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NONEXISTENCE RESULTS FOR SYSTEMS OF ELLIPTIC AND PARABOLIC DIFFERENTIAL INEQUALITIES IN EXTERIOR DOMAINS OF $\mathbb{R}^n$

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We present a unified approach for the investigation of nonexistence results of systems of elliptic and parabolic differential inequalities. Our results accord with those on elliptic differential inequalities given by Bidaut-Véron and Pohozaev. The results on systems of parabolic differential inequalities are new.

1. Introduction

In this paper, we study the nonexistence of nonnegative solutions to systems of the following elliptic and parabolic differential inequalities

$$\begin{cases} \Delta u + |x|^a v^p \le 0 & \text{in } \bar{D}^c, \\ \Delta v + |x|^b u^q \le 0 & \text{in } \bar{D}^c, \\ u(x) \ge f(x), \quad v(x) \ge g(x) & \text{on } \partial D, \end{cases} \tag{1-1}$$

and

$$\begin{cases} \Delta u - \partial_t u + |x|^a v^p \le 0 & \text{in } \bar{D}^c \times (0,\infty), \\ \Delta v - \partial_t v + |y|^b u^q \le 0 & \text{in } \bar{D}^c \times (0,\infty), \\ u(x,t) \ge f(x), \quad v(x,t) \ge g(x) & \text{on } \partial D \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \bar{D}^c, \end{cases} \tag{1-2}$$

where D is a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \ge 3$ containing the origin, and $\bar{D}^c = \mathbb{R}^n \setminus \bar{D}$. The exponents satisfy $a, b > -2$ and $p, q > 1$, and $f(x)$, $g(x)$ are $L^1(\partial D)$ nonnegative and positive somewhere functions, and $u_0(x)$, $v_0(x)$ are nonnegative functions.

It is well known that the nonexistence theorems for elliptic equations started from the seminal work by Gidas and Spruck [1981], where they proved the following results for the semilinear problem

$$\Delta u + u^p = 0, \quad \text{in } \mathbb{R}^n. \tag{1-3}$$

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If

$$1 < p < \frac{n+2}{n-2}, \tag{1-4}$$

then the only nonnegative solution of (1-3) is identically zero.

In 1986, Ni and Serrin showed that the exponent $(n+2)/(n-2)$ in (1-4) is critical; namely, if $p \geq (n+2)/(n-2)$, then there exist nontrivial positive solutions to (1-3). We refer to the papers [Ni and Serrin 1986a; 1986b] for more information.

In the study of equation (1-3) in the exterior domain $\mathbb{R}^n \setminus \overline{B_1(0)}$ instead of the entire Euclidean space $\mathbb{R}^n$, some incredible phenomena arise. Here $B_r(0)$ is the ball of radius r centered at the origin. This marvelous result is due to Bidaut-Véron [1989]: if

$$1 < p \leq \frac{n}{n-2}, \tag{1-5}$$

then the only nonnegative solution of (1-3) in the exterior domain is identically zero. Actually Bidaut-Véron [1989] obtained more generalized results on the problem $\Delta_m u + u^p = 0$ in the exterior domain under additional restrictions on m and p. Here to compare with Gidas and Spruck's result profitably, we only list the nonexistence result for $m = 2$. However, if $p > n/(n-2)$, the nonexistence result does not hold any more. A simple counterexample is given by the function

$$u(x) = \lambda |x|^{-2/(p-1)},$$

which is a well-defined solution to (1-3) in $\mathbb{R}^n \setminus \overline{B_1(0)}$, where

$$\lambda = (p-1)^{-2/(p-1)} \Big[2(n-2) \Big(p - \frac{n}{n-2} \Big) \Big]^{1/(p-1)}.$$

Let us turn our attention to the elliptic differential inequality case; namely, consider the problem

$$\Delta u + u^p \leq 0, \quad \text{in } \mathbb{R}^n, \tag{1-6}$$

with $n > 2$. Ni and Serrin [1986a] proved that if

$$1 < p \leq \frac{n}{n-2}, \tag{1-7}$$

then the only nonnegative solution of (1-6) is identically zero. For more elliptic differential inequality cases, we refer to the papers [Caristi et al. 2008; 2009; Mitidieri and Pohozaev 1998; 2001].

Bidaut-Véron and Pohozaev [2001] showed that in the exterior domain $\mathbb{R}^n \setminus \overline{B_1(0)}$, if under the same condition (1-7) as in the entire Euclidean space, then the only nonnegative solution of (1-6) in the exterior domain $\mathbb{R}^n \setminus \overline{B_1(0)}$ is identically zero. It is easy to see that in the inequality case, the critical exponents arising from the entire Euclidean space and exterior domain settings are the same. The difference between the entire Euclidean space and exterior domain vanishes when we move our focus from equation to differential inequality problems.

Now, let us provide some motivations from the point of view of parabolic equations. The study of critical exponents of the parabolic equation also has a long story. When D is empty, in a celebrated paper, Fujita [1966] proved that for the problem

$$\begin{cases} \partial_t u - \Delta u = u^p & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{1-8}$$

(1) If $1 < p < 1+2/n$ and $u_0 > 0$, then (1-8) possesses no global positive solution.

(2) If $p > 1+2/n$ and u_0 is smaller than a small Gaussian, then (1-8) has global solutions.

Usually, we call $1+2/n$ the Fujita exponent. The sharpness of $p = 1+2/n$ is more difficult. Several authors independently showed that $p = 1+2/n$ belongs to the blowup case; we refer to the papers [Aronson and Weinberger 1978; Hayakawa 1973; Kobayashi et al. 1977]. Let us replace $\mathbb{R}^n$ by $\bar{D}^c$ in (1-8) (here D is a bounded nonempty domain), and we have an additional boundary condition $u|_{\partial D} \equiv f(x) \geq 0$. If the boundary condition $f(x) \equiv 0$, Bandle and Levine [1989] proved the Fujita exponent is still $p = 1+2/n$ for (1-8). But, if the boundary condition $f(x)$ is not identically zero, Zhang found that the Fujita exponent for (1-8) will jump from $1+2/n$ to a much bigger value $1+2/(n-2)$; see [Zhang 2001].

Laptev [2003] considered the scalar case of (1-2) with the nonzero boundary condition

$$\partial_t u - \Delta u \geq |x|^a u^p, \qquad \bar{D}^c \times (0,\infty), \tag{1-9}$$

and obtained that if $1 < p < (n+1+a)/(n-1)$, then (1-9) has no nontrivial nonnegative global solutions.

Motivated by the above literature, we investigate systems of elliptic and parabolic differential inequalities. First, let us explain in which sense solutions of (1-2) are defined.

Definition 1.1. A nonnegative pair (u, v) is called a weak nonnegative global solution of the inequality system (1-2), if

(i) $\nabla_x u, \nabla_x v \in L^2_{loc}(\bar{D}^c)$;

(ii) For all compactly supported $\psi \in C^2(\bar{D}^c \times [0,\infty)) \cap C^1(D^c \times [0,\infty))$ vanishing on $\partial D \times [0,\infty)$, and for all $\tau \in [0,\infty)$, we have

$$\begin{cases} \int_0^\tau \int_{\bar{D}^c} [u\Delta\psi + u\partial_t\psi + |y|^a v^p(y,s)\psi]\, dy\, ds \\ \qquad - \int_0^\tau \int_{\partial D} f(\partial\psi/\partial n^+)\, dS_y\, ds - \int_{\bar{D}^c} u(x,\cdot)\psi(x,\cdot)|_0^\tau\, dx \leq 0, \\ \int_0^\tau \int_{\bar{D}^c} [v\Delta\psi + v\partial_t\psi + |y|^b u^q(y,s)\psi]\, dy\, ds \\ \qquad - \int_0^\tau \int_{\partial D} g(\partial\psi/\partial n^+)\, dS_y\, ds - \int_{\bar{D}^c} v(x,\cdot)\psi(x,\cdot)|_0^\tau\, dx \leq 0. \end{cases} \tag{1-10}$$

Here, n^+ means the outward unit normal of ∂D, relative to D^c, which is defined almost everywhere.

Throughout, when we say that (u, v) is a global positive solution of (1-2), we mean that $u, v \geq 0$ and $u(x,t)$, $v(x,t)$ are not identically zero for each $t > 0$.

Here are our main results:

Theorem 1.2. *Assume $p \geq q > 1$. If*

$$\max\left\{\frac{2p(1+q)+bp+a}{pq-1}, \frac{2q(1+p)+aq+b}{pq-1}\right\} > n, \tag{1-11}$$

then there exist no global positive solutions to (1-2).

Corollary 1.3. *Assume $p \geq q > 1$. If*

$$\max\left\{\frac{2p(1+q)+bp+a}{pq-1}, \frac{2q(1+p)+aq+b}{pq-1}\right\} > n, \tag{1-12}$$

then there exist no positive solutions to (1-1).

Theorem 1.2 and Corollary 1.3 require that D is not empty, since we technically depend on Proposition 2.1. Corollary 1.3 was also obtained by Bidaut-Véron and Pohozaev [2001]. We claim that our technique is quite different from the one in that work, where they investigated various elliptic inequalities, and their technique is to multiply the elliptic inequalities (1-1) by functions $u^\alpha\varphi$, $v^\beta\varphi$ and to obtain the integral estimates with respect to the polynomials of u, v near infinity, where φ has compact support in $\bar{D}^c$, and $\alpha, \beta < 0$. However, we mainly investigate the parabolic differential inequalities. As a byproduct, we obtain the same result for the elliptic problem. Our method, motivated by [Zhang 1998; 1999; 2001], is to show that the integrals of I_R, J_R in (2-9) and (2-10) will blow up in some selected fixed domain.

We also improve the result obtained by Laptev [2003]. When $u = v$, $p = q$, $a = b$, the system (1-2) is reduced to the scalar case (1-9). From Theorem 1.2, it is easy to obtain that if $1 < p < (n+a)/(n-2)$, then (1-9) admits no nontrivial nonnegative global solutions. Our exponent $(n+a)/(n-2)$ here is strictly bigger than $(n+1+a)/(n-1)$ which is obtained by Laptev. We claim that our method is also different from Laptev's, since his method is based on the test function approach, which was developed by Mitidieri and Pohozaev [1998; 2001].

Notation. The letters $C, C', C_0, C_1, \ldots$ denote positive constants whose values are unimportant and may vary at different occurrences.

2. Proof of Theorem 1.2

In this section, we show the proof of Theorem 1.2. Since every positive solution (u, v) of the elliptic problem (1-1) can also be considered as a global nontrivial positive solution of the parabolic inequality system (1-2), it suffices to show that the parabolic system (1-2) has no global positive solution, provided that

$$\max\left\{\frac{2p(1+q)+bp+a}{pq-1}, \frac{2q(1+p)+aq+b}{pq-1}\right\} > n.$$

Before presenting the proof, let us cite a result which is proved in [Zhang 2001].

Proposition 2.1. *Let $\zeta_i = \zeta_i(x,t)$, $i = 1, 2$ be the solution of the linear problem*

$$\begin{cases} \Delta\zeta - \partial_t\zeta = 0 & \text{in } \bar{D}^c \times (0,\infty), \\ \zeta(x,t) = f(x) \quad (\text{respectively } g(x)) & \text{on } \partial D \times (0,\infty), \\ \zeta(x,0) = 0 & \text{in } \bar{D}^c. \end{cases} \tag{2-1}$$

If $f(x)$, $g(x)$ are nonnegative and positive somewhere, then there exist positive constants C and R_0 such that

$$\zeta_1(x,t), \zeta_2(x,t) \geq \frac{C}{R^{n-2}}, \quad \text{if } R_0 \leq R \leq |x| \leq 2R,\ R^{4n} \leq t. \tag{2-2}$$

Now we step into the proof of Theorem 1.2.

Proof of Theorem 1.2. Let

$$\omega_1(x,t) := u(x,t) - \zeta_1(x,t), \quad \omega_2(x,t) := v(x,t) - \zeta_2(x,t), \tag{2-3}$$

From (1-2) and (2-1), we derive that ω_1, ω_2 satisfy the following problems:

$$\begin{cases} \Delta\omega_1 - \partial_t\omega_1 + |x|^a(\omega_2+\zeta_2)^p \leq 0, & \text{in } \bar{D}^c \times (0,\infty) \\ \omega_1(x,t) \geq 0, & \text{on } \partial D \times (0,\infty), \\ \omega_1(x,0) = u_0(x), & \text{in } \bar{D}^c, \end{cases} \tag{2-4}$$

and

$$\begin{cases} \Delta\omega_2 - \partial_t\omega_2 + |x|^b(\omega_1+\zeta_1)^q \leq 0, & \text{in } \bar{D}^c \times (0,\infty) \\ \omega_2(x,t) \geq 0, & \text{on } \partial D \times (0,\infty), \\ \omega_2(x,0) = v_0(x), & \text{in } \bar{D}^c. \end{cases} \tag{2-5}$$

Moreover, applying the maximum principle, we know that ω_1, ω_2 are nonnegative functions.

Since

$$|x|^a(\omega_1+\zeta_1)^q \geq |x|^a\omega_1^q + |x|^a\zeta_1^q, \quad |x|^b(\omega_2+\zeta_2)^p \geq |x|^b\omega_2^p + |x|^b\zeta_2^p,$$

we obtain that

$$\Delta\omega_1 - \partial_t\omega_1 + |x|^a\omega_2^p + |x|^a\zeta_2^p \leq 0, \tag{2-6}$$

and

$$\Delta\omega_2 - \partial_t\omega_2 + |x|^b\omega_1^q + |x|^b\zeta_1^q \leq 0. \tag{2-7}$$

Introduce two functions $\varphi, \eta \in C^\infty[0, \infty)$ which satisfy the following conditions:

(i) $0 \le \varphi \le 1$; $\quad \varphi(r) = 1,\ r \in [2, 3]$; $\quad \varphi(r) = 0,\ r \in [0, 1) \cup (4, \infty)$;

(ii) $|\varphi'(r)| \le C$; $\quad \varphi'(1) = \varphi'(4) = 0$; $\quad |\varphi''(r)| \le C$;

(iii) $0 \le \eta \le 1$; $\quad \eta(t) = 1,\ t \in \left[0, \frac{1}{4}\right]$; $\quad \eta(t) = 0,\ t \in [1, \infty)$; $\quad -C \le \eta'(t) \le 0$.

Since D is bounded, we can choose $R > 0$ large enough so that $D \subset B_R(0)$. Denote

$$\varphi_R(x) := \varphi\Big(\frac{|x|}{R}\Big), \quad \eta_R(t) := \eta\Big(\frac{t - R^{4n}}{R^2}\Big).$$

It is obvious that

$$\left|\frac{\partial \varphi_R}{\partial r}\right| \le \frac{C}{R}, \quad \left|\frac{\partial^2 \varphi_R}{\partial r^2}\right| \le \frac{C}{R^2}, \quad -\frac{C}{R^2} \le \eta_R'(t) \le 0, \tag{2-8}$$

and also

$$\frac{\partial \varphi_R(x)}{\partial r} = 0 \quad \text{for } |x| = R \text{ or } |x| = 4R.$$

Denote

$$Q_R := [B_{4R}(0) \setminus B_R(0)] \times [R^{4n}, R^{4n} + R^2],$$

and also

$$\psi_R(x, t) := \varphi_R(x)\eta_R(t).$$

Let us estimate the following two integrals:

$$I_R := \int_{Q_R} |x|^a \omega_2^p(x, t)\psi_R^{q'}(x, t)\, dx\, dt, \tag{2-9}$$

and

$$J_R := \int_{Q_R} |x|^b \omega_1^q(x, t)\psi_R^{q'}(x, t)\, dx\, dt, \tag{2-10}$$

where q' is Hölder conjugate to q, satisfying $1/q + 1/q' = 1$.

Since $\omega_1(x, t)$ is a nonnegative solution of (2-6), we obtain

$$I_R + \int_{Q_R} |x|^a \zeta_2^p \psi_R^{q'}(x, t)\, dx\, dt \le \int_{Q_R} [\partial_t \omega_1 - \Delta \omega_1]\psi_R^{q'}(x, t)\, dx\, dt,$$

Integration by parts yields

$$\begin{aligned}
I_R + \int_{Q_R} |x|^a \zeta_2^p \psi_R^{q'}(x,t)\,dx\,dt \le & \int_{B_{4R(0)\setminus B_R(0)}} \omega_1(x,\cdot)\psi_R^{q'}(x,\cdot)\Big|_{R^{4n}}^{R^{4n}+R^2} dx \\
& - q'\int_{Q_R} \omega_1(x,t)\varphi_R^{q'}(x)\eta_R^{q'-1}(t)\eta_R'(t)\,dx\,dt \\
& + \int_{R^{4n}}^{R^{4n}+R^2}\int_{\partial B_{4R}(0)} \omega_1(x,t)\frac{\partial\varphi_R^{q'}(x)}{\partial n}\eta_R^{q'}(t)\,dS_x\,dt \\
& - \int_{R^{4n}}^{R^{4n}+R^2}\int_{\partial B_{4R}(0)} \psi_R^{q'}\frac{\partial\omega_1}{\partial n}(x,t)\,dS_x\,dt \\
& - \int_{R^{4n}}^{R^{4n}+R^2}\int_{\partial B_R(0)} \omega_1(x,t)\frac{\partial\varphi_R^{q'}(x)}{\partial n}\eta_R^{q'}(t)\,dS_x\,dt \\
& + \int_{R^{4n}}^{R^{4n}+R^2}\int_{\partial B_R(0)} \psi_R^{q'}\frac{\partial\omega_1}{\partial n}(x,t)\,dS_x\,dt \\
& - \int_{Q_R} \omega_1(x,t)\Delta\varphi_R^{q'}(x)\eta_R^{q'}(t)\,dx\,dt.
\end{aligned}$$

Noting here that $-\partial/\partial n = \partial/\partial n^+$, $\omega_1(x, R^{4n}) \ge 0$,

$$\frac{\partial\varphi_R^{q'}}{\partial n} = q'\varphi_R^{q'-1}\frac{\partial\varphi_R}{\partial n} = 0 \quad \text{on } \partial B_R(0)\cup\partial B_{4R}(0)$$

and $\psi_R(x,t) = 0$ on the lateral boundary of Q_R, we obtain

$$\begin{aligned}
\text{(2-11)}\quad I_R + \int_{Q_R} |x|^a \zeta_2^p \psi_R^{q'}(x,t)dx dt \le & -q'\int_{Q_R} \omega_1(x,t)\varphi_R^{q'}(x)\eta_R^{q'-1}(t)\eta_R'(t)dxdt \\
& - \int_{Q_R} \omega_1(x,t)\Delta\varphi_R^{q'}(x)\eta_R^{q'}(t)dxdt.
\end{aligned}$$

Since $\Delta\varphi_R^{q'} = q'\varphi_R^{q'-1}\Delta\varphi_R + q'(q'-1)\varphi_R^{q'-2}|\nabla\varphi_R|^2$, combining with (2-11), we get

$$\begin{aligned}
\text{(2-12)}\quad I_R + \int_{Q_R} |x|^a \zeta_2^p \psi_R^{q'}\,dx\,dt \le & -q'\int_{Q_R} \omega_1(x,t)\varphi_R^{q'}(x)\eta_R^{q'-1}(t)\eta_R'(t)\,dx\,dt \\
& - q'\int_{Q_R} \omega_1(x,t)\varphi_R^{q'-1}(x)\Delta\varphi_R(x)\eta_R^{q'}(t)\,dx\,dt.
\end{aligned}$$

By the definition of φ_R and η_R, and applying Proposition 2.1, for large R, we obtain

$$\begin{aligned}
\int_{Q_R} |x|^a \zeta_2^p \psi_R^{q'}(x,t)\,dx\,dt & \ge \int_{R^{4n}}^{R^{4n}+R^2/4}\int_{B_{3R}(0)\setminus B_{2R}(0)} |x|^a \zeta_2^p\,dx\,dt \\
& \ge CR^{n+2+a-p(n-2)}.
\end{aligned}$$

It follows from (2-12) that

$$I_R + CR^{n+2+a-p(n-2)} \le -q' \int_{Q_R} \omega_1(x,t)\varphi_R^{q'}(x)\eta_R^{q'-1}(t)\eta_R'(t)\,dx\,dt - q' \int_{Q_R} \omega_1(x,t)\varphi_R^{q'-1}(x)\Delta\varphi_R(x)\eta_R^{q'}(t)\,dx\,dt. \tag{2-13}$$

Noting that φ_R is radial, we obtain $\Delta\varphi_R = \varphi_R'' + (n-1)/r\varphi_R'$. For large enough R,

$$|\Delta\varphi_R| \le \frac{C}{R^2}, \quad \text{for } x \in B_{4R}(0) \setminus B_R(0). \tag{2-14}$$

Combining (2-13) and (2-14), we obtain

$$\begin{aligned} I_R &+ CR^{n+2+a-p(n-2)} \\ &\le \frac{C}{R^2} \int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} \omega_1(x,t)\varphi_R^{q'}(x)\eta_R^{q'-1}(t)\,dx\,dt \\ &\quad + \frac{C}{R^2} \int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} \omega_1(x,t)\varphi_R^{q'-1}(x)\eta_R^{q'}(t)\,dx\,dt. \end{aligned}$$

According to the assumptions that $\varphi_R(x), \eta_R(t) \le 1$ and $\psi_R(x,t) = \varphi_R(x)\eta_R(t)$, we have $\varphi_R^{q'}(x)\eta_R^{q'-1}(t) \le \psi_R^{q'-1}(x,t)$, and $\varphi_R^{q'-1}(x)\eta_R^{q'}(t) \le \psi_R^{q'-1}(x,t)$. Applying the Hölder inequality to the above, we obtain

$$\begin{aligned} I_R &+ CR^{n+2+a-p(n-2)} \\ &\le \frac{C}{R^2} \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^b \omega_1^q \psi_R^{q(q'-1)}(x,t)\,dx\,dt \right]^{1/q} \\ &\qquad \times \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^{-bq'/q}\,dx\,dt \right]^{1/q'} \\ &\quad + \frac{C}{R^2} \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^b \omega_1^q \psi_R^{q(q'-1)}(x,t)\,dx\,dt \right]^{1/q} \\ &\qquad \times \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^{-bq'/q}\,dx\,dt \right]^{1/q'}. \end{aligned}$$

Hence

$$\begin{aligned} I_R &+ CR^{n+2+a-p(n-2)} \\ &\le C \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^b \omega_1^q \psi_R^{q'}(x,t)\,dx\,dt \right]^{\frac{1}{q}} R^{\frac{(n+2)(q-1)-b}{q}-2} \\ &\quad + C \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^b \omega_1^q \psi_R^{q'}(x,t)\,dx\,dt \right]^{\frac{1}{q}} R^{\frac{(n+2)(q-1)-b}{q}-2}, \end{aligned}$$

which yields

$$I_R + CR^{n+2+a-p(n-2)} \le C J_R^{\frac{1}{q}} R^{\frac{(n+2)(q-1)-b}{q}-2}, \tag{2-15}$$

where we have used the definition of J_R in (2-10).

Using the same arguments with J_R, we obtain an analogous inequality for J_R. Since $\omega_2(x,t)$ is a solution of (2-7), we have

$$J_R + \int_{Q_R} |x|^b \zeta_1^q \psi_R^{q'}(x,t)\,dx\,dt \le \int_{Q_R} [\partial_t \omega_2 - \Delta \omega_2] \psi_R^{q'}(x,t)\,dx\,dt.$$

It follows that

$$\begin{aligned} J_R + CR^{n+2+b-q(n-2)} &\le \frac{C}{R^2} \int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} \omega_2(x,t) \varphi_R^{q'} \eta_R^{q'-1}(t)\,dx\,dt \\ &\quad + \frac{C}{R^2} \int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} \omega_2(x,t) \varphi_R^{q'-1}(x) \eta_R^{q'}(t)\,dx\,dt. \end{aligned}$$

Applying the Hölder inequality, we obtain

$$\begin{aligned} &J_R + CR^{n+2+b-q(n-2)} \\ &\quad \le \frac{C}{R^2} \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^a \omega_2^p(x,t) \psi_R^{p(q'-1)}(x,t)\,dx\,dt \right]^{1/p} \\ &\qquad \times \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^{-ap'/p}\,dx\,dt \right]^{1/p'} \\ &\qquad + \frac{C}{R^2} \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^a \omega_2^p(x,t) \psi_R^{p(q'-1)}(x,t)\,dx\,dt \right]^{1/p} \\ &\qquad \times \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^{-ap'/p}\,dx\,dt \right]^{1/p'}, \end{aligned}$$

where $1/p + 1/p' = 1$. Using that $\psi_R(x,t) \le 1$, $p \ge q$ and $\psi_R^{p(q'-1)} \le \psi_R^{q(q'-1)} = \psi_R^{q'}$, we obtain

$$\begin{aligned} J_R + CR^{n+2+b-q(n-2)} &\le \frac{C}{R^2} \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^a \omega_2^p(x,t) \psi_R^{q'}(x,t)\,dx\,dt \right]^{1/p} \\ &\quad \times \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^{-ap'/p}\,dx\,dt \right]^{1/p'} \\ &\quad + \frac{C}{R^2} \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^a \omega_2^p(x,t) \psi_R^{q'}(x,t)\,dx\,dt \right]^{1/p} \\ &\quad \times \left[\int_{R^{4n}}^{R^{4n}+R^2} \int_{B_{4R}(0)\setminus B_R(0)} |x|^{-ap'/p}\,dx\,dt \right]^{1/p'}. \end{aligned}$$

which is

$$J_R + CR^{n+2+b-q(n-2)} \le C I_R^{\frac{1}{p}} R^{\frac{(n+2)(p-1)-a}{p}-2}. \tag{2-16}$$

Case 1: If

$$\frac{2p(q+1)+bp+a}{pq-1} = \max\left\{\frac{2p(q+1)+bp+a}{pq-1}, \frac{2q(1+p)+aq+b}{pq-1}\right\} > n.$$

Combining (2-15) and (2-16), we obtain

$$J_R + C_0 R^{n+2+b-q(n-2)} \le C_1 J_R^{\frac{1}{pq}} R^{\frac{(n+2)(pq-1)-b-aq}{pq}-\frac{2(1+p)}{p}}. \tag{2-17}$$

Denote

$$\begin{aligned} k_0 &:= n+2+b-q(n-2), \\ k_1 &:= \frac{(n+2)(pq-1)-b-aq}{pq} - \frac{2(1+p)}{p}. \end{aligned} \tag{2-18}$$

From (2-17), we obtain

$$J_R \ge \left(\frac{C_0}{C_1}\right)^{pq} R^{k_0(pq)-k_1(pq)}. \tag{2-19}$$

Substituting (2-19) into the left-hand side of (2-17), we obtain

$$J_R \ge \frac{C_0^{(pq)^q}}{C_1^{pq+(pq)^2}} R^{k_0(pq)^2-k_1(pq)^2-k_1(pq)}.$$

Repeating the above procedure, we obtain for any integer $j > 1$

$$J_R \ge \frac{C_0^{(pq)^j}}{C_1^{pq+\cdots+(pq)^j}} R^{k_0(pq)^j-k_1[pq+\cdots+(pq)^j]}. \tag{2-20}$$

The exponent of R in the right-hand side of (2-20) gives

$$\begin{aligned} k_0(pq)^j - k_1[pq+\cdots+(pq)^j] &= k_0(pq)^j - k_1 pq\frac{(pq)^j-1}{pq-1} \\ &= (pq)^j\left[k_0 - \frac{k_1 pq}{pq-1}\right] + \frac{k_1 pq}{pq-1}. \end{aligned}$$

From (2-20), we obtain

$$\begin{aligned} J_R &\ge C_2^{(pq)^j} R^{(pq)^j[k_0-k_1pq/(pq-1)]} R^{k_1pq/(pq-1)} \\ &= (C_2 R^{k_0-k_1pq/(pq-1)})^{(pq)^j} R^{k_1pq/(pq-1)}. \end{aligned} \tag{2-21}$$

Combining with (2-18), we obtain

$$\begin{aligned}k_0-\frac{k_1pq}{pq-1}&=n+2+b-q(n-2)-\left[\frac{(n+2)(pq-1)-b-aq}{pq}-\frac{2(1+p)}{p}\right]\frac{pq}{pq-1}\\&=\frac{2pq(1+q)+bpq+aq-nq(pq-1)}{pq-1}.\end{aligned}$$

Obviously, if

$$\frac{2p(1+q)+bp+a}{pq-1}>n,$$

then $k_0-k_1pq/(pq-1)>0$. Whence, if R is chosen large enough, we have $C_2R^{k_0-k_1pq/(pq-1)}>1$.

From (2-21), for fixed R, letting $j\to\infty$, we obtain

$$J_R=\int_{Q_R}|x|^b\omega_1^q(x,t)\psi_R^{q'}(x,t)\,dx\,dt=\infty. \tag{2-22}$$

However, the above contradicts (2-17), since (2-17) implies that $J_R\le CR^{k_1pq/(pq-1)}$. Moreover, by the definition of J_R, (2-22) means that $u(x,t)$ has to blow up when $t\le R^{4n}+R^2$.

Case 2: If

$$\frac{2q(1+p)+aq+b}{pq-1}=\max\left\{\frac{2p(q+1)+bp+a}{pq-1},\frac{2q(1+p)+aq+b}{pq-1}\right\}>n,$$

one can argue in the same way as with I_R and obtain the same contradiction. Hence, we finish the proof. □

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References

[Aronson and Weinberger 1978] D. G. Aronson and H. F. Weinberger, "Multidimensional nonlinear diffusion arising in population genetics", *Adv. in Math.* **30**:1 (1978), 33–76. MR Zbl

[Bandle and Levine 1989] C. Bandle and H. A. Levine, "On the existence and nonexistence of global solutions of reaction-diffusion equations in sectorial domains", *Trans. Amer. Math. Soc.* **316**:2 (1989), 595–622. MR Zbl

[Bidaut-Véron 1989] M.-F. Bidaut-Véron, "Local and global behavior of solutions of quasilinear equations of Emden–Fowler type", *Arch. Rational Mech. Anal.* **107**:4 (1989), 293–324. MR Zbl

[Bidaut-Véron and Pohozaev 2001] M.-F. Bidaut-Véron and S. Pohozaev, "Nonexistence results and estimates for some nonlinear elliptic problems", *J. Anal. Math.* **84** (2001), 1–49. MR Zbl

[Caristi et al. 2008] G. Caristi, L. D'Ambrosio, and E. Mitidieri, "Liouville theorems for some nonlinear inequalities", *Tr. Mat. Inst. Steklova* **260**:1 (2008), 97–118. In Russian; translated in *Proc. Steklov Inst. Math.* **260**:1 (2008), 90–111. MR Zbl

[Caristi et al. 2009] G. Caristi, E. Mitidieri, and S. I. Pohozaev, "Liouville theorems for quasilinear elliptic inequalities", *Dokl. Acad. Nauk.* **424**:6 (2009), 741–747. In Russian; translation in *Dokl. Math.* **79**:1 (2009) 118–124. MR Zbl

[Fujita 1966] H. Fujita, "On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$", *J. Fac. Sci. Univ. Tokyo Sect. I* **13** (1966), 109–124. MR Zbl

[Gidas and Spruck 1981] B. Gidas and J. Spruck, "Global and local behavior of positive solutions of nonlinear elliptic equations", *Comm. Pure Appl. Math.* **34**:4 (1981), 525–598. MR Zbl

[Hayakawa 1973] K. Hayakawa, "On nonexistence of global solutions of some semilinear parabolic differential equations", *Proc. Japan Acad.* **49** (1973), 503–505. MR Zbl

[Kobayashi et al. 1977] K. Kobayashi, T. Sirao, and H. Tanaka, "On the growing up problem for semilinear heat equations", *J. Math. Soc. Japan* **29**:3 (1977), 407–424. MR

[Laptev 2003] G. G. Laptev, "Nonexistence results for higher-order evolution partial differential inequalities", *Proc. Amer. Math. Soc.* **131**:2 (2003), 415–423. MR Zbl

[Mitidieri and Pohozaev 1998] E. Mitidieri and S. I. Pohozaev, "Absence of global positive solutions of quasilinear elliptic inequalities", *Dokl. Akad. Nauk* **359**:4 (1998), 456–460. In Russian; translated in *Dokl. Math.* **57**:2 (1998), 250–253. MR Zbl

[Mitidieri and Pohozaev 2001] E. Mitidieri and S. I. Pohozaev, "A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities", *Tr. Mat. Inst. Steklova* **234** (2001), 3–383. In Russian; translation in *Proc. Steklov Inst. Math.* **234**:3 (2001), 1–362. MR Zbl

[Ni and Serrin 1986a] W.-M. Ni and J. Serrin, "Existence and non-existence theorems for ground states of quasilinear partial differential equations: the anomalous case", *Atti Conv. Lincei* **77** (1986), 231–257.

[Ni and Serrin 1986b] W.-M. Ni and J. Serrin, "Nonexistence theorems for singular solutions of quasilinear partial differential equations", *Comm. Pure Appl. Math.* **39**:3 (1986), 379–399. MR Zbl

[Zhang 1998] Q. S. Zhang, "Blow up and global existence of solutions to an inhomogeneous parabolic system", *J. Differential Equations* **147**:1 (1998), 155–183. MR Zbl

[Zhang 1999] Q. S. Zhang, "Blow-up results for nonlinear parabolic equations on manifolds", *Duke Math. J.* **97**:3 (1999), 515–539. MR Zbl

[Zhang 2001] Q. S. Zhang, "A general blow-up result on nonlinear boundary-value problems on exterior domains", *Proc. Roy. Soc. Edinburgh Sect. A* **131**:2 (2001), 451–475. MR Zbl

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