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**IRREDUCIBILITY OF THE MODULI SPACE
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ODD DEGREE ON A VERY GENERAL QUINTIC SURFACE**

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The moduli space $M(c_2)$ of stable rank-two vector bundles of degree one on a very general quintic surface $X \subset \mathbb{P}^3$ is irreducible for all $c_2 \geq 4$ and empty otherwise. On the other hand, for a very general sextic surface, the moduli space at $c_2 = 11$ has at least two irreducible components.

1. Introduction

Let $X \subset \mathbb{P}_{\mathbb{C}}^3$ be a very general quintic hypersurface. Let $M(c_2) := M_X(2, 1, c_2)$ denote the moduli space [Huybrechts and Lehn 1997] of stable rank 2 vector bundles on X of degree 1 with $c_2(E) = c_2$. Let $\bar{M}(c_2) := \bar{M}_X(2, 1, c_2)$ denote the moduli space of stable rank 2 torsion-free sheaves on X of degree 1 with $c_2(E) = c_2$. Recall that $\bar{M}(c_2)$ is projective, and $M(c_2) \subset \bar{M}(c_2)$ is an open set, whose complement is called the *boundary*. Let $\overline{M(c_2)}$ denote the closure of $M(c_2)$ inside $\bar{M}(c_2)$. This might be a strict inclusion, as will in fact be the case for $c_2 \leq 10$.

In [Mestrano and Simpson 2011] we showed that $M(c_2)$ is irreducible for $4 \leq c_2 \leq 9$, and empty for $c_2 \leq 3$. In [Mestrano and Simpson 2013] we showed that the open subset $M(10)^{\text{sn}} \subset M(10)$ of bundles with seminatural cohomology is irreducible. Nijssse [1995] showed that $M(c_2)$ is irreducible for $c_2 \geq 16$.

In the present paper, we complete the proof of irreducibility for the remaining intermediate values of c_2 .

Theorem 1.1. *For any $c_2 \geq 4$, the moduli space of bundles $M(c_2)$ is irreducible.*

For $c_2 \geq 11$, the moduli space of torsion-free sheaves $\bar{M}(c_2)$ is irreducible. On the other hand, $\bar{M}(10)$ has two irreducible components: the closure $\overline{M(10)}$ of the irreducible open set $M(10)$; and the smallest stratum $M(10, 4)$ of the double dual

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stratification corresponding to torsion-free sheaves whose double dual has $c'_2 = 4$. Similarly $\bar{M}(c_2)$ has several irreducible components when $5 \leq c_2 \leq 9$, too.

The moduli space $\bar{M}(c_2)$ is good for $c_2 \geq 10$, generically smooth of the expected dimension $4c_2 - 20$, whereas for $4 \leq c_2 \leq 9$, the moduli space $M(c_2)$ is not good. For $c_2 \leq 3$ it is empty.

Yoshioka [1997; 1999; 2001], Gómez [1997] and others have shown that the moduli space of stable torsion-free sheaves with irreducible Mukai vector (which contains, in particular, the case of bundles of rank 2 and degree 1) is irreducible, over an abelian or K3 surface. Those results use the triviality of the canonical bundle, leading to a symplectic structure and implying among other things that the moduli spaces are smooth [Mukai 1984]. Notice that the case of K3 surfaces includes degree 4 hypersurfaces in \mathbb{P}^3 .

We were motivated to look at a next case, of bundles on a quintic or degree 5 hypersurface in \mathbb{P}^3 where $K_X = \mathcal{O}_X(1)$ is ample but not by very much. This paper is the third in a series starting with [Mestrano and Simpson 2011; 2013] dedicated to Professor Maruyama who, along with Gieseker, pioneered the study of moduli of bundles on higher dimensional varieties [Gieseker 1977; 1988; Maruyama 1973; 1975; 1982]. Recall that the moduli space of stable bundles is irreducible for $c_2 \gg 0$ on any smooth projective surface [Gieseker and Li 1994; Li 1993; O'Grady 1993; 1996], but there exist surfaces, such as smooth hypersurfaces in \mathbb{P}^3 of sufficiently high degree [Mestrano 1997], where the moduli space is not irreducible for intermediate values of c_2 .

Our theorem shows that the irreducibility of the moduli space of bundles $M(c_2)$, for all values of c_2 , can persist into the range where K_X is ample. On the other hand, the fact that $\bar{M}(10)$ has two irreducible components, means that if we consider all torsion-free sheaves, then the property of irreducibility in the good range has already started to fail in the case of a quintic hypersurface.

We furthermore show in Section 11 below that irreducibility fails for stable vector bundles on surfaces of degree $d = 6$. This improves the result of [Mestrano 1997] where nonirreducibility had been obtained on surfaces of degree $d \geq 27$.

A possible application of the irreducibility theorem to the case of Calabi–Yau varieties could be envisioned by noting that a general hyperplane section of a quintic threefold in \mathbb{P}^4 will be a quintic surface $X \subset \mathbb{P}^3$.

Outline of the proof. The starting point is O'Grady's [1993; 1996] method of deformation to the boundary, as exploited by Nijssse [1995] in the case of a very general quintic hypersurface. We use in particular some of the intermediate results of Nijssse who showed, for example, that $\bar{M}(c_2)$ is connected for $c_2 \geq 10$.

Application of these techniques is made possible by the explicit description of the moduli spaces $M(c_2)$ for $4 \leq c_2 \leq 9$ and the partial result for $M(10)$ obtained in [Mestrano and Simpson 2011; 2013].

Our approach therefore has a botanical flavor. The information gleaned from the descriptions in [Mestrano and Simpson 2011] allows us to understand the boundary components. It turns out that the bigger components growing out of these will correspond to bundles with seminatural cohomology, so that the result of [Mestrano and Simpson 2013] applies. We should stress that it is not a priori clear something like this should happen — the possibility of getting to the proof in this way becomes accessible only through an understanding of the components at lower levels. This will present a challenge for generalization to other surfaces.

The boundary $\partial\bar{M}(c_2) := \bar{M}(c_2) - M(c_2)$ is the set of points corresponding to torsion-free sheaves which are not locally free. We just endow $\partial\bar{M}(c_2)$ with its reduced scheme structure. There might in some cases be a better nonreduced structure which one could put on the boundary or onto some strata, but that won't be necessary for our argument and we don't worry about it here.

We can further refine the decomposition

$$\bar{M}(c_2) = M(c_2) \sqcup \partial\bar{M}(c_2)$$

by the *double dual stratification* [O'Grady 1996]. Let $M(c_2; c'_2)$ denote the locally closed subset, again with its reduced scheme structure, parametrizing sheaves F which fit into an exact sequence

$$0 \rightarrow F \rightarrow F^{**} \rightarrow S \rightarrow 0$$

such that $F \in \bar{M}(c_2)$ and S is a coherent sheaf of finite length $d = c_2 - c'_2$ hence $c_2(F^{**}) = c'_2$. Notice that $E = F^{**}$ is also stable so it is a point in $M(c'_2)$. The stratum can be nonempty only when $c'_2 \geq 4$, which shows by the way that $\bar{M}(c_2)$ is empty for $c_2 \leq 3$. The boundary now decomposes into locally closed subsets

$$\partial\bar{M}(c_2) = \coprod_{4 \leq c'_2 < c_2} M(c_2; c'_2).$$

Let $\overline{M(c_2, c'_2)}$ denote the closure of $M(c_2, c'_2)$ in $\bar{M}(c_2)$. Notice that we don't know anything about the position of this closure with respect to the stratification; its boundary will not in general be a union of strata. We can similarly denote by $\overline{M(c_2)}$ the closure of $M(c_2)$ inside $\bar{M}(c_2)$, a subset which might well be strictly smaller than $\bar{M}(c_2)$.

The construction $F \mapsto F^{**}$ provides, by the definition of the stratification, a well-defined map

$$M(c_2; c'_2) \rightarrow M(c'_2).$$

The fiber over $E \in M(c'_2)$ is the Grothendieck Quot-scheme $\text{Quot}(E; d)$ of quotients of E of length $d := c_2 - c'_2$.

It follows from Li's theorem [Li 1993, Proposition 6.4] that if $M(c'_2)$ is irreducible, then $M(c_2; c'_2)$ and hence $\overline{M}(c_2; c'_2)$ are irreducible, with $\dim(M(c_2; c'_2)) = \dim(M(c'_2)) + 3(c_2 - c'_2)$. See Corollary 4.3 below. From the previous papers [Mestrano and Simpson 2011; 2013], we know the dimensions of $M(c'_2)$, so we can fill in the dimensions of the strata, as will be summarized in Table 2. Furthermore, by [Mestrano and Simpson 2011] and Li's theorem, the strata $M(c_2; c'_2)$ are irreducible whenever $c'_2 \leq 9$.

Nijssse [1995] proves that $\overline{M}(c_2)$ is connected whenever $c_2 \geq 10$, using O'Grady's [1993; 1996] techniques. This is discussed and we review the proof in [Mestrano and Simpson 2016]. By [Mestrano and Simpson 2011], the moduli space $\overline{M}(c_2)$ is *good*, that is to say it is generically reduced of the expected dimension $4c_2 - 20$, whenever $c_2 \geq 10$. In particular, the dimension of the Zariski tangent space, minus the dimension of the space of obstructions, is equal to the dimension of the moduli space. The Kuranishi theory of deformation spaces implies that $\overline{M}(c_2)$ is locally a complete intersection. Hartshorne's [1962] connectedness theorem now says that if two different irreducible components of $\overline{M}(c_2)$ meet at some point, then they intersect in a codimension 1 subvariety. This intersection has to be contained in the singular locus.

The singular locus in $M(c_2)$ contains a subvariety denoted $V(c_2)$, which is the set of bundles E with $h^0(E) > 0$. It is the image of the space Σ_{c_2} of extensions

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow J_P(1) \rightarrow 0,$$

where P satisfies Cayley–Bacharach for quadrics. For $c_2 \geq 10$, $V(c_2)$ is irreducible of dimension $3c_2 - 11$. For $c_2 \geq 11$ one can see directly that the closure of $V(c_2)$ meets the boundary. For $c_2 = 10$, bundles in $V(10)$ almost have seminatural cohomology, in the sense that any deformation moving away from $V(10)$ will have seminatural cohomology, so $V(10)$ is contained only in the irreducible component constructed in [Mestrano and Simpson 2013], and that component meets the boundary. On the other hand, any other irreducible components of the singular locus have strictly smaller dimension [Mestrano and Simpson 2011, Corollary 7.1].

These properties of the singular locus, together with the connectedness statement of [Nijssse 1995], allow us to show that any irreducible component of $\overline{M}(c_2)$ meets the boundary. O'Grady proves furthermore an important lemma, that the intersection with the boundary must have pure codimension 1.

We explain the strategy for proving irreducibility of $M(10)$ and $M(11)$ below, but it will perhaps be easiest to explain first why this implies irreducibility of $M(c_2)$ for $c_2 \geq 12$. Based on O'Grady's method, this is the same strategy as was used by Nijssse who treated the cases $c_2 \geq 16$.

Suppose $c_2 \geq 12$ and $Z \subset \bar{M}(c_2)$ is an irreducible component. Suppose inductively we know that $M(c_2 - 1)$ is irreducible. Then $\partial Z := Z \cap \partial \bar{M}(c_2)$ is a nonempty subset in Z of codimension 1, thus of dimension $4c_2 - 21$. However, by looking at [Table 2](#), the boundary $\partial \bar{M}(c_2)$ is a union of the stratum $M(c_2, c_2 - 1)$ of dimension $4c_2 - 21$, plus other strata of strictly smaller dimension. Therefore, ∂Z must contain $M(c_2, c_2 - 1)$. But, the general torsion-free sheaf parametrized by a point of $M(c_2, c_2 - 1)$ is the kernel F of a general surjection $E \rightarrow S$ from a stable bundle E general in $M(c_2 - 1)$, to a sheaf S of length 1. We claim that F is a smooth point of the moduli space $\bar{M}(c_2)$. Indeed, if F were a singular point then there would exist a nontrivial coobstruction $\phi : F \rightarrow F(1)$; see [[Langer 2008](#); [Mestrano and Simpson 2011](#); [Zuo 1991](#)]. This would have to come from a nontrivial coobstruction $E \rightarrow E(1)$ for E , but that cannot exist because a general E is a smooth point since $M(c_2 - 1)$ is good. Thus, F is a smooth point of the moduli space. It follows that a given irreducible component of $M(c_2, c_2 - 1)$ is contained in at most one irreducible component of $\bar{M}(c_2)$. On the other hand, by the induction hypothesis $M(c_2 - 1)$ is irreducible, so $M(c_2, c_2 - 1)$ is irreducible. This gives the induction step, that $M(c_2)$ is irreducible.

The strategy for $M(10)$ is similar. However, due to the fact that the moduli spaces $M(c'_2)$ are not good for $c'_2 \leq 9$, in particular they tend to have dimensions bigger than the expected dimensions, there are several boundary strata which can come into play. Luckily, we know that the $M(c'_2)$, hence all of the strata $M(10, c'_2)$, are irreducible for $c'_2 \leq 9$.

The dimension of $M(10)$, equal to the expected one, is 20. Looking at the row $c_2 = 10$ in [Table 2](#) below, one may see that there are three strata $M(10, 9)$, $M(10, 8)$ and $M(10, 6)$ with dimension 19. These can be irreducible components of the boundary ∂Z if we follow the previous argument. More difficult is the case of the stratum $M(10, 4)$ which has dimension 20. A general point of $M(10, 4)$ is not in the closure of $M(10)$, in other words $M(10, 4)$, which is closed since it is the lowest stratum, constitutes a separate irreducible component of $\bar{M}(10)$. Now, if $Z \subset M(10)$ is an irreducible component, ∂Z could contain a codimension 1 subvariety of $M(10, 4)$.

The next idea is to use the main result of [[Mestrano and Simpson 2013](#)], that the moduli space $M(10)^{\text{sn}}$ of bundles with seminatural cohomology, is irreducible. To prove that $M(10)$ is irreducible, it therefore suffices to show that a general point of any irreducible component Z , has seminatural cohomology. From [[Mestrano and Simpson 2013](#)] there are two conditions that need to be checked: $h^0(E) = 0$ and $h^1(E(1)) = 0$. The first condition is automatic for a general point, since the locus $V(10)$ of bundles with $h^0(E) > 0$ has dimension $3 \cdot 10 - 11 = 19$ so cannot contain a general point of Z . For the second condition, it suffices to note that a general sheaf F in any of the strata $M(10, 9)$, $M(10, 8)$ and $M(10, 6)$ has $h^1(F(1)) = 0$;

and to show that the subspace of sheaves F in $M(10, 4)$ with $h^1(F(1)) > 0$ has codimension ≥ 2 . This latter result is treated in [Section 7](#), using the dimension results of Ellingsrud and Lehn for the scheme of quotients of a locally free sheaf, generalizing Li's theorem. This is how we will show irreducibility of $M(10)$.

The full moduli space of torsion-free sheaves $\overline{M}(10)$ has two different irreducible components, the closure $\overline{M}(10)$ and the lowest stratum $M(10, 4)$. This distinguishes the case of the quintic surface from the cases of abelian and K3 surfaces, where the full moduli spaces of stable torsion-free sheaves were irreducible [[Yoshioka 1999](#); [2001](#); [Gómez 1997](#)].

For $M(11)$, the argument is almost the same as for $c_2 \geq 12$. However, there are now two different strata of codimension 1 in the boundary: $M(11, 10)$ coming from the irreducible variety $M(10)$, and $M(11, 4)$ which comes from the other 20-dimensional component $M(10, 4)$ of $\overline{M}(10)$. To show that these two can give rise to at most a single irreducible component in $M(11)$, completing the proof, we will note that they do indeed intersect, and furthermore that the intersection contains smooth points.

After the end of the proof of [Theorem 1.1](#), the last two sections of the paper treat some related considerations.

In [Section 10](#) we provide a correction and improvement to [[Mestrano and Simpson 2011](#), Lemma 5.1] and answer that paper's Question 5.1. Recall from there that a coobstruction may be interpreted as a sort of Higgs field with values in the canonical bundle K_X ; it has a spectral surface $Z \subset \text{Tot}(K_X)$. The question was to bound the irregularity of a resolution of singularities of the spectral surface Z . We show in [Lemma 10.1](#) that the irregularity vanishes.

Example on a sextic. At the end of the paper in [Section 11](#), we show [Theorem 1.1](#) is sharp as far as the degree 5 of the very general hypersurface is concerned. In the case of bundles on very general hypersurfaces X^6 of degree 6, we show in [Theorem 11.4](#) that the moduli space $M_{X^6}(2, 1, 11)$ of stable rank two bundles of degree 1 and $c_2 = 11$ has at least two irreducible components. This improves the result of [[Mestrano 1997](#)], bringing from 27 down to 6 the degree of a very general hypersurface on which there exist two irreducible components. We expect that there will be several irreducible components in any degree ≥ 6 but that isn't shown here.

2. Preliminary facts

The moduli space $\overline{M}(c_2)$ is locally a fine moduli space. The obstruction to existence of a Poincaré universal sheaf on $\overline{M}(c_2) \times X$ is an interesting question but not considered in the present paper. A universal family exists étale-locally over $\overline{M}(c_2)$ so for local questions we may consider $\overline{M}(c_2)$ as a fine moduli space.

c_2	4	5	6	7	8	9
$\dim(M)$	2	3	7	9	13	16
$\dim(\text{obs})$	6	3	3	3	1	1
$h^1(E(1))$	0	1	0	0	0	0
generically	sm	sm	sm	nr	sm	nr

Table 1. Moduli spaces for $c_2 \leq 9$.

The Zariski tangent space to $\bar{M}(c_2)$ at a point E is $\text{Ext}^1(E, E)$. If E is locally free, this is the same as $H^1(\text{End}(E))$. The *space of obstructions* $\text{obs}(E)$ is by definition the kernel of the surjective map

$$\text{Tr} : \text{Ext}^2(E, E) \rightarrow H^2(\mathcal{O}_X).$$

The *space of coobstructions* is the dual $\text{obs}(E)^*$ which is, by Serre duality with $K_X = \mathcal{O}_X(1)$, equal to $\text{Hom}^0(E, E(1))$, the space of maps $\phi : E \rightarrow E(1)$ such that $\text{Tr}(\phi) = 0$ in $H^0(\mathcal{O}_X(1)) \cong \mathbb{C}^4$. Such a map is called a *coobstruction*.

Since a torsion-free sheaf E of rank two and odd degree can have no rank-one subsheaves of the same slope, all semistable sheaves are stable, and Gieseker and slope stability are equivalent. If E is a stable sheaf then $\text{Hom}(E, E) = \mathbb{C}$ so the space of trace-free endomorphisms is zero. Notice that $H^1(\mathcal{O}_X) = 0$ so we may disregard the trace-free condition for $\text{Ext}^1(E, E)$. An Euler characteristic calculation gives

$$\dim(\text{Ext}^1(E, E)) - \dim(\text{obs}(E)) = 4c_2 - 20,$$

and this is called the *expected dimension* of the moduli space. The moduli space is said to be *good* if the dimension is equal to the expected dimension.

Lemma 2.1. *If the moduli space is good, then it is locally a complete intersection.*

Proof. Kuranishi theory expresses the local analytic germ of the moduli space $\bar{M}(c_2)$ at E , as $\Phi^{-1}(0)$ for a holomorphic map of germs $\Phi : (\mathbb{C}^a, 0) \rightarrow (\mathbb{C}^b, 0)$ where $a = \dim(\text{Ext}^1(E, E))$ (resp. $b = \dim(\text{obs}(E))$). Hence, if the moduli space has dimension $a - b$, it is a locally complete intersection. \square

We investigated closely the structure of the moduli space for $c_2 \leq 9$ in [Mestrano and Simpson 2011].

Proposition 2.2. *The moduli space $M(c_2)$ is empty for $c_2 \leq 3$. For $4 \leq c_2 \leq 9$, the moduli space $M(c_2)$ is irreducible. It has dimension strictly bigger than the expected one, for $4 \leq c_2 \leq 8$, and for $c_2 = 9$ it is generically nonreduced but with dimension equal to the expected one; it is also generically nonreduced for $c_2 = 7$. The dimensions of the moduli spaces, the dimensions of the spaces of obstructions at a general point, and the dimensions $h^1(E(1))$ for a general bundle E in $M(c_2)$, are given in Table 1 above.*

The proof of [Proposition 2.2](#) will be given in the next section, with a review of the cases $c_2 \leq 9$ from the paper [\[Mestrano and Simpson 2011\]](#).

We also proved that the moduli space is good for $c_2 \geq 10$, known by Nijssse [\[1995\]](#) for $c_2 \geq 13$.

Proposition 2.3. *For $c_2 \geq 10$, the moduli space $M(c_2)$ is good. The singular locus $M(c_2)^{\text{sing}}$ is the union of the locus $V(c_2)$ consisting of bundles with $h^0(E) > 0$, which has dimension $3c_2 - 11$, plus other pieces of dimension ≤ 13 which in particular have codimension ≥ 6 .*

Proof. Following O’Grady’s and Nijssse’s terminology $V(c_2)$ denotes the locus which is the image of the moduli space of bundles together with a section, called Σ_{c_2} or sometimes $\{E, P\}$. See [\[Mestrano and Simpson 2011, Theorem 7.1\]](#). Any pieces of the singular locus corresponding to bundles which are not in $V(c_2)$, have dimension ≤ 13 by [\[Mestrano and Simpson 2011, Corollary 5.1\]](#) (see [Lemma 10.1](#) below for a correction and improvement of this statement). \square

The case $c_2 = 10$ is an important central point in the classification, where the case-by-case treatment gives way to a general picture. In [\[Mestrano and Simpson 2013\]](#), we proved the following partial result that will be used in the present paper to complete the proof of irreducibility.

Proposition 2.4. *Let $M(10)^{\text{sn}} \subset M(10)$ denote the open subset of bundles $E \in M(10)$ which have *seminatural cohomology*, that is, where for any m at most one of $h^i(E(m))$ is nonzero for $i = 0, 1, 2$. Then $E \in M(10)^{\text{sn}}$ if and only if $h^0(E) = 0$ and $h^1(E(1)) = 0$. The moduli space $M(10)^{\text{sn}}$ is irreducible.*

Proof. See [\[Mestrano and Simpson 2013\]](#), Theorem 0.2 and Corollary 3.5. \square

3. Review of $c_2 \leq 9$

Our strategy of proof uses in a fundamental way an understanding of the irreducible components for $c_2 \leq 9$ that were studied in [\[Mestrano and Simpson 2011\]](#). The discussion of these moduli spaces went by a sometimes exhaustive classification of cases Lemmas 7.3, 7.4 there. In retrospect we can give more uniform proofs of some parts. For this reason, and for the reader’s convenience, it is worthwhile to review here some of the arguments leading to the proof of [Proposition 2.2](#). This section may, however, be skipped or perused lightly on the first reading.

There is a change of notation with respect to that work. There we considered bundles of degree -1 . The bundle of degree 1 denoted here by E is the same as the bundle denoted by $E(1)$ there. Thus Lemma 5.2 there speaks of $h^1(E)$ in our notation. The present notation was already in effect in [\[Mestrano and Simpson 2013\]](#). Fortunately, the indexing by second Chern class remains the same in both cases.

Following O’Grady, we denote by $V(c_2) \subset M(c_2)$ the subvariety of bundles such that $h^0(E) > 0$. For $c_2 \leq 9$ the Euler characteristic argument of [Mestrano and Simpson 2011, §6.1] tells us that $h^0(E) > 0$ for any E , so $V(c_2)$ is the full moduli space.

It will be useful to consider the moduli space Σ_{c_2} consisting of pairs (E, η) where $E \in M(c_2)$ and $\eta \in H^0(E)$ is a nonzero section. The pairs are taken up to isomorphism, i.e., up to scaling of the section, so the fiber of the map $\Sigma_{c_2} \rightarrow V(c_2)$ over a bundle E is the projective space $\mathbb{P}H^0(E)$.

Each irreducible component of Σ_{c_2} has dimension $\geq 3c_2 - 11$, see [O’Grady 1996; Nijssse 1995] or [Mestrano and Simpson 2011, Corollary 3.1].

A point of Σ_{c_2} may also be considered as an extension of the form

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow J_{P/X}(1) \rightarrow 0,$$

again up to isomorphism. We therefore employ the notation $\{E, P\} := \Sigma_{c_2}$, too.

Such an extension exists, with E a bundle, if and only if $P \subset X$ is locally a complete intersection of length c_2 and satisfies the Cayley–Bacharach condition for quadrics denoted CB(2). See [Barth 1977; Griffiths and Harris 1978; Reider 1988] and the references for the Hartshorne–Serre correspondence discussed in [Arrondo 2007] for the origins of this principle.

Denote by $\{P\}$ the Hilbert scheme of l.c.i. subschemes P that satisfy CB(2). The map $\{E, P\} \rightarrow \{P\}$ has fibers described as follows: the fiber over P is a dense open subset¹ of the projective space of all extensions $\mathbb{P} \text{Ext}^1(J_{P/X}(1), \mathcal{O}_X)$; its dimension by duality is $h^1(J_{P/X}(1)) - 1$.

Consider c the number of conditions imposed by P on quadrics. This is related to $h^1(E(1))$ by the exact sequences

$$H^0(\mathcal{O}_X(2)) \rightarrow H^0(\mathcal{O}_P(2)) \rightarrow H^1(J_{P/X}(2)) \rightarrow 0$$

and

$$0 \rightarrow H^1(E(1)) \rightarrow H^1(J_{P/X}(2)) \rightarrow H^2(\mathcal{O}_X(1)) \rightarrow 0,$$

where $H^2(E(1)) = H^0(E(1))^* = 0$ by stability, and $H^2(\mathcal{O}_X(1)) = H^2(K_X) = \mathbb{C}$. The number c is the rank of the evaluation map of $H^0(\mathcal{O}_X(2))$ on P , so $h^1(J_{P/X}(2)) = c_2 - c$, and by the second exact sequence we have $h^1(E(1)) = c_2 - c - 1$.

The number $c_2 - c - 1$ is also equal to the dimension of the fiber of the map from the space of extensions $\{E, P\}$ to the Hilbert scheme of subschemes $\{P\}$. As stated previously, the space of extensions $\{E, P\}$ fibers over the moduli space of bundles $\{E\}$ with fiber $\mathbb{P}H^0(E)$ of dimension $h^1(J_{P/X}(1))$.

¹It is the open subset of extensions such that E is locally free, nonempty because of the conditions on P .

The locus $V(c_2)$, image of Σ_{c_2} , is the main piece of the set of potentially obstructed bundles, that is to say bundles for which the space of obstructions is nonzero.

The other pieces are of smaller dimension. There was an error in the proof of this dimension estimate, Lemma 5.1 and hence Corollary 5.1 in [Mestran and Simpson 2011]. These will be corrected and improved in a separate section at the end of the present paper, see Lemma 10.1 below.

Using the Cayley–Bacharach condition. Recall that a 0-dimensional subscheme $P \subset \mathbb{P}^3$ satisfies the *Cayley–Bacharach condition* $\text{CB}(n)$ if, for any subscheme $P' \subset P$ with length $\ell(P') = \ell(P) - 1$, a section $f \in H^0(\mathcal{O}_{\mathbb{P}^3}(n))$ vanishing on P' must also vanish on P . When $P \subset X$ this is the condition governing the existence of an extension of $J_{P/X}(n-1)$ by \mathcal{O}_X that is locally free. For the study of Σ_{c_2} we are therefore interested in subschemes satisfying $\text{CB}(2)$.

See [Mestran and Simpson 2011; 2013] and the survey [Mestran and Simpson 2016] for details on the basic techniques we use to analyze the Cayley–Bacharach condition.

If $U \subset \mathbb{P}^3$ is a divisor, usually for us a plane, and P a subscheme, there is a *residual subscheme* P' for P with respect to U . In the case of distinct points it is just the complement of $P \cap U$, but more generally it has a schematic meaning with $\ell(P') + \ell(P \cap U) = \ell(P)$. If P satisfies $\text{CB}(n)$ and U has degree m then the residual P' satisfies $\text{CB}(n-m)$.

The following fact will be used often: if P' is the residual of P with respect to U , and if $Z \subset \mathbb{P}^3$ is a subvariety, then the length of $Z \cap P$ at any point is at least equal to the length of $Z \cap P'$. So for example if P' has 3 points in a line (schematically), then P does too.

It is easy to see that the Cayley–Bacharach condition $\text{CB}(2)$ cannot be satisfied by ≤ 3 points, so the moduli space is empty for $c_2 \leq 3$. Here is a case-by-case review of the cases $4 \leq c_2 \leq 9$.

For $c_2 = 4, 5$. Here the subscheme P is either 4 or 5 points contained in a line. Both of these configurations impose $c = 3$ conditions on quadrics, since $h^0(\mathcal{O}_{\mathbb{P}^1}(2)) = 3$. This gives values of $4 - 3 - 1 = 0$ and $5 - 3 - 1 = 1$ for $h^1(E(1))$ respectively. The moduli space is generically smooth and its dimension is equal to $c_2 - 2$ by [Mestran and Simpson 2011, Lemma 7.7]. This may be seen directly from the more explicit descriptions we shall give in Section 7 below. We get the dimension of the space of coobstructions by subtracting the expected dimension. This completes the proof of Proposition 2.2 for the columns $c_2 = 4, 5$.

For $c_2 = 6, 7$. In both cases, the Euler characteristic argument of [Mestran and Simpson 2011, Section 6.1] gives $h^0(E) = 2$, hence $h^0(J_{P/X}(1)) = 1$ and P is

contained in a unique plane U . By Lemma 5.5 there, the space of obstructions has dimension 3.

For $c_2 = \ell(P) = 6$, see Proposition 7.4 there that we now review. The number c of conditions imposed on quadrics has to be ≤ 5 , in particular P is contained in a planar conic $Y \subset U$. However, $c \leq 4$ may be ruled out by the size of P and the Cayley–Bacharach condition; see the second paragraph of §7.5 there. It follows that the dimension of $\{E, P\}$ equals the dimension of $\{P\}$, and as noted above this dimension is $\geq 3c_2 - 11 = 7$.

Look at the family of length 6 subschemes $P \subset X \cap Y$ such that all points of P are located either at smooth points of Y , or at smooth points of $X \cap U$. Such a subscheme is uniquely determined by its multiplicities at each point, so given Y the set of choices of P is discrete and if we generalize Y , the subscheme P generalizes. Therefore, this defines a set of irreducible components of dimension equal to the dimension of the space of choices of Y , that is 8. For U fixed and Y general, the choice of P is equivalent to the choice of complementary set of 4 points in $Y \cap X$; but since any 4 points in the plane lie on a conic, the monodromy action as we move Y can take any choice of 4 points to any other one. Therefore, this family is a single irreducible component of dimension 8.

The remaining locus of P containing a point where Y is singular and U is tangent to X , has dimension ≤ 5 . For example if there is one such point, then the space of choices of U has dimension 2; the space of choices of Y has dimension 2; and by the precise estimate of [Briřançon et al. 1981, Proposition 4.3], noting that Y has multiplicity 2 at the singular point, the space of choices of P has dimension ≤ 1 . For more points, we get one further dimension of the space of choices of P for each other point but more than 1 new condition imposed by the tangencies. Therefore, the locus of subschemes not fitting into the situation of the previous paragraph has dimension < 7 , and it cannot produce a new irreducible component.

This completes the discussion for $c_2 = 6$: we have an irreducible component of $\{E, P\}$ of dimension 8 whose general point consists of a choice of 6 out of the 10 intersection points in $X \cap Y$ for a plane conic Y . Since $h^0(E) = 2$ the dimension of $\{E\}$ is 7. For the table, notice that $h^1(E(1)) = 6 - 5 - 1 = 0$. Comparing dimension, expected dimension $4 \cdot 6 - 20 = 4$ and the dimension 3 of the space of obstructions, we find that the moduli space is generically smooth with vanishing obstruction maps.

Consider now the case $c_2 = 7$. See [Mestrano and Simpson 2011, Proposition 7.3] to be reviewed as follows. As previously from the second paragraph of §7.5 the same work, the case $c \leq 4$ may be ruled out. If $c = 5$, then P would be contained in a plane conic $Y \subset U$, but using the same arguments as before the dimension of the space of choices of P would be ≤ 8 ; however any irreducible component of $\{E, P\}$ has dimension $\geq 3 \cdot 7 - 11 = 10$ and the fiber of the map to $\{P\}$ has dimension 1, so a family of subschemes P of dimension ≤ 8 cannot contribute an

irreducible component. Therefore we may suppose $c = 6$, the dimensions of $\{E, P\}$ and $\{P\}$ are the same and are ≥ 10 . For a given plane U the space of choices of subscheme $P \subset X \cap U$ of length 7 has dimension 7 by [Brianchon et al. 1981]. The space of choices of P such that $U \cap X$ is singular (i.e., U tangent to X), therefore has dimension ≤ 9 and cannot contribute. If U is a plane such that $X \cap U$ is smooth, the Hilbert scheme of $P \subset X \cap U$ is irreducible and a general point corresponds to choosing 7 distinct points. We conclude that $\{E, P\}$ is irreducible of dimension 10 with general point consisting of a general subscheme $P \subset U \cap X$ of length 7 that indeed satisfies CB(2) imposing $c = 6$ conditions on quadrics.

Notice that since $h^0(E) = 2$ the map $\{E, P\} \rightarrow \{E\}$ is a fibration with fibers \mathbb{P}^1 so the corresponding irreducible component of the moduli space has dimension 9 as filled into the table. At a general point where P imposes $c = 6$ conditions on quadrics, we get $h^1(E(1)) = 7 - 6 - 1 = 0$. From [Mestranò and Simpson 2011, Proposition 7.3], by comparing dimensions the moduli space is generically nonreduced. This treats the column $c_2 = 7$.

For $c_2 = 8$. See the discussion in [Mestranò and Simpson 2011, Section 6.2] and Theorem 7.2 there which will now be reviewed with some improvement in the arguments allowing us to bypass certain case-by-case considerations.

Any component of $\{E, P\}$ has dimension $\geq 3 \cdot 8 - 11 = 13$.

The following technique, involving the residual subscheme recalled above, will be useful.

Lemma 3.1. *Suppose $U \subset \mathbb{P}^3$ is a plane, and let P' denote the residual subscheme for P with respect to U . If nonempty, P' satisfies CB(1), so $\ell(P') \geq 3$ and in case of equality P' is collinear.*

Let c be the number of conditions imposed on quadrics in \mathbb{P}^3 passing through P , and let n be the number of additional conditions on these quadrics needed to insure their vanishing on U . Suppose $10 - c \geq n + 1$. Then there exists a quadric containing P of the form $U \cup U'$ where U' is another plane, containing P . In particular, $P' \subset U'$. If $10 - c \geq n + 2$ then P' is contained in a line, and if $10 - c \geq n + 3$ then $P \subset U$.

Proof. The first paragraph is a restatement of the basic property of the residual subscheme. Note that one or two points, or three noncollinear points, cannot be CB(1).

In the second paragraph, we could define n as the dimension of the image of

$$H^0(J_{P/\mathbb{P}^3}(2)) \rightarrow H^0(\mathcal{O}_U(2)).$$

If $10 - c \geq n + 1$, then it means that we can impose n additional conditions (say, vanishing at general points of U) on the $(10 - c)$ -dimensional space quadrics $H^0(J_{P/\mathbb{P}^3}(2))$, to get one that vanishes on U . This quadric has the form $U \cup U'$ of

the union of U with another plane U' . By definition the residual is contained in U' . If $10 - c \geq n + 2$, then the U' move in a 2-dimensional family so they cut out a line containing P' . If $10 - c \geq n + 3$, the family of U' cuts out a point; however P' satisfying CB(1) cannot be a single point, so in this case it is empty and $P \subset U$. \square

Look at the value of c at a general point of an irreducible component. The case $c \leq 5$ may be ruled out (using a simpler version of the subsequent arguments), so we may assume either $c = 6$ or $c = 7$. If $c = 6$ then the fiber of $\{E, P\} \rightarrow \{P\}$ has dimension 1 and $\{P\}$ has dimension ≥ 12 , whereas if $c = 7$ then the irreducible component of $\{E, P\}$ is the same as that of $\{P\}$, and $\{P\}$ has dimension ≥ 13 .

It follows that a general P is not contained in any multiple of a plane. Indeed, the space of the $m.U$ has dimension 3 whereas for any one, the dimension of the space of length 8 subschemes $P \subset X \cap m.U$ is ≤ 8 by [Briřanęon et al. 1981]. (Here and below, by $m.U$ we denote the m -tuple scheme structure on U .)

Lemma 3.2. *In a given irreducible component, a general P does not contain a collinear subscheme of length ≥ 3 in a line.*

Proof. Start by noting that P is not contained in $U \cup L$ for a plane U and a line L . The space of quadrics containing $U \cup L$ has dimension 2, whereas $c \leq 7$ so there would be a third quadric containing P . One can see that it would have to contain L , so it defines a plane conic $Y \subset U$, meeting L , and $P \subset Y \cup L$. But the dimension of the space of choices of Y, L is 3 for the plane, 5 for the conic, 1 for the intersection point with L and then 2 for the direction of L making 11. Given Y, L the choice of P is discrete (except in some degenerate cases²). The set of such P can therefore not be dense in an irreducible component.

We now show that P cannot have three points collinear in a line R , assuming to the contrary that it does. Choose a point $p \in P$ not contained in R (possible by the paragraph above the lemma). Let U be the plane spanned by p and R . Vanishing on $P \cap R$ and at p imposes 4 conditions on conics of U .

In the case $c = 6$, by Lemma 3.1 with $n \leq 2$ so $4 = 10 - c \geq n + 2$, the residual P' of P with respect to U is contained in a line L , and we get $P \subset U \cup L$, contradicting the first paragraph.

In the case $c = 7$, by Lemma 3.1 with $n \leq 2$, so $3 = 10 - c \geq n + 1$, we get $P \subset U \subset U'$. Both U and U' must contain points not touching R . The residual P' of P with respect to U has length ≥ 4 , indeed if it were to consist of 3 points they would have to be collinear by the CB(1) property but that would give $P \subset U \cup L$.

²Since P is not contained in a double plane, Y is not a double line; in the other cases, singularities of $X \cap (Y \cup L)$ are always contained in planar singularities of multiplicity 2 so by [Briřanęon et al. 1981] the dimension of the space of P increases by 1 at any such point; but existence of the singularity imposes at least one additional condition decreasing the dimension of the space of Y, L .

If U' doesn't contain R , the intersection $P \cap (U' \cup R)$ has length³ at least 7, but since $U' \cup R$ is cut out by quadrics the CB(2) property of P says that in fact $P \subset (U' \cup R)$ contradicting the first paragraph of the proof.

Suppose $R \subset U'$. Given a residual point lying along R , it cannot correspond to a subscheme leaving R in a direction different from U' . For in that case, we could let U_2 be the plane contacting this direction, different from U or U' , and applying Lemma 3.1 again would give $P \subset U_2 \cup U_3$ contradicting the fact that both U and U' contain points of P not on R . So, any point of P' along R corresponds to a point of extra contact with U' . We conclude that the residual subscheme of $P \cap U'$ with respect to $R \subset U'$, has length ≥ 2 . Therefore, $n = 1$ conditions suffice to imply vanishing of quadrics on U' so by Lemma 3.1 this time with $3 = 10 - c \geq n + 2$ we find that the residual of P with respect to U' is contained in a line. This again gives P contained in a plane plus a line, contradicting the first paragraph of the proof. \square

We may now show that the case $c = 6$ doesn't contribute a general point of an irreducible component. Choose 3 points of P defining a plane U and apply Lemma 3.1 adding $n \leq 3$ extra conditions: we get at least one quadric in our family that has the form $U \cup U'$. Now if $U \cap P$ has length 5, then the residual would have length 3 and satisfy CB(1); therefore it would have to be collinear, contradicting the previous lemma. It follows that $U \cap P$ and $U' \cap P$ both have length 4. But then, it actually sufficed to add $n \leq 2$ conditions so we get a line containing the residual, again contradicting Lemma 3.2. This finishes ruling out the possibility of an irreducible component whose general point imposes $c \leq 6$ conditions on quadrics.

Therefore assume $c = 7$. Now $\{E, P\}$ and $\{P\}$ have the same dimension which is ≥ 13 . There is a vector space of dimension $10 - c = 3$ of quadrics passing through P . Let H_1, H_2, H_3 denote the elements of a basis of this space.

Here the proof divides into an analysis of two distinct cases; these were called (a) and (b) in [Mestran and Simpson 2011] referring to the two cases of Proposition 7.1 from there. Case (a) is when $H_1 \cap H_2 \cap H_3$ has dimension 0. It is a subscheme

³An algebraic argument is needed for the piece of P located at $R \cap U'$; letting A denote its coordinate algebra, u the equation of U' , f the equation of U and g the equation of another plane through R , our hypothesis is $f u A = 0$ and the local piece of $P \cap (U' \cup R)$ corresponds to A/guA . Considering the exact sequence

$$A/guA \rightarrow A/(fA + gA) \oplus A/uA \rightarrow \mathbb{C} \rightarrow 0,$$

we see that if the required inequality $\ell(A/guA) \geq \ell(A/(fA + gA)) + \ell(fA)$ didn't hold we would have $guA = (fA + gA) \cap uA$ and $fA \cong A/uA$, hence also $uA \cong A/fA$. The exact sequence

$$0 \rightarrow guA \rightarrow uA \rightarrow A/(fA + gA)$$

becomes $0 \rightarrow g(A/fA) \rightarrow A/fA \xrightarrow{u} A/(fA + gA)$, which would give that multiplication by u on $A/(fA + gA)$ is injective, but that isn't possible since A has finite length.

of length 8 so we get

$$P = H_1 \cap H_2 \cap H_3.$$

A general such subscheme satisfies CB(2), and I. Dolgachev pointed out to us that these are called ‘‘Cayley octads’’. We shall treat the Cayley octads of case (a) secondly, since that will use one part of the discussion of case (b).

Case (b): This is when the subscheme $Y = H_1 \cap H_2 \cap H_3$ contains a pure 1-dimensional subscheme Y_1 . Notice that Y_1 is a union of components of the curve⁴ $H_1 \cap H_2$. On the other hand, by Lasker’s theorem [Eisenbud et al. 1996, p. 314] if Y_1 were equal to $H_1 \cap H_2$ then there couldn’t be a third quadric vanishing on Y_1 . Therefore, Y_1 is a curve of degree ≤ 3 .

We will now show that Y_1 doesn’t contain a line. Suppose to the contrary that $R \subset Y_1$ is a line. Then all quadrics in our family contain R .

Choose a point p of P not on R , let U be the plane through R and p , and apply Lemma 3.1 with $n = 2$ to get $P \subset U \cup U'$. If $P \cap U'$ has length ≥ 5 , it doesn’t have four collinear points so it imposes 5 conditions on conics; hence we can apply Lemma 3.1 with $n = 1$ and get three residual points in a line, contradicting Lemma 3.2. Therefore $P \cap U$ has length ≥ 4 , however since $P \cap R$ has length ≤ 2 by Lemma 3.2, the residual of $P \cap U$ with respect to R has length ≥ 2 . Now, vanishing on R and on $P \cap U$ imposes 5 conditions on conics of U . Thus we may again apply Lemma 3.1 with $n = 1$ and get a residual consisting of 3 collinear points contradicting Lemma 3.2. This completes the proof that Y_1 does not contain a line.

That rules out almost all of the cases listed in [Mestrano and Simpson 2011, Lemma 7.4].

A next case is if Y_1 is a conic in a plane U . Then, it suffices to impose a single condition, $n = 1$ in Lemma 3.1, so $3 = 10 - c \geq n + 2$ and the residual subscheme consists of at least 3 points in a line. This contradicts Lemma 3.2, so Y_1 cannot be a plane conic.

The only remaining possibility for our curve of degree three, is that Y_1 could be a rational cubic curve not contained in a plane. It has to be a rational normal cubic, in particular smooth. The restriction of $\mathcal{O}_{\mathbb{P}^3}(2)$ to the rational curve has degree 6 so it has seven linearly independent sections; our three-dimensional family of quadrics is therefore the family of all quadrics passing through Y_1 . They define Y_1 schematically, in particular $P \subset Y_1$.

This case will be of interest for our treatment of case (a) below. We have that P is a length 8 subscheme of the intersection $Y_1 \cap X$. For given Y_1 the space of choices of P is discrete, and as Y_1 moves any P becomes general. The family of such subschemes may therefore be identified with a covering of the space of choices

⁴Note that H_i cannot all vanish on some plane, otherwise by CB(1) for the residual P would have to be contained in the plane as we saw previously.

of rational normal cubic Y_1 . The covering is determined, over a general point, by the choice of 8 out of the 15 points in $Y_1 \cap X$, or equivalently by the choice of the 7 complementary points.

The space of choices of Y_1 has dimension 12 (see [Mestrano and Simpson 2011, §6.2]). Therefore, this family cannot constitute an irreducible component of $\{P\}$. This completes the proof that case (b) cannot happen at a general point of an irreducible component.

Case (a): We start this discussion by continuing to look at the above 12-dimensional family of subschemes consisting of points in $X \cap Y_1$ for a smooth rational normal cubic curve Y_1 .

We claim that the family of subschemes, and hence of bundles, obtained in this way is irreducible. This may be seen as follows. Any 6 points from $X \cap Y_1$ determine the rational normal cubic Y_1 , so if we move a set of 6 points around to a different set, we get back to the same rational normal curve and this shows that the monodromy action includes permutations sending any subset of 6 points to any other one. On the other hand, there is a rational normal curve with first order tangency to X , and moving it a little bit induces a permutation of two points keeping the other points fixed. Therefore, the subgroup of the symmetric group contains a transposition. Now since it is 6-tuply transitive, it contains all the transpositions. Thus, the monodromy group is the full symmetric group and any group of 8 points can be moved to any other one. This shows that the family is irreducible.

As was pointed out at the end of Section 6.2 in [Mestrano and Simpson 2011], the space of obstructions at a general point in our family has dimension 1. The expected dimension is $4c_2 - 20 = 12$, so the Zariski tangent space to the moduli space has dimension 13; however, as noted above, any irreducible component has dimension ≥ 13 because of the existence of the extension. Therefore, a general point of our 12-dimensional family lies in a smooth open subset of a unique 13-dimensional irreducible component of the moduli space $\{E\}$ (notice here that the spaces $\{E, P\}$ and $\{P\}$ are also the same). As our 12-dimensional family is irreducible by the previous paragraph, this determines a canonical irreducible component of the moduli space.

This discussion corrects an error of notation in the second paragraph of the proof of Lemma 7.6 of [Mestrano and Simpson 2011], where it was stated that the irreducible 12-dimensional family of Cayley–Bacharach subschemes on the rational normal cubic was inside the type (a) subspace of the moduli space; but that family is clearly of type (b). Those phrases should be replaced by the argument of the previous paragraph showing that our 12-dimensional family is contained in a unique 13-dimensional irreducible component of the moduli space, whose general point is of type (a).

We now turn to consideration of the full set of irreducible components, whose general points are of type (a), that is to say bundles determined by Cayley octad subschemes P (since we showed in the previous part that type (b) cannot lead to a general point of a component).

The argument given in [Mestrano and Simpson 2011, §7.4], using the incidence variety suggested by A. Hirschowitz, shows that the existence of a canonically defined irreducible component implies irreducibility of the moduli space.

Let us recall here briefly how this works. We look at the full incidence scheme $\{X, P\}$ parametrizing smooth quintic hypersurfaces X together with l.c.i. subschemes $P \subset X$ of length 8 satisfying CB(2) of type (a). For a given $P \subset \mathbb{P}^3$ the space of quintics X containing it is a projective space and these all have the same dimension. So the fibration $\{X, P\} \rightarrow \{P\}$ is smooth, over the base that is an open subset in the Grassmannian $\text{Grass}(3, 10)$ of 3-dimensional subspaces of $H^0(\mathcal{O}_{\mathbb{P}^3}(2))$. Thus, the full incidence variety $\{X, P\}$ is irreducible. There is a dense open subset of the space of quintics $\{X\}$, over which the sets of irreducible components of the fibers don't change locally. Thus, the fundamental group of this open set acts on the set of irreducible components of the fiber $\{P\}_X$ over a basepoint $X \in \{X\}$. This action is transitive, by irreducibility of the full incidence variety. On the other hand, we have described above a canonically defined irreducible component of $\{P\}_X$, containing the nearby generalizations of our 12-dimensional family of subschemes of a rational normal cubic curve. Since it is canonically defined, this component is preserved by the monodromy action. Transitivity now implies that $\{P\}_X$ has only a single irreducible component.

This completes the proof of irreducibility for $c_2 = 8$. The generic space of obstructions has dimension 1. That was seen for points on the rational normal cubic curve, at the end of §6.2 of [Mestrano and Simpson 2011]; however the moduli space has dimension 13, equal to the expected dimension plus 1, so the space of obstructions remains 1-dimensional at a general point.

As the dimension of the moduli space is equal to the expected dimension plus the dimension of the space of obstructions, we get that the moduli space is generically smooth, and in fact that was already the case at a point of the 12-dimensional family of subschemes on a rational normal cubic. Since $c = 7$ at a general point we have $h^1(E(1)) = 8 - 7 - 1 = 0$, to complete the corresponding column of our table.

For $c_2 = 9$. For the column $c_2 = 9$, see [Mestrano and Simpson 2011, Theorem 6.1 and Proposition 7.2], for the dimension 16 and general obstruction space of dimension 1. The proof of Proposition 7.2 there starts out by ruling out, for a general point of an irreducible component, all cases of Proposition 7.1 there except case (d), for which $c = 8$. Thus $h^1(E(1)) = 9 - 8 - 1 = 0$ for a general bundle, as we shall also see below.

We give here an alternate argument by dimension count to show that a general bundle in any irreducible component consists of a collection of 9 out of the 20 points on a degree 4 elliptic curve, intersection of two quadrics, intersected with X .

The expected dimension of $\{E\}$ is $4c_2 - 20 = 16$, and a general E determines a unique⁵ extension hence a unique subscheme P of length 9. The dimension of any irreducible component of $\{E, P\}$ is ≥ 16 (notice that it coincides with the value of $3c_2 - 11$ too).

We first rule out the possibility that $c \leq 7$ for a general point. If there were a three-dimensional family of quadrics passing through P then they cannot intersect transversally in a zero-dimensional subscheme, since that would have length only 8 and so be unable to contain P . But if the intersection of the three quadrics has a component of positive dimension, then arguing much as in the previous section we can get a contradiction. Indeed, the space of length 9 subschemes contained in the intersection of X with two planes has dimension $\leq 3 + 3 + 9 = 15 < 16$, so any time [Lemma 3.1](#) applies we immediately obtain a contradiction. The remaining case of points on a rational normal curve is ruled out by dimension.

We may therefore assume $c = 8$, from which it follows that any irreducible component of $\{P\}$ has dimension ≥ 16 . It follows that a general P contains at least 7 points in general position on X . Let us explain the details of this argument, since this kind of dimension count has already been used several times above. Let $\mathbf{H} \subset \{P\}$ denote some component of the Hilbert scheme of subschemes we are interested in, that is to say l.c.i. subschemes $P \subset X$ of length 9 satisfying CB(2). Let

$$\mathbf{I} \subset \mathbf{H} \times X$$

be the incidence subscheme, whose fiber over a point $h \in \mathbf{H}$ is the subscheme P_h thereby parametrized. Suppose p_1, \dots, p_k is a collection of distinct points in X , and let $\mathbf{H}(p_1, \dots, p_k) \subset \mathbf{H}$ be the closed subscheme parametrizing those P that contain p_1, \dots, p_k . It may be inductively defined as follows: we have the incidence subvariety $\mathbf{I}(p_1, \dots, p_k) \subset \mathbf{H}(p_1, \dots, p_k) \times X$, and for a point p_{k+1} distinct from the other ones,

$$\mathbf{H}(p_1, \dots, p_k, p_{k+1}) := \text{pr}_2^{-1}(p_{k+1}) \cap \mathbf{I}(p_1, \dots, p_k).$$

By induction we show that for general points p_i , $\mathbf{H}(p_1, \dots, p_k)$ is nonempty of dimension $\geq 16 - 2k$ whenever $k \leq 7$. Assume it is known for $k - 1$ but not true for k . That means that the map $\mathbf{I}(p_1, \dots, p_{k-1}) \rightarrow X$ maps onto a closed subvariety; in other words, there is a curve $C \subset X$ depending on p_1, \dots, p_{k-1} and containing all of the subschemes parametrized by points of $\mathbf{H}(p_1, \dots, p_{k-1})$. But then the space of

⁵An easy dimension count rules out the possibility that P be contained in a plane.

such subschemes has dimension $\leq 9 - (k - 1)$ (by [Briřanęon et al. 1981]), contradicting our inductive hypothesis since $9 - (k - 1) < 16 - 2(k - 1)$ as $(k - 1) < 16 - 9 = 7$.

After the 7 points in general position there remain two points. We may conclude that the dimension of a family of subschemes P , once the set theoretical locations of the points are known, is ≤ 2 .

We now claim that if P is general, then for a general element H of our family of quadrics passing through P , the intersection $H \cap X$ is smooth. The proof is by a dimension count of the complementary family. If the $H \cap X$ is always singular, then the singular point is a basepoint (of the linear system on X), of which there are finitely many, so it is fixed. Thus, all the H are tangent to X at some point. The space of 2-dimensional linear systems tangent to $x \in X$ is a Grassmannian $\text{Grass}(2, \mathbb{C}^7)$ of dimension 10. As the point moves in X we have a 12-dimensional space of choices of the linear system; and each one of these fixes the set-theoretical location of the points of P so by the previous paragraph, the corresponding space of P has dimension ≤ 2 , so altogether we obtain that the family not satisfying our claimed condition has dimension ≤ 14 . Since any component has dimension ≥ 16 it follows that the complementary family cannot constitute a component, which proves the claim.

Suppose $V := H^0(J_{P/\mathbb{P}^3}(2)) \subset \mathbb{C}^{10} = H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ is the two-dimensional space of quadrics passing through our general point P . Then any deformation of the subspace $V \subset \mathbb{C}^{10}$ lifts to a deformation of P . This is because, by the previous claim, we can choose a general element of V corresponding to a quadric H_1 such that $H_1 \cap X$ is smooth. As the smooth curve deforms, our subscheme P of $(H_1 \cap X) \cap H_2$ becomes general since it is uniquely determined just by its multiplicities at each point.

From the above discussion it follows that a general point P in any irreducible component is obtained by choosing 9 out of the 20 points of $(H_1 \cap H_2) \cap X$ for a general pair of quadrics H_1, H_2 . But since any 8 points determine the subspace $\langle H_1, H_2 \rangle$, the monodromy action on the set of 20 intersection points is 8-tuply transitive. By going around a curve $H_1 \cap H_2$ with a single simple tangent point to X , we get a transposition in the monodromy group; hence it contains all transpositions and it is the full symmetric group. Therefore, the set of choices of 9 points forms a single orbit under the monodromy group. This completes the proof that there is only one irreducible component of dimension 16.

The space of obstructions at a general point has dimension 1, see the discussion above Theorem 6.1 in [Mestrano and Simpson 2011]. This completes our review of the proof of Proposition 2.2.

For $c_2 \geq 10$. We will not be further reviewing the partial result of the case $c_2 = 10$ that was treated in [Mestrano and Simpson 2013], giving irreducibility of the open subset of the moduli space corresponding to seminatural cohomology as was stated

in [Proposition 2.4](#) above, since the argument is more involved and it is the subject of a distinct paper.

On the other hand, it will be useful to discuss in more detail the structure of $V(c_2)$.

Lemma 3.3. *For $c_2 \geq 11$, $V(c_2)$ is irreducible of dimension $3c_2 - 11$ and its general point corresponds to a set of points P in general position with respect to quadrics. The closure of $V(c_2)$ meets the boundary.*

Proof. See [[Mestrano and Simpson 2011](#), Corollary 7.1], showing that for $c_2 \geq 11$, Σ_{c_2} contains an open dense subset $\Sigma_{c_2}^{10}$ consisting of collections P such that any colength 1 subscheme of P imposes vanishing of all quadrics. This is an open subset of the Hilbert scheme of all subschemes P of length c_2 so it is smooth, and it further contains an open dense subscheme where the points of P are distinct. The latter is an open subset of the symmetric product of X so it is irreducible.

The closure of $V(c_2)$ intersects the boundary, as was discussed in the proof of [[Nijssse 1995](#), Proposition 3.2]. Indeed, choose a collection P_0 of distinct points that impose vanishing of quadrics but that doesn't satisfy CB(2). Deform this collection in a family P_t such that the general P_t (for $t \neq 0$) satisfies CB(2). Since all elements of the family impose the same number of conditions on quadrics, the space of Ext groups varies in a bundle with respect to the parameter t and we may choose a family of extensions such that the general one is locally free. But the special one is not locally free since P_0 didn't satisfy CB(2). This family gives a curve in $\Sigma_{c_2}^{10}$ with parameter $t \neq 0$, whose limiting sheaf at $t = 0$ is not locally free: we have a deformation to the boundary. \square

Lemma 3.4. *For $c_2 = 10$, $V(10)$ is irreducible of dimension $3c_2 - 11 = 19$ and its general point corresponds to a subscheme P composed of 10 general points on a smooth intersection with a quadric $Y = X \cap H$. A general bundle in $V(10)$ has $h^1(E(1)) = 0$ so any deformation moving away from $V(10)$ will have seminatural cohomology, and only the irreducible component of $M(10)$ constructed in [[Mestrano and Simpson 2013](#)] contains $V(10)$.*

Proof. See [[Nijssse 1995](#), Lemma 3.1]. General elements of any irreducible component correspond to subschemes P not contained in a plane, so the irreducible components of $V(10)$ correspond to those of Σ_{10} having the same dimension.

By [[Mestrano and Simpson 2011](#), Corollary 7.1], Σ_{10} is pure of dimension 19. The stratum Σ_{10}^8 consisting of extensions where P lies in the intersection of two quadrics, has dimension < 19 . Indeed, the subscheme P is determined by the two-dimensional subspace of quadrics⁶ and this has dimension 16, to which we should add 1 for the space of choices of extension: it comes out strictly less than 19.

⁶Unless they share a common plane but that case may also be dealt with by a dimension count: 3 for the choice of plane, plus 4 for the choice of line, plus at most 7 for the choice of points in the

Similarly, the dimension of the stratum Σ_{10}^7 is strictly less than 19, and the strata Σ_{10}^c for $c \leq 6$ may be ruled out using our previous line of argument with [Lemma 3.1](#).

We conclude that the stratum Σ_{10}^9 is dense in Σ_{10} . Here the extension class is determined (up to scaling) so $\{E, P\}$ and $\{P\}$ are the same, and $\{P\}$ is an open subset of the space $\{H, P\}$ parametrizing quadrics H together with $P \subset H \cap X$. The open subset is given by the conditions that no other quadrics vanish on P , and that P satisfies CB(2). But the space $\{H, P\}$ is irreducible.

Thus, $V(10)$ is irreducible and its general point parametrizes collections of 10 general points on a general smooth quadric section $Y = X \cap H$. One may now calculate with the standard exact sequence that for a general $E \in V(10)$, we have $h^1(E(1)) = 0$.

Recall by [\[Mestrano and Simpson 2013, Corollary 3.5\]](#) that the condition of having seminatural cohomology, for bundles in $M(10)$, is equivalent to the conjunction of two conditions⁷ $h^1(E(1)) = 0$ and $h^0(E) = 0$. Bundles in $V(10)$ clearly don't satisfy the second condition because $V(10)$ is the locus where $h^0(E) > 0$. However, we have seen that a general point of $V(10)$ satisfies the first condition. On the other hand $V(10)$ is pure of dimension 19 whereas any component of $M(10)$ has dimension ≥ 20 . Therefore, in any irreducible component of $M(10)$ containing $V(10)$, the general point has $h^0(E) = 0$, but also $h^1(E(1)) = 0$ since it is a generization of the general point of $V(10)$ that satisfies this condition. Therefore, any irreducible component of $M(10)$ containing V parametrizes, generically, bundles with seminatural cohomology.

It now follows from the main result of [\[Mestrano and Simpson 2013\]](#) (stated as [Proposition 2.4](#) above) that any irreducible component of $M(10)$ containing $V(10)$ must be the unique component constructed there. \square

4. The double dual stratification

Turn now to the proof of the main theorem on the moduli spaces for $c_2 \geq 10$. Our subsequent proofs will make use of O'Grady's [\[1993; 1996\]](#) techniques, as they were recalled and used by Nijssse [\[1995\]](#). The main idea is to look at the boundary of the moduli spaces. His first main observation is the following:

Lemma 4.1 [\[O'Grady 1996, Proposition 3.3\]](#). *The boundary of any irreducible component (or indeed, of any closed subset) of $M(c_2)$ has pure codimension 1, if it is nonempty.*

plane since they would otherwise all be in the plane and then we could ignore the choice of line, plus 1 for the choice of extension class, comes out to strictly less than 19.

⁷We use duality and Euler characteristic to rewrite the conditions of [\[Mestrano and Simpson 2013, Corollary 3.5\]](#).

The boundary is divided up into Uhlenbeck strata corresponding to the “number of delta-like singular instantons”, which in the geometric picture corresponds to the number of points where the torsion-free sheaf is not a bundle, counted with correct multiplicities. A boundary stratum denoted $M(c_2, c_2 - d)$ parametrizes torsion-free sheaves F fitting into an exact sequence of the form

$$0 \rightarrow F \rightarrow E \xrightarrow{\sigma} S \rightarrow 0$$

where $E \in M(c_2 - d)$ is a stable locally free sheaf of degree 1 and $c_2(E) = c_2 - d$, and S is a finite coherent sheaf of length d so that $c_2(F) = c_2$. In this case $E = F^{**}$. We may think of $M(c_2, c_2 - d)$ as the moduli space of pairs (E, σ) . Forgetting the quotient σ gives a smooth map

$$M(c_2, c_2 - d) \rightarrow M(c_2 - d),$$

sending F to its double dual. The fiber over E is the Grothendieck Quot scheme $\text{Quot}(E, d)$ parametrizing quotients σ of E of length d .

Since we are dealing with sheaves of degree 1, all semistable points are stable and our objects have no nonscalar automorphisms. Hence the moduli spaces are fine, with a universal family existing étale-locally and well defined up to a scalar automorphism. We may view the double-dual map as being the relative Grothendieck Quot scheme of quotients of the universal object E^{univ} on $M(c_2 - d) \times X$ over $M(c_2 - d)$. Furthermore, locally on the Quot scheme the quotients are localized near a finite set of points, and we may trivialize the bundle E^{univ} near these points, so $M(c_2, c_2 - d)$ has a covering by, say, analytic open sets which are trivialized as products of open sets in the base $M(c_2 - d)$ with open sets in $\text{Quot}(E, d)$ for any single choice of E . This is all to say that the map $M(c_2, c_2 - d) \rightarrow M(c_2 - d)$ may be viewed as a fibration in a fairly strong sense, with fiber $\text{Quot}(E, d)$.

Li [1993, Proposition 6.4] shows that $\text{Quot}(E, d)$ is irreducible with a dense open subset U parametrizing quotients which are given by a collection of d quotients of length 1 supported at distinct points of X :

Theorem 4.2 [Li 1993]. *Suppose E is a locally free sheaf of rank 2 on X . Then for any $d > 0$, $\text{Quot}(E, d)$ is an irreducible scheme of dimension $3d$, containing a dense open subset parametrizing quotients $E \rightarrow S$ such that $S \cong \bigoplus \mathbb{C}_{y_i}$, where \mathbb{C}_{y_i} is a skyscraper sheaf of length 1 supported at $y_i \in X$, and the y_i are distinct. This dense open set maps to $X^{(d)} - \text{diag}$ (the space of choices of distinct d -tuple of points in X), with fiber over $\{y_i\}$ equal to $\prod_{i=1}^d \mathbb{P}(E_{y_i})$.*

Proof. See Proposition 6.4 in the appendix of [Li 1993]. Notice right away that U is an open subset of $\text{Quot}(F, d)$, and that U fibers over the set $X^{(d)} - \text{diag}$ of distinct d -tuples of points (y_1, \dots, y_d) (up to permutations). The fiber over a d -tuple (y_1, \dots, y_d) is the product of projective lines $\mathbb{P}(F_{y_i})$ of quotients of the vector

c_2	e.d.	$\dim(M)$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
4	-4	2	-	-	-	-	-	-	-	-
5	0	3	5	-	-	-	-	-	-	-
6	4	7	6	8	-	-	-	-	-	-
7	8	9	10	9	11	-	-	-	-	-
8	12	13	12	13	12	14	-	-	-	-
9	16	16	16	15	16	15	17	-	-	-
10	20	20	19	19	18	19	18	20	-	-
11	24	24	23	22	22	21	22	21	23	-
12	28	28	27	26	25	25	24	25	24	26
≥ 13	$4c_2 - 20$	$4c_2 - 20$	$4c_2 - 21$	$\leq 4c_2 - 22$						

Table 2. Dimensions of strata.

spaces F_{y_i} . As $X^{(d)} - \text{diag}$ has dimension $2d$, and $\prod_{i=1}^d \mathbb{P}(F_{y_i})$ has dimension d , we get that U is a smooth open variety of dimension $3d$.

This theorem may also be viewed as a consequence of a more precise bound established by Ellingsrud and Lehn [1999], which will be stated as **Theorem 7.6** below, needed for our arguments in **Section 7**. □

Corollary 4.3. *We have*

$$\dim(M(c_2; c'_2)) = \dim(M(c'_2)) + 3(c_2 - c'_2).$$

If $M(c'_2)$ is irreducible, then $M(c_2; c'_2)$ and hence $\overline{M(c_2; c'_2)}$ are irreducible.

Proof. The fibration $M(c_2; c'_2) \rightarrow M(c'_2)$ has fiber the Quot scheme whose dimension is $3(c_2 - c'_2)$ by the previous proposition. Furthermore, these Quot schemes are irreducible so if the base is irreducible, so is the total space. □

Corollary 4.3 allows us to fill in the dimensions of the strata $M(c_2; c'_2)$ in **Table 2**, starting from the dimensions of the moduli spaces given by **Propositions 2.2** and **2.3**. The entries in the second column are the expected dimension $4c_2 - 20$; in the third column the dimension of $M := M(c_2)$; and in the following columns, $\dim M(c_2, c_2 - d)$ for $d = 1, 2, \dots$. The rule is to add 3 as you go diagonally down and to the right by one.

The first remark useful for interpreting this information is that any irreducible component of $\overline{M}(c_2)$ must have dimension at least equal to the expected dimension $4c_2 - 20$. In particular, a stratum with strictly smaller dimension must be a part of at least one irreducible component containing a bigger stratum. For $c_2 \geq 11$, we have

$$\dim(M(c_2, c'_2)) < \dim(\overline{M}(c_2)) = 4c_2 - 20.$$

Hence, for $c_2 \geq 11$ the closures $\overline{M(c_2, c'_2)}$ cannot themselves form irreducible components of $\overline{M}(c_2)$, in other words the irreducible components of $\overline{M}(c_2)$ are the

same as those of $M(c_2)$. Notice, on the other hand, that $\overline{M}(10)$ contains two pieces of dimension 20, the locally free sheaves in $M(10)$ and the sheaves in $M(10, 4)$ whose double duals come from $M(4)$.

Recall from [Proposition 2.2](#) that the moduli spaces $M(c_2)$ are irreducible for $c_2 = 4, \dots, 9$. It follows from [Corollary 4.3](#) that the strata $M(c_2, c'_2)$ are irreducible, for any $c'_2 \leq 9$. In particular, the piece $\overline{M}(10, 4)$ is irreducible, and its general point, representing a not locally free sheaf, is not confused with any point of $\overline{M}(10)$. Since the other strata of $\overline{M}(10)$ all have dimension < 20 , it follows that $\overline{M}(10, 4)$ is an irreducible component of $\overline{M}(10)$. One similarly gets from the table that $\overline{M}(c_2)$ has several irreducible components when $5 \leq c_2 \leq 9$.

5. Hartshorne's connectedness theorem

Hartshorne proves a connectedness theorem for locally complete intersections. Here is the version that we need.

Theorem 5.1 [[Hartshorne 1962](#)]. *Suppose Z is a locally complete intersection of dimension d . Then, any nonempty intersection of two irreducible components of Z has pure dimension $d - 1$.*

Proof. See [[Hartshorne 1962](#); [Sawant 2011](#)]. □

Corollary 5.2. *If the moduli space \overline{M} is good and has two different irreducible components Z_1 and Z_2 meeting at a point z , then $Z_1 \cap Z_2$ has codimension 1 at z and the singular locus $\text{Sing}(\overline{M})$ contains z and has codimension 1 at z .*

Proof. If \overline{M} is good, then by [Lemma 2.1](#) it is a locally complete intersection so Hartshorne's theorem applies: $Z_1 \cap Z_2$ has pure codimension 1. The intersection of two irreducible components is necessarily contained in the singular locus. □

We draw the following conclusions.

Corollary 5.3. *Suppose, for $c_2 \geq 10$, that two different irreducible components Z_1 and Z_2 of \overline{M} meet at a point z . Then z is on the boundary.*

Proof. If z is not on the boundary, then by the previous corollary it is in a component of the singular locus having codimension 1 in M . We have seen in [[Mestrano and Simpson 2011](#), [Theorem 7.1](#)] that for $c_2 \geq 10$, a piece of $\text{Sing}(M)$ having codimension 1 in $M(c_2)$ has to be in $V(c_2)$, cf., [Proposition 2.3](#) above. On the other hand $V(c_2)$ is irreducible, see [Lemmas 3.3](#) and [3.4](#), so any such component of $\text{Sing}(M)$ has to be equal to $V(c_2)$.

Recall that $\dim(V(c_2)) = 3c_2 - 11$ whereas the dimension of the moduli space is $4c_2 - 20$, thus for $c_2 \geq 11$ the singular locus has codimension ≥ 2 , so the present situation could only occur for $c_2 = 10$.

But now by [Lemma 3.4](#), $V(10)$ is contained in only one irreducible component of M , the one whose general point parametrizes bundles with seminatural cohomology. So, two distinct components cannot meet along $V(10)$. \square

Next, recall one of Nijssse’s theorems, connectedness of the moduli space.

Theorem 5.4 [[Nijssse 1995](#)]. *For $c_2 \geq 10$, the moduli space $\overline{M}(c_2)$ is connected.*

Proof. See [[Nijssse 1995](#)], Proposition 3.2. We have reviewed the argument in [[Mestrano and Simpson 2016](#), Theorem 18.8]. \square

Corollary 5.5. *Suppose Z is an irreducible component of $\overline{M}(c_2)$ for $c_2 \geq 10$. Then Z meets the boundary in a nonempty subset of codimension ≤ 1 .*

Proof. The codimension 1 property is given by [Lemma 4.1](#), so we just have to show that \overline{Z} contains a boundary point.

For $c_2 \geq 10$, the first boundary stratum $M(c_2, c_2 - 1)$ has codimension 1, so it must meet at least one irreducible component of $\overline{M}(c_2)$, call it Z_0 . Of course if $Z = Z_0$ we are done. Suppose $Z \subset M(c_2)$ is another irreducible component with $c_2 \geq 10$. By the connectedness of $\overline{M}(10)$, there exist a sequence of irreducible components $Z_0, \dots, Z_k = \overline{Z}$ such that $Z_i \cap Z_{i+1}$ is nonempty. By [Corollary 5.3](#), $Z_{k-1} \cap Z_k$ is contained in the boundary. \square

6. Seminaturality along the 19-dimensional boundary strata

To treat the case $c_2 = 10$, we will apply the main result of our previous paper.

Proposition 6.1. *Suppose Z is an irreducible component of $M(10)$. Suppose that \overline{Z} contains a point corresponding to a torsion-free sheaf F with $h^1(F(1)) = 0$. Then Z is the unique irreducible component containing the open set of bundles with seminatural cohomology, constructed in [[Mestrano and Simpson 2013](#)].*

Proof. The locus $V(c_2)$ of bundles with $h^0(E) \neq 0$ has dimension ≤ 19 , so a general point of Z must have $h^0(E) = 0$. The hypothesis implies that a general point has $h^1(E(1)) = 0$. Thus, there is a nonempty dense open subset $Z' \subset Z$ parametrizing bundles with $h^0(E) = 0$ and $h^1(E(1)) = 0$. By [[Mestrano and Simpson 2013](#), Corollary 3.5], these bundles have seminatural cohomology. Thus, our open set is $Z' = M(10)^{\text{sn}}$, the moduli space of bundles with seminatural cohomology, shown to be irreducible in the main Theorem 0.2 of the same work recalled as [Proposition 2.4](#) above. \square

Using [Proposition 6.1](#), and since we know by [Corollary 5.5](#) that any irreducible component Z meets the boundary in a codimension 1 subset, in order to prove irreducibility of $M(10)$, it suffices to show that the torsion-free sheaves F parametrized by general points on the various irreducible components of the boundary of $\overline{M}(10)$ have $h^1(F(1)) = 0$.

The dimension is $\dim(Z) = 20$, so the boundary components will have dimension 19. Looking at the line $c_2 = 10$ in [Table 2](#), we notice that there are three 19-dimensional boundary pieces, and a 20-dimensional piece which must constitute a different irreducible component. Consider first the 19-dimensional pieces,

$$M(10, 9), M(10, 8) \text{ and } M(10, 6).$$

Recall that $M(10, 10 - d)$ consists generically of torsion-free sheaves F fitting into an exact sequence

$$(6-1) \quad 0 \rightarrow F \rightarrow F^{**} \rightarrow S \rightarrow 0,$$

where F^{**} is a general point in the moduli space of stable bundles with $c_2 = 10 - d$, and S is a general quotient of length d .

Proposition 6.2. *For a general point F in either of the three boundary pieces $M(10, 9)$, $M(10, 8)$ or $M(10, 6)$, we have $h^1(F(1)) = 0$.*

Proof. Notice that $\chi(F^{**}(1)) = 15 - c_2(F^{**}) \geq 6$ and by stability $h^2(F^{**}(1)) = h^0(F^{**}(-1)) = 0$, so $F^{**}(1)$ has at least six linearly independent sections. In particular, for a general quotient S of length 1, 2 or 4, consisting of the direct sum $S = \bigoplus S_x$ of general length 1 quotients $E_x \rightarrow S_x$ at 1, 2 or 4 distinct general points x , the map

$$H^0(F^{**}(1)) \rightarrow H^0(S)$$

will be surjective.

For a general point F^{**} in either $M(9)$, $M(8)$ or $M(6)$, we have $h^1(F^{**}(1)) = 0$. These results from [\[Mestrano and Simpson 2011\]](#) were recalled in [Proposition 2.2](#), [Table 1](#), and reviewed in [Section 3](#). The long exact sequence associated to (6-1) now gives $h^1(F(1)) = 0$. \square

This treats the 19-dimensional irreducible components of the boundary. There remains the piece $\overline{M}(10, 4)$ which has dimension 20. This is a separate irreducible component. It could meet $\overline{M}(10)$ along a 19-dimensional divisor, and we would like to show that $h^1(F(1)) = 0$ for the sheaves parametrized by this divisor. In particular, we are no longer in a completely generic situation so some further discussion is needed. This will be the topic of the next section.

7. The lowest stratum

The lowest stratum is $M(10, 4)$, which is therefore closed. We would like to understand the points in $\overline{M}(10) \cap M(10, 4)$. These are singular, so our main tool will be to look at where the singular locus of $\overline{M}(10)$ meets $M(10, 4)$. Denote this by

$$M(10, 4)^{\text{sing}} := \text{Sing}(\overline{M}(10)) \cap M(10, 4).$$

In what follows, we give a somewhat explicit description of the lowest moduli space $M(4)$.

Lemma 7.1. *For $E \in M(4)$ we have $h^1(E) = 0$, $h^0(E) = h^2(E) = 3$, $h^0(E(1)) = 11$ and $h^1(E(1)) = h^2(E(1)) = 0$.*

Proof. Choosing an element $s \in H^0(E)$ gives an exact sequence

$$(7-1) \quad 0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow J_{P/X}(1) \rightarrow 0.$$

In [Mestrano and Simpson 2011] we have seen that $P \subset X \cap L$ is a subscheme of length 4 in the intersection of X with a line $L \subset \mathbb{P}^3$. As P spans L , the space of linear forms vanishing on P is the same as the space of linear forms vanishing on L , so $H^0(J_{P/X}(1)) \cong \mathbb{C}^2$. In the long exact sequence associated to (7-1), note that $H^1(\mathcal{O}_X) = 0$, giving

$$0 \rightarrow H^0(\mathcal{O}_X) \rightarrow H^0(E) \rightarrow H^0(J_{P/X}(1)) \rightarrow 0,$$

hence $H^0(E) \cong \mathbb{C}^3$. By duality, $H^2(E) \cong \mathbb{C}^3$, and the Euler characteristic of E is 6, so $H^1(E) = 0$.

For $E(1)$, note that $H^2(E(1)) = 0$ by stability and duality, and (7-1) gives an exact sequence

$$0 \rightarrow H^1(E(1)) \rightarrow H^1(J_{P/X}(2)) \rightarrow H^2(\mathcal{O}_X(1)) \rightarrow 0.$$

On the other hand, $H^1(J_{P/X}(2)) \cong \mathbb{C}$ corresponding to the length 4 of P , minus the dimension 3 of the space of sections of $\mathcal{O}_P(2)$ coming from global quadrics (since the space of quadrics on L has dimension 3). This gives $H^1(E(1)) = 0$. The Euler characteristic then gives $h^0(E(1)) = 11$. This is also seen in the first part of the exact sequence, where $H^0(\mathcal{O}_X(1)) = \mathbb{C}^4$ and $H^0(J_{P/X}(2)) \cong \mathbb{C}^7$. \square

If $p \in \mathbb{P}^3$, let $G \cong \mathbb{C}^3$ be the space of linear generators of the ideal of p , that is to say $G := H^0(J_{p/\mathbb{P}^3}(1))$, and consider the natural exact sequence of sheaves on \mathbb{P}^3

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^3} \otimes G^* \rightarrow \mathcal{R}_p \rightarrow 0.$$

Here the cokernel sheaf \mathcal{R}_p is a reflexive sheaf of degree 1, and $c_2(\mathcal{R}_p)$ is the class of a line. The restriction $\mathcal{R}_p|_X$ therefore has $c_2 = 5$. If $p \in X$, it is torsion-free but not locally free, giving a point in $M(5, 4)$. It turns out that these sheaves account for all of $M(4)$ and $M(5)$.

Theorem 7.2. *Suppose $E \in M(4)$. Then there is a unique point $p \in X$ such that E is generated by global sections outside of p , and $\mathcal{R}_p|_X$ is isomorphic to the subsheaf of E generated by global sections. This fits into an exact sequence*

$$0 \rightarrow \mathcal{R}_p|_X \rightarrow E \rightarrow S \rightarrow 0,$$

where S has length 1, in particular $E \cong (\mathcal{R}_p|_X)^{**}$. The correspondence $E \leftrightarrow p$ establishes an isomorphism $M(4) \cong X$.

For $E' \in M(5)$, there exists a unique point $p \in \mathbb{P}^3 - X$ such that $E' \cong \mathcal{R}_p|_X$. This correspondence establishes an isomorphism $\overline{M}(5) \cong \mathbb{P}^3$ such that the boundary component $M(5, 4) \cap \overline{M}(5)$ is exactly $X \subset \mathbb{P}^3$. Note however that $M(5, 4)$ itself has dimension strictly bigger than 3 and constitutes another irreducible component of $\overline{M}(5)$.

Proof. Consider the exact sequence (7-1). The space $H^0(J_{P/X}(1))$ consists of linear forms on X (or equivalently, on \mathbb{P}^3), which vanish along P . However, a linear form which vanishes on P also vanishes on L . In particular, elements of $H^0(J_{P/X}(1))$ generate $J_{X \cap L/X}(1)$, which has colength 1 in $J_{P/X}(1)$.

Let $R \subset E$ be the subsheaf generated by global sections, and let S be the cokernel in the exact sequence

$$0 \rightarrow R \rightarrow E \rightarrow S \rightarrow 0.$$

We also have the exact sequence

$$0 \rightarrow J_{X \cap L/X}(1) \rightarrow J_{P/X}(1) \rightarrow S \rightarrow 0$$

so S has length 1. It is supported on a point p . The sheaf R is generated by three global sections so we have an exact sequence

$$0 \rightarrow \ker \rightarrow \mathcal{O}_X^3 \rightarrow R \rightarrow 0.$$

The kernel is a saturated subsheaf, hence locally free, and by looking at its degree we have $\ker = \mathcal{O}_X(-1)$. Thus, R is the cokernel of a map $\mathcal{O}_X(-1) \rightarrow \mathcal{O}_X^3$ given by three linear forms; these linear forms are a basis for the space of forms vanishing at the point p . We see that R is the restriction to X of the sheaf \mathcal{R}_p described above, hence $E \cong (\mathcal{R}_p|_X)^{**}$. The map $E \mapsto p$ gives a map $M(4) \rightarrow X$, with inverse $p \mapsto (\mathcal{R}_p|_X)^{**}$.

The second paragraph, about $\overline{M}(5)$, is not actually needed later and we leave it to the reader. □

Even though the moduli space $M(4)$ is smooth, it has much more than the expected dimension, and the space of coobstructions is nontrivial. It will be useful to understand the coobstructions, because if $F \in M(10, 4)$ is a torsion-free sheaf with $F^{**} = E$ then coobstructions for F come from coobstructions for E which preserve the subsheaf $F \subset E$.

Lemma 7.3. *Suppose $E \in M(4)$. A general coobstruction $\phi : E \rightarrow E(1)$ has generically distinct eigenvalues with an irreducible spectral variety in $\text{Tot}(K_X)$.*

Proof. It suffices to write down a map $\phi : E \rightarrow E(1)$ with generically distinct eigenvalues and irreducible spectral variety. To do this, we construct a map $\phi_R :$

$R \rightarrow R(1)$ using the expression $R = \mathcal{R}_p|_X$. The exact sequence defining \mathcal{R}_p extends to the Koszul resolution, a long exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^3}^3 \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^3 \rightarrow J_{p/\mathbb{P}^3}(2) \rightarrow 0.$$

Thus \mathcal{R}_p may be viewed as the image of the middle map. Without loss of generality, p is the origin in an affine system of coordinates (x, y, z) for $\mathbb{A}^3 \subset \mathbb{P}^3$, and the coordinate functions are the three coefficients of the maps on the left and right in the Koszul sequence. The 3×3 matrix in the middle is

$$K := \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}.$$

Any 3×3 matrix of constants Φ gives a composed map

$$\phi_R : \mathcal{R}_p \hookrightarrow \mathcal{O}_{\mathbb{P}^3}(1)^3 \xrightarrow{\Phi} \mathcal{O}_{\mathbb{P}^3}(1)^3 \rightarrow \mathcal{R}_p(1).$$

Use the first two columns of K to give a map $k : \mathcal{O}_{\mathbb{P}^3}^2 \rightarrow \mathcal{R}_p$ which is an isomorphism over an open set. On the other hand, the projection onto the first two coordinates gives a map $q : \mathcal{R}_p \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^2$ which is, again, an isomorphism over an open set. The composition of these two is the map given by the upper 2×2 square of K ,

$$qk = K_{2,2} := \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}.$$

We can now analyze the map ϕ_R by noting that $q\phi_R k = K_{2,3}\Phi K_{3,2}$ where $K_{2,3}$ and $K_{3,2}$ are respectively the upper 2×3 and left 3×2 blocks of K . Over the open set where q and k are isomorphisms,

$$q\phi_R q^{-1} = q\phi_R k(qk)^{-1} = K_{2,3}\Phi K_{3,2}K_{2,2}^{-1}.$$

Now

$$K_{3,2}K_{2,2}^{-1} = \begin{pmatrix} 0 & z \\ -z & 0 \\ y & -x \end{pmatrix} \cdot \begin{pmatrix} 0 & -1/z \\ 1/z & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -x/z & -y/z \end{pmatrix}.$$

Suppose

$$\Phi = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \psi \\ \chi & \theta & \rho \end{pmatrix}.$$

Then

$$\begin{aligned}
q\phi_Rq^{-1} &= K_{2,3}\Phi K_{3,2}K_{2,2}^{-1} \\
&= \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \psi \\ \chi & \theta & \rho \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -x/z & -y/z \end{pmatrix} \\
&= \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \end{pmatrix} \cdot \begin{pmatrix} \alpha - \gamma x/z & \beta - \gamma y/z \\ \delta - \psi x/z & \epsilon - \psi y/z \\ \chi - \rho x/z & \theta - \rho y/z \end{pmatrix} \\
&= \begin{pmatrix} z\delta - \psi x - y\chi + \rho xy/z & z\epsilon - \psi y + y\theta - \rho y^2/z \\ -z\alpha + \gamma x - x\chi + \rho x^2/z & -z\beta + \gamma y + x\theta - \rho xy/z \end{pmatrix}.
\end{aligned}$$

Notice that the trace of this matrix is

$$\mathrm{Tr}(\phi) = x(\theta - \psi) + y(\gamma - \chi) + z(\delta - \beta),$$

which is a section of $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ vanishing at p . A coobstruction should have trace zero, so we should impose three linear conditions

$$\theta = \psi, \quad \chi = \gamma, \quad \delta = \beta,$$

which together just say that Φ is a symmetric matrix. Our expression simplifies to

$$q\phi_Rq^{-1} = \begin{pmatrix} \beta z - \psi x - \gamma y + \rho xy/z & \epsilon z - \rho y^2/z \\ -\alpha z + \rho x^2/z & -\beta z + \psi x + \gamma y - \rho xy/z \end{pmatrix}.$$

Now, restrict \mathcal{R}_p to X to get the sheaf R , take its double dual to get $E = R^{**}$, and consider the induced map $\phi : E \rightarrow E(1)$. Over the intersection of our open set with X , this will have the same formula. We can furthermore restrict to the curve $Y \subset X$ given by the intersection with the plane $y = 0$. Note that X is in general position subject to the condition that it contains the point p . Setting $y = 0$ the above matrix becomes

$$(q\phi q^{-1})|_{y=0} = \begin{pmatrix} \beta z - \psi x & \epsilon z \\ -\alpha z + \rho x^2/z & -\beta z + \psi x \end{pmatrix}.$$

Choose for example $\beta = \psi = 0$ and $\alpha = \rho = \epsilon = 1$, giving the matrix whose determinant is

$$\det \begin{pmatrix} 0 & z \\ x^2/z - z & 0 \end{pmatrix} = z^2 - x^2 = (z+x)(z-x).$$

The eigenvalues of $\phi|_Y$ are therefore $\pm\sqrt{(z+x)(z-x)}$, generically distinct. For a general choice of the surface X , our curve $Y = X \cap (y = 0)$ will intersect the planes $x = z$ and $x = -z$ transversally, so the two eigenvalues of $\phi|_Y$ are permuted when going around points in the ramification locus different from p . This provides an

explicit example of ϕ for which the spectral variety is irreducible, completing the proof of the lemma. We included the detailed calculations because they look to be useful if one wants to write down explicitly the spectral varieties. \square

Turn now to the study of the boundary component $M(10, 4)$ consisting of torsion-free sheaves in $\bar{M}(10)$ which come from bundles in $M(4)$. A point in $M(10, 4)$ consists of a torsion-free sheaf F in an exact sequence of the form (6-1)

$$0 \rightarrow F \rightarrow E \xrightarrow{\sigma} S \rightarrow 0,$$

where $E = F^{**}$ is a point in $M(4)$, and S is a length 6 quotient.

The basic description of the space of obstructions as dual to the space of K_X -twisted endomorphisms still holds for torsion-free sheaves. Thus, the obstruction space for F is $\text{Hom}^o(F, F(1))^*$. A coobstruction is a map $\phi : F \rightarrow F(1) = F \otimes K_X$ with $\text{Tr}(\phi) = 0$, which is a kind of *Higgs field*. Since the moduli space is good, a point F is in $\text{Sing}(\bar{M}(10))$ if and only if the obstruction space is nonzero, that is to say, if and only if there exists a nonzero trace-free $\phi : F \rightarrow F(1)$.

To give a map ϕ is the same thing as to give a map $\varphi : E \rightarrow E(1)$ compatible with the quotient map $E \rightarrow S$, in other words fitting into a commutative square with σ , for an induced map $\varphi_S : S \rightarrow S$. The maps φ , coobstructions for E , were studied in Lemma 7.3 above.

Let $\mathbb{P}(E) \rightarrow X$ denote the Grothendieck projective space bundle. A point in $\mathbb{P}(E)$ is a pair (x, s) , where $x \in X$ and $s : E_x \rightarrow S_x$ is a length one quotient of the fiber. Suppose we are given a map $\varphi : E \rightarrow E(1)$. We can consider the *internal spectral variety*

$$\text{Sp}_E(\varphi) \subset \mathbb{P}(E),$$

defined as the set of points $(x, s) \in \mathbb{P}(E)$ such that there is a commutative diagram

$$\begin{array}{ccc} E_x & \xrightarrow{\varphi(x)} & E_x \\ \downarrow & & \downarrow \\ S_x & \longrightarrow & S_x. \end{array}$$

The term “internal” signifies that it is a subvariety of $\mathbb{P}(E)$ as opposed to the classical spectral variety which is a subvariety of the total space of K_X . Here, we have only given $\text{Sp}_E(\varphi)$ a structure of closed subset of $\mathbb{P}(E)$, hence of reduced subvariety. It would be interesting to give it an appropriate scheme structure which could be nonreduced in case φ is nilpotent, but that will not be needed here.

Corollary 7.4. *Suppose $E \in M(4)$ and $\varphi : E \rightarrow E(1)$ is a general coobstruction. Then the internal spectral variety $\text{Sp}_E(\varphi)$ has a single irreducible component of dimension 2. A quotient $E \rightarrow S$ consisting of a disjoint sum of length one quotients $s_i : E_{x_i} \rightarrow S_i$ with $S = \bigoplus S_i$ and the points x_i disjoint, is compatible with φ if and only if the points $(x_i, s_i) \in \mathbb{P}(E)$ lie on the internal spectral variety $\text{Sp}_E(\varphi)$.*

Proof. Notice if $z \in X$ is a point such that $\varphi(z) = 0$, then the whole fiber $\mathbb{P}(E)_z \cong \mathbb{P}^1$ is in $\mathrm{Sp}_E(\varphi)$. In particular, if such a point exists then the map $\mathrm{Sp}_E(\varphi) \rightarrow X$ will not be finite.

A first remark is that the zero-set of φ is 0-dimensional. Indeed, if φ vanished along a divisor D , then $D \in |\mathcal{O}_X(n)|$ for $n \geq 1$ and $\varphi : F \rightarrow F(1-n)$. This is possible only if $n = 1$ and $\varphi : F \rightarrow F$ is a scalar endomorphism (since F is stable). However, the trace of the coobstruction vanishes, so the scalar φ would have to be zero, which we are assuming is not the case.

At an isolated point z with $\varphi(z) = 0$, the fiber of the projection $\mathrm{Sp}_E(\varphi) \rightarrow X$ contains the whole $\mathbb{P}(E_z) = \mathbb{P}^1$. However, these can contribute at most irreducible components of dimension ≤ 1 (although we conjecture that in fact these fibers are contained in the closure of the 2-dimensional component so that $\mathrm{Sp}_E(\varphi)$ is irreducible).

Away from such fibers, the internal spectral variety is isomorphic to the external one, a two-sheeted covering of X , and by [Lemma 7.3](#), for a general φ the monodromy of this covering interchanges the sheets, so it is irreducible. Thus, $\mathrm{Sp}_E(\varphi)$ has a single irreducible component of dimension 2, and it maps to X by a generically finite (2 to 1) map.

The second statement, that a quotient consisting of a direct sum of length one quotients, is compatible with φ if and only if the corresponding points lie on $\mathrm{Sp}_E(\varphi)$, is immediate from the definition. \square

Definition 7.5. A triple (E, φ, σ) where $E \in M(4)$, $\varphi : E \rightarrow E(1)$ is a nonnilpotent map, and $\sigma = \bigoplus s_x$ is a quotient composed of six length 1 quotients over distinct points, compatible with φ as in the previous [Corollary 7.4](#), leads to an obstructed point $F = F_{(E, \varphi, \sigma)} \in M(10, 4)^{\mathrm{sing}}$ obtained by setting $F := \ker(\sigma)$. Such a point will be called *usual*.

Ellingsrud and Lehn have given a very nice description of the Grothendieck quotient scheme of a bundle of rank r on a smooth surface. It extends the basic idea of Li's theorem which we already stated as [Theorem 4.2](#) above, and will allow us to count dimensions of strata in $M(10, 4)$.

Theorem 7.6 [[Ellingsrud and Lehn 1999](#)]. *The quotient scheme parametrizing quotients of a locally free sheaf \mathcal{O}_X^r of rank r on a smooth surface X , located at a given point $x \in X$, and of length ℓ , is irreducible of dimension $r\ell - 1$.*

Proof. See [[Ellingsrud and Lehn 1999](#)]. We have given the local version of the statement here. \square

In our case, $r = 2$ so the dimension of the local quotient scheme is $2\ell - 1$.

A given quotient $E \rightarrow S$ decomposes as a direct sum of quotients $E \rightarrow S_i$ located at distinct points $x_i \in X$. Order these by decreasing length, and define the *length vector* of S to be the sequence (ℓ_1, \dots, ℓ_k) of lengths $\ell_i = \ell(S_i)$ with $\ell_i \geq \ell_{i+1}$.

This leads to a stratification of the Quot scheme into strata labeled by length vectors. By [Ellingsrud and Lehn 1999], the dimension of the space of quotients supported at a single (but not fixed) point x_i and having length ℓ_i , is $2\ell_i + 1$, giving the following dimension count.

Corollary 7.7. *For a fixed bundle E of rank 2, the dimension of the stratum associated to length vector (ℓ_1, \dots, ℓ_k) in the Quot-scheme of quotients $E \rightarrow S$ with total length $\ell = \sum_{i=1}^k \ell_i$, is*

$$\sum (2\ell_i + 1) = 2\ell + k.$$

Recall that the moduli space $M(4)$ has dimension 2, so the dimension of the stratum of $M(10, 4)$ corresponding to a vector (ℓ_1, \dots, ℓ_k) is $14 + k$. In particular, $M(10, 4)$ has a single stratum $(1, 1, 1, 1, 1, 1)$ of dimension 20, corresponding to quotients which are direct sums of length one quotients supported at distinct points, and a single stratum $(2, 1, 1, 1, 1)$ of length 19. This yields the following corollary.

Corollary 7.8. *If $Z' \subset M(10, 4)$ is any 19-dimensional irreducible subvariety, then either Z' is equal to the stratum $(2, 1, 1, 1, 1)$, or else the general point on Z' consists of a direct sum of six length 1 quotients supported over six distinct points of X .*

Proposition 7.9. *The singular locus $M(10, 4)^{\text{sing}}$ has only one irreducible component of dimension 19. This irreducible component has a nonempty dense open subset consisting of the usual points (Definition 7.5). For a usual point, the coobstruction φ is unique up to a scalar, so this open set may be viewed as the moduli space of usual triples (E, φ, σ) , which is irreducible.*

Proof. Suppose $Z' \subset M(10, 4)^{\text{sing}}$ is an irreducible component. Consider the two cases given by Corollary 7.8.

(i) If Z' contains an open set consisting of points which are direct sums of six length 1 quotients supported on distinct points of X , then this open set parametrizes usual triples. Furthermore, a point in this open set corresponds to a choice of (E, φ) together with six points on the internal spectral variety $\text{Sp}_E(\varphi)$. We count the dimension of this piece as follows.

Let $M'(4)$ denote the moduli space of pairs (E, φ) with $E \in M(4)$ and φ a nonzero coobstruction for E . The space of coobstructions for any $E \in M(4)$, has dimension 6 and the family of these spaces forms a vector bundle over $M(4)$ (more precisely, a twisted vector bundle twisted by the obstruction class for existence of a universal family over $M(4)$). Thus, the moduli space of pairs has a fibration $M'(4) \rightarrow M(4)$ whose fibers are \mathbb{P}^5 . In particular, $M'(4)$ is a smooth irreducible variety of dimension 7.

For a general such (E, φ) the moduli space of usual triples has dimension ≤ 12 , with a unique 12-dimensional piece corresponding to a general choice of 6 points on the unique 2-dimensional irreducible component of $\text{Sp}_E(\varphi)$. This gives the 19-dimensional component of $M(10, 4)^{\text{sing}}$ mentioned in the proposition.

Suppose (E, φ) is not general, that is to say, contained in some subvariety of $M'(4)$ of dimension ≤ 6 . Then, as φ is nonzero, even though we no longer can say that it is irreducible, in any case the internal spectral variety $\text{Sp}_E(\varphi)$ has dimension 2 so the space of choices of 6 general points on it has dimension ≤ 12 , and this contributes at most subvarieties of dimension ≤ 18 in $M(10, 4)^{\text{sing}}$. This shows that in the first case (i) of [Corollary 7.8](#), we obtain the conclusion of the proposition.

(ii) Suppose Z' is equal to the stratum of $M(10, 4)$ corresponding to length vector $(2, 1, 1, 1, 1)$. In this case, we show that a general point of Z' has no nonzero coobstructions, contradicting the hypothesis that $Z' \subset M(10, 4)^{\text{sing}}$ and showing that this case cannot occur.

Fix $E \in M(4)$. The space of coobstructions of E has dimension 6. Suppose $E \rightarrow S_1$ is a quotient of length 2. If it is just the whole fiber of E over x_1 , then it is automatically compatible with any coobstruction. However, these quotients contribute only a 2-dimensional subspace of the space of such quotients which has dimension 5 by [\[Ellingsrud and Lehn 1999\]](#). Thus, these points don't contribute general points. On the other hand, a general quotient of length 2 corresponds to an infinitesimal tangent vector in $\mathbb{P}(E)$, and the condition that this vector be contained in $\text{Sp}_E(\varphi)$ imposes two conditions on φ . Therefore, the space of coobstructions compatible with S_1 has dimension ≤ 4 . Next, given a nonzero coobstruction in that subspace, a general quotient $E \rightarrow S_2$ of length 1 will not be compatible, so imposing compatibility with S_1 and S_2 leads to a space of coobstructions of dimension ≤ 3 . Continuing in this way, we see that imposing the condition of compatibility of φ with a general quotient $S = S_1 \oplus \cdots \oplus S_5$ in the stratum $(2, 1, 1, 1, 1)$ leads to $\varphi = 0$. Thus, a general point of this stratum has no nonzero coobstructions as we have claimed, and this case (ii) cannot occur.

Hence, the only case from [Corollary 7.8](#) which can contribute a 19-dimensional stratum, contributes the single irreducible component described in the statement of the proposition. One may note that φ is uniquely determined for a general set of six points on its internal spectral variety, since the first 5 points are general in $\mathbb{P}(E)$ and impose linearly independent conditions. \square

Corollary 7.10. *Suppose $M(10, 4) \cap \overline{M(10)}$ is nonempty. Then it is the unique 19-dimensional irreducible component of usual triples in $M(10, 4)^{\text{sing}}$ identified by [Proposition 7.9](#).*

Proof. By Hartshorne's theorem, the intersection $M(10, 4) \cap \overline{M(10)}$ has pure dimension 19 if it is nonempty. This could also be seen from O'Grady's lemma that the boundary of $\overline{M(10)}$ has pure dimension 19. However, any point in this intersection is singular. By [Proposition 7.9](#), the singular locus $M(10, 4)^{\text{sing}}$ has only one irreducible component of dimension 19, and it is the closure of the space of usual triples. \square

If the intersection $M(10, 4) \cap \overline{M(10)}$ is nonempty, the torsion-free sheaves F parametrized by general points satisfy $h^1(F(1)) = 0$. We show this by a dimension estimate using [Ellingsrud and Lehn 1999]. The more precise information about $M(10, 4)^{\text{sing}}$ given in Proposition 7.9, while not really needed for the proof at $c_2 = 10$, will be useful in treating the case of $c_2 = 11$ in Section 9.

Proposition 7.11. *The subspace of $M(10, 4)$ consisting of points F such that $h^1(F(1)) \geq 1$ has codimension ≥ 2 .*

Proof. Use the exact sequence $0 \rightarrow F \rightarrow E \rightarrow S \rightarrow 0$, where $E \in M(4)$. One has $h^1(E(1)) = 0$ for all $E \in M(4)$, see Lemma 7.1. Therefore, $h^1(F(1)) = 0$ is equivalent to saying that the map

$$(7-2) \quad H^0(E(1)) \rightarrow H^0(S(1)) \cong \mathbb{C}^6$$

is surjective.

Considering the theorem of Ellingsrud and Lehn [1999], there are two strata to be looked at: the case of a direct sum of six quotients of length 1 over distinct points, to be treated below; and the case of a direct sum of four quotients of length 1 and one quotient of length 2. However, this latter stratum already has codimension 1, and it is irreducible. So, for this stratum it suffices to note that a general quotient $E \rightarrow S$ in it leads to a surjective map (7-2), which may be seen by a classical general position argument, placing first the quotient of length 2.

Consider now the stratum of quotients which are the direct sum of six length 1 quotients s_i at distinct points $x_i \in X$. Fix the bundle E . The space of choices of the six quotients (x_i, s_i) has dimension 18. We claim that the space of choices such that (7-2) is not surjective, has codimension ≥ 2 .

Note that $h^0(E(1)) = 11$. Given six quotients (x_i, s_i) , if the map (7-2) (with $S = \bigoplus S_i$) is not surjective, then its kernel has dimension ≥ 6 , so if we choose five additional points $(y_j, t_j) \in \mathbb{P}(E)$ with $t_j : E_{y_j} \rightarrow T_j$ for T_i of length 1, the total evaluation map

$$(7-3) \quad H^0(E(1)) \rightarrow \bigoplus_{i=1}^6 S_i(1) \oplus \bigoplus_{j=1}^5 T_j(1)$$

has a nontrivial kernel. Consider the variety

$$W := \{(u, \dots, (x_i, s_i), \dots, (y_j, t_j), \dots) \mid 0 \neq u \in H^0(E(1)), s_i(u) = 0, t_j(u) = 0\}$$

with the nonzero section u taken up to multiplication by a scalar.

Let $Q'_6(E)$ and $Q'_5(E)$ denote the open subsets of the quotient schemes of length 6 and length 5 quotients of E respectively, open subsets consisting of quotients which are direct sums of length one quotients over distinct points. Let $K \subset Q'_6(E)$ denote the locus of quotients $E \rightarrow S$ such that the kernel sheaf F has $h^1(F(1)) \geq 1$.

It is a proper closed subset, since it is easy to see that a general quotient $E \rightarrow S$ leads to a surjection (7-2). The above argument with (7-3) shows that $K \times Q'_5(E) \subset p(W)$, where $p : W \rightarrow Q'_6(E) \times Q'_5(E)$ is the projection forgetting the first variable u . Our goal is to show that K has dimension ≤ 16 .

We claim that W has dimension ≤ 32 and has a single irreducible component of dimension 32. To see this, start by noting that the choice of u lies in the projective space \mathbb{P}^{10} associated to $H^0(E(1)) \cong \mathbb{C}^{11}$.

For a section u which is special in the sense that its scheme of zeros has positive dimension, the locus of choices of (x_i, s_i) and (y_j, t_j) has dimension ≤ 22 , but might have several irreducible components depending on whether the points are on the zero-set of u or not. However, the space of sections u which are special in this sense is equal to the space of pairs $u' \in H^0(E)$, $u'' \in H^0(\mathcal{O}_X(1))$ up to scalars for both pieces, and this has dimension $2 + 3 = 5$, which is much smaller than the dimension of the space of all sections u . Therefore, these pieces don't contribute anything of dimension higher than 27.

For a section u which is not special in the sense of the previous paragraph, the space of choices of a single length 1 quotient (x, s) which vanishes on the section, has a single irreducible component of dimension 2. It might possibly have some pieces of dimension 1 corresponding to quotients located at the zeros of u (although we don't think so). Hence, the space of choices of point in W lying over the section u has dimension ≤ 22 and has a single irreducible component of dimension 22.

Putting these together over \mathbb{P}^{10} , the dimension of W is ≤ 32 and it has a single irreducible component of dimension 32, as claimed. Its image $p(W)$ therefore also has dimension ≤ 32 , and has at most one irreducible component of dimension 32. Denote this component, if it exists, by $p(W)'$.

Suppose now that K had an irreducible component K' of dimension 17. Then $K' \times Q'_5(E) \subset p(W)$, but $\dim(Q'_5(E)) = 15$ so $p(W)'$ would exist and would be equal to $K' \times Q'_5(E)$. However, $p(W)'$ is symmetric under permutation of the 11 different variables (x, s) and (y, t) , but that would then imply that $P(W)'$ was the whole of $Q'_6(E) \times Q'_5(E)$ which is not the case. Therefore, K must have codimension ≥ 2 . This completes the proof of the proposition. \square

Corollary 7.12. *Suppose $M(10, 4) \cap \overline{M(10)}$ is nonempty. Then a general point of this intersection corresponds to a torsion-free sheaf with $h^1(F(1)) = 0$.*

Proof. By Hartshorne's or O'Grady's theorem, if the intersection is nonempty then it has pure dimension 19. However, the space of torsion-free sheaves $F \in M(10, 4)$ with $h^1(F(1)) > 0$ has dimension ≤ 18 by Proposition 7.11. Thus, a general point in any irreducible component of $M(10, 4) \cap \overline{M(10)}$ must have $h^1(F(1)) = 0$. In fact there can be at most one irreducible component, by Corollary 7.10. \square

8. Irreducibility for $c_2 = 10$

Corollary 8.1. *Suppose Z is an irreducible component of $M(10)$. Then, for a general point F in any irreducible component of the intersection of \bar{Z} with the boundary, we have $h^1(F(1)) = 0$.*

Proof. By O’Grady’s lemma, the intersection of \bar{Z} with the boundary has pure dimension 19. By considering the line $c_2 = 10$ in the [Table 2](#), this subset must be a union of some of the irreducible subsets $\overline{M}(10, 9)$, $\overline{M}(10, 8)$, $\overline{M}(10, 6)$, and the unique 19-dimensional irreducible component of $M(10, 4)^{\text{sing}}$ given by [Proposition 7.9](#). Combining [Proposition 6.2](#) and [Corollary 7.12](#), we conclude that any one of these irreducible components of the intersection of \bar{Z} with the boundary contains a point F such that $h^1(F(1)) = 0$. \square

Corollary 8.2. *Suppose Z is an irreducible component of $M(10)$. Then the bundle E parametrized by a general point of Z has seminatural cohomology, and Z is the closure of the irreducible open set $M(10)^{\text{sn}}$.*

Proof. The closure \bar{Z} of Z meets the boundary in a nonempty subset, by [Corollary 5.5](#). By the previous [Corollary 8.1](#), there exists a point F in \bar{Z} with $h^1(F(1)) = 0$; thus the general bundle E in Z also satisfies $h^1(E(1)) = 0$. By [Proposition 6.1](#), the irreducible moduli space $M(10)^{\text{sn}}$ of bundles with seminatural cohomology is an open set of Z . \square

Theorem 8.3. *The moduli space $M(10)$ of stable bundles of degree 1 and $c_2 = 10$, is irreducible.*

Proof. By [Corollary 8.2](#), any irreducible component of $M(10)$ contains a dense open set parametrizing bundles with seminatural cohomology. By the main theorem of [\[Mestrano and Simpson 2013\]](#), there is only one such irreducible component. \square

Theorem 8.4. *The full moduli space of stable torsion-free sheaves $\bar{M}(10)$ of degree 1 and $c_2 = 10$, has two irreducible components, $\overline{M}(10)$ and $M(10, 4)$ meeting along the irreducible component of usual triples in $M(10, 4)^{\text{sing}}$. These two components have the expected dimension, 20, hence the moduli space is good and connected.*

Proof. Recall that we know $M(10, 4)$ is irreducible by the results of [\[Mestrano and Simpson 2011\]](#). Also $M(10)$ is irreducible. Any component has dimension ≥ 20 , and by looking at the dimensions in [Table 2](#), these are the only two possible irreducible components. Since they have dimension 20 which is the expected dimension, it follows that the moduli space is good.

It remains to be proven that these two components do indeed intersect in a nonempty subset, which then by [Corollary 7.10](#) has to be the irreducible component of usual triples in $M(10, 4)^{\text{sing}}$. Notice that [Corollary 7.10](#) did not say that the intersection was necessarily nonempty, since it started from the hypothesis that

there was a meeting point. It is a consequence of Nijssse's connectedness theorem that the intersection is nonempty, but this may be seen more concretely as follows.

Consider the stratum $M(10, 5)$. Recall from [Mestran and Simpson 2011] that the moduli space $M(5)$ consists of bundles which fit into an exact sequence of the form

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow J_{P/X}(1) \rightarrow 0,$$

such that $P = L \cap X$ for $L \subset \mathbb{P}^3$ a line. In what follows, choose L general so that P consists of 5 distinct points.

The space of extensions $\text{Ext}^1(J_{P/X}(1), \mathcal{O}_X)$ is dual to $\text{Ext}^1(\mathcal{O}_X, J_{P/X}(2)) = H^1(J_{P/X}(2))$. We have the exact sequence

$$H^0(\mathcal{O}_X(2)) \rightarrow H^0(\mathcal{O}_P(2)) \rightarrow H^1(J_{P/X}(2)) \rightarrow 0.$$

However, $H^0(\mathcal{O}_X(2)) = H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ and the map to $H^0(\mathcal{O}_P(2))$ factors through $H^0(\mathcal{O}_L(2))$, the space of degree two forms on $L \cong \mathbb{P}^1$, which has dimension 3. Hence, the cokernel $H^1(J_{P/X}(2))$ has dimension 2. The extension classes which correspond to bundles, are the linear forms on $H^1(J_{P/X}(2))$ which don't vanish on any of the images of the lines in $H^0(\mathcal{O}_P(2))$ corresponding to the 5 different points. Since X is general, the collection of 5 points $X \cap L$ is not in a special position in \mathbb{P}^1 , so the images of the lines are distinct in the two-dimensional space $H^1(J_{P/X}(2))$. So we can find a family of extension classes whose limiting point is an extension which vanishes on one of the lines corresponding to a point in P . This gives a degeneration towards a torsion-free sheaf with a single not locally free point, still sitting in a nontrivial extension of the above form. We conclude that the limiting bundle is still stable, so we have constructed a degeneration from a point of $M(5)$, to the single boundary stratum $M(5, 4)$.

Notice that the dimension of $M(5, 4)$ is bigger than that of $M(5)$, so the set of limiting points is a strict subvariety of $M(5, 4)$. We have $\overline{M}(5) = M(5) \cup M(5, 4)$, and we have shown that the closures of these two strata have nonempty intersection. This fact is also a consequence of the more explicit description of $\overline{M}(5)$ stated in Theorem 7.2 above (but where the proof was left to the reader).

Moving up to $c_2 = 10$, it follows that the closure of the stratum $M(10, 5)$ intersects $M(10, 4)$. However, $M(10, 4)$ is closed, and the remaining strata of the boundary have dimension ≤ 19 , so all of the other strata in the boundary, in particular $M(10, 5)$, are contained in the closure of the locus of bundles $\overline{M}(10)$. Thus, $\overline{M}(10, 5) \subset \overline{M}(10)$, but $M(10, 4) \cap \overline{M}(10, 5) \neq \emptyset$, proving that the intersection $M(10, 4) \cap \overline{M}(10)$ is nonempty. \square

Physics discussion. From this fact, we see that there are degenerations of stable bundles in $M(10)$, near to boundary points in $M(10, 4)$. Donaldson's Yang–Mills connections then degenerate towards Uhlenbeck boundary points, connections

where 6 delta-like singular instantons appear. However, these degenerations go not to all points in $M(10, 4)$ but only to ones which are in the irreducible subvariety $M(10, 4)^{\text{sing}} \subset M(10, 4)$ consisting of points on the internal spectral variety of a nonzero Higgs field $\varphi : E \rightarrow E \otimes K_X$. It gives a constraint of a global nature on the 6-tuples of singular instantons which can appear in Yang–Mills connections on a stable bundle $F \in M(10)$. It would be interesting to understand the geometry of the Higgs field which shows up, somewhat virtually, in the limit.

9. Irreducibility for $c_2 \geq 11$

Consider next the moduli space $\overline{M}(11)$ of stable torsion-free sheaves of degree one and $c_2 = 11$. The moduli space is good, of dimension 24. From Table 2, the dimensions of the boundary strata are all ≤ 23 , so the set of irreducible components of $\overline{M}(11)$ is the same as the set of irreducible components of $M(11)$. Suppose Z is an irreducible component. By Corollary 5.5, Z meets the boundary in a nonempty subset of codimension 1, i.e., dimension 23. From Table 2, the only two possibilities are $M(11, 10)$ and $M(11, 4)$. Note that $M(11, 4)$ is closed since it is the lowest stratum; it is irreducible by Li’s theorem and irreducibility of $M(4)$. The stratum $M(11, 10)$ is irreducible because of Theorem 8.3.

Lemma 9.1. *The intersection $M(11, 4) \cap \overline{M}(11, 10)$ is a nonempty subset containing, in particular, points which are torsion-free sheaves F' entering into an exact sequence of the form*

$$0 \rightarrow F' \rightarrow F \rightarrow S_x \rightarrow 0,$$

where F is a usual point of $M(10, 4)^{\text{sing}}$, $x \in X$ is a general point, and $F \rightarrow S_x$ is a general length one quotient.

Proof. Theorem 8.4 shows that the intersection $M(10, 4) \cap \overline{M}(10)$ is nonempty. It is the unique 19-dimensional irreducible component of $M(10, 4)^{\text{sing}}$, containing the usual points. Starting with a general point $F \in M(10, 4) \cap \overline{M}(10)$ and taking an additional general length 1 quotient S_x , the subsheaf F' gives a point in $M(11, 4) \cap \overline{M}(11, 10)$. □

Let $Y \subset M(10, 4)$ be the unique 19-dimensional irreducible component of the singular locus $M(10, 4)^{\text{sing}}$. It contains a dense open set where the quotient S is a direct sum of six quotients (x_i, s_i) of length 1. Choose a quasifinite surjection $Y' \rightarrow Y$ such that (x_i, s_i) are well defined as functions $Y' \rightarrow \mathbb{P}(E)$.

Forgetting the quotients and considering only the bundle E gives a map $Y' \rightarrow M(4)$. Fix a bundle E in the image of $Y' \rightarrow M(4)$. Let Y'_E denote the fiber of Y' over E , which has dimension ≥ 17 .

We claim that for any $0 \leq k \leq 5$, there exists a choice of k out of the 6 points such that the map $Y'_E \rightarrow \mathbb{P}(E)^k$ is surjective. For $k = 0$ this is automatic, so assume that

$k \leq 4$ and it is known for k ; we need to show that it is true for $k + 1$ points. Reorder so that the k points to be chosen, are the first ones. For a general point $q \in \mathbb{P}(E)^k$, let $Y'_{E,q}$ denote the fiber of $Y'_E \rightarrow \mathbb{P}(E)^k$ over q . We have $\dim(Y'_{E,q}) \geq 17 - 3k$. We get an injection

$$Y'_{E,q} \rightarrow \mathbb{P}(E)^{6-k}.$$

Suppose that the image mapped into a proper subvariety of each factor; then it would map into a subvariety of dimension $\leq 2(6 - k)$, which would give $\dim(Y'_{E,q}) \leq 12 - 2k$. However, for $k \leq 4$ we have $12 - 2k < 17 - 3k$, a contradiction. Therefore, at least one of the projections must be a surjection $Y'_{E,q} \rightarrow \mathbb{P}(E)$. Adding this point to our list, gives a list of $k + 1$ points such that the map $Y'_E \rightarrow \mathbb{P}(E)^{k+1}$ is surjective. This completes the induction, yielding the following lemma.

Lemma 9.2. *Suppose $Y \subset M(10, 4)$ is as above. Then for a fixed bundle $E \in M(4)$ corresponding to some points in Y , and for a general point in the fiber Y_E over E , some 5 out of the 6 quotients correspond to a general point of $\mathbb{P}(E)^5$.*

Lemma 9.3. *Suppose F is the torsion-free sheaf parametrized by a general point of Y , and let F' be defined by an exact sequence*

$$0 \rightarrow F' \rightarrow F \xrightarrow{(x_7, s_7)} S_7 \rightarrow 0,$$

where S_7 has length 1 and (x_7, s_7) is general (with respect to the choice of F) in $\mathbb{P}(E)$. Then F' has no nontrivial coobstructions: $\text{Hom}(F', F'(1)) = 0$.

Proof. The space of coobstructions for the bundle E has dimension 6. Imposing a condition of compatibility with a general length-1 quotient (x_i, s_i) cuts down the dimension of the space of coobstructions by at least 1.

By Lemma 9.2 above, we may assume after reordering that the first five points $(x_1, s_1), \dots, (x_5, s_5)$ constitute a general vector in $\mathbb{P}(E)^5$. Adding the 7-th general point given by the statement of the proposition, we obtain a general point $(x_1, s_1), \dots, (x_5, s_5), (x_7, s_7)$ in $\mathbb{P}(E)^6$. As this 6-tuple of points is general with respect to E , it imposes vanishing on the 6-dimensional space of coobstructions, giving $\text{Hom}(F', F'(1)) = 0$. \square

Corollary 9.4. *There exists a point*

$$F' \in \overline{M(11, 10)} \cap M(11, 4)$$

in the boundary of $\overline{M(11)}$, such that F' is a smooth point of $\overline{M(11)}$.

Proof. By Lemma 9.3, choosing a general quotient (x_7, s_7) gives a torsion-free sheaf F' with no coobstructions, hence corresponding to a smooth point of $\overline{M(11)}$. By construction we have $F' \in \overline{M(11, 10)} \cap M(11, 4)$. \square

Theorem 9.5. *The moduli space $\overline{M(11)}$ is irreducible.*

Proof. Suppose Z is an irreducible component. Then Z meets the boundary in a codimension 1 subset; but by looking at Table 2, there are only two possibilities: $\overline{M(11, 10)}$ and $M(11, 4)$. The coobstructions vanish for general points of $M(10, 4)$ since those correspond to 6 general quotients of length 1, and the coobstructions vanish for general points of $M(10)$ by goodness. It follows that there are no coobstructions at general points of $\overline{M(11, 10)}$ or $M(11, 4)$, so each of these is contained in at most a single irreducible component of $\overline{M(11)}$. However, in the previous corollary, there is a unique irreducible component containing F' , which shows that the irreducible components containing $\overline{M(11, 10)}$ and $M(11, 4)$ must be the same. Hence, $\overline{M(11)}$ has only one irreducible component. \square

Remark. Sarbeswar Pal has pointed out to us a simplified proof for $c_2 \geq 11$, avoiding the use of Lemma 9.1. He observes from the connectedness property and goodness of the moduli of torsion-free sheaves, that any change of irreducible component must occur along a codimension 1 piece of the singular locus. However, general points of the boundary components are smooth points of the full moduli space, by an easier version of the previous discussion, so we can conclude that the singular locus has codimension ≥ 2 . We have nonetheless presented our original proof since it gives some additional geometrical information on the intersection of the two boundary strata.

The cases $c_2 \geq 12$ are now easy to treat.

Theorem 9.6. *For any $c_2 \geq 12$, the moduli space $\overline{M}(c_2)$ of stable torsion-free sheaves of degree 1 and second Chern class c_2 , is irreducible.*

Proof. By Corollary 5.5, any irreducible component of $\overline{M}(c_2)$ meets the boundary in a subset of codimension 1. However, for $c_2 \geq 12$, the only stratum of codimension 1 is $M(c_2, c_2 - 1)$. By induction on c_2 , starting at $c_2 = 11$, we may assume that $M(c_2, c_2 - 1)$ is irreducible. Furthermore, if E is a general point of $M(c_2 - 1)$, then E admits no coobstructions, since $M(c_2 - 1)$ is good. Hence, a general point F in $M(c_2, c_2 - 1)$, which is the kernel of a general length-1 quotient $E \rightarrow S$, doesn't admit any coobstructions either. Therefore, $\overline{M}(c_2)$ is smooth at a general point of $M(c_2, c_2 - 1)$. Thus, there is a unique irreducible component containing $M(c_2, c_2 - 1)$, which completes the proof that $\overline{M}(c_2)$ is irreducible. \square

We have finished proving our main statement, Theorem 1.1 of the introduction: *for any $c_2 \geq 4$, the moduli space $M(c_2)$ of stable vector bundles of degree 1 and second Chern class c_2 on a very general quintic hypersurface $X \subset \mathbb{P}^3$ is nonempty and irreducible.*

For $4 \leq c_2 \leq 9$, this is shown in [Mestrano and Simpson 2011]. For $c_2 = 10$ it is Theorem 8.3, for $c_2 = 11$ it is Theorem 9.5, and $c_2 \geq 12$ it is Theorem 9.6. Note that for $c_2 \geq 16$ it is Nijse's [1995] theorem.

It was shown in [Mestran and Simpson 2011] that the moduli space is good for $c_2 \geq 10$ (shown by Nijse for $c_2 \geq 13$), and from Table 1 we see that it isn't good for $4 \leq c_2 \leq 9$. The moduli space of torsion-free sheaves $\bar{M}(c_2)$ is irreducible for $c_2 \geq 11$, as may be seen by looking at the dimensions of boundary strata in Table 2. Whereas $M(4) = \bar{M}(4)$ is irreducible, the dimensions of the strata in Table 2 imply that $\bar{M}(c_2)$ has several irreducible components for $5 \leq c_2 \leq 9$, although we haven't answered the question as to their precise number. By Theorem 8.4, $\bar{M}(10)$ has two irreducible components $\bar{M}(10)$ and $M(10, 4)$.

10. An irregularity estimate

In this section we provide a correction and improvement to [Mestran and Simpson 2011, Lemma 5.1] and hence Corollary 5.1 there. There was an error in the proof given in there.

Lemma 10.1. *Suppose X is a very general quintic hypersurface in \mathbb{P}^3 . Suppose $s \in H^0(\mathcal{O}_X(2))$ is a section which is not the square of a section of $\mathcal{O}_X(1)$. It defines an irreducible spectral covering $Z \subset \text{Tot}(K_X)$ consisting of square roots of s . Let \tilde{Z} be a resolution of singularities of Z . Then the irregularity of \tilde{Z} is zero, that is to say, $H^0(\tilde{Z}, \Omega_{\tilde{Z}}^1) = 0$. Hence the dimension of $\text{Pic}^0(\tilde{Z})$ is zero.*

Proof. The divisor D of zeros of s is reduced since s isn't a square and in view of the fact that $\mathcal{O}_X(1)$ generates $\text{Pic}(X)$. Therefore the map $Z \rightarrow X$ is ramified with simple ramification along the smooth points of D . The involution of multiplication by -1 acts in the fibers. Choose an equivariant resolution of singularities $\tilde{Z} \rightarrow Z$ with an involution $\sigma : \tilde{Z} \rightarrow \tilde{Z}$ covering the given involution of Z . The irregularity of \tilde{Z} is independent of the choice of resolution, so we would like to show that $H^0(\tilde{Z}, \Omega_{\tilde{Z}}^1) = 0$.

The map $p : \tilde{Z} \rightarrow X$ induces an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow p_*(\mathcal{O}_{\tilde{Z}}) \rightarrow Q \rightarrow 0,$$

with Q a rank 1 torsion-free sheaf on X . The double dual Q^{**} is a line bundle L . Using the involution σ , the above exact sequence splits: Q is the anti-invariant part. Multiplying together sections of Q gives a map

$$Q \otimes Q \rightarrow \mathcal{O}_X,$$

which extends by Hartogs to a map

$$L \otimes L \rightarrow \mathcal{O}_X.$$

Look locally near a smooth point of D where X has coordinates (x, y) such that D is given by $y = 0$, and \tilde{Z} has coordinates (x, z) with $y = z^2$. As a $\mathbb{C}\{x, y\}$ -module, Q or equivalently L is generated by z . The image of the multiplication map is

therefore the submodule generated by $z^2 = y$. It is an isomorphism outside of D , and to get an isomorphism it suffices to look off of codimension 2. This shows that

$$L \otimes L \xrightarrow{\sim} \mathcal{O}_X(-D) \cong \mathcal{O}_X(-2),$$

hence $L \cong \mathcal{O}_X(-1)$. It means that L is generated by the linear functions along the fibers of $K_X \rightarrow X$, restricted back to \tilde{Z} .

Consider similarly the decomposition into invariant and anti-invariant pieces

$$p_*(\Omega_{\tilde{Z}}^1) = \mathcal{F}^+ \oplus \mathcal{F}^-.$$

These sheaves are torsion-free, and we have a map $\Omega_X^1 \rightarrow \mathcal{F}^+$. Again with the local coordinates x, y for X and x, z for \tilde{Z} near a smooth point of D as above, we have that $\Omega_{\tilde{Z}}^1$ is generated by dx and dz . As a module over $\mathbb{C}\{x, y\}$, \mathcal{F}^+ is generated by dx and zdz or equivalently dx and dy . This shows that the map $\Omega_X^1 \rightarrow \mathcal{F}^+$ is an isomorphism on smooth points of D . Since \mathcal{F}^+ is torsion-free and Ω_X^1 is locally free, it follows that this map is an isomorphism. We may therefore write

$$p_*(\Omega_{\tilde{Z}}^1) = \Omega_X^1 \oplus \mathcal{F}^-.$$

Consider now the map $\Omega_X^1 \otimes \mathcal{Q} \rightarrow \mathcal{F}^-$. Let $\mathcal{G} := (\mathcal{F}^-)^{**}$ be the double dual, and the previous map induces a map

$$\Omega_X^1 \otimes L \rightarrow \mathcal{G}.$$

Consider again the situation at a smooth point of D using local coordinates. Note that \mathcal{G} is generated by zdx and dz , whereas $\Omega_X^1 \otimes L$ is generated by zdx and $zdy = z^2dz = ydz$. Recalling that $L = \mathcal{O}_X(-1)$, we get an exact sequence

$$0 \rightarrow \Omega_X^1(-1) \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow 0,$$

where \mathcal{B} is a sheaf supported on D , locally near the smooth points being isomorphic to \mathcal{O}_D . This says that \mathcal{G} and $\Omega_X^1(-1)$ are related by an elementary transformation. In particular, we get

$$0 \rightarrow \mathcal{G} \rightarrow \Omega_X^1(-1)(D) = \Omega_X^1(1).$$

The irregularity of X vanishes so $H^0(\Omega_X^1) = 0$. Hence,

$$H^0(\tilde{Z}, \Omega_{\tilde{Z}}^1) \cong H^0(X, p_*\Omega_{\tilde{Z}}^1) \xrightarrow{\sim} H^0(X, \mathcal{F}^-) \hookrightarrow H^0(X, \mathcal{G}) \hookrightarrow H^0(X, \Omega_X^1(1)).$$

We have finally shown that there is an injection

$$H^0(\tilde{Z}, \Omega_{\tilde{Z}}^1) \hookrightarrow H^0(X, \Omega_X^1(1)).$$

One may show⁸ the right-hand space of sections vanishes, completing the proof. \square

⁸For convenience, here is the argument. The canonical exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}^3}^1 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^4 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow 0$$

Therefore Corollary 5.1 of [Mestrano and Simpson 2011] holds, with the improved bound that the dimension is ≤ 9 . Along the way we have answered [Mestrano and Simpson 2011, Question 5.1]: in the notation from there, $A = 0$.

11. Example on a degree 6 hypersurface

In this section we shall start in the direction of considering hypersurfaces of higher degree, and consider briefly the case of hypersurfaces of degree 6. In particular, the notation differs from that in effect previously.

Here, $X \subset \mathbb{P}^3$ is a very general hypersurface of degree 6, which will be denoted $X = X^6$ in the statements of the main corollaries, for precision. We have $K_X = \mathcal{O}_X(2)$. We consider stable rank 2 vector bundles E of degree 1 and more precisely with $\det(E) = \mathcal{O}_X(1)$, and some specified value of c_2 .

Assume $h^0(E) > 0$. Then there is a section, corresponding to a morphism $s : \mathcal{O}_X \rightarrow E$. The zeros of s are in codimension 2; otherwise it would extend to $\mathcal{O}_X(1) \rightarrow E$, contradicting stability. Therefore, s fits into an exact sequence of the usual form

$$(11-1) \quad 0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow J_{P/X}(1) \rightarrow 0,$$

where $P \subset X$ is a locally complete intersection subscheme of dimension 0. By the general theory, P satisfies the condition $\text{CB}(L^{-1} \otimes M \otimes K_X)$, where $L = \mathcal{O}_X$ and $M = \mathcal{O}_X(1)$. In other words, P is a $\text{CB}(3)$ subscheme.

Notice that $c_2(\mathcal{O}_X \oplus \mathcal{O}_X(1)) = 0$ by the product formula for Chern polynomials; therefore in the above extension, we have $c_2(E) = |P|$.

In our examples, we will consider the case $c_2 = 11$, and give two different kinds of 11-point $\text{CB}(3)$ subschemes.

Before getting to these, let us note some general things about the deformation theory. Our bundle satisfies $E^* = E(-1)$, so

$$\text{End}(E) = E^* \otimes E \cong E \otimes E(-1).$$

gives rise to

$$0 \rightarrow H^0(\Omega_{\mathbb{P}^3}^1(1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3}^4) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(1)),$$

in which the right map is an isomorphism, so $H^0(\Omega_{\mathbb{P}^3}^1(1)) = 0$. We also get $H^1(\mathbb{P}^3, \Omega_{\mathbb{P}^3}^1(-4)) = 0$; thus the exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}^3}^1(-4) \rightarrow \Omega_{\mathbb{P}^3}^1(1) \rightarrow \Omega_{\mathbb{P}^3}^1(1)|_X \rightarrow 0$$

implies $H^0(\Omega_{\mathbb{P}^3}^1(1)|_X) = 0$. Now using $H^1(\mathcal{O}_X(n)) = 0$, the exact sequence

$$0 \rightarrow N_{X/\mathbb{P}^3}^*(1) = \mathcal{O}_X(-4) \rightarrow \Omega_{\mathbb{P}^3}^1(1)|_X \rightarrow \Omega_X^1(1) \rightarrow 0$$

gives $H^0(\Omega_X^1(1)) = 0$.

The decomposition $\text{End}(E) = \text{End}^0(E) \oplus \mathcal{O}_X$ into the trace-free plus the central part, corresponds to the decomposition

$$E \otimes E(-1) = \text{Sym}^2(E)(-1) \oplus \bigwedge^2(E)(-1).$$

Let us denote for short $V := \text{Sym}^2(E)(-1)$. The deformation theory of E as a bundle with fixed determinant is governed by $H^*(V)$. Notice that if E is stable, it has no endomorphisms except the scalars, so $H^0(V) = 0$. We may also apply Serre duality, noting that V is self-dual and recalling $K_X = \mathcal{O}_X(2)$. The space of infinitesimal deformations of E is

$$\text{Def}(E) = H^1(V) \cong H^1(V(2))^*,$$

and the space of obstructions is

$$\text{Obs}(E) = H^2(V) \cong H^0(V(2))^*.$$

Let $2P$ denote the subscheme defined by the square of the ideal of P , so $J_{2P/X} = (J_{P/X})^2$. We have an exact sequence

$$(11-2) \quad 0 \rightarrow E(-1) \rightarrow V \rightarrow J_{2P/X}(1) \rightarrow 0,$$

and hence

$$(11-3) \quad 0 \rightarrow E(1) \rightarrow V(2) \rightarrow J_{2P/X}(3) \rightarrow 0.$$

Points on the rational normal cubic. The first case is when $C \subset \mathbb{P}^3$ is a general rational normal cubic, and $P \subset X \cap C$ is a collection of 11 points. This exists since $C \cap X$ consists of 18 distinct points and we may choose 11 of them.

Notice that $C \cong \mathbb{P}^1$ and $\mathcal{O}_{\mathbb{P}^3}(1)|_C = \mathcal{O}_C(3p)$ for any point $p \in C$, that is to say it is a line bundle of degree 3. Thus, $\mathcal{O}_{\mathbb{P}^3}(3)|_C = \mathcal{O}_C(9p)$ has degree 9. If $P' \subset P$ is any collection of 10 points, a section of $\mathcal{O}_{\mathbb{P}^3}(3)$ vanishing on P' must vanish on C , hence it must vanish on P . The sections of $\mathcal{O}_X(3)$ are all restrictions of sections of $\mathcal{O}_{\mathbb{P}^3}(3)$, so this proves that P satisfies the property CB(3).

The space of extensions of $J_{P/X}(1)$ by \mathcal{O}_X is dual to $H^1(J_{P/X}(3))$, which in turn is the cokernel of

$$(11-4) \quad H^0(\mathcal{O}_X(3)) \rightarrow H_0(P, \mathcal{O}_P(3)) \cong \mathbb{C}^{11}.$$

As we have seen above, a section of $H^0(\mathcal{O}_X(3))$ vanishing on P corresponds to a section of $H^0(\mathcal{O}_{\mathbb{P}^3}(3))$ vanishing on C . One may calculate by hand that the map

$$\mathbb{C}^{20} = H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \rightarrow H^0(\mathcal{O}_C(9p)) = \mathbb{C}^{10}$$

is surjective. Indeed, the image of $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ consists of the sections which may be written as $1, t, t^2, t^3$ for an affine coordinate t on $C \cong \mathbb{P}^1$ with pole at the point p . Then, monomials of degree 3 in these sections give all of the monomials $1, t, \dots, t^9$.

From this surjectivity we get that the kernel is \mathbb{C}^{10} . Thus, the kernel of the map (11-4) is \mathbb{C}^{10} so the image of the map also has dimension 10. Finally, we get that the cokernel of (11-4) has dimension 1. We have shown that $\text{Ext}^1(J_{P/X}, \mathcal{O}_X)$ has dimension 1. Therefore, a given subscheme P gives rise to only one bundle since scaling of the extension class doesn't change the isomorphism class of the bundle.

For the other direction, we claim that $h^0(E) = 1$. Consider the exact sequence

$$0 \rightarrow H^0(\mathcal{O}_X) \rightarrow H^0(E) \rightarrow H^0(J_{P/X}(1)) \rightarrow H^1(\mathcal{O}_X) = 0.$$

Given a section of $H^0(\mathcal{O}_X(1))$ vanishing on P , it comes from a section of $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ which, by the same argument as previously, vanishes on C . If the section is nonzero, that would say that C is contained in a plane, which however is not the case. Therefore, $H^0(J_{P/X}(1)) = 0$ and $\mathbb{C} \cong H^0(\mathcal{O}_X) \xrightarrow{\sim} H^0(E)$. We get $h^0(E) = 1$ as claimed.

In particular, for a given bundle E , the choice of section s is unique up to a scalar, so the subscheme P is uniquely determined.

By these arguments, we conclude that the space of bundles E in this case is isomorphic to the space of choices of subscheme $P \subset C \cap X$.

Now, given $P \subset C \cap X$ of length 11, we claim that C is the only rational normal curve passing through P . Indeed, suppose C' were another one. Note that C' is cut out by conics. If $Q \subset \mathbb{P}^3$ is a conic containing C' then $Q \cap C$ is either equal to C , or has length 6; the latter case can't happen so $C \subset Q$. Thus, any conic containing C' also contains C , which shows that $C = C'$.

The dimension of the space of subschemes P in this case is therefore equal to the dimension of the space $\text{PGL}(4)/\text{PGL}(2)$ of rational normal cubic curves, which is $15 - 3 = 12$. This completes the proof of the following proposition:

Proposition 11.1. *The space of bundles E fitting into an exact sequence of the form (11-1), where P is a length 11 subscheme of $C \cap X$ for C a rational normal cubic in \mathbb{P}^3 , has dimension 12.*

Lemma 11.2. *Suppose E is a bundle fitting into an exact sequence of the form (11-1), where P is a length 11 subscheme of $C \cap X$ for C a general rational normal cubic in \mathbb{P}^3 . Then $h^1(\text{End}^0(E)) = h^1(V) = 12$.*

Proof. Use the exact sequence (11-2). The first step is to calculate $h^1(E(-1))$. Note that (11-1) gives the following sequence, using that $h^1(\mathcal{O}_X(n)) = 0$ for any n as well as $H^2(J_{P/X}(n)) = H^2(\mathcal{O}_X(n))$:

$$0 \rightarrow H^1(E(-1)) \rightarrow H^1(J_{P/X}) \rightarrow H^2(\mathcal{O}_X(-1)) \rightarrow H^2(E(-1)) \rightarrow H^2(\mathcal{O}_X) \rightarrow 0.$$

Now $H^2(E(-1))$ is dual to $H^0(E(2))$ which itself fits into the sequence

$$0 \rightarrow H^0(\mathcal{O}_X(2)) \rightarrow H^0(E(2)) \rightarrow H^0(J_{P/X}(3)) \rightarrow 0.$$

We have $H^0(J_{P/X}(3)) \cong H^0(J_{C/\mathbb{P}^3}(3)) = \ker(H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \rightarrow H^0(\mathcal{O}_C(9P)))$. The latter map is surjective from \mathbb{C}^{20} to \mathbb{C}^{10} so its kernel has dimension 10. This gives $h^0(J_{P/X}(3)) = 10$. Also $h^0(\mathcal{O}_X(2)) = 10$ so $h^2(E(-1)) = h^0(E(2)) = 20$. We have $h^2(\mathcal{O}_X) = h^0(\mathcal{O}_X(2)) = 10$ and $h^2(\mathcal{O}_X(-1)) = h^0(\mathcal{O}_X(3)) = 20$. Finally, $H^1(J_{P/X})$ is just \mathbb{C}^{11} modulo $H^0(\mathcal{O}_X) = \mathbb{C}$ so $h^1(J_{P/X}) = 10$. The alternating sum from the above sequence vanishes, saying now that

$$h^1(E(-1)) - 10 + 20 - 20 + 10 = 0,$$

so $h^1(E(-1)) = 0$.

The long exact sequence associated to (11-2) starting with $H^1(E(-1)) = 0$ now gives

$$0 \rightarrow H^1(V) \rightarrow H^1(J_{2P/X}(1)) \rightarrow H^2(E(-1)) \rightarrow H^2(V) \rightarrow H^2(\mathcal{O}_X(1)) \rightarrow 0.$$

As we have seen above, $h^2(E(-1)) = 20$. It is also easy to see that $h^0(J_{2P/X}(1)) = 0$ (we will in fact see this for $J_{2P/X}(3)$ below), so noting that the length of $2P$ is 33 we get $h^1(J_{2P/X}(1)) = 33 - h^0(\mathcal{O}_X(1)) = 29$. Putting these together and using $h^2(\mathcal{O}_X(1)) = h^0(\mathcal{O}_X(1)) = 4$ we get

$$h^1(V) - 29 + 20 - h^2(V) + 4 = 0,$$

so $h^1(V) - h^2(V) = 5$. This is the expected dimension of the moduli space.

Next, by duality $h^2(V) = h^0(V(2))$ which we can calculate using the sequence (11-3). We have

$$0 \rightarrow H^0(E(1)) \rightarrow H^0(V(2)) \rightarrow H^0(J_{2P/X}(3)).$$

We claim that $H^0(J_{2P/X}(3)) = 0$. To see this, consider a smooth quadric surface $Q \subset \mathbb{P}^3$ containing C . We have $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ and C is a divisor of bidegree $(1, 2)$ on Q . On the other hand, $\mathcal{O}_Q(1)$ has bidegree $(1, 1)$. Suppose we have a section u of $H^0(\mathcal{O}_X(3)) = H^0(\mathcal{O}_{\mathbb{P}^3}(3))$ vanishing on the $2P$ (recall that $2P$ is the subscheme of X defined by the square of the ideal of P). We have seen already above that vanishing on P implies that it vanishes on C . Therefore $u|_Q$ is a section of the bundle of bidegree $(3, 3) - (1, 2) = (2, 1)$. The intersection of $2P$ with Q consists of a collection of double points transverse to C at the points of P , so it imposes again a single condition on the section u considered as a section of $\mathcal{O}_Q(2, 1)$. The restriction of $\mathcal{O}_Q(2, 1)$ to C is a line bundle on $C \cong \mathbb{P}^1$ of degree equal to the intersection number $(2, 1) \cdot (1, 2) = 5$. Therefore, a section of $\mathcal{O}_Q(2, 1)$ which vanishes on 11 points has to vanish. This says that our section of bidegree $(2, 1)$ again vanishes on C , so it is a section of a bundle of bidegree $(1, -1)$; but that is not effective so

this section has to vanish. This proves that our section $u|_Q$ vanishes. Therefore, u may be viewed as a section of $\mathcal{O}_{\mathbb{P}^3}(3)(-Q) = \mathcal{O}_{\mathbb{P}^3}(1)$. The remaining pieces of the double points composing $2P$ give conditions of vanishing again at all the points of P for this section of $\mathcal{O}_{\mathbb{P}^3}(1)$, but as C is not contained in a plane, it implies that the section vanishes. This completes the proof that $H^0(J_{2P/X}(3)) = 0$. We conclude from the previous exact sequence that

$$h^2(V) = h^0(V(2)) = h^0(E(1)).$$

Now use the sequence

$$0 \rightarrow H^0(\mathcal{O}_X(1)) \rightarrow H^0(E(1)) \rightarrow H^0(J_{P/X}(2)) \rightarrow 0.$$

As usual, $H^0(J_{P/X}(2))$ is isomorphic to the kernel of the restriction map

$$\mathbb{C}^{10} = H^0(\mathcal{O}_X(2)) \rightarrow H^0(\mathcal{O}_C(6P)) = \mathbb{C}^7$$

and this restriction map is surjective, so its kernel has dimension 3. We get

$$h^0(E(1)) = 4 + 3 = 7.$$

Thus, $h^2(V) = 7$, and putting this together with the formula that the expected dimension is 5, we have finally shown $h^1(V) = 12$. This proves the lemma. \square

Even though there is a 7-dimensional obstruction space, we have constructed a 12-dimensional family; it follows that all of the obstructions vanish and a general point lies in a generically smooth irreducible component of dimension 12.

Corollary 11.3. *The space of bundles E fitting into an exact sequence of the form (11-1), where P is a length 11 subscheme of $C \cap X$ for C a rational normal cubic in \mathbb{P}^3 , consists of a single 12-dimensional generically smooth irreducible component of the moduli space $M_{X^6}(2, 1, 11)$ of stable bundles of rank 2, degree 1 and $c_2 = 11$ on our degree 6 hypersurface $X = X^6$.*

Proof. In order to understand how many irreducible components are produced by this construction, we should investigate the monodromy of the set of choices of 11 out of the 18 points of $C \cap X$, as C moves. A choice of 6 points determines the rational normal cubic C , so any 6 points can be moved to any 6 other ones. Therefore, the monodromy action is 6-tuply transitive. On the other hand, it contains a transposition, since we can move C around a choice of curve that is simply tangent to X at one point. Therefore, the monodromy group contains all transpositions, hence it is the full symmetric group on 18 elements. It acts transitively on the set of choices of 11 out of the 18 intersection points, so our construction produces a single irreducible component. \square

Points on a plane. The other construction we have found for CB(3) subschemes is to take 11 points in a plane. Let H be a plane in general position with respect to X , and let $Y = X \cap H$. Let P consist of a general collection of 11 points in Y .

Suppose $P' \subset P$ is a subset of 10 points. The map $H^0(\mathcal{O}_H(3)) \rightarrow H^0(\mathcal{O}_{P'}(3))$ is injective (since Y is a curve of degree 6 in the plane H), so a general collection of 10 points imposes independent conditions on $H^0(\mathcal{O}_H(3))$. As $h^0(\mathcal{O}_H(3)) = 10$, it means that $H^0(J_{P'/H}(3)) = 0$, hence a section of $H^0(\mathcal{O}_{\mathbb{P}^3}(3))$ vanishing on P' , has to vanish on H . In particular it vanishes on P , proving the CB(3) property for P . This also gives the formula

$$H^0(J_{P/X}(3)) \cong H^0(\mathcal{O}_X(2)) = \mathbb{C}^{10}.$$

Consider next the space of choices of extension (11-1). As

$$\dim(\text{Ext}^1(J_{P/X}(1), \mathcal{O}_X)) = h^1(J_{P/X}(3)) = 11 - 20 + h^0(J_{P/X}(3)) = 1,$$

whereas scalar multiples of an extension class give the same bundle, it means that for a given P there is a single corresponding bundle. On the other hand, we have $h^0(J_{P/X}(1)) = 1$ since P is contained in a plane, so $h^0(E) = 2$. This means that for a given bundle E , the space of choices of section s (modulo scaling) leading to the subscheme P , has dimension 1. Hence the dimension of the space of bundles obtained by this construction is one less than the dimension of the space of subschemes:

$$\dim\{E\} = \dim\{P\} - 1.$$

Count now the dimension of the space of choices of P : there is a three-dimensional space of choices of the plane H , and for each one we have an 11-dimensional space of choices of the subscheme P of 11 points in Y . This gives $\dim\{P\} = 3 + 11 = 14$, so $\dim\{E\} = 13$. Altogether, we have constructed a 13-dimensional family of stable bundles. It follows that this family must be in at least one irreducible component distinct from the 12-dimensional component constructed above. This proves the following theorem:

Theorem 11.4. *For a very general degree 6 hypersurface $X^6 \subset \mathbb{P}^3$, the moduli space $M_{X^6}(2, 1, 11)$ contains a generically smooth 12-dimensional component from Corollary 11.3, and contains at least one irreducible component of dimension ≥ 13 . In particular, it is not irreducible.*

The general bundle in our 13-dimensional family may be viewed as an elementary transformation [Maruyama 1973; 1982]. A general line bundle L of degree 11 on Y has a 2-dimensional space of sections and the two sections generate L . If $j : Y \hookrightarrow X$ denotes the inclusion then we get a bundle E , elementary transformation of \mathcal{O}_X^2 ,

fitting into exact sequences

$$\begin{aligned} 0 \rightarrow E(-1) \rightarrow \mathcal{O}_X^2 \rightarrow j_*(L) \rightarrow 0, \\ 0 \rightarrow \mathcal{O}_X^2 \rightarrow E \rightarrow j_*(L^*)(1) \rightarrow 0. \end{aligned}$$

This shows that E determines Y and L . Since Y has genus 10, the space of choices of hyperplane plus choice of L has dimension $3 + 10 = 13$. One may see that these bundles are the same as the previous ones, indeed the zeros of a section of our elementary transformation E are the same as those of the corresponding section of L . This gives an alternate canonical viewpoint on our second construction of bundles that should be useful for understanding the obstruction map.

We conjecture that the rational normal case and the planar case cover all of $M_{X^6}(2, 1, 11)$. More precisely:

Conjecture 11.5. The 13-dimensional family constructed in the present subsection constitutes a full irreducible component of $M_{X^6}(2, 1, 11)$; this component is nonreduced and obstructed. Together with the 12-dimensional generically smooth component constructed in the previous subsection, these are the only irreducible components of $M_{X^6}(2, 1, 11)$. In particular, $h^0(E) > 0$ for any stable bundle with $c_2 = 11$.

There doesn't seem to be an easy direct proof of the property $h^0(E) > 0$. The Euler characteristic consideration does give $h^0(E(1)) > 0$ so any E has to be in an extension of $\mathcal{O}(-1)$ by $J_{P/X}(2)$ with P satisfying CB(5). If this conjecture is true, it would imply that any CB(5) subscheme of length 21 contained in X^6 , would have to be contained in a quadric hypersurface. We didn't find a proof of that, but we couldn't find any length-21 subschemes of X^6 satisfying CB(5) that weren't contained in quadric hypersurfaces either, leading to the conjecture.

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Large-scale rigidity properties of the mapping class groups	1
BRIAN H. BOWDITCH	
Bach-flat isotropic gradient Ricci solitons	75
ESTEBAN CALVIÑO-LOUZAO, EDUARDO GARCÍA-RÍO, IXCHEL GUTIÉRREZ-RODRÍGUEZ and RAMÓN VÁZQUEZ-LORENZO	
Contact stationary Legendrian surfaces in \mathbb{S}^5	101
YONG LUO	
Irreducibility of the moduli space of stable vector bundles of rank two and odd degree on a very general quintic surface	121
NICOLE MESTRANO and CARLOS SIMPSON	
A capillary surface with no radial limits	173
COLM PATRIC MITCHELL	
Initial-seed recursions and dualities for d -vectors	179
NATHAN READING and SALVATORE STELLA	
Codimensions of the spaces of cusp forms for Siegel congruence subgroups in degree two	207
ALOK SHUKLA	
Nonexistence results for systems of elliptic and parabolic differential inequalities in exterior domains of \mathbb{R}^n	245
YUHUA SUN	