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DAVID E. BARRETT AND DUSTY E. GRUNDMEIER

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We characterize sums of CR functions from competing CR structures in two scenarios. In one scenario the structures are conjugate and we are adding to the theory of pluriharmonic boundary values. In the second scenario the structures are related by projective duality considerations. In both cases we provide explicit vector field-based characterizations for two-dimensional circular domains satisfying natural convexity conditions.

1. Introduction

The Dirichlet problem for pluriharmonic functions is a natural problem in several complex variables with a long history going back at least to Amoroso [1912], Severi [1931], Wirtinger [1927], and others. It was known early on that the problem is not solvable for general boundary data, so we may try to characterize the admissible boundary values with a system of tangential partial differential operators. This was first done for the ball by Bedford [1974]; see Section 2.1 for details. More precisely, given a bounded domain Ω with smooth boundary S, we seek a system $\mathcal{L}$ of partial differential operators tangential to S such that a function $u \in \mathcal{C}^\infty(S, \mathbb{C})$ satisfies $\mathcal{L}u = 0$ if and only if there exists $U \in \mathcal{C}^\infty(\overline{\Omega})$ such that $U|_S = u$ and $\partial\bar{\partial}U = 0$. The problem may also be considered locally.

While natural in its own right, this problem also arises in less direct fashion in many areas of complex analysis and geometry. For instance, this problem plays a fundamental role in Graham's work [1983] on the Bergman Laplacian, Lee's work [1988] on pseudo-Einstein structures, and Case, Chanillo, and Yang's work [Case et al. 2016] on CR Paneitz operators. From another point of view, the existence of nontrivial restrictions on pluriharmonic boundary values points to the need to look elsewhere (such as to the Monge–Ampère equations studied in [Bedford and Taylor 1976]) for Dirichlet problems solvable for general boundary data.

The pluriharmonic boundary value problem is closely related to the problem of characterizing sums of CR functions from different, competing CR structures;

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indeed, when the competing CR structures are conjugate then these problems coincide (in simply connected settings); see Propositions 3 and 4 below. Another natural construction leading to competing CR structures arises from the study of projective duality (see Section 3 or [Barrett 2016] for precise definitions).

In each of these two scenarios, we precisely characterize sums of CR functions from the two competing CR structures in the setting of two-dimensional circular domains satisfying appropriate convexity conditions. For conjugate structures we assume strong pseudoconvexity; our result appears as Theorem A below. In the projective duality scenario we assume strong convexity (the correct assumption without the circularity assumption would be strong $\mathbb{C}$-convexity, but these notions coincide in the circular case; see Section 3.1), and the main result appears as Theorem B below (with an expanded version appearing later in Section 3.2). Our techniques for these two related problems are interconnected to a surprising extent, and the reader will notice that the projective dual scenario actually turns out to have more structure and symmetry.

Theorem A. *Let $S \subset \mathbb{C}^2$ be a strongly pseudoconvex circular hypersurface. Then there exist nowhere-vanishing tangential vector fields X, Y on S satisfying the following conditions*:

(1-1a) *If u is a smooth function on a relatively open subset of S, then u is CR if and only if $Xu = 0$.*

(1-1b) *If u is a smooth function on a relatively open subset of S, then u is CR if and only if $Y\bar{u} = 0$.*

(1-1c) *If S is compact, then a smooth function u on S is a pluriharmonic boundary value (in the sense of Proposition 3 below) if and only if $XXYu = 0$.*

(1-1d) *A smooth function u on a relatively open subset of S is a pluriharmonic boundary value (in the sense of Proposition 4 below) if and only if $XXYu = 0 = \overline{XXY}u$.*

Theorem B. *Let $S \subset \mathbb{C}^2$ be a strongly convex circular hypersurface. Then there exist nowhere-vanishing tangential vector fields X, T on S satisfying the following conditions*:

(1-2a) *If u is a smooth function on a relatively open subset of S, then u is CR if and only if $Xu = 0$.*

(1-2b) *If u is a smooth function on a relatively open subset of S, then u is dual-CR if and only if $Tu = 0$.*

(1-2c) *If S is compact, then a smooth function u on S is the sum of a CR function and a dual-CR function if and only if $XXTu = 0$.*

(1-2d) *If S is simply connected (but not necessarily compact), then a smooth function u on S is the sum of a CR function and a dual-CR function if and only if $XXTu = 0 = TTXu$.*

This paper is organized as follows. In Section 2 we focus on the case of conjugate CR structures (the pluriharmonic case). In Section 3 we study the competing CR structures coming from projective duality. In Section 4 we prove Theorem B, while Theorem A is proved in Section 5. The final Section 6 includes a discussion of uniqueness issues.

2. Conjugate structures

2.1. *Results on the ball.* Early work focused on the case of the ball B^n in $\mathbb{C}^n$. In particular, Nirenberg observed that there is no second-order system of differential operators tangent to S^3 that exactly characterize pluriharmonic functions (see Section 6.2 for more details). Third-order characterizations were developed by Bedford in the global case and Audibert in the local case (which requires stronger conditions). To state these results, we define the tangential operators

$$L_{kl} = z_k \frac{\partial}{\partial \bar{z}_l} - z_l \frac{\partial}{\partial \bar{z}_k}, \qquad \overline{L_{kl}} = \bar{z}_k \frac{\partial}{\partial z_l} - \bar{z}_l \frac{\partial}{\partial z_k} \quad \text{for } 1 \le k, l \le n. \tag{2-1}$$

Theorem 1 [Bedford 1974]. *Let u be smooth on S^{2n-1}. Then*

$$\overline{L_{kl}}\,\overline{L_{kl}}\,L_{kl}u = 0$$

for $1 \le k, l \le n$ if and only if u extends to a pluriharmonic function on B^n.

Theorem 2 [Audibert 1977]. *Let S be a relatively open subset of S^{2n-1}, and let u be smooth on S. Then*

$$L_{jk}L_{lm}\overline{L_{rs}}u = 0 = \overline{L_{jk}}\,\overline{L_{lm}}\,L_{rs}u$$

for $1 \le j, k, l, m, r, s \le n$ if and only if u extends to a pluriharmonic function on a one-sided neighborhood of S.

For a treatment of both of these results along with further details and examples, see §18.3 of [Rudin 1980].

2.2. *Other results.* Laville [1977; 1984] also gave a fourth order operator to solve the global problem. Bedford and Federbush [1974] solved the local problem in the more general setting where $b\Omega$ has nonzero Levi form at some point. Later Bedford [1980] used the induced boundary complex $(\partial\bar{\partial})_b$ to solve the local problem in certain settings. In Lee's work [1988] on pseudo-Einstein structures, he gives a characterization for abstract CR manifolds using third order pseudohermitian covariant derivatives. Case, Chanillo, and Yang [Case et al. 2016] study when the kernel of the CR Paneitz operator characterizes CR-pluriharmonic functions.

2.3. ***Relation to decomposition on the boundary.*** Outside the proof of Theorem 30 below, all forms, functions, and submanifolds will be assumed $\mathcal{C}^\infty$-smooth.

Proposition 3. *Let $S \subset \mathbb{C}^n$ be a compact, connected and simply connected real hypersurface, and let Ω be the bounded domain with boundary S. Then for $u : S \to \mathbb{C}$ the following conditions are equivalent*:

(2-2a) *u extends to a (smooth) function U on $\overline{\Omega}$ that is pluriharmonic on Ω;*

(2-2b) *u is the sum of a CR function and a conjugate-CR function.*

Proof. In the proof that (2-2a) implies (2-2b), the CR term is the restriction to S of an antiderivative for ∂U on a simply connected one-sided neighborhood of S, and the conjugate-CR term is the restriction to S of an antiderivative for $\bar{\partial} U$ on a one-sided neighborhood of S (adjusting one term by a constant as needed).

To see that (2-2b) implies (2-2a), we use the global CR extension result [Hörmander 1990, Theorem 2.3.2] to extend the terms to holomorphic and conjugate-holomorphic functions, respectively; U is then the sum of the extensions. □

Proposition 4. *Let $S \subset \mathbb{C}^n$ be a simply connected, strongly pseudoconvex real hypersurface. Then for $u : S \to \mathbb{C}$ the following conditions are equivalent*:

(2-3a) *there is an open subset W of $\mathbb{C}^n$ with $S \subset bW$ (with W lying locally on the pseudoconvex side of S) so that u extends to a (smooth) function U on $W \cup S$ that is pluriharmonic on W;*

(2-3b) *u is the sum of a CR function and a conjugate-CR function.*

Proof. The proof follows the proof of Proposition 3 above, replacing the global CR extension result by the Hans Lewy local CR extension result as stated in [Boggess 1991, Section 14.1, Theorem 1]. □

3. Projective dual structures

3.1. ***Projective dual hypersurfaces.*** Let $S \subset \mathbb{C}^n$ be an oriented real hypersurface with defining function ρ. Then S is said to be *strongly $\mathbb{C}$-convex* if S is locally equivalent via a projective transformation (that is, via an automorphism of projective space) to a strongly convex hypersurface; this condition is equivalent to either of the following two equivalent conditions:

(3-1a) the second fundamental form for S is positive definite on the maximal complex subspace $H_z S$ of each $T_z S$;

(3-1b) the complex tangent (affine) hyperplanes for S lie to one side (the "concave side") of S near the point of tangency with minimal order of contact.

Theorem 5. *When S is compact and strongly $\mathbb{C}$-convex the complex tangent hyperplanes for S are in fact disjoint from the domain bounded by S.*

Proof. [Andersson et al. 2004, §2.5]. □

We note that strongly $\mathbb{C}$-convex hypersurfaces are also strongly pseudoconvex.

A circular hypersurface (that is, a hypersurface invariant under rotations $z \mapsto e^{i\theta} z$) is strongly $\mathbb{C}$-convex if and only if it is strongly convex [Černe 2002, Proposition 3.7].

The proper general context for the notion of strong $\mathbb{C}$-convexity is in the study of real hypersurfaces in complex projective space $\mathbb{CP}^n$ (see for example [Barrett 2016] and [Andersson et al. 2004]).

We specialize now to the two-dimensional case.

Lemma 6. *Let $S \subset \mathbb{C}^2$ be a compact strongly $\mathbb{C}$-convex hypersurface enclosing the origin. Then there is a uniquely determined map*

$$\mathfrak{D} : S \to \mathbb{C}^2 \setminus \{0\}, \quad z \mapsto w(z) = (w_1(z), w_2(z))$$

satisfying

(3-2a) $z_1 w_1 + z_2 w_2 = 1$ *on S;*

(3-2b) *the vector field*

$$Y \stackrel{\text{def}}{=} w_2 \frac{\partial}{\partial z_1} - w_1 \frac{\partial}{\partial z_2}$$

is tangent to S. Moreover, Y annihilates conjugate-CR functions on any relatively open subset of S.

Proof. It is easy to check that (3-2a) and (3-2b) force

$$w_1(z) = \frac{\frac{\partial \rho}{\partial z_1}}{z_1 \frac{\partial \rho}{\partial z_1} + z_2 \frac{\partial \rho}{\partial z_2}}, \quad w_2(z) = \frac{\frac{\partial \rho}{\partial z_2}}{z_1 \frac{\partial \rho}{\partial z_1} + z_2 \frac{\partial \rho}{\partial z_2}}.$$

establishing uniqueness. Existence follows provided that the denominators do not vanish, but the vanishing of the denominators occurs precisely when the complex tangent line for S at z passes through the origin, and Theorem 5 above guarantees that this does not occur under the given hypotheses. □

Remark 7. It is clear from the proof that the conclusions of Lemma 6 also hold under the assumption that S is a (not necessarily compact) hypersurface satisfying

(3-3) no complex tangent line for S passes through the origin.

Remark 8. Any tangential vector field annihilating conjugate-CR functions will be a scalar multiple of Y.

Remark 9. The complex line tangent to S at z is given by

$$\{\zeta \in \mathbb{C}^2 : w_1(z)\zeta_1 + w_2(z)\zeta_2 = 1\}. \tag{3-4}$$

Remark 10. The maximal complex subspace $H_z S$ of each $T_z S$ is annihilated by the form $w_1\,dz_1 + w_2\,dz_2$.

Proposition 11. *For S strongly $\mathbb{C}$-convex satisfying (3-3), the map $\mathfrak{D}$ is a local diffeomorphism onto an immersed strongly $\mathbb{C}$-convex hypersurface S^*, with each maximal complex subspace $H_z S$ of $T_z S$ mapped (not $\mathbb{C}$-linearly) by $\mathfrak{D}'_z$ onto the corresponding maximal complex subspace of $H_{w(z)} S^*$. For S strongly $\mathbb{C}$-convex and compact, S^* is an embedded strongly $\mathbb{C}$-convex hypersurface and $\mathfrak{D}$ is a diffeomorphism.*

Proof. [Barrett 2016, §6], [Andersson et al. 2004, §2.5]. □

For S strongly $\mathbb{C}$-convex satisfying (3-3) we may extend $\mathfrak{D}$ to a smooth map on an open set in $\mathbb{C}^2$; the extended map $\mathfrak{D}^\star$ will be a local diffeomorphism in some neighborhood U of S. We may then define vector fields $\partial/\partial w_1$, $\partial/\partial w_2$, $\partial/\partial \overline{w}_1$, $\partial/\partial \overline{w}_2$ on U by applying $((\mathfrak{D}^\star)^{-1})'$ to the corresponding vector fields on $\mathfrak{D}^\star(U)$; these newly defined vector fields will depend on the choice of the extension $\mathfrak{D}^\star$.

Lemma 12. *The nonvanishing vector field*

$$V \stackrel{\text{def}}{=} z_2 \frac{\partial}{\partial w_1} - z_1 \frac{\partial}{\partial w_2}$$

is tangent to S and is independent of the choice of the extension $\mathfrak{D}^\star$.

Proof. From (3-2a) we have

$$0 = d(z_1 w_1 + z_2 w_2) = z_1\,dw_1 + z_2\,dw_2 + w_1\,dz_1 + w_2\,dz_2 \quad \text{on } T_z S.$$

From Remark 10 we deduce that the null space in $T_z\mathbb{C}^2$ of $z_1\,dw_1 + z_2\,dw_2$ is precisely the maximal complex subspace $H_z S$ of $T_z S$ (and moreover the null space in $(T_z\mathbb{C}^2) \otimes \mathbb{C}$ of $z_1\,dw_1 + z_2\,dw_2$ is precisely $(H_z S) \otimes \mathbb{C}$). If we apply $z_1\,dw_1 + z_2\,dw_2$ to V we obtain

$$z_1 \cdot V w_1 + z_2 \cdot V w_2 = z_1 \cdot z_2 - z_2 \cdot z_1 = 0$$

showing that V takes values in $(H_z S) \otimes \mathbb{C}$ and is thus tangential.

If an alternate tangential vector field $\tilde{V}$ is constructed with the use of an alternate extension $\widetilde{\mathfrak{D}}^\star$ of $\mathfrak{D}$, then

$$\tilde{V} w_j = \pm z_{3-j} = V w_j, \quad \tilde{V} \overline{w}_j = 0 = V \overline{w}_j$$

along S, so $\tilde{V} = V$ along S. □

Definition 13. A function u on a relatively open subset of S will be called *dual-CR* if $\overline{V} u = 0$.

Example 14. If S is the unit sphere in $\mathbb{C}^2$, then $w(z) = \bar{z}$ and the set of dual-CR functions on S coincides with the set of conjugate-CR functions on S.

The set of dual-CR functions will only rarely coincide with the set of conjugate-CR functions as we see from the following two related results.

Theorem 15. *If S is a compact strongly $\mathbb{C}$-convex hypersurface in $\mathbb{C}^2$, then the set of dual-CR functions on S will coincide with the set of conjugate-CR functions on S if and only if S is a complex affine image of the unit sphere.*

Theorem 16. *If S is a strongly $\mathbb{C}$-convex hypersurface in $\mathbb{C}^2$, then the set of dual-CR functions on S will coincide with the set of conjugate-CR functions on S if and only if S is locally the image of a relatively open subset of the unit sphere by a projective transformation.*

For proofs of these results see [Jensen 1983], [Detraz and Trépreau 1990], and [Bolt 2008].

Remark 17. The constructions of the vector fields Y and V transform naturally under complex affine mappings of S. The construction of the dual-CR structure transforms naturally under projective transformation of S. (See for example [Barrett 2016, §6].)

Lemma 18. *Relations of the form*

$$V = \chi Y + \sigma \bar{Y}, \quad Y = \kappa V + \xi \bar{V}$$

hold along S with σ and ξ nowhere vanishing.

Proof. This follows from the following facts:

- V, $\bar{V}$, Y and $\bar{Y}$ all take values in the two-dimensional space $(H_z S) \otimes \mathbb{C}$;
- V and $\bar{V}$ are $\mathbb{C}$-linearly independent, as are Y and $\bar{Y}$;
- the map $\mathcal{D}'_z : (H_z S) \otimes \mathbb{C} \to (H_z S^*) \otimes \mathbb{C}$ is not $\mathbb{C}$-linear (see Proposition 11). □

Lemma 19. *If f_1, f_2 are CR functions and g_1, g_2 are dual-CR functions on a connected relatively open subset W of S with $f_1+g_1 = f_2+g_2$, then $g_2-g_1 = f_1-f_2$ is constant.*

Proof. From Lemma 18 we deduce that the directional derivatives of $g_2-g_1 = f_1-f_2$ vanish in every direction belonging to the maximal complex subspace of TS. Applying one Lie bracket we find that in fact all directional derivatives along S of $g_2 - g_1 = f_1 - f_2$ vanish. □

Corollary 20. *If W is a simply connected relatively open subset of S and u is a function on W that is locally decomposable as the sum of a CR function and a dual-CR function, then u is decomposable on all of W as the sum of a CR function and a dual-CR function.*

3.2. ***Circular hypersurfaces in*** $\mathbb{C}^2$**.** We begin the section with an expanded restatement of the main theorem in the projective setting.

Theorem B [expanded statement]. *Let $S \subset \mathbb{C}^2$ be a strongly* ($\mathbb{C}$-)*convex circular hypersurface. Then there exist scalar functions ϕ and ψ on S so that the vector fields*

$$X = V + \phi \overline{V}, \tag{3-5a}$$

$$T = Y + \psi \overline{Y} \tag{3-5b}$$

satisfy the following conditions.

(3-6a) *If u is a smooth function on a relatively open subset of S, then u is CR if and only if $Xu = 0$; equivalently, X is a nonvanishing scalar multiple $\alpha\overline{Y}$ of $\overline{Y}$.*

(3-6b) *If u is a smooth function on a relatively open subset of S, then u is dual-CR if and only if $Tu = 0$; equivalently, T is a nonvanishing scalar multiple $\beta\overline{V}$ of $\overline{V}$.*

(3-6c) *If S is compact, then a smooth function u on S is the sum of a CR function and a dual-CR function if and only if $XXTu = 0$.*

(3-6d) *If S is simply connected (but not necessarily compact), then a smooth function u on S is the sum of a CR function and a dual-CR function if and only if $XXTu = 0 = TTXu$.*

As we shall see the vector field X in Theorem B will also work as the vector field X in Theorem A.

Example 21 (cf. [Audibert 1977]). The function z_1/w_2 satisfies $XXT(z_1/w_2) = 0$ but is not globally defined. Since $TTX(z_1/w_2) = 2 \neq 0$, this function is not locally the sum of a CR function and a dual-CR function.

Conditions (3-5a), (3-6a) and (3-6b) uniquely determine X and T. See Section 6.1 for some discussion of what can happen without condition (3-5a).

4. Proof of Theorem B

To prove Theorem B we start by consulting Lemma 18 and note that (3-5a), (3-6a) and (3-6b) will hold if we set

$$\alpha = 1/\bar{\xi}, \quad \beta = 1/\bar{\sigma}, \quad \phi = \bar{\kappa}/\bar{\xi}, \quad \psi = \overline{\chi}/\bar{\sigma};$$

it remains to check (3-6c) and (3-6d).

We note for future reference and the reader's convenience that

$$
\begin{array}{llll}
Xw_1 = z_2, & Xw_2 = -z_1, & \bar{Y}w_1 = \bar{\xi} z_2, & \bar{Y}w_2 = -\bar{\xi} z_1, \\
X\overline{w}_1 = \phi \bar{z}_2, & X\overline{w}_2 = -\phi \bar{z}_1, & Xz_1 = \bar{Y}z_1 = 0, & Xz_2 = \bar{Y}z_2 = 0, \\
X\bar{z}_1 = \alpha \overline{w}_2, & X\bar{z}_2 = -\alpha \overline{w}_1, & Tz_1 = w_2, & Tz_2 = -w_1, \\
\bar{V}z_1 = \bar{\sigma} w_2, & \bar{V}z_2 = -\bar{\sigma} w_1, & T\bar{z}_1 = \psi \overline{w}_2, & T\bar{z}_2 = -\psi \overline{w}_1, \\
Tw_1 = \bar{V}w_1 = 0, & Tw_2 = \bar{V}w_2 = 0, & T\overline{w}_1 = \beta z_2, & T\overline{w}_2 = -\beta z_1.
\end{array}
\tag{4-1}
$$

Lemma 22.

$$
\begin{aligned}
[Y, \bar{Y}] &= \bar{\xi}\Big(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}\Big) - \xi\Big(\bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_2}\Big) \\
[V, \bar{V}] &= \bar{\sigma}\Big(w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2}\Big) - \sigma\Big(\overline{w}_1 \frac{\partial}{\partial \overline{w}_1} + \overline{w}_2 \frac{\partial}{\partial \overline{w}_2}\Big).
\end{aligned}
$$

Proof. The first statement follows from

$$
[Y, \bar{Y}] = (Y\overline{w}_2)\frac{\partial}{\partial \bar{z}_1} - (Y\overline{w}_1)\frac{\partial}{\partial \bar{z}_2} - (\bar{Y}w_2)\frac{\partial}{\partial z_1} + (\bar{Y}w_1)\frac{\partial}{\partial z_2}
$$

along with (4-1).

The proof of the second statement is similar. □

We note that the assumption that S is circular has not been used so far in this section. We now bring it into play by introducing the real tangential vector field

$$
R \overset{\text{def}}{=} i\Big(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2}\Big)
$$

generating the rotations of $z \mapsto e^{i\theta} z$ of S.

Lemma 23. *The following equalities hold.*

(4-2a) $\bar{\xi} = \xi$.

(4-2b) $\bar{\sigma} = \sigma$.

(4-2c) $\bar{\alpha} = \alpha$.

(4-2d) $\bar{\beta} = \beta$.

(4-2e) $R = -i\Big(w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2} - \overline{w}_1 \frac{\partial}{\partial \overline{w}_1} - \overline{w}_2 \frac{\partial}{\partial \overline{w}_2}\Big)$.

(4-2f) $[Y, \bar{Y}] = -i\xi R$.

(4-2g) $[V, \bar{V}] = i\sigma R$.

(4-2h) $[X, Y] = iR - (Y\alpha)\bar{Y}$.

Proof. We start by considering the tangential vector field

$$
[Y, \bar{Y}] + i\xi R = (\bar{\xi} - \xi)\Big(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}\Big);
$$

if (4-2a) fails, then $z_1\,\partial/\partial z_1 + z_2\,\partial/\partial z_2$ is a nonvanishing holomorphic tangential vector field on some nonempty relatively open subset of S, contradicting the strong pseudoconvexity of S.

To prove (4-2e) we first note from Lemma 6 that $w(e^{i\theta}z) = e^{-i\theta}w(z)$; differentiation with respect to θ yields (4-2e).

The proof of (4-2a) now may be adapted to prove (4-2b). (4-2c) and (4-2d) follow immediately.

Using Lemma 22 in combination with (4-2a) and (4-2b) we obtain (4-2f) and (4-2g).

From (3-6a) and (4-2f) we obtain (4-5b). □

Lemma 24. $$[X, T] = iR.$$

Proof. On the one hand,

$$\begin{aligned}[X, T] = [V + \phi\bar{V}, \beta\bar{V}] &= ((V + \phi\bar{V})\beta - \beta(\bar{V}\phi))\bar{V} + i\beta\sigma R\\ &= ((V + \phi\bar{V})\beta - \beta(\bar{V}\phi))\bar{V} + iR.\end{aligned}$$

On the other hand,

$$\begin{aligned}[X, T] = [\alpha\bar{Y}, Y + \psi\bar{Y}] &= (\alpha(\bar{Y}\psi) - (Y + \psi\bar{Y})\alpha)\bar{Y} + i\alpha\xi R\\ &= (\alpha(\bar{Y}\psi) - (Y + \psi\bar{Y})\alpha)\bar{Y} + iR.\end{aligned}$$

Since $\bar{V}$ and $\bar{Y}$ are linearly independent, it follows that $[X, T] = iR$. □

Lemma 25. *The following equalities hold.*

(4-3a) $[R, Y] = -2iY$.

(4-3b) $[R, \bar{Y}] = 2i\bar{Y}$.

(4-3c) $[R, V] = 2iV$.

(4-3d) $[R, \bar{V}] = -2i\bar{V}$.

(4-3e) $[R, X] = 2iX$.

(4-3f) $[R, \bar{X}] = -2i\bar{X}$.

(4-3g) $[R, T] = -2iT$.

(4-3h) $[R, \bar{T}] = 2i\bar{T}$.

(4-3i) $R\alpha = 0$.

(4-3j) $R\beta = 0$.

Proof. (4-3a), (4-3b), (4-3c) and (4-3d) follow from direct calculation.

For (4-3g) first note that writing $T = \beta\bar{V}$ and using (4-3d) we see that $[R, T]$ is a scalar multiple of T. Then writing

$$[R, T] = [R, Y + \psi\bar{Y}] = -2iY + (\text{multiple of } \bar{Y}),$$

we conclude using (3-5a) that $[R, T] = -2iT$. The proof of (4-3e) is similar, and (4-3f) and (4-3h) follow by conjugation.

Using (3-6a) along with (4-3b) and (4-3e) we obtain (4-3i); (4-3j) is proved similarly. □

Lemma 26. *$XXf = 0$ if and only if $f = f_1 w_1 + f_2 w_2$ with f_1, f_2 CR.*

Proof. From (3-6a) and (4-1) it is clear that $XX(f_1 w_1 + f_2 w_2) = 0$ if f_1 and f_2 are CR.

For the other direction, suppose that $XXf = 0$. Then if we set

$$f_1 \stackrel{\text{def}}{=} z_1 f + w_2 Xf, \qquad f_2 \stackrel{\text{def}}{=} z_2 f - w_1 Xf,$$

it is clear that $f = f_1 w_1 + f_2 w_2$; with the use of (3-6a) and (4-1) it is also easy to check that f_1 and f_2 are CR. □

Lemma 27. *Suppose that $XXTu = 0$ so that by Lemma 26 we may write $Tu = f_1 w_1 + f_2 w_2$ with f_1, f_2 CR. Then*

$$TTXu = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}. \tag{4-4}$$

In particular, $TTXu$ is CR.

The nontangential derivatives appearing in (4-4) may be interpreted using the Hans Lewy local CR extension result mentioned in the proof of Proposition 4, or else by rewriting them in terms of tangential derivatives (as in the last step of the proof below).

Proof.

$$\begin{aligned}
TTXu &= TXTu + T[T, X]u \\
&= TX(f_1 w_1 + f_2 w_2) - iTRu && \text{(Lemma 24)} \\
&= T(f_1 z_2 - f_2 z_1) - iRTu - i[T, R]u && \text{(3-6a), (4-1)} \\
&= T(f_1 z_2 - f_2 z_1) - iR(f_1 w_1 + f_2 w_2) + 2Tu && \text{(4-3g)} \\
&= (Tf_1)z_2 - f_1 w_1 - (Tf_2)z_1 - f_2 w_2 \\
&\quad - i(Rf_1)w_1 - f_2 w_2 - i(Rf_2)w_2 - f_2 w_2 \\
&\quad + 2(f_1 w_1 + f_2 w_2) && \text{(4-1), (4-2e)} \\
&= (z_2 T - i w_1 R) f_2 - (z_1 T + i w_2 R) f_2 \\
&= (z_2 Y - i w_1 R) f_2 - (z_1 Y + i w_2 R) f_2 \\
&= \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}.
\end{aligned}$$ □

Lemma 28. *The following statements hold.*

(4-5a) *The operator XT maps CR functions to CR functions.*

(4-5b) *The operator XY maps CR functions to CR functions.*

(4-5c) *The operator TX maps dual-CR functions to dual-CR functions.*

(4-5d) *The operator $\overline{XY}$ maps conjugate-CR functions to conjugate-CR functions.*

Proof. To prove (4-5a) and (4-5b) note that for u CR we have $XTu = XYu = -z_1\,\partial u/\partial z_1 - z_2\,\partial u/\partial z_2$, which is also CR. The other proofs are similar. □

Proof of (3-6d). To get the required lower bound on the null spaces, it will suffice to show that XXT and TTX annihilate CR functions and dual-CR functions. This follows from (3-6a) and (3-6b) along with (4-5a) and (4-5c).

For the other direction, if $XXTu = 0 = TTXu$, then from Lemma 27 we have a closed 1-form $\omega \stackrel{\text{def}}{=} f_2\,dz_1 - f_1\,dz_2$ on S where f_1 and f_2 are CR functions satisfying $Tu = f_1 w_1 + f_2 w_2$. Since S is simply connected we may write $\omega = df$ with f CR. Then from (3-5a) we have

$$Tf = Yf = w_2 f_2 + w_1 f_1 = Tu.$$

Thus u is the sum of the CR function f and the dual-CR function $u - f$. □

To set up the proof of the global result (3-6c) we introduce the form

$$\nu \stackrel{\text{def}}{=} (z_2\,dz_1 - z_1\,dz_2) \wedge dw_1 \wedge dw_2 \tag{4-6}$$

and the $\mathbb{C}$-bilinear pairing

$$\langle\langle \mu, \eta \rangle\rangle \stackrel{\text{def}}{=} \int_S \mu\eta \cdot \nu \tag{4-7}$$

between functions on S (but see Technical Remark 32 below).

Lemma 29. $\langle\langle T\gamma, \eta\rangle\rangle = -\langle\langle \gamma, T\eta\rangle\rangle$.

Proof. In the sequence of equalities below we will use

- the definition (4-7) of the pairing $\langle\langle \cdot\,, \cdot \rangle\rangle$,
- the Leibniz rule $\iota_T(\varphi_1 \wedge \varphi_2) = (\iota_T\varphi_1) \wedge \varphi_2 + (-1)^{\deg \varphi_1}\varphi_1 \wedge (\iota_T\varphi_2)$ for the interior product ι_T,
- the fact that S is integral for 4-forms,
- Stokes' theorem,
- the rules (4-1),
- the relation (3-2a).

$$
\begin{aligned}
\langle\langle T\gamma, \eta\rangle\rangle + \langle\langle \gamma, T\eta\rangle\rangle &= \int_S T(\gamma\eta)\cdot\nu \\
&= \int_S \iota_T d(\gamma\eta)\cdot\nu \\
&= \int_S d(\gamma\eta)\cdot\iota_T\nu \\
&= \int_S d(\gamma\eta\cdot\iota_T\nu) - \int_S \gamma\eta\cdot d(\iota_T\nu) \\
&= 0 - \int_S \gamma\eta\cdot d\big(\iota_T((z_2\,dz_1 - z_1\,dz_2)\wedge dw_1\wedge dw_2)\big) \\
&= -\int_S \gamma\eta\cdot d((z_2\cdot Tz_1 - z_1\cdot Tz_2)\cdot dw_1\wedge dw_2) \\
&\qquad + \int_S \gamma\eta\cdot d((z_2\,dz_1 - z_1\,dz_2)\cdot Tw_1\wedge dw_2) \\
&\qquad - \int_S \gamma\eta\cdot d((z_2\,dz_1 - z_1\,dz_2)\wedge dw_1\cdot Tw_2) \\
&= -\int_S \gamma\eta\cdot d((z_2w_2 + z_1w_1)\,dw_1\wedge dw_2) + 0 - 0 \\
&= -\int_S \gamma\eta\cdot d(dw_1\wedge dw_2) \\
&= 0. \qquad \square
\end{aligned}
$$

Theorem 30. *Let μ be a CR function on a compact strongly $\mathbb{C}$-convex hypersurface S. Then $\mu = 0$ if and only if $\langle\langle\mu, \eta\rangle\rangle = 0$ for all dual-CR η on S.*

Proof. [Barrett 2016, (4.3d) from Theorem 3]. (Note also definition enclosing [Barrett 2016, (4.2)].) □

Proof of (3-6c). Assume that $XXTu = 0$. Noting that S is simply connected, from (3-6d) it suffices to prove that $TTXu = 0$. From Lemma 27 we know that $TTXu$ is CR. By Theorem 30 it will suffice to show that

$$\langle\langle TTXu, \eta\rangle\rangle = 0$$

for dual-CR η. But from Lemma 29 we have, as required,

$$\langle\langle TTXu, \eta\rangle\rangle = -\langle\langle TXu, T\eta\rangle\rangle = 0. \qquad \square$$

Remark 31. From symmetry of formulas in Lemmas 6 and 12 we have

$$X_{S^*} = \mathfrak{D}_* T_S, \quad T_{S^*} = \mathfrak{D}_* X_S, \quad S^{**} = S.$$

These facts serve to explain why the formulas throughout this section appear in dual pairs.

Technical Remark 32. In [Barrett 2016] the pairing (4-7) applies not to functions μ, ν but rather to forms $\mu(z)\,(dz_1 \wedge dz_2)^{2/3}$, $\mu(w)\,(dw_1 \wedge dw_2)^{2/3}$; the additional notation is important there for keeping track of invariance properties under projective transformation but is not needed here.

Note also that (4-7) coincides (up to a constant) with the pairing (3.1.8) in [Andersson et al. 2004] with $s = w_1\,dz_1 + w_2\,dz_2$.

5. Proof of Theorem A

For the reader's convenience we restate the main theorem in the conjugate setting.

Theorem A. *Let $S \subset \mathbb{C}^2$ be a strongly pseudoconvex circular hypersurface. Then there exist nowhere-vanishing tangential vector fields X, Y on S satisfying the following conditions*:

(5-1a) *If u is a smooth function on a relatively open subset of S, then u is CR if and only if $Xu = 0$.*

(5-1b) *If u is a smooth function on a relatively open subset of S, then u is CR if and only if $Y\bar{u} = 0$.*

(5-1c) *If S is compact, then a smooth function u on S is a pluriharmonic boundary value* (*in the sense of Proposition 3 below*) *if and only if $XXYu = 0$.*

(5-1d) *A smooth function u on a relatively open subset of S is a pluriharmonic boundary value* (*in the sense of Proposition 4 below*) *if and only if*

$$XXYu = 0 = \overline{XXY}u.$$

It is not possible in general to have $Y = \overline{X}$.

Lemma 33. *Suppose that $XXYu = 0$ so that by Lemma 26 we may write*

$$Yu = f_1 w_1 + f_2 w_2$$

with f_1, f_2 CR. Then

$$\overline{XXY}u = \alpha\left(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}\right). \tag{5-2}$$

In particular, $\alpha^{-1}\,\overline{XXY}u$ is CR.

Proof. We have

$$\begin{aligned}
\overline{XXY}u &= \overline{XYX}u + \bar{X}[\bar{X},\bar{Y}]u \\
&= \overline{XY}(\alpha(f_1w_1+f_2w_2)) + \bar{X}(-iR-(\bar{Y}\alpha)Y)u \quad \text{(3-6a), (4-2c), (4-5b)} \\
&= \bar{X}(\alpha\bar{Y}(f_1w_1+f_2w_2)) - i\bar{X}Ru \\
&= \bar{X}(f_1z_2-f_2z_1) - iR\bar{X}u - i[\bar{X},R]u \quad \text{(3-6a), (4-1)} \\
&= \bar{X}(f_1z_2-f_2z_1) - iR(\alpha(f_1w_1+f_2w_2)) + 2\bar{X}u \quad \text{(3-6a), (4-3f)} \\
&= (\bar{X}f_1)\cdot z_2 - f_1\cdot\alpha w_1 - (\bar{X}f_2)\cdot z_1 - f_2\cdot\alpha w_2 \\
&\quad -i\alpha((Rf_1)\cdot w_1 - f_1\cdot(iw_1) + (Rf_2)\cdot w_2 - f_2\cdot(iw_2)) \quad \text{(4-1),(4-3i),} \\
&\quad +2\alpha(f_1w_1+f_2w_2) \quad \text{(4-2e), (3-6a)} \\
&= (\bar{X}f_1)\cdot z_2 - (\bar{X}f_2)\cdot z_1 - i\alpha((Rf_1)\cdot w_1 + (Rf_2)\cdot w_2) \\
&= \alpha\big((z_2Y - iw_1R)f_1 - (z_1Y + iw_2R)f_2\big) \\
&= \alpha\Big(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}\Big). \qquad \square
\end{aligned}$$

Proof of (1-1d). To get the required lower bound on the null spaces, it will suffice to show that XXY and $\overline{XXY}$ annihilate CR functions and conjugate-CR functions. This follows from (1-1a) along with (4-5b) and (4-5d).

For the other direction, if $XXYu = 0 = \overline{XXY}u$, then from Lemma 27 we have a closed 1-form $\tilde{\omega} \stackrel{\text{def}}{=} f_2\,dz_1 - f_1\,dz_2$ on the open subset of S where f_1 and f_2 are CR functions satisfying $Yu = f_1w_1 + f_2w_2$. Restricting our attention to a simply connected subset, we may write $\omega = df$ with f CR. Then we have

$$Yf = w_2f_2 + w_1f_1 = Yu.$$

Thus u is the sum of the CR function f and the conjugate-CR function $u - f$.

The general case follows by localization. $\square$

Lemma 34. div $Y \stackrel{\text{def}}{=} \partial w_2/\partial z_1 - \partial w_1/\partial z_2$ *and* div $\bar{Y} \stackrel{\text{def}}{=} \partial\bar{w}_2/\partial\bar{z}_1 - \overline{\partial\bar{w}_1/\partial\bar{z}_2}$ *vanish on* S.

Proof. Since S is circular, any defining function ρ for S will satisfy

$$\operatorname{Im}\Big(z_1\frac{\partial\rho}{\partial z_1} + z_2\frac{\partial\rho}{\partial z_2}\Big) = -\frac{R\rho}{2} = 0.$$

Adjusting our choice of defining function we may arrange that

$$z_1\frac{\partial\rho}{\partial z_1} + z_2\frac{\partial\rho}{\partial z_2} \equiv 1$$

in some neighborhood of S. Then from the proof of Lemma 6 we have

$$\frac{\partial w_2}{\partial z_1} - \frac{\partial w_1}{\partial z_2} = \frac{\partial^2 \rho}{\partial z_1 \partial z_2} - \frac{\partial^2 \rho}{\partial z_2 \partial z_1} = 0. \tag{5-3}$$

The remaining statement follows by conjugation. □

Lemma 35.
$$\int_S (X\gamma)\eta \,\frac{dS}{\alpha} = -\int_S \gamma(X\eta)\,\frac{dS}{\alpha}$$

Proof.
$$\begin{aligned}\int_S (X\gamma)\eta \,\frac{dS}{\alpha} &= \int_S (\bar{Y}\gamma)\eta \, dS && \text{(3-6a)}\\ &= -\int_S \gamma(\bar{Y}\eta)\, dS && \text{(Lemma 34)}\\ &= -\int_S \gamma(X\eta)\,\frac{dS}{\alpha} && \text{(3-6a)}\end{aligned}$$

(The integration by parts above may be justified by applying the divergence theorem on a tubular neighborhood of S and passing to a limit.) □

Proof of (1-1c). Assume that $XXYu = 0$. Noting that S is simply connected, from (1-1d) it suffices to prove that $\overline{XXY}u = 0$. From Lemma 27 we know that $\alpha^{-1}\overline{XXY}u$ is CR. The desired conclusion now follows from

$$\begin{aligned}\int_S |\overline{XXY}u|^2 \,\frac{dS}{\alpha^2} &= \int_S \alpha^{-1}\overline{XXY}u \cdot XXY\bar{u}\,\frac{dS}{\alpha}\\ &= -\int_S X(\alpha^{-1}\overline{XXY}u)\cdot XY\bar{u}\,\frac{dS}{\alpha} && \text{(Lemma 35)}\\ &= -\int_S 0\cdot XY\bar{u}\,\frac{dS}{\alpha} && \text{(Lemma 33)}\\ &= 0.\end{aligned}$$
□

6. Further comments

6.1. *Remarks on uniqueness.*

Proposition 36. *Suppose that in the setting of Theorem B we have vector fields $\tilde{X}$, $\tilde{T}$ satisfying (suitably modified) (3-6a) and (3-6b). Then $\tilde{X}\tilde{X}\tilde{T}$ annihilates CR functions and dual-CR functions if and only if there are CR functions f_1, f_2 and f_3 so that $f_1 w_1 + f_2 w_2$ and f_3 are nonvanishing and*

$$\tilde{X} = f_3(f_1 w_1 + f_2 w_2)^2 X, \quad \tilde{T} = \frac{1}{f_1 w_1 + f_2 w_2} T.$$

Proof. From (3-6a) and (3-6b) we have $\tilde{X} = \gamma X$, $\tilde{T} = \eta T$ with nonvanishing scalar functions γ and η.

Suppose that $\tilde{X}\tilde{X}\tilde{T}$ annihilates CR functions and dual-CR functions. By routine computation we have

$$\tilde{X}\tilde{X}\tilde{T} = \gamma^2\eta XXT + \gamma\big((2\gamma(X\eta) + \eta(X\gamma))XT + X(\gamma(X\eta))T\big).$$

The operator $(2\gamma(X\eta)+\eta(X\gamma))XT + X(\gamma(X\eta))T$ must in particular annihilate CR functions. But if f is CR, then using Lemma 24 we have

$$\big((2\gamma(X\eta)+\eta(X\gamma))XT+X(\gamma(X\eta))T\big)f=\big(i(2\gamma(X\eta)+\eta(X\gamma))R+X(\gamma(X\eta))T\big)f$$

Since R and T are $\mathbb{C}$-linearly independent and f is arbitrary it follows that

$$X(\gamma\eta^2) = 2\gamma(X\eta) + \eta(X\gamma) = 0, \quad X(\gamma(X\eta)) = 0.$$

We set $f_3 = \gamma\eta^2$, which is CR and nonvanishing. Then the second equation above yields

$$-f_3 \cdot XX(\eta^{-1}) = X(f_3\eta^{-2}(X\eta)) = X(\gamma(X\eta)) = 0,$$

and hence $XX(\eta^{-1}) = 0$. From Lemma 26 we have $\eta = 1/(f_1w_1 + f_2w_2)$ with f_1 and f_2 CR. The result now follows.

The converse statement follows by reversing steps. □

Proposition 37. *Suppose that in the setting of Theorem A we have vector fields $\tilde{X}$, $\tilde{T}$ satisfying (suitably modified) (1-1a) and (1-1b). Then $\tilde{X}\tilde{X}\tilde{Y}$ annihilates CR functions and conjugate-CR functions if and only if there are CR functions f_1, f_2 and f_3 so that $f_1w_1 + f_2w_2$ and f_3 are nonvanishing and*

$$\tilde{X} = f_3(f_1w_1 + f_2w_2)^2 X, \quad \tilde{Y} = \frac{1}{f_1w_1+f_2w_2}Y.$$

The proof is similar to that of Proposition 36, using (4-2h) in place of Lemma 26.

6.2. *Nirenberg-type result.*

Proposition 38. *Given a point p on a strongly pseudoconvex hypersurface $S \subset \mathbb{C}^2$, any 2-jet at p of a $\mathbb{C}$-valued function on S is the 2-jet of the restriction to S of a pluriharmonic function on $\mathbb{C}^2$.*

Proof. After performing a standard local biholomorphic change of coordinates we may reduce to the case where $p = 0$ and S is described near 0 by an equation of the form

$$y_2 = z_1\bar{z}_1 + O(\|(z_1, x_2)\|)^3.$$

The projection $(z_1, x_2 + iy_2) \mapsto (z_1, x_2)$ induces a bijection between 2-jets at 0 along S and 2-jets at 0 along $\mathbb{C} \times \mathbb{R}$. It suffices now to note that the 2-jet

$$A + Bz_1 + C\bar{z}_1 + Dx_2 + Ez_1^2 + F\bar{z}_1^2 + Gz_1\bar{z}_1 + Hz_1x_2 + I\bar{z}_1x_2 + Jx_2^2$$

is induced by the pluriharmonic polynomial

$$A + Bz_1 + C\bar{z}_1 + \frac{D - iG}{2} z_2 + \frac{D + iG}{2}\bar{z}_2 + Ez_1^2 + F\bar{z}_1^2 + Hz_1z_2 + I\bar{z}_1\bar{z}_2 + J\bar{z}_2^2. \quad \square$$

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DAVID E. BARRETT
DEPT. OF MATHEMATICS
UNIVERSITY OF MICHIGAN ANN ARBOR
ANN ARBOR, MI 48109-1043
UNITED STATES

barrett@umich.edu

DUSTY E. GRUNDMEIER
DEPT. OF MATHEMATICS
HARVARD UNIVERSITY
1 OXFORD ST
CAMBRIDGE, MA 02138
UNITED STATES

deg@math.harvard.edu

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