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Let $(M^5, \alpha, g_\alpha, J)$ be a 5-dimensional Sasakian Einstein manifold with contact 1-form α, associated metric g_α and almost complex structure J, and let L be a contact stationary Legendrian surface in M^5. We will prove that L satisfies the equation

$$-\Delta^\nu H + (K-1)H = 0,$$

where Δ^ν is the normal Laplacian with respect to the metric g on L induced from g_α and K is the Gauss curvature of (L, g).

Using this equation and a new Simons' type inequality for Legendrian surfaces in the standard unit sphere $\mathbb{S}^5$, we prove an integral inequality for contact stationary Legendrian surfaces in $\mathbb{S}^5$. In particular, we prove that if L is a contact stationary Legendrian surface in $\mathbb{S}^5$ and B is the second fundamental form of L, with $S = |B|^2$, $\rho^2 = S - 2H^2$ and

$$0 \le S \le 2,$$

then we have either $\rho^2 = 0$ and L is totally umbilic or $\rho^2 \neq 0$, $S = 2$, $H = 0$ and L is a flat minimal Legendrian torus.

1. Introduction

Let $(M^{2n+1}, \alpha, g_\alpha, J)$ be a $2n+1$ dimensional contact metric manifold with contact structure α, associated metric g_α and almost complex structure J. Assume that (L, g) is an n-dimensional compact Legendrian submanifold of M^{2n+1} with metric g induced from g_α. The volume of L is defined by

$$V(L) = \int_L d\mu, \tag{1-1}$$

where $d\mu$ is the volume form of g. A contact stationary Legendrian submanifold of M^{2n+1} is a Legendrian submanifold of M^{2n+1} which is a stationary point of V with respect to Legendrian deformations. That is we call a Legendrian submanifold

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$L \subseteq M^{2n+1}$ a contact stationary Legendrian submanifold, if for any Legendrian deformations $L_t \subseteq M^{2n+1}$ with $L_0 = L$ we have

$$\left.\frac{dV(L_t)}{dt}\right|_{t=0} = 0.$$

Remark 1.1. L_t is a Legendrian deformation of $L := L_0$, if L_t is a Legendrian submanifold for every t.

The E–L equation for a contact stationary Legendrian submanifold L is [Iriyeh 2005; Castro et al. 2006]

$$\operatorname{div}_g(JH) = 0, \tag{1-2}$$

where div_g is the divergence with respect to g and H is the mean curvature vector of L in M^{2n+1}.

Remark 1.2. The notion of a contact stationary Legendrian submanifold was first defined by Iriyeh [2005] and Castro et al. [2006] independently, where they used the name of Legendrian minimal Legendrian submanifold and contact minimal Legendrian submanifold, respectively. In this paper we prefer to use the name of contact stationary Legendrian submanifold.

The study of contact stationary Legendrian submanifolds is motivated by the study of Hamiltonian minimal Lagrangian (briefly, HSL) submanifolds, which was first studied by Oh [1990; 1993]. An HSL submanifold in a Kähler manifold is a Lagrangian submanifold which is a stationary point of the Volume functional under Hamiltonian deformations. By [Reckziegel 1988], Legendrian submanifolds in a Sasakian manifold M^{2n+1} can be seem as links of Lagrangian submanifolds in the cone CM^{2n+1}, which is a Kähler manifold with proper metric and complex structure (see Section 2). In fact, a close relation between contact stationary Legendrian submanifolds and HSL submanifolds was found by Iriyeh [2005] and Castro et al. [2006]. Precisely, they independently proved that $C(L)$ is an HSL submanifold in $\mathbb{C}^n (n \geq 2)$ if and only if L is a contact stationary Legendrian submanifold in $\mathbb{S}^{2n-1}$ and L is a contact stationary Legendrian submanifold in $\mathbb{S}^{2n+1} (n \geq 1)$ if and only if $\Pi(L)$ is an HSL submanifold in $\mathbb{CP}^n$, where $\Pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is the Hopf fibration.

From the definition we see that minimal Legendrian submanifolds are a special kind of contact stationary Legendrian submanifold. Another special kind of contact stationary Legendrian submanifold are Legendrian submanifolds with parallel mean curvature vector fields in the normal bundle. The study of (nonminimal) contact stationary Legendrian submanifolds of $\mathbb{S}^{2n+1}$ is relatively recent endeavor. For $n = 1$, by [Iriyeh 2005], contact stationary Legendrian curves in $\mathbb{S}^3$ are the so called (p, q) curves discovered by Schoen and Wolfson [2001], where p and q are relatively prime integers. For $n = 2$, since a harmonic 1-forms on a 2-sphere must be trivial, contact stationary Legendrian 2-spheres in $\mathbb{S}^5$ must be minimal and so must be the equatorial 2-spheres by Yau's result [1974]. There are a lot of contact stationary doubly periodic surfaces form $\mathbb{R}^2$ to $\mathbb{S}^5$ by lifting Hélein and Romon's examples [2002] and more contact stationary Legendrian surfaces (mainly tori) are constructed in [Mironov 2003; 2008; Iriyeh 2005; Hélein and Romon 2005; Ma 2005; Ma and Schmies 2006; Butscher and Corvino 2012]. And general dimension examples are constructed in [Oh 1993; Mironov 2004; Dong and Han 2007; Dong 2007; Butscher 2009; Joyce et al. 2011; Lee 2012; Chen et al. 2012]. See also [Ono 2005; Hunter and McIntosh 2011; Kajigaya 2013] for other studies of contact stationary Legendrian submanifolds.

In this paper we will study pinching properties of contact stationary Legendrian surfaces in $\mathbb{S}^5$. To do this we first prove an equation satisfied by contact stationary Legendrian surfaces in a Sasakian Einstein manifold, which we hope will be useful in analyzing analytic properties of contact stationary Legendrian surfaces.

Theorem 1.3. *Let L be a contact stationary Legendrian surface in a 5-dimensional Sasakian Einstein manifold $(M^5, \alpha, g_\alpha, J)$, then L satisfies the following equation*:

$$-\Delta^\nu H + (K-1)H = 0, \tag{1-3}$$

where Δ^ν is the normal Laplacian with respect to the metric g on L induced from g_α and K is the Gauss curvature of (L, g).

We recall that the well-known Clifford torus is

$$T_{\mathrm{Clif}} = \mathbb{S}^1\Big(\frac{1}{\sqrt{2}}\Big) \times \mathbb{S}^1\Big(\frac{1}{\sqrt{2}}\Big) \subseteq \mathbb{S}^5. \tag{1-4}$$

In the theory of minimal surfaces, the following Simons' integral inequality and pinching theorem due to Simons [1968], Lawson [1969] and Chern et al. [1970] are well known.

Theorem 1.4. *Let M be a compact minimal surface in a unit sphere $\mathbb{S}^3$ and B be the second fundamental form of M in $\mathbb{S}^3$. Set $S = |B|^2$, then we have*

$$\int_M S(2-S)\, d\mu \le 0.$$

In particular, *if*

$$0 \le S \le 2,$$

then either $S = 0$ and M is totally geodesic, *or $S = 2$ and M is the Clifford torus T_{Clif}*, *which is defined by* (1-4).

The above integral inequality was proved by Simons [1968] in his celebrated paper and the classification result was given by Chern et al. [1970] and Lawson [1969], independently.

For minimal surfaces in a sphere with higher codimension, a corresponding integral inequality was proved by Benko et al. [1979] and Kozlowski and Simon [1984]. In order to state their result, we first record an example.

Example. The Veronese surface is a minimal surface in $\mathbb{S}^4 \subseteq \mathbb{R}^5$ defined by

$$u : \mathbb{S}^2(\sqrt{3}) \subseteq \mathbb{R}^3 \to \mathbb{S}^4(1) \subseteq \mathbb{R}^5$$

$$(x, y, z) \mapsto (u_1, u_2, u_3, u_4, u_5)$$

where

$$u_1 = \frac{yz}{\sqrt{3}}, \quad u_2 = \frac{xz}{\sqrt{3}}, \quad u_3 = \frac{xy}{\sqrt{3}}, \quad u_4 = \frac{x^2 - y^2}{2\sqrt{3}}, \quad u_5 = \frac{x^2 + y^2 - 2z^2}{6}.$$

Here, u defines an isometric immersion of $\mathbb{S}^2(\sqrt{3})$ into $\mathbb{S}^4(1)$, and it maps two points (x, y, z) and $(-x, -y, -z)$ of $\mathbb{S}^2(\sqrt{3})$ into the same point of $\mathbb{S}^4(1)$, and so it imbeds the real projective plane into $\mathbb{S}^4(1)$.

We have

Theorem 1.5 [Benko et al. 1979]. *Let M be a minimal surface in an n-dimensional sphere $\mathbb{S}^n$, then*

$$\int_M S\left(2 - \tfrac{3}{2}S\right) d\mu \leq 0. \tag{1-5}$$

In particular, if

$$0 \leq S \leq \tfrac{4}{3},$$

then either $S = 0$ and M is totally geodesic, or $S = \frac{4}{3}$, $n = 4$ and M is the Veronese surface.

The above classification for minimal surfaces in a sphere with $S = \frac{4}{3}$ was also shown by Chern et al. [1970].

We see that the (first) pinching constant for minimal surfaces in $\mathbb{S}^3$ is 2, but it is $\frac{4}{3}$ for minimal surfaces of higher codimension. This is an interesting phenomenon and we think it is due to the complexity of the normal bundle, because for minimal Legendrian surfaces in $\mathbb{S}^5$, the (first) pinching constant is also 2.

Theorem 1.6 [Yamaguchi et al. 1976]. *If M is a minimal Legendrian surface of the unit sphere $\mathbb{S}^5$ and $0 \leq S \leq 2$, then S is identically 0 or 2.*

Remark 1.7. For higher dimensional case of this theorem we refer to [Dillen and Vrancken 1990].

All of these results are based on calculating the Laplacian of S and then getting Simons' type equalities or inequalities, a powerful method which was originated by Simons [1968]. The minimal condition is used to cancel some terms in the resulting calculation and to some extent it is important. In this note we prove a Simons' type inequality (Lemma 3.8) for Legendrian surfaces in $\mathbb{S}^5$, without minimal condition. By using (1-3) and this Simons' type inequality we get

Theorem 1.8. *Let $L : \Sigma \to \mathbb{S}^5$ be a contact stationary Legendrian surface, where $\mathbb{S}^5$ is the unit sphere with standard contact structure and metric (as given at the end of Section 2). Then we have*

$$\int_L \rho^2\big(3 - \tfrac{3}{2}S + 2H^2\big)\, d\mu \le 0,$$

where $\rho^2 := S - 2H^2$. In particular, if

$$0 \le S \le 2,$$

then either $\rho^2 = 0$ and L is totally umbilic, or $\rho^2 \ne 0$, $S = 2$, $H = 0$ and L is a flat minimal Legendrian torus.

Remark 1.9. Because minimal Legendrian surfaces are contact stationary Legendrian surfaces and satisfy $\rho^2 = S$, and because totally umbilic minimal surfaces are totally geodesic, we see that Theorem 1.6 is a corollary of Theorem 1.8.

Integral inequality and gap phenomenon for submanifolds satisfying a fourth order quasielliptic nonlinear equation was first studied by Li [2001; 2002a; 2002b] who proved several gap theorems for Willmore submanifolds in a sphere. These results are partial motivations of our paper.

We end this introduction by recalling a classification theorem of flat minimal Legendrian tori in $\mathbb{S}^5$. For a constant θ let T_θ be the 2-torus in $\mathbb{S}^5$ defined by

$$T_\theta = \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_i| = \tfrac{1}{3}, i = 1, 2, 3 \text{ and } \sum_i \arg z_i = \theta\Big\}.$$

T_θ is called the generalized Clifford torus and it is a flat minimal Legendrian torus in $\mathbb{S}^5$. Its projection under the Hopf map $\pi : \mathbb{S}^5 \to \mathbb{CP}^2$ is a flat minimal Lagrangian torus, which is also called a generalized Clifford torus. It is proved in [Ludden et al. 1975] that a flat minimal Lagrangian torus in $\mathbb{CP}^2$ must be $\mathbb{S}^1 \times \mathbb{S}^1$. By the correspondence of minimal Lagrangian surfaces in $\mathbb{CP}^2$ and minimal Legendrian surfaces in $\mathbb{S}^5$ (see [Reckziegel 1988]), we see that a flat minimal Legendrian torus in $\mathbb{S}^5$ must be a generalized Clifford torus. For more details we refer to [Haskins 2004, p. 853].

The rest of this paper is organized as follows: In Section 2 we collect some basic material from Sasakian geometry, which will be used in the next section. In Section 3 we prove our main results, Theorems 1.3 and 1.8.

2. Preliminaries on contact geometry

In this section we recall some basic material from contact geometry. For more information we refer to [Blair 2002].

Contact manifolds.

Definition 2.1. A contact manifold M is an odd dimensional manifold with a one form α such that $\alpha \wedge (d\alpha)^n \neq 0$, where $\dim M = 2n+1$.

Assume now that (M, α) is a given contact manifold of dimension $2n+1$. Then α defines a $2n$-dimensional vector bundle over M, where the fiber at each point $p \in M$ is given by

$$\xi_p = \operatorname{Ker} \alpha_p.$$

Since $\alpha \wedge (d\alpha)^n$ defines a volume form on M, we see that

$$\omega := d\alpha$$

is a closed nondegenerate 2-form on $\xi \oplus \xi$ and hence it defines a symplectic product on ξ such that $(\xi, \omega|_{\xi \oplus \xi})$ becomes a symplectic vector bundle. A consequence of this fact is that there exists an almost complex bundle structure

$$\tilde{J} : \xi \to \xi$$

compatible with $d\alpha$, i.e., a bundle endomorphism satisfying

(1) $\tilde{J}^2 = -\operatorname{id}_\xi$,
(2) $d\alpha(\tilde{J}X, \tilde{J}Y) = d\alpha(X, Y)$ for all $X, Y \in \xi$,
(3) $d\alpha(X, \tilde{J}X) > 0$ for $X \in \xi \setminus 0$.

Since M is an odd dimensional manifold, ω must be degenerate on TM, and so we obtains a line bundle η over M with fibers

$$\eta_p := \{V \in T_pM | \omega(V, W) = 0, \forall W \in \xi_p\}.$$

Definition 2.2. The Reeb vector field $\boldsymbol{R}$ is the section of η such that $\alpha(\boldsymbol{R}) = 1$.

Thus α defines a splitting of TM into a line bundle η with the canonical section $\boldsymbol{R}$ and a symplectic vector bundle $(\xi, \omega|\xi \oplus \xi)$. We denote the projection along η by π, i.e.,

$$\pi : TM \to \xi, \quad \pi(V) := V - \alpha(V)\boldsymbol{R}.$$

Using this projection we extend the almost complex structure $\tilde{J}$ to a section $J \in \Gamma(T^*M \otimes TM)$ by setting

$$J(V) = \tilde{J}(\pi(V)),$$

for $V \in TM$.

We call J an almost complex structure of the contact manifold M.

Definition 2.3. Let (M, α) be a contact manifold, a submanifold L of (M, α) is called an isotropic submanifold if $T_xL \subseteq \xi_x$ for all $x \in L$.

For algebraic reasons the dimension of an isotropic submanifold of a $2n+1$ dimensional contact manifold can not be bigger than n.

Definition 2.4. An isotropic submanifold $L \subseteq (M, \alpha)$ of maximal possible dimension n is called a Legendrian submanifold.

Sasakian manifolds. Let (M, α) be a contact manifold, with the almost complex structure J and Reeb field $\boldsymbol{R}$. A Riemannian metric g_α defined on M is said to be associated, if it satisfies the following three conditions:

(1) $g_\alpha(\boldsymbol{R}, \boldsymbol{R}) = 1$.

(2) $g_\alpha(V, \boldsymbol{R}) = 0$, $\forall V \in \xi$.

(3) $\omega(V, JW) = g_\alpha(V, W)$, $\forall V, W \in \xi$.

We should mention here that on any contact manifold there exists an associated metric on it, because we can construct one in the following way. We introduce a bilinear form b by

$$b(V, W) := \omega(V, JW),$$

then the tensor

$$g := b + \alpha \otimes \alpha$$

defines an associated metric on M.

Sasakian manifolds are the odd dimensional analogue of Kähler manifolds. They are defined as follows.

Definition 2.5. A contact manifold (M, α) with an associated metric g_α is called Sasakian, if the cone CM equipped with the following extended metric $\bar{g}$

$$(CM, \bar{g}) = (\mathbb{R}_+ \times M, dr^2 + r^2 g_\alpha) \tag{2-1}$$

is Kähler with respect to the following canonical almost complex structure J on $TCM = \mathbb{R} \oplus \langle \boldsymbol{R} \rangle \oplus \xi$:

$$J(r\partial r) = \boldsymbol{R}, \; J(\boldsymbol{R}) = -r\partial r.$$

Furthermore if g_α is Einstein, M is called a Sasakian Einstein manifold.

We record several lemmas which are well known in Sasakian geometry. These lemmas will be used in the next section.

Lemma 2.6. *Let (M, α, g_α, J) be a Sasakian manifold. Then*

$$\overline{\nabla}_X \boldsymbol{R} = -JX, \tag{2-2}$$

and

$$(\overline{\nabla}_X J)(Y) = g(X, Y)\boldsymbol{R} - \alpha(Y)X, \tag{2-3}$$

for $X, Y \in TM$, where $\overline{\nabla}$ is the Levi–Civita connection on (M, g_α).

Lemma 2.7. *Let L be a Legendrian submanifold in a Sasakian Einstein manifold (M, α, g_α, J), then the mean curvature form $\omega(H, \cdot)|_L$ defines a closed one form on L.*

For a proof of this lemma we refer to [Lê 2004, Proposition A.2] or [Smoczyk 2003, Lemma 2.8]. In fact they proved this result under the weaker assumption that (M, α, g_α, J) is a weakly Sasakian Einstein manifold, where weakly Einstein means that g_α is Einstein only when restricted to the contact hyperplane $\operatorname{Ker}\alpha$.

Lemma 2.8. *Let L be a Legendrian submanifold in a Sasakian manifold (M,α,g_α,J) and B be the second fundamental form of L in M. Then we have*

$$g_\alpha(B(X, Y), \boldsymbol{R}) = 0, \tag{2-4}$$

for any $X, Y \in TL$.

Proof. For any $X, Y \in TL$,

$$\begin{aligned}\langle B(X, Y), \boldsymbol{R}\rangle &= \langle \overline{\nabla}_X Y, \boldsymbol{R}\rangle \\ &= -\langle Y, \overline{\nabla}_X \boldsymbol{R}\rangle \\ &= \langle Y, JX\rangle \\ &= \omega(X, Y) \\ &= d\alpha(X, Y) \\ &= 0,\end{aligned}$$

where in the third equality we used (2-2). □

In particular this lemma implies that the mean curvature H of L is orthogonal to the Reeb field $\boldsymbol{R}$.

Lemma 2.9. *For any $Y, Z \in \operatorname{Ker}\alpha$, we have*

$$g_\alpha(\overline{\nabla}_X(JY), Z) = g_\alpha(J\overline{\nabla}_X Y, Z). \tag{2-5}$$

Proof. Note that

$$(\overline{\nabla}_X J)Y = \overline{\nabla}_X(JY) - J\overline{\nabla}_X Y.$$

Therefore by using (2-3) we have

$$\begin{aligned}\langle \overline{\nabla}_X(JY), Z\rangle &= \langle (\overline{\nabla}_X J)Y, Z\rangle + \langle J\overline{\nabla}_X Y, Z\rangle \\ &= \langle J\overline{\nabla}_X Y, Z\rangle,\end{aligned}$$

for any $Y, Z \in \operatorname{Ker}\alpha$. □

A canonical example of Sasakian Einstein manifolds is the standard odd dimensional sphere $\mathbb{S}^{2n+1}$.

The standard sphere $\mathbb{S}^{2n+1}$. Let $\mathbb{C}^n = \mathbb{R}^{2n+2}$ be the Euclidean space with coordinates $(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1})$ and $\mathbb{S}^{2n+1}$ be the standard unit sphere in $\mathbb{R}^{2n+2}$. Define

$$\alpha_0 = \frac{1}{2}\sum_{j+1}^{n+1}(x_j dy_j - y_j dx_j),$$

then

$$\alpha := \alpha_0|_{\mathbb{S}^{2n+1}}$$

defines a contact one form on $\mathbb{S}^{2n+1}$. Assume that g_0 is the standard metric on $\mathbb{R}^{2n+2}$ and J_0 is the standard complex structure of $\mathbb{C}^n$. We define

$$g_\alpha = g_0|_{\mathbb{S}^{2n+1}} \quad \text{and} \quad J = J_0|_{\mathbb{S}^{2n+1}},$$

then $(\mathbb{S}^{2n+1}, \alpha, g_\alpha, J)$ is a Sasakian Einstein manifold with associated metric g_α. Its contact hyperplane is characterized by

$$\operatorname{Ker}\alpha_x = \{Y \in T_x\mathbb{S}^{2n+1} | \langle Y, Jx\rangle = 0\}.$$

3. Proof of the theorems

Several lemmas. In this part we assume that (M, α, g_α, J) is a Sasakian manifold. We show several lemmas which are analogous to results in Kähler geometry.

The first lemma shows $\omega = d\alpha$ when restricted to the contact hyperplane $\operatorname{Ker}\alpha$ behaves as the Kähler form on a Kähler manifold.

Lemma 3.1. *Let $X, Y, Z \in \operatorname{Ker}\alpha$, then*

$$\overline{\nabla}_X\omega(Y, Z) = 0, \tag{3-1}$$

where $\overline{\nabla}$ is the derivative with respect to g_α.

Proof.

$$\begin{aligned}\overline{\nabla}_X\omega(Y, Z) &= X(\omega(Y, Z)) - \omega(\overline{\nabla}_X Y, Z) - \omega(Y, \overline{\nabla}_X Z)\\ &= -Xg_\alpha(Y, JZ) - \omega(\overline{\nabla}_X Y, Z) - \omega(Y, \overline{\nabla}_X Z)\\ &= -g_\alpha(\overline{\nabla}_X Y, JZ) - g_\alpha(Y, \overline{\nabla}_X JZ) + g_\alpha(\overline{\nabla}_X Y, JZ) + g_\alpha(Y, J\overline{\nabla}_X Z)\\ &= 0,\end{aligned}$$

where in the third equality we used $g_\alpha(Y, \overline{\nabla}_X JZ) = g_\alpha(Y, J\overline{\nabla}_X Z)$, which is a direct corollary of (2-3). □

Now let L be a Legendrian submanifold of M. We have a natural identification of $NL \cap \operatorname{Ker}\alpha$ with T^*L, where NL is the normal bundle of L and T^*L is the cotangent bundle.

Definition 3.2. $\tilde{\omega} : NL \cap \operatorname{Ker}\alpha \to T^*L$ is the bundle isomorphism defined by

$$\tilde{\omega}_p(v_p) = (v_p \rfloor \omega_p)|_{T_pL},$$

where $p \in L$ and $v_p \in (NL \cap \operatorname{Ker}\alpha)_p$.

Recall that $\omega(\boldsymbol{R}) = 0$ and $g_\alpha(V, W) = \omega(V, JW)$ for any $V, W \in \xi$, hence $\tilde{\omega}$ defines an isomorphism.

We have

Lemma 3.3. *Let $V \in \Gamma(NL \cap \operatorname{Ker}\alpha)$. Then*

$$\tilde{\omega}(\Delta^\nu V - \langle \Delta^\nu V, \boldsymbol{R}\rangle \boldsymbol{R} + V) = \Delta(\tilde{\omega}(V)), \quad \textit{i.e.},$$

$$(\Delta^\nu V + V)\rfloor\omega = \Delta(V\rfloor\omega), \tag{3-2}$$

where Δ is the Laplace–Beltrami operator on (L, g).

Remark 3.4. This kind of lemma in the context of symplectic geometry was proved by Oh [1990, Lemma 3.3]. Our proof follows his argument with only slight modifications.

Proof. We first show that

$$\nabla_X(\tilde{\omega}(V)) = \tilde{\omega}\big(\nabla^\nu_X V - \langle \nabla^\nu_X V, \boldsymbol{R}\rangle \boldsymbol{R}\big) \tag{3-3}$$

for any $X \in TL$. Equality (3-3) is equivalent to

$$\nabla_X(\tilde{\omega}(V))(Y) = \tilde{\omega}\big(\nabla^\nu_X V - \langle \nabla^\nu_X V, \boldsymbol{R}\rangle \boldsymbol{R}\big)(Y) \tag{3-4}$$

for any $Y \in TL$.

$$\begin{aligned}
\nabla_X(\tilde{\omega}(V))(Y) &= \nabla_X(\tilde{\omega}(V)(Y)) - \tilde{\omega}(V)(\nabla_X Y)\\
&= \overline{\nabla}_X(\omega(V, Y)) - \tilde{\omega}(V)(\nabla_X Y)\\
&= \omega(\nabla^\nu_X V, Y) + \omega(V, \nabla_X Y) - \omega(V, \nabla_X Y)\\
&= \omega(\nabla^\nu_X V, Y)\\
&= \tilde{\omega}\big(\nabla^\nu_X V - \langle \nabla^\nu_X V, \boldsymbol{R}\rangle \boldsymbol{R}\big)(Y),
\end{aligned}$$

where in the third equality we used $\overline{\nabla}_X\omega = 0$, when restricted to $\operatorname{Ker}\alpha$, which is proved in Lemma 3.1.

Let $p \in L$ and choose an orthonormal frame $\{E_1, \dots, E_n\}$ on TL such that $\nabla_{E_i} E_j(p) = 0$, then the general Laplacian Δ can be written as

$$\Delta\psi(p) = \sum_{i=1}^n \nabla_{E_i}\nabla_{E_i}\psi(p),$$

where ψ is a tensor on L. Therefore

$$
\begin{aligned}
(\tilde{\omega}^{-1} \circ \Delta \cdot \tilde{\omega}(V))(p) \\
&= \Big(\tilde{\omega}^{-1} \circ \sum_{i=1}^{n} \nabla_{E_i} \nabla_{E_i} \tilde{\omega}(V)\Big)(p) \\
&= \sum_{i=1}^{n} (\tilde{\omega}^{-1} \nabla_{E_i} \tilde{\omega} \cdot \tilde{\omega}^{-1} \nabla_{E_i} \tilde{\omega}(V))(p) \\
&= \sum_{i=1}^{n} \big(\tilde{\omega}^{-1} \nabla_{E_i} \tilde{\omega}\big(\nabla^{\nu}_{E_i} V - \langle \nabla^{\nu}_{E_i} V, \boldsymbol{R}\rangle \boldsymbol{R}\big)\big)(p) \\
&= \sum_{i=1}^{n} \nabla^{\nu}_{E_i} \big(\nabla^{\nu}_{E_i} V - \langle \nabla^{\nu}_{E_i} V, \boldsymbol{R}\rangle \boldsymbol{R}\big) - \big\langle \nabla^{\nu}_{E_i} \big(\nabla^{\nu}_{E_i} V - \langle \nabla^{\nu}_{E_i} V, \boldsymbol{R}\rangle \boldsymbol{R}\big), \boldsymbol{R}\big\rangle \boldsymbol{R} \\
&= \Delta^{\nu} V - \langle \Delta^{\nu} V, \boldsymbol{R}\rangle \boldsymbol{R} + V,
\end{aligned}
$$

where in the third and fourth equalities we used (3-3) and in the last equality we used equality (2-2). □

Proof of Theorem 1.3. We see that for any function s defined on L,

$$
\begin{aligned}
0 = \int_L s \operatorname{div} JH \, d\mu &= \int_L g(JH, \nabla s) \, d\mu \\
= \int_L \omega(H, \nabla s) \, d\mu &= \int_L \langle \omega \rfloor H, \omega \rfloor \nabla s \rangle \, d\mu \\
= \int_L \langle \omega \rfloor H, \, ds \rangle &= \int_L \delta(\omega \rfloor H) s \, d\mu.
\end{aligned}
$$

Therefore the E–L equation for L is equivalent to

$$\delta(\omega \rfloor H) = 0, \tag{3-5}$$

where δ is the adjoint operator of d on L.

By Lemma 2.7 we see that L satisfies

$$\Delta_h(\omega \rfloor H) = 0, \tag{3-6}$$

where $\Delta_h := \delta d + d\delta$ is the Hodge–Laplace operator. That is the mean curvature form of L is a harmonic one form.

To proceed on, we need the following Weitzenböck formula

Lemma 3.5. *Let M be an n dimensional oriented Riemannian manifold. If $\{V_i\}$ is a local orthonormal frame field and $\{\omega^i\}$ is its dual coframe field, then*

$$\Delta_h = -\sum_i D^2_{V_i V_i} + \sum_{ij} \omega^i \wedge i(V_j) R_{V_i V_j},$$

where $D^2_{XY} \equiv D_X D_Y - D_{D_X Y}$ *represents the covariant derivatives*, $\Delta_d = d\delta + \delta d$ *is the Hodge–Laplace and* $R_{XY} = -D_X D_Y + D_Y D_X + D_{[X,Y]}$ *is the curvature tensor.*

Remark 3.6. For a detailed discussion on the Weitzenböck formula we refer to Wu [1988].

Using the Weitzenböck formula we have

$$-\Delta(\omega \rfloor H) + \sum_{ij} \omega^i \wedge i(V_j) R_{V_i V_j}(\omega \rfloor H) = 0, \tag{3-7}$$

where $\{V_i\}$ is a local orthogonal frame field and $\{\omega^i\}$ is its dual coframe field on L.

Denote $\omega \rfloor H$ by $\theta_H = \sum_k \theta_k \omega^k$, we have

$$\begin{aligned}
\sum_{ij} \omega^i \wedge i(V_j) R_{V_i V_j} \theta_H &= \sum_{ij} R_{V_i V_j} \theta_H(V_j) \omega^i \\
&= \sum_{ijk} R_{V_i V_j} \omega^k(V_j) \theta_k \omega^i \\
&= -\sum_{ijk} \omega^k(R_{V_i V_j} V_j) \theta_k \omega^i \\
&= -\sum_{ijk} \langle R_{V_i V_j} V_j, V_k \rangle \theta_k \omega^i \\
&= -\sum_{ij} \langle R_{V_i V_j} V_j, V_i \rangle \theta_i \omega^i \\
&= K \theta_H.
\end{aligned}$$

That is

$$\sum_{ij} \omega^i \wedge i(V_j) R_{V_i V_j}(\omega \rfloor H) = K \omega \rfloor H. \tag{3-8}$$

Recall that $H \in NL \cap \operatorname{Ker} \alpha$, using (3-2) on H we get

$$\Delta(\omega \rfloor H) = (\Delta^\nu H + H) \rfloor \omega. \tag{3-9}$$

Combining (3-7)–(3-9), we have

$$0 = -\Delta^\nu H \rfloor \omega - H + K \omega \rfloor H = (-\Delta^\nu H + (K-1) H) \rfloor \omega,$$

which implies that

$$-\Delta^\nu H + (K-1) H = f \boldsymbol{R} \tag{3-10}$$

for some function f on L.

The next lemma is one of our key observations which states that a Legendrian submanifold in a Sasakian manifold is contact stationary if and only if $\langle \Delta^\nu H, \boldsymbol{R} \rangle = 0$.

Lemma 3.7. *Let $L \subseteq (M^{2n+1}, \alpha, g_\alpha, J)$ be a contact stationary Legendrian submanifold. Then $\Delta^\nu H$ is orthogonal to $\boldsymbol{R}$.*

Proof. For any point $p \in L$, we choose a local orthonormal frame $\{E_i : i = 1, \dots, n\}$ of L such that $\nabla_{E_i} E_j(p) = 0$. We have at p (in the following computation we adopt the Einstein summation convention)

$$\begin{aligned}
\langle \Delta^\nu H, \boldsymbol{R}\rangle &= \langle \nabla^\nu_{E_i}\nabla^\nu_{E_i} H, \boldsymbol{R}\rangle \\
&= E_i\langle \nabla^\nu_{E_i} H, \boldsymbol{R}\rangle - \langle \nabla^\nu_{E_i} H, \overline{\nabla}_{E_i}\boldsymbol{R}\rangle \\
&= E_i\langle \nabla^\nu_{E_i} H, \boldsymbol{R}\rangle + \langle \nabla^\nu_{E_i} H, JE_i\rangle \\
&= E_i(E_i\langle H, \boldsymbol{R}\rangle - \langle H, \overline{\nabla}_{E_i}\boldsymbol{R}\rangle) + \langle \nabla^\nu_{E_i} H, JE_i\rangle \\
&= E_i\langle H, JE_i\rangle + \langle \nabla^\nu_{E_i} H, JE_i\rangle \\
&= 2\langle \nabla^\nu_{E_i} H, JE_i\rangle + \langle H, \overline{\nabla}_{E_i} JE_i\rangle \\
&= 2\langle \nabla^\nu_{E_i} H, JE_i\rangle + \langle H, J\overline{\nabla}_{E_i} E_i\rangle \\
&= 2\langle \nabla^\nu_{E_i} H, JE_i\rangle \\
&= 2\langle \overline{\nabla}_{E_i} H, JE_i\rangle \\
&= -2\langle J\overline{\nabla}_{E_i} H, E_i\rangle \\
&= -2\langle \overline{\nabla}_{E_i} JH, E_i\rangle \\
&= -2\langle \nabla_{E_i} JH, E_i\rangle \\
&= -2\,\mathrm{div}_g(JH) \\
&= 0.
\end{aligned}$$

Note that in this computation we used Equation (2-3) and Lemmas 2.8 and 2.9 several times and the last equality holds because L is contact stationary. □

Therefore we have

$$(-\Delta^\nu H + (K-1)H) \perp \boldsymbol{R}$$

by this lemma and Lemma 2.8, which shows $f \equiv 0$, i.e.,

$$-\Delta^\nu H + (K-1)H = 0,$$

and we are done with the proof of Theorem 1.3.

Proof of Theorem 1.8. Let L be a Legendrian surface in $\mathbb{S}^5$ with the induced metric g. Let $\{e_1, e_2\}$ be an orthogonal frame on L such that $\{e_1, e_2, Je_1, Je_2, \boldsymbol{R}\}$ is an orthonormal frame on $\mathbb{S}^5$.

In the following we use indices i, j, k, l, s, t, m and β and γ such that

$$1 \le i, j, k, l, s, t, m \le 2, \quad 1 \le \beta, \gamma \le 3, \quad \gamma^* = \gamma + 2 \quad \text{and} \quad \beta^* = \beta + 2.$$

Let B be the second fundamental form of L in $\mathbb{S}^5$ and define

$$h^k_{ij} = g_\alpha(B(e_i, e_j), Je_k), \tag{3-11}$$

$$h^3_{ij} = g_\alpha(B(e_i, e_j), \boldsymbol{R}). \tag{3-12}$$

Then

$$h^k_{ij} = h^j_{ik} = h^i_{kj}, \tag{3-13}$$

$$h^3_{ij} = 0. \tag{3-14}$$

The Gauss equations and Ricci equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_s (h^s_{ik}h^s_{jl} - h^s_{il}h^s_{jk}) \tag{3-15}$$

$$R_{ik} = \delta_{ik} + 2\sum_s H^s h^s_{ik} - \sum_{s,j} h^s_{ij}h^s_{jk}, \tag{3-16}$$

$$2K = 2 + 4H^2 - S, \tag{3-17}$$

$$\begin{aligned} R_{3412} &= \sum_i (h^1_{i1}h^2_{i2} - h^1_{i2}h^2_{i1}) \\ &= \det h^1 + \det h^2, \end{aligned} \tag{3-18}$$

where h^1 and h^2 are the second fundamental forms with respect to the directions Je_1 and Je_2.

In addition we have the following Codazzi equations and Ricci identities

$$h^\beta_{ijk} = h^\beta_{ikj}, \tag{3-19}$$

$$h^\beta_{ijkl} - h^\beta_{ijlk} = \sum_m h^\beta_{mj}R_{mikl} + \sum_m h^\beta_{mi}R_{mjkl} + \sum_\gamma h^\gamma_{ij}R_{\gamma^*\beta^*kl}. \tag{3-20}$$

Using these equations, we can get the following Simons' type inequality:

Lemma 3.8. *Let L be a Legendrian surface in $\mathbb{S}^5$. Then we have*

$$\begin{aligned} \frac{1}{2}\Delta\sum_{i,j,\beta}(h^\beta_{ij})^2 \geq |\nabla^T h|^2 - 2|\nabla^T H|^2 - 2|\nabla^\nu H|^2 + \sum_{i,j,k,\beta}(h^\beta_{ij}h^\beta_{kki})_j \\ + S - 2H^2 + 2(1+H^2)\rho^2 - \rho^4 - \frac{1}{2}S^2, \end{aligned} \tag{3-21}$$

where $|\nabla^T h|^2 = \sum_{i,j,k,s}(h^s_{ijk})^2$ and $|\nabla^T H|^2 = \sum_{i,s}(H^s_i)^2$.

Proof. Using equations (3-15)–(3-20), we have

$$
\begin{aligned}
(3\text{-}22)\quad \frac{1}{2}\Delta\sum_{i,j,\beta}(h^\beta_{ij})^2
&= \sum_{i,j,k,\beta}(h^\beta_{ijk})^2+\sum_{i,j,k,\beta}h^\beta_{ij}h^\beta_{kijk}\\
&=|\nabla h|^2-4|\nabla^\nu H|^2+\sum_{i,j,k,\beta}(h^\beta_{ij}h^\beta_{kki})_j+\sum_{i,j,l,k,\beta}h^\beta_{ij}(h^\beta_{lk}R_{lijk}+h^\beta_{il}R_{lj})\\
&\qquad+\sum_{i,j,k,\beta,\gamma}h^\beta_{ij}h^\gamma_{ki}R_{\gamma^*\beta^*jk}\\
&=|\nabla h|^2-4|\nabla^\nu H|^2+\sum_{i,j,k,s}(h^s_{ij}h^s_{kki})_j+2K\rho^2-2(\det h^1+\det h^2)^2\\
&\geq|\nabla h|^2-4|\nabla^\nu H|^2+\sum_{i,j,k,\beta}(h^\beta_{ij}h^\beta_{kki})_j+2(1+H^2)\rho^2-\rho^4-\tfrac{1}{2}S^2,
\end{aligned}
$$

where $\rho^2:=S-2H^2$ and in the above calculations we used the identities

$$
\sum_{i,j,k,l,\beta}h^\beta_{ij}(h^\beta_{lk}R_{lijk}+h^\beta_{il}R_{lj})=2K\rho^2,
$$
$$
\sum_{i,j,k,\beta,\gamma}h^\beta_{ij}h^\gamma_{ki}R_{\gamma^*\beta^*jk}=-2(\det h^1+\det h^2)^2,
$$

where in the first equality we used $R_{lijk}=K(\delta_{lj}\delta_{ik}-\delta_{lk}\delta_{ij})$ and $R_{lj}=K\delta_{lj}$ in a proper coordinate, because L is a surface.

Note that

$$
\begin{aligned}
(3\text{-}23)\qquad |\nabla h|^2=\sum_{i,j,k,\beta}(h^\beta_{ijk})^2&=|\nabla^T h|^2+\sum_{i,j,k}(h^3_{ijk})^2\\
&=|\nabla^T h|^2+\sum_{i,j,k}(h^k_{ij})^2=|\nabla^T h|^2+S,
\end{aligned}
$$

where in the third equality we used

$$
\begin{aligned}
h^3_{ijk}&=\langle\overline{\nabla}_{e_k}B(e_i,e_j),\boldsymbol{R}\rangle=-\langle B(e_i,e_j),\overline{\nabla}_{e_k}\boldsymbol{R}\rangle\\
&=\langle B(e_i,e_j),Je_k\rangle=h^k_{ij}.
\end{aligned}
$$

Similarly we have

$$
(3\text{-}24)\qquad |\nabla^\nu H|^2=|\nabla^T H|^2+H^2.
$$

Combing (3-22), (3-23) and (3-24) we get (3-21). □

Now we prove an integral equality for L, by using (1-3).

Lemma 3.9. *Let $L : \Sigma \to \mathbb{S}^5$ be a contact stationary Legendrian surface, where $\mathbb{S}^5$ is the unit sphere with standard contact structure and metric. Then*

$$\int_L |\nabla^\nu H|^2 \, d\mu = -\int_L (K-1) H^2 \, d\mu, \tag{3-25}$$

where $|\nabla^\nu H|^2 = \sum_{\beta,i} (H_i^\beta)^2$.

Proof. By using (1-3) we have

$$\begin{aligned} |\nabla^\nu H|^2 &= \sum_{\beta,i} (H_i^\beta)^2 \\ &= \sum_{\beta,i} (H_i^\beta H^\beta)_i - \sum_\beta H^\beta \Delta^\nu H^\beta \\ &= \sum_{\beta,i} (H_i^\beta H^\beta)_i - (K-1) H^2. \end{aligned} \tag{3-26}$$

We get (3-25) by integrating over (3-26). □

Integrating over (3-21) and using $|\nabla^T h|^2 \ge 3|\nabla^T H|^2$ (see Lemma A.1) we get

$$\begin{aligned} 0 &\ge \int_L \big[(|\nabla^T h|^2 - 2|\nabla^T H|^2) - 2|\nabla^\nu H|^2 + S - 2H^2 + 2(1+H^2)\rho^2 - \rho^4 - \tfrac{1}{2}S^2\big]\, d\mu \\ &\ge \int_L \big[-2|\nabla^\nu H|^2 + S - 2H^2 + 2(1+H^2)\rho^2 - \rho^4 - \tfrac{1}{2}S^2\big]\, d\mu \\ &= \int_L (2-\rho^2)\rho^2 d\mu + \int_L 2H^2\rho^2 + 2(K-1)H^2 - 2H^2 + S - \tfrac{1}{2}S^2 \, d\mu \\ &= \int_L (2-\rho^2)\rho^2 d\mu + \int_L 2H^2\rho^2 + (4H^2 - S)H^2 - 2H^2 + S - \tfrac{1}{2}S^2 \, d\mu \\ &= \int_L (2-\rho^2)\rho^2 d\mu + \int_L H^2 S - 2H^2 + S - \tfrac{1}{2}S^2 \, d\mu \\ &= \int_L (2-\rho^2)\rho^2 d\mu + \int_L H^2(S-2) + \tfrac{1}{2}S(2-S)\, d\mu \\ &= \int_L (2-\rho^2)\rho^2 + (2-S)\big(\tfrac{1}{2}S - H^2\big)\, d\mu \\ &= \int_L \rho^2(2-\rho^2) + \tfrac{1}{2}\rho^2(2-S)\, d\mu \\ &= \int_L \tfrac{3}{2}\rho^2(2-S) + 2H^2\rho^2 \, d\mu, \end{aligned}$$

where in the second equality we used the Gauss equation $2K = 2 + 4H^2 - S$.

Therefore we obtain the desired integral inequality

$$\int_L \rho^2\big(3 - \tfrac{3}{2}S + 2H^2\big)\,d\mu \le 0.$$

Particularly if $0 \le S \le 2$, we must have $\rho^2 = 0$ and L is totally umbilic or $\rho^2 \ne 0$, which implies $S = 2$, $H = 0$ and L is a flat minimal Legendrian torus. Thus we have proved Theorem 1.8.

Appendix

In this section we prove the following lemma.

Lemma A.1. *Let L be a Legendrian surface in $\mathbb{S}^5$, and assume that $|\nabla^T h|^2$ and $|\nabla^T H|^2$ are defined as in Lemma 3.8. Then we have*

$$|\nabla^T h|^2 \ge 3|\nabla^T H|^2.$$

Proof. We construct the flowing symmetric tracefree tensor:

$$F^s_{ijk} = h^s_{ijk} - \frac{1}{2}(H^s_i\delta_{jk} + H^s_j\delta_{ik} + H^s_k\delta_{ji}). \tag{A-27}$$

Then it is easy to see that

$$|F|^2 = |\nabla^T h|^2 - 3|\nabla^T H|^2,$$

and we get $|\nabla^T h|^2 \ge 3|\nabla^T H|^2$. □

Final discussions. To end this paper we propose several questions which we will study in the future.

Problem 1. Is any umbilical contact stationary Legendrian surface in $\mathbb{S}^5$ with $0 \le S \le 2$ totally geodesic?

Problem 2. Assume that L is a closed csL submanifold in $\mathbb{S}^{2n+1}$, satisfying $0 \le S \le n$, then is L totally geodesic or $S = n$?

Problem 3. Is any contact stationary Legendrian surface in $\mathbb{S}^5$ with second fundamental form of constant length minimal?

Problem 4. What is the second gap for minimal Legendrian submanifolds in a sphere?

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