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We construct examples of Bach-flat gradient Ricci solitons in neutral signature which are neither half conformally flat nor conformally Einstein.

1. Introduction

Let (M, g) be a pseudo-Riemannian manifold. Let $f \in C^\infty(M)$. We say that (M, g, f) is a *gradient Ricci soliton* if the equation

$$(1) \quad \text{Hes}_f + \rho = \lambda g,$$

is satisfied for some $\lambda \in \mathbb{R}$, where ρ is the *Ricci tensor*, and $\text{Hes}_f = \nabla df$ is the *Hessian tensor* acting on f . A gradient Ricci soliton is said to be *trivial* if the potential function f is constant, since (1) reduces to the Einstein equation $\rho = \lambda g$. Gradient Ricci solitons have been extensively investigated and their classification under geometric conditions is a problem of current interest. We refer to [Cao 2010] for more information.

The gradient Ricci soliton equation codifies geometric information of (M, g) in terms of the Ricci curvature and the second fundamental form of the level sets of the potential function f . The fact that the Ricci tensor completely determines the curvature tensor in the locally conformally flat case has yielded some results in this situation [Cao and Chen 2012; Munteanu and Sesum 2013; Petersen and Wylie 2010]. Any locally conformally flat gradient Ricci soliton is locally a warped product in the Riemannian setting [Fernández-López and García-Río 2014]. The higher signature case, however, allows other possibilities when the level sets of the potential function are degenerate hypersurfaces [Brozos-Vázquez et al. 2013]. Four-dimensional half conformally flat (i.e., self-dual or anti-self-dual) gradient Ricci solitons have been investigated in the Riemannian and neutral signature cases [Brozos-Vázquez and García-Río 2016; Chen and Wang 2015]. While they are locally conformally flat in the Riemannian situation, neutral signature allows other examples given by Riemannian extensions of affine gradient Ricci solitons.

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Let W be the Weyl conformal curvature tensor of (M, g) . The Bach tensor, $\mathfrak{B}_{ij} = \nabla^k \nabla^\ell W_{kij\ell} + \frac{1}{2} \rho^{k\ell} W_{kij\ell}$, is conformally invariant in dimension 4. Bach-flat metrics contain half conformally flat and conformally Einstein metrics as special cases [Besse 1987]. Hence, a natural problem is to classify Bach-flat gradient Ricci solitons. The Riemannian case was investigated in [Cao et al. 2014; Cao and Chen 2013] both in the shrinking and steady cases. In all situations the Bach-flat condition reduces to the locally conformally flat one under some natural assumptions.

Our main purpose in this paper is to construct new examples of Bach-flat gradient Ricci solitons in neutral signature. The corresponding potential functions have degenerate level set hypersurfaces and their underlying structure is never locally conformally flat, in sharp contrast with the Riemannian situation. These metrics are realized on the cotangent bundle $T^*\Sigma$ of an affine surface (Σ, D) , and they may be viewed as perturbations of the classical Riemannian extensions introduced by Patterson and Walker [1952].

Here is a brief guide to some of the most important results of this paper. In Theorem 3.1 we show that, for any affine surface (Σ, D) admitting a parallel nilpotent $(1, 1)$ -tensor field T , the modified Riemannian extension $(T^*\Sigma, g_{D,T,\Phi})$ is Bach-flat. Moreover we show that Bach-flatness is independent of the deformation tensor field Φ , thus providing an infinite family of Bach-flat metrics for any initial data (Σ, D, T) . Affine surfaces admitting a parallel nilpotent $(1, 1)$ -tensor field T are characterized in Proposition 3.3 by the recurrence of the symmetric part of the Ricci tensor, being $\ker T$ a parallel one-dimensional distribution whose integral curves are geodesics.

The previous construction is used in Theorem 4.3 to show that, for any smooth function $h \in \mathcal{C}^\infty(\Sigma)$, there exist appropriate deformation tensor fields Φ such that $(T^*\Sigma, g_{D,T,\Phi}, f = h \circ \pi)$ is a steady gradient Ricci soliton if and only if $dh(\ker T) = 0$. This provides infinitely many examples of Bach-flat gradient Ricci solitons in neutral signature.

Theorems 5.1 and 6.1 show that $(T^*\Sigma, g_{D,T,\Phi})$ is generically strictly Bach-flat, i.e., neither half conformally flat nor conformally Einstein. Moreover, Theorem 5.1 is used in Proposition 5.2 to construct new examples of anti-self-dual metrics. Turning to gradient Ricci solitons, we show in Theorem 5.4 the existence of anti-self-dual steady gradient Ricci solitons which are not locally conformally flat.

The paper is organized as follows. Some basic results on the Bach tensor and gradient Ricci solitons are introduced in Section 2, as well as a sketch of the construction of modified Riemannian extensions $g_{D,\Phi,T}$. We use these metrics in Section 3 to show that, for any parallel tensor field T on (Σ, D) , $g_{D,\Phi,T}$ is Bach-flat if and only if T is either a multiple of the identity or nilpotent (see Theorem 3.1). In Section 4 we show that for each initial data (Σ, D, T) there are an infinite number of Bach-flat steady gradient Ricci solitons (see Theorem 4.3). Nontriviality

of the examples is obtained after an examination of the half conformally flat condition (see Section 5) and the conformally Einstein property (see Section 6) of the modified Riemannian extensions introduced in Section 2. Finally, we specialize this construction in Section 7 to provide some illustrative examples.

2. Preliminaries

Let (M^n, g) be a pseudo-Riemannian manifold with Ricci curvature ρ and scalar curvature τ . Let W denote the Weyl conformal curvature tensor and define

$$W[\rho](X, Y) = \sum_{ij} \varepsilon_i \varepsilon_j W(E_i, X, Y, E_j) \rho(E_i, E_j),$$

where $\{E_i\}$ is a local orthonormal frame and $\varepsilon_i = g(E_i, E_i)$. Then the *Bach tensor* is defined (see [Bach 1921]) by

$$(2) \quad \mathfrak{B} = \operatorname{div}_1 \operatorname{div}_4 W + \frac{n-3}{n-2} W[\rho],$$

where div is the divergence operator.

Let $\mathfrak{S} = \rho - \tau/(2(n-1)) g$ denote the Schouten tensor of (M, g) . Let \mathfrak{C} be the *Cotton tensor*, $\mathfrak{C}_{ijk} = (\nabla_i \mathfrak{S})_{jk} - (\nabla_j \mathfrak{S})_{ik}$; it provides a measure of the lack of symmetry on the covariant derivative of the Schouten tensor. Since $\operatorname{div}_4 W = -(n-3)/(n-2) \mathfrak{C}$, the Bach and the Cotton tensors of any four-dimensional manifold are related by $\mathfrak{B} = \frac{1}{2}(-\operatorname{div}_1 \mathfrak{C} + W[\rho])$.

The Bach tensor, which is trace-free and conformally invariant in dimension $n = 4$, has been broadly investigated in the literature, both from the geometrical and physical viewpoints (see, for example, [Chen and He 2013; Derdzinski 1983; Dunajski and Tod 2014]). It is the gradient of the L^2 functional of the Weyl curvature on compact manifolds. The field equations of conformal gravity are equivalent to setting the Bach tensor equal to zero and it is also central in the study of the Bach flow, a geometric flow which is quadratic on the curvature and whose fixed points are the vacuum solutions of conformal Weyl gravity [Bakas et al. 2010].

Besides the half conformally flat metrics and the conformally Einstein ones, there are few known examples of strictly Bach-flat manifolds, meaning the ones which are neither half conformally flat nor conformally Einstein (see, for example, [Abbena et al. 2013; Hill and Nurowski 2009; Leistner and Nurowski 2010]). Motivated by this lack of examples, we first construct new explicit four-dimensional Bach-flat manifolds of neutral signature.

Riemannian extensions. In order to introduce the family of metrics under consideration, we recall that a pseudo-Riemannian manifold (M, g) is a *Walker manifold* if there exists a parallel null distribution \mathcal{D} on M . Walker metrics, also called Brinkmann waves in the literature, have been widely investigated in the Lorentzian

setting (pp-waves being a special class among them). They appear in many geometrical situations showing a specific behavior without Riemannian counterpart (see [Brozos-Vázquez et al. 2009]).

Let (M, g, \mathcal{D}) be a four-dimensional Walker manifold of neutral signature and \mathcal{D} of maximal rank. Then there are local coordinates $(x^1, x^2, x_{1'}, x_{2'})$ so that the metric g is given (see [Walker 1950]) by

$$(3) \quad g = 2 dx^i \circ dx_{i'} + g_{ij} dx^i \circ dx^j,$$

where “ \circ ” denotes the symmetric product $\omega_1 \circ \omega_2 := \frac{1}{2}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1)$ and (g_{ij}) is a 2×2 symmetric matrix whose entries are functions of all the variables. Moreover, the parallel degenerate distribution is given by $\mathcal{D} = \text{span}\{\partial_{x_{1'}}, \partial_{x_{2'}}\}$.

A special family of four-dimensional Walker metrics is provided by the Riemannian extensions of affine connections to the cotangent bundle of an affine surface. Next we briefly sketch their construction. Let $T^*\Sigma$ be the cotangent bundle of a surface Σ and let $\pi : T^*\Sigma \rightarrow \Sigma$ be the projection. Let $\tilde{p} = (p, \omega)$ denote a point of $T^*\Sigma$, where $p \in \Sigma$ and $\omega \in T_p^*\Sigma$. Local coordinates (x^i) in an open set \mathcal{U} of Σ induce local coordinates $(x^i, x_{i'})$ in $\pi^{-1}(\mathcal{U})$, where one sets $\omega = \sum x_{i'} dx^i$. The evaluation functions on $T^*\Sigma$ play a central role in the construction. They are defined as follows. For each vector field X on Σ , the *evaluation of X* is the real valued function $\iota X : T^*\Sigma \rightarrow \mathbb{R}$ given by $\iota X(p, \omega) = \omega(X_p)$. Vector fields on $T^*\Sigma$ are characterized by their action on evaluations ιX and one defines the complete lift to $T^*\Sigma$ of a vector field X on Σ by $X^C(\iota Z) = \iota[X, Z]$, for all vector fields Z on Σ . Moreover, a $(0, s)$ -tensor field on $T^*\Sigma$ is characterized by its action on complete lifts of vector fields on Σ .

Next, let D be a torsion-free affine connection on Σ . The *Riemannian extension* g_D is the neutral signature metric g_D on $T^*\Sigma$ characterized by the identity $g_D(X^C, Y^C) = -\iota(D_X Y + D_Y X)$ (see [Patterson and Walker 1952]). They are expressed in the induced local coordinates $(x^i, x_{i'})$ as follows:

$$(4) \quad g_D = 2 dx^i \circ dx_{i'} - 2x_{k'} {}^D\Gamma_{ij}^k dx^i \circ dx^j,$$

where ${}^D\Gamma_{ij}^k$ denote the Christoffel symbols of D . The geometry of $(T^*\Sigma, g_D)$ is strongly related to that of (Σ, D) . Recall that the curvature of any affine surface is completely determined by its Ricci tensor ρ^D . Moreover, the symmetric and skew-symmetric parts given by $\rho_{\text{sym}}^D(X, Y) = \frac{1}{2}\{\rho^D(X, Y) + \rho^D(Y, X)\}$ and $\rho_{\text{sk}}^D(X, Y) = \frac{1}{2}\{\rho^D(X, Y) - \rho^D(Y, X)\}$ play a distinguished role.

Let Φ be a symmetric $(0, 2)$ -tensor field on Σ . Then the *deformed Riemannian extension*, $g_{D, \Phi} = g_D + \pi^* \Phi$, is a first perturbation of the Riemannian extension. A second one is obtained as follows. Let $T = T_i^k dx^i \otimes \partial_{x^k}$ be a $(1, 1)$ -tensor field on Σ . Its evaluation ιT defines a one-form on $T^*\Sigma$ characterized by $\iota T(X^C) = \iota(TX)$. The *modified Riemannian extension* $g_{D, \Phi, T}$ is the neutral signature metric

on $T^*\Sigma$ defined (see [Calviño-Louzao et al. 2009]) by

$$(5) \quad g_{D,\Phi,T} = \iota T \circ \iota T + g_D + \pi^* \Phi,$$

where Φ is a symmetric $(0, 2)$ -tensor field on Σ . In local coordinates one has

$$g_{D,\Phi,T} = 2 dx^i \circ dx_{i'} + \left\{ \frac{1}{2} x_{r'} x_{s'} (T_i^r T_j^s + T_j^r T_i^s) - 2 x_{k'}^D \Gamma_{ij}^k + \Phi_{ij} \right\} dx^i \circ dx^j.$$

The case when T is a multiple of the identity ($T = c \text{Id}$, $c \neq 0$) is of special interest. It was shown in [Calviño-Louzao et al. 2009] that for any affine surface (Σ, D) , the modified Riemannian extension $g_{D,\Phi,c \text{Id}}$ is an Einstein metric on $T^*\Sigma$ if and only if the deformation tensor Φ is the symmetric part of the Ricci tensor of (Σ, D) . Moreover, a slight generalization of the modified Riemannian extension allows a complete description of self-dual Walker metrics as follows.

Theorem 2.1 [Calviño-Louzao et al. 2009; Díaz-Ramos et al. 2006]. A four-dimensional Walker metric is self-dual if and only if it is locally isometric to the cotangent bundle $T^*\Sigma$ of an affine surface (Σ, D) , with metric tensor

$$g = \iota X (\iota \text{id} \circ \iota \text{id}) + \iota \text{id} \circ \iota T + g_D + \pi^* \Phi,$$

where X , T , D and Φ are a vector field, a $(1, 1)$ -tensor field, a torsion-free affine connection and a symmetric $(0, 2)$ -tensor field on Σ , respectively.

As a matter of notation, we will write $\partial_k = \partial/\partial x^k$ and $\partial_{k'} = \partial/\partial x_{k'}$, unless we want to emphasize some special coordinates. We will let

$$\phi_k = (\partial/\partial x^k)\phi \quad \text{and} \quad \phi_{k'} = (\partial/\partial x_{k'})\phi$$

denote the corresponding first derivatives of a smooth function ϕ .

Gradient Ricci solitons and affine gradient Ricci solitons. Let (M, g, f) be a gradient Ricci soliton. The level set hypersurfaces of the potential function play a distinguished role in analyzing the geometry of gradient Ricci solitons. Hence we say that the soliton is *nonisotropic* if ∇f is nowhere lightlike (i.e., $\|\nabla f\|^2 \neq 0$), and that the soliton is *isotropic* if $\|\nabla f\|^2 = 0$, but $\nabla f \neq 0$.

Nonisotropic gradient Ricci solitons lead to local warped product decompositions in the locally conformally flat and half conformally flat cases, and their geometry resembles the Riemannian situation [Brozos-Vázquez and García-Río 2016; Brozos-Vázquez et al. 2013]. The isotropic case is, however, in sharp contrast with the positive definite setting since ∇f gives rise to a Walker structure. Self-dual gradient Ricci solitons which are not locally conformally flat are isotropic and steady ($\lambda = 0$ in (1)). Moreover, they are described in terms of Riemannian extensions as follows.

Theorem 2.2 [Brozos-Vázquez and García-Río 2016]. Let (M, g, f) be a four-dimensional self-dual gradient Ricci soliton of neutral signature which is not locally

conformally flat. Then (M, g) is locally isometric to the cotangent bundle $T^*\Sigma$ of an affine surface (Σ, D) equipped with a modified Riemannian extension $g_{D,\Phi,0}$.

Moreover any such gradient Ricci soliton is steady and the potential function is given by $f = h \circ \pi$ for some $h \in C^\infty(\Sigma)$ satisfying the affine gradient Ricci soliton equation

$$(6) \quad \text{Hes}_h^D + 2\rho_{\text{sym}}^D = 0,$$

for any symmetric $(0, 2)$ -tensor field Φ on Σ .

The previous result relates the affine geometry of (Σ, D) and the pseudo-Riemannian geometry of $(T^*\Sigma, g_{D,\Phi,0})$, allowing the construction of an infinite family of steady gradient Ricci solitons on $T^*\Sigma$ for any initial data (Σ, D, h) satisfying (6). It is important to remark here that the existence of affine gradient Ricci solitons imposes some restrictions on (Σ, D) , as shown in [Brozos-Vázquez et al. 2018] in the locally homogeneous case.

3. Bach-flat modified Riemannian extensions

The use of modified Riemannian extensions with $T = c \text{Id}$ allowed the construction of many examples of self-dual Einstein metrics [Calviño-Louzao et al. 2009]. One of the crucial facts in understanding the metrics $g_{D,\Phi,c\text{Id}}$ is that the $(1, 1)$ -tensor field $T = c \text{Id}$ is parallel with respect to the connection D . Hence, a natural generalization arises by considering arbitrary tensor fields T which are parallel with respect to the affine connection D .

Let (Σ, D, T) be a torsion-free affine surface equipped with a parallel $(1, 1)$ -tensor field T . Parallelizability of T guarantees the existence of local coordinates (x^1, x^2) on Σ so that

$$T \partial_1 = T_1^1 \partial_1 + T_1^2 \partial_2,$$

and

$$T \partial_2 = T_2^1 \partial_1 + T_2^2 \partial_2,$$

for some real constants T_i^j . Let $(T^*\Sigma, g_{D,\Phi,T})$ be the modified Riemannian extension given by (5). Further note that D and Φ are taken with full generality. Thus, the corresponding Christoffel symbols ${}^D\Gamma_{ij}^k$ and the coefficient functions Φ_{ij} are arbitrary smooth functions of the coordinates (x^1, x^2) .

Our first main result concerns the construction of Bach-flat metrics:

Theorem 3.1. Let (Σ, D, T) be a torsion-free affine surface equipped with a parallel $(1, 1)$ -tensor field T . Let Φ be an arbitrary symmetric $(0, 2)$ -tensor field on Σ . Then the Bach tensor of $(T^*\Sigma, g_{D,\Phi,T})$ vanishes if and only if T is either a multiple of the identity or nilpotent.

Proof. In order to compute the Bach tensor of $(T^*\Sigma, g_{D,\Phi,T})$, first of all observe that being T -parallel imposes some restrictions on the components T_i^j as well as on the Christoffel symbols of the connection D :

$$(7) \quad DT = 0 \Rightarrow \begin{cases} T_2^1 {}^D\Gamma_{11}^2 - T_1^2 {}^D\Gamma_{12}^1 = 0, \\ T_2^1 {}^D\Gamma_{12}^2 - T_1^2 {}^D\Gamma_{22}^1 = 0, \\ T_1^2 {}^D\Gamma_{11}^1 + (T_2^2 - T_1^1) {}^D\Gamma_{11}^2 - T_1^2 {}^D\Gamma_{12}^2 = 0, \\ T_2^1 {}^D\Gamma_{11}^1 + (T_2^2 - T_1^1) {}^D\Gamma_{12}^1 - T_2^1 {}^D\Gamma_{12}^2 = 0, \\ T_1^2 {}^D\Gamma_{12}^1 + (T_2^2 - T_1^1) {}^D\Gamma_{12}^2 - T_1^2 {}^D\Gamma_{22}^2 = 0, \\ T_2^1 {}^D\Gamma_{12}^1 + (T_2^2 - T_1^1) {}^D\Gamma_{22}^1 - T_2^1 {}^D\Gamma_{22}^2 = 0. \end{cases}$$

Then, expressing the Bach tensor $\mathfrak{B}_{ij} = \mathfrak{B}(\partial_i, \partial_j)$ in induced coordinates $(x^i, x_{i'})$, a long but straightforward calculation shows that

$$(8) \quad (\mathfrak{B}_{ij}) = \left(\begin{array}{cc|c} \mathfrak{B}_{11} & \mathfrak{B}_{12} & \tilde{\mathfrak{B}} \\ \mathfrak{B}_{12} & \mathfrak{B}_{22} & \\ \hline & \tilde{\mathfrak{B}} & 0 \end{array} \right),$$

where

$$\tilde{\mathfrak{B}} = \frac{1}{6}((T_1^1 - T_2^2)^2 + 4T_2^1 T_1^2) \cdot (T_1^1 + T_2^2) \cdot \begin{pmatrix} T_1^1 - T_2^2 & 2T_1^2 \\ 2T_2^1 & T_2^2 - T_1^1 \end{pmatrix}$$

and where the coefficients \mathfrak{B}_{11} , \mathfrak{B}_{12} and \mathfrak{B}_{22} can be written in terms of $\mathfrak{d} = \det(T)$ and $\mathfrak{t} = \text{tr}(T)$ as follows:

$$\begin{aligned} \mathfrak{B}_{11} = & -\frac{1}{6}\{10\mathfrak{d}^3 - 2(\mathfrak{t}^2 + 13T_2^2\mathfrak{t} - 15(T_2^2)^2)\mathfrak{d}^2 + (5\mathfrak{t} - T_2^2)(\mathfrak{t} - T_2^2)\mathfrak{t}^2\mathfrak{d} - (\mathfrak{t} - T_2^2)^2\mathfrak{t}^4\}x_1'^2 \\ & - \frac{1}{6}\{(T_1^2)^2(30\mathfrak{d}^2 + \mathfrak{t}^2\mathfrak{d} - \mathfrak{t}^4)\}x_2'^2 \\ & - \frac{1}{3}\{(13\mathfrak{t} - 30T_2^2)\mathfrak{d}^2 + (3\mathfrak{t} - T_2^2)\mathfrak{t}^2\mathfrak{d} - (\mathfrak{t} - T_2^2)\mathfrak{t}^4\}T_1^2 x_1'x_2' \\ & - \frac{1}{3}\{({}^D\Gamma_{11}^1 + 2{}^D\Gamma_{12}^2)(\mathfrak{t} - 2T_2^2) + 2T_1^2 {}^D\Gamma_{22}^2\}(\mathfrak{t}^2 - 4\mathfrak{d})\mathfrak{t}x_1' \\ & - \frac{1}{3}\{({}^D\Gamma_{11}^2(\mathfrak{t} - 2T_2^2) + 2T_1^2 {}^D\Gamma_{12}^2\}(\mathfrak{t}^2 - 4\mathfrak{d})\mathfrak{t}x_2' \\ & - \frac{1}{6}\{10\mathfrak{d}^2 + (3\mathfrak{t}^2 - 22T_2^2\mathfrak{t} + 14(T_2^2)^2)\mathfrak{d} - (\mathfrak{t}^2 - 4T_2^2\mathfrak{t} + 2(T_2^2)^2)\mathfrak{t}^2\}\Phi_{11} \\ & - \frac{1}{3}\{(11\mathfrak{t} - 14T_2^2)\mathfrak{d} - 2(\mathfrak{t} - T_2^2)\mathfrak{t}^2\}T_1^2\Phi_{12} \\ & + \frac{1}{3}\{\mathfrak{t}^2 - 7\mathfrak{d}\}(T_1^2)^2\Phi_{22} \\ & - \frac{2}{3}(\partial_2 {}^D\Gamma_{11}^2 - \partial_1 {}^D\Gamma_{12}^2)(4\mathfrak{d} - \mathfrak{t}^2), \end{aligned}$$

$$\begin{aligned}
\mathfrak{B}_{12} = & -\frac{1}{6}\{(13\mathfrak{t} - 30T_2^2)\mathfrak{d}^2 + (3\mathfrak{t} - T_2^2)\mathfrak{t}^2\mathfrak{d} - (\mathfrak{t} - T_2^2)\mathfrak{t}^4\}T_2^1x_1^2 \\
& + \frac{1}{6}\{(17\mathfrak{t} - 30T_2^2)\mathfrak{d}^2 - (2\mathfrak{t} + T_2^2)\mathfrak{t}^2\mathfrak{d} + T_2^2\mathfrak{t}^4\}T_1^2x_2^2 \\
& + \frac{1}{6}\{20\mathfrak{d}^3 + 4(4\mathfrak{t}^2 - 15T_2^2\mathfrak{t} + 15(T_2^2)^2)\mathfrak{d}^2 \\
& \quad - (3\mathfrak{t}^2 + 2T_2^2\mathfrak{t} - 2(T_2^2)^2)\mathfrak{t}^2\mathfrak{d} + 2(\mathfrak{t} - T_2^2)T_2^2\mathfrak{t}^4\}x_1'x_2' \\
& - \frac{1}{3}\{^D\Gamma_{12}^1(\mathfrak{t} - 2T_2^2) + 2T_1^2{}^D\Gamma_{22}^1\}(\mathfrak{t}^2 - 4\mathfrak{d})\mathfrak{t}x_1' \\
& - \frac{1}{3}\{^D\Gamma_{12}^2(\mathfrak{t} - 2T_2^2) + 2T_1^2{}^D\Gamma_{22}^2\}(\mathfrak{t}^2 - 4\mathfrak{d})\mathfrak{t}x_2' \\
& - \frac{1}{6}\{(11\mathfrak{t} - 14T_2^2)\mathfrak{d} - 2(\mathfrak{t} - T_2^2)\mathfrak{t}^2\}T_2^1\Phi_{11} \\
& + \frac{1}{6}\{4\mathfrak{d}^2 + (6\mathfrak{t}^2 - 28T_2^2\mathfrak{t} + 28(T_2^2)^2)\mathfrak{d} - (\mathfrak{t} - 2T_2^2)^2\mathfrak{t}^2\}\Phi_{12} \\
& + \frac{1}{6}\{(3\mathfrak{t} - 14T_2^2)\mathfrak{d} + 2T_2^2\mathfrak{t}^2\}T_1^1\Phi_{22} \\
& - \frac{1}{3}\{(\partial_2{}^D\Gamma_{11}^1 - \partial_1{}^D\Gamma_{12}^1 - \partial_2{}^D\Gamma_{12}^2 + \partial_1{}^D\Gamma_{22}^2)(\mathfrak{t}^2 - 4\mathfrak{d})\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{B}_{22} = & -\frac{1}{6}\{30\mathfrak{d}^2 - \mathfrak{t}^4 + \mathfrak{t}^2\mathfrak{d}\}(T_2^1)^2x_1^2 \\
& - \frac{1}{6}\{10\mathfrak{d}^3 + 2(\mathfrak{t}^2 - 17T_2^2\mathfrak{t} + 15(T_2^2)^2)\mathfrak{d}^2 + (4\mathfrak{t} + T_2^2)T_2^2\mathfrak{t}^2\mathfrak{d} - (T_2^2)^2\mathfrak{t}^4\}x_2^2 \\
& + \frac{1}{3}\{(17\mathfrak{t} - 30T_2^2)\mathfrak{d}^2 - (2\mathfrak{t} + T_2^2)\mathfrak{t}^2\mathfrak{d} + T_2^2\mathfrak{t}^4\}T_2^1x_1'x_2' \\
& - \frac{1}{3}\{^D\Gamma_{22}^1(\mathfrak{t} - 2T_2^2) + 2T_2^1{}^D\Gamma_{22}^2\}(\mathfrak{t}^2 - 4\mathfrak{d})\mathfrak{t}x_1' \\
& + \frac{1}{3}\{^D\Gamma_{22}^2(\mathfrak{t} - 2T_2^2) - 2T_2^1{}^D\Gamma_{22}^1\}(\mathfrak{t}^2 - 4\mathfrak{d})\mathfrak{t}x_2' \\
& - \frac{1}{3}(7\mathfrak{d} - \mathfrak{t}^2)(T_2^1)^2\Phi_{11} \\
& + \frac{1}{3}\{(3\mathfrak{t} - 14T_2^2)T_2^1\mathfrak{d} + 2T_2^1T_2^2\mathfrak{t}^2\}\Phi_{12} \\
& - \frac{1}{6}\{10\mathfrak{d}^2 - (5\mathfrak{t}^2 + 6T_2^2\mathfrak{t} - 14(T_2^2)^2)\mathfrak{d} + \mathfrak{t}^4 - 2(T_2^2)^2\mathfrak{t}^2\}\Phi_{22} \\
& - \frac{2}{3}(\partial_2{}^D\Gamma_{12}^1 - \partial_1{}^D\Gamma_{22}^1)(\mathfrak{t}^2 - 4\mathfrak{d}).
\end{aligned}$$

Suppose first that the Bach tensor of $(T^*\Sigma, g_{D_{\Phi,T}})$ vanishes. We start analyzing the case $T_2^1 = 0$. In this case, the expression of \mathfrak{B} in (8) reduces to

$$(9) \quad \tilde{\mathfrak{B}} = \frac{1}{6}(T_1^1 - T_2^2)^2 \cdot (T_1^1 + T_2^2) \cdot \begin{pmatrix} T_1^1 - T_2^2 & 2T_1^2 \\ 0 & T_2^2 - T_1^1 \end{pmatrix}.$$

If $T_2^2 = T_1^1$, we differentiate the component \mathfrak{B}_{11} in (8) twice with respect to x_2' to obtain $T_1^2T_1^1 = 0$. Thus, either $T_1^2 = 0$ and T is a multiple of the identity, or $T_1^1 = 0$ and, in such a case, T is determined by $T\partial_1 = T_1^2\partial_2$ and therefore it is nilpotent. If $T_2^2 \neq T_1^1$, then (9) implies that $T_2^2 = -T_1^1$. In this case, we differentiate

the component \mathfrak{B}_{22} in (8) twice with respect to $x_{2'}$ and obtain $T_1^1 = 0$. Thus, as before, T is nilpotent.

Next we analyze the case $T_2^1 \neq 0$. We use (7) to express

$$\begin{aligned} D\Gamma_{11}^1 &= \frac{T_1^1 - T_2^2}{T_2^1} D\Gamma_{12}^1 + \frac{T_1^2}{T_2^1} D\Gamma_{22}^1, & D\Gamma_{11}^2 &= \frac{T_1^2}{T_2^1} D\Gamma_{12}^1, \\ D\Gamma_{12}^2 &= \frac{T_1^2}{T_2^1} D\Gamma_{22}^1, & D\Gamma_{22}^2 &= D\Gamma_{12}^1 - \frac{T_1^1 - T_2^2}{T_2^1} D\Gamma_{22}^1. \end{aligned}$$

Considering the component $\tilde{\mathfrak{B}}_{11}$ in (8),

$$\tilde{\mathfrak{B}}_{11} = \frac{1}{6} (T_1^1 - T_2^2) \cdot (T_1^1 + T_2^2) \cdot ((T_1^1 - T_2^2)^2 + 4T_2^1 T_1^2),$$

we analyze separately the vanishing of each one of the three factors in $\tilde{\mathfrak{B}}_{11}$.

Assume that $T_2^2 = T_1^1$. In this case, component $\tilde{\mathfrak{B}}_{12}$ in (8) reduces to $\tilde{\mathfrak{B}}_{12} = \frac{8}{3} T_2^1 (T_1^1)^2 T_1^1$; since we are assuming that $T_2^1 \neq 0$, then either $T_1^2 = 0$ or $T_1^1 \neq 0$ and $T_1^1 = 0$. If $T_1^1 = 0$, the only nonzero component of the Bach tensor is given by

$$\mathfrak{B}_{22} = -(T_2^1)^2 (T_1^1)^2 (3(T_1^1)^2 x_{1'}^2 + \Phi_{11}),$$

from where it follows that $T_1^1 = 0$ and hence T is determined by $T\partial_2 = T_2^1\partial_1$ and is nilpotent. If $T_1^2 \neq 0$ and $T_1^1 = 0$, then we differentiate the component \mathfrak{B}_{12} in (8) with respect to $x_{1'}$ and $x_{2'}$ to get $T_2^1 T_1^2 = 0$, which is not possible since both T_2^1 and T_1^2 are non-null.

Suppose now that $T_2^2 = -T_1^1$. In this case, we differentiate the component \mathfrak{B}_{22} in (8) twice with respect to $x_{1'}$ and as a consequence we obtain $T_2^1 (T_2^1 T_1^1 + (T_1^1)^2) = 0$; since we are assuming $T_2^1 \neq 0$, it follows that $T_1^2 = -(T_1^1)^2 / T_2^1$. Thus, the $(1, 1)$ -tensor field T is given by $T\partial_1 = T_1^1\partial_1 - (T_1^1)^2 / T_2^1 \partial_2$ and $T\partial_2 = T_2^1\partial_1 - T_1^1\partial_2$, and therefore it is nilpotent as well.

Finally, suppose that $(T_1^1 - T_2^2)^2 + 4T_2^1 T_1^2 = 0$; since $T_2^1 \neq 0$, this is equivalent to $T_1^2 = -(T_1^1 - T_2^2)^2 / (4T_2^1)$. Now, we differentiate the component \mathfrak{B}_{22} in (8) twice with respect to $x_{1'}$ to obtain $T_2^1 (T_1^1 + T_2^2) = 0$. Thus, we have $T_2^2 = -T_1^1$ and T is given by $T\partial_1 = T_1^1\partial_1 - (T_1^1)^2 / T_2^1 \partial_2$ and $T\partial_2 = T_2^1\partial_1 - T_1^1\partial_2$, which again implies that T is nilpotent.

To conclude the proof we show the “only if” part. If T is a multiple of the identity, then $(T^*\Sigma, g_{D,\Phi,T})$ is self-dual by Theorem 2.1 and therefore it has vanishing Bach tensor. Thus, we suppose T is parallel and nilpotent and, in this case, we can choose a system of coordinates (x^1, x^2) such that T is determined by $T\partial_1 = \partial_2$ and $T\partial_2 = 0$. Hence, examining (8), clearly $\tilde{\mathfrak{B}} = 0$ and, since $\mathfrak{d} = \mathfrak{t} = 0$, one easily checks that $\mathfrak{B}_{11} = \mathfrak{B}_{12} = \mathfrak{B}_{22} = 0$, showing that the Bach tensor of $(T^*\Sigma, g_{D,\Phi,T})$ vanishes. \square

Remark 3.2. We emphasize that even though the Bach tensor of the metrics $g_{D,\Phi,T}$ depends on the choice of Φ (as shown in the proof of Theorem 3.1), the existence of Bach-flat metrics in Theorem 3.1 is independent of the symmetric $(0, 2)$ -tensor field Φ , thus providing an infinite family of examples for each initial data (Σ, D, T) . Moreover, note that the metrics $g_{D,\Phi,T}$ are generically nonisometric for different deformation tensor fields Φ .

The Bach-flat modified Riemannian extensions in Theorem 3.1 obtained from a $(1, 1)$ -tensor field of the form $T = c \text{Id}$ are not of interest for our purposes since they all are half conformally flat (see Theorem 2.1). Hence, in what follows we focus on the case when T is a parallel nilpotent $(1, 1)$ -tensor field and refer to $g_{D,\Phi,T}$ as a *nilpotent Riemannian extension*.

Affine connections supporting parallel nilpotent tensors. It is shown in the proof of Theorem 3.1 that the existence of a parallel nilpotent tensor field T on a torsion-free affine surface (Σ, D) imposes some restrictions on D .

Proposition 3.3. Let (Σ, D, T) be a torsion-free affine surface equipped with a nilpotent $(1, 1)$ -tensor field T . If T is parallel, then:

- (i) $\ker T$ is a parallel one-dimensional distribution whose integral curves are geodesics of (Σ, D) .
- (ii) The symmetric part of the Ricci tensor, ρ_{sym}^D , is zero or of rank one and recurrent, i.e.,

$$D\rho_{\text{sym}}^D = \eta \otimes \rho_{\text{sym}}^D,$$

for some one-form η .

Proof. Let (Σ, D) be a torsion-free affine surface admitting a parallel nilpotent $(1, 1)$ -tensor field T . Then there exist suitable coordinates (x^1, x^2) where $T\partial_1 = \partial_2$, $T\partial_2 = 0$ and it follows from (7) that the Christoffel symbols of D satisfy

$$(10) \quad {}^D\Gamma_{12}^1 = 0, \quad {}^D\Gamma_{12}^2 = {}^D\Gamma_{11}^1, \quad {}^D\Gamma_{22}^1 = 0, \quad {}^D\Gamma_{22}^2 = 0.$$

In such a case the one-dimensional distribution $\ker T (= \text{span}\{\partial_2\})$ is parallel and ∂_2 is a geodesic vector field, thus showing (i). Moreover, the Ricci tensor of any affine connection given by (10) satisfies

$$\rho^D = \begin{pmatrix} \partial_2 {}^D\Gamma_{11}^2 - \partial_1 {}^D\Gamma_{11}^1 & \partial_2 {}^D\Gamma_{11}^1 \\ -\partial_2 {}^D\Gamma_{11}^1 & 0 \end{pmatrix},$$

from where it follows that the symmetric and the skew-symmetric parts of the Ricci tensor are given by

$$\rho_{\text{sym}}^D = (\partial_2 {}^D\Gamma_{11}^2 - \partial_1 {}^D\Gamma_{11}^1) dx^1 \circ dx^1, \quad \rho_{\text{sk}}^D = \partial_2 {}^D\Gamma_{11}^1 dx^1 \wedge dx^2.$$

Hence ρ_{sym}^D is either zero or of rank one. Moreover, a straightforward calculation of the covariant derivative of the symmetric part of the Ricci tensor gives

$$\begin{aligned} (D_{\partial_1} \rho_{\text{sym}}^D)(\partial_1, \partial_1) &= \partial_{12} {}^D\Gamma_{11}^2 - \partial_{11} {}^D\Gamma_{11}^1 - 2 {}^D\Gamma_{11}^1 (\partial_2 {}^D\Gamma_{11}^2 - \partial_1 {}^D\Gamma_{11}^1), \\ (D_{\partial_2} \rho_{\text{sym}}^D)(\partial_1, \partial_1) &= \partial_{22} {}^D\Gamma_{11}^2 - \partial_{12} {}^D\Gamma_{11}^1, \end{aligned}$$

with the other components being zero. This shows that ρ_{sym}^D is recurrent, i.e., $D\rho_{\text{sym}}^D = \eta \otimes \rho_{\text{sym}}^D$, with recurrence one-form

$$(11) \quad \eta = \left\{ \partial_1 \ln \rho_{\text{sym}}^D(\partial_1, \partial_1) - 2 {}^D\Gamma_{11}^1 \right\} dx^1 + \partial_2 \ln \rho_{\text{sym}}^D(\partial_1, \partial_1) dx^2,$$

which proves (ii). \square

Remark 3.4. It follows from the expression of ρ_{sk}^D in the proof of Proposition 3.3 that any connection given by (10) has symmetric Ricci tensor if and only if $\partial_2 {}^D\Gamma_{11}^1 = 0$, in which case ρ^D is recurrent. Now, it follows from the work of Wong [1964] that any such connection can be described in suitable coordinates (\bar{u}^1, \bar{u}^2) by

$$D_{\partial_{\bar{u}^1}} \partial_{\bar{u}^1} = \bar{u} \Gamma_{11}^2 (\bar{u}^1, \bar{u}^2) \partial_{\bar{u}^2},$$

where $\bar{u} \Gamma_{11}^2 (\bar{u}^1, \bar{u}^2)$ is an arbitrary function satisfying $\partial_{\bar{u}^2} \bar{u} \Gamma_{11}^2 (\bar{u}^1, \bar{u}^2) \neq 0$. Further, the only nonzero component of the Ricci tensor is $\rho^D(\partial_{\bar{u}^1}, \partial_{\bar{u}^1}) = \partial_{\bar{u}^2} \bar{u} \Gamma_{11}^2$, and the recurrence one-form ω is given by

$$(12) \quad \omega = \partial_{\bar{u}^1} (\ln \partial_{\bar{u}^2} \bar{u} \Gamma_{11}^2) d\bar{u}^1 + \partial_{\bar{u}^2} (\ln \partial_{\bar{u}^2} \bar{u} \Gamma_{11}^2) d\bar{u}^2.$$

Further assume that T is a parallel nilpotent $(1, 1)$ -tensor field on (Σ, D) . Then a straightforward calculation shows that its expression in the coordinates (\bar{u}^1, \bar{u}^2) is given by $T \partial_{\bar{u}^1} = T_1^2 \partial_{\bar{u}^2}$ and $T \partial_{\bar{u}^2} = 0$, for some $T_1^2 \in \mathbb{R}$, $T_1^2 \neq 0$. Hence, considering the modified coordinates $(u^1, u^2) = (\bar{u}^1, (T_1^2)^{-1} \bar{u}^2)$ one has that $T \partial_{u^1} = \partial_{u^2}$ and $T \partial_{u^2} = 0$, and the connection is determined by the only nonzero Christoffel symbol ${}^u\Gamma_{11}^2$. Moreover, it follows from the expression of the recurrence one-form ω that $\omega(\ker T) = 0$ if and only if $\partial_{22} {}^u\Gamma_{11}^2 = 0$.

4. Bach-flat gradient Ricci solitons

Let Φ be a symmetric $(0, 2)$ -tensor field on (Σ, D, T) . One uses the nilpotent structure T to construct an associated symmetric $(0, 2)$ -tensor field $\widehat{\Phi}$ given by $\widehat{\Phi}(X, Y) = \Phi(TX, TY)$, for all vector fields X, Y on Σ . Further, let (x^1, x^2) be local coordinates where $T \partial_1 = \partial_2$, $T \partial_2 = 0$ and let $\Phi = \Phi_{ij} dx^i \otimes dx^j$. Then $\widehat{\Phi}$ expresses as $\widehat{\Phi} = \widehat{\Phi}_{ij} dx^i \otimes dx^j = \Phi_{22} dx^1 \otimes dx^1$.

Einstein nilpotent Riemannian extensions.

Theorem 4.1. Let (Σ, D, T) be an affine surface equipped with a parallel nilpotent $(1, 1)$ -tensor field T and let Φ be a symmetric $(0, 2)$ -tensor field on Σ . Then $(T^*\Sigma, g_{D,\Phi,T})$ is Einstein (indeed, Ricci-flat) if and only if $\widehat{\Phi} = -2\rho_{\text{sym}}^D$.

Proof. Let (x^1, x^2) be local coordinates on Σ so that $T\partial_1 = \partial_2$, $T\partial_2 = 0$, and consider the induced coordinates $(x^1, x^2, x_{1'}, x_{2'})$ on $T^*\Sigma$. A straightforward calculation shows that the Ricci tensor of any nilpotent Riemannian extension $g_{D,\Phi,T}$ is determined by

$$\rho(\partial_1, \partial_1) = \Phi(\partial_2, \partial_2) + 2\rho_{\text{sym}}^D(\partial_1, \partial_1),$$

the other components being zero. Hence the Ricci operator is nilpotent and $g_{D,\Phi,T}$ has zero scalar curvature. Moreover, the Ricci tensor vanishes if and only if $\Phi(\partial_2, \partial_2) + 2\rho_{\text{sym}}^D(\partial_1, \partial_1) = 0$. The result now follows. \square

Remark 4.2. The Weyl tensor of a pseudo-Riemannian manifold is harmonic if and only if the Cotton tensor vanishes. Let (Σ, D, T) be an affine surface equipped with a parallel nilpotent $(1, 1)$ -tensor field T and let Φ be a symmetric $(0, 2)$ -tensor field on Σ . Let (x^1, x^2) be local coordinates on Σ so that $T\partial_1 = \partial_2$, $T\partial_2 = 0$, and consider the induced coordinates $(x^1, x^2, x_{1'}, x_{2'})$ on $T^*\Sigma$. A straightforward calculation shows that the Cotton tensor of $(T^*\Sigma, g_{D,\Phi,T})$ is given by

$$\mathfrak{C}(\partial_1, \partial_2, \partial_1) = -\{\partial_2 \Phi(\partial_2, \partial_2) + 2\partial_2 \rho_{\text{sym}}^D(\partial_1, \partial_1)\},$$

the other components being zero. Hence $(T^*\Sigma, g_{D,\Phi,T})$ has harmonic Weyl tensor if and only if $\widehat{D}\widehat{\Phi} = -2\widehat{\eta} \otimes \rho_{\text{sym}}^D$, where $\widehat{\eta}(X) = \eta(TX)$, η being the recurrence one-form given in (11), and $\widehat{D}\widehat{\Phi}(X, Y; Z) = D\Phi(TX, TY; TZ)$.

Gradient Ricci solitons on nilpotent Riemannian extensions. From Theorem 2.2, recall that the affine gradient Ricci soliton equation $\text{Hes}_h^D + 2\rho_{\text{sym}}^D = 0$ determines the potential function of any self-dual gradient Ricci soliton which is not locally conformally flat, independently of the deformation tensor Φ . The next theorem shows that, in contrast with the previous situation, for any $h \in C^\infty(\Sigma)$ with $dh(\ker T) = 0$, one may use the symmetric $(0, 2)$ -tensor field $\text{Hes}_h^D + 2\rho_{\text{sym}}^D$ to determine a deformation tensor field Φ so that the resulting nilpotent Riemannian extension is a Bach-flat steady gradient Ricci soliton with potential function $f = h \circ \pi$.

Theorem 4.3. Let (Σ, D, T) be an affine surface equipped with a parallel nilpotent $(1, 1)$ -tensor field T and let Φ be a symmetric $(0, 2)$ -tensor field on Σ . Let $h \in C^\infty(\Sigma)$ be a smooth function. Then $(T^*\Sigma, g_{D,\Phi,T}, f = h \circ \pi)$ is a Bach-flat gradient Ricci soliton if and only if $dh(\ker T) = 0$ and

$$(13) \quad \widehat{\Phi} = -\text{Hes}_h^D - 2\rho_{\text{sym}}^D.$$

Moreover the soliton is steady and isotropic.

Proof. Let (x^1, x^2) be local coordinates on Σ so that $T\partial_1 = \partial_2$, $T\partial_2 = 0$, and consider the induced coordinates $(x^1, x^2, x_{1'}, x_{2'})$ on $T^*\Sigma$. Setting $f = h \circ \pi$, one has that $\text{Hes}_f(\partial_1, \partial_{1'}) + \rho(\partial_1, \partial_{1'}) = \lambda g(\partial_1, \partial_{1'})$ leads to $\lambda = 0$, which shows that the soliton is steady. A straightforward calculation shows that the remaining nonzero terms in the gradient Ricci soliton equation are given by

$$\text{Hes}_f(\partial_2, \partial_2) + \rho(\partial_2, \partial_2) = \partial_{22}h,$$

$$\text{Hes}_f(\partial_1, \partial_2) + \rho(\partial_1, \partial_2) = \partial_{12}h - {}^D\Gamma_{11}^1 \partial_2 h,$$

$$\begin{aligned} \text{Hes}_f(\partial_1, \partial_1) + \rho(\partial_1, \partial_1) &= x_{2'} \partial_2 h - {}^D\Gamma_{11}^2 \partial_2 h + \partial_{11}h - {}^D\Gamma_{11}^1 \partial_1 h \\ &\quad + \Phi_{22} + 2\partial_2 {}^D\Gamma_{11}^2 - 2\partial_1 {}^D\Gamma_{11}^1. \end{aligned}$$

It immediately follows from the equation $(\text{Hes}_f + \rho)(\partial_1, \partial_1) = 0$ that $\partial_2 h = 0$, which shows that $dh(\ker T) = 0$. The only remaining equation now becomes

$$\begin{aligned} \text{Hes}_f(\partial_1, \partial_1) + \rho(\partial_1, \partial_1) &= \partial_{11}h - {}^D\Gamma_{11}^1 \partial_1 h + \Phi_{22} + 2\partial_2 {}^D\Gamma_{11}^2 - 2\partial_1 {}^D\Gamma_{11}^1 \\ &= \Phi(\partial_2, \partial_2) + \text{Hes}_h^D(\partial_1, \partial_1) + 2\rho_{\text{sym}}^D(\partial_1, \partial_1), \end{aligned}$$

from which (13) follows. Moreover, it also follows from the form of the potential function that $\nabla f = h'(x^1)\partial_{1'}$, and thus $\|\nabla f\|^2 = 0$ (equivalently the level hypersurfaces of the potential function are degenerate submanifolds of $T^*\Sigma$), which shows that the soliton is isotropic. \square

Remark 4.4. The potential functions of the gradient Ricci solitons in Theorem 4.3 are of the form $f = h \circ \pi$ for some $h \in C^\infty(\Sigma)$. Next we show that this is indeed the case if the Ricci tensor of (Σ, D) is nonsymmetric.

Let $(T^*\Sigma, g_{D, \Phi, T}, f)$ be a gradient Ricci soliton with potential function $f \in C^\infty(T^*\Sigma)$. Take local coordinates $(x^1, x^2, x_{1'}, x_{2'})$ on $T^*\Sigma$ as in the proof of Theorem 4.3. Since $\text{Hes}_f(\partial_{i'}, \partial_{j'}) = \partial_{i'j'} f(x^1, x^2, x_{1'}, x_{2'})$, it follows from the expression of the Ricci tensor in Theorem 4.1 and the metric tensor (5), that the potential function is determined by $f = \iota X + h \circ \pi$, for some $h \in C^\infty(\Sigma)$ and some vector field X on Σ , where ιX is the evaluation map acting on X .

Further set $X = A(x^1, x^2)\partial_1 + B(x^1, x^2)\partial_2$ in the local coordinates (x^1, x^2) above, for some $A, B \in C^\infty(\Sigma)$. Then $\text{Hes}_f(\partial_2, \partial_{1'}) = \partial_2 A(x^1, x^2)$, from where it follows that $X = A(x^1)\partial_1 + B(x^1, x^2)\partial_2$. Considering the component $\text{Hes}_f(\partial_2, \partial_{2'}) = -A''(x^1) + \partial_2 B(x^1, x^2)$, one has that $X = A(x^1)\partial_1 + (P(x^1) + x^2 A'(x^1))\partial_2$ for some smooth function $P(x^1)$. Next the component

$$\begin{aligned} \text{Hes}_f(\partial_1, \partial_{2'}) &= A(x^1) {}^D\Gamma_{11}^2 - x_{2'} A(x^1) \\ &\quad + {}^D\Gamma_{11}^1 (P(x^1) + x^2 A'(x^1)) + P'(x^1) + x^2 A''(x^1) \end{aligned}$$

shows that $A = 0$ and it reduces to $\text{Hes}_f(\partial_1, \partial_{2'}) = P'(x^1) + P(x^1) {}^D\Gamma_{11}^1$. A solution $P(x^1)$ of the equation $P'(x^1) + P(x^1) {}^D\Gamma_{11}^1 = 0$ either vanishes identically (and

hence $X = 0$) or it is nowhere zero, in which case $\partial_2 {}^D\Gamma_{11}^1 = 0$ (see the proof of Theorem 6.1). In the latter case Proposition 3.3 shows that the Ricci tensor of (Σ, D) is symmetric and thus recurrent of rank one. Therefore Theorem 4.3 describes all possible gradient Ricci solitons on $(T^*\Sigma, g_{D,\Phi,T})$ whenever ρ_{sk}^D is nonzero.

Remark 4.5. The tensor field $\mathbb{D}_{ijk} = \mathfrak{C}_{ijk} + W_{ijk\ell}\nabla_\ell f$ introduced in [Cao and Chen 2013] plays an essential role in analyzing the geometry of Bach-flat gradient Ricci solitons. Local conformal flatness in [Cao et al. 2014; Cao and Chen 2013] follows from $\mathbb{D} = 0$, which is obtained under some natural assumptions.

Gradient Ricci solitons in Theorem 4.3 satisfy $\nabla f = h'(x^1)\partial_1$. Then, a straightforward calculation shows that \mathbb{D} is completely determined by

$$\mathbb{D}_{121} = -2h'(x^1)\partial_2 {}^D\Gamma_{11}^1(x^1, x^2),$$

the other components being zero. Hence it follows from the proof of Proposition 3.3 that the tensor field \mathbb{D} vanishes if and only if the Ricci tensor ρ^D is symmetric. However Theorem 5.1 shows that $(T^*\Sigma, g_{D,\Phi,T})$ is never locally conformally flat.

5. Half conformally flat nilpotent Riemannian extensions

The existence of a null distribution \mathcal{D} on a four-dimensional manifold (M, g) of neutral signature defines a natural orientation on M : the one which, for any basis u, v of \mathcal{D} , makes the bivector $u \wedge v$ self-dual (see [Derdzinski 2008]). We consider on $T^*\Sigma$ the orientation which agrees with $\mathcal{D} = \ker \pi_*$, and thus self-duality and anti-self-duality are not interchangeable. The following result shows that they are essentially different for nilpotent Riemannian extensions.

Theorem 5.1. Let (Σ, D, T) be an affine surface equipped with a parallel nilpotent $(1, 1)$ -tensor field T . Then

- (i) $(T^*\Sigma, g_{D,\Phi,T})$ is never self-dual for any deformation tensor field Φ .
- (ii) If $(T^*\Sigma, g_{D,\Phi,T})$ is anti-self-dual, then D is either a flat connection or (Σ, D) is recurrent with symmetric Ricci tensor of rank one.

In the later case there exist local coordinates (u^1, u^2) where the only nonzero Christoffel symbol is ${}^u\Gamma_{11}^2$ and the tensor field T is given by $T\partial_{u^1} = \partial_{u^2}$, $T\partial_{u^2} = 0$. Moreover, $(T^*\Sigma, g_{D,\Phi,T})$ is anti-self-dual if and only if the symmetric $(0, 2)$ -tensor field Φ satisfies the equations

$$\begin{aligned} \widehat{D}\widehat{\Phi} &= -2\widehat{\omega} \otimes \rho^D, \\ (14) \quad 0 &= \frac{1}{2}\widehat{\Phi} \otimes \widehat{\Phi}(\partial_1, \partial_1, \partial_1, \partial_1) + 2(\widehat{\Phi} \otimes \rho^D)(\partial_1, \partial_1, \partial_1, \partial_1) \\ &\quad + D^2\Phi(\partial_1, \partial_1; T\partial_1, T\partial_1) + D^2\Phi(T\partial_1, T\partial_1; \partial_1, \partial_1) \\ &\quad - 2D^2\Phi(\partial_1, T\partial_1; T\partial_1, \partial_1), \end{aligned}$$

where

$$\widehat{D\Phi}(X, Y, Z) = D\Phi(TX, TY; TZ),$$

ω is the recurrence one-form given by $D\rho^D = \omega \otimes \rho^D$, and $\widehat{\omega}(X) = \omega(TX)$.

Proof. A direct computation using the expression of the anti-self-dual curvature operator of any four-dimensional Walker metric obtained in [Díaz-Ramos et al. 2006] shows that, for any nilpotent Riemannian extension $g_{D,\Phi,T}$, W^- takes the form

$$(15) \quad W^- = \frac{1}{2} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

thus showing that the anti-self-dual Weyl curvature operator W^- is nilpotent and hence $(T^*\Sigma, g_{D,\Phi,T})$ is never self-dual, which proves (i).

Next we show (ii). Let (M, g) be a four-dimensional Walker metric (3) and set the metric components $g_{11} = a$, $g_{12} = c$ and $g_{22} = b$, where g_{ij} are functions of the Walker coordinates $(x^1, x^2, x_{1'}, x_{2'})$. Then the self-dual Weyl curvature operator takes the form (see [Díaz-Ramos et al. 2006])

$$(16) \quad W^+ = \begin{pmatrix} W_{11}^+ & W_{12}^+ & W_{11}^+ + \frac{\tau}{12} \\ -W_{12}^+ & \frac{\tau}{6} & -W_{12}^+ \\ -W_{11}^+ - \frac{\tau}{12} & -W_{12}^+ & -W_{11}^+ - \frac{\tau}{6} \end{pmatrix},$$

where

$$(17) \quad W_{11}^+ = \frac{1}{12} (6ca_1b_2 - 6a_1b_{1'} - 6ba_1c_2 + 12a_1c_{2'} - 6ca_2b_1 + 6a_2b_{2'} \\ + 6ba_2c_1 + 6a_{1'}b_1 - 6a_{2'}b_2 - 12a_{2'}c_1 + 6ab_1c_2 - 6ab_2c_1 \\ + 12b_2c_{1'} - 12b_{1'}c_2 - a_{11} - 12c^2a_{11} - 12bca_{12} + 24ca_{12'} \\ - 3b^2a_{22} + 12ba_{22'} - 12a_{2'}c_{2'} - 3a^2b_{11} + 12ab_{11'} - b_{22} \\ - 12b_{1'1'} + 12acc_{11} - 2c_{12} + 6abc_{12} - 24cc_{11'} - 12ac_{12'} \\ - 12bc_{21'} + 24c_{1'2'}),$$

and

$$(18) \quad W_{12}^+ = \frac{1}{4} (-2ca_{11} - ba_{12} + 2a_{12'} + ab_{12} - 2b_{21'} + ac_{11} - 2cc_{12} \\ - 2c_{11'} - bc_{22} + 2c_{22'}).$$

Since any anti-self-dual metric is Bach-flat, we proceed as in the proof of Theorem 3.1 considering local coordinates (x^1, x^2) on the surface Σ such that T is determined by $T\partial_1 = \partial_2$ and $T\partial_2 = 0$. Since T is parallel, the Christoffel

symbols must satisfy (10), i.e.,

$${}^D\Gamma_{12}^1 = 0, \quad {}^D\Gamma_{12}^2 = {}^D\Gamma_{11}^1, \quad {}^D\Gamma_{22}^1 = 0, \quad {}^D\Gamma_{22}^2 = 0.$$

Next, we analyze the self-dual Weyl curvature operator, which is completely determined by the scalar curvature and its components W_{11}^+ and W_{12}^+ already described in equations (17) and (18). The scalar curvature is zero by Theorem 4.1, and $W_{12}^+ = -2\partial_2 {}^D\Gamma_{11}^1$, from where it follows that the Ricci tensor ρ^D is symmetric of rank one and recurrent (see Remark 3.4). Take local coordinates (u^1, u^2) as in Remark 3.4 so that the only nonzero Christoffel symbol is ${}^u\Gamma_{11}^2$ and $T\partial_{u^1} = \partial_{u^2}$, $T\partial_{u^2} = 0$. Finally, we compute the component W_{11}^+ given by (17) in the coordinates $(u^1, u^2, u_{1'}, u_{2'})$ of $T^*\Sigma$, obtaining

$$\begin{aligned} W_{11}^+ = & (\partial_2 \Phi_{22} + 2\partial_{22} {}^u\Gamma_{11}^2) u_{2'} - \frac{1}{2}(\Phi_{22})^2 - 2\Phi_{22}\partial_2 {}^u\Gamma_{11}^2 - \partial_2 \Phi_{22} {}^u\Gamma_{11}^2 \\ & + 2\partial_{12}\Phi_{12} - \partial_{22}\Phi_{11} - \partial_{11}\Phi_{22}. \end{aligned}$$

Thus $(T^*\Sigma, g_{D,\Phi,T})$ is anti-self-dual if and only if

$$\begin{aligned} \partial_2 \Phi_{22} + 2\partial_{22} {}^u\Gamma_{11}^2 &= 0, \\ \frac{1}{2}(\Phi_{22})^2 + 2\Phi_{22}\partial_2 {}^u\Gamma_{11}^2 + \partial_2 \Phi_{22} {}^u\Gamma_{11}^2 &= 2\partial_{12}\Phi_{12} - \partial_{22}\Phi_{11} - \partial_{11}\Phi_{22}, \end{aligned}$$

from where (14) follows. \square

Anti-self-dual gradient Ricci solitons. Self-dual gradient Ricci solitons which are not locally conformally flat are described in Theorem 2.2. In contrast, no explicit examples of strictly anti-self-dual gradient Ricci solitons were previously reported. In this section we use nilpotent Riemannian extensions to construct anti-self-dual isotropic gradient Ricci solitons. In this case, Theorem 5.1 shows that (Σ, D) must have symmetric Ricci tensor.

Proposition 5.2. Let (Σ, D, T, Φ) be an affine surface with symmetric Ricci tensor equipped with a parallel nilpotent $(1, 1)$ -tensor field T and a parallel symmetric $(0, 2)$ -tensor field Φ . Then $(T^*\Sigma, g_{D,\Phi,T})$ is anti-self-dual if and only if $\widehat{\omega} = 0$ and $\widehat{\Phi} = 0$, where ω is the recurrence one-form given by (12).

Proof. If the Ricci tensor ρ^D is symmetric of rank one and Φ is parallel, then the equations in Theorem 5.1 reduce to $\widehat{\omega} = 0$ and $\widehat{\Phi} = 0$, which proves the result. If (Σ, D) is a flat surface then a straightforward calculation shows that anti-self-duality is equivalent to $\widehat{\Phi} = 0$, being Φ a parallel tensor. \square

Since the deformation tensor Φ of any gradient Ricci soliton in Theorem 4.3 must satisfy $\widehat{\Phi} = -\text{Hes}_h^D - 2\rho_{\text{sym}}^D$, the condition $\widehat{\Phi} = 0$ in the previous proposition restricts the consideration of Ricci solitons on $(T^*\Sigma, g_{D,\Phi,T})$ to those originated by affine gradient Ricci solitons on (Σ, D) .

Proposition 5.3. Let (Σ, D, T) be an affine surface equipped with a parallel nilpotent $(1, 1)$ -tensor field T and let $h \in C^\infty(\Sigma)$. Then:

- (i) (Σ, D, T, h) is an affine gradient Ricci soliton with $dh(\ker T) = 0$ if and only if $(T^*\Sigma, g_{D, \widehat{\Phi}, T}, f = h \circ \pi)$ is a Bach-flat steady gradient Ricci soliton for any symmetric $(0, 2)$ -tensor field Φ .
- (ii) (Σ, D, T, h) is a nonflat affine gradient Ricci soliton with $dh(\ker T) = 0$ if and only if the recurrence one-form η in (11) satisfies $\widehat{\eta} = 0$.

Proof. Since T is nilpotent, $\widehat{\Phi}(TX, TY) = 0$ for any $(0, 2)$ -tensor field Φ . Hence (13) shows that $(T^*\Sigma, g_{D, \widehat{\Phi}, T}, f = h \circ \pi)$ is a gradient Ricci soliton if and only if (Σ, D, T, h) is an affine gradient Ricci soliton with $dh(\ker T) = 0$, which shows (i).

Next take local coordinates (x^1, x^2) on Σ so that $T\partial_1 = \partial_2$, $T\partial_2 = 0$. Since the Christoffel symbols ${}^D\Gamma_{ij}^k$ are given by (10), using the expression of ρ_{sym}^D in Proposition 3.3, one has $(\text{Hes}_h^D + 2\rho_{\text{sym}}^D)(\partial_2, \partial_2) = \partial_{22}h$. Thus $h(x^1, x^2) = x^2P(x^1) + Q(x^1)$ for some $P, Q \in C^\infty(\Sigma)$. Hence $dh(\ker T) = 0$ holds if and only if $P = 0$. Since $h(x^1, x^2) = Q(x^1)$, one has that $(\text{Hes}_h^D + 2\rho_{\text{sym}}^D)(\partial_1, \partial_2) = 0$, and the only remaining equation is

$$0 = (\text{Hes}_h^D + 2\rho_{\text{sym}}^D)(\partial_1, \partial_1) = Q'' + 2(\partial_2 {}^D\Gamma_{11}^2 - \partial_1 {}^D\Gamma_{11}^1) = Q'' + 2\rho^D(\partial_1, \partial_1).$$

Therefore, the integrability condition becomes $\partial_2\rho^D(\partial_1, \partial_1) = 0$. Hence, it follows from (11) that (Σ, D, T, h) is an affine gradient Ricci soliton with $dh(\ker T) = 0$ if and only if the symmetric part of the Ricci tensor ρ_{sym}^D is recurrent with recurrence one-form η satisfying $\eta(\ker T) = 0$. Assertion (ii) now follows. \square

A direct application of the previous propositions gives the desired examples.

Theorem 5.4. Let (Σ, D, T, Φ) be an affine surface with symmetric Ricci tensor equipped with a parallel nilpotent $(1, 1)$ -tensor field T and a parallel symmetric $(0, 2)$ -tensor field Φ .

- (i) (Σ, D, h) is an affine gradient Ricci soliton with $dh(\ker T) = 0$ if and only if $(T^*\Sigma, g_{D, \widehat{\Phi}, T}, f = h \circ \pi)$ is an anti-self-dual steady gradient Ricci soliton which is not locally conformally flat.
- (ii) (Σ, D, h) is an affine gradient Ricci soliton with $dh(\ker T) = 0$ if and only if there exist local coordinates (u^1, u^2) on Σ so that the only nonzero Christoffel symbol is given by ${}^u\Gamma_{11}^2 = P(u^1) + u^2Q(u^1)$ and the potential function $h(u^1)$ is determined by $h''(u^1) = -2Q(u^1)$, for any $P, Q \in C^\infty(\Sigma)$.

Proof. $(T^*\Sigma, g_{D, \widehat{\Phi}, T}, f = h \circ \pi)$ is a gradient Ricci soliton by Proposition 5.3(i). Anti-self-duality now follows from Proposition 5.2 and Proposition 5.3(ii), showing assertion (i).

Assertion (ii) follows from Proposition 5.3(ii) and the expression of the recurrence form ω in (12). Take local coordinates (u^1, u^2) on Σ as in the proof of

Proposition 5.3(ii). Then it follows from (12) that $\widehat{\omega} = 0$ if and only if $\partial_{22}{}^u \Gamma_{11}^2 = 0$. Thus,

$${}^u \Gamma_{11}^2(u^1, u^2) = P(u^1) + u^2 Q(u^1),$$

for some $P, Q \in C^\infty(\Sigma)$ and $h''(u^1) = -2Q(u^1)$. \square

6. Conformally Einstein nilpotent Riemannian extensions

A pseudo-Riemannian manifold (M^n, g) is said to be *(locally) conformally Einstein* if every point $p \in M$ has an open neighborhood \mathcal{U} and a positive smooth function φ defined on \mathcal{U} such that $(\mathcal{U}, \bar{g} = \varphi^{-2}g)$ is Einstein. Brinkmann [1924] showed that a manifold is conformally Einstein if and only if the equation

$$(19) \quad (n-2) \text{Hes}_\varphi + \varphi \rho - \frac{1}{n} \{(n-2)\Delta\varphi + \varphi \tau\} g = 0$$

has a positive solution. Despite its apparent simplicity, the integration of the conformally Einstein equation is surprisingly difficult (see [Kühnel and Rademacher 2008] for more information). It was shown in [Gover and Nagy 2007; Kozameh et al. 1985] that any four-dimensional conformally Einstein manifold satisfies

$$(20) \quad (i) \quad \mathfrak{C} + W(\cdot, \cdot, \cdot, \nabla\sigma) = 0, \quad (ii) \quad \mathfrak{B} = 0,$$

where the conformal metric is given by $\bar{g} = e^{2\sigma} g$.

Conditions (i) and (ii) above are also sufficient to be conformally Einstein if (M, g) is *weakly-generic* (i.e., the Weyl tensor viewed as a map $TM \rightarrow \bigotimes^3 TM$ is injective). Since nilpotent Riemannian extensions are not weakly generic (see the expression of W^- in the proof of Theorem 5.1), we will analyze the conformally Einstein equation (19), seeking solutions on nilpotent Riemannian extensions $(T^*\Sigma, g_{D,\Phi,T})$.

Theorem 6.1. Let (Σ, D, T) be a torsion-free affine surface equipped with a parallel nilpotent $(1, 1)$ -tensor field T . Then any solution of (19) is of the form $\varphi = \iota X + \phi \circ \pi$ for some vector field X on Σ such that $X \in \ker T$ and $\text{tr}(DX) = 0$.

Moreover $(T^*\Sigma, g_{D,\Phi,T})$ is conformally Einstein if and only if one of the following holds:

- (i) The conformally Einstein equation (19) admits a solution $\varphi = \phi \circ \pi$ for some $\phi \in C^\infty(\Sigma)$ with $d\phi(\ker T) = 0$, and the deformation tensor Φ is determined by

$$\phi \widehat{\Phi} + 2(\text{Hes}_\phi^D + \phi \rho_{\text{sym}}^D) = 0.$$

- (ii) The conformally Einstein equation (19) admits a solution $\varphi = \iota X + \phi \circ \pi$ for some $\phi \in C^\infty(\Sigma)$ and some nonzero vector field X on Σ such that $X \in \ker T$ and $\text{tr}(DX) = 0$.

In this case, the Ricci tensor ρ^D is symmetric of rank one and recurrent. Moreover, there are local coordinates (u^1, u^2) on Σ so that

$$\varphi(u^1, u^2, u_{1'}, u_{2'}) = \kappa u_{2'} + \phi(u^1, u^2)$$

is a solution of (19) if and only if

$$d\phi(T\partial_1) = \frac{\kappa}{2}\Phi(T\partial_1, T\partial_1),$$

$$\begin{aligned} \text{Hes}_\phi^D(\partial_1, \partial_1) + \phi \rho^D(\partial_1, \partial_1) &= -\frac{1}{2}(\phi + 2\kappa {}^u\Gamma_{11}^2)\Phi(T\partial_1, T\partial_1) \\ &\quad + \frac{\kappa}{2}\{2(D_{\partial_1}\Phi)(T\partial_1, \partial_1) - (D_{T\partial_1}\Phi)(\partial_1, \partial_1)\}. \end{aligned}$$

Proof. Let (x^1, x^2) be local coordinates on Σ so that $T\partial_1 = \partial_2$, $T\partial_2 = 0$, and consider the induced coordinates $(x^1, x^2, x_{1'}, x_{2'})$ on $T^*\Sigma$. Since T is parallel, we obtain directly from (7) that

$${}^D\Gamma_{12}^1 = 0, \quad {}^D\Gamma_{12}^2 = {}^D\Gamma_{11}^1, \quad {}^D\Gamma_{22}^1 = 0, \quad {}^D\Gamma_{22}^2 = 0.$$

In order to analyze the conformally Einstein equation (19), consider the symmetric $(0, 2)$ -tensor field

$$\mathcal{E} = 2\text{Hes}_\varphi + \varphi \rho - \frac{1}{4}\{2\Delta\varphi + \varphi \tau\}g$$

and set $\mathcal{E} = 0$. Let $\mathcal{E}_{ij} = \mathcal{E}(\partial_i, \partial_j)$ and let $\varphi \in C^\infty(T^*\Sigma)$ be a solution of (19). Then one computes

$$\mathcal{E}_{33} = 2\partial_{1'1'}\varphi, \quad \mathcal{E}_{34} = 2\partial_{1'2'}\varphi, \quad \mathcal{E}_{44} = 2\partial_{2'2'}\varphi,$$

to show that any solution of (19) must be of the form

$$(21) \quad \varphi(x^1, x^2, x_{1'}, x_{2'}) = A(x^1, x^2)x_{1'} + B(x^1, x^2)x_{2'} + \psi(x^1, x^2),$$

for some smooth functions A, B and ψ depending only on the coordinates (x^1, x^2) . This shows that any solution of the conformally Einstein equation on $(T^*\Sigma, g_{D,\Phi,T})$ is of the form

$$\varphi = \iota X + \psi \circ \pi,$$

where ιX is the evaluation of a vector field $X = A\partial_1 + B\partial_2$ on Σ , $\psi \in C^\infty(\Sigma)$ and $\pi : T^*\Sigma \rightarrow \Sigma$ is the projection.

Now, the conformally Einstein condition given in (19) can be expressed in matrix form as follows:

$$(22) \quad (\mathcal{E}_{ij}) = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} & \partial_1 A - \partial_2 B & 2({}^D\Gamma_{11}^2 A + {}^D\Gamma_{11}^1 B + \partial_1 B - A x_{2'}) \\ * & \mathcal{E}_{22} & 2\partial_2 A & -\partial_1 A + \partial_2 B \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$$

where positions with $*$ are not written since the matrix is symmetric, and where

$$\begin{aligned}
\mathcal{E}_{11} = & -(\partial_1 A - \partial_2 B - 4 {}^D\Gamma_{11}^1 A)x_2^2 \\
& + \{A\Phi_{22} + 2(\partial_{11}A - {}^D\Gamma_{11}^2 \partial_2 A + {}^D\Gamma_{11}^1 \partial_2 B + A\partial_2 {}^D\Gamma_{11}^2 - B\partial_2 {}^D\Gamma_{11}^1)\}x_{1'} \\
& - \{B\Phi_{22} + 2A\Phi_{12} - 2(\partial_{11}B + {}^D\Gamma_{11}^2 \partial_1 A - {}^D\Gamma_{11}^1 \partial_1 B \\
& + (\partial_1 {}^D\Gamma_{11}^2 - 2 {}^D\Gamma_{11}^1 {}^D\Gamma_{11}^2)A + (\partial_1 {}^D\Gamma_{11}^1 - 2({}^D\Gamma_{11}^1)^2)B + \partial_2 \psi\}x_{2'} \\
& + 2\partial_2 Ax_{1'}x_{2'} \\
& - (\partial_1 A + \partial_2 B)\Phi_{11} + 2({}^D\Gamma_{11}^2 A + {}^D\Gamma_{11}^1 B)\Phi_{12} + (2 {}^D\Gamma_{11}^2 B + \psi)\Phi_{22} \\
& - A\partial_1 \Phi_{11} + B\partial_2 \Phi_{11} - 2B\partial_1 \Phi_{12} \\
& + 2\partial_{11}\psi - 2 {}^D\Gamma_{11}^1 \partial_1 \psi - 2 {}^D\Gamma_{11}^2 \partial_2 \psi - 2(\partial_1 {}^D\Gamma_{11}^1 - \partial_2 {}^D\Gamma_{11}^2)\psi,
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{12} = & 2(\partial_{12}A - {}^D\Gamma_{11}^1 \partial_2 A + A\partial_2 {}^D\Gamma_{11}^1)x_{1'} \\
& + 2(\partial_{12}B + {}^D\Gamma_{11}^1 \partial_1 A + A\partial_2 {}^D\Gamma_{11}^2)x_{2'} \\
& - (\partial_1 A + \partial_2 B)\Phi_{12} + 2 {}^D\Gamma_{11}^1 B\Phi_{22} - A\partial_2 \Phi_{11} - B\partial_1 \Phi_{22} \\
& + 2\partial_{12}\psi - 2 {}^D\Gamma_{11}^1 \partial_2 \psi,
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_{22} = & 2\partial_{22}Ax_{1'} + 2(\partial_{22}B + 2A\partial_2 {}^D\Gamma_{11}^1)x_{2'} \\
& - (\partial_1 A + \partial_2 B + 2 {}^D\Gamma_{11}^1 A)\Phi_{22} - 2A\partial_2 \Phi_{12} + A\partial_1 \Phi_{22} - B\partial_2 \Phi_{22} + 2\partial_{22}\psi.
\end{aligned}$$

First, we use component

$$\mathcal{E}_{14} = 2({}^D\Gamma_{11}^2 A + {}^D\Gamma_{11}^1 B + \partial_1 B - Ax_{2'})$$

in (22); note that $\partial_2 \mathcal{E}_{14} = -2A$, and so $A(x^1, x^2) = 0$, which shows that $X \in \ker T$. Now component \mathcal{E}_{13} in (22) gives $\partial_2 B = 0$, which implies $B(x^1, x^2) = P(x^1)$ for some smooth function P depending only on the coordinate x^1 , i.e., the vector field $X = B\partial_2$ satisfies $\text{tr}(DX) = 0$.

At this point, the conformal function φ has the coordinate expression

$$\varphi(x^1, x^2, x_{1'}, x_{2'}) = P(x^1)x_{2'} + \psi(x^1, x^2)$$

and the possible nonzero components in (22) are \mathcal{E}_{11} , \mathcal{E}_{12} , \mathcal{E}_{22} and \mathcal{E}_{14} . Considering the component $\mathcal{E}_{14} = 2(P'(x^1) + {}^D\Gamma_{11}^1(x^1, x^2)P(x^1))$, we distinguish two cases depending on whether the function P vanishes identically or not. Indeed, if $P(x^1)$ is a solution of the equation $\mathcal{E}_{14} = 0$, then

$$\partial_1(P(x^1)e^{\int {}^D\Gamma_{11}^1(x^1, x^2) dx^1}) = e^{\int {}^D\Gamma_{11}^1(x^1, x^2) dx^1} \{P'(x^1) + P(x^1){}^D\Gamma_{11}^1(x^1, x^2)\} = 0,$$

which shows that $P(x^1)e^{\int {}^D\Gamma_{11}^1(x^1, x^2) dx^1} = \mathcal{Q}(x^2)$ for some smooth function $\mathcal{Q}(x^2)$. Now, if the function $\mathcal{Q}(x^2)$ vanishes at some point, then $P(x^1) = 0$ at each point. Otherwise, if $\mathcal{Q}(x^2)$ is not equal to 0 at each point, neither is $P(x^1)$.

First, suppose that $P(x^1) \equiv 0$, and hence $\varphi = \psi \circ \pi$. In this case, component \mathcal{E}_{22} in (22) yields $\partial_{22}\psi = 0$, which implies $\psi(x^1, x^2) = Q(x^1)x^2 + \phi(x^1)$ for some smooth functions Q and ϕ depending only on the coordinate x^1 . Now, the only components in (22) which could be non-null are

$$\begin{aligned}\mathcal{E}_{11} &= 2Qx_{2'} + (Q\Phi_{22} + 2Q'' - 2{}^D\Gamma_{11}^1Q' - 2(\partial_1{}^D\Gamma_{11}^1 - \partial_2{}^D\Gamma_{11}^2)Q)x_2 \\ &\quad + \phi\Phi_{22} + 2\phi'' - 2{}^D\Gamma_{11}^1\phi' - 2(\partial_1{}^D\Gamma_{11}^1 - \partial_2{}^D\Gamma_{11}^2)\phi - 2{}^D\Gamma_{11}^2Q, \\ \mathcal{E}_{12} &= 2(Q' - {}^D\Gamma_{11}^1Q).\end{aligned}$$

Now, $\partial_{2'}\mathcal{E}_{11} = 2Q$ implies $Q = 0$, thus showing that $d\varphi(\ker T) = 0$. Then $\mathcal{E}_{12} = 0$ and the component \mathcal{E}_{11} reduces to

$$\mathcal{E}_{11} = \phi\Phi_{22} + 2\phi'' - 2{}^D\Gamma_{11}^1\phi' - 2(\partial_1{}^D\Gamma_{11}^1 - \partial_2{}^D\Gamma_{11}^2)\phi.$$

Since $\varphi(x^1, x^2, x_{1'}, x_{2'}) = \phi(x^1)$, ϕ must be non-null and we obtain that $\mathcal{E}_{11} = 0$ is equivalent to

$$\begin{aligned}\Phi_{22} &= -\frac{2}{\phi}\{\phi'' - {}^D\Gamma_{11}^1\phi' - (\partial_1{}^D\Gamma_{11}^1 - \partial_2{}^D\Gamma_{11}^2)\phi\}, \\ &= -\frac{2}{\phi}\{\text{Hes}_\phi^D(\partial_1, \partial_1) + \phi\rho_{\text{sym}}^D(\partial_1, \partial_1)\},\end{aligned}$$

from where (i) is obtained.

Finally, we analyze the case in which the function $P(x^1)$ does not vanish identically. Since $\mathcal{E}_{14} = 2(P'(x^1) + {}^D\Gamma_{11}^1(x^1, x^2)P(x^1))$, we have $\partial_2{}^D\Gamma_{11}^1 = 0$. Now it follows from Remark 3.4 that the Ricci tensor ρ^D is symmetric of rank one and recurrent. Specialize the local coordinates (u^1, u^2) on Σ so that the only nonzero Christoffel symbol of D is ${}^u\Gamma_{11}^2(u^1, u^2)$ and $T\partial_{u^1} = \partial_{u^2}$, $T\partial_{u^2} = 0$. Then any solution of the conformally Einstein equation takes the form

$$\varphi(u^1, u^2, u_{1'}, u_{2'}) = \mathcal{A}(u^1)u_{2'} + \phi(u^1, u^2).$$

Now, considering the component \mathcal{E}_{41} of the conformally Einstein equation in the new coordinates (u^1, u^2) , one has $\mathcal{E}_{41} = 2\mathcal{A}'(u^1)$, which shows that $\varphi(u^1, u^2, u_{1'}, u_{2'}) = \kappa u_{2'} + \phi(u^1, u^2)$ for some $\kappa \neq 0$. Considering now the component

$$\begin{aligned}\mathcal{E}_{11} &= (2\partial_2\phi - \kappa\Phi_{22})u_{2'} + 2\partial_{11}\phi - 2\partial_2\phi{}^u\Gamma_{11}^2 \\ &\quad + 2\phi\partial_2{}^u\Gamma_{11}^2 + \phi\Phi_{22} + 2\kappa\Phi_{22}{}^u\Gamma_{11}^2 + \kappa\partial_2\Phi_{11} - 2\kappa\partial_1\Phi_{12}\end{aligned}$$

it follows that the conformally Einstein equation reduces to

$$\begin{aligned}\kappa\Phi_{22} &= 2\partial_2\phi, \\ (\phi + 2\kappa{}^u\Gamma_{11}^2)\Phi_{22} &= -2(\text{Hes}_\phi^D(\partial_{u^1}, \partial_{u^1}) + \phi\rho^D(\partial_{u^1}, \partial_{u^1})) + \kappa(2\partial_1\Phi_{12} - \partial_2\Phi_{11}),\end{aligned}$$

from where (ii) is obtained. \square

7. Examples

Nilpotent Riemannian extensions with flat base. Let (Σ, D) be a flat torsion-free affine surface. Take local coordinates on Σ so that all Christoffel symbols vanish. Let T be a parallel nilpotent $(1, 1)$ -tensor field. Since T is parallel, its components T_i^j are necessarily constant on the given coordinates. Hence one may further specialize the local coordinates (x^1, x^2) , by using a linear transformation, so that $T\partial_1 = \partial_2$, $T\partial_2 = 0$ and all the Christoffel symbols ${}^D\Gamma_{ij}^k$ remain identically zero. Now Theorem 3.1 shows that $(T^*\Sigma, g_{D,\Phi,T})$ is *Bach-flat* for any symmetric $(0, 2)$ -tensor field Φ on Σ . Moreover it follows from Theorem 4.3 that $(T^*\Sigma, g_{D,\Phi,T}, f = h \circ \pi)$ is a *steady gradient Ricci soliton* for any $h \in C^\infty(\Sigma)$ with $dh \circ T = 0$ and any symmetric $(0, 2)$ -tensor field Φ such that $\Phi_{22}(x^1, x^2) = -h''(x^1)$.

Further note from Remark 4.5 that the steady gradient Ricci soliton

$$(T^*\Sigma, g_{D,\Phi,T}, f = h \circ \pi)$$

satisfies $\mathbb{D} = 0$. Moreover, since $\Phi_{22} = -h''(x^1)$, one has that $(T^*\Sigma, g_{D,\Phi,T})$ is in the conformal class of an Einstein metric (just considering the conformal metric $\bar{g} = \phi^{-2}g_{D,\Phi,T}$ determined by the equation $\phi''(x^1) - \frac{1}{2}\phi(x^1)h''(x^1) = 0$).

Remark 7.1. Set $\Sigma = \mathbb{R}^2$ with usual coordinates (x^1, x^2) and put $T\partial_1 = \partial_2$, $T\partial_2 = 0$. For any smooth function $h(x^1)$ consider the deformation tensor Φ given by $\Phi_{22}(x^1, x^2) = -h''(x^1)$ (the other components being zero). Then, the nonzero Christoffel symbols of $g_{D,\Phi,T}$ are given by

$$\Gamma_{11}^2 = -x_{2'} = -\Gamma_{12'}^1, \quad \Gamma_{11}^{2'} = -h''(x^1)x_{2'}, \quad \Gamma_{12}^{2'} = -\frac{1}{2}h^{(3)}(x^1) = -\Gamma_{22}^{1'}.$$

Hence a curve $\gamma(t) = (x^1(t), x^2(t), x_{1'}(t), x_{2'}(t))$ is a geodesic if and only if

$$\begin{aligned} \ddot{x}^1(t) &= 0, & \ddot{x}^2(t) - x_{2'}(t) \dot{x}^1(t)^2 &= 0, \\ \ddot{x}_{1'}(t) + 2x_{2'}(t) \dot{x}^1(t) \dot{x}_{2'}(t) + \frac{1}{2}h^{(3)}(x^1(t)) \dot{x}^2(t)^2 &= 0, \\ \ddot{x}_{2'}(t) - h''(x^1(t))x_{2'}(t) \dot{x}^1(t)^2 - h^{(3)}(x^1(t)) \dot{x}^1(t) \dot{x}^2(t) &= 0. \end{aligned}$$

Thus $x^1(t) = at + b$ for some $a, b \in \mathbb{R}$ and

$$\begin{aligned} \ddot{x}^2(t) - a^2 x_{2'}(t) &= 0, \\ \ddot{x}_{2'}(t) - h''(at + b)a^2 x_{2'}(t) - h^{(3)}(at + b)a \dot{x}^2(t) &= 0, \\ \ddot{x}_{1'}(t) + 2a x_{2'}(t) \dot{x}_{2'}(t) + \frac{1}{2}h^{(3)}(at + b) \dot{x}^2(t)^2 &= 0. \end{aligned}$$

Now the first two equations above are linear and thus $x^2(t)$ and $x_{2'}(t)$ are globally defined. Finally, since $\ddot{x}_{1'}(t) + 2a x_{2'}(t) \dot{x}_{2'}(t) + \frac{1}{2}h^{(3)}(at + b) \dot{x}^2(t)^2 = 0$ is also linear on $x_{1'}(t)$, one has that geodesics are globally defined.

Then it follows from Theorem 4.3 that $(T^*\mathbb{R}^2, g_{D,\Phi,T}, f = h \circ \pi)$ is a *geodesically complete steady gradient Ricci soliton, which is conformally Einstein* by Theorem 6.1.

Nilpotent Riemannian extensions with nonrecurrent base. Let $(T^*\Sigma, g_{D,\Phi,T}, f = h \circ \pi)$ be a nontrivial Bach-flat steady gradient Ricci soliton as in Theorem 4.3. Further assume that the Ricci tensor ρ^D is nonsymmetric, i.e., $\rho_{sk}^D \neq 0$ (equivalently $\partial_2^D \Gamma_{11}^1 \neq 0$ as shown in the proof of Proposition 3.3). Then it follows from Theorem 5.1 that $(T^*\Sigma, g_{D,\Phi,T})$ is not half conformally flat.

Theorem 6.1 shows that $(T^*\Sigma, g_{D,\Phi,T})$ is conformally Einstein if and only if there exists a positive $\phi \in C^\infty(\Sigma)$ with $d\phi \circ T = 0$ such that

$$\phi \widehat{\Phi} + 2(\text{Hes}_\phi^D + \phi \rho_{\text{sym}}^D) = 0.$$

Hence it follows from Theorem 4.3 that $\text{Hes}_h^D = \frac{2}{\phi} \text{Hes}_\phi^D$, which means

$$\left(2\frac{\phi'}{\phi} - h'\right)^D \Gamma_{11}^1 = 2\frac{\phi''}{\phi} - h''.$$

Taking derivatives with respect to x^2 and, since $\partial_2^D \Gamma_{11}^1 \neq 0$, the equation above splits into

$$2\frac{\phi'}{\phi} - h' = 0, \quad \text{and} \quad 2\frac{\phi''}{\phi} - h'' = 0,$$

which only admits constant solutions. Summarizing the above one has the following: *Let (Σ, D, T) be an affine surface with nonsymmetric Ricci tensor (i.e., $\rho_{sk}^D \neq 0$). Then any Bach-flat gradient Ricci soliton $(T^*\Sigma, g_{D,\Phi,T}, f = h \circ \pi)$ is neither half conformally flat nor conformally Einstein.*

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
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