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A CAPILLARY SURFACE WITH NO RADIAL LIMITS

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In 1996, Kirk Lancaster and David Siegel investigated the existence and behavior of radial limits at a corner of the boundary of the domain of solutions of capillary and other prescribed mean curvature problems with contact angle boundary data. They provided an example of a capillary surface in a unit disk D which has no radial limits at $(0,0) \in \partial D$. In their example, the contact angle, γ, cannot be bounded away from zero and π. Here we consider a domain Ω with a convex corner at $(0,0)$ and find a capillary surface $z = f(x,y)$ in $\Omega \times \mathbb{R}$ which has no radial limits at $(0,0) \in \partial\Omega$ such that γ is bounded away from 0 and π.

Let Ω be a domain in $\mathbb{R}^2$ with locally Lipschitz boundary and $\mathcal{O} = (0,0) \in \partial\Omega$ such that $\partial\Omega \setminus \{\mathcal{O}\}$ is a C^4 curve and $\Omega \subset B_1(0,1)$, where $B_\delta(\mathcal{N})$ is the open ball in $\mathbb{R}^2$ of radius δ about $\mathcal{N} \in \mathbb{R}^2$. Denote the unit exterior normal to Ω at $(x,y) \in \partial\Omega$ by $\nu(x,y)$ and let polar coordinates relative to $\mathcal{O}$ be denoted by r and θ. We shall assume there exist $\delta^* \in (0,2)$ and $\alpha \in (0, \frac{\pi}{2})$ such that $\partial\Omega \cap B_{\delta^*}(\mathcal{O})$ consists of the line segments

$$\partial^+\Omega^* = \{(r\cos(\alpha), r\sin(\alpha)) : 0 \le r \le \delta^*\}$$

and

$$\partial^-\Omega^* = \{(r\cos(-\alpha), r\sin(-\alpha)) : 0 \le r \le \delta^*\}.$$

Set $\Omega^* = \Omega \cap B_{\delta^*}(\mathcal{O})$. Let $\gamma : \partial\Omega \to [0,\pi]$ be given. Let $(x^\pm(s), y^\pm(s))$ be arclength parametrizations of $\partial^\pm\Omega$ with $(x^+(0), y^+(0)) = (x^-(0), y^-(0)) = (0,0)$ and set $\gamma^\pm(s) = \gamma(x^\pm(s), y^\pm(s))$.

Consider the capillary problem of finding a function $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$ satisfying

$$\operatorname{div}(Tf) = \tfrac{1}{2} f \quad \text{in } \Omega \tag{1}$$

$$Tf \cdot \nu = \cos(\gamma) \quad \text{on } \partial\Omega \setminus \{\mathcal{O}\}, \tag{2}$$

where $Tf = \nabla f / \sqrt{1 + |\nabla f|^2}$. We are interested in the existence of the radial limits $Rf(\cdot)$ of a solution f of (1) and (2), where

$$Rf(\theta) = \lim_{r \to 0^+} f(r\cos\theta, r\sin\theta), \qquad -\alpha < \theta < \alpha$$

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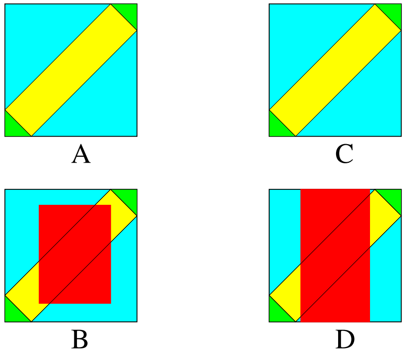

Figure 1. The Concus–Finn rectangle (A and C) with regions R (yellow), $D_2^{\pm}$ (blue) and $D_1^{\pm}$ (green); the restrictions on γ in [Lancaster and Siegel 1996] (red region in B) and in [Crenshaw et al. 2017] (red region in D).

and $Rf(\pm\alpha) = \lim_{\partial^{\pm}\Omega^* \ni \boldsymbol{x} \to \mathcal{O}} f(\boldsymbol{x})$, $\boldsymbol{x} = (x, y)$, which are the limits of the boundary values of f on the two sides of the corner if these exist.

Proposition 1 [Crenshaw et al. 2017]. *Let f be a bounded solution to (1) satisfying (2) on $\partial^{\pm}\Omega^* \setminus \{\mathcal{O}\}$ which is discontinuous at $\mathcal{O}$. If $\alpha > \pi/2$ then $Rf(\theta)$ exists for all $\theta \in (-\alpha, \alpha)$. If $\alpha \le \pi/2$ and there exist constants $\underline{\gamma}^{\pm}, \bar{\gamma}^{\pm}$, $0 \le \underline{\gamma}^{\pm} \le \bar{\gamma}^{\pm} \le \pi$, satisfying*

$$\pi - 2\alpha < \underline{\gamma}^{+} + \underline{\gamma}^{-} \le \bar{\gamma}^{+} + \bar{\gamma}^{-} < \pi + 2\alpha$$

so that $\underline{\gamma}^{\pm} \le \gamma^{\pm}(s) \le \bar{\gamma}^{\pm}$ for all s, $0 < s < s_0$, for some s_0, then again $Rf(\theta)$ exists for all $\theta \in (-\alpha, \alpha)$.

Lancaster and Siegel [1996] proved this theorem with the additional restriction that γ be bounded away from 0 and π; Figure 1 illustrates these cases.

They also proved the following:

Proposition 2 [Lancaster and Siegel 1996, Theorem 3]. *Let Ω be the disk of radius 1 centered at $(1, 0)$. Then there exists a solution to $Nf = \frac{1}{2}f$ in Ω, $|f| \le 2$, $f \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus O)$, $O = (0, 0)$ such that no radial limits $Rf(\theta)$ exist ($\theta \in [-\pi/2, \pi/2]$).*

In this case, $\alpha = \frac{\pi}{2}$; if γ is bounded away from 0 and π, then Proposition 1 would imply that $Rf(\theta)$ exists for each $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and therefore the contact angle $\gamma = \cos^{-1}(Tf \cdot \nu)$ in Proposition 2 is not bounded away from 0 and π.

In our case, the domain Ω has a convex corner of size 2α at $\mathcal{O}$ and we wish to investigate the question of whether an example like that in Proposition 2 exists in this case when γ is bounded away from 0 and π. In terms of the Concus–Finn rectangle, the question is whether, given $\epsilon > 0$, there is a solution $f \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \{\mathcal{O}\})$ of

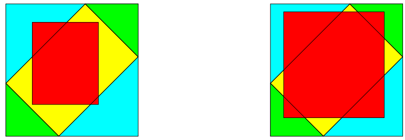

Figure 2. The Concus–Finn rectangle. When γ remains in the red region in E, $Rf(\cdot)$ exists; γ in Theorem 1 remains in the red region in F.

(1) and (2) such that no radial limits $Rf(\theta)$ exist ($\theta \in [-\alpha, \alpha]$) and $\left|\gamma - \frac{\pi}{2}\right| \le \alpha + \epsilon$; this is illustrated in Figure 2.

Theorem 1. *For each* $\epsilon > 0$, *there is a domain* Ω *as described above and a solution* $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$ *of* (1) *such that the contact angle*

$$\gamma = \cos^{-1}(Tf \cdot \nu) : \partial\Omega \setminus \{\mathcal{O}\} \to [0, \pi]$$

satisfies $\left|\gamma - \frac{\pi}{2}\right| \le \alpha + \epsilon$ *and there exists a sequence* $\{r_j\}$ *in* $(0, 1)$ *with* $\lim_{j\to\infty} r_j = 0$ *such that*

$$(-1)^j f(r_j, 0) > 1 \quad \textit{for each } j \in \mathbb{N}.$$

Assuming Ω *and* γ *are symmetric with respect to the line* $\{(x, 0) : x \in \mathbb{R}\}$, *this implies that no radial limit*

$$Rf(\theta) \stackrel{\text{def}}{=} \lim_{r\downarrow 0} f(r\cos(\theta), r\sin(\theta)) \tag{3}$$

exists for any $\theta \in [-\alpha, \alpha]$.

We remark that our theorem is an extension of [Lancaster and Siegel 1996, Theorem 3] to contact angle data in a domain with a convex corner. As in [Lancaster 1989; Lancaster and Siegel 1996], we first state and prove a localization lemma; this is analogous to [Lancaster 1989, Lemma] and [Lancaster and Siegel 1996, Lemma 2].

Lemma 1. *Let* $\Omega \subseteq \mathbb{R}^2$ *be as above,* $\epsilon > 0$, $\eta > 0$ *and* $\gamma_0 : \partial\Omega \setminus \{\mathcal{O}\} \to [0, \pi]$ *such that* $\left|\gamma_0 - \frac{\pi}{2}\right| \le \alpha + \epsilon$. *For each* $\delta \in (0, 1)$ *and* $h \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$ *which satisfies* (1) *and* (2) *with* $\gamma = \gamma_0$, *there exists a solution*

$$g \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$$

of (1) *such that* $\lim_{\overline{\Omega}\ni(x,y)\to(0,0)} g(x, y) = +\infty$,

$$\sup_{\Omega_\delta} |g - h| < \eta \quad \textit{and} \quad \left|\gamma_g - \frac{\pi}{2}\right| \le \alpha + \epsilon, \tag{4}$$

where $\Omega_\delta = \overline{\Omega} \setminus B_\delta(\mathcal{O})$ *and* $\gamma_g = \cos^{-1}(Tg \cdot \nu) : \partial\Omega \setminus \{\mathcal{O}\} \to [0, \pi]$ *is the contact angle which the graph of* g *makes with* $\partial\Omega \times \mathbb{R}$.

Proof. Let $\epsilon, \eta, \delta, \Omega, h$ and γ_0 be given. For $\beta \in (0,\delta)$, let $g_\beta \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$ satisfy (1) and (2) with $\gamma = \gamma_\beta$, where

$$\gamma_\beta = \begin{cases} \frac{\pi}{2} - \alpha - \epsilon & \text{on } \overline{B_\beta(\mathcal{O})} \\ \gamma_0 & \text{on } \overline{\Omega} \setminus B_\beta(\mathcal{O}). \end{cases}$$

As in the proof of [Lancaster and Siegel 1996, Theorem 3], g_β converges to h pointwise and uniformly in the C^1 norm on $\overline{\Omega}_\delta$ as β tends to zero. Fix $\beta > 0$ small enough that $\sup_{\Omega_\delta} |g - h| < \eta$.

Set $\Sigma = \{(r\cos(\theta), r\sin(\theta)) : r > 0, -\alpha \le \theta \le \alpha\}$. Now define $w : \Sigma \to \mathbb{R}$ by

$$w(r\cos\theta, r\sin\theta) = \frac{\cos\theta - \sqrt{k^2 - \sin^2\theta}}{k\kappa r},$$

where $k = \sin\alpha \sec\left(\frac{\pi}{2} - \alpha - \epsilon\right) = \sin\alpha \csc(\alpha + \epsilon)$. As in [Concus and Finn 1970], there exists a $\delta_1 > 0$ such that $\operatorname{div}(Tw) - \frac{1}{2}w \ge 0$ on $\Sigma \cap B_{\delta_1}(\mathcal{O})$, $Tw \cdot \nu = \cos(\frac{\pi}{2} - \alpha - \epsilon)$ on $\partial\Sigma \cap B_{\delta_1}(\mathcal{O})$, and $\lim_{r \to 0^+} w(r\cos\theta, r\sin\theta) = \infty$ for each $\theta \in [-\alpha, \alpha]$. We may assume $\delta_1 \le \delta^*$. Let

$$M = \sup_{\Omega \cap \partial B_{\delta_1}(\mathcal{O})} |w - g_\beta| \quad \text{and} \quad w_\beta = w - M.$$

Since $\operatorname{div}(Tw_\beta) - \frac{1}{2}w_\beta \ge \frac{M}{2} \ge 0 = \operatorname{div}(Tg_\beta) - \frac{1}{2}g_\beta$ in $\Omega \cap B_{\delta_1}(\mathcal{O})$, $w_\beta \le g_\beta$ on $\Omega \cap \partial B_{\delta_1}(\mathcal{O})$ and $Tg_\beta \cdot \nu \ge Tw_\beta \cdot \nu$ on $\partial\Omega \cap B_{\delta_1}(\mathcal{O})$, we see that $g_\beta \ge w_\beta$ on $\Omega \cap \partial B_{\delta_1}(\mathcal{O})$. ☐

We may now prove Theorem 1.

Proof. We shall construct a sequence f_n of solutions of (1) and a sequence $\{r_n\}$ of positive real numbers such that $\lim_{n\to\infty} r_n = 0$, $f_n(x, y)$ is even in y and

$$(-1)^j f_n(r_j, 0) > 1 \quad \text{for each } j = 1, \dots, n.$$

Let $\gamma_0 = \frac{\pi}{2}$ and $f_0 = 0$. Set $\eta_1 = 1$ and $\delta_1 = \delta_0$. From Lemma 1, there exists $f_1 \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$ which satisfies (1) such that $\sup_{\Omega_{\delta_1}} |f_1 - f_0| < \eta_1$, $\left|\gamma_1 - \frac{\pi}{2}\right| \le \alpha + \epsilon$ and $\lim_{\Omega \ni (x,y) \to \mathcal{O}} f_1(x, y) = -\infty$, where $\gamma_1 = \cos^{-1}(Tf_1 \cdot \nu)$. Then there exists $r_1 \in (0, \delta_1)$ such that $f_1(r_1, 0) < -1$.

Now set $\eta_2 = -(f_1(r_1, 0) + 1) > 0$ and $\delta_2 = r_1$. From Lemma 1, there exists $f_2 \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$ which satisfies (1) such that $\sup_{\Omega_{\delta_2}} |f_2 - f_1| < \eta_2$, $\left|\gamma_2 - \frac{\pi}{2}\right| \le \alpha + \epsilon$ and $\lim_{\Omega \ni (x,y) \to \mathcal{O}} f_2(x, y) = \infty$, where $\gamma_2 = \cos^{-1}(Tf_2 \cdot \nu)$. Then there exists $r_2 \in (0, \delta_2)$ such that $f_2(r_2, 0) > 1$. Since $(r_1, 0) \in \Omega_{\delta_2}$,

$$f_1(r_1, 0) + 1 < f_2(r_1, 0) - f_1(r_1, 0) < -(f_1(r_1, 0) + 1)$$

and so $f_2(r_1, 0) < -1$.

Next set $\eta_3 = \min\{-(f_2(r_1,0)+1), f_2(r_2,0)-1\} > 0$ and $\delta_3 = r_2$. From Lemma 1, there exists $f_3 \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$ which satisfies (1) such that $\sup_{\Omega_{\delta_3}} |f_3 - f_2| < \eta_3$, $\left|\gamma_3 - \frac{\pi}{2}\right| \le \alpha + \epsilon$ and $\lim_{\Omega \ni (x,y) \to \mathcal{O}} f_3(x,y) = -\infty$, where $\gamma_3 = \cos^{-1}(Tf_3 \cdot \nu)$. Then there exists $r_3 \in (0, \delta_3)$ such that $f_3(r_3, 0) < -1$. Since $(r_1, 0), (r_2, 0) \in \Omega_{\delta_2}$, we have

$$f_2(r_1,0)+1 < f_3(r_1,0) - f_2(r_1,0) < -(f_2(r_1,0)+1)$$

and

$$-(f_2(r_2,0)-1) < f_3(r_2,0) - f_2(r_2,0) < f_2(r_2,0)-1;$$

hence $f_3(r_1,0) < -1$ and $1 < f_3(r_2,0)$.

Continuing to define f_n and r_n inductively, we set

$$\eta_{n+1} = \min_{1 \le j \le n} |f_n(r_j,0) - (-1)^j| \quad \text{and} \quad \delta_{n+1} = \min\left\{r_n, \tfrac{1}{n}\right\}.$$

From Lemma 1, there exists $f_{n+1} \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$ satisfying (1) such that $\sup_{\Omega_{\delta_{n+1}}} |f_{n+1} - f_n| < \eta_{n+1}$, $\left|\gamma_{n+1} - \frac{\pi}{2}\right| \le \alpha + \epsilon$ and $\lim_{\Omega \ni (x,y) \to \mathcal{O}} f_{n+1}(x,y) = (-1)^{n+1}\infty$, where $\gamma_{n+1} = \cos^{-1}(Tf_{n+1} \cdot \nu)$. Then there exists $r_{n+1} \in (0, \delta_{n+1})$ such that $(-1)^{n+1} f_{n+1}(r_{n+1}, 0) > 1$. For each $j \in \{1, \ldots, n\}$ which is an even number, we have

$$-(f_n(r_j,0)-1) < f_{n+1}(r_j,0) - f_n(r_j,0) < f_n(r_j,0)-1$$

and so $1 < f_{n+1}(r_j,0)$. For each $j \in \{1, \ldots, n\}$ which is an odd number, we have

$$f_n(r_j,0)+1 < f_{n+1}(r_j,0) - f_n(r_j,0) < -(f_n(r_j,0)+1)$$

and so $f_{n+1}(r_j,0) < -1$.

As in [Lancaster and Siegel 1996; Siegel 1980], there is a subsequence of $\{f_n\}$, still denoted $\{f_n\}$, which converges pointwise and uniformly in the C^1 norm on $\overline{\Omega}_\delta$ for each $\delta > 0$ as $n \to \infty$ to a solution $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \mathcal{O})$ of (1). For each $j \in \mathbb{N}$ which is even, $f_n(r_j,0) > 1$ for each $n \in \mathbb{N}$ and so $f(r_j,0) \ge 1$. For each $j \in \mathbb{N}$ which is odd, $f_n(r_j,0) < -1$ for each $n \in \mathbb{N}$ and so $f(r_j,0) \le -1$. Therefore

$$\lim_{r \to 0^+} f(r,0) \text{ does not exist, even as an infinite limit,}$$

and so $Rf(0)$ does not exist.

Since Ω is symmetric with respect to the x-axis and $\gamma_n(x,y)$ is an even function of y, $f(x,y)$ is an even function of y. Now suppose that there exists $\theta_0 \in [-\alpha, \alpha]$ such that $Rf(\theta_0)$ exists; then $\theta_0 \ne 0$. From the symmetry of f, $Rf(-\theta_0)$ must also exist and $Rf(-\theta_0) = Rf(\theta_0)$. Set

$$\Omega' = \{(r\cos\theta, r\sin\theta) : 0 < r < \delta_0, -\theta_0 < \theta < \theta_0\} \subset \Omega.$$

Since f has continuous boundary values on $\partial\Omega'$, $f \in C^0(\overline{\Omega}')$ and so $Rf(0)$ does exist, which is a contradiction. Thus $Rf(\theta)$ does not exist for any $\theta \in [-\alpha, \alpha]$. $\square$

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