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JIACHENG SUN

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It is well known that Hawking mass is nonnegative for a stable sphere of constant mean curvature (CMC) in a three-manifold of nonnegative scalar curvature. R. Bartnik proposed the rigidity problem of the Hawking mass of stable CMC spheres. We show partial rigidity results of Hawking mass for stable CMC spheres in asymptotic flat (AF) manifolds with nonnegative scalar curvature. If the Hawking mass of a nearly round stable CMC surface vanishes, then the surface must be the standard sphere in $\mathbb{R}^3$ and the interior of the surface is flat. Similar results also hold for asymptotic hyperbolic manifolds. A complete AF manifold having small or large isoperimetric surface with zero Hawking mass must be flat. We use the mean field equation and monotonicity of Hawking mass as well as rigidity results of Y. Shi in our proof.

1. Introduction

One of the most important tasks in general relativity is to understand the mass of spacetime. The first attempt on this topic is the positive mass theorem, which says that the mass of an asymptotic flat manifold is nonnegative if the scalar curvature is nonnegative, and the mass vanishes if and only if the manifold is isometric to standard Euclidean space. Another important attempt is the Penrose inequality, which tells us that the mass is no less than $\sqrt{A/16\pi}$ when there is a horizon, where A is the area of the outmost minimal surface, and the equality holds if and only if the manifold is isometric to Schwarzschild space. From the Penrose inequality we see the impact of boundary behavior is also remarkable. This motivates us to study quasilocal mass for a compact manifold with boundary.

Brown–York mass is a well defined quasilocal mass for a domain with convex boundary, which characterizes the deviation of mean curvature compared with a Euclidean metric, whose positivity and rigidity is proved by [Shi and Tam 2002]. Another important quasilocal mass is Hawking mass, which played a key role in proving the Penrose inequality in [Bray 1997] and [Huisken and Ilmanen 2001].

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Because the Willmore functional of a surface can be arbitrarily large, we cannot expect positivity for an arbitrary surface. But for a stable CMC sphere in a nonnegative scalar curvature manifold, the Hawking mass is nonnegative [Christodoulou and Yau 1988].

Bartnik [2002, p. 235] proposed the rigidity problem of Hawking mass, i.e., what can we say about the ambient manifold when the Hawking mass vanishes for some surface. This paper is devoted to a partial result for the rigidity problem if the surface is nearly round. We study the eigenvalue and eigenfunctions of the Jacobi operator for a stable CMC surface with zero Hawking mass, then transfer the rigidity problem to a mean field type equation with respect to the second eigenvalue 6 of the standard S^2 under some restriction. If the equation has only the zero solution, then the rigidity of Hawking mass holds. We get the local uniqueness by studying the spherical harmonics on S^2 carefully and also iteration methods. If the solution is small in some sense, we can get the power decay of both the kernel part of $\Delta+6$ and also the orthogonal part. But we believe that the equation has only the zero solution with the integral restriction.

The main term in Hawking mass is the Willmore functional. In $\mathbb{R}^3$ the Willmore functional is constant 4π if and only if the surface is round sphere. So we can detect the curvature of ambient space by the Willmore functional. For this reason, we expect that manifolds with zero Hawking mass surface may have some flatness properties.

Theorem 1. *Let (M, g) be a complete Riemannian three-manifold with scalar curvature $R(g) \geq 0$ and $\Omega \subset M$ be a domain with boundary $\Sigma = \partial\Omega$. If Σ is a nearly round stable CMC sphere in M with $m_H(\Sigma) = 0$, then Ω isometric to a Euclidean ball in $\mathbb{R}^3$. In particular, Σ is isometric to the standard S^2 in $\mathbb{R}^3$. In this paper, nearly round is in the sense that Gauss curvature satisfies*

$$\left| \frac{|\Sigma|}{4\pi} K_\Sigma - 1 \right|_{C^0} < \epsilon_0 \tag{1-1}$$

for some universal constant $\epsilon_0 \ll 1$.

The hyperbolic case of the above rigidity is the following:

Theorem 2. *Let (M, g) be a complete Riemannian three-manifold with scalar curvature $R(g) \geq -6$ and $\Omega \subset M$ be a domain with boundary $\Sigma = \partial\Omega$. If Σ is a nearly round stable CMC sphere in M with $m_H(\Sigma) = 0$, then Ω isometric to a hyperbolic ball in $\mathbb{H}^3$.*

By the examples of A. Carlotto and R. Schoen [2016] there are manifolds with nonnegative scalar curvature which are flat in a half space of $\mathbb{R}^3$, so we can only expect flatness inside the surface with zero Hawking mass for stable CMC surfaces.

But we can get global flatness for isoperimetric surfaces of sphere type:

Theorem 3. *Let (M, g) be a complete AF three-manifold with scalar curvature $R(g) \geq 0$. If there exists a nearly round isoperimetric sphere Σ with $m_H(\Sigma) = 0$, then (M, g) is isometric to $(\mathbb{R}^3, \delta)$.*

This theorem also has a hyperbolic version:

Theorem 4. *Let (M, g) be a complete AH three-manifold with scalar curvature $R(g) \geq -6$. If there exists a nearly round isoperimetric sphere Σ with $m_H(\Sigma) = 0$, then (M, g) is isometric to $(\mathbb{H}^3, g_{\mathbb{H}})$.*

We already know from [Chodosh et al. 2016] that large surfaces of the canonical stable CMC foliation in [Huisken and Yau 1996; Qing and Tian 2007] are isoperimetric and close to the coordinate spheres. So we can get the rigidity result for large isoperimetric surfaces. For rigidity of small isoperimetric surfaces, we use the monotonicity of Hawking mass and also a rigidity result of Y. Shi.

Theorem 5. *Let (M, g) be a complete AF three-manifold with scalar curvature $R(g) \geq 0$ which has no compact minimal surface. If there is an small enough isoperimetric surface Σ with $m_H^+(\Sigma) = 0$ (see definition in Section 5.3), then (M, g) is isometric to $\mathbb{R}^3$.*

Structure of this paper. In Section 2, we give the basic definitions. In Section 3, we prove the rigidity of Hawking mass for nearly round stable CMC spheres. We transform the rigidity problem to a mean field-type equation, and prove the local uniqueness of the zero solution. By doing so, we get that a surface with zero Hawking mass must be the standard S^2 and then use the rigidity of [Shi and Tam 2002; 2007] to finish the proof. In Section 4, we prove the global properties of manifolds with nearly round isoperimetric surfaces having zero Hawking mass. This directly implies the rigidity for large isoperimetric surfaces in the canonical stable CMC foliation by Huisken and Yau [1996] and Qing and Tian [2007]. In Section 5, we prove the rigidity for small isoperimetric surfaces by using the monotonicity of Hawking mass. This relies on the fact that the topology of a small isoperimetric surface must be a sphere. In Appendix A.1 we give the spherical harmonics and computations for the square of second order spherical harmonics. In Appendix A.2 we sketch a proof of the existence of isoperimetric surfaces for all volumes in AF three-manifolds. In Appendix A.3 we sketch the proof of continuity of isoperimetric profile for AF manifolds which is important to prove the right continuity of I'_+.

2. Preliminaries

We give some basic notations to present our result. Let $\Sigma \subset (M, g)$ be a surface with unit normal vector field n, second fundamental form A and mean curvature H.

Definition. The Willmore functional of Σ is defined by:

$$W(\Sigma)=\frac{1}{4}\int_\Sigma H^2, \tag{2-1}$$

when $R(g)\geq 0$ and

$$W(\Sigma)=\frac{1}{4}\int_\Sigma (H^2-4). \tag{2-2}$$

when $R(g)\geq -6$.

The Willmore functional appears in various areas, such as bending energy of elastic membranes. It appears naturally in general relativity in the form of the Hawking mass of a surface:

Definition. The Hawking mass of Σ is defined by

$$m_H(\Sigma)=\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\left(16\pi-\int_\Sigma H^2\right), \tag{2-3}$$

when $R(g)\geq 0$ and

$$m_H(\Sigma)=\frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\left(16\pi-\int_\Sigma (H^2-4)\right). \tag{2-4}$$

when $R(g)\geq -6$.

Definition. If H is constant along Σ, we say Σ is a CMC surface; the Jacobi operator of a CMC surface Σ is the second variation of area:

$$L_\Sigma=-\Delta_\Sigma-(|A|^2+\mathrm{Ric}(n,n)). \tag{2-5}$$

A CMC surface Σ is *stable* if the first eigenvalue of L_Σ on mean-zero functions is nonnegative:

$$\Lambda_1(L_\Sigma)=\inf\left\{\int_\Sigma fL_\Sigma f:\int_\Sigma f=0,\ \int_\Sigma f^2=1\right\}\geq 0, \tag{2-6}$$

i.e., it satisfies the following stability condition:

$$\int_\Sigma (|A|^2+\mathrm{Ric}(n,n))f^2\leq\int_\Sigma|\nabla f|^2 \tag{2-7}$$

for all $f\in C_c^\infty(\Sigma)$ and $\int_\Sigma f=0$.

Remark. The above definition of eigenvalue in mean-zero functions is different from the eigenvalue defined in the ordinary way by min-max construction:

$$\lambda_1(L_\Sigma)=\inf\left\{\int_\Sigma fL_\Sigma f:\int_\Sigma fu_0=0,\ \int_\Sigma f^2=1\right\} \tag{2-8}$$

where u_0 is the zeroth eigenfunction of L_Σ. By definition we have

$$\Lambda_1(L_\Sigma) \le \lambda_1(L_\Sigma). \tag{2-9}$$

We also want to study isoperimetric surfaces in AF (resp. AH) three-manifolds. We will always use the bracket to denote the asymptotic hyperbolic case after related asymptotic flat situations.

Definition. A complete connected three-manifold (M, g) is called AF (resp. AH), if there exists a constant $C > 0$ and a compact set K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_R(0)$ for some $R > 0$, and in standard coordinates the metric g has the following properties:

$$g = \delta + h \quad (\text{resp. } g = g_{\mathbb{H}} + h) \tag{2-10}$$

and

$$|h_{ij}| + r|\partial h_{ij}| + r^2|\partial^2 h_{ij}| \le C r^{-\tau} \tag{2-11}$$

$\tau \in \left(\frac{1}{2}, 1\right]$ (resp. $\tau = 3$), where r and ∂ denote the Euclidean distance and standard derivative operator on $\mathbb{R}^3$ respectively. The region $M \setminus K$ is called the end of M. The standard hyperbolic space $(\mathbb{H}^3, g_{\mathbb{H}})$ is

$$g_{\mathbb{H}} = \frac{1}{1+r^2}\, dr^2 + r^2 g_{S^2}. \tag{2-12}$$

We also need the following definition of isoperimetric surface:

Definition. Given a complete Riemannian 3-manifold (M, g), its isoperimetric profile with volume V is defined as

$$I(V) = \inf\left\{ \mathcal{H}^2(\partial^*\Omega) : \begin{array}{c} \Omega \subset M \text{ is a Borel set with finite perimeter} \\ \text{and } \mathcal{H}^3_g(\Omega) = V \end{array} \right\}, \tag{2-13}$$

where $\mathcal{H}^2$ is a 2-dimensional Hausdorff measure for the reduced boundary of Ω denoted by $\partial^*\Omega$. A Borel set $\Omega \subset M$ of finite perimeter such that $\mathcal{H}^3_g(\Omega) = V$ and $I(V) = \mathcal{H}^2(\partial^*\Omega)$ is called an isoperimetric region of (M, g) of volume V. The surface $\partial\Omega$ is called an isoperimetric surface.

3. Rigidity of Hawking mass for nearly round stable CMC surfaces

It was shown in [Christodoulou and Yau 1988] that the Hawking mass is nonnegative for a stable CMC sphere. It is proved by using a Hersch-type test function in the stability condition and the nonnegativity of scalar curvature. Since we need to study the equality case, we prove it here for completeness.

Lemma 6 [Christodoulou and Yau 1988]. *Let (M, g) be a Riemannian three-manifold with scalar curvature $R(g) \ge 0$, if Σ is a stable CMC sphere in M, then $m_H(\Sigma) \ge 0$.*

Proof. By [Li and Yau 1982] there exists a conformal $\varphi : \Sigma \to S^2 \subseteq \mathbb{R}^3$ with $\int_\Sigma \varphi = 0$. We can plug these test functions in stability condition, and using that

$$\int_\Sigma |\nabla \varphi_i|^2 \geq \int_\Sigma (|A|^2 + \mathrm{Ric}(n,n))\varphi_i^2 \tag{3-1}$$

for a surface conformal to $S^2 \subseteq \mathbb{R}^3$,

$$\int_\Sigma |\nabla \varphi_i|^2 \, d\mu_\Sigma = \int_{S^2} |\nabla x_i|^2 \, d\mu_{S^2} = -\int_{S^2} x_i \Delta x_i \, d\mu_{S^2} = 2\int_{S^2} x_i^2 \, d\mu_{S^2} = \tfrac{8}{3}\pi. \tag{3-2}$$

Thus we can get

$$8\pi \geq \int_\Sigma |A|^2 + \mathrm{Ric}(n,n). \tag{3-3}$$

By Gauss's equation

$$K_\Sigma = \frac{R}{2} - \mathrm{Ric}(n,n) + \frac{1}{2}(H^2 - |A|^2). \tag{3-4}$$

So we have

$$|A|^2 + \mathrm{Ric}(n,n) = \frac{R}{2} - K_\Sigma + \frac{1}{2}(H^2 + |A|^2) = \frac{1}{2}(R + |A^0|^2) + \frac{3}{4}H^2 - K_\Sigma, \tag{3-5}$$

where we have used that $|A|^2 = |A^0|^2 + \frac{1}{2}H^2$. We get

$$8\pi \geq \frac{1}{2}\int_\Sigma (R + |A^0|^2) + \frac{3}{4}\int_\Sigma H^2 - \int_\Sigma K_\Sigma, \tag{3-6}$$

so we obtain

$$16\pi - \int_\Sigma H^2 \geq \frac{2}{3}\int_\Sigma (R + |A^0|^2) \geq 0 \tag{3-7}$$

□

We can get an analogous result for the hyperbolic case; see also [Chodosh 2016]:

Lemma 7. *Let (M, g) be a Riemannian three-manifold with scalar curvature $R(g) \geq -6$. If Σ is a stable CMC sphere in M, then $m_H(\Sigma) \geq 0$.*

Now we start to study stable CMC surfaces with zero Hawking mass. First we can get a spectral characterization of them. We need the following lemma in [El Soufi and Ilias 1992], which gives a optimal estimate of the second eigenvalue of the Schrödinger operator. It also gives part of the rigidity of the second eigenvalue which is the case for a Jacobi operator on a stable CMC sphere.

Lemma 8 [El Soufi and Ilias 1992]. *For any continuous function q on surface Σ,*

$$\lambda_1(-\Delta_\Sigma + q)|\Sigma| \leq 2A_c(\Sigma) + \int_\Sigma q. \tag{3-8}$$

The equality holds if and only if Σ admits a conformal map into the standard S^2 whose components are the first eigenfunctions. If Σ is of genus zero, then the equality implies that Σ is conformal to the standard S^2 in $\mathbb{R}^3$ and q is given by the energy density of a Möbius transform, where λ_1 is the first eigenvalue of $-\Delta_\Sigma + q$ in the sense of (2-8), $A_c(\Sigma)$ is the conformal volume in [Li and Yau 1982] and for a sphere, $A_c(\Sigma) = 4\pi$.

By the above lemma we have the following characterization of zero-Hawking mass stable CMC spheres.

Proposition 9. *Let (M, g) be a complete Riemannian three-manifold with scalar curvature $R(g) \geq 0$ (resp. $R(g) \geq -6$). If Σ is a stable CMC sphere with $m_H(\Sigma) = 0$ and area $|\Sigma| = 4\pi$, then the second eigenvalue $\lambda_1(-\Delta_\Sigma + K_\Sigma) = 3$, with three eigenfunctions $\varphi_1, \varphi_2, \varphi_3$, $\int_\Sigma \varphi_i = 0$, and $\sum_{i=1}^3 {\varphi_i}^2 = 1$. In particular, $|\nabla\varphi|^2 = 3 - K_\Sigma$, which is independent of eigenfunctions.*

Proof. From the above proof of Lemma 6 we can see if $m_H(\Sigma) = 0$ on Σ, we have $\int_\Sigma H^2 = 16\pi$ (resp. $\int_\Sigma (H^2 - 4) = 16\pi$), $R = 0$ (resp. $R = -6$), $A^0 = 0$ on Σ. The area $|\Sigma| = 4\pi$, then $H = 2$ (resp. $H = 2\sqrt{2}$), the Jacobi operator becomes

$$L_\Sigma = -\Delta_\Sigma + K_\Sigma - 3. \tag{3-9}$$

By the stability of Σ and Lemma 8, we have

$$0 \leq 4\pi\,\Lambda_1(L_\Sigma) \leq 4\pi\lambda_1(L_\Sigma) \leq 8\pi + \int_\Sigma (K_\Sigma - 3) = 0, \tag{3-10}$$

so all the equalities hold, in particular

$$\lambda_1(-\Delta_\Sigma + K_\Sigma) = 3, \tag{3-11}$$

with three eigenfunctions $\varphi_1, \varphi_2, \varphi_3$, $\int_\Sigma \varphi_i = 0$, and $\sum_{i=1}^3 {\varphi_i}^2 = 1$, so

$$-\Delta_\Sigma\varphi + K_\Sigma\varphi - 3\varphi = 0. \tag{3-12}$$

By $|\varphi|^2 = \sum_{i=1}^3 {\varphi_i}^2 = 1$, we have

$$0 = \Delta_\Sigma|\varphi|^2 = 2\varphi\Delta_\Sigma\varphi + 2|\nabla.\varphi|^2 \tag{3-13}$$

Taking inner product of φ with (3-12), we get

$$|\nabla\varphi|^2 = 3 - K_\Sigma. \tag{3-14}$$

□

Remark. We see from the above lemma that the first eigenvalue and eigenfunctions of the Schrödinger operator $-\Delta_\Sigma + K_\Sigma$ equal those of the standard S^2. We expect that the metric is isometric to the standard metric on S^2.

In the following, we will always use Σ to denote a stable CMC surface with zero Hawking mass. Let $\varphi : \Sigma \to S^2 \subseteq \mathbb{R}^3$ be the conformal map in Proposition 9 with $\int_\Sigma \varphi = 0$. Denote the metric on Σ by $g = e^u g_0$, with g_0 the standard metric on S^2. By definition of the conformal map φ,

$$e^{-u} = \tfrac{1}{2}|\nabla\varphi|^2. \tag{3-15}$$

The standard formula for Gauss curvature under a conformal change of metric gives

$$K_\Sigma = e^{-u}\bigl(1 - \tfrac{1}{2}\Delta_{g_0} u\bigr). \tag{3-16}$$

So (3-14) gives

$$\Delta_{g_0} u = 6 - 6e^u. \tag{3-17}$$

Also the volume-preserving variation implies

$$\int_{S^2} x_i e^u = 0. \tag{3-18}$$

So for this stable CMC surface with zero Hawking mass Σ,

$$K_\Sigma - 1 = e^{-u}(1 - 3 + 3e^u) - 1 = 2(1 - e^{-u}). \tag{3-19}$$

This means that if u is C^0 close to 0, then K_Σ is C^0 close to 1, which implies Σ is nearly round. If we can prove (3-17), (3-18) admit only the zero solution, then the stable CMC surface with vanishing Hawking mass is isometric to the standard S^2.

Equations of the same type as (3-17) have been studied in various aspects, such as prescribed Gaussian curvature [Kazdan and Warner 1974], the mean field model and the Chern–Simons–Higgs model. This kind of equation may have bifurcation when approaching the eigenvalues of S^2, so it may lose compactness. Ding, Jost, Li and Wang [Ding et al. 1997; 1998] have studied the equation at the first eigenvalue. Li [1999] has initiated the study of the existence of solutions by computing the Leray–Schauder topological degree. Lin [2000] computed the degree on S^2 and surface of any genus [Chen and Lin 2003], but there is little work on the uniqueness of this kind of equation at second eigenvalue of S^2. In fact, because the bifurcation occurs after the first eigenvalue, it is hard to guarantee the uniqueness globally, but we can get local uniqueness of the constant solution for (3-17). That's why we put the nearly round condition in our results. We use the Lyapunov–Schmidt decomposition as in [Neves and Tian 2009] to estimate the kernel of $\Delta_{g_0} + 6$ and the orthogonal part separately.

Lemma 10. *Let u satisfy*

$$\Delta_{g_0} u = 6(1 - e^u) \tag{3-20}$$

on standard S^2. There exists a universal constant $\delta_0 > 0$ such that if $\sup|u| < \delta_0$, *then $u \equiv 0$.*

Proof. In the following, the constant C is universal, and may differ from line to line. Denote $E_2 = \ker\{\Delta_{g_0} + 6\}$, which is the second eigenspace of $-\Delta_{g_0}$ on the standard S^2. It is well known that $E_2 = \operatorname{span}\{Y_{2,-2}, Y_{2,-1}, Y_{2,0}, Y_{2,1}, Y_{2,2}\}$ (see appendix below). Let P_2 be the projection operator on E_2. Consider the decomposition $u = u_1 + u_2$, where $u_1 \in E_2^{\perp}$, and $u_2 \in E_2$. Then

$$\Delta_{g_0} u_1 + 6u_1 = 6(1 + u - e^u), \tag{3-21}$$

$$\Delta_{g_0} u_2 + 6u_2 = 0. \tag{3-22}$$

As $(\Delta_{g_0} + 6)^{-1}$ is bounded from L^2 to $W^{2,2}$ on $E_2^{\perp}$, we have

$$|u_1|_{W^{2,2}} \le C|1 + u - e^u|_{L^2}. \tag{3-23}$$

We can assume

$$\sup |u| \le \delta < 1. \tag{3-24}$$

Then from (3-23) and the Sobolev embedding, we have

$$|u_1|_{L^\infty} \le C|u^2|_{L^2} \le C\delta^2. \tag{3-25}$$

Also from equation (3-17), we know

$$|\Delta_{g_0} u + 6u + 3u^2|_{L^2} = 6|1 + u + \tfrac{1}{2}u^2 - e^u|_{L^2} \le C|u^3|_{L^2} \le C\delta^3. \tag{3-26}$$

By (3-25), we can get

$$|u_1^2|_{L^2} \le C\delta^4.$$

By the decomposition of $u_2 = u - u_1$ we have

$$|\Delta_{g_0} u + 6u + 3u_2^2|_{L^2} \le 2|u^3|_{L^2} + 6|u_1 u|_{L^2} + 3|u_1^2|_{L^2} \le C\delta^3. \tag{3-27}$$

In order to get the estimate of u_2, we project the above equation to E_2. Then

$$|P_2 u_2^2|_{L^2} \le C\delta^3. \tag{3-28}$$

By Lemma 11 below and (3-28) we have

$$|u_2|_{L^\infty} \le C|u_2|_{L^2} \le C\delta^{3/2} \tag{3-29}$$

Combining (3-25) and (3-29), we improve the initial assumption (3-24):

$$\sup |u| \le C|u|_{L^2} < C\delta^{3/2}. \tag{3-30}$$

Taking $\delta_0 = \frac{1}{2}C^{-2}$ and iterate the procedure, we get

$$\sup |u| \le C_0|u|_{L^2} < C^{-2}(C^2\delta_0)^{(3/2)^k} = C^{-2}\left(\tfrac{1}{2}\right)^{(3/2)^k}, \tag{3-31}$$

and let $k \to \infty$, we get the desired result. □

Lemma 11. *For all $u_2 \in E_2$,*

(3-32) $$|P_2 u_2^2|_{L^2} = \tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}|u_2|_{L^2}^2.$$

Proof. Let

(3-33) $$u_2 = \sum_{i=-2}^{2} \lambda_i Y_{2,i},$$

where $Y_{2,i}$ are the second-order spherical harmonics (see Appendix A.1). By computations and projecting u_2^2 to E_2, we have

(3-34) $$\begin{aligned} P_2 u_2^2 = {} & \tfrac{1}{14}\sqrt{\tfrac{5}{\pi}}[2(\lambda_0^2 - \lambda_{-2}^2 - \lambda_2^2) + \lambda_1^2 + \lambda_{-1}^2]Y_{2,0} \\ & + \tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}\big(\sqrt{3}\lambda_{-1}\lambda_1 - 2\lambda_{-2}\lambda_0\big)Y_{2,-2} \\ & + \tfrac{1}{14}\sqrt{\tfrac{5}{\pi}}[\sqrt{3}(\lambda_1^2 - \lambda_{-1}^2) - 4\lambda_0\lambda_2]Y_{2,2} \\ & + \tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}[\lambda_{-1}\lambda_0 + \sqrt{3}(\lambda_{-2}\lambda_1 - \lambda_{-1}\lambda_2)]Y_{2,-1} \\ & + \tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}[\lambda_0\lambda_1 + \sqrt{3}(\lambda_{-2}\lambda_{-1} + \lambda_1\lambda_2)]Y_{2,1}. \end{aligned}$$

Thus

$$\begin{aligned} |P_2 u_2^2|_{L^2}^2 = {} & \big(\tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}\big)^2\big\{\tfrac{1}{4}(2\lambda_0^2 - 2\lambda_{-2}^2 - 2\lambda_2^2 + \lambda_{-1}^2 + \lambda_1^2)^2 \\ & + (\sqrt{3}\lambda_{-1}\lambda_1 - 2\lambda_{-2}\lambda_0)^2 + \tfrac{1}{4}[\sqrt{3}(\lambda_1^2 - \lambda_{-1}^2) - 4\lambda_0\lambda_2]^2 \\ & + [\lambda_{-1}\lambda_0 + \sqrt{3}(\lambda_{-2}\lambda_1 - \lambda_{-1}\lambda_2)]^2 \\ & + [\lambda_0\lambda_1 + \sqrt{3}(\lambda_{-2}\lambda_{-1} + \lambda_1\lambda_2)]^2\big\} \end{aligned}$$

(3-35) $$= \big(\tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}\big)^2\bigg(\sum_{i=-2}^{2}\lambda_i^2\bigg)^2 = \big(\tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}\big)^2|u_2|_{L^2}^2. \qquad \square$$

The following rigidity result is a kind of positive mass theorem in the compact case (see [Miao 2002], [Shi and Tam 2002], and [Hang and Wang 2006]):

Lemma 12. *Let (M, g) be a compact, orientable Riemannian 3-manifold with scalar curvature $R(g) \geq 0$ and ∂M isometric to a round S^2 with mean curvature $H = 2$. Then (M, g) is isometric to the unit ball in $(\mathbb{R}^3, \delta)$.*

To prove Theorem 2 we need a rigidity result for the hyperbolic case of the sphere; see Theorem 3.8 in [Shi and Tam 2007].

Lemma 13. *Let (M, g) be a compact orientable Riemannian 3-manifold with scalar curvature $R(g) \geq -6$ and ∂M isometric to a round S^2 with mean curvature $H = 2\sqrt{2}$. Then (M, g) is isometric to the unit ball in hyperbolic space $\mathbb{H}^3$.*

After Lemmas 10, 12 and 13, now we are in the position to prove Theorems 1 and 2.

Proof of Theorems 1 and 2. If $m_H(\Sigma) = 0$ on a nearly round stable CMC surface Σ, without loss of generality, assume $|\Sigma| = 4\pi$, and then $H = 2$ (resp. $H = 2\sqrt{2}$)

$$L_\Sigma = -\Delta_\Sigma + K - 3.$$

By Lemma 10 we get the nearly round stable CMC surface Σ is the standard S^2 in $\mathbb{R}^3$. Then by Lemma 12 (resp. Lemma 13), we conclude that Ω isometric to a unit ball in $\mathbb{R}^3(\mathbb{H}^3_{-1})$. □

Theorem 1 (resp. Theorem 2) and Lemma 6 (resp. Lemma 7) can help us to understand the Willmore functional in manifolds with scalar curvature $R(g) \geq 0$ (resp. $R(g) \geq -6$).

Corollary 14. *Let (M, g) be a complete Riemannian three-manifold with scalar curvature $R(g) \geq 0$ (resp. $R(g) \geq -6$), $\Sigma = \partial\Omega$ a stable CMC sphere. Then $W(\Sigma) \leq 4\pi$. If Σ is nearly round, then equality holds if and only if Σ is the standard S^2 and Ω isometric to unit ball in $\mathbb{R}^3$ (resp. $\mathbb{H}^3$).*

4. Rigidity of Hawking mass for nearly round isoperimetric surfaces

Theorem 1 can be used to prove rigidity of isoperimetric surfaces in AF manifolds. By the manifold constructed by A. Carlotto and R. Schoen [2016]; see also [Chodosh et al. 2016]:

Example. There is an asymptotically flat Riemannian metric g on $\mathbb{R}^3$ with nonnegative scalar curvature and positive mass and such that $g = \delta$ on $\mathbb{R}^2 \times (0, +\infty)$.

We can only expect flatness inside the surface with zero Hawking mass for stable CMC surface. In order to prove Theorem 3 we need the following isoperimetric inequality of [Shi 2016], which also plays a key role in proving the existence of isoperimetric surfaces for all volumes in AF three-manifolds. It says that if there exists a Euclidean ball in an AF manifold with nonnegative scalar curvature, then the AF manifold must be $\mathbb{R}^3$.

Lemma 15 [Shi 2016]. *Suppose (M, g) is an AF manifold with scalar curvature $R(g) \geq 0$. Then for any $V > 0$,*

$$I(V) \leq (36\pi)^{1/3} V^{2/3}. \tag{4-1}$$

There is a $V_0 > 0$ with

$$I(V_0) = (36\pi)^{1/3} V_0^{2/3} \tag{4-2}$$

if and only if (M, g) is isometric to $\mathbb{R}^3$.

Also there is an analogous result for an isoperimetric profile on AH manifolds; see Proposition 3.3 in [Ji et al. 2016].

Lemma 16 [Ji et al. 2016]. *Suppose* (M, g) *is an AH manifold with scalar curvature* $R(g) \geq -6$. *Then for any* $V > 0$,

$$I(V) \leq I_{\mathbb{H}}(V). \tag{4-3}$$

There is a $V_0 > 0$ *with*

$$I(V_0) = I_{\mathbb{H}}(V_0) \tag{4-4}$$

if and only if (M, g) *is isometric to* $(\mathbb{H}^3, g_{\mathbb{H}})$.

Now we can prove the rigidity of nearly round isoperimetric surfaces:

Proof of Theorems 3 and 4. If there is an nearly round isoperimetric surface Σ with $m_H(\Sigma) = 0$, and we assume $|\Sigma| = 4\pi$, then $H = 2$. By Theorem 1, the isoperimetric region is a Euclidean ball of volume $\frac{4}{3}\pi$. So we have

$$4\pi = I\big(\tfrac{4}{3}\pi\big) = (36\pi)^{1/3}\big(\tfrac{4}{3}\pi\big)^{2/3}. \tag{4-5}$$

By the rigidity part of Lemma 15, we conclude that (M, g) is isometric to $\mathbb{R}^3$. Theorem 4 follows similarly from Theorem 2 and Lemma 16. □

In fact, large surfaces of the canonical stable CMC foliation in [Huisken and Yau 1996; Qing and Tian 2007] are isoperimetric and close to the coordinate spheres.

Corollary 17. *Let* (M, g) *be an AF three-manifold with scalar curvature* $R(g) \geq 0$. *Then the Hawking mass of all the large enough surfaces in the canonical stable CMC foliation in [Huisken and Yau 1996; Qing and Tian 2007] are positive unless* (M, g) *is isometric to* $\mathbb{R}^3$.

5. Rigidity of Hawking mass for small isoperimetric surfaces

For rigidity of small isoperimetric surfaces, we need to prove that such a surface is a sphere when the volume is small enough.

5.1. ***Topology of small isoperimetric surface.*** It is shown in [Ros 2005] that for a compact manifold without boundary, the isoperimetric surface is a topological sphere when the enclosing volume is small enough to be contained in a geodesic ball. For AF manifolds we still have this property; the proof follows that of the compact case in that work and relies on the behavior at infinity.

Lemma 18. *If* (M, g) *is a complete AF three-manifold without boundary*, *then there exits a* $\delta_0 > 0$, *such that for all volume* $V \leq \delta_0$ *the isoperimetric region is convex and contained in a small neighborhood of some point of M. In particular*,

$$I(V) \sim (36\pi)^{1/3} V^{2/3} \quad \textit{when } V \to 0. \tag{5-1}$$

Proof. Let $\{\Sigma_n\}$ be a sequence of isoperimetric surfaces with second fundamental form A_n and volume $V_n \to 0$. There are two possibilities:

Case 1: $\{|A_n|\}$ is unbounded. Assume $r_n = \max |A_n| = |A_n|(x_n)$; by scaling Σ_n homothetically to $\Sigma'_n = r_n \Sigma_n$ with metric $g_n = r_n^2 g$, also $r_n \to \infty$, $x_n \in \Sigma'_n$, second fundamental form of Σ'_n satisfies $\max |A'_n| = |A'_n(x_n)| = 1$. We have $(M, x_n, g_n) \to (\mathbb{R}^3, 0, \delta)$ smoothly; the limit manifold is standard $\mathbb{R}^3$ because the manifold is AF. Thus Σ'_n is a sequence of stable CMC surfaces with bounded curvature, and locally Σ'_n consists of a certain number of sheets as in Figure 13 of [Ros 2005]. Each one of the sheets is a graph over a bounded planar domain with bounded derivatives.

If two of the sheets become arbitrarily close near some point when $n \to \infty$, then we can modify the surface to get a new one with smaller area and the same volume. For details, see page 196 of [Ros 2005]. More precisely, if the two sheets of the surface become arbitrarily close near some point, we can reduce area without modifying the enclosed volume, which contradicts the minimizing property of Σ_n. There are three cases:

(1) if there is a thin slab, then we can cut part of the volume to one end and reduce the area;

(2) if there is a thin defect of the region, then we can fill part of the defect with volume gained from deforming a far-away portion of the boundary to reduce area;

(3) if there are two close thin defects in the region, then we can reduce the area by moving the part between two defects to one of the defects.

Hence by compactness results [Pérez and Ros 2002], up to a subsequence, $\Sigma'_n \to \Sigma'$ smoothly with multiplicity one and $\Sigma' \subset \mathbb{R}^3$ is a surface of constant mean curvature $H_{\Sigma'}$ properly embedded in $\mathbb{R}^3$ endowed with a standard metric δ, $0 \in \Sigma'$, $|A'(0)|^2 = 1$. The fact that Σ_n are isoperimetric surfaces implies that Σ' is a stable CMC surface. By [da Silveira 1987] and the stability condition, we can conclude that Σ' is either a union of planes or a sphere. That the curvature at the origin is one implies Σ' is a unit sphere. Going back to Σ_n, for n large enough, the mean curvature H_{Σ_n} of Σ_n is large enough, such that

$$\tfrac{1}{2} H_{\Sigma_n}^2 + \mathrm{Ric}(n, n) > 0. \tag{5-2}$$

If Σ_n is not connected, since the mean curvature of the isoperimetric surface Σ_n is the same (see Appendix A.4) for each component Σ_n^i, as $|A_n^i|^2 \geq \frac{1}{2} H_{\Sigma_n^i}^2$,

$$|A_n^i|^2 + \mathrm{Ric}(n, n) > 0 \tag{5-3}$$

on every component Σ_n^i. On the other hand, we can construct a variation f_i on Σ_n^i which is constant and $\sum_i \int_{\Sigma_n^i} f_i = 0$ in the stability condition of the isoperimetric inequality. This gives

$$0 \geq \sum_i f_i^2 \int_{\Sigma_n^i} |A_n|^2 + \mathrm{Ric}(n, n), \tag{5-4}$$

a contradiction. So for large n, we know Σ_n is connected and thus a sphere.

Case 2: $\{|A_n|\}$ is bounded. Scale Σ_n to enclose volume 1. By the above argument we get the limit consists of pairwise disjoint planes enclosing volume 1, a contradiction. So the lemma follows. □

By the above lemma, the rigidity follows from Theorem 3. But it can also be proved by the monotonicity of Hawking mass with respect to the volume of the connected isoperimetric surface. This method relies on the connectedness of isoperimetric surface which was used by Bray [1997]. Bray needed the connectedness of isoperimetric surface when proving monotonicity of Hawking mass.

5.2. ***Properties of I.*** The isoperimetric profile I contains important geometric information of the manifold. It is nondecreasing in the outside of horizon. It is concave if the manifold has nonnegative Ricci curvature. The existence and regularity properties of isoperimetric regions for *all* volumes for AF is proved by combining [Shi 2016] with [Carlotto et al. 2016]; we sketch the proof in Appendix A.2 for completeness.

The continuity and differentiability of I for AF manifold is proved as in [Flores and Nardulli 2014] for manifolds with bounded geometry (Ricci curvature and volume of unit geodesic ball bounded below):

Lemma 19. *Given (M, g) is an AF manifold and $V \in (0, \infty)$, let $\Omega \subset M$ be an isoperimetric region with* $\mathrm{vol}(\Omega) = V$ *and denote $\partial\Omega$ by Σ. The isoperimetric profile has the following regularity*:

(a) *It is continuous and has left and right derivatives at V, $I'_+(V) \leq H_\Sigma \leq I'_-(V)$ and $I'_+(V)$ and $I'_-(V)$ are right and left continuous respectively.*

(b) *The inequality $I''(V)I(V)^2 + \int_\Sigma(\mathrm{Ric}(n, n) + |A_\Sigma|^2) \leq 0$ holds in the sense of comparison functions, i.e., for every $V_0 \geq 0$, there is a smooth function $I_{V_0}(V) \geq I(V)$, $I_{V_0}(V_0) = I(V_0)$ and $I''_{V_0}(V) I_{V_0}(V)^2 + \int_\Sigma(\mathrm{Ric}(n,n) + |A_\Sigma|^2) \leq 0$.*

Proof. The continuity of I is proved in Appendix A.3 by adding and subtracting a small geodesic ball to the isoperimetric regions under the condition of bounded geometry. We only prove (b) which implies the differentiability of I. For every $V_0 > 0$, assume Ω_0 is the isoperimetric region with volume V_0 and $\Sigma_0 = \partial\Omega_0$ is the isoperimetric surface with unit outer normal n_0, second fundamental form A_0 and mean curvature H_0. In order to get an upper bound of I'' we do a unit normal variation on Σ_0. Let Σ_t denote the surface by flowing Σ_0 out with unit speed along the normal n_0 for time t. Since Σ_0 is a smooth embedded surface, there exits a $\delta > 0$ such that Σ_t exists for any $t \in (-\delta, \delta)$. Let $I_{V_0}(t) = \text{area}(\Sigma_t)$. By the first and second variational formula for area we have:

$$I'_{V_0}(t) = \int_{\Sigma_t} H \, d\mu, \tag{5-5}$$

$$V'(t) = I_{V_0}(t), \tag{5-6}$$

$$H'(t) = -|A|^2 - \text{Ric}(n, n). \tag{5-7}$$

We can also parametrize this isoperimetric surface by its volume as $\Sigma(V)$, and $I_{V_0}(V) = \text{area}(\Sigma(V))$. By definition of $\Sigma(V_0)$, $I_{V_0}(V) \geq I(V)$, $I_{V_0}(V_0) = I(V_0)$, so

$$I'_{V_0}(V) = \frac{\int_{\Sigma(V)} H \, d\mu}{I_{V_0}(V)} = H. \tag{5-8}$$

The second derivative of I_{V_0} is

$$\begin{aligned} I''_{V_0}(V) &= \frac{\int_{\Sigma(V)} (H^2 - |A|^2 - \text{Ric}(n, n)) \, d\mu}{I^2_{V_0}(V)} - \frac{H}{I^2_{V_0}(V)} \int_{\Sigma_t} H \, d\mu \\ &= -\frac{\int_{\Sigma(V)} |A|^2 + \text{Ric}(n, n) \, d\mu}{I^2_{V_0}(V)}. \end{aligned} \tag{5-9}$$

For an AF three-manifold, Ricci curvature is bounded blow. Thus there exists $k \in R$ such that $\text{Ric} \geq kg$, and it follows that

$$I''_{V_0}(V) \leq -\frac{k}{I_{V_0}(V)}. \tag{5-10}$$

If $k \geq 0$, then $I_{V_0}(V)$ is concave, and by Lemma 20 below we can get the concaveness of $I(V)$, and then the conclusion follows. In particular, I'_+, I'_- are both nonincreasing functions, they are right and left continuous respectively and I'' exists almost everywhere.

If $k < 0$, let $\lambda = \lambda(k, a, b) := k/(2\delta(a, b))$, where $\delta(a, b) = \min\{I(V) : V \in [a, b]\}$ is strictly positive by continuity of I. For every $V_0 \in [a, b]$,

$$I_{V_0}(V) + \lambda V^2 \geq I(V) + \lambda V^2,$$

so we get $I_{V_0}(V) + \lambda V^2$ is concave. We can argue as above to get the same conclusion. □

In the proof above, we used the following properties of concave functions:

Lemma 20. (a) [Morgan and Johnson 2000] *Let $f : (a, b) \to R$ be a continuous function. Then f is concave if and only if for every $x_0 \in (a, b)$ there exists an* *open interval* $I_{x_0} \subseteq (a, b)$ *of* x_0 *and a concave smooth function* $g_{x_0} : I_{x_0} \to R$ *such that* $g_{x_0}(x_0) = f(x_0)$ *and* $g_{x_0}(x) \geq f(x)$ *for every* $x_0 \in I_{x_0}$.

(b) *If* $f : (a, b) \to R$ *is a concave function, then* f'_+ *and* f'_- *are monotonic nonincreasing functions and also right and left continuous respectively. Moreover,* f'' *exists almost everywhere.*

Proof. (a) If f is concave, just take g to be linear. If f is not concave, then there exists $\epsilon > 0$, such that $f_\epsilon(x) = f(x) - \epsilon x^2$ is not concave. So we can choose $x_1, x_3 \in (a, b)$, such that the graph of $f_\epsilon(x)$ lies below line $l(x)$ from $(x_1, f_\epsilon(x_1))$ to $(x_3, f_\epsilon(x_3))$. Assume $f_\epsilon(x) - l(x)$ attains its minimum at $x_2 \in (x_1, x_3)$.

By hypothesis, there is a concave smooth $g_{x_2}(x) \geq f(x)$, and $g_{x_2}(x_2) = f(x_2)$. Then $g_\epsilon(x) = g_{x_2}(x) - \epsilon x^2 \geq f_\epsilon(x)$, $g_\epsilon(x_2) = f_\epsilon(x_2)$, so we have that $g_\epsilon(x) - l(x)$ also attain its minimum at $x_2 \in (x_1, x_3)$, which implies $g''_\epsilon(x_2) \geq 0$, but $g''_\epsilon(x_2) = g''_{x_2}(x_2) - 2\epsilon \leq -2\epsilon$, a contradiction.

(b) It is well known that f'_+ and f'_- are monotonic nonincreasing and f'' exists almost everywhere, so we just prove the right continuity of f'_+, and left continuity of f'_- follows similarly. For any $x_0 \in (a, b)$, by monotonicity of f'_+ have

$$\lim_{x \to x_0^+} f'_+(x) \leq f'_+(x_0). \tag{5-11}$$

On the other hand,

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0^+} \frac{\int_{x_0}^x f'_+(t)\,dt}{x - x_0} \tag{5-12}$$

where we have used the stronger versions of the fundamental theorem of calculus [Walker 1977]

$$f(x) - f(x_0) = \int_{x_0}^x f'_+(t)\,dt \tag{5-13}$$

whenever f is continuous and $f'_+ \in L^1$. Again by the monotonicity we have

$$f'_+(t) \leq \lim_{x \to x_0^+} f'_+(x). \tag{5-14}$$

Combining with (5-12) and (5-14), we get

$$f'_+(x_0) \leq \lim_{x \to x_0^+} \frac{\int_{x_0}^x \lim_{x \to x_0^+} f'_+(x)\,dt}{x - x_0} = \lim_{x \to x_0^+} f'_+(x). \tag{5-15}$$

Then (5-11) and (5-15) give the right continuity of f'_+. $\square$

5.3. ***Monotonicity of m_H^+.*** For differentiable points of I we have $H(V) = I'(V)$, so we can replace H with I' in Hawking mass in order to simplify Hawking mass to be a function of only volume. But I may not be differentiable for every volume, and there is a jump for H from I'_+ to I'_- at volumes which are not differentiable. By the compactness of isoperimetric surfaces, see [Meeks et al. 2014], there is a surface which achieves the minimal (maximal) mean curvature enclosing the same volume. So we can define the maximal Hawking mass as:

Definition. Let (M, g) be an AF three-manifold with nonnegative scalar curvature, $\Sigma \subset M$ be a isoperimetric surface of volume V. Then the maximal Hawking mass is $m_H^+(V) = \sqrt{I(V)}(16\pi - I(V)I'_+(V)^2)$.

When I is not differentiable, m_H^+ is the maximal Hawking mass, and it reduces to the ordinary Hawking mass at the differentiable points of I. We have the following result on the monotonicity of m_H^+:

Lemma 21 [Bray 1997]. *Let (M, g) be an AF three-manifold with nonnegative scalar curvature. Assume for every $V > 0$ there is a connected isoperimetric surface enclosing volume V, and also $I(V)$ is increasing. Then $m_H^+(V)$ is nondecreasing.*

Proof. By Gauss's equation,

$$K = \frac{R}{2} - \mathrm{Ric}(n, n) + \frac{1}{2}(H^2 - |A|^2). \tag{5-16}$$

So we have

$$|A|^2 + \mathrm{Ric}(n, n) = \frac{R}{2} - K + \frac{1}{2}(H^2 + |A|^2) \tag{5-17}$$

by $|A|^2 = |A^0|^2 + \frac{1}{2}H^2$, and $R \geq 0$, so we have

$$I''_{V_0}(V) = -\frac{\int_{\Sigma(V)} |A|^2 + \mathrm{Ric}(n, n)\, d\mu}{I^2_{V_0}(V)} \leq \frac{\int_{\Sigma(V)} K - \frac{3}{4}H^2\, d\mu}{I^2_{V_0}(V)}. \tag{5-18}$$

By the connectedness of $\Sigma(V)$, we have

$$\int_{\Sigma(V)} K\, d\mu = 2\pi\chi(\Sigma(V)) \leq 4\pi. \tag{5-19}$$

Then

$$I''_{V_0}(V) \leq \frac{16\pi - 3I'_{V_0}(V)^2 I_{V_0}(V)}{4I^2_{V_0}(V)}. \tag{5-20}$$

As we have proved that $I'_+(V)$ is right continuous, so is maximal Hawking mass. Thus it is sufficient to prove $m_H^+(V)$ is weak nondecreasing, i.e., for any $[a, b] \in (0, \infty)$, $\int_a^b m_H^+(V)\phi'(V)\, dV \leq 0$ for all smooth nonnegative $\phi \in C_c^\infty(a, b)$, $\phi \geq 0$.

The reason to do so is that $m_H^+(V)$ has only countable jump points. Let the difference quotient be defined by

$$\Delta^h F(V) = \frac{1}{h}(F(V+h) - F(V)).$$

Then

$$\begin{aligned}
(5\text{-}21)\quad & \int_a^b m_H^+(V)\phi'(V)\,dV \\
&= \int_a^b \sqrt{I(V)}(16\pi - I(V)I'_+(V)^2)\phi'(V)\,dV \\
&= \lim_{h\to 0^+}\int_a^b \sqrt{I(V)}(16\pi - I(V)\Delta^h I(V)^2)\Delta^h\phi(V)\,dV \\
&= -\lim_{h\to 0^+}\int_a^b \Delta^{-h}\{\sqrt{I(V)}(16\pi - I(V)\Delta^h I(V)^2)\}\phi(V)\,dV \\
&= \lim_{h\to 0^+}\int_a^b \Big\{\phi I^{3/2}\Big\{\Delta^{-h}(\Delta^h I)^2 - I'\frac{16\pi - 3I'^2 I}{2I^2}\Big\}\Big\}\,dV,
\end{aligned}$$

where we use the fact that $I'_+ = I'_-$ almost everywhere.

Since $I_{V_0}(V_0) = I(V_0)$, and $I_{V_0}(V) \ge I(V)$, also $I(V)$ is increasing, we get $\Delta^{-h}(\Delta^h I)^2(V_0) \le \Delta^{-h}(\Delta^h I_{V_0})^2(V_0)$, and $I'_{V_0} \ge 0$, so

$$\begin{aligned}
(5\text{-}22)\quad & \int_a^b m_H^+(V)\phi'(V)\,dV \\
&\le \lim_{h\to 0^+}\int_a^b \Big\{\phi I_{V_0}^{3/2}\{\Delta^{-h}(\Delta^h I_{V_0})^2 - I'_{V_0}\}\frac{16\pi - 3I'^2_{V_0}I_{V_0}}{2I^2_{V_0}}\Big\}\,dV_{V_0} \\
&= \int_a^b 2\phi I_{V_0}^{3/2} I'_{V_0}\Big\{I''_{V_0} - \frac{16\pi - 3I'^2_{V_0}I_{V_0}}{4I^2_{V_0}}\Big\}\,dV_{V_0} \le 0,
\end{aligned}$$

where we used (5-20) and Fatou's lemma for the last equality. Hence, $m_H^+(V)$ is nondecreasing. □

Remark. (1) Hawking mass is also monotonic along the stable CMC foliation as long as the area is nondecreasing; the proof is the same as above.

(2) We can see that the monotonicity of maximal Hawking mass relies heavily on the connectedness of the isoperimetric surface. If the isoperimetric surface has more than one components, Bray [1997] considers the sum of three halves of the area of the components

$$F(V) = \inf\Big\{\sum_i \text{area}(\Sigma_i)^{3/2} : \{\Sigma_i\} \text{ enclose volume } V \text{outside the horizons}\Big\}$$

under the condition the components are disjoint with each other. Then he proved the mass

$$m^{+}(V) = F(V)^{1/3}(36\pi - {F'_{+}}^{2})/144\pi^{3/2}$$

is nondecreasing. In fact, for F he got the estimate

$$F''(V) \leq \frac{36\pi - F'(V)^2}{6F(V)}, \tag{5-23}$$

and then the proof follows as above. The minimizing surfaces are CMC generally with different mean curvatures on each component. When the minimizer of F has only one component it must be an isoperimetric surface. We already know that for large enough volume in AF manifolds the isoperimetric surfaces are spheres close to coordinate spheres and $m^{+}(V) = m_H^{+}(V)$; their limits are the ADM mass of the manifold when volume goes to infinity.

Now we are in a position to prove the rigidity of small isoperimetric surfaces:

Proof of Theorem 5. First we claim that

$$\lim_{V\to 0} m_H^{+}(V) = 0. \tag{5-24}$$

In fact, by Lemma 18 we know the isoperimetric surface is of sphere type when the volume is small enough. Combined with Lemma 6, we get

$$\lim_{V\to 0} m_H^{+}(V) \geq 0. \tag{5-25}$$

By definition,

$$m_H^{+}(V) = \sqrt{I(V)}(16\pi - I(V){I'_{+}(V)}^{2}) \leq 16\pi\sqrt{I(V)}, \tag{5-26}$$

which implies

$$\lim_{V\to 0} m_H^{+}(V) \leq 0. \tag{5-27}$$

Thus the claim follows by (5-25) and (5-27).

If there exists an isoperimetric surface Σ with volume $0 < V_0 \leq \delta_0$, such that $m_H^{+}(V_0) = 0$, then by monotonicity of Lemma 21 for m_H^{+} and (5-24), we get

$$m_H^{+}(V) \equiv 0, \quad \text{for any } V \in [0, V_0]. \tag{5-28}$$

Thus

$$I(V){I'_{+}(V)}^{2} \equiv 16\pi \quad \text{on } [0, V_0]. \tag{5-29}$$

Since I is continuous by Lemma 19, we get

$$I'_{+}(V) = I'(V) \quad \text{on } [0, V_0]. \tag{5-30}$$

Since there are no compact minimal surfaces, I is increasing, and

$$I' = \sqrt{\frac{16\pi}{I}}. \tag{5-31}$$

Since $I(0) = 0$, we have

$$I(V) = (36\pi)^{1/3} V^{2/3} \quad \text{on } [0, V_0]. \tag{5-32}$$

Then by Lemma 15 above we conclude that (M, g) is isometric to $\mathbb{R}^3$. □

Appendix

A.1. ***Spherical harmonics on S^2.*** Write

$$\Delta_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

The eigenvalues of $-\Delta_{S^2}$ are $\lambda = l(l+1)$, $l = 0, 1, 2, \dots$; the eigenfunctions are

$$Y_l^m(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\sin^{|m|}\theta\, P_l^{|m|}(\cos\theta)e^{im\varphi},$$

where $m = -l, \dots, l$, and the $P_l(x)$ are the Legendre polynomials, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$. The reduction formula is

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

The real form of spherical harmonics are

$l = 0$

$$Y_{0,0} = \tfrac{1}{2}\sqrt{\tfrac{1}{\pi}},$$

$l = 1$

$$Y_{1,0} = \sqrt{\tfrac{3}{4\pi}}\cos\theta, \quad Y_{1,-1} = \sqrt{\tfrac{3}{4\pi}}\sin\theta\sin\varphi, \quad Y_{1,1} = \sqrt{\tfrac{3}{4\pi}}\sin\theta\cos\varphi,$$

$l = 2$

$$Y_{2,-2} = \tfrac{1}{4}\sqrt{\tfrac{15}{\pi}}\sin^2\theta\sin 2\varphi, \quad Y_{2,-1} = \tfrac{1}{4}\sqrt{\tfrac{15}{\pi}}\sin 2\theta\sin\varphi, \quad Y_{2,0} = \tfrac{1}{4}\sqrt{\tfrac{5}{\pi}}(3\cos^2\theta - 1),$$
$$Y_{2,1} = \tfrac{1}{4}\sqrt{\tfrac{15}{\pi}}\sin 2\theta\cos\varphi, \qquad Y_{2,2} = \tfrac{1}{4}\sqrt{\tfrac{15}{\pi}}\sin^2\theta\cos 2\varphi,$$

$l=4$

$$Y_{4,-4}=\tfrac{3}{16}\sqrt{\tfrac{35}{\pi}}\sin^4\theta\sin 4\varphi,\qquad Y_{4,-3}=\tfrac{3}{4}\sqrt{\tfrac{35}{2\pi}}\sin^3\theta\cos\theta\sin 3\varphi,$$
$$Y_{4,-2}=\tfrac{3}{8}\sqrt{\tfrac{5}{\pi}}\sin^2\theta(7\cos^2\theta-1)\sin 2\varphi,\qquad Y_{4,-1}=\tfrac{3}{8}\sqrt{\tfrac{10}{\pi}}\sin\theta\cos\theta(7\cos^2\theta-3)\sin\varphi,$$
$$Y_{4,0}=\tfrac{3}{16}\sqrt{\tfrac{1}{\pi}}(35\cos^4\theta-30\cos^2\theta+3),\qquad Y_{4,1}=\tfrac{3}{8}\sqrt{\tfrac{10}{\pi}}\sin\theta\cos\theta(7\cos^2\theta-3)\cos\varphi,$$
$$Y_{4,2}=\tfrac{3}{8}\sqrt{\tfrac{5}{\pi}}\sin^2\theta(7\cos^2\theta-1)\cos 2\varphi,\qquad Y_{4,3}=\tfrac{3}{4}\sqrt{\tfrac{35}{2\pi}}\sin^3\theta\cos\theta\cos 3\varphi,$$
$$Y_{4,4}=\tfrac{3}{16}\sqrt{\tfrac{35}{\pi}}\sin^4\theta\cos 4\varphi.$$

To compute u_2^2, we need to decompose the following terms into different order spherical harmonics:

$$Y_{2,0}^2=\tfrac{3}{7}\sqrt{\tfrac{1}{\pi}}Y_{4,0}+\tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}Y_{2,0}+\tfrac{1}{4\pi},$$
$$Y_{2,-2}^2=-\tfrac{1}{2}\sqrt{\tfrac{5}{7\pi}}Y_{4,4}+\tfrac{1}{14}\sqrt{\tfrac{1}{\pi}}Y_{4,0}-\tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}Y_{2,0}+\tfrac{1}{4\pi},$$
$$Y_{2,2}^2=\tfrac{1}{2}\sqrt{\tfrac{5}{7\pi}}Y_{4,4}+\tfrac{1}{14}\sqrt{\tfrac{1}{\pi}}Y_{4,0}-\tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}Y_{2,0}+\tfrac{1}{4\pi},$$
$$Y_{2,-1}^2=-\tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}Y_{4,2}-\tfrac{2}{7}\sqrt{\tfrac{1}{\pi}}Y_{4,0}-\tfrac{1}{14}\sqrt{\tfrac{15}{\pi}}Y_{2,2}+\tfrac{1}{14}\sqrt{\tfrac{5}{\pi}}Y_{2,0}+\tfrac{1}{4\pi},$$
$$Y_{2,1}^2=\tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}Y_{4,2}-\tfrac{2}{7}\sqrt{\tfrac{1}{\pi}}Y_{4,0}+\tfrac{1}{14}\sqrt{\tfrac{15}{\pi}}Y_{2,2}+\tfrac{1}{14}\sqrt{\tfrac{5}{\pi}}Y_{2,0}+\tfrac{1}{4\pi},$$
$$Y_{2,-2}Y_{2,2}=\tfrac{1}{2}\sqrt{\tfrac{5}{7\pi}}Y_{4,-4},\qquad Y_{2,-2}Y_{2,0}=\tfrac{1}{14}\sqrt{\tfrac{15}{\pi}}Y_{4,-2}-\tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}Y_{2,-2},$$
$$Y_{2,2}Y_{2,0}=\tfrac{1}{14}\sqrt{\tfrac{15}{\pi}}Y_{4,2}-\tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}Y_{2,2},\qquad Y_{2,-1}Y_{2,0}=\tfrac{1}{7}\sqrt{\tfrac{15}{2\pi}}Y_{4,-1}+\tfrac{1}{14}\sqrt{\tfrac{5}{\pi}}Y_{2,-1},$$
$$Y_{2,1}Y_{2,0}=\tfrac{1}{7}\sqrt{\tfrac{15}{2\pi}}Y_{4,1}+\tfrac{1}{14}\sqrt{\tfrac{5}{\pi}}Y_{2,1},\qquad Y_{2,-1}Y_{2,1}=\tfrac{1}{7}\sqrt{\tfrac{5}{\pi}}Y_{4,-2}+\tfrac{1}{14}\sqrt{\tfrac{15}{\pi}}Y_{2,-2},$$
$$Y_{2,-2}Y_{2,-1}=-\tfrac{1}{2}\sqrt{\tfrac{5}{14\pi}}Y_{4,3}-\tfrac{1}{14}\sqrt{\tfrac{5}{2\pi}}Y_{4,1}+\tfrac{1}{14}\sqrt{\tfrac{15}{\pi}}Y_{2,1},$$
$$Y_{2,2}Y_{2,1}=\tfrac{1}{2}\sqrt{\tfrac{5}{14\pi}}Y_{4,3}-\tfrac{1}{14}\sqrt{\tfrac{5}{2\pi}}Y_{4,1}+\tfrac{1}{14}\sqrt{\tfrac{15}{\pi}}Y_{2,1},$$
$$Y_{2,-2}Y_{2,1}=\tfrac{1}{2}\sqrt{\tfrac{5}{14\pi}}Y_{4,-3}-\tfrac{1}{14}\sqrt{\tfrac{5}{2\pi}}Y_{4,-1}+\tfrac{1}{14}\sqrt{\tfrac{15}{\pi}}Y_{2,-1},$$
$$Y_{2,2}Y_{2,-1}=\tfrac{1}{2}\sqrt{\tfrac{5}{14\pi}}Y_{4,-3}+\tfrac{1}{14}\sqrt{\tfrac{5}{2\pi}}Y_{4,-1}+\tfrac{1}{14}\sqrt{\tfrac{15}{\pi}}Y_{2,-1}.$$

A.2. *Existence of isoperimetric surface for all volumes.*

Lemma 22 [Carlotto et al. 2016]. *Let (M, g) be a three-manifold with nonnegative scalar curvature, maybe with horizon. Then the isoperimetric surface for all volumes exists.*

Proof. By Theorem 2.1 of [Ritoré and Rosales 2004], we can prove this in the same way as in [Eichmair and Metzger 2013]; for every $V > 0$, there exists an isoperimetric region Ω and a radius $r \geq 0$ such that

$$|\Omega|_g + \tfrac{4}{3}\pi r^3 = V, \quad |\partial\Omega|_g + 4\pi r^2 = I(V). \tag{A-1}$$

By the isoperimetric inequality of Shi [2016] on nonnegative scalar curvature manifolds, we get for every $r > 0$ that there is a bounded region Ω' with finite perimeter $|\partial\Omega'|_g$ lying arbitrary far out in the asymptotic flat region of (M, g) such that

$$|\partial\Omega'|_g = 4\pi r^2, \quad |\Omega'|_g > \tfrac{4}{3}\pi r^3. \tag{A-2}$$

If $r > 0$ in (A-1), then there is an Ω' that satisfies (A-2). We consider the region $\Omega \cup \Omega'$. Then

$$|\Omega|_g + |\Omega'|_g > V, \quad |\partial\Omega|_g + |\partial\Omega'|_g = I(V). \tag{A-3}$$

But by the definition of I and the above equality we get

$$I(|\Omega|_g + |\Omega'|_g) \leq |\partial\Omega|_g + |\partial\Omega'|_g = I(V). \tag{A-4}$$

By the fact that I is strictly increasing [Chodosh 2016], we have

$$I(|\Omega|_g + |\Omega'|_g) > I(V), \tag{A-5}$$

a contradiction. Thus $r = 0$, which implies that Ω is the isoperimetric region of volume V. □

A.3. ***Continuity of I.***

Lemma 23. *I is continuous on AF three-manifold.*

Proof. The proof is from [Flores and Nardulli 2014] for bounded geometry, where they don't have existence of isoperimetric surfaces. We need to prove the upper semicontinuity and lower semicontinuity for I, i.e., for any $V_0 > 0$,

$$\limsup_{V \to V_0^+} I(V) \leq I(V_0), \quad \limsup_{V \to V_0^-} I(V) \leq I(V_0), \tag{A-6}$$

$$I(V_0) \leq \liminf_{V \to V_0^+} I(V), \quad I(V_0) \leq \liminf_{V \to V_0^-} I(V). \tag{A-7}$$

Upper semicontinuity of I: Given $V_0 > 0$, there is isoperimetric region Ω_0 such that $\mathrm{vol}(\Omega_0) = V_0$, $\mathrm{area}(\partial\Omega_0) = I(V_0)$. For any $V \uparrow V_0$, we can subtract a small geodesic ball $B_r(p)$ such that $\mathrm{vol}(B_r(p)) = V_0 - V$, $\mathrm{vol}(\Omega_0 \backslash B_r(p)) = V$. Thus

$$I(V) \leq \mathrm{area}(\partial\Omega_0) + \mathrm{area}(\partial B_r(p)) = I(V_0) + \mathrm{area}(\partial B_r(p)). \tag{A-8}$$

This implies

$$\limsup_{V\to V_0^+} I(V) \le \operatorname{area}(\partial\Omega_0) + \lim_{V\to V_0^+} \operatorname{area}(\partial B_r(p)) = I(V_0). \tag{A-9}$$

For any $V \downarrow V_0$, we can add a small geodesic ball $B_r(p)$, such that $\operatorname{vol}(B_r(p)) = V - V_0$, $\operatorname{vol}(\Omega_0 \bigcup B_r(p)) = V$. Thus

$$I(V) \le \operatorname{area}(\partial\Omega_0) + \operatorname{area}(\partial B_r(p)) = I(V_0) + \operatorname{area}(\partial B_r(p)). \tag{A-10}$$

This implies

$$\limsup_{V\to V_0^-} I(V) \le \operatorname{area}(\partial\Omega_0) + \lim_{V\to V_0^-} \operatorname{area}(\partial B_r(p)) = I(V_0). \tag{A-11}$$

So we get the upper semicontinuity of I from (A-9) and (A-11).

Lower semicontinuity of I: for $V \uparrow V_0$, there exists an isoperimetric region Ω such that $\operatorname{vol}(\Omega) = V$. Adding a small geodesic ball $B_r(p)$ such that $\operatorname{vol}(B_r(p)) = V_0 - V$,

$$I(V_0) \le \operatorname{area}(\partial\Omega) + \operatorname{area}(\partial B_r(p)) = I(V) + \operatorname{area}(\partial B_r(p)). \tag{A-12}$$

This implies

$$I(V_0) \le \liminf_{V\to V_0^-} I(V) + \lim_{V\to V_0^-} \operatorname{area}(\partial B_r(p)) \le \liminf_{V\to V_0^-} I(V). \tag{A-13}$$

For $V \downarrow V_0$, subtract a small geodesic ball $B_r(p)$ such that $\operatorname{vol}(B_r(p)) = V - V_0$, so that

$$I(V_0) \le \operatorname{area}(\partial\Omega) + \operatorname{area}(\partial B_r(p)) = I(V) + \operatorname{area}(\partial B_r(p)). \tag{A-14}$$

This implies

$$I(V_0) \le \liminf_{V\to V_0^-} I(V) + \lim_{V\to V_0^-} \operatorname{area}(\partial B_r(p)) \le \liminf_{V\to V_0^-} I(V). \tag{A-15}$$

The lower semicontinuity follows from (A-13) and (A-15). □

A.4. *Mean curvature of isoperimetric surface.*

Lemma 24. *The mean curvatures of all the components for an isoperimetric surface are the same.*

Proof. We know that an isoperimetric surface is stable CMC and the mean curvature is same on each component. This follows by the stability condition when choosing a piecewise constant variation function on each component. Assume $\Sigma = \Sigma_1 \bigcup \Sigma_2$ is an isoperimetric surface with disjoint components Σ_1 and Σ_2. If the mean curvature of Σ_1 and Σ_2 are constants H_1 and H_2, respectively, let

$$f = \begin{cases} -|\Sigma_2| & \text{on } \Sigma_1 \\ \ \ |\Sigma_1| & \text{on } \Sigma_2. \end{cases} \tag{A-16}$$

As Σ is an isoperimetric surface, so the first variation formula

$$0=\int_{\Sigma} fH=\int_{\Sigma_1\bigcup\Sigma_2} fH$$

$$=-|\Sigma_2|H_1|\Sigma_1|+|\Sigma_1|H_2|\Sigma_2|=|\Sigma_1||\Sigma_2|(H_2-H_1). \tag{A-17}$$

So $H_1=H_2$, which implies mean curvature on each component is the same. □

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JIACHENG SUN
BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH
PEKING UNIVERSITY
QUANZHAI
BEIJING, 100871
CHINA
sunxujason@pku.edu.cn

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