordinate on $A$. Then there exists a canonical isomorphism*

$$\widehat{\mathrm{D}}^{(\infty)}_{A/R,\sigma} \xrightarrow{\sim} \widehat{\mathrm{D}}^{(\infty)}_{A/R}.$$

Note that the hypothesis is always satisfied when $A$ is a localization of $R[x]$. More generally, when $A$ is smooth over $R$, a usual coordinate is the same thing as an étale coordinate.

**Remark.** (1) In theorem 2 of the introduction to [Pulita 2017], Pulita shows that, in the $p$-adic world, some differential modules have the natural structure of a $\sigma$-module. The main idea is to realize a formal solution of the differential module as formal solution of some $\sigma$-module. This is analogous to the way we derive the formal deformation theorem from the formal density lemma. It seems that the first result in this direction was obtained by André and Di Vizio [2004].

(2) In [Gros and Le Stum 2014], Michel Gros and Bernard Le Stum were able to prove a quantum Simpson's correspondence. Assume $R = \mathbb{Z}[q]$, where $q$ is a $p$-th root of unity with $p$ prime, $A$ is the polynomial ring and $\sigma(x) = qx$. Then the category of $A$-modules endowed with a quasinilpotent $q$-derivation is equivalent to the category of $A$-modules endowed with an $A$-*linear* quasinilpotent endomorphism. This is derived from an isomorphism quite analogous to (12) which reads

$$\widehat{D}_q \simeq \mathrm{End}_{\widehat{Z}_q}(\widehat{Z_q A}).$$

(3) We want to insist on the fact that the isomorphism of Corollary 7.5 is not an isomorphism of rings. Actually, neither side of this isomorphism is a ring. In order to achieve this goal, it would be necessary to refine the topology. After the remarks following Corollary 7.7, we will give some elementary but nontrivial examples over the complex numbers, and refer the reader to forthcoming articles for a detailed study of the ultrametric and the $q$-characteristic zero cases.

We give now some explicit formulas in order to express the twisted derivation from one world as a twisted differential operator in another world:

**Proposition 7.6.** *Assume that $\tau$ is some other $R$-endomorphism of $A$ and $x$ is also a $\tau$-coordinate. Then we have*

$$\partial_\sigma = \sum_{k=1}^{\infty} \left( \prod_{i=1}^{k-1} (\sigma(x) - \tau^i(x)) \right) \partial_\tau^{[k]}.$$

*Proof.* This follows directly by duality from Lemma 7.3.                    □

**Corollary 7.7.** *Assume that $x$ is also a usual coordinate on A. Then*

$$\partial_\sigma = \sum_{k=1}^{\infty} (\sigma(x) - x)^{k-1} \partial^{[k]}$$

*and*

$$\sigma = \sum_{k=0}^{\infty} (\sigma(x) - x)^k \partial^{[k]}.$$

*Proof.* The first assertion follows directly from the proposition and the other one is deduced from the equality $\sigma = 1 - y_\sigma \partial_\sigma$.                    □

**Remarks.** (1) In the case $\sigma(x) = qx$, the formulas read

(13)       $$\partial_\sigma = \sum_{k=1}^{\infty} (q-1)^{k-1} x^{k-1} \partial^{[k]} \quad \text{and} \quad \sigma = \sum_{k=0}^{\infty} (q-1)^k x^k \partial^{[k]}.$$

(2) In the case $\sigma(x) = x + h$, the formulas read

(14)       $$\partial_\sigma = \sum_{k=1}^{\infty} h^{k-1} \partial^{[k]} \quad \text{and} \quad \sigma = \sum_{k=0}^{\infty} h^k \partial^{[k]}.$$

**Examples.** (1) Consider the differential equation

$$\frac{ds}{dz} = s$$

over the complex numbers. If we use the second formula of (13), we see that the corresponding $q$-difference equation is given by

$$s(qz) = e^{(q-1)z} s(z).$$

If we use (14) instead, we obtain the difference equation

$$s(z + h) = e^h s(z).$$

(2) Consider now the differential equation

$$\frac{ds}{dz} = \frac{a}{z} s$$

for some $a \in \mathbb{C}$. Then the $q$-difference equation will be given by

$$s(qz) = q^a s(z),$$

and the difference equation is

$$s(z + h) = \left(\frac{z+h}{z}\right)^a s(z).$$

We may also go the other way and express the usual derivation in term of twisted derivations. We only do the classical cases:

**Corollary 7.8.** *Assume that x is also a usual coordinate on A. Then*

(1) *If $\sigma(x) = qx$, we have*

$$\partial = \sum_{k=1}^{\infty} (1-q)^{k-1}(k-1)_q! x^{k-1} \partial_\sigma^{[k]}.$$

(2) *If $\sigma(x) = x + h$, we have*

$$\partial = \sum_{k=1}^{\infty} (-1)^{k-1} h^{k-1}(k-1)! \partial_\sigma^{[k]}.$$

*Proof.* In the formula of Proposition 7.6 we replace $\tau$ by $\sigma$ and $\sigma$ by Id respectively, in order to obtain

$$\partial = \sum_{k=1}^{\infty} \prod_{i=1}^{k-1}(x - q^i x)\partial_\sigma^{[k]} = \sum_{k=1}^{\infty} \prod_{i=1}^{k-1}(1 - q^i)x^{k-1}\partial_\sigma^{[k]}$$

$$= \sum_{k=1}^{\infty}(1-q)^{k-1}\prod_{i=1}^{k-1}(i)_q x^{k-1}\partial_\sigma^{[k]} = \sum_{k=1}^{\infty}(1-q)^{k-1}(k-1)_q! x^{k-1}\partial_\sigma^{[k]}$$

in the first case, and

$$\partial = \sum_{k=1}^{\infty} \prod_{i=1}^{k-1}(-ih)\partial_\sigma^{[k]} = \sum_{k=1}^{\infty}(-h)^{k-1}(k-1)!\partial_\sigma^{[k]}$$

in the other one. $\qquad\square$

We may also apply Proposition 7.6 in order to make more precise the statement of Proposition 4.13.

**Corollary 7.9.** *Assume that x is also a $\sigma^m$-coordinate for A over R, then we have*

$$\partial_{\sigma^m} = \sum_{k=1}^{m} \left( \prod_{i=1}^{k-1}(\sigma^m(x) - \sigma^i(x)) \right) \partial_\sigma^{[k]}.$$

One can give a more concrete formula in the quantum situation:

**Corollary 7.10.** *Assume that x is also a $\sigma^m$-coordinate for A over R and that $\sigma(x) = qx$. Then we have*

$$\partial_{\sigma^m} = \sum_{k=1}^{m} q^{\frac{k(k-1)}{2}}(q-1)^{k-1}(m-1)_q \cdots (m-k+1)_q x^{k-1}\partial_\sigma^{[k]}.$$

*Proof.* We compute the coefficient of $\partial_\sigma^{[k]}$ for $1 \leq k \leq m$:

$$\prod_{i=1}^{k-1}(q^m x - q^i x) = \left(\prod_{i=1}^{k-1} q^i\right)\left(\prod_{i=1}^{k-1}(q^{m-i} - 1)\right) x^{k-1}$$

$$= q^{\frac{k(k-1)}{2}}(q-1)^{k-1}\left(\prod_{i=1}^{k-1}(m-i)_q\right) x^{k-1}. \qquad \square$$

**Example.** We have $\partial_{\sigma^2} = \partial_\sigma + q(q-1)x\partial_\sigma^{[2]}$.

## 8. Formal confluence

We explain here how one can use the quantum Weyl algebra in order to approximate a usual differential operator. We assume that $A$ is a quantum $R$-algebra which means that we are given a twisted coordinate $x$ on $A$ with $\sigma(x) = qx + h$ with $q, h \in R$.

Recall from [Le Stum and Quirós 2015b] that the twisted Weyl $R$-algebra $D_\sigma$ has an increasing filtration by free $A$-modules of finite rank

$$\text{Fil}^m D_\sigma = \bigoplus_{k \leq m} A\partial_\sigma^k$$

that is called the *order filtration*. But it also has a decreasing filtration by free $A$-modules (of infinite rank)

$$K_\sigma^m = \bigoplus_{k \geq m} A\partial_\sigma^k$$

that we called the *ideal filtration*. We will consider the completion

$$\widehat{D}_\sigma = \varprojlim D_\sigma / K_\sigma^{m+1}$$

that also comes with its *ideal filtration* and we have $D_\sigma \subset \widehat{D}_\sigma$. Note that $\widehat{D}_\sigma$ is the set of all infinite sums $\sum_0^\infty z_k \partial_\sigma^k$ with $z_k \in A$. In other words, there exists an isomorphism of $A$-modules

$$\widehat{D}_\sigma = \prod_{k \in \mathbb{N}} A\partial_\sigma^k.$$

The $A$-module $\widehat{D}_\sigma$ is not a ring in general. However, the result holds in the finite quantum characteristic case as we can check right now:

**Proposition 8.1.** *If $A$ is $q$-flat and $q$-char$(A) = p > 0$, then multiplication is continuous for the ideal topology on $D_{A/R,\sigma}$ and turns $\widehat{D}_{A/R,\sigma}$ into an $R$-algebra.*

*Proof.* From the equality (see the remark following Theorem 6.3)

$$D_\sigma / \partial_\sigma^p \simeq \bar{D}_\sigma,$$

and the fact that $\bar{D}_\sigma$ is a ring, we deduce that the (left) ideal generated by $\partial_\sigma^p$ is a two-sided ideal. Since multiplication is automatically continuous for the $\partial_\sigma^p$-adic filtration, it follows that it is also continuous for the ideal filtration (which is the filtration by left ideals generated by the powers of $\partial_\sigma$). $\qquad\square$

**Lemma 8.2.** *The composite map*

$$D_{A/R,\sigma} \twoheadrightarrow \bar{D}_{A/R,\sigma} \hookrightarrow D_{A/R,\sigma}^{(\infty)}$$

*is compatible with the ideal filtrations. Moreover, if $R$ is $q$-divisible and if $q$ -char$(R) = 0$, then $\widehat{D}_{A/R,\sigma} \simeq \widehat{D}_{A/R,\sigma}^{(\infty)}$.*

This applies in particular when $R$ is a $\mathbb{Q}$-algebra and $\sigma = \mathrm{Id}$.

*Proof.* The first assertion follows from Corollary 6.2 and the second follows from Theorem 6.3. $\qquad\square$

We can now state our fist confluence theorem (recall that a usual coordinate is the same thing as an étale coordinate when $A$ is smooth over $R$):

**Theorem 8.3** (formal quantum confluence 1). *Let $R$ be a $\mathbb{Q}$-algebra, $A$ a $q$-$R$-algebra for some $q \in R$. Assume that the quantum coordinate on $A$ is also a usual coordinate. Then there exists a canonical map of filtered $A$-modules*

$$(15) \qquad\qquad D_{A/R,\sigma} \to \widehat{D}_{A/R}.$$

*Moreover, if $R$ is $q$-divisible, we have*

(1) *If $q$ -char$(A) = 0$, then the map (15) is injective and $D_{A/R,\sigma}$ is dense in $\widehat{D}_{A/R}$. Actually, the inclusion morphism $D_{A/R,\sigma} \subset \widehat{D}_{A/R}$ is strict for the ideal filtrations in the sense that $D_{A/R,\sigma} \cap \widehat{K}^m = K_\sigma^m$ for all $m \in \mathbb{N}$.*

(2) *If $q$ -char$(A) = p > 0$, then the map (15) induces an isomorphism*

$$(D_{A/R,\sigma}/\partial_\sigma^p \simeq) \quad \bar{D}_{A/R,\sigma} \simeq D_{A/R}/\partial^p.$$

*Proof.* The map (15) is simply the composite

$$D_\sigma \to D_\sigma^{(\infty)} \hookrightarrow \widehat{D}_\sigma^{(\infty)} \simeq \widehat{D}^{(\infty)} \simeq \widehat{D}_{A/R},$$

where the next to the last map is the formal deformation isomorphism of Corollary 7.5 and the last one comes from Lemma 8.2 applied to the case $\sigma = \mathrm{Id}$ since $R$ is a $\mathbb{Q}$-algebra.

If we assume that $R$ is $q$-divisible and that $q$ -char$(A) = 0$, then Lemma 8.2 tells us that $D_\sigma = D_\sigma^{(\infty)}$ as filtered rings (for the ideal filtrations) and we can use Corollary 7.5 again.

Finally, when $R$ is $q$-divisible but $q$ -char$(A) = p > 0$, then $(p)_q = 0$ in $A$ and all $(m)_q \in R^\times$ for $m < p$. We can use the last assertion of Theorem 6.3 and the fact that the isomorphism of Corollary 7.5 is strictly compatible with the filtrations. $\quad\square$

As in [Le Stum and Quirós 2015b], we denote by $A[T]_\sigma$ the twisted polynomial ring associated to $\sigma$: this is the noncommutative polynomial ring in $T$ over $A$ with the commutation rule $Tx = \sigma(x)T$. Recall also that the twisted coordinate $x$ is said to be *strong* if $x - \sigma(x) \in A^\times$.

**Corollary 8.4.** *If $R$ is a $q$-divisible $\mathbb{Q}$-algebra, $q$ -char$(A) = 0$, $x$ is strong and is also a $\sigma^n$-coordinate for all $n \in \mathbb{N}$, then the $A$-linear map*

$$(16) \qquad A[T]_\sigma \to \widehat{\mathrm{D}}_{A/R}, \quad T^n \mapsto \sum_{k=0}^\infty \frac{1}{k!}(\sigma^n(x) - x)^k \partial^k$$

*has dense image.*

*Proof.* Compose the isomorphism of theorem 6.13 of [Le Stum and Quirós 2015b] with the map of Theorem 8.3. The formula comes from Corollary 7.7. □

**Example.** The theorem (and its corollary) applies in particular when $q$ is not a root of unity in some field $K$ of characteristic zero, $R$ is an algebra containing $K$, $A = R[x, x^{-1}]$ and $\sigma(x) = qx$. In this situation, $\mathrm{D} := \mathrm{D}_{A/R}$ is the noncommutative ring $R[x, x^{-1}, \partial]$ with the commutation rule $\partial x = x\partial + 1$ and we may see $\widehat{\mathrm{D}}$ (which is *not* a ring) as the set of infinite sums $R[x, x^{-1}][\![\partial]\!]$. Also, $\mathrm{D}_\sigma := \mathrm{D}_{A/R,\sigma}$ is the noncommutative ring $R[x, x^{-1}, \partial_\sigma]$ with the commutation rule $\partial_\sigma x = qx\partial_\sigma + 1$. The map of the theorem satisfies

$$\partial_\sigma \mapsto \sum_{k=1}^\infty \frac{1}{k!}(q-1)^{k-1}x^{k-1}\partial^k,$$

and the map of the corollary is given by

$$T^n \mapsto \sum_{k=0}^\infty \frac{1}{k!}(q^n - 1)^k x^k \partial^k.$$

In the next section, we will have to move between $\sigma$ and powers of $\sigma$ (or more precisely, the other way around, between $\sigma$ and roots of $\sigma$).

**Proposition 8.5.** *Let $m \in \mathbb{N} \setminus \{0\}$. If $(m)_q! \in R^\times$, then there exists a unique $A$-linear ring homomorphism*

$$\mathrm{D}_{A/R,\sigma^m} \xrightarrow{\iota_{\sigma,m}} \mathrm{D}_{A/R,\sigma}, \quad \partial_{\sigma^m} \longmapsto \sum_{k=1}^m \frac{1}{(k)_q!}\left(\prod_{i=1}^{k-1}(\sigma^m(x) - \sigma^i(x))\right)\partial_\sigma^k$$

*Moreover*, *the diagram*

(17)

$$
\begin{array}{ccccc}
D_{A/R,\sigma^m} & \longrightarrow & D^{(\infty)}_{A/R,\sigma^m} & \hookrightarrow & \mathrm{End}_R(A) \\
\downarrow {\scriptstyle \iota_{\sigma,m}} & & \uparrow & & \| \\
D_{A/R,\sigma} & \longrightarrow & D^{(\infty)}_{A/R,\sigma} & \hookrightarrow & \mathrm{End}_R(A)
\end{array}
$$

*is commutative.*

*Proof.* We may assume (see the second remark following Theorem 6.3) that we are in the generic situation

$$ R = \mathbb{Q}(t)[s], \quad A = R[x], \quad q = t \quad \text{and} \quad h = s. $$

But then the left horizontal arrows in diagram (17) are bijective. In other words, we can identify the twisted Weyl $R$-algebras with the rings of twisted differential operators of infinite level and use Corollary 7.9 and Corollary 6.2. □

**Remarks.** (1) In the case $h = 0$, we have thanks to Corollary 7.10 the more concrete formula:

$$ \partial_{\sigma^m} \mapsto \sum_{k=1}^{m} \frac{q^{\frac{k(k-1)}{2}} (q-1)^{k-1}}{(k)_q} \binom{m-1}{k-1}_q x^{k-1} \partial_\sigma^k. $$

(2) Assume that $R$ is $q$-divisible. If $q$-char$(R) = p > 0$, then the hypothesis is satisfied if and only if $m < p$. Of course if $q$-char$(R) = 0$, then the hypothesis is satisfied for any $m \in \mathbb{N}$.

(3) Just to give an idea of the geometric intuition, the geometric counterpart of the twisted localization hypothesis should be the requirement to work on a Zariski open subset of "the quantum line".

**Proposition 8.6.** *Let* $m, n \in \mathbb{N} \setminus \{0\}$. *If* $(mn)_q!$ *is invertible, then*

$$ \iota_{\sigma,mn} = \iota_{\sigma^m,n} \circ \iota_{\sigma,m}. $$

*Proof.* After a base change, we may assume that all $q$-integers are invertible and identify twisted Weyl $R$-algebras with rings of twisted differential operators of infinite level. The assertion then becomes a triviality. □

**Proposition 8.7.** *If* $(m)_q!$ *is invertible in R, then the following diagram commutes*:

$$
\begin{array}{ccccc}
T & A[T]_{\sigma^m} & \longrightarrow & D_{A/R,\sigma^m}, & T \mapsto 1 + (\sigma^m(x) - x)\partial_{\sigma^m} \\
\downarrow & \downarrow & & \downarrow {\scriptstyle \iota_{\sigma,m}} & \\
T^m & A[T]_\sigma & \longrightarrow & D_{A/R,\sigma}, & T \mapsto 1 + (\sigma(x) - x)\partial_\sigma
\end{array}
$$

*Proof.* Again, we may assume that all $q$-integers are invertible in $R$ and use the second remark following Theorem 6.13 of [Le Stum and Quirós 2015b]. □

## 9. Formal confluence in positive quantum characteristic

In this section, we extend the formal confluence theorem to the case of positive quantum characteristic. In order to do that, it is necessary to use the $S$-twisted theory, where $S$ is a filtering (for division) set of positive integers. Thus, we assume here that $A$ is an $S$-twisted $R$-algebra in the sense of [Le Stum and Quirós 2015b] (it is endowed with a compatible system of $n$-th roots $\sigma_n$ of $\sigma$ for all $n \in S$).

We recall from [Le Stum and Quirós 2015a] that a *system of roots* of $q \in R$ is a compatible family $\underline{q} := \{q_n\}_{n \in S}$ of $n$-th roots of $q$. We call the system $\underline{q}$ *admissible* if for all $n \in S$,

$$(n)_{q^n} \in R^\times.$$

This is a natural condition in order to define the *$q$-rational number* $(r)_q$ for any $r \in \mathbb{N}(1/S)$. More precisely, if $r = m/n$ with $m \in \mathbb{N}$ and $n \in S$, then

$$(r)_q := \frac{(m)_{q_n}}{(n)_{q_n}}$$

only depends on $r$ and not on the choice of $m$ and $n$. It is convenient to introduce the following terminology:

**Definition 9.1.** Let $\{q_n\}_{n \in S}$ be a system of roots of $q$ in $R$. Then $R$ is said to be *$\underline{q}$-divisible* if $R$ is $q_n$-divisible for all $n \in S$.

When the system is admissible, there exists a nice equivalent definition:

**Lemma 9.2.** *Let $\{q_n\}_{n \in S}$ be an admissible system of roots of $q$. Then $R$ is $\underline{q}$-divisible if and only if for all $r \in \mathbb{N}(1/S)$,*

$$(r)_q \in R^\times \cup \{0\}.$$

*Proof.* If $r = m/n$ with $m \in \mathbb{N}$ and $n \in S$, we have $(r)_q = (m)_{q_n}/(n)_{q_n}$. It follows that $(r)_q \in R^\times$ (resp. $= 0$) if and only if $(m)_{q_n} \in R^\times$ (resp. $= 0$). □

**Remarks.** (1) If $1 - q \in R^\times$, then $\underline{q}$ is admissible. In particular, if $q \in K$ where $K$ is a subfield of $R$ and $q \neq 0$, then $\underline{q}$ is admissible.

(2) If $\underline{q} \subset K$ where $K$ is a subfield of $R$, then $R$ is $\underline{q}$-divisible.

**Lemma 9.3.** *For a system $\underline{q} := \{q_n\}_{n \in S}$ of roots of $q$, the following are equivalent*:

(1) *$\underline{q}$ is admissible and for all $r \in \mathbb{N}(1/S) \cap [0, 1]$, we have $(r)_q \in R^\times$.*

(2) *For all $n \in S$, we have $(n)_{q_n}! \in R^\times$.*

*Proof.* First of all, the second condition also implies that $q$ is admissible and we may therefore make this assumption. If $r = m/n$ with $m \in \mathbb{N}$ and $n \in S$, we know that $(r)_q \in R^\times$ if and only if $(m)_{q_n} \in R^\times$. Also, we have $r \leq 1$ if and only $m \leq n$. Thus, the second condition, which means that $(m)_{q_n} \in R^\times$ whenever $m \leq n$, is equivalent to the requirement that $(r)_q \in R^\times$ for $r \leq 1$. $\qquad\square$

**Definition 9.4.** Such a system of roots will be called *strongly admissible*.

**Lemma 9.5.** *If $q$ is an admissible* (*resp. a strongly admissible*) *system of roots, and we write for all $n \in S$, $p_n := q_n$ -char$(R)$, then we have for all $n \in S$,*

$$p_n \nmid n \text{ (resp. } p_n > n) \quad \text{or} \quad p_n = 0.$$

*When $R$ is $q$-divisible, the converse is true.*

*Proof.* By definition, $q_n$ is admissible (resp. strongly admissible) if and only if $(n)_{q_n} \in R^\times$ (resp. $(n)_{q_n}! \in R^\times$). By definition also, we have $p_n \neq 0$ and $p_n \mid n$ (resp. $p_n \leq n$) if and only if $(n)_{q_n} = 0$ (resp. $(n)_{q_n}! = 0$). We see that these two conditions are mutually exclusive in general and that they are exhaustive when $R$ is $q_n$-divisible. $\qquad\square$

There usually exist strongly admissible systems as the next result shows:

**Proposition 9.6.** *Assume that $R$ is $q$-divisible and that $q$ -char$(R) = p > 0$. Then if $q$ is a system of $p^n$-th root of $q$, it is strongly admissible.*

*Proof.* It follows from Proposition 1.16 of [Le Stum and Quirós 2015a] that for all $n \in S$, we have $q_{p^n}$ -char$(R) = p^{n+1} > p^n$ and we can apply Lemma 9.5. $\qquad\square$

**Definition 9.7.** An $x \in A$ is called an *S-twisted coordinate*, or *rooted twisted coordinate*, (resp. an *S-quantum coordinate*, or *rooted quantum coordinate*,) if $x$ is a twisted (resp. quantum) coordinate for all $(A, \sigma_n)$. We call it *strong* if $x - \sigma_n(x) \in A^\times$ for all $n \in S$.

Thus, by definition, $x$ is an $S$-quantum coordinate if and only if for all $n \in S$,

$$(18) \qquad \sigma_n(x) = q_n x + h_n \quad \text{with} \quad q_n, h_n \in R.$$

When this is the case, we might also say $q$-coordinate and call $A$ a *rooted quantum R-algebra*, an *S-quantum R-algebra* or a *$q$-R-algebra*. We call it *strong* when $x$ is strong.

**Lemma 9.8.** *Assume that $x$ is simultaneously a twisted coordinate and a rooted quantum coordinate on $A$ so that* (18) *holds. Then $x$ is a quantum coordinate on $A$ and if we write $\sigma(x) = qx + h$ with $q, h \in R$, then $q := \{q_n\}_{n \in S}$ is a system of roots of $q$. If this system is admissible, we have for all $n \in \mathbb{N}$,*

$$h_n = \left(\frac{1}{n}\right)_q h.$$

*Proof.* If we are given $m \in \mathbb{N}$ and $n \in S$, we know that $\sigma_n^m$ only depends on $r := m/n$ and so does $\sigma_n^m(x) = q_n^m x + (m)_{q_n} h_n$. It immediately follows that $\underline{q}$ is a system of roots. Moreover, the case $m = n$ implies that $\sigma(x) = qx + h$ with $q = q_n^n$ and $h = (n)_{q^n} h_n$ from which we derive the other assertions. $\square$

**Examples.** (1) If we are given a system $\underline{q}$ of roots in $R$, we can endow $A := R[x]$ or $A = R[x, x^{-1}]$ with, for all $n \in S$,

$$\sigma_n(x) = q_n x.$$

In the later case, $x$ is a strong $\underline{q}$-coordinate if and only if $1 - q \in R^\times$ (and then $\underline{q}$ is also admissible).

(2) More generally, if we are given an *admissible* system $\underline{q}$ of roots of $q \in R$ and some $h \in R$, we can endow $R[x]$ with, for all $n \in S$,

$$\sigma_n(x) = q_n x + \left(\frac{1}{n}\right)_q h.$$

**Definition 9.9.** Assume that $x$ is a $\underline{q}$-coordinate on $A$ with $\underline{q}$ strongly admissible. A *rooted twisted differential A-module* is an $A$-module $M$ endowed with a family of $\partial_{A,\sigma_n}$-derivations $\partial_{M,n}$ for all $n \in S$ such that whenever $n, n' \in S$ with $n \mid n'$, we have for all $s \in M$,

$$\partial_{M,n}(s) = \sum_{k=1}^{n'/n} \frac{1}{(k)_{q_{n'}}!} \left( \prod_{i=1}^{k-1} (\sigma_n(x) - \sigma_{n'}^i(x)) \right) \partial_{M,n'}^k(s).$$

They form a category that we will denote by $\partial_{A,\underline{\sigma}}\text{-Mod}^{\text{root}}$.

**Definition 9.10.** If $A$ is a $\underline{q}$-$R$-algebra with $\underline{q}$ strongly admissible, then the *rooted twisted Weyl algebra* of $A$ is

$$D_{A/R,\underline{\sigma}} := \varinjlim D_{A/R,\sigma_n},$$

with transition maps (from Proposition 8.5) for $n, n' \in S$, with $n \mid n'$ given by

$$D_{A/R,\sigma_n} \xrightarrow{\iota_{\sigma_{n'},n'/n}} D_{A/R,\sigma_{n'}}, \quad \partial_{\sigma_n} \longmapsto \sum_{k=1}^{n'/n} \frac{1}{(k)_{q_{n'}}!} \left( \prod_{i=1}^{k-1} (\sigma_n(x) - \sigma_{n'}^i(x)) \right) \partial_{\sigma_{n'}}^k.$$

**Proposition 9.11.** *Assume that $A$ is a $\underline{q}$-$R$-algebra with $\underline{q}$ strongly admissible. Then the rooted twisted differential A-modules form an abelian category with sufficiently many injective and projective objects. Actually, if $M$ is a $D_{A/R,\underline{\sigma}}$-module, then the maps*

$$M \xrightarrow{\partial_{M,n}} M, \quad s \longmapsto \partial_{M,n}s$$

*turn M into a rooted twisted differential A-module and we obtain an equivalence (an isomorphism) of categories*

$$D_{A/R,\underline{\sigma}}\text{-Mod} \simeq \partial_{A,\underline{\sigma}}\text{-Mod}^{\text{root}}.$$

*Proof.* This is a consequence of proposition 5.6 of [Le Stum and Quirós 2015b]. □

Recall from the same work that $A[T^{1/S}]_{\underline{\sigma}}$ denotes the noncommutative Puiseux polynomial ring with the commutation rule for all $m \in \mathbb{N}$, for all $n \in S$,

$$T^{m/n}x = \sigma_n^m(x)T^{m/n}.$$

One can also consider the notion of $\underline{\sigma}_A$-*module*: this is an $A$-module endowed with a compatible family of $\sigma_{A,n}$-linear endomorphisms $\sigma_{M,n}$. There exists an equivalence (an isomorphism) between the category of $A[T^{1/S}]_{\underline{\sigma}}$-modules and the category of $\underline{\sigma}_A$-modules.

**Proposition 9.12.** *Assume that $x$ is a $\underline{q}$-coordinate on $A$ with $\underline{q}$ strongly admissible. Then there exists a unique $A$-linear homomorphism of rings*

$$(19) \qquad A[T^{1/S}]_{\underline{\sigma}} \to D_{A/R,\underline{\sigma}}, \quad T^{1/n} \mapsto 1 + (\sigma_n(x) - x)\partial_{\sigma_n}$$

*inducing a functor*

$$(20) \qquad \partial_{A,\underline{\sigma}}\text{-Mod} \to \underline{\sigma}_A\text{-Mod}.$$

*If $x$ is a strong $\underline{q}$-coordinate, then the map (19) is an isomorphism and the functor (20) is an equivalence.*

*Proof.* This will follow from Theorem 6.13 of [Le Stum and Quirós 2015b] once we know that the various maps for the different $n$ are all compatible. More precisely, we have to check that the differential operators $1 + (\sigma_n(x) - x)\partial_{\sigma_n}$ form a system of roots in $D_{\underline{\sigma}}$. But this follows from Proposition 8.7. □

**Theorem 9.13** (formal quantum confluence 2). *Let $R$ be a $\mathbb{Q}$-algebra, $\underline{q}$ a strongly admissible system of roots in $R$ and $A$ a $\underline{q}$-$R$-algebra. Then there exists a canonical $A$-linear map*

$$D_{A/R,\underline{\sigma}} \to \widehat{D}_{A/R}.$$

*If $R$ is $\underline{q}$-divisible and $\underline{q}$ is infinite, then the map (16) has dense image.*

*Proof.* First of all, the map (16) is obtained by taking the direct limit of the maps (15) for all the $\sigma_n$. Now, we set $p_n := q_n\text{-char}(R)$ for all $n \in S$ and we apply the confluence Theorem 8.3 to $\sigma_n$. If $p_n = 0$, we are done. Otherwise, the theorem tells

us that the bottom map in the following commutative diagram is surjective:

$$
\begin{array}{ccc}
D_{\sigma} & \longrightarrow & \widehat{D} \\
\uparrow & & \downarrow \\
D_{\sigma_n} & \longrightarrow\!\!\!\!\longrightarrow & D/\partial^{p_n}
\end{array}
$$

And we proved in Lemma 9.5 that for all $n \in S$, we have $p_n > n$. Since $\underline{q}$ is infinite, we have $p_n \to \infty$ and we see that the image of the upper map is dense. $\qquad\square$

**Corollary 9.14.** *Assume moreover that $x$ is a strong $\underline{q}$-coordinate on $A$. Then the following $A$-linear map has dense image*:

$$
A[T^{1/S}]_{\underline{\sigma}} \to \widehat{D}_{A/R}, \qquad T^{m/n} \mapsto \sum_{k=0}^{\infty} \frac{1}{k!}(\sigma_n^m(x) - x)^k \partial^k
$$

*Proof.* The formula comes from Corollary 7.7. $\qquad\square$

**Example.** Theorem 9.13 (and its corollary) applies in particular when $q$ is a primitive $p$-th root of unity in some algebraically closed field $K$ of characteristic zero, $\underline{q}$ is a system of $p^n$-th roots of $q$ in $K$, $R$ is an algebra containing $K$, $A = R[x, x^{-1}]$ and $\sigma_n(x) = q_n x$ (we should actually write $\sigma_{p^n}$ and $q_{p^n}$ but we will try to make the notation easier to use).

As we already saw, in this situation, $D := D_{A/R}$ is the noncommutative ring $R[x, x^{-1}, \partial]$ with the commutation rule $\partial x = x\partial + 1$ and we may see $\widehat{D}$ (which is *not* a ring) as $R[x, x^{-1}][\![\partial]\!]$. We also want to understand the left-hand side and see that $D_n := D_{A/R, \sigma_n}$ is the noncommutative ring $R[x, x^{-1}, \partial_n]$ with the commutation rule $\partial_n x = q_n x \partial_n + 1$. The transition maps are given by the rather tricky formulas

$$
\partial_n \mapsto \sum_{k=1}^{p^{n'-n}} \frac{q_{n'}^{\frac{k(k-1)}{2}}(q_{n'} - 1)^{k-1}}{(k)_{q_{n'}}} \binom{p^{n'-n} - 1}{k - 1}_{q_{n'}} x^{k-1} \partial_{n'}^k,
$$

and the map of the theorem satisfies

$$
\partial_n \mapsto \sum_{k=1}^{\infty} \frac{1}{k!}(q_n - 1)^{k-1} x^{k-1} \partial^k.
$$

Also, if we set $q^{m/n} := q_n^m$, the map of the corollary is given by

$$
T^r \mapsto \sum_{k=1}^{\infty} \frac{1}{k!}(q^r - 1)^k x^k \partial^k.
$$

**Remark.** Alexei Belov-Kanel and Maxim Kontsevich [2007] proved that the Jacobian conjecture is stably equivalent to the Dixmier conjecture [1968]. According to

the latter, any endomorphism of a Weyl algebra over $\mathbb{C}$ is an automorphism. And this is still a conjecture even in dimension one. On the other hand, when $q$ is a primitive $p$-th root of unity, then the quantum Weyl algebra $D_\sigma$ does *not* satisfy the Dixmier conjecture but it is an Azumaya algebra. In particular, checking that an endomorphism is an automorphism can be done on the center. As explained by Backelin [2011], it is appealing to attack the Dixmier conjecture through quantum Weyl algebras and one can hope that Theorem 9.13 might provide a tool for this quest.

## References

[Adams 1928/29] C. R. Adams, "On the linear ordinary $q$-difference equation", *Ann. of Math.* (2) **30**:1-4 (1928/29), 195–205. MR Zbl

[André 2001] Y. André, "Différentielles non commutatives et théorie de Galois différentielle ou aux différences", *Ann. Sci. École Norm. Sup.* (4) **34**:5 (2001), 685–739. MR Zbl

[André and Di Vizio 2004] Y. André and L. Di Vizio, "$q$-difference equations and $p$-adic local monodromy", pp. 55–111 in *Analyse complexe, systèmes dynamiques, sommabilité des séries divergentes et théories galoisiennes, I*, edited by M. Loday-Richaud, Astérisque **296**, Société Mathématique de France, Paris, 2004. MR Zbl

[Backelin 2011] E. Backelin, "Endomorphisms of quantized Weyl algebras", *Lett. Math. Phys.* **97**:3 (2011), 317–338. MR Zbl

[Belov-Kanel and Kontsevich 2007] A. Belov-Kanel and M. Kontsevich, "The Jacobian conjecture is stably equivalent to the Dixmier conjecture", *Mosc. Math. J.* **7**:2 (2007), 209–218. MR Zbl

[Birkhoff 1911] G. D. Birkhoff, "General theory of linear difference equations", *Trans. Amer. Math. Soc.* **12**:2 (1911), 243–284. MR Zbl

[Bourbaki 1970] N. Bourbaki, *Algèbre, chapitres 1 à 3*, Hermann, Paris, 1970. Translated as *Algebra, chapters 1-3*, Addison-Wesley, 1973. MR Zbl

[Bourbaki 2012] N. Bourbaki, *Algèbre, chapitre 8: Modules et anneaux semisimples*, 2nd revised ed., Springer, Berlin, 2012. MR Zbl

[Carmichael 1911] R. D. Carmichael, "Linear difference equations and their analytic solutions", *Trans. Amer. Math. Soc.* **12**:1 (1911), 99–134. MR Zbl

[Carmichael 1912] R. D. Carmichael, "The general theory of linear $q$-difference equations", *Amer. J. Math.* **34**:2 (1912), 147–168. MR Zbl

[Cohn 1965] R. M. Cohn, *Difference algebra*, Interscience, New York, 1965. MR Zbl

[Di Vizio 2004] L. Di Vizio, "Introduction to $p$-adic $q$-difference equations (weak Frobenius structure and transfer theorems)", pp. 615–675 in *Geometric aspects of Dwork theory, I, II*, edited by A. Adolphson et al., de Gruyter, Berlin, 2004. MR Zbl arXiv

[Di Vizio et al. 2003] L. Di Vizio, J.-P. Ramis, J. Sauloy, and C. Zhang, "Équations aux $q$-différences", *Gaz. Math.* 96 (2003), 20–49. MR Zbl

[Dixmier 1968] J. Dixmier, "Sur les algèbres de Weyl", *Bull. Soc. Math. France* **96** (1968), 209–242. MR Zbl

[Duval and Roques 2008] A. Duval and J. Roques, "Familles fuchsiennes d'équations aux ($q$-) différences et confluence", *Bull. Soc. Math. France* **136**:1 (2008), 67–96. MR Zbl

[Gros and Le Stum 2014] M. Gros and B. Le Stum, "Une neutralisation explicite de l'algèbre de Weyl quantique complétée", *Comm. Algebra* **42**:5 (2014), 2163–2170. MR Zbl

[Grothendieck 1967] A. Grothendieck, "Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas IV", *Inst. Hautes Études Sci. Publ. Math.* **32** (1967), 5–333. MR Zbl

[Guichard 1887] C. Guichard, "Sur la résolution de l'équation aux différences fines $G(x+1)-G(x)=H(x)$", *Ann. Sci. École Norm. Sup.* (3) **4** (1887), 361–380. MR Zbl

[Hardouin 2010] C. Hardouin, "Iterative $q$-difference Galois theory", *J. Reine Angew. Math.* **644** (2010), 101–144. MR Zbl

[Iyer and McCune 2002] U. N. Iyer and T. C. McCune, "Quantum differential operators on $\mathbb{K}[x]$", *Internat. J. Math.* **13**:4 (2002), 395–413. MR Zbl

[Karoubi and Suárez-Álvarez 2003] M. Karoubi and M. Suárez-Álvarez, "Twisted Kähler differential forms", *J. Pure Appl. Algebra* **181**:2-3 (2003), 279–289. MR Zbl

[Le Stum and Quirós 2015a] B. Le Stum and A. Quirós, "On quantum integers and rationals", pp. 107–130 in *Trends in number theory* (Seville, 2013), edited by F. Chamizo et al., Contemp. Math. **649**, American Mathematical Society, Providence, RI, 2015. MR Zbl

[Le Stum and Quirós 2015b] B. Le Stum and A. Quirós, "Twisted calculus", preprint, 2015. arXiv

[Lunts and Rosenberg 1997] V. A. Lunts and A. L. Rosenberg, "Differential operators on noncommutative rings", *Selecta Math.* (*N.S.*) **3**:3 (1997), 335–359. MR Zbl

[Mebkhout and Narváez Macarro 1998] Z. Mebkhout and L. Narváez Macarro, "Le théorème de continuité de la division dans les anneaux d'opérateurs différentiels", *J. Reine Angew. Math.* **503** (1998), 193–236. MR Zbl

[Pulita 2008] A. Pulita, "$p$-adic confluence of $q$-difference equations", *Compos. Math.* **144**:4 (2008), 867–919. MR Zbl

[Pulita 2017] A. Pulita, "Infinitesimal deformation of $p$-adic differential equations on Berkovich curves", *Math. Ann.* **368**:1-2 (2017), 111–164. MR Zbl

[Sauloy 2000] J. Sauloy, "Systèmes aux $q$-différences singuliers réguliers: classification, matrice de connexion et monodromie", *Ann. Inst. Fourier* (*Grenoble*) **50**:4 (2000), 1021–1071. MR Zbl

[Sauloy 2003] J. Sauloy, "Galois theory of Fuchsian $q$-difference equations", *Ann. Sci. École Norm. Sup.* (4) **36**:6 (2003), 925–968. MR Zbl

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