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CODIMENSIONS OF THE SPACES OF CUSP FORMS FOR SIEGEL CONGRUENCE SUBGROUPS IN DEGREE TWO

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We give a computational algorithm for describing the one-dimensional cusps of the Satake compactifications for the Siegel congruence subgroups in the case of degree two for arbitrary levels. As an application of the results thus obtained, we calculate the codimensions of the spaces of cusp forms in the spaces of modular forms of degree two with respect to Siegel congruence subgroups of levels not divisible by 8. We also construct a linearly independent set of Klingen–Eisenstein series with respect to the Siegel congruence subgroup of an arbitrary level.

1. Introduction

One of the most basic questions about the spaces of modular forms is to ask for the dimensions and the codimensions of the spaces of cusp forms. For the spaces of Siegel modular forms of degree two with respect to the full modular group $\mathrm{Sp}(4, \mathbb{Z})$ the answers have been well known for several decades. However, while the answers for the spaces of modular forms with respect to Siegel congruence subgroups are not so clear, several special cases have been treated in the literature. Dimensions of the spaces of cusp forms with respect to $\Gamma_0(p)$ have been computed by Hashimoto [1983] for weights $k \geq 5$. For $\Gamma_0(2)$, Ibukiyama [1991] gave the structure of the ring of Siegel modular forms of degree 2. Poor and Yuen [2007] computed the dimensions of cusp forms for weights $k = 2, 3, 4$ with respect to $\Gamma_0(p)$ in the case of a small prime p. In [Poor and Yuen 2013] Poor and Yuen described the one-dimensional and zero-dimensional cusps of the Satake compactifications for the paramodular subgroups in the degree two case and calculated the codimensions of cusp forms. More recently Böcherer and Ibukiyama [2012] have given a formula for calculating the codimensions of the spaces of cusp forms in the spaces of modular forms of degree two with respect to Siegel congruence subgroups of

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square-free levels. In this paper we generalize their result and give a formula for the codimensions of the spaces of cusp forms in the spaces of modular forms of degree two with respect to Siegel congruence subgroups of level N with $8 \nmid N$. Another application of the results presented here will appear in a forthcoming work on Klingen–Eisenstein series. The method used to find the codimensions of the spaces of cusp forms makes use of a result from the theory of Satake compactification. The cusp structure of the Satake compactification encodes information about the codimensions of cusp forms and works of several authors indicate that it is an important object worth investigating.

2. Notation

We shall use the following notation throughout this paper unless otherwise stated. We realize the group GSp(4) as

$$\mathrm{GSp}(4) := \{g \in \mathrm{GL}(4) \mid {}^t g J g = \lambda(g) J \text{ for some } \lambda(g) \in \mathrm{GL}(1)\},$$

with $J = \left[\begin{smallmatrix} & J_1 \\ -J_1 & \end{smallmatrix}\right]$ and $J_1 = \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]$. We note that this version of GSp(4) is isomorphic to the classical version of GSp(4) and we denote this isomorphism by the map $\jmath$ which interchanges the first two rows and the first two columns of any matrix. Sp(4) is the subgroup of GSp(4) consisting of matrices with multiplier $\lambda = 1$. By $Q(\mathbb{Q})$ and $P(\mathbb{Q})$ we will denote the Klingen and Siegel parabolic subgroups of $\mathrm{GSp}(4, \mathbb{Q})$, respectively, consisting of the matrices of the form

$$Q(\mathbb{Q}) = \left\{ \left[\begin{smallmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{smallmatrix}\right] \,\middle|\, * \in \mathbb{Q} \right\} \quad \text{and} \quad P(\mathbb{Q}) = \left\{ \left[\begin{smallmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{smallmatrix}\right] \,\middle|\, * \in \mathbb{Q} \right\}.$$

We define

$$s_1 := \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \quad s_2 := \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix}.$$

We define the Siegel congruence subgroup as

$$\Gamma_0(N) = \Gamma_0^4(N) := \left\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ a & b & * & * \\ c & d & * & * \end{bmatrix} \in \mathrm{Sp}(4, \mathbb{Z}) \,\middle|\, a, b, c, d \equiv 0 \mod N \right\}.$$

We will denote by $\Gamma(N)$ the usual principal congruence subgroup of $\mathrm{Sp}(4, \mathbb{Z})$. Next we define $\Delta(\mathbb{Z}/N\mathbb{Z}) := \{g \mod N \mid g \in \Gamma_0(N)\}$, $\Gamma_\infty(\mathbb{Z}) := Q(\mathbb{Q}) \cap \mathrm{Sp}(4, \mathbb{Z})$.

We will use $\Gamma_0^2(N) := \left\{\left[\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{Z}) \mid c \equiv 0 \bmod N\right\}$ to denote the Hecke congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$ and $\Gamma_\infty^2(\mathbb{Z}) := \left\{\pm\left[\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right] \mid b \in \mathbb{Z}\right\}$. For $Z \in \mathbb{H}_2 := \{z \in M_2(\mathbb{C}) \mid {}^t z = z, \operatorname{Im} z > 0\}$, and for any $m = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{Sp}(4, \mathbb{Z})$ we define $m\langle Z\rangle :=$

$(AZ+B)(CZ+D)^{-1}$, $j(m,Z):=CZ+D$ and $m\langle Z\rangle^*=\tilde{\tau}$ for $m\langle Z\rangle=\left[\begin{smallmatrix}\tilde{\tau} & \tilde{z}\\ \tilde{z} & \tilde{\tau}'\end{smallmatrix}\right]$. We will denote by $C_0(N)$ and $C_1(N)$ the number of zero and one-dimensional cusps for $\Gamma_0(N)$ respectively, i.e.,

$$C_0(N)=\#(\Gamma_0(N)\backslash\operatorname{GSp}(4,\mathbb{Q})/P(\mathbb{Q})),\quad C_1(N)=\#(\Gamma_0(N)\backslash\operatorname{GSp}(4,\mathbb{Q})/Q(\mathbb{Q})).$$

Let

$$\omega_1(q)=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\quad\text{for } q=\begin{bmatrix} * & * & * & *\\ & a & b & *\\ & c & d & *\\ & & & *\end{bmatrix}\in Q(\mathbb{Q}),$$

and let ι_1 be an embedding map

$$\iota_1\left(\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right)=\begin{bmatrix} 1 & * & * & *\\ & a & b & *\\ & c & d & *\\ & & & 1\end{bmatrix}$$

from $\operatorname{SL}(2,\mathbb{Q})$ to $Q(\mathbb{Q})$. For $g\in\operatorname{GSp}(4,\mathbb{Q})$, we define $\Gamma_g:=\omega_1(g^{-1}\Gamma_0(\mathrm{N})g\cap Q(\mathbb{Q}))$.

3. A brief overview of the main results

We recall cusps in the degree one case. Let Γ be a congruence subgroup of $\operatorname{SL}(2,\mathbb{Z})$ which acts on the complex upper half plane $\mathbb{H}$ by the usual action. In order to compactify $\Gamma\backslash\mathbb{H}$ we adjoin $\mathbb{Q}\cup\{\infty\}$ to $\mathbb{H}$ to define the extended plane $\mathbb{H}^*=\mathbb{H}\cup\mathbb{Q}\cup\{\infty\}$ and take the quotient $X(\Gamma)=\Gamma\backslash\mathbb{H}^*$. Then a cusp of $X(\Gamma)$ is a Γ-equivalence class of points in $\mathbb{Q}\cup\{\infty\}$. As $\operatorname{SL}(2,\mathbb{Z})$ acts transitively on $\mathbb{Q}\cup\{\infty\}$ there is just one cusp of the modular curve $X(1)=\operatorname{SL}(2,\mathbb{Z})\backslash\mathbb{H}^*$. It is well known that cusps of $X(\Gamma_0^2(N))$ correspond to the double coset decompositions of $\Gamma_0^2(N)\backslash\operatorname{SL}(2,\mathbb{Z})/\Gamma_\infty^2(\mathbb{Z})$, for example see [Diamond and Shurman 2005, Proposition 3.8.5] or [Miyake 1989, §4.2].

The theory of Satake compactification is explained in [Satake 1957/58a]. A quick review can be found in [Poor and Yuen 2013, Section 3]. In fact similar to the degree one case, the one-dimensional cusps for the Siegel congruence subgroup $\Gamma_0(N)$, in the degree two case, correspond to the double coset decompositions $\Gamma_0(N)\backslash\operatorname{Sp}(4,\mathbb{Z})/\Gamma_\infty(\mathbb{Z})$ and also equivalently to $\Gamma_0(N)\backslash\operatorname{GSp}(4,\mathbb{Q})/Q(\mathbb{Q})$. Similarly the zero-dimensional cusps correspond to the double coset decompositions $\Gamma_0(N)\backslash\operatorname{GSp}(4,\mathbb{Q})/P(\mathbb{Q})$. It turns out that for even weights $k>4$, the codimension of cusp forms can be obtained by using the Satake's theorem; see [Satake 1957/58b] if the structure of zero-dimensional cusps and one-dimensional cusps are known.

We prove the following result concerning one-dimensional cusps in the case when $N=p^n$ for some prime p and $n\geq 1$. In fact, the one-dimensional cusps for $\Gamma_0(p^n)$ are inverses of the representatives listed below.

Theorem 3.1. *Assume $n \geq 1$. A complete and minimal system of representatives for the double cosets $Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/\Gamma_0(p^n)$ is given by*

$$1, \quad s_1 s_2, \quad g_1(p, \gamma, r) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \gamma p^r & & & 1 \end{bmatrix}$$

$$g_2(p, s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ & p^s & & 1 \end{bmatrix}, \quad g_3(p, \delta, r, s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ \delta p^r & p^s & & 1 \end{bmatrix}$$

where $1 \leq s, r \leq n-1$, $s < r < 2s$ and where γ, δ run through elements in $(\mathbb{Z}/p^{f_1}\mathbb{Z})^\times$ and $(\mathbb{Z}/p^{f_2}\mathbb{Z})^\times$, respectively, with $f_1 = \min(r, n-r)$ and $f_2 = \min(2s - r, n - r)$. The total number of representatives given above is

$$C_1(p^n) = \begin{cases} \frac{p^{n/2+1} + p^{n/2} - 2}{p-1} & \text{if } n \text{ is even}, \\ \frac{2(p^{n+1/2} - 1)}{p-1} & \text{if } n \text{ is odd}. \end{cases} \tag{3-1}$$

Some remarks.

(i) We note that alternatively one can get the following system of complete and minimal representatives for the double cosets $Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/\Gamma_0(p^n)$.

$$g_1(\gamma, x) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ x\gamma & & & 1 \end{bmatrix}, \quad 1 \leq \gamma \leq N, \ \gamma \mid N,$$

$$g_3(\gamma, \delta, y) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ \delta & & 1 & \\ y\gamma & \delta & & 1 \end{bmatrix}, \quad 1 < \delta < \gamma \leq N, \ \gamma \mid N, \ \delta \mid N, \ \delta \mid \gamma, \ \gamma \mid \delta^2;$$

where $N = p^n$ and for fixed γ and δ we set

$$M = \gcd\Big(\gamma, \frac{N}{\gamma}\Big), \quad L = \gcd\Big(\frac{\delta^2}{\gamma}, \frac{N}{\gamma}\Big),$$

and x and y vary through all the elements of $(\mathbb{Z}/M\mathbb{Z})^\times$ and $(\mathbb{Z}/L\mathbb{Z})^\times$, respectively. Here we interpret $(\mathbb{Z}/\mathbb{Z})^\times$ as an empty set. Clearly $g_1(N, x)$ is equivalent to the representative 1 in $Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/\Gamma_0(p^n)$ and one can show that $g_1(1, x)$ is equivalent to the representative $s_1 s_2$ (see Lemma 5.7). One can also easily show that $g_3(N, p^s, 1)$ is equivalent to the representative $g_2(p, s)$.

(ii) One can write yet another formulation for a complete and minimal system of representatives for the double cosets $Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/\Gamma_0(p^n)$ as follows:

$$g_0(\gamma, \delta, y) := \begin{bmatrix} 1 & & & \\ & 1 & & \\ \delta & & 1 & \\ y\gamma & \delta & & 1 \end{bmatrix}, \quad 1 \le \delta \le \gamma \le N,\ \gamma \mid N,\ \delta \mid N,\ \delta \mid \gamma,\ \gamma \mid \delta^2,$$

with y, γ, δ and N as in the first remark. This is clear on observing that the definition of $g_0(\gamma, \delta, y)$ is different from $g_3(\gamma, \delta, y)$ only when $\delta = \gamma$ and in that case the set of representatives $g_0(\gamma, \gamma, y)$ is equivalent to $g_1(\gamma, x)$.

The above result can be extended by using the strong approximation theorem and the Chinese remainder theorem to arbitrary N. We have the following lemma.

Lemma 3.2. *Assume $N = \prod_{i=1}^m p_i^{n_i}$. Then, the number of inequivalent representatives for the double cosets $Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/\Gamma_0(\mathrm{N})$ is given by $C_1(N) = \prod_{i=1}^m C_1(p_i^{n_i})$.*

We have the following corollary of Theorem 3.1 based on Lemma 3.2.

Corollary 3.3. *Assume $N = \prod_{i=1}^m p_i^{n_i}$. A complete and minimal system of representatives for the double cosets $Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/ \Gamma_0(N)$ is given by*

$$g_1(\gamma, x) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ x\gamma & & & 1 \end{bmatrix}, \quad 1 \le \gamma \le N,\ \gamma \mid N,$$

$$g_3(\gamma, \delta, y) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ \delta & & 1 & \\ y\gamma & \delta & & 1 \end{bmatrix}, \quad 1 < \delta < \gamma \le N,\ \gamma \mid N,\ \delta \mid N,\ \delta \mid \gamma,\ \gamma \mid \delta^2,$$

where for fixed γ and δ we have

$$x = M + \zeta \prod_{p_i \nmid M,\ p_i \mid N} p_i^{n_i}, \quad y = L + \theta \prod_{p_i \nmid L,\ p_i \mid N} p_i^{n_i},$$

with $M = \gcd(\gamma, N/\gamma)$, $L = \gcd(\delta^2/\gamma, N/\gamma)$, ζ and θ varies through all the elements of $(\mathbb{Z}/M\mathbb{Z})^\times$ and $(\mathbb{Z}/L\mathbb{Z})^\times$, respectively. Here we interpret $(\mathbb{Z}/\mathbb{Z})^\times$ as an empty set.

Essentially the representatives listed above are obtained by appropriately lifting the representatives of $Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/ \Gamma_0(p_i^{n_i})$ for each prime factor p_i of N. The last statement will be made explicit in the proof of the corollary. We also note that x and y are defined in such a way that $\gcd(x, N) = 1$ and $\gcd(y, N) = 1$.

We remark that the one-dimensional cusps for $\Gamma_0(N)$ are given by the inverses of the representatives listed above in Corollary 3.3.

Corollary 3.4. (1) *Let f_1 be an elliptic cusp form of even weight k with $k \geq 6$ and level N. Let $\jmath(g)$ be a one-dimensional cusp for $\Gamma_0(N)$ of the form $g_1(\gamma, x)^{-1}$. Then*

$$E_g(Z) = \sum_{\jmath(\xi) \in (\jmath(g) Q(\mathbb{Q}) \jmath(g^{-1}) \cap \Gamma_0(N)) \backslash \Gamma_0(N)} f_1(g^{-1}\xi\langle Z\rangle^*) \det(j(g^{-1}\xi, Z))^{-k}$$

defines a Klingen–Eisenstein series of level N with respect to the Siegel congruence subgroup $\Gamma_0(N)$.

(2) *Let $\jmath(h)$ be a one-dimensional cusp for $\Gamma_0(N)$ of the form $g_3(\gamma, \delta, y)^{-1}$. Let $f_2 \in S_k(\Gamma_{\jmath(h)})$ with even weight k such that $k \geq 6$. Then*

$$E_h(Z) = \sum_{\jmath(\xi) \in (\jmath(h) Q(\mathbb{Q}) \jmath(h^{-1}) \cap \Gamma_0(N)) \backslash \Gamma_0(N)} f_2(h^{-1}\xi\langle Z\rangle^*) \det(j(h^{-1}\xi, Z))^{-k}$$

defines a Klingen–Eisenstein series of level N with respect to the Siegel congruence subgroup $\Gamma_0(N)$.

As $\jmath(g)$ and $\jmath(h)$ run through all one-dimensional cusps of the form $g_1(\gamma, x)^{-1}$ and $g_3(\gamma, \delta, y)^{-1}$ respectively, and for some fixed g and h, as f_1 and f_2 run through a basis of $S_k(\Gamma_0(N))$ and $S_k(\Gamma_{\jmath(h)})$ respectively, the Klingen–Eisenstein series thus obtained are linearly independent.

The number of zero-dimensional cusps $C_0(p^n)$ for odd prime p was calculated by Markus Klein in his thesis [2004, Korollar 2.28]:

$$C_0(p^n) = 2n + 1 + 2\left(\sum_{j=1}^{n-1} \phi(p^{\min(j,n-j)}) + \sum_{j=1}^{n-2} \sum_{i=j+1}^{n-1} \phi(p^{\min(j,n-i)})\right). \tag{3-2}$$

It is the same as

$$C_0(p^n) = \begin{cases} 3 & \text{if } n = 1, \\ 2p+3 & \text{if } n = 2, \\ -2n - 1 + 2p^{n/2} + 8\frac{p^{n/2}-1}{p-1} & \text{if } n \geq 4 \text{ is even}, \\ -2n - 1 + 6p^{(n-1)/2} + 8\frac{p^{(n-1)/2}-1}{p-1} & \text{if } n \geq 3 \text{ is odd}. \end{cases} \tag{3-3}$$

The above formula remains valid if $p = 2$ and $n = 1$. The above result also remains true for $p = 2$ and $n = 2$ as calculated by Tsushima (cf. [Tsushima 2003]). Hence, assume $8 \nmid N$ and if $N = \prod_{i=1}^{m} p_i^{n_i}$ then following an argument similar to the one given in the proof of Lemma 3.2 we obtain

$$C_0(N) = \prod_{i=1}^{m} C_0(p_i^{n_i}). \tag{3-4}$$

Finally, by using Satake's [1957/58b] theorem and the formula for $C_0(N)$ and $C_1(N)$ described above we obtain the following result:

Theorem 3.5. *Let* $N \geq 1$, $8 \nmid N$ *and* $k \geq 6$, *even. Then*

$$\dim M_k(\Gamma_0(N)) - \dim S_k(\Gamma_0(N)) = C_0(N) + \Bigl(\sum_{\gamma \mid N} \phi(\gcd(\gamma, N/\gamma))\Bigr) \dim S_k(\Gamma_0^2(N)) + \sum_{\substack{1<\delta<\gamma,\ \gamma \mid N, \\ \delta \mid \gamma,\ \gamma \mid \delta^2}} \sum{}' \dim S_k(\Gamma_g) \tag{3-5}$$

where $C_0(N)$ *is given by* (3-4) *if* $N > 1$, $C_0(1) = 1$, ϕ *is Euler's totient function, and for a fixed* γ *and* δ, *the summation* $\sum'$ *is carried out such that* g *runs through every one-dimensional cusp of the form* $g_3(\gamma, \delta, y)$, *with* y *taking all possible values as in Corollary 3.3.*

Some remarks.

(i) We note that Markus Klein did not consider the case $4 \mid N$ for calculating the number of zero-dimensional cusps in his thesis. Tsushima provided the result for $N = 4$. Since we refer to their results for the number of zero-dimensional cusps we have this restriction in our theorem. We hope to return to this case in the future.

(ii) The above result in the special case of square-free N reduces to the dimension formula given in [Böcherer and Ibukiyama 2012] for even $k \geq 6$. They also treat the case $k = 4$ for square-free N.

In Section 4 we briefly review Satake compactification and cusps. Thereafter in Section 5 we give proofs of the main results. We remark that the proof of Theorem 3.1 is entirely algorithmic and essentially uses elementary number theory to establish the result.

4. Cusps of $\Gamma_0(N)$

We recall a few basic facts related to the Satake compactification $S(\Gamma\backslash\mathbb{H}_2)$ of $\Gamma\backslash\mathbb{H}_2$ (see [Satake 1957/58a; 1957/58b; Böcherer and Ibukiyama 2012; Poor and Yuen 2013]). Here Γ is a congruence subgroup of $\mathrm{Sp}(4, \mathbb{Z})$. We will be interested in $S(N) := S(\Gamma_0(N)\backslash\mathbb{H}_2)$. By $\mathrm{Bd}(N)$ we denote the boundary of $S(N)$. The one-dimensional components of $\mathrm{Bd}(N)$ are modular curves and are called the one-dimensional cusps. The one-dimensional cusps intersect on the zero-dimensional cusps.

We define $M_k(\mathrm{Bd}(N))$ to be the space of modular forms on $\mathrm{Bd}(N)$ which consists of modular forms of weight k on the one-dimensional boundary components such that they are compatible on each intersection point. In the following we make the above description more explicit. Let $\mathrm{GSp}(4,\mathbb{Q}) = \bigsqcup_{i=1}^{l} \Gamma_0(\mathrm{N}) g_i Q(\mathbb{Q})$.

Then the one-dimensional cusps bijectively correspond to $\{g_i\}$. Let $\Gamma_i = \omega_1(g_i^{-1}\Gamma_0(\mathrm{N})g_i \cap Q(\mathbb{Q}))$. In this situation the one-dimensional cusp g_i can be associated to the modular curve $\Gamma_i\backslash\mathbb{H}_1$. The zero-dimensional cusps of $\Gamma_i\backslash\mathbb{H}_1$ correspond to the representatives h_j of $\Gamma_i\backslash \mathrm{SL}(2,\mathbb{Z})/\Gamma_\infty^2(\mathbb{Z})$. In fact, h_j can be identified with the zero-dimensional cusp of $S(N)$ that corresponds to $\Gamma_0(N)g_i\iota_1(h_j)P(\mathbb{Q})$. If $\Gamma_0(N)g_i\iota_1(h_j)P(\mathbb{Q}) = \Gamma_0(N)g_r\iota_1(h_j)P(\mathbb{Q})$ for two inequivalent one-dimensional cusps g_i and g_r then it means that these two one-dimensional cusps intersect at a zero-dimensional cusp. Next we define a map Φ from $M_k(\Gamma_0(N))$ to $M_k(\Gamma_i)$ by $(\Phi(F))(z) = \lim_{\lambda\to\infty} F\left(\left[\begin{smallmatrix} z & \\ & i\lambda \end{smallmatrix}\right]\right)$ for $z \in \mathbb{H}_1$. Then we define $\tilde{\Phi} : M_k(\Gamma_0(N)) \to M_k(\mathrm{Bd}(N))$ by $F \to (\Phi(F|_k \jmath(g_i)))_{1\le i\le l}$. Here $|_k$ denotes the usual slash operator defined as $F|_k \jmath(g) = \det(CZ+D)^{-k} F(\jmath(g)\langle Z\rangle)$ for $\jmath(g) = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and $\jmath(g)\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$ with $g \in \mathrm{Sp}(4,\mathbb{R})$. Any element $(f_i)_{1\le i\le l}$ in the image of $\tilde{\Phi}$ satisfies the condition that $f_i|_k h_1 = f_j|_k h_2$ whenever $\Gamma_0(N)g_i\iota_1(h_1)P(\mathbb{Q}) = \Gamma_0(N)g_j\iota_1(h_2)P(\mathbb{Q})$; where $h_1, h_2 \in \mathrm{SL}(2,\mathbb{Q})$ and $1 \le i, j \le l$. It essentially means that f_i and f_j, which are modular forms on the one-dimensional cusps g_i and g_j respectively, are compatible on the intersection points of these cusps.

5. Proofs

Proof of Theorem 3.5.

Proof. By Satake's theorem (see [Satake 1957/58b]) it follows that the codimension of the space of cusp forms is $\dim M_k(\mathrm{Bd}(N))$. We recall that by definition $f \in M_k(\mathrm{Bd}(N))$ means: f is a modular form of weight k on the boundary components of $S(N)$ such that f takes the same value on each intersection point of the boundary components. If $f \in S_k(\Gamma_i)$ on a boundary component $\Gamma_i\backslash\mathbb{H}_1$ corresponding to a one-dimensional cusp, say g_i, then f vanishes at every cusp of g_i and in particular f takes the same value zero at every intersection point of the boundary components. Hence $f \in M_k(\mathrm{Bd}(N))$. We note that for any representatives g_1 of the form $g_1(\gamma, x)$ with $g_1(\gamma, x)$ defined as in Corollary 3.3, we have $\omega_1(g_1^{-1}\Gamma_0(\mathrm{N})g_1 \cap Q(\mathbb{Q})) = \Gamma_0^2(N)$ and similarly for any representatives g_3 of the form $g_3(\gamma,\delta,y)$ a simple calculation shows that $\Gamma_{g_3} = \omega_1(g_3^{-1}\Gamma_0(\mathrm{N})g_3 \cap Q(\mathbb{Q})) \subset \Gamma_0^2(\delta)$. It follows that each one-dimensional cusp of the form $g_1(\gamma, x)$ contributes $\dim S_k(\Gamma_0^2(N))$ linearly independent cusp forms and this accounts for the second term in the formula (3-5). For a fixed δ and γ such that $1 < \delta < \gamma$, $\gamma \mid N$, $\delta \mid \gamma$, $\gamma \mid \delta^2$ and for a fixed y, the one-dimensional cusp g of the form $g_3(\gamma,\delta,y)$ contributes $\dim S_k(\Gamma_g)$ cusp forms. These contributions account for the last term in the summation formula (3-5). We remark that the last two terms in the summation formula (3-5) count the Klingen–Eisenstein series associated to each one-dimensional cusp as defined in

Corollary 3.4. Finally, since $k > 4$ and even there exists a basis of Eisenstein series that is supported at a single cusp. The total number of zero-dimensional cusps $C_0(N)$ accounts for all such cases and these are in fact Siegel–Eisenstein series. □

In the following we determine a complete and minimal system of representatives for the double cosets $Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/\Gamma_0(p^n)$ and prove Theorem 3.1. For that we begin with first stating and proving several lemmas. In the following we write $g \sim h \iff Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})h\Gamma_0(p^n)$, for $g, h \in \mathrm{GSp}(4, \mathbb{Q})$.

Lemma 5.1. *Let p be a prime number and let x_1, x_2 be integers such that x_1, x_2 and p are pairwise coprime. Further, assume y_1, y_2 to be integers such that y_1, y_2 and p are pairwise coprime with* $\gcd(x_1, y_1) = 1$. *Let $x = x_1 x_2^{-1} p^{-r}$ and $y = y_1 y_2^{-1} p^{-s}$ with $r, s \geq 0,\ x_2 \neq 0,\ y_2 \neq 0$. Let $n \geq 1$. Let*

$$g = s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Then we have the following results.

(1) *If $s > r$, then there exist integers η_1 and η_2 which are coprime to p such that*

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & & 1 & \\ \eta_1 p^s & & & 1 \\ \eta_2 p^{-r+2s} & \eta_1 p^s & & & 1 \end{bmatrix} \Gamma_0(p^n).$$

(2) *If $s \leq r < n$, then there exists a nonzero integer x_3 coprime to p such that*

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ x_3 p^r & & & 1 \end{bmatrix} \Gamma_0(p^n),$$

and if $s \leq r$ and $r \geq n$, then $Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})1\Gamma_0(p^n)$.

Proof. We have

$$s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \sim \begin{bmatrix} -x & & -y & -1 \\ & & -1 & \\ & 1 & y^2/x & y/x \\ & & & -1/x \end{bmatrix} s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1$$

$$= \begin{bmatrix} 1 & & & \\ & & -1 & \\ -y/x & 1 & & \\ 1/x & & y/x & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & & & \\ & & 1 & \\ -y/x & & & 1 \\ 1/x & -y/x & & 1 \end{bmatrix} s_2.$$

Now we prove the first part of the lemma.

Case 1: $s > r$ Assume $s > r$. Let $\gcd(y_2, x_2) = \tau$. Let l_1, l_2 be integers such that

$$l_1 x_2 y_1 + l_2 p^{s-r} x_1 y_2 = \tau.$$

Let

$$d_1 = \frac{l_1^2 x_2 y_2 p^s}{\tau}, \quad d_2 = \frac{l_1 l_2 x_2 y_2 p^s}{\tau}.$$

It follows that

$$d_1 x_2 y_1 + d_2 x_1 y_2 p^{s-r} = l_1 p^s x_2 y_2.$$

Then we have

$$\begin{bmatrix} 1 & & & \\ & & -1 & \\ -\frac{p^r x_2 y_1}{p^s x_1 y_2} & 1 & & \\ \frac{p^r x_2}{x_1} & & \frac{p^r x_2 y_1}{p^s x_1 y_2} & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} \frac{\tau}{p^{-r+s} x_1 y_2} & & l_1 & \\ & & \frac{p^{-r+s} x_1 y_2}{\tau} & \\ & -\frac{\tau}{p^{-r+s} x_1 y_2} & -d_1 & -l_1 \\ & & & \frac{p^{-r+s} x_1 y_2}{\tau} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & -1 & \\ -\frac{p^r x_2 y_1}{p^s x_1 y_2} & 1 & & \\ \frac{p^r x_2}{x_1} & & \frac{p^r x_2 y_1}{p^s x_1 y_2} & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -\frac{l_1 p^r x_2 y_1}{p^s x_1 y_2} + \frac{\tau}{p^{-r+s} x_1 y_2} & -l_1 & & \\ -\frac{x_2 y_1}{\tau} & -\frac{p^{-r+s} x_1 y_2}{\tau} & & \\ -\frac{l_1 p^r x_2}{x_1} + \frac{d_1 p^r x_2 y_1}{p^s x_1 y_2} & d_1 & \frac{l_1 p^r x_2 y_1}{p^s x_1 y_2} - \frac{\tau}{p^{-r+s} x_1 y_2} & -l_1 \\ \frac{p^s x_2 y_2}{\tau} & & -\frac{x_2 y_1}{\tau} & \frac{p^{-r+s} x_1 y_2}{\tau} \end{bmatrix}$$

$$\sim \begin{bmatrix} l_2 & -l_1 & & \\ -\frac{x y_1}{\tau} & -\frac{p^{-r+s} x_1 y_2}{\tau} & & \\ -d_2 & d_1 & -l_2 & -l_1 \\ \frac{p^s x_2 y_2}{\tau} & & -\frac{x_2 y_1}{\tau} & \frac{p^{-r+s} x_1 y_2}{\tau} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ -\frac{l_1 p^s x_2 y_2}{\tau} & & 1 & \\ \frac{p^{-r+2s} x_1 x_2 y_2^2}{\tau^2} & -\frac{l_1 p^s x_2 y_2}{\tau} & & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_2 & -l_1 & & \\ -\frac{x_2 y_1}{\tau} & -\frac{p^{-r+s} x_1 y_2}{\tau} & & \\ & & -l_2 & -l_1 \\ -\frac{l_1 p^s x_2^2 y_1 y_2}{\tau^2} - \frac{l_2 p^{-r+2s} x_1 x_2 y_2^2}{\tau^2} + \frac{p^s x_2 y_2}{\tau} & & -\frac{x_2 y_1}{\tau} & \frac{p^{-r+s} x_1 y_2}{\tau} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ -\frac{l_1 p^s x_2 y_2}{\tau} & & 1 & \\ \frac{p^{-r+2s} x_1 x_2 y_2^2}{\tau^2} & -\frac{l_1 p^s x_2 y_2}{\tau} & & 1 \end{bmatrix} \begin{bmatrix} l_2 & -l_1 & & \\ -\frac{x_2 y_1}{\tau} & -\frac{p^{-r+s} x_1 y_2}{\tau} & & \\ & & -l_2 & -l_1 \\ & & -\frac{x_2 y_1}{\tau} & \frac{p^{-r+s} x_1 y_2}{\tau} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ -\frac{l_1 p^s x_2 y_2}{\tau} & & 1 & \\ \frac{p^{-r+2s} x_1 x_2 y_2^2}{\tau^2} & -\frac{l_1 p^s x_2 y_2}{\tau} & & 1 \end{bmatrix}.$$

This completes the proof of the first part of the lemma with $\eta_1 = -l_1 x_2 y_2/\tau$ and $\eta_2 = x_1 x_2 y_2^2/\tau^2$.

Now we prove the second part of the lemma.

Case 2: $s \le r$ Assume $s \le r$ and $\gcd(x_1, y_1) = 1$. Let $\gcd(y_2, x_2) = \tau$. Let l_1 and l_2 be integers such that

$$l_1 x_2 y_1 p^{r-s} + l_2 x_1 y_2 = \tau.$$

Let d_1 and d_2 be integers such that

$$d_1 y_1 + d_2 x_1 p^s = -l_1 p^s.$$

Let c_1 and c_2 be integers such that

$$c_1 \frac{x_1 y_2}{\tau} + c_2 p^{\max(0,n-s)} = \frac{-d_1 y_2}{p^s}.$$

Let

$$\beta = \frac{x_2}{\tau}(d_2 \tau - c_1 y_1).$$

It is easy to see that $\beta \in \mathbb{Z}$, as $\tau = \gcd(y_2, x_2)$. Further, if $r < n$ then we make the following choices. Let l_3 and l_4 be integers such that

$$l_3 \frac{x_2 y_2}{\tau} + l_4 p^{n-r} = \beta.$$

Let x_3 and x_4 be integers such that

$$x_3 \left(l_3 p^{r-s} \frac{x_2 y_1}{\tau} \right) - x_4 p^{n-r} = -\frac{x_2 y_2}{\tau}.$$

Otherwise, if $r \ge n$ then let

$$l_3 = l_4 = x_3 = x_4 = 0.$$

With the above choices in place, we obtain

$$
\begin{bmatrix} \frac{\tau}{x_1y_2} & & l_1 & \\ & & \frac{x_1y_2}{\tau} & \\ & -\frac{\tau}{x_1y_2} & d_1y_2 & -l_1 \\ & & & \frac{x_1y_2}{\tau} \end{bmatrix}
\begin{bmatrix} 1 & & & \\ & & -1 & \\ -\frac{p^rx_2y_1}{p^sx_1y_2} & 1 & & \\ \frac{p^rx_2}{x_1} & & \frac{p^rx_2y_1}{p^sx_1y_2} & 1 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} -\frac{l_1p^rx_2y_1}{p^sx_1y_2}+\frac{\tau}{x_1y_2} & -l_1 & & \\ -\frac{p^rx_2y_1}{p^s\tau} & -\frac{x_1y_2}{\tau} & & \\ -\frac{l_1p^rx_2}{x_1}-\frac{d_1p^rx_2y_1}{p^sx_1} & -d_1y_2 & \frac{l_1p^rx_2y_1}{p^sx_1y_2}-\frac{\tau}{x_1y_2} & -l_1 \\ \frac{p^rx_2y_2}{\tau} & & -\frac{p^rx_2y_1}{p^s\tau} & \frac{x_1y_2}{\tau} \end{bmatrix}
$$

$$
\sim \begin{bmatrix} l_2 & -l_1 & & \\ -\frac{p^rx_2y_1}{p^s\tau} & -\frac{x_1y_2}{\tau} & & \\ d_2p^rx_2 & -d_1y_2 & -l_2 & -l_1 \\ \frac{p^rx_2y_2}{\tau} & & -\frac{p^rx_2y_1}{p^s\tau} & \frac{x_1y_2}{\tau} \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & l_3 & & \\ & 1 & & \\ & c_1p^s & 1 & -l_3 \\ & & & 1 \end{bmatrix}
\begin{bmatrix} l_2 & -l_1 & & \\ -\frac{p^rx_2y_1}{p^s\tau} & -\frac{x_1y_2}{\tau} & & \\ d_2p^rx_2 & -d_1y_2 & -l_2 & -l_1 \\ \frac{p^rx_2y_2}{\tau} & & -\frac{p^rx_2y_1}{p^s\tau} & \frac{x_1y_2}{\tau} \end{bmatrix}
$$

$$
\sim \begin{bmatrix} -\frac{l_3p^rx_2y_1}{p^s\tau}+l_2 & -\frac{l_3x_1y_2}{\tau}-l_1 & & \\ -\frac{p^rx_2y_1}{p^s\tau} & -\frac{x_1y_2}{\tau} & & \\ d_2p^rx_2-\frac{c_1p^rx_2y_1}{\tau}-\frac{l_3p^rx_2y_2}{\tau} & -\frac{c_1p^sx_1y_2}{\tau}-d_1y_2 & \frac{l_3p^rx_2y_1}{p^s\tau}-l_2 & -\frac{l_3x_1y_2}{\tau}-l_1 \\ \frac{p^rx_2y_2}{\tau} & & -\frac{p^rx_2y_1}{p^s\tau} & \frac{x_1y_2}{\tau} \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^rx_3 & & & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} -\frac{l_3p^rx_2y_1}{p^s\tau}+l_2 & -\frac{l_3x_1y_2}{\tau}-l_1 & & \\ -\frac{p^rx_2y_1}{p^s\tau} & -\frac{x_1y_2}{\tau} & & \\ d_2p^rx_2-\frac{c_1p^rx_2y_1}{\tau}-\frac{l_3p^rx_2y_2}{\tau} & -\frac{c_1p^sx_1y_2}{\tau}-d_1y_2 & \frac{l_3p^rx_2y_1}{p^s\tau}-l_2 & -\frac{l_3x_1y_2}{\tau}-l_1 \\ \frac{l_3(p^r)^2x_2x_3y_1}{p^s\tau}-l_2p^rx_3+\frac{p^rx_2y_2}{\tau} & \frac{l_3p^rx_1x_3y_2}{\tau}+l_1p^rx_3 & -\frac{p^rx_2y_1}{p^s\tau} & \frac{x_1y_2}{\tau} \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^rx_3 & & & 1 \end{bmatrix},
$$

and the second part of the lemma follows. □

Lemma 5.2. *Assume p to be a prime number and n be a positive integer. Let*

$$x = \frac{x_1}{p^r x_2}, \quad y = \frac{y_1}{p^s y_2} \quad \textit{and} \quad z = \frac{z_1}{p^t z_2},$$

where r, s, t are nonnegative integers, x_1, x_2, y_2, z_2 are nonzero integers and y_1, z_1 are integers. Let any two nonzero elements from the set $\{x_1, y_1, z_1, x_2, y_2, z_2, p\}$ be mutually coprime except, possibly, when both belong to $\{x_2, y_2, z_2\}$. Let

$$g = s_1 s_2 s_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Then there exist $x', y' \in \mathbb{Q}$ such that

$$Q(\mathbb{Q}) g \Gamma_0(p^n) = Q(\mathbb{Q}) s_1 s_2 \begin{bmatrix} 1 & & y' & \\ & 1 & x' & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \Gamma_0(p^n).$$

Proof. We have

$$\begin{aligned}
g &\sim s_1 s_2 s_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 \\
&= \begin{bmatrix} 1 & & & \\ & 1 & & \\ -x & & 1 & \\ & -x & & 1 \end{bmatrix} s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & z & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} x & & 1 & \\ & x & & 1 \\ & & \frac{1}{x} & \\ & & & \frac{1}{x} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ -x & & 1 & \\ & -x & & 1 \end{bmatrix} s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & z & y \\ & & 1 & \\ & & & 1 \end{bmatrix} (s_1)^{-1}
\end{aligned}$$

(5-1)
$$= s_2 s_1 s_2 s_1 s_2 \begin{bmatrix} 1 & x^{-1} & & \\ & 1 & & \\ & & 1 & -x^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & \\ & 1 & z & y \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Case 1: $y \neq 0, z \neq 0$ Let us first consider the case when $y \neq 0$ and $z \neq 0$. Let $\alpha = \gcd(x_2 y_2, z_2)$. Further, if $s > t - r$, then let

$$(5\text{-}2)\qquad \begin{cases} -d_1 p^{n+t} + d_2 x_1 & = 1 \quad \text{with } d_1, d_2 \in \mathbb{Z}, \\ d_3 & = d_1 x_2 y_2 z_2 p^{n+s+r+t}, \\ c_1 & = (x_2 y_2 z_1 p^{r+s-t} + x_1 y_1 z_2)\alpha^{-1}, \\ \tau & = \gcd(d_3, c_1), \\ y_4 & = \alpha \tau x_1^{-1} z_2^{-1} p^{-s}, \\ -d_4 \tau p^n + d_5 x_1 & = 1 \quad \text{with } d_4, d_5 \in \mathbb{Z}, \\ d & = d_4 x_2 y_2 z_2 \alpha^{-1} p^{n+s+r}. \end{cases}$$

Otherwise, if $s \leq t - r$ then let

$$(5\text{-}3)\qquad \begin{cases} -d_1 p^{n+s+r} + d_2 x_1 & = 1 \quad \text{with } d_1, d_2 \in \mathbb{Z}, \\ d_3 & = d_1 x_2 y_2 z_2 p^{n+s+r+t}, \\ c_1 & = (x_2 y_2 z_1 + x_1 y_1 z_2 p^{-r-s+t})\alpha^{-1}, \\ \tau & = \gcd(d_3, c_1), \\ y_4 & = \alpha \tau x_1^{-1} z_2^{-1} p^{r-t}, \\ -d_4 \tau p^n + d_5 x_1 & = 1 \quad \text{with } d_4, d_5 \in \mathbb{Z}, \\ d & = d_4 x_2 y_2 z_2 \alpha^{-1} p^{n+t}. \end{cases}$$

Now, if $\tau > 1$ then we replace c_1 by $c_1 \tau^{-1}$. Then, if needed on appropriately adjusting d_4 and d_5, we pick integers a_1 and b such that

$$a_1 d - b c_1 = 1.$$

Next, we set

$$\begin{aligned} a &= a_1 + b z_1 z_2^{-1} p^{-t}, \\ c &= c_1 + d z_1 z_2^{-1} p^{-t}, \\ x_4 &= \frac{c p^r x_2 y_4}{x_1} + \frac{p^r x_2 y_1}{p^s x_1} + \frac{d y_1 y_4}{p^s y_2}. \end{aligned}$$

Then

$$g \sim s_2 s_1 s_2 s_1 s_2 \begin{bmatrix} 1 & x^{-1} & & \\ & 1 & & \\ & & 1 & -x^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & \\ & 1 & z & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \text{(from (5-1))}$$

$$\sim \begin{bmatrix} & & & 1 \\ & & 1 & -\frac{p^r x_2}{x_1} \\ & -1 & -\frac{z_1}{p^t z_2} & -\frac{y_1}{p^s y_2} \\ -1 & -\frac{p^r x_2}{x_1} & -\frac{p^r x_2 z_1}{p^t x_1 z_2} - \frac{y_1}{p^s y_2} & -\frac{p^r x_2 y_1}{p^s x_1 y_2} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & -\frac{p^r x_2}{x_1} \\ & -1 & -\frac{z_1}{p^t z_2} & -\frac{y_1}{p^s y_2} \\ -1 & -\frac{p^r x_2}{x_1} & -\frac{p^r x_2 z_1}{p^t x_1 z_2} - \frac{y_1}{p^s y_2} & -\frac{p^r x_2 y_1}{p^s x_1 y_2} \end{bmatrix}$$

$$\sim \begin{bmatrix} & & & 1 \\ & -b & a - \frac{b z_1}{p^t z_2} & -\frac{a p^r x_2}{x_1} - \frac{b y_1}{p^s y_2} \\ & -d & c - \frac{d z_1}{p^t z_2} & -\frac{c p^r x_2}{x_1} - \frac{d y_1}{p^s y_2} \\ -1 & -\frac{p^r x_2}{x_1} & -\frac{p^r x_2 z_1}{p^t x_1 z_2} - \frac{y_1}{p^s y_2} & -\frac{p^r x_2 y_1}{p^s x_1 y_2} \end{bmatrix}$$

$$\sim \begin{bmatrix} & & 1 & \\ 1 & & \frac{y_4}{y_2} & \\ & & & 1 \\ & -1 & -x_4 & -\frac{y_4}{y_2} \end{bmatrix}$$

$$\begin{bmatrix} & -b & a - \frac{b z_1}{p^t z_2} & -\frac{a p^r x_2}{x_1} - \frac{b y_1}{p^s y_2} - \frac{y_4}{y_2} \\ 1 & \frac{p^r x_2}{x_1} + \frac{d y_4}{y_2} & -\frac{c y_4}{y_2} + \frac{p^r x_2 z_1}{p^t x_1 z_2} + \frac{d y_4 z_1}{p^t y_2 z_2} + \frac{y_1}{p^s y_2} & \frac{c p^r x_2 y_4}{x_1 y_2} - x_4 + \frac{p^r x_2 y_1}{p^s x_1 y_2} + \frac{d y_1 y_4}{p^s y_2^2} \\ & & & 1 \\ & -d & c - \frac{d z_1}{p^t z_2} & -\frac{c p^r x_2}{x_1} - \frac{d y_1}{p^s y_2} \end{bmatrix}$$

$$\sim \begin{bmatrix} & & 1 & \\ 1 & & \frac{y_4}{y_2} & \\ & & & 1 \\ & -1 & -\frac{x_4}{y_2} & -\frac{y_4}{y_2} \end{bmatrix} \begin{bmatrix} & -b & a_1 & -a_1 d_5 x_2 p^r \\ 1 & d_5 x_2 p^r & & \\ & & & 1 \\ & -d & c_1 & -c_1 d_5 x_2 p^r \end{bmatrix} \sim s_1 s_2 \begin{bmatrix} 1 & & y' & \\ & 1 & x' & y' \\ & & 1 & \\ & & & 1 \end{bmatrix},$$

with

$$x' = \frac{x_4}{y_2} \quad \text{and} \quad y' = \frac{y_4}{y_2}.$$

This completes the proof in the case when both y and z are nonzero.

Case 2: $y = 0$, $z \neq 0$. If $y = 0$, $z \neq 0$, then we set $y_1 = 0$, $y_2 = 1$ and $s = 0$. It is easy to see that the previous proof remains valid for this case as well.

Case 3: $y \neq 0$, $z = 0$. If $z = 0$, $y \neq 0$ then we set $z_1 = 0$, $z_2 = 1$ and $t = 0$; and it is easy to see that the proof given in the first case remains valid for this case as well.

Case 4: $y = 0, z = 0$. Finally, we consider the case when both y and z are zero. In this case we make the following choices. Let

$$c = x_1, \quad d = -p^n$$

and select integers a and b such that $ad - bc = 1$. If $r \geq n$, then set

$$y_4 = \frac{x_2 p^{r-n}}{x_1}, \quad x_4 = x_2 y_4 p^r, \quad e = b x_2 p^{r-n}, \quad f = 0.$$

Otherwise if $r < n$ then set

$$y_4 = \frac{-a x_2 p^r}{x_1}, \quad x_4 = x_2 y_4 p^r, \quad e = 0, \quad f = -b x_2 p^r.$$

Now we have

$$\begin{aligned}
g &\sim s_2 s_1 s_2 s_1 s_2 \begin{bmatrix} 1 & x^{-1} & & \\ & 1 & & \\ & & 1 & -x^{-1} \\ & & & 1 \end{bmatrix} \qquad \text{(from (5-1))} \\
&\sim \begin{bmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & -\frac{p^r x_2}{x_1} \\ & -1 & & \\ -1 & -\frac{p^r x_2}{x_1} & & \end{bmatrix} \\
&= \begin{bmatrix} & & 1 & \\ 1 & & y_4 & \\ & & & 1 \\ & -1 & -x_4 & -y_4 \end{bmatrix} \begin{bmatrix} & -b & a & -\frac{a p^r x_2}{x_1} - y_4 \\ 1 & d y_4 + \frac{p^r x_2}{x_1} & -c y_4 & \frac{c p^r x_2 y_4}{x_1} - x_4 \\ & & & 1 \\ & -d & c & -\frac{c p^r x_2}{x_1} \end{bmatrix} \\
&= \begin{bmatrix} & & 1 & \\ 1 & & y_4 & \\ & & & 1 \\ & -1 & -x_4 & -y_4 \end{bmatrix} \begin{bmatrix} & -b & a & e \\ 1 & f & -p^{-n+r} x_2 & \\ & & & 1 \\ & p^n & x_1 & -p^r x_2 \end{bmatrix} \\
&\sim \begin{bmatrix} & & 1 & \\ 1 & & y_4 & \\ & & & 1 \\ & -1 & -x_4 & -y_4 \end{bmatrix} \sim s_1 s_2 \begin{bmatrix} 1 & & y' & \\ & 1 & x' & y' \\ & & 1 & \\ & & & 1 \end{bmatrix}
\end{aligned}$$

with $x' = x_4$ and $y' = y_4$. This completes the proof of the lemma. □

Lemma 5.3. *Assume n and s to be positive integers. Also let p be a prime number and η be an integer such that* $\gcd(\eta, p) = 1$. *Then*

$$Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ & p^s & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ \eta p^s & & 1 & \\ & \eta p^s & & 1 \end{bmatrix} \Gamma_0(p^n).$$

Proof.

If $s \geq n$, then of course both sides equal $Q(\mathbb{Q})1\Gamma_0(p^n)$. Therefore, in the following we assume that $s < n$. Next we consider the case when $n \geq 2s$ and make the following choices. Since $\gcd(\eta, p) = 1$, there exist integers α_1 and β_1 such that $\alpha_1\eta + \beta_1 p^{n-s} = 1$. Further, we also have $\gcd(\alpha_1, p^{n-s}) = 1$, so there exist β_2' and β_3' such that $\alpha_1\beta_2' + \beta_3' p^{n-s} = 1$. Let $\beta_2 = \beta_1\beta_2'$ and $\beta_3 = -\beta_1\beta_3'$. Now set

$$a = \frac{1 - \beta_1 p^{n-s}}{\eta} = \alpha_1, \quad b = p^{n-2s}\beta_3, \quad c = p^n, \quad d = \eta + \beta_2 p^{n-s}.$$

We also check that

$$\begin{aligned} ad - bc &= \alpha_1(\eta + \beta_2 p^{n-s}) - \beta_3 p^{2n-2s} \\ &= 1 - \beta_1 p^{n-s} + \alpha_1\beta_2 p^{n-s} - \beta_3 p^{2n-2s} \\ &= 1 - \beta_1 p^{n-s}(1 - \alpha_1\beta_2' - \beta_3' p^{n-s}) \\ &= 1. \end{aligned}$$

On the other hand if $2s > n$, then we make the following choices. Since $\gcd(\eta, p) = 1$, there exist integers α_1 and β_1 such that $\alpha_1\eta + \beta_1 p^{n-s} = 1$. Further, we also have $\gcd(\alpha_1, p^s) = 1$, so there exist β_2' and β_3' such that $\alpha_1\beta_2' + \beta_3' p^s = 1$. Let $\beta_2 = \beta_1\beta_2'$ and $\beta_3 = -\beta_1\beta_3'$. Now set

$$a = \frac{1 - \beta_1 p^{n-s}}{\eta} = \alpha_1, \quad b = \beta_3, \quad c = p^n \quad d = \eta + \beta_2 p^{n-s}.$$

Next we note

$$\begin{aligned} ad - bc = \alpha_1(\eta + \beta_2 p^{n-s}) - \beta_3 p^n &= 1 - \beta_1 p^{n-s} + \alpha_1\beta_2 p^{n-s} - \beta_3 p^n \\ &= 1 - \beta_1 p^{n-s} + \alpha_1\beta_1\beta_2' p^{n-s} + \beta_1\beta_3' p^n \\ &= 1 - \beta_1 p^{n-s}(1 - \alpha_1\beta_2' - \beta_3' p^{n-s}) \\ &= 1. \end{aligned}$$

Now the lemma follows from the following calculations:

$$\begin{aligned}
&Q(\mathbb{Q})\begin{bmatrix}1&&&\\&1&&\\p^s&&1&\\&p^s&&1\end{bmatrix}\Gamma_0(p^n)\\
&=Q(\mathbb{Q})\begin{bmatrix}1&&&\\&a&b&\\&c&d&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\p^s&&1&\\&p^s&&1\end{bmatrix}\Gamma_0(p^n)\\
&=Q(\mathbb{Q})\begin{bmatrix}1&&&\\&1&&\\\eta p^s&&1&\\&\eta p^s&&1\end{bmatrix}\begin{bmatrix}1&&&\\bp^s&a&b&\\dp^s-\eta p^s&c&d&\\-b\eta(p^s)^2&-a\eta p^s+p^s&-b\eta p^s&1\end{bmatrix}\Gamma_0(p^n)\\
&=Q(\mathbb{Q})\begin{bmatrix}1&&&\\&1&&\\\eta p^s&&1&\\&\eta p^s&&1\end{bmatrix}\Gamma_0(p^n).
\end{aligned}$$

□

Lemma 5.4. *Assume s, r and n to be positive integers with $0 < s \le n$. Also let p be a prime number and η_1, η_2 be integers such that $\gcd(\eta_1, p) = \gcd(\eta_2, p) = 1$. Then*

$$Q(\mathbb{Q})\begin{bmatrix}1&&&\\&1&&\\p^s&&1&\\\eta_2 p^r&p^s&&1\end{bmatrix}\Gamma_0(p^n)=Q(\mathbb{Q})\begin{bmatrix}1&&&\\&1&&\\\eta_1 p^s&&1&\\\eta_2 p^r&\eta_1 p^s&&1\end{bmatrix}\Gamma_0(p^n).$$

Proof.
Let

$$z_1=\begin{bmatrix}1&&&\\&1&&\\p^s&&1&\\\eta_2 p^r&p^s&&1\end{bmatrix}\quad\text{and}\quad z_2=\begin{bmatrix}1&&&\\&1&&\\\eta_1 p^s&&1&\\\eta_2 p^r&\eta_1 p^s&&1\end{bmatrix}.$$

Then we note that

$$z_2^{-1}\begin{bmatrix}1&&&\\&a&b&\\&c&d&\\&&&1\end{bmatrix}z_1=\begin{bmatrix}1&&&\\bp^s&a&b&\\dp^s-p^s\eta_1&c&d&\\-b(p^s)^2\eta_1&-ap^s\eta_1+p^s&-bp^s\eta_1&1\end{bmatrix}.$$

Now, the result follows by proceeding as in the proof of Lemma 5.3. □

Lemma 5.5. *Assume n to be a positive integer and s to be a nonnegative integer. Also let p be a prime number. Let $y_1, y_2 \in \mathbb{Z}$ such that p, y_1, y_2 are pairwise coprime. Then we have the following results.*

(1) *If $s < n$ then there exists an integer b_1 with* $\gcd(b_1, p) = 1$, *such that*

$$Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ \frac{y_1}{y_2} p^s & & 1 & \\ & \frac{y_1}{y_2} p^s & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ b_1 p^s & & 1 & \\ & b_1 p^s & & 1 \end{bmatrix} \Gamma_0(p^n).$$

(2) *If $s \geq n$ then*

$$Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ \frac{y_1}{y_2} p^s & & 1 & \\ & \frac{y_1}{y_2} p^s & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) 1 \Gamma_0(p^n).$$

Proof. Let α_1 and β_1 be integers such that

$$\alpha_1 p^s y_1 + \beta_1 y_2 = 1.$$

If $0 < s < n$ then set:

$$\alpha = \alpha_1, \quad \beta = \beta_1, \quad b_1, b_2 \in \mathbb{Z} \quad \text{such that} \quad b_1 \beta + b_2 p^{n-s} = y_1, \ b = b_1 p^s.$$

Otherwise, if $s \geq n$ then set:

$$\alpha = \alpha_1, \quad \beta = \beta_1, \quad b_2 = y_1 p^{s-n}, \quad b_1 = 0, \quad b = 0.$$

If $s = 0$ then set:

$$\alpha = \begin{cases} \alpha_1 - y_2 & \text{if } p \mid \beta_1, \\ \alpha_1 & \text{if } p \nmid \beta_1, \end{cases} \qquad \beta = \begin{cases} \beta_1 + y_1 & \text{if } p \mid \beta_1, \\ \beta_1 & \text{if } p \nmid \beta_1, \end{cases}$$

$b_1, b_2 \in \mathbb{Z}$ such that $b_1 \beta + b_2 p^n = y_1, \ b = b_1$. We note that, for each of the cases considered above, i.e., whenever $s \geq 0$, the following holds:

$$-b\beta + p^s y_1 = (-b_1 \beta + y_1) p^s = b_2 p^n.$$

Then,

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ \frac{y_1}{y_2}p^s & & 1 & \\ & \frac{y_1}{y_2}p^s & & 1 \end{bmatrix} \sim \begin{bmatrix} y_2^{-1} & & -\alpha & \\ & y_2^{-1} & & -\alpha \\ & & y_2 & \\ & & & y_2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ \frac{y_1}{y_2}p^s & & 1 & \\ & \frac{y_1}{y_2}p^s & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \beta & & -\alpha & \\ & \beta & & -\alpha \\ p^s y_1 & & y_2 & \\ & p^s y_1 & & y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ b & & 1 & \\ & b & & 1 \end{bmatrix} \begin{bmatrix} \beta & & -\alpha & \\ & \beta & & -\alpha \\ -b\beta + p^s y_1 & & \alpha b + y_2 & \\ & -b\beta + p^s y_1 & & \alpha b + y_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ b_1 p^s & & 1 & \\ & b_1 p^s & & 1 \end{bmatrix}.$$

This completes the proof of the lemma. □

Lemma 5.6. *Assume n to be a positive integer, r to be a nonnegative integer and p to be a prime number. Let $x_1, x_2 \in \mathbb{Z}$ such that p, x_1, x_2 are pairwise coprime. Then we have the following results.*

(1) *If $r < n$, then there exists an integer c_1 with $\gcd(c_1, p) = 1$ such that*

$$Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \frac{x_1}{x_2}p^r & & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ c_1 p^r & & & 1 \end{bmatrix} \Gamma_0(p^n).$$

(2) *If $r \geq n$, then*

$$Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \frac{x_1}{x_2}p^r & & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) 1 \Gamma_0(p^n).$$

Proof. Let α_1 and β_1 be integers such that $\alpha_1 p^r x_1 + \beta_1 x_2 = 1$. If $0 < r < n$ then set:

$$\alpha = \alpha_1, \quad \beta = \beta_1, \quad c_1, c_2 \in \mathbb{Z} \quad \text{such that} \quad c_1\beta + c_2 p^{n-r} = x_1, \; c = c_1 p^r.$$

Otherwise, if $r \geq n$, then set:

$$\alpha = \alpha_1, \quad \beta = \beta_1, \quad c_2 = x_1 p^{r-n}, \quad c_1 = 0, \quad c = 0.$$

If $r = 0$ then set:

$$\alpha = \begin{cases} \alpha_1 - x_2 & \text{if } p \mid \beta_1, \\ \alpha_1 & \text{if } p \nmid \beta_1, \end{cases} \qquad \beta = \begin{cases} \beta_1 + x_1 & \text{if } p \mid \beta_1, \\ \beta_1 & \text{if } p \nmid \beta_1, \end{cases}$$

$c_1, c_2 \in \mathbb{Z}$ such that $c_1\beta + c_2 p^n = x_1,\ c = c_1$. We note that for $r \geq 0$,

$$-c\beta + p^r x_1 = (-c_1\beta + x_1)p^r = c_2 p^n.$$

Then

$$\begin{aligned}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \frac{x_1}{x_2}p^r & & & 1 \end{bmatrix} &\sim \begin{bmatrix} x_2^{-1} & & & -\alpha \\ & 1 & & \\ & & 1 & \\ & & & x_2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \frac{x_1}{x_2}p^r & & & 1 \end{bmatrix} \\
&= \begin{bmatrix} \beta & & & -\alpha \\ & 1 & & \\ & & 1 & \\ p^r x_1 & & & x_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ c & & & 1 \end{bmatrix} \begin{bmatrix} \beta & & & -\alpha \\ & 1 & & \\ & & 1 & \\ -\beta c + p^r x_1 & & & \alpha c + x_2 \end{bmatrix} \sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ c_1 p^r & & & 1 \end{bmatrix}.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 5.7. *Assume n to be a positive integer and p to be a prime number. Let x, y be nonzero integers coprime to p. Then we have the following results.*

(1)

$$Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ x & & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) s_1 s_2 \Gamma_0(p^n).$$

(2)

$$Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ y & & 1 & \\ & y & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) s_1 s_2 \Gamma_0(p^n).$$

(3)

$$Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ y & & 1 & \\ x & y & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) s_1 s_2 \Gamma_0(p^n).$$

Proof. Let k_1 and k_2 be integers such that $k_1 x + k_2 p^n = 1$. Then we have

$$\begin{bmatrix}1&&&\\&1&&\\&&1&\\x&&&1\end{bmatrix} \sim \begin{bmatrix}1&&&\\&1&&\\&&1&\\x&&&1\end{bmatrix}\begin{bmatrix}&-k_1&1&\\1&&&\\&&&1\\&k_1x-1&-x&\end{bmatrix} = \begin{bmatrix}1&&&k_1\\&1&&\\&&1&\\&&&1\end{bmatrix} s_1 s_2 \sim s_1 s_2.$$

This completes the proof of the first part of lemma.

Now, let l_1 and l_2 be integers such that $l_1 y + l_2 p^n = -1$. Then,

$$\begin{bmatrix}1&&&\\&1&&\\y&&1&\\&y&&1\end{bmatrix} \sim \begin{bmatrix}1&&&\\&1&&\\y&&1&\\&y&&1\end{bmatrix}\begin{bmatrix}l_1&&1&\\&l_1&&1\\-l_1y-1&&-y&\\&-l_1y-1&&-y\end{bmatrix}$$

$$= s_2\begin{bmatrix}1&l_1&&\\&1&&\\&&1&-l_1\\&&&1\end{bmatrix} s_1 s_2 \sim s_1 s_2.$$

This completes the proof of the second part of lemma. Finally we have

$$\begin{bmatrix}1&&&\\&1&&\\y&&1&\\x&y&&1\end{bmatrix} \sim \begin{bmatrix}1&&&\\&1&&\\y&&1&\\x&y&&1\end{bmatrix}\begin{bmatrix}l_1&&1&\\l_1^2x&l_1&&1\\-l_1y-1&&-y&\\-l_1^2xy-l_1x&-l_1y-1&-x&-y\end{bmatrix}$$

$$= \begin{bmatrix}1&&&\\&l_1^2x&1&\\&-1&&\\&&&1\end{bmatrix}\begin{bmatrix}1&l_1&&\\&1&&\\&&1&-l_1\\&&&1\end{bmatrix} s_1 s_2 \sim s_1 s_2.$$

This completes the proof of the last part of lemma. □

Lemma 5.8. *Assume n and r to be integers such that $0 < r < n$. Let p be a prime number and $x, y \in \mathbb{Z}$ such that $\gcd(x, p) = \gcd(y, p) = 1$. Let*

$$g_1(x, p, r) = \begin{bmatrix}1&&&\\&1&&\\&&1&\\p^r x&&&1\end{bmatrix} \quad \textit{and} \quad g_1(y, p, r) = \begin{bmatrix}1&&&\\&1&&\\&&1&\\p^r y&&&1\end{bmatrix}.$$

Then

$$Q(\mathbb{Q})g_1(x, p, r)\Gamma_0(p^n) = Q(\mathbb{Q})g_1(y, p, r)\Gamma_0(p^n),$$

if and only if

$$x \equiv y \mod p^f$$

where $f = \min(r, n-r)$.

Proof. It is clear that $g_1(x,p,r)\Gamma_0(p^n) \sim g_1(y,p,r)\Gamma_0(p^n)$ if and only if there exists an element

$$q = \begin{bmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & \frac{1}{t} \end{bmatrix} \begin{bmatrix} 1 & l & \mu & k \\ & 1 & & \mu \\ & & 1 & -l \\ & & & 1 \end{bmatrix} \in Q(\mathbb{Q}),$$

such that $g_1(y,p,r)^{-1} q g_1(x,p,r) \in \Gamma_0(p^n)$. We have

$$g_1(y,p,r)^{-1} q g_1(x,p,r)$$
$$= \begin{bmatrix} kp^r tx + t & lt & \mu t & kt \\ -(bl - a\mu)p^r x & a & b & -bl + a\mu \\ -(dl - c\mu)p^r x & c & d & -dl + c\mu \\ -(kp^r ty - \frac{1}{t})p^r x - p^r ty & -lp^r ty & -\mu p^r ty & -kp^r ty + \frac{1}{t} \end{bmatrix}.$$

Suppose $g_1(y,p,r)^{-1} q g_1(x,p,r) \in \Gamma_0(p^n)$. Then we must have $t = \pm 1$. We also need the condition that

$$-\Big(kp^r ty - \frac{1}{t}\Big)p^r x - p^r ty \equiv 0 \mod p^n \Longrightarrow t^2 y - x \equiv 0 \mod p^f$$
$$\Longrightarrow y - x \equiv 0 \mod p^f.$$

Conversely, we show that if $y - x \equiv 0 \mod p^f$, then $g_1(x,p,r)$ and $g_1(y,p,r)$ lie in the same double coset. Suppose $x - y = k_2 p^f$. As $\gcd(p^{n-r-f}, xyp^{r-f}) = 1$, there exist integers k and k_2 such that $kxyp^{r-f} + k_1 p^{n-r-f} = k_2$. So we obtain

$$-(kp^r y - 1)p^r x - p^r y = k_1 p^n.$$

Therefore

$$g_1(y,p,r)^{-1} \begin{bmatrix} 1 & & & k \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} g_1(x,p,r)$$
$$= \begin{bmatrix} kp^r x + 1 & & & k \\ & 1 & & \\ & & 1 & \\ -(kp^r y - 1)p^r x - p^r y & & & -kp^r y + 1 \end{bmatrix} \in \Gamma_0(p^n).$$

This means that $g_1(x,p,r)$ and $g_1(y,p,r)$ lie in the same double coset. This completes the proof of the lemma. □

Lemma 5.9. *Assume s, r and n to be integers such that $n \geq 1$, $0 < s < n$. Let p be a prime number and $x, y \in \mathbb{Z}$ such that $\gcd(x, p) = \gcd(y, p) = 1$. Let*

$$g_3(p,x,r,s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ xp^r & p^s & & 1 \end{bmatrix}, \quad g_2(p,s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ & p^s & & 1 \end{bmatrix}$$

$$g_3(p,y,r,s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ yp^r & p^s & & 1 \end{bmatrix}.$$

(1) *If $r < n$ and $0 < s < r < 2s$ and $f = \min(2s - r, n - r)$, then*

$$Q(\mathbb{Q})g_3(p,x,r,s)\Gamma_0(p^n) = Q(\mathbb{Q})g_3(p,y,r,s)\Gamma_0(p^n) \iff x \equiv y \mod p^f.$$

(2) *If $2s \leq r$, then*

$$Q(\mathbb{Q})g_3(p,x,r,s)\Gamma_0(p^n) = Q(\mathbb{Q})g_2(p,s)\Gamma_0(p^n).$$

(3) *If $r \geq n$, then*

$$Q(\mathbb{Q})g_3(p,x,r,s)\Gamma_0(p^n) = Q(\mathbb{Q})g_2(p,s)\Gamma_0(p^n).$$

Proof. It is clear that $Q(\mathbb{Q})g_3(p,x,r,s)\Gamma_0(p^n) = Q(\mathbb{Q})g_3(p,y,r,s)\Gamma_0(p^n)$ if and only if there exists an element

$$q = \begin{bmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & 1/t \end{bmatrix} \begin{bmatrix} 1 & l & \mu & k \\ & 1 & & \mu \\ & & 1 & -l \\ & & & 1 \end{bmatrix} \in Q(\mathbb{Q})$$

such that $g_3(p,y,r,s)^{-1} q g_3(p,x,r,s) \in \Gamma_0(p^n)$. Suppose

$$g_3(p,y,r,s)^{-1} q g_3(p,x,r,s) \in \Gamma_0(p^n).$$

Then, comparing the multiplier of the matrices on both sides, we see that $ad - bc = 1$. Then by writing the matrix on the left explicitly it also follows that $t = \pm 1$. We can assume that $t = 1$. Now,

$$g_3(p, y, r, s)^{-1} q g_3(p, x, r, s)$$
$$= \begin{bmatrix} kp^r x + \mu p^s + 1 & kp^s + l & \mu & k \\ -(bl - a\mu) p^r x + bp^s & -(bl - a\mu) p^s + a & b & -bl + a\mu \\ -(dl - c\mu + kp^s) p^r x - (\mu p^s - d) p^s - p^s & -(dl - c\mu + kp^s) p^s - lp^s + c & -\mu p^s + d & -dl + c\mu - kp^s \\ (blp^s - a\mu p^s - kp^r y + 1) p^r x - (\mu p^r y + bp^s) p^s - p^r y & -lp^r y + (blp^s - a\mu p^s - kp^r y + 1) p^s - ap^s & -\mu p^r y - bp^s & blp^s - a\mu p^s - kp^r y + 1 \end{bmatrix},$$

and then looking at the lowest left entry we get

$$\begin{aligned} (blp^s - a\mu p^s - kp^r y + 1) p^r x - (\mu p^r y + bp^s) p^s - p^r y &\equiv 0 \mod p^n \\ \Longrightarrow p^r (x - y) + (bl - a\mu) x p^{r+s} - kxyp^{2r} - \mu y p^{r+s} + bp^{2s} &\equiv 0 \mod p^n \\ \Longrightarrow x - y + (bl - a\mu) x p^s - kxyp^r - \mu y p^s + bp^{2s-r} &\equiv 0 \mod p^{n-r} \\ \Longrightarrow x - y &\equiv 0 \mod p^f . \end{aligned}$$

Conversely, we show that if $y - x \equiv 0 \mod p^f$ then $g_3(p, x, r, s)$ and $g_3(p, y, r, s)$ lie in the same double-coset. If $f = n - r$, then let $x - y = k_1 p^{n-r}$.

$$\begin{aligned} Q(\mathbb{Q}) g_3(p, x, r, s) \Gamma_0(p^n) &= Q(\mathbb{Q}) g_3(p, y, r, s) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^r x - p^r y & & & 1 \end{bmatrix} \Gamma_0(p^n) \\ &= Q(\mathbb{Q}) g_3(p, y, r, s) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^n k_1 & & & 1 \end{bmatrix} \Gamma_0(p^n) \\ &= Q(\mathbb{Q}) g_3(p, y, r, s) \Gamma_0(p^n). \end{aligned}$$

On the other hand if $f = 2s - r$ or equivalently $2s \le n$, then let $x - y = k_2 p^{2s-r}$.

$$Q(\mathbb{Q}) g_3(p, x, r, s)\Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & k_2 & \\ & & 1 & \\ & & & 1 \end{bmatrix} g_3(p, x, r, s)\Gamma_0(p^n)$$

$$= Q(\mathbb{Q}) g_3(p, y, r, s) \begin{bmatrix} 1 & & & \\ k_2 p^s & 1 & k_2 & \\ & & 1 & \\ & & -k_2 p^s & 1 \end{bmatrix} \Gamma_0(p^n)$$

$$= Q(\mathbb{Q}) g_3(p, y, r, s)\Gamma_0(p^n).$$

This means that $g_3(p, x, r, s)$ and $g_3(p, y, r, s)$ lie in the same double coset and the first part of lemma follows. Next,

$$Q(\mathbb{Q}) g_3(p, x, r, s)\Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & p^{r-2s}x & \\ & & 1 & \\ & & & 1 \end{bmatrix} g_3(p, x, r, s)\Gamma_0(p^n)$$

$$= Q(\mathbb{Q}) g_2(p, s) \begin{bmatrix} 1 & & & \\ p^{r-2s} p^s x & 1 & p^{r-2s}x & \\ & & 1 & \\ & & -p^{r-2s} p^s x & 1 \end{bmatrix} \Gamma_0(p^n)$$

$$= Q(\mathbb{Q}) g_2(p, s)\Gamma_0(p^n).$$

This completes the proof of the second part of lemma. Finally, the last part of the lemma follows from the calculation

$$g_2(p, s)^{-1} g_3(p, x, r, s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^r x & & & 1 \end{bmatrix} \in \Gamma_0(p^n). \qquad \square$$

Proof of Theorem 3.1.

Proof. First we prove completeness. We begin by writing

$$\text{(5-4)} \quad \mathrm{GSp}(4, \mathbb{Q}) = Q(\mathbb{Q}) \sqcup Q(\mathbb{Q}) s_1 \begin{bmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{bmatrix}$$

$$\sqcup Q(\mathbb{Q}) s_1 s_2 \begin{bmatrix} 1 & & * & \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix} \sqcup Q(\mathbb{Q}) s_1 s_2 s_1 \begin{bmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{bmatrix},$$

by using the Bruhat decomposition. We consider all the different possibilities.

First Cell: If $g \in Q(\mathbb{Q})$, then, of course, $Q(\mathbb{Q})g\Gamma_0(p^n)$ is represented by 1.

Second Cell: Assume that g is in the second cell. Then we may assume that

$$g = s_1 \begin{bmatrix} 1 & \frac{x_1}{x_2} & & \\ & 1 & & \\ & & 1 & -\frac{x_1}{x_2} \\ & & & 1 \end{bmatrix}, \quad x_1, x_2 \in \mathbb{Z} \text{ and } \gcd(x_1, x_2) = 1.$$

As $\gcd(x_1, x_2) = 1$, there exist integers l_1 and l_2 such that $-l_1 x_1 + l_2 x_2 = 1$.

$$\begin{aligned} Q(\mathbb{Q})g\Gamma_0(p^n) &= Q(\mathbb{Q}) \begin{bmatrix} \frac{1}{x_2} & l_1 & & \\ & x_2 & & \\ & & \frac{1}{x_2} & -l_1 \\ & & & x_2 \end{bmatrix} g\Gamma_0(p^n) \\ &= Q(\mathbb{Q}) \begin{bmatrix} l_1 & \frac{l_1 x_1}{x_2} + \frac{1}{x_2} & & \\ x_2 & x_1 & & \\ & & -l_1 & \frac{l_1 x_1}{x_2} + \frac{1}{x_2} \\ & & x_2 & -x_1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) 1 \Gamma_0(p^n). \end{aligned}$$

Third Cell: Next let g be an element in the third cell. We may assume that

$$g = s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad x, y \in Q(\mathbb{Q}).$$

The following calculation shows that we can replace x, y by $x + 1$ and $y + 1$ respectively.

$$s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \sim s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & 1 & \\ & 1 & 1 & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix} \sim s_1 s_2 \begin{bmatrix} 1 & & y+1 & \\ & 1 & x+1 & y+1 \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Let x_1, x_2, x_3 and p be pairwise coprime. Also assume y_1, y_2, y_3, p to be pairwise coprime. Let $x = x_3 p^{r_1}/x_2$ with $r_1 > 0$. Then the above calculation shows that we can change x to $x + 1 = (x_3 p^{r_1} + x_2)/x_2$. So we can always assume x to be of the form $x_1/(x_2 p^r)$ for some $r \geq 0$. Similarly, we can also assume y to be of the form $y_1/(y_2 p^s)$ with $s \geq 0$. Next, suppose $\tau = \gcd(x_1, y_1) > 1$. Then replacing $x = x_1/(x_2 p^r)$ by $x + \tau_1$, with τ_1 being the largest factor of y_1 that is coprime to τ, we can also assume that $\gcd(x_1, y_1) = 1$.

Now we consider all the different possibilities that may arise. First of all, it is clear that, if both x and y are in $\mathbb{Z}$, i.e., $x_2 = y_2 = 1$, $r = 0$, $s = 0$, then

$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})s_1s_2\Gamma_0(p^n)$. Next, if $x \in \mathbb{Z}$ but $y \notin \mathbb{Z}$, then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})s_1s_2\begin{bmatrix}1 & y & & \\ & 1 & & y \\ & & 1 & \\ & & & 1\end{bmatrix}\Gamma_0(p^n)$$

$$= Q(\mathbb{Q})\begin{bmatrix}1 & & & \\ y & 1 & & \\ & & 1 & \\ & & -y & 1\end{bmatrix}s_1s_2\Gamma_0(p^n)$$

$$= Q(\mathbb{Q})s_1\begin{bmatrix}1 & y^{-1} & & \\ & 1 & & \\ & & 1 & -y^{-1} \\ & & & 1\end{bmatrix}s_1s_2\Gamma_0(p^n)$$

$$= Q(\mathbb{Q})\begin{bmatrix}1 & & & \\ & 1 & & \\ y^{-1} & & 1 & \\ & y^{-1} & & 1\end{bmatrix}\Gamma_0(p^n) = Q(\mathbb{Q})\begin{bmatrix}1 & & & \\ & 1 & & \\ \frac{y_2p^s}{y_1} & & 1 & \\ & \frac{y_2p^s}{y_1} & & 1\end{bmatrix}\Gamma_0(p^n).$$

We note that the third equality follows from the following matrix identity:

$$\begin{bmatrix}1 & & & \\ y & 1 & & \\ & & 1 & \\ & & -y & 1\end{bmatrix} = \begin{bmatrix}-y^{-1} & 1 & & \\ & y & & \\ & & y^{-1} & 1 \\ & & & -y\end{bmatrix}s_1\begin{bmatrix}1 & y^{-1} & & \\ & 1 & & \\ & & 1 & -y^{-1} \\ & & & 1\end{bmatrix}.$$

But, now from Lemmas 5.5 and 5.3 it follows that if $0 < s < n$, then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})\begin{bmatrix}1 & & & \\ & 1 & & \\ p^s & & 1 & \\ & p^s & & 1\end{bmatrix}\Gamma_0(p^n),$$

and if $s \geq n$ then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})1\Gamma_0(p^n).$$

Further, If $s = 0$ then from the Lemma 5.5 and 5.7 it follows that

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})s_1s_2\Gamma_0(p^n),$$

which is one of the listed representatives in the statement of the theorem. Therefore we are done in this case.

Now consider the case when $x \notin \mathbb{Z}$ and $y \in \mathbb{Z}$. Then we have

$$\begin{aligned} Q(\mathbb{Q})g\Gamma_0(p^n) &= Q(\mathbb{Q})s_1s_2\begin{bmatrix}1&&&\\&1&x&\\&&1&\\&&&1\end{bmatrix}\Gamma_0(p^n) \\ &= Q(\mathbb{Q})\begin{bmatrix}x&&&1\\&1&&\\&&1&\\&&&1/x\end{bmatrix}s_1s_2\begin{bmatrix}1&&&\\&1&x&\\&&1&\\&&&1\end{bmatrix}\Gamma_0(p^n) \\ &= Q(\mathbb{Q})\begin{bmatrix}1&&&\\&1&&\\&&1&\\x^{-1}&&&1\end{bmatrix}\begin{bmatrix}&-1&&\\1&&&\\&&&1\\&&-1&\end{bmatrix}\Gamma_0(p^n) \\ &= Q(\mathbb{Q})\begin{bmatrix}1&&&\\&1&&\\&&1&\\x^{-1}&&&1\end{bmatrix}\Gamma_0(p^n). \end{aligned}$$

Now it follows from Lemmas 5.6 and 5.7 that if $r = 0$, then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})s_1s_2\Gamma_0(p^n),$$

and if $r \geq n$, then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})1\Gamma_0(p^n).$$

Further, if $0 < r < n$, then Lemma 5.6 yields that

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})\begin{bmatrix}1&&&\\&1&&\\&&1&\\c_1p^r&&&1\end{bmatrix}\Gamma_0(p^n)$$

for some integer c_1 such that $\gcd(c_1, p) = 1$. Then it follows from Lemma 5.8 that g lies in the same double coset as one of the elements listed in the statement of the theorem.

Next, suppose $x \notin \mathbb{Z}$ and $y \notin \mathbb{Z}$. If $s = r = 0$ then from Lemmas 5.1 and 5.7 it follows that

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})s_1s_2\Gamma_0(p^n).$$

Further it follows from Lemma 5.1 that if $s \leq r$ and $r \geq n$, then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})1\Gamma_0(p^n);$$

otherwise, if $s \le r < n$, then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})g_1(x_3, p, r)\Gamma_0(p^n)$$

for some nonzero integer x_3 coprime to p. But then these cases have already been considered. Hence, we are left with the case when $s > r$, and then if $s \ge n$ from Lemma 5.1 we get

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})1\Gamma_0(p^n),$$

and we are done. Otherwise, still assuming $s > r$ but $s < n$, we get

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & & 1 & \\ \eta_1 p^s & & & 1 \\ \eta_2 p^{-r+2s} & \eta_1 p^s & & 1 \end{bmatrix} \Gamma_0(p^n),$$

where $\eta_1, \eta_2 \in \mathbb{Z}$ and $\gcd(\eta_i, p) = 1$ for $i = 1, 2$. In view of Lemma 5.4 it further reduces to

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & & 1 & \\ p^s & & & 1 \\ \eta_2 p^{-r+2s} & p^s & & 1 \end{bmatrix} \Gamma_0(p^n).$$

Now, the result follows from Lemma 5.9 and we are done in this case as well.

Fourth Cell: Next we consider an element g from the fourth cell and let

$$g = s_1 s_2 s_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

If $x \in \mathbb{Z}$ then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & z + 2xy & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \Gamma_0(p^n),$$

and we are reduced to the case of the third cell. Therefore let us assume that $x \notin \mathbb{Z}$. If necessary on multiplication by a suitable matrix from right, we can assume that $x = x_1/(p^r x_2)$, $y = y_1/(p^s y_2)$ and $z = z_1/(p^{r_1} z_2)$ where $x_i, y_i, z_i \in \mathbb{Z}$, for $i = 1, 2$; r, s, r_1 are nonnegative integers, x_1, x_2, p are mutually coprime integers; y_1, y_2, p are mutually coprime integers and z_1, z_2, p are also mutually coprime integers. We can further adjust x_1, y_1 and z_1 by multiplication by a proper matrix from the right, such that any two nonzero elements selected from the set $\{x_1, y_1, z_1, x_2, y_2, z_2, p\}$ are mutually coprime except, possibly, when both the chosen elements belong to

$\{x_2, y_2, z_2\}$. Then by the virtue of Lemma 5.2 once again we are reduced to the case of the third cell. This proves that the representatives listed in the theorem constitute a complete set of double coset representatives.

Disjointness. Now we prove that the double cosets represented by the representatives listed in the theorem are disjoint. It is clear that two elements w_1 and w_2 represent the same double coset if and only if there exists an element

$$q = \begin{bmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & (ad-bc)t^{-1} \end{bmatrix} \begin{bmatrix} 1 & l & \mu & k \\ & 1 & & \mu \\ & & 1 & -l \\ & & & 1 \end{bmatrix} \in Q(\mathbb{Q})$$

such that $w_2^{-1} q w_1 \in \Gamma_0(p^n)$. On comparing the multiplier on both sides we conclude that $ad - bc = 1$. Then it is also clear that we can assume $t = 1$. Also clearly q must be a matrix with integral entries. Now we consider all pairs of different representatives for checking disjointness.

$\boldsymbol{w_1 = g_3(p, \alpha, r, s)}$, $\boldsymbol{w_2 = g_3(p, \beta, v, w)}$: Let

$$w_1 = g_3(p,\alpha,r,s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ \alpha p^r & p^s & & 1 \end{bmatrix} \quad \text{and} \quad w_2 = g_3(p,\beta,v,w) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^w & & 1 & \\ \beta p^v & p^w & & 1 \end{bmatrix},$$

with α, β integers coprime to p and $r, s, v, w \in \mathbb{Z}$ such that $0 < s < r < 2s$, $0 < v < w < 2v$, $0 < s,\ r < n,\ 0 < w,\ v < n$. We see that

$$w_2^{-1} q w_1 = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ -(dl - c\mu + kp^w)\alpha p^r - (\mu p^w - d)p^s - p^w & -(dl - c\mu + kp^w)p^s - lp^w + c & * & * \\ -(\beta kp^v - blp^w + a\mu p^w - 1)\alpha p^r - (\beta\mu p^v + bp^w)p^s - \beta p^v & -\beta lp^v - (\beta kp^v - blp^w + a\mu p^w - 1)p^s - ap^w & * & * \end{bmatrix}$$

Suppose $s > w$. If $w_2^{-1} q w_1 \in \Gamma_0(p^n)$, then looking at the bottom two entries of the second column we conclude that p must divide both a and c. But this contradicts that $ad - bc = 1$. Similarly, if $s < w$, by looking at first two entries of the third row we get that $p \mid d$ and $p \mid c$ contradicting $ad - bc = 1$. Therefore, we assume $s = w$. Now looking at the bottommost entry of the first column we conclude that if $r \neq v$, then the valuation of this element can not be n. Therefore, if $r \neq v$ or $s \neq w$, then $g_3(p, \alpha, r, s)$ and $g_3(p, \beta, v, w)$ lie in different double cosets. If $r = v$ and $s = w$, then Lemma 5.9 describes the condition for $g_3(p, \alpha, r, s)$ and $g_3(p, \beta, v, w)$

to lie in the same double coset. We conclude that such representatives listed in the theorem represent disjoint double cosets.

$\boldsymbol{w_1 = g_3(p, \alpha, r, s)}$, $\boldsymbol{w_2 = g_2(p, w)}$: Let $w_1 = g_3(p, \alpha, r, s)$ and $w_2 = g_2(p, w)$. Assume $w_2^{-1} q w_1 \in \Gamma_0(p^n)$. Then we see that

$$w_2^{-1} q w_1 = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ -(dl - c\mu + kp^w)\alpha p^r - (\mu p^w - d)p^s - p^w & -(dl - c\mu + kp^w)p^s - lp^w + c & * & * \\ (blp^w - a\mu p^w + 1)\alpha p^r - bp^s p^w & (blp^w - a\mu p^w + 1)p^s - ap^w & * & * \end{bmatrix}$$

and it is clear that $p \mid c$ and if $s > w$ then and $p \mid a$ or else if $s < w$ then $p \mid d$. In any case $p \mid ad - bc = 1$ which is a contradiction. Hence, we further assume $s = w$. Now, as $s < r < 2s$, looking at the last entry of the first column we see that the valuation of the element $(blp^s - a\mu p^s + 1)\alpha p^r - b(p^s)^2$ is r. Since $r < n$, we conclude that $g_3(p, \alpha, r, s)$ and $g_2(p, w)$ lie in different double cosets.

$\boldsymbol{w_1 = g_3(p, \alpha, r, s), w_2 = g_1(p, \beta, v)}$: Let $w_1 = g_3(p, \alpha, r, s)$ and $w_2 = g_1(p, \beta, v)$. Assume $w_2^{-1} q w_1 \in \Gamma_0(p^n)$. Then we see that

$$w_2^{-1} q w_1 = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ -(dl - c\mu)\alpha p^r + dp^s & -(dl - c\mu)p^s + c & * & * \\ -\beta\mu p^s p^v - (\beta kp^v - 1)\alpha p^r - \beta p^v & -\beta lp^v - (\beta kp^v - 1)p^s & * & * \end{bmatrix}.$$

Clearly, $p \mid c$. Since $r > s$, p also divides d and it contradicts the condition $ad - bc = 1$. Therefore $g_3(p, \alpha, r, s)$ and $w_2 = g_1(p, \beta, v)$ lie in different double cosets.

$\boldsymbol{w_1 = g_2(p, s)}$, $\boldsymbol{w_2 = g_1(p, \beta, v)}$: Let $w_1 = g_2(p, s)$ and $w_2 = g_1(p, \beta, v)$. Assume $w_2^{-1} q w_1 \in \Gamma_0(p^n)$. Then we see that

$$w_2^{-1} q w_1 = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ dp^s & -(dl - c\mu)p^s + c & * & * \\ -\beta\mu p^s p^v - \beta p^v & -\beta lp^v - (\beta kp^v - 1)p^s & * & * \end{bmatrix}.$$

Once again we see that p divides both c and d which is a contradiction to the condition $ad - bc = 1$. Therefore $g_2(p, s)$ and $w_2 = g_1(p, \beta, v)$ lie in different double cosets.

$\boldsymbol{w_1 = g_2(p,s),\ w_2 = g_2(p,w)}$: Let $w_1 = g_2(p,s)$ and $w_2 = g_1(p,w)$. Let us assume that $w_2^{-1} q w_1 \in \Gamma_0(p^n)$. Then we see that

$$w_2^{-1} q w_1 = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ dp^s & -(dl - c\mu)p^s + c & * & * \\ -\beta\mu p^s p^w - \beta p^w & -\beta l p^w - (\beta k p^w - 1)p^s & * & * \end{bmatrix}.$$

Once again we see that $p \mid c$ and if $s > w$, then $p \mid a$ or else if $s < w$, then $p \mid d$. In any case $p \mid ad - bc = 1$, which is a contradiction. Therefore $g_2(p,s)$ and $g_2(p,w)$ lie in different double cosets.

$\boldsymbol{w_1 = g_1(p,\alpha,r),\ w_2 = g_1(p,\beta,v)}$: Let $w_1 = g_1(p,\alpha,r)$ and $w_2 = g_1(p,\beta,v)$. Let us assume that $w_2^{-1} q w_1 \in \Gamma_0(p^n)$. Then we see that

$$w_2^{-1} q w_1 = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ -(dl - c\mu)\alpha p^r & c & * & * \\ -(\beta k p^v - 1)\alpha p^r - \beta p^v & -\beta l p^v & * & * \end{bmatrix}.$$

Since $r, v < n$, we see that if $r \neq v$, then valuation of $-(\beta k p^v - 1)\alpha p^r - \beta p^v$ is less than n. Therefore $g_1(p,\alpha,r)$ and $g_1(p,\beta,v)$ lie in different double cosets. This completes the proof of disjointness.

The number of representatives. Finally, we calculate the total number of inequivalent representatives. First let n be even, say $n = 2m$ for some positive integer m. Then

$$\begin{aligned} &\#(Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(p^{2m})) \\ &\quad = 2 + 2m - 1 + \sum_{r=1}^{2m-1} \phi(p^{\min(r,2m-r)}) + \sum_{s=1}^{2m-1} \sum_{r=s+1}^{\min(2s-1,2m-1)} \phi(p^{\min(2s-r,2m-r)}) \\ &\quad = \frac{p^{m+1} + p^m - 2}{p-1}. \end{aligned}$$

Similarly, if n is odd, say $n = 2m+1$, then

$$\begin{aligned} &\#(Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(p^{2m+1})) \\ &\qquad = 2 + 2m + \sum_{r=1}^{2m} \phi(p^{\min(r,2m+1-r)}) + \sum_{s=1}^{2m} \sum_{r=s+1}^{\min(2s-1,2m)} \phi(p^{\min(2s-r,2m+1-r)}) \\ &\qquad = \frac{2(p^{m+1} - 1)}{p-1}. \end{aligned}$$

Thus on combining these we obtain the formula (3-1) for the number of one-dimensional cusps. □

Proof of Lemma 3.2.

Proof. We note that the representatives for $Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(N)$ may be obtained from the representatives of $Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(p_i^{n_i})$ for $i=1$ to m. This observation is essentially based on the following two well known facts.

(1) The natural projection map from $\mathrm{Sp}(4,\mathbb{Z})$ to $\mathrm{Sp}(4,\mathbb{Z}/N\mathbb{Z})$ is surjective.

(2) $\mathrm{Sp}(4,\mathbb{Z}/\prod_p p^e\mathbb{Z}) \xrightarrow{\sim} \prod_p \mathrm{Sp}(4,\mathbb{Z}/p^e\mathbb{Z})$.

In fact, we have

$$\begin{aligned}\mathrm{Sp}(4,\mathbb{Z})/\Gamma_0(\mathrm{N}) &\xrightarrow{\sim} (\mathrm{Sp}(4,\mathbb{Z})/\Gamma(N))/(\Gamma_0(\mathrm{N})/\Gamma(N))\\ &\xrightarrow{\sim} \mathrm{Sp}(4,\mathbb{Z}/N\mathbb{Z})/\Delta(\mathbb{Z}/N\mathbb{Z}).\end{aligned}$$

Clearly, $\Delta(\mathbb{Z}/N\mathbb{Z})=\prod_{i=1}^m \Delta(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$, and the following diagram is commutative.

$$\begin{array}{ccc}
\mathrm{Sp}(4,\mathbb{Z}/N\mathbb{Z}) & \xrightarrow[\psi]{\sim} & \prod_{i=1}^m \mathrm{Sp}(4,\mathbb{Z}/p_i^{n_i}\mathbb{Z}) \\
\Cup & & \Cup \\
g & & (g_1,\dots,g_m) \\
\downarrow & & \downarrow \\
g\Delta(\mathbb{Z}/N\mathbb{Z}) & & (g_1\Delta(\mathbb{Z}/p_1^{n_1}\mathbb{Z}),\dots,g_m\Delta(\mathbb{Z}/p_m^{n_m}\mathbb{Z})) \\
\Cap & & \Cap \\
A=\mathrm{Sp}(4,\mathbb{Z}/N\mathbb{Z})/\Delta(\mathbb{Z}/N\mathbb{Z}) & \xrightarrow[\phi]{\sim} & \prod_{i=1}^m \mathrm{Sp}(4,\mathbb{Z}/p_i^{n_i}\mathbb{Z})/\Delta(\mathbb{Z}/p_i^{n_i}\mathbb{Z})=B
\end{array}$$

Next we show that the left action by $\Gamma_\infty(\mathbb{Z})$ is compatible with the isomorphisms described in the commutative diagram above. In fact, $\Gamma_\infty(\mathbb{Z})$ acts on both sides as follows:

- on A: via

$$\Gamma_\infty(\mathbb{Z}) \to \Gamma_\infty(\mathbb{Z}/N\mathbb{Z}), \quad \gamma \to \bar{\gamma}$$

- on B: via

$$\begin{array}{ccccc}\Gamma_\infty(\mathbb{Z}) \to & \Gamma_\infty(\mathbb{Z}/N\mathbb{Z}) & \xrightarrow{\sim} & \displaystyle\prod_{i=1}^m \Gamma_\infty(\mathbb{Z}/p_i^{n_i}\mathbb{Z}) \\ \gamma \to & \bar{\gamma} & \xrightarrow{\sim} & (\gamma_1,\gamma_2,\dots,\gamma_{m-1},\gamma_m).\end{array}$$

Let $g\in \mathrm{Sp}(4,\mathbb{Z}/N\mathbb{Z})$, $a=g\Delta(\mathbb{Z}/N\mathbb{Z})\in A$ and $\gamma\in\Gamma_\infty(\mathbb{Z})$. Then it is easy to check that $\phi(\gamma a)=\gamma(\phi(a))$. Therefore we obtain,

$$\begin{aligned}Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(N) &\xrightarrow{\sim} (Q(\mathbb{Q})\cap \mathrm{Sp}(4,\mathbb{Z}))\backslash \mathrm{Sp}(4,\mathbb{Z})/\Gamma_0(\mathrm{N})\\ &\xrightarrow{\sim} \Gamma_\infty(\mathbb{Z}/N\mathbb{Z})\backslash \mathrm{Sp}(4,\mathbb{Z}/N\mathbb{Z})/\Delta(\mathbb{Z}/N\mathbb{Z})\\ &\xrightarrow{\sim} \prod_{i=1}^m \Gamma_\infty(\mathbb{Z}/p_i^{n_i}\mathbb{Z})\backslash \mathrm{Sp}(4,\mathbb{Z}/p_i^{n_i}\mathbb{Z})/\Delta(\mathbb{Z}/p_i^{n_i}\mathbb{Z}).\end{aligned}$$

Now the result follows from Theorem 3.1. □

Proof of Corollary 3.3.

Proof. It is easy to check that the double cosets represented by the listed representatives are disjoint. Let $\alpha(N)$ denote the total number of representatives listed in the statement of the corollary. We note that for $N = p^n$, with p a prime and $n \geq 1$, the number of listed representatives are the same as given by Theorem 3.1 (moreover, the set of representatives in this case will be seen to be equivalent to the set of representatives given by Theorem 3.1 if one applies Lemma 5.8 and Lemma 5.9 and works out the details). We will show that for any pair of coprime positive integers R and S we have $\alpha(RS) = \alpha(R)\alpha(S)$. Then it will follow that the listed representatives form a complete set because for any N their number will agree with the number given in Lemma 3.2. We have

$$\begin{aligned}
\alpha(RS) &= 1 + \sum_{\substack{\gamma|RS\\ 1<\gamma\leq RS}} \phi\Big(\gcd\Big(\gamma, \frac{RS}{\gamma}\Big)\Big) + \sum_{\substack{\gamma|RS\\ 1<\gamma\leq RS}} \sum_{\substack{\delta|\gamma,\, \gamma|\delta^2\\ \gamma>\delta}} \phi\Big(\gcd\Big(\frac{\delta^2}{\gamma}, \frac{RS}{\gamma}\Big)\Big)\\
&= 1 + \sum_{\substack{\gamma|RS\\ 1<\gamma\leq RS}} \sum_{\substack{\delta|\gamma,\, \gamma|\delta^2\\ \gamma\geq\delta}} \phi\Big(\gcd\Big(\frac{\delta^2}{\gamma}, \frac{RS}{\gamma}\Big)\Big)\\
&= 1 + \sum_{\substack{\gamma_1|R,\, \gamma_2|S\\ 1<\gamma_1\gamma_2\leq RS}} \sum_{\substack{\delta|\gamma_1\gamma_2,\, \gamma_1\gamma_2|\delta^2\\ \gamma_1\gamma_2\geq\delta}} \phi\Big(\gcd\Big(\frac{\delta^2}{\gamma_1\gamma_2}, \frac{RS}{\gamma_1\gamma_2}\Big)\Big)\\
&\qquad + \sum_{\substack{\gamma_1|R\\ 1<\gamma_1\leq R}} \sum_{\substack{\delta|\gamma_1,\, \gamma_1|\delta^2\\ \gamma_1\geq\delta}} \phi\Big(\gcd\Big(\frac{\delta^2}{\gamma_1}, \frac{RS}{\gamma_1}\Big)\Big)\\
&\qquad + \sum_{\substack{\gamma_2|S\\ 1<\gamma_1\leq S}} \sum_{\substack{\delta|\gamma_2,\, \gamma_2|\delta^2\\ \gamma_2\geq\delta}} \phi\Big(\gcd\Big(\frac{\delta^2}{\gamma_2}, \frac{RS}{\gamma_2}\Big)\Big)\\
&= \Bigg(1 + \sum_{\substack{\gamma_1|R\\ 1<\gamma_1\leq R}} \sum_{\substack{\delta_1|\gamma_1,\, \gamma_1|\delta_1^2\\ \gamma_1\geq\delta_1}} \phi\Big(\gcd\Big(\frac{\delta_1^2}{\gamma_1}, \frac{R}{\gamma_1}\Big)\Big)\Bigg)\Bigg(1 + \sum_{\substack{\gamma_2|S\\ 1<\gamma_2\leq S}} \sum_{\substack{\delta_2|\gamma_2,\, \gamma_2|\delta_2^2\\ \gamma_2\geq\delta_2}} \phi\Big(\gcd\Big(\frac{\delta_2^2}{\gamma_2}, \frac{S}{\gamma_2}\Big)\Big)\Bigg)\\
&= \alpha(R)\alpha(S).
\end{aligned}$$

This completes the proof.

Alternatively, instead of the above counting argument the corollary could also be proved by giving an explicit bijection between sets of representatives for $\prod_{i=1}^{m} Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/\, \Gamma_0(p_i^{n_i})$ and $Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/\, \Gamma_0(N)$. For this we recall the remark (ii) after Theorem 3.1 and note that for $m = 1$, i.e., for $N = p_1^{n_1}$, a complete set of representatives for $Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(N)$ is also given by

$$g_0(\gamma,\delta,y) := \begin{bmatrix} 1 & & & \\ & 1 & & \\ \delta & & 1 & \\ y\gamma & \delta & & 1 \end{bmatrix}, \quad 1 \le \delta \le \gamma \le N,\ \gamma \mid N,\ \delta \mid N,\ \delta \mid \gamma,\ \gamma \mid \delta^2,$$

with y being the same as in the statement of the corollary.

Now suppose $N = \prod_{i=1}^m N_i$ with $N_i = p_i^{n_i}$ and $m > 1$. We define the map

$$\phi : \prod_{j=1}^m Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(p_j^{n_j}) \to Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(N)$$
$$(g_0(\gamma_j,\delta_j,y_j))_{j=1,\dots,m} \to g_0(\gamma,\delta,y),$$

where γ_j and δ_j are factors of N_j such that $1 \le \delta_j \le \gamma_j \le N_j$, $\delta_j \mid \gamma_j$, $\gamma_j \mid \delta_j^2$ and $\gamma = \prod_{j=1}^m \gamma_j$, $\delta = \prod_{j=1}^m \delta_j$. Also, $y_j = L_j + \theta_j$ with $L_j = \gcd(\delta_j^2/\gamma_j, N_j/\gamma_j)$ and $\theta_j = 0$ if $L_j = 1$ otherwise $\theta_j \in (\mathbb{Z}/L_j\mathbb{Z})^\times$. Also, $y = L + \theta\beta$ with $L = \prod_{j=1}^m L_j$, $\beta = \prod_{p_i \nmid L,\ p_i \mid N} p_i^{n_i}$, $\theta = \sum_{j=1}^m \alpha_j (L/L_j)\theta_j$ and α_j is such that $\alpha_j \beta (L/L_j) \equiv 1 \mod L_j$.

It is clear that $L = \gcd(\delta^2/\gamma, N/\gamma)$. If a prime p divides L then it must divide some L_k with $1 \le k \le m$. Assume this to be the case. Then from the definition of θ it follows that $\beta\theta \equiv \theta_k \mod L_k$. As $p \mid L_k$ and $\theta_k \in (\mathbb{Z}/L_k\mathbb{Z})^\times$, it is clear that $p \nmid \theta$. Therefore $\theta \in (\mathbb{Z}/L\mathbb{Z})^\times$ as desired.

Next we show that ϕ is injective. For this let us assume that

$$\phi((g_0(\gamma_j,\delta_j,y_j))_{j=1,\dots,m}) = \phi((g_0(\gamma_j',\delta_j',y_j'))_{j=1,\dots,m}) = g_0(\gamma,\delta,y).$$

Then $\gamma = \prod_{j=1}^m \gamma_j = \prod_{j=1}^m \gamma_j'$ implies $\gamma_j = \gamma_j'$ for all j. Similarly $\delta_j = \delta_j'$ for all j. Hence $L_j = L_j'$ for all j. Moreover, we have $y \equiv \beta\theta \equiv \theta_j \equiv y_j \mod L_j$. Similarly $y \equiv \beta\theta \equiv \theta_j' \equiv y_j' \mod L_j'$. This gives $y_j \equiv y_j' \mod L_j$ for all j. Now Lemmas 5.8 and 5.9 imply that $g_0(\gamma_j,\delta_j,y_j)$ is equivalent to $g_0(\gamma_j',\delta_j',y_j')$ in $Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(p_j^{n_j})$. This shows that ϕ is injective.

Finally we prove that ϕ is surjective. Assume that $g_0(\gamma,\delta,y)$ is a given representative of $Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(N)$. We define γ_j and δ_j as the highest power of p_j that divides γ and δ respectively. Then we define $L_j = \gcd(\delta_j^2/\gamma_j, N_j/\gamma_j)$. Also let θ_j be defined as $\beta\theta \mod L_j$ and let $y_j = L_j + \theta_j$. It is enough to define θ_j modulo L_j because Lemmas 5.8 and 5.9 imply that if $y_j \equiv y_j' \mod L_j$ then $g_0(\gamma_j,\delta_j,y_j)$ is equivalent to $g_0(\gamma_j,\delta_j,y_j')$ in $Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(p_j^{n_j})$. Hence we have uniquely defined the representative $g_0(\gamma_j,\delta_j,y_j)$ up to equivalence in $Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(p_j^{n_j})$. It can be checked that $\phi((g_0(\gamma_j,\delta_j,y_j))_{j=1,\dots,m}) = g_0(\gamma,\delta,y)$ up to equivalence in $Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(N)$. Therefore ϕ is surjective and we are done. □

Proof of Corollary 3.4. Since $k \geq 6$ and even the Klingen–Eisenstein series defined in the statement of the Corollary have nice convergence properties. Let $\jmath(\alpha)$ and $\jmath(\beta)$ be two one-dimensional cusps for $\Gamma_0(N)$. Let

$$E_\alpha(z) = \sum_{\jmath(\xi)\in(\jmath(\alpha)Q(\mathbb{Q})\jmath(\alpha^{-1})\cap\Gamma_0(N))\backslash\Gamma_0(N)} f_\alpha(\alpha^{-1}\xi\langle Z\rangle^*)\det(j(\alpha^{-1}\xi, Z))^{-k},$$

be a Klingen–Eisenstein series associated to α. We have

$$\begin{aligned}(E_\alpha|_k\beta)(Z) &= \sum f_\alpha(\alpha^{-1}\xi\langle\beta\langle Z\rangle\rangle^*)\det(j(\alpha^{-1}\xi, \beta\langle Z\rangle))^{-k}\det(j(\beta, Z))^{-k}\\ &= \sum f_\alpha(\alpha^{-1}\xi\beta\langle Z\rangle^*)\det(j(\alpha^{-1}\xi\beta, Z))^{-k},\end{aligned}$$

where the sums are taken over $\jmath(\xi) \in (\jmath(\alpha)Q(\mathbb{Q})\jmath(\alpha^{-1}) \cap \Gamma_0(N))\backslash\Gamma_0(N)$. Next consider $\Phi(E_\alpha|_k\beta)(z) = \lim_{\lambda\to\infty}(E_\alpha|_k\beta)\left(\left[\begin{smallmatrix} z & \\ & i\lambda\end{smallmatrix}\right]\right)$ where Φ is the Siegel Φ operator defined earlier. The limit can be evaluated term by term because of nice convergence properties of the Eisenstein series. It follows from the proof of [Klingen 1990, Proposition 5, Chapter 5], that on taking the limit the only surviving terms are those with $\jmath(\alpha^{-1})\jmath(\xi)\jmath(\beta) \in Q(\mathbb{Q})$ with $\jmath(\xi) \in \Gamma_0(N)$. If $\jmath(\alpha)$ and $\jmath(\beta)$ are inequivalent cusps, then clearly no term survives and $\Phi(E_\alpha|_k\beta)(z) = 0$, whereas we see that $\Phi(E_\alpha|_k\alpha)(z) = f_\alpha(z)$. We have shown that each Eisenstein series is supported on a unique one-dimensional cusp. Further for a fixed one-dimensional cusp all the associated Klingen–Eisenstein series are clearly linearly independent. The corollary is now evident. □

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