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**A CAPILLARY SURFACE WITH NO RADIAL LIMITS**

COLM PATRIC MITCHELL

## A CAPILLARY SURFACE WITH NO RADIAL LIMITS

COLM PATRIC MITCHELL

**In 1996, Kirk Lancaster and David Siegel investigated the existence and behavior of radial limits at a corner of the boundary of the domain of solutions of capillary and other prescribed mean curvature problems with contact angle boundary data. They provided an example of a capillary surface in a unit disk  $D$  which has no radial limits at  $(0, 0) \in \partial D$ . In their example, the contact angle,  $\gamma$ , cannot be bounded away from zero and  $\pi$ . Here we consider a domain  $\Omega$  with a convex corner at  $(0, 0)$  and find a capillary surface  $z = f(x, y)$  in  $\Omega \times \mathbb{R}$  which has no radial limits at  $(0, 0) \in \partial \Omega$  such that  $\gamma$  is bounded away from 0 and  $\pi$ .**

Let  $\Omega$  be a domain in  $\mathbb{R}^2$  with locally Lipschitz boundary and  $\mathcal{O} = (0, 0) \in \partial \Omega$  such that  $\partial \Omega \setminus \{\mathcal{O}\}$  is a  $C^4$  curve and  $\Omega \subset B_1(0, 1)$ , where  $B_\delta(\mathcal{N})$  is the open ball in  $\mathbb{R}^2$  of radius  $\delta$  about  $\mathcal{N} \in \mathbb{R}^2$ . Denote the unit exterior normal to  $\Omega$  at  $(x, y) \in \partial \Omega$  by  $\nu(x, y)$  and let polar coordinates relative to  $\mathcal{O}$  be denoted by  $r$  and  $\theta$ . We shall assume there exist  $\delta^* \in (0, 2)$  and  $\alpha \in (0, \frac{\pi}{2})$  such that  $\partial \Omega \cap B_{\delta^*}(\mathcal{O})$  consists of the line segments

$$\partial^+ \Omega^* = \{(r \cos(\alpha), r \sin(\alpha)) : 0 \leq r \leq \delta^*\}$$

and

$$\partial^- \Omega^* = \{(r \cos(-\alpha), r \sin(-\alpha)) : 0 \leq r \leq \delta^*\}.$$

Set  $\Omega^* = \Omega \cap B_{\delta^*}(\mathcal{O})$ . Let  $\gamma : \partial \Omega \rightarrow [0, \pi]$  be given. Let  $(x^\pm(s), y^\pm(s))$  be arclength parametrizations of  $\partial^\pm \Omega$  with  $(x^+(0), y^+(0)) = (x^-(0), y^-(0)) = (0, 0)$  and set  $\gamma^\pm(s) = \gamma(x^\pm(s), y^\pm(s))$ .

Consider the capillary problem of finding a function  $f \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \{\mathcal{O}\})$  satisfying

$$(1) \quad \operatorname{div}(Tf) = \frac{1}{2} f \quad \text{in } \Omega$$

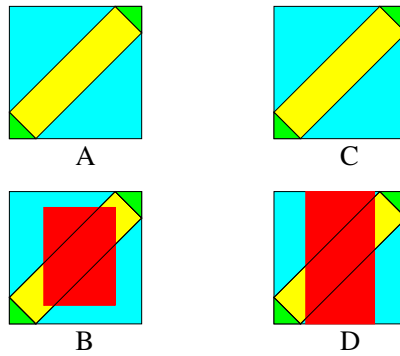
$$(2) \quad Tf \cdot \nu = \cos(\gamma) \quad \text{on } \partial \Omega \setminus \{\mathcal{O}\},$$

where  $Tf = \nabla f / \sqrt{1 + |\nabla f|^2}$ . We are interested in the existence of the radial limits  $Rf(\cdot)$  of a solution  $f$  of (1) and (2), where

$$Rf(\theta) = \lim_{r \rightarrow 0^+} f(r \cos \theta, r \sin \theta), \quad -\alpha < \theta < \alpha$$

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**Figure 1.** The Concus–Finn rectangle (A and C) with regions  $R$  (yellow),  $D_2^\pm$  (blue) and  $D_1^\pm$  (green); the restrictions on  $\gamma$  in [Lancaster and Siegel 1996] (red region in B) and in [Crenshaw et al. 2017] (red region in D).

and  $Rf(\pm\alpha) = \lim_{\partial^\pm\Omega^* \ni \mathbf{x} \rightarrow \mathcal{O}} f(\mathbf{x})$ ,  $\mathbf{x} = (x, y)$ , which are the limits of the boundary values of  $f$  on the two sides of the corner if these exist.

**Proposition 1** [Crenshaw et al. 2017]. *Let  $f$  be a bounded solution to (1) satisfying (2) on  $\partial^\pm\Omega^* \setminus \{\mathcal{O}\}$  which is discontinuous at  $\mathcal{O}$ . If  $\alpha > \pi/2$  then  $Rf(\theta)$  exists for all  $\theta \in (-\alpha, \alpha)$ . If  $\alpha \leq \pi/2$  and there exist constants  $\underline{\gamma}^\pm, \bar{\gamma}^\pm$ ,  $0 \leq \underline{\gamma}^\pm \leq \bar{\gamma}^\pm \leq \pi$ , satisfying*

$$\pi - 2\alpha < \underline{\gamma}^+ + \underline{\gamma}^- \leq \bar{\gamma}^+ + \bar{\gamma}^- < \pi + 2\alpha$$

so that  $\underline{\gamma}^\pm \leq \gamma^\pm(s) \leq \bar{\gamma}^\pm$  for all  $s$ ,  $0 < s < s_0$ , for some  $s_0$ , then again  $Rf(\theta)$  exists for all  $\theta \in (-\alpha, \alpha)$ .

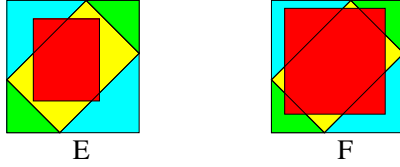
Lancaster and Siegel [1996] proved this theorem with the additional restriction that  $\gamma$  be bounded away from 0 and  $\pi$ ; Figure 1 illustrates these cases.

They also proved the following:

**Proposition 2** [Lancaster and Siegel 1996, Theorem 3]. *Let  $\Omega$  be the disk of radius 1 centered at  $(1, 0)$ . Then there exists a solution to  $Nf = \frac{1}{2}f$  in  $\Omega$ ,  $|f| \leq 2$ ,  $f \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \mathcal{O})$ ,  $\mathcal{O} = (0, 0)$  such that no radial limits  $Rf(\theta)$  exist ( $\theta \in [-\pi/2, \pi/2]$ ).*

In this case,  $\alpha = \frac{\pi}{2}$ ; if  $\gamma$  is bounded away from 0 and  $\pi$ , then Proposition 1 would imply that  $Rf(\theta)$  exists for each  $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  and therefore the contact angle  $\gamma = \cos^{-1}(Tf \cdot \nu)$  in Proposition 2 is not bounded away from 0 and  $\pi$ .

In our case, the domain  $\Omega$  has a convex corner of size  $2\alpha$  at  $\mathcal{O}$  and we wish to investigate the question of whether an example like that in Proposition 2 exists in this case when  $\gamma$  is bounded away from 0 and  $\pi$ . In terms of the Concus–Finn rectangle, the question is whether, given  $\epsilon > 0$ , there is a solution  $f \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \{\mathcal{O}\})$  of



**Figure 2.** The Concus–Finn rectangle. When  $\gamma$  remains in the red region in E,  $Rf(\cdot)$  exists;  $\gamma$  in [Theorem 1](#) remains in the red region in F.

(1) and (2) such that no radial limits  $Rf(\theta)$  exist ( $\theta \in [-\alpha, \alpha]$ ) and  $|\gamma - \frac{\pi}{2}| \leq \alpha + \epsilon$ ; this is illustrated in [Figure 2](#).

**Theorem 1.** For each  $\epsilon > 0$ , there is a domain  $\Omega$  as described above and a solution  $f \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \{\mathcal{O}\})$  of (1) such that the contact angle

$$\gamma = \cos^{-1}(Tf \cdot \nu) : \partial\Omega \setminus \{\mathcal{O}\} \rightarrow [0, \pi]$$

satisfies  $|\gamma - \frac{\pi}{2}| \leq \alpha + \epsilon$  and there exists a sequence  $\{r_j\}$  in  $(0, 1)$  with  $\lim_{j \rightarrow \infty} r_j = 0$  such that

$$(-1)^j f(r_j, 0) > 1 \quad \text{for each } j \in \mathbb{N}.$$

Assuming  $\Omega$  and  $\gamma$  are symmetric with respect to the line  $\{(x, 0) : x \in \mathbb{R}\}$ , this implies that no radial limit

$$(3) \quad Rf(\theta) \stackrel{\text{def}}{=} \lim_{r \downarrow 0} f(r \cos(\theta), r \sin(\theta))$$

exists for any  $\theta \in [-\alpha, \alpha]$ .

We remark that our theorem is an extension of [[Lancaster and Siegel 1996](#), Theorem 3] to contact angle data in a domain with a convex corner. As in [[Lancaster 1989](#); [Lancaster and Siegel 1996](#)], we first state and prove a localization lemma; this is analogous to [[Lancaster 1989](#), Lemma] and [[Lancaster and Siegel 1996](#), Lemma 2].

**Lemma 1.** Let  $\Omega \subseteq \mathbb{R}^2$  be as above,  $\epsilon > 0$ ,  $\eta > 0$  and  $\gamma_0 : \partial\Omega \setminus \{\mathcal{O}\} \rightarrow [0, \pi]$  such that  $|\gamma_0 - \frac{\pi}{2}| \leq \alpha + \epsilon$ . For each  $\delta \in (0, 1)$  and  $h \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \{\mathcal{O}\})$  which satisfies (1) and (2) with  $\gamma = \gamma_0$ , there exists a solution

$$g \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \{\mathcal{O}\})$$

of (1) such that  $\lim_{\bar{\Omega} \ni (x,y) \rightarrow (0,0)} g(x, y) = +\infty$ ,

$$(4) \quad \sup_{\Omega_\delta} |g - h| < \eta \quad \text{and} \quad \left| \gamma_g - \frac{\pi}{2} \right| \leq \alpha + \epsilon,$$

where  $\Omega_\delta = \bar{\Omega} \setminus B_\delta(\mathcal{O})$  and  $\gamma_g = \cos^{-1}(Tg \cdot \nu) : \partial\Omega \setminus \{\mathcal{O}\} \rightarrow [0, \pi]$  is the contact angle which the graph of  $g$  makes with  $\partial\Omega \times \mathbb{R}$ .

*Proof.* Let  $\epsilon, \eta, \delta, \Omega, h$  and  $\gamma_0$  be given. For  $\beta \in (0, \delta)$ , let  $g_\beta \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$  satisfy (1) and (2) with  $\gamma = \gamma_\beta$ , where

$$\gamma_\beta = \begin{cases} \frac{\pi}{2} - \alpha - \epsilon & \text{on } \overline{B_\beta(\mathcal{O})} \\ \gamma_0 & \text{on } \overline{\Omega} \setminus B_\beta(\mathcal{O}). \end{cases}$$

As in the proof of [Lancaster and Siegel 1996, Theorem 3],  $g_\beta$  converges to  $h$  pointwise and uniformly in the  $C^1$  norm on  $\overline{\Omega}_\delta$  as  $\beta$  tends to zero. Fix  $\beta > 0$  small enough that  $\sup_{\Omega_\delta} |g - h| < \eta$ .

Set  $\Sigma = \{(r \cos(\theta), r \sin(\theta)) : r > 0, -\alpha \leq \theta \leq \alpha\}$ . Now define  $w : \Sigma \rightarrow \mathbb{R}$  by

$$w(r \cos \theta, r \sin \theta) = \frac{\cos \theta - \sqrt{k^2 - \sin^2 \theta}}{k \kappa r},$$

where  $k = \sin \alpha \sec(\frac{\pi}{2} - \alpha - \epsilon) = \sin \alpha \csc(\alpha + \epsilon)$ . As in [Concus and Finn 1970], there exists a  $\delta_1 > 0$  such that  $\operatorname{div}(Tw) - \frac{1}{2}w \geq 0$  on  $\Sigma \cap B_{\delta_1}(\mathcal{O})$ ,  $Tw \cdot \nu = \cos(\frac{\pi}{2} - \alpha - \epsilon)$  on  $\partial\Sigma \cap B_{\delta_1}(\mathcal{O})$ , and  $\lim_{r \rightarrow 0^+} w(r \cos \theta, r \sin \theta) = \infty$  for each  $\theta \in [-\alpha, \alpha]$ . We may assume  $\delta_1 \leq \delta^*$ . Let

$$M = \sup_{\Omega \cap \partial B_{\delta_1}(\mathcal{O})} |w - g_\beta| \quad \text{and} \quad w_\beta = w - M.$$

Since  $\operatorname{div}(Tw_\beta) - \frac{1}{2}w_\beta \geq \frac{M}{2} \geq 0 = \operatorname{div}(Tg_\beta) - \frac{1}{2}g_\beta$  in  $\Omega \cap B_{\delta_1}(\mathcal{O})$ ,  $w_\beta \leq g_\beta$  on  $\Omega \cap \partial B_{\delta_1}(\mathcal{O})$  and  $Tg_\beta \cdot \nu \geq Tw_\beta \cdot \nu$  on  $\partial\Omega \cap B_{\delta_1}(\mathcal{O})$ , we see that  $g_\beta \geq w_\beta$  on  $\Omega \cap \partial B_{\delta_1}(\mathcal{O})$ .  $\square$

We may now prove [Theorem 1](#).

*Proof.* We shall construct a sequence  $f_n$  of solutions of (1) and a sequence  $\{r_n\}$  of positive real numbers such that  $\lim_{n \rightarrow \infty} r_n = 0$ ,  $f_n(x, y)$  is even in  $y$  and

$$(-1)^j f_n(r_j, 0) > 1 \quad \text{for each } j = 1, \dots, n.$$

Let  $\gamma_0 = \frac{\pi}{2}$  and  $f_0 = 0$ . Set  $\eta_1 = 1$  and  $\delta_1 = \delta_0$ . From [Lemma 1](#), there exists  $f_1 \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$  which satisfies (1) such that  $\sup_{\Omega_{\delta_1}} |f_1 - f_0| < \eta_1$ ,  $|\gamma_1 - \frac{\pi}{2}| \leq \alpha + \epsilon$  and  $\lim_{\Omega \ni (x, y) \rightarrow \mathcal{O}} f_1(x, y) = -\infty$ , where  $\gamma_1 = \cos^{-1}(Tf_1 \cdot \nu)$ . Then there exists  $r_1 \in (0, \delta_1)$  such that  $f_1(r_1, 0) < -1$ .

Now set  $\eta_2 = -(f_1(r_1, 0) + 1) > 0$  and  $\delta_2 = r_1$ . From [Lemma 1](#), there exists  $f_2 \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$  which satisfies (1) such that  $\sup_{\Omega_{\delta_2}} |f_2 - f_1| < \eta_2$ ,  $|\gamma_2 - \frac{\pi}{2}| \leq \alpha + \epsilon$  and  $\lim_{\Omega \ni (x, y) \rightarrow \mathcal{O}} f_2(x, y) = \infty$ , where  $\gamma_2 = \cos^{-1}(Tf_2 \cdot \nu)$ . Then there exists  $r_2 \in (0, \delta_2)$  such that  $f_2(r_2, 0) > 1$ . Since  $(r_1, 0) \in \Omega_{\delta_2}$ ,

$$f_1(r_1, 0) + 1 < f_2(r_1, 0) - f_1(r_1, 0) < -(f_1(r_1, 0) + 1)$$

and so  $f_2(r_1, 0) < -1$ .

Next set  $\eta_3 = \min\{-(f_2(r_1, 0) + 1), f_2(r_2, 0) - 1\} > 0$  and  $\delta_3 = r_2$ . From [Lemma 1](#), there exists  $f_3 \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$  which satisfies (1) such that  $\sup_{\Omega_{\delta_3}} |f_3 - f_2| < \eta_3$ ,  $|\gamma_3 - \frac{\pi}{2}| \leq \alpha + \epsilon$  and  $\lim_{\Omega_{\delta_3}(x, y) \rightarrow \mathcal{O}} f_3(x, y) = -\infty$ , where  $\gamma_3 = \cos^{-1}(Tf_3 \cdot \nu)$ . Then there exists  $r_3 \in (0, \delta_3)$  such that  $f_3(r_3, 0) < -1$ . Since  $(r_1, 0), (r_2, 0) \in \Omega_{\delta_2}$ , we have

$$f_2(r_1, 0) + 1 < f_3(r_1, 0) - f_2(r_1, 0) < -(f_2(r_1, 0) + 1)$$

and

$$-(f_2(r_2, 0) - 1) < f_3(r_2, 0) - f_2(r_2, 0) < f_2(r_2, 0) - 1;$$

hence  $f_3(r_1, 0) < -1$  and  $1 < f_3(r_2, 0)$ .

Continuing to define  $f_n$  and  $r_n$  inductively, we set

$$\eta_{n+1} = \min_{1 \leq j \leq n} |f_n(r_j, 0) - (-1)^j| \quad \text{and} \quad \delta_{n+1} = \min\{r_n, \frac{1}{n}\}.$$

From [Lemma 1](#), there exists  $f_{n+1} \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{\mathcal{O}\})$  satisfying (1) such that  $\sup_{\Omega_{\delta_{n+1}}} |f_{n+1} - f_n| < \eta_{n+1}$ ,  $|\gamma_{n+1} - \frac{\pi}{2}| \leq \alpha + \epsilon$  and  $\lim_{\Omega_{\delta_{n+1}}(x, y) \rightarrow \mathcal{O}} f_{n+1}(x, y) = (-1)^{n+1} \infty$ , where  $\gamma_{n+1} = \cos^{-1}(Tf_{n+1} \cdot \nu)$ . Then there exists  $r_{n+1} \in (0, \delta_{n+1})$  such that  $(-1)^{n+1} f_{n+1}(r_{n+1}, 0) > 1$ . For each  $j \in \{1, \dots, n\}$  which is an even number, we have

$$-(f_n(r_j, 0) - 1) < f_{n+1}(r_j, 0) - f_n(r_j, 0) < f_n(r_j, 0) - 1$$

and so  $1 < f_{n+1}(r_j, 0)$ . For each  $j \in \{1, \dots, n\}$  which is an odd number, we have

$$f_n(r_j, 0) + 1 < f_{n+1}(r_j, 0) - f_n(r_j, 0) < -(f_n(r_j, 0) + 1)$$

and so  $f_{n+1}(r_j, 0) < -1$ .

As in [[Lancaster and Siegel 1996](#); [Siegel 1980](#)], there is a subsequence of  $\{f_n\}$ , still denoted  $\{f_n\}$ , which converges pointwise and uniformly in the  $C^1$  norm on  $\overline{\Omega}_\delta$  for each  $\delta > 0$  as  $n \rightarrow \infty$  to a solution  $f \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \mathcal{O})$  of (1). For each  $j \in \mathbb{N}$  which is even,  $f_n(r_j, 0) > 1$  for each  $n \in \mathbb{N}$  and so  $f(r_j, 0) \geq 1$ . For each  $j \in \mathbb{N}$  which is odd,  $f_n(r_j, 0) < -1$  for each  $n \in \mathbb{N}$  and so  $f(r_j, 0) \leq -1$ . Therefore

$$\lim_{r \rightarrow 0^+} f(r, 0) \text{ does not exist, even as an infinite limit,}$$

and so  $Rf(0)$  does not exist.

Since  $\Omega$  is symmetric with respect to the  $x$ -axis and  $\gamma_n(x, y)$  is an even function of  $y$ ,  $f(x, y)$  is an even function of  $y$ . Now suppose that there exists  $\theta_0 \in [-\alpha, \alpha]$  such that  $Rf(\theta_0)$  exists; then  $\theta_0 \neq 0$ . From the symmetry of  $f$ ,  $Rf(-\theta_0)$  must also exist and  $Rf(-\theta_0) = Rf(\theta_0)$ . Set

$$\Omega' = \{(r \cos \theta, r \sin \theta) : 0 < r < \delta_0, -\theta_0 < \theta < \theta_0\} \subset \Omega.$$

Since  $f$  has continuous boundary values on  $\partial\Omega'$ ,  $f \in C^0(\overline{\Omega}')$  and so  $Rf(0)$  does exist, which is a contradiction. Thus  $Rf(\theta)$  does not exist for any  $\theta \in [-\alpha, \alpha]$ .  $\square$

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