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We prove that if a nonselfadjoint dual operator algebra admitting a normal virtual diagonal and an injective von Neumann algebra are close enough for the Kadison–Kastler metric, then they are similar. The bound explicitly depends on the norm of the normal virtual diagonal. This is inspired by E. Christensen's work on perturbation of operator algebras and is related to a conjecture of G. Pisier on nonselfadjoint amenable operator algebras.

1. Introduction

The starting point of this paper is the conjunction of perturbation theory of operator algebras and a conjecture on amenable nonselfadjoint operator algebras. Let us first recall this conjecture and propose a dual version of it, then we will explain the connection with our main result.

A conjecture raised by G. Pisier asserts that a nonselfadjoint amenable operator algebra $\mathcal{A}$ should be similar to a nuclear C^*-algebra (i.e., there is an invertible operator S such that $S\mathcal{A}S^{-1}$ is a C^*-algebra). Recently, this conjecture has been proved for commutative amenable operator algebras in [Marcoux and Popov 2016]. It generalizes [Choi 2013; Willis 1995]; see also [Marcoux 2008] for more details around this conjecture. A nonseparable counter-example to Pisier's conjecture has been found [Choi et al. 2014] but the separable case remains open.

In his memoir, B.E. Johnson [1972] characterized amenability of Banach algebras by the existence of a virtual diagonal. Recall that injectivity for von Neumann algebras can be characterized by the existence of a normal virtual diagonal (in the sense of E.G. Effros [1988], see Section 2C below for details). Therefore, a dual version of Pisier's conjecture would be:

Conjecture. *A unital dual operator algebra admitting a normal virtual diagonal should be similar to an injective von Neumann algebra. In that case, it is expected that the similarity constant is controlled by a nondecreasing function of the norm of*

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the normal virtual diagonal. Note that one advantage of this conjecture is to avoid the separability question.

In 1972, R.V. Kadison and D. Kastler defined a metric d on the collection of all subspaces of the bounded operators on a fixed Hilbert space (see Section 2A). They conjectured [1972] that sufficiently close C^*-algebras are necessarily unitarily conjugated. A great amount of work around this conjecture has been done since then (see [Christensen et al. 2012] for a nice introduction on this topic). Notably, E. Christensen proved the conjecture for the class of type I von Neumann algebras [Christensen 1975] and for the class of injective von Neumann algebras [Christensen 1977]. Very recently, Kadison and Kastler's conjecture has been proved for the class of separable nuclear C^*-algebras in [Christensen et al. 2012] (see also [Christensen et al. 2010b]). The recent paper [Cameron et al. 2014] is an important breakthrough beyond amenability. Let us state Christensen's first result on perturbation of injective von Neumann algebras (this result has subsequently been improved in [Christensen 1980]):

Theorem 1 [Christensen 1977, Theorem 4.1]. *Let $\mathcal{M},\mathcal{N}$ be two von Neumann subalgebras of a fixed $\mathbb{B}(H)$. We suppose that $\mathcal{M}$ has Schwartz's property (P) and $\mathcal{N}$ has the extension property. If $d(\mathcal{M},\mathcal{N}) < \frac{1}{169}$, then there is a unitary U in the von Neumann algebra generated by $\mathcal{M}\cup\mathcal{N}$ such that $U\mathcal{M}U^* = \mathcal{N}$. Moreover, $\|U - I_H\| \leq 19d(\mathcal{M},\mathcal{N})^{1/2}$.*

It is now well known, after the work of A. Connes [1976; 1978] and U. Haagerup [1985], that Schwartz's property (P), the extension property and injectivity (and thus the existence of a normal virtual diagonal) are equivalent conditions for von Neumann algebras.

The aforementioned conjecture leads to the following question: can we replace, in the preceding theorem, $\mathcal{M}$ by a unital nonselfadjoint dual operator algebra admitting a normal virtual diagonal? In other words, is the selfadjointness hypothesis on $\mathcal{M}$ necessary? Indeed, assume for a moment that our conjecture is true, then there would be an invertible S such that $S\mathcal{M}S^{-1}$ is an injective von Neumann algebra. Moreover,

$$d(\mathcal{M}, S\mathcal{M}S^{-1}) \leq 2(1+\|S\|\|S^{-1}\|)\|S-I_H\|$$

(and this last quantity is controlled by a nondecreasing function of the norm of the normal virtual diagonal). Hence, if $d(\mathcal{M},\mathcal{N})$ is small enough such that the following strict inequality holds

$$d(\mathcal{N}, S\mathcal{M}S^{-1}) \leq d(\mathcal{M},\mathcal{N}) + 2(1+\|S\|\|S^{-1}\|)\|S-I_H\| < \tfrac{1}{169},$$

then (from Theorem 1 above) the injective von Neumann algebras $\mathcal{N}$ and $S\mathcal{M}S^{-1}$ would be unitarily conjugated, so $\mathcal{M}$ and $\mathcal{N}$ would be similar. Therefore, it is not incongruous to try to replace $\mathcal{M}$ by a unital dual operator algebra admitting a normal virtual diagonal.

In this paper, we prove:

Theorem 2. *Let $\mathcal{M}, \mathcal{N} \subset \mathbb{B}(H)$ be two unital w^*-closed operator algebras. Suppose that $\mathcal{M}$ admits a normal virtual diagonal u and $\mathcal{N}$ is an injective von Neumann algebra. If $d(\mathcal{M}, \mathcal{N}) < 1/(656\|u\|)$, then there exists an invertible operator S in the w^*-closed algebra generated by $\mathcal{M} \cup \mathcal{N}$ such that $S\mathcal{M}S^{-1} = \mathcal{N}$. Moreover, $\|S - I_H\| \leq 656\|u\| d(\mathcal{M}, \mathcal{N})$.*

The proof of Theorem 2 is the consequence of Theorem 5.2 and Lemma 5.4. Note that von Neumann algebras enjoy a self-improvement phenomenon; if a von Neumann algebra admits a normal virtual diagonal then it admits a normal virtual diagonal of norm 1, see [Haagerup 1985; Effros 1988; Effros and Kishimoto 1987] (self-improvement phenomena are frequent for selfadjoint algebras, for instance nuclearity constant and exactness constant). This may explain why in Theorem 1 the bound is a universal constant, whereas in Theorem 2, the bound depends on the feature of the nonselfadjoint algebra involved. Moreover, from Theorem 7.4.18 (1) in [Blecher and Le Merdy 2004] and Remark 2.1 below, if a unital dual operator algebra admits a normal virtual diagonal of norm 1, then it is necessarily a von Neumann algebra (no similarity is needed in this extreme case). Hence, Theorem 1 corresponds exactly to the case $\|u\|$ equals 1 in Theorem 2 (as the unitary U is obtained by taking the polar decomposition of S, see Lemma 2.7 in [Christensen 1975]). Our bound in this special case is not as good as Christensen's one, but the important point is that we have been able to remove the selfadjointness hypothesis on $\mathcal{M}$. This is not a minor modification; knowing that nonselfadjoint algebras are less rigid than selfadjoint ones (no order structure for instance) and fewer tools are available (no continuous or Borel functional calculus), our proof requires new ingredients from operator space theory in particular the normal Haagerup tensor product of dual operator spaces.

Now let us sketch the main lines of our proof. There are three steps (as in Christensen's work [1977]):

Step 1. Find a linear isomorphism, between the two algebras, which is close to the identity representation.

Step 2. Find an algebra homomorphism close to the previous linear isomorphism.

Step 3. Prove this algebra homomorphism is similar to the identity representation.

For the first step, as $\mathcal{N}$ is injective, one just has to take the restriction to $\mathcal{M}$ of a completely contractive projection onto $\mathcal{N}$. This gives a linear isomorphism T from $\mathcal{M}$ onto $\mathcal{N}$ which is close to the identity representation of $\mathcal{M}$. But in order to apply certain averaging procedure for Step 2, we need a w^*-continuous linear isomorphism. For this, Christensen used Tomiyama's decomposition into normal and singular parts of bounded linear maps defined on von Neumann algebras. But when $\mathcal{M}$ is nonselfadjoint, such decomposition is not available. Hence, we have to consider the normal part of T^{-1}. This w^*-continuous linear isomorphism from $\mathcal{N}$ onto $\mathcal{M}$ is not necessarily completely positive, and moreover the target algebra $\mathcal{M}$ is not necessarily selfadjoint, thus we can not use Christensen's averaging trick [1977, Lemma 3.3] to accomplish the second step. The idea is to turn to Banach algebras results and operator spaces tools. More precisely, we will use a dual operator space version of a B.E. Johnson theorem [1988] on almost multiplicative maps. Indeed, the issue here is that we need to preserve the w^*-continuity, but we cannot use the normal projective tensor product of dual Banach spaces (as we could not check its associativity, see Section 3). This second step will force us to work with the normal Haagerup tensor product of dual operator spaces.

Finally, the third step, which is related to a more general problem on neighboring representations (already mentioned in [Kadison and Kastler 1972]), is done by an averaging technique. However, because of the second step, we have had to work in the operator space category and as a consequence we had to assume that the algebras $\mathbb{M}_n(\mathcal{M})$ nearly embed in $\mathbb{M}_n(\mathcal{N})$ uniformly in n (see the notion of near cb-inclusion defined in Section 2A). As an intermediate result, we prove a perturbation theorem with a near cb-inclusion assumption (see Theorem 5.2). Therefore, our final task is to notice that the existence of a normal virtual diagonal is an "automatic near cb-inclusion" condition (see Lemma 5.4).

To conclude this introduction, let us mention that an engaging objective would be to prove an analog of Theorem 2 when both algebras are nonselfadjoint (for details see Remark 5.6). We also should mention that after the writing and circulation of our paper, L. Dickson has obtained an improvement of our Theorem 2 (see [Dickson 2014, Theorem 6.1.1]). His result is interesting because he was able to get rid of the normal virtual diagonal hypothesis. His proof uses a variant of Johnson's result on almost multiplicative maps (like our proof) and also the characterization of injective von Neumann algebras as the w^*-closure of a net of finite-dimensional subalgebras. This is a strong approximation property characterization, but such a characterization is far from being available for nonselfadjoint operator algebras admitting normal virtual diagonal. Hence, unfortunately, we can not use Dickson's techniques for our perturbation problem (mentioned in Remark 5.6) when both algebras are nonselfadjoint.

2. Preliminaries

For background on completely bounded maps, operator space theory and nonselfadjoint algebra theory, the reader is referred to [Blecher and Le Merdy 2004; Effros and Ruan 2000; Paulsen 2002; Pisier 2003], especially Section 2.7 in [Blecher and Le Merdy 2004] for background on dual operator algebras.

2A. ***Perturbation theory.*** We first recall definitions and notations commonly used in perturbation theory of operator algebras (see, e.g., [Christensen et al. 2010a]). Let H be a Hilbert space, and $\mathbb{B}(H)$ be the von Neumann algebra of all bounded operators on H. Let $\mathcal{E}, \mathcal{F}$ be two subspaces of $\mathbb{B}(H)$. We denote by d the Kadison–Kastler metric, i.e., $d(\mathcal{E}, \mathcal{F})$ denotes the Hausdorff distance between the unit balls of $\mathcal{E}$ and $\mathcal{F}$. More explicitly,

$$d(\mathcal{E}, \mathcal{F}) = \inf\bigl\{\gamma > 0 : \text{for all } x \in B_{\mathcal{E}},\ \text{there exists } x' \in B_{\mathcal{F}},\ \|x - x'\| < \gamma \text{ and } \\ \text{for all } y \in B_{\mathcal{F}},\ \text{there exists } y' \in B_{\mathcal{E}},\ \|y - y'\| < \gamma\bigr\},$$

where $B_{\mathcal{E}}$ (respectively, $B_{\mathcal{F}}$) denotes the unit ball of $\mathcal{E}$ (respectively, $\mathcal{F}$). Let $\gamma > 0$, then we write $\mathcal{E} \subseteq^{\gamma} \mathcal{F}$ if for any x in the unit ball of $\mathcal{E}$, there exists y in $\mathcal{F}$ such that

$$\|x - y\| \leq \gamma.$$

We also write $\mathcal{E} \subset^{\gamma} \mathcal{F}$ if there exists $\gamma' < \gamma$ such that $\mathcal{E} \subseteq^{\gamma'} \mathcal{F}$. We will also need the notion of *near* cb-*inclusion*. As usual in operator space theory, $\mathbb{M}_n(\mathcal{E})$, the subspace of $n \times n$ matrices with coefficients in $\mathcal{E}$ is normed by the identification $\mathbb{M}_n(\mathcal{E}) \subset \mathbb{M}_n(\mathbb{B}(H)) = \mathbb{B}(\ell_n^2 \otimes H)$. We write

$$\mathcal{E} \subseteq^{\gamma}_{\mathrm{cb}} \mathcal{F}$$

if $\mathbb{M}_n(\mathcal{E}) \subseteq^{\gamma} \mathbb{M}_n(\mathcal{F})$, for all n.

2B. ***The normal projective tensor product and normal Haagerup tensor product.*** For dual operator spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $(\mathcal{X} \otimes_h \mathcal{Y})^*_\sigma$ the space of all completely bounded bilinear forms which are separately w^*-continuous (see [Blecher and Le Merdy 2004, Paragraph 1.5.4] for the definition of completely bounded bilinear maps). The normal Haagerup tensor product, denoted $\otimes_{\sigma h}$, can be defined as

$$\mathcal{X} \otimes_{\sigma h} \mathcal{Y} = ((\mathcal{X} \otimes_h \mathcal{Y})^*_\sigma)^*, \tag{2-1}$$

see [Blecher and Le Merdy 2004, Paragraph 1.6.8]. The normal Haagerup tensor product is characterized by the following universal property: $\mathcal{X} \otimes \mathcal{Y}$ is w^*-dense in $\mathcal{X} \otimes_{\sigma h} \mathcal{Y}$ and, for any dual operator space $\mathbb{Z}$, for any w^*-continuous completely contractive bilinear map $B : \mathcal{X} \times \mathcal{Y} \to \mathbb{Z}$, there exists a (unique) w^*-continuous completely contractive linear map $\tilde{B} : \mathcal{X} \otimes_{\sigma h} \mathcal{Y} \to \mathbb{Z}$ such that $\tilde{B}(x \otimes y) = B(x, y)$, for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. We will also need the normal projective tensor product $\widehat{\otimes}_\sigma$ of dual Banach spaces. If $\mathcal{X}$ and $\mathcal{Y}$ are dual Banach spaces,

$$\mathcal{X} \widehat{\otimes}_\sigma \mathcal{Y} = ((\mathcal{X} \widehat{\otimes} \mathcal{Y})^*_\sigma)^*,$$

where $(\mathcal{X} \widehat{\otimes} \mathcal{Y})^*_\sigma$ denotes the space of all bounded bilinear forms on $\mathcal{X} \times \mathcal{Y}$ which are separately w^*-continuous. The normal projective tensor product enjoys a similar universal property to the normal Haagerup tensor product, but for separately w^*-continuous bounded bilinear maps instead of separately w^*-continuous completely bounded (for von Neumann algebras, the projective normal tensor product appeared for instance in [Effros 1988] under the name binormal projective tensor product). These two tensor products are "functorial" in the sense that, if $L_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$, are bounded (respectively, completely bounded) w^*-continuous linear maps between dual Banach spaces (dual operator spaces), then there is a unique bounded (completely bounded) w^*-continuous linear map

$$L_1 \widehat{\otimes}_\sigma L_2 : \mathcal{X}_1 \widehat{\otimes}_\sigma \mathcal{X}_2 \to \mathcal{Y}_1 \widehat{\otimes}_\sigma \mathcal{Y}_2$$

$(L_1 \otimes_{\sigma h} L_2 : \mathcal{X}_1 \otimes_{\sigma h} \mathcal{X}_2 \to \mathcal{Y}_1 \otimes_{\sigma h} \mathcal{Y}_2)$ extending $L_1 \otimes L_2$. Moreover,

$$\|L_1 \widehat{\otimes}_\sigma L_2\| \leq \|L_1\| \|L_2\|$$

$(\|L_1 \otimes_{\sigma h} L_2\|_{\mathrm{cb}} \leq \|L_1\|_{\mathrm{cb}} \|L_2\|_{\mathrm{cb}})$.

The main difference between these two tensor products is that the normal Haagerup tensor product is associative (see Lemma 2.2 in [Blecher and Kashyap 2008]), whereas the normal projective tensor product does not seem to be associative in general (this difference will have an important consequence for us in Section 3).

2C. ***Normal virtual diagonals and normal virtual h-diagonals.*** Normal virtual diagonals appeared implicitly in [Haagerup 1985] and explicitly in [Effros 1988] (see p. 147 thereof). In this paper, we also need the notion of normal virtual h-diagonal (called reduced normal virtual diagonal in [Effros 1988], see also [Blecher and Le Merdy 2004, Paragraph 7.4.8] for more details). Let us just recall this notion. Replacing the normal Haagerup tensor product by the normal projective tensor product in the following, one can analogously obtain the definition of normal virtual diagonal. Let $\mathcal{M}$ be a unital dual operator algebra, and let us recall the $\mathcal{M}$-bimodule structure of $\mathcal{M} \otimes_{\sigma h} \mathcal{M}$. Letting $\psi \in (\mathcal{M} \otimes_h \mathcal{M})^*_\sigma$ and $a, b, c, d \in \mathcal{M}$,

$$\langle b \cdot \psi \cdot a, c \otimes d \rangle = \psi(ac, db).$$

Hence by duality, one can define actions of $\mathcal{M}$ on $\mathcal{M} \otimes_{\sigma h} \mathcal{M} = ((\mathcal{M} \otimes_h \mathcal{M})^*_\sigma)^*$. One can check that these actions are determined on the elementary tensors by

$$a \cdot (c \otimes d) \cdot b = ac \otimes db.$$

On a dual operator algebra, the multiplication is a separately w^*-continuous completely contractive bilinear map [Blecher and Le Merdy 2004, Proposition 2.7.4 (1)]. Consequently, it induces a w^*-continuous complete contraction,

$$\mathrm{m}_{\sigma h} : \mathcal{M} \otimes_{\sigma h} \mathcal{M} \to \mathcal{M}.$$

A *normal virtual h-diagonal* for $\mathcal{M}$ is an element $u \in \mathcal{M} \otimes_{\sigma h} \mathcal{M}$ satisfying

(C1) $m \cdot u = u \cdot m$ for any $m \in \mathcal{M}$,

(C2) $\mathrm{m}_{\sigma h}(u) = 1$.

Note that condition (C2) implies that the norm of a normal virtual h-diagonal is always greater than or equal to 1.

Remark 2.1. Note that the inclusion $(\mathcal{X} \otimes_h \mathcal{Y})^*_\sigma \subset (\mathcal{X} \widehat{\otimes} \mathcal{Y})^*_\sigma$ induces, by duality, a contraction from $\mathcal{M} \widehat{\otimes}_\sigma \mathcal{M}$ into $\mathcal{M} \otimes_{\sigma h} \mathcal{M}$ and this contraction sends normal virtual diagonals into normal virtual h-diagonals. Consequently, if $\mathcal{M}$ admits a normal virtual diagonal, it admits a normal virtual h-diagonal.

3. B.E. Johnson's theorem revisited

The aim of this section is to find a solution to the second step mentioned in the Introduction. Johnson [1988] proved that an approximately multiplicative map defined on an amenable Banach algebra is close to an actual algebra homomorphism. His result is the Banach algebraic version of an earlier result due to D. Kazhdan [1982] for amenable groups (see also [Burger et al. 2013]). If L is a linear map between operator algebras $\mathcal{M}$ and $\mathcal{N}$, we denote by $L^\vee : \mathcal{M} \times \mathcal{M} \to \mathcal{N}$ the bilinear map defined by

$$L^\vee(x, y) = L(xy) - L(x)L(y).$$

This enables us to measure the defect of multiplicativity of L.

In our present case, we have to take into account the dual operator space structure of our algebras. Starting from a w^*-continuous linear map from $\mathcal{M}$ into $\mathcal{N}$, we must obtain a w^*-continuous algebra homomorphism. This will force us to work in the category of operator spaces. The proof of Theorem 3.1 in [Johnson 1988] is by induction, the algebra homomorphism is the limit (in operator norm) of a sequence of linear maps with multiplicativity defect tending to zero. The problem is that these linear maps are defined using the w^*-topology of the target algebra (see equation $(*)$ in the proof of [Johnson 1988, Theorem 3.1]). Here, to justify the w^*-continuity of these linear maps, we must consider a trilinear map defined on $\mathcal{M} \times \mathcal{M} \times \mathcal{M}$; see (3-1) below. But the normal projective tensor product does not seem associative. To circumvent this difficulty, we will instead work with the normal Haagerup tensor product, which is associative [Blecher and Kashyap 2008, Lemma 2.2]. As a consequence, we have to control the cb-norm of the bilinear map $L^\vee$.

Remark 3.1. Actually, this difficulty concerning the associativity of the normal projective tensor product has already been encountered in disguise. The main issue in [Effros 1988] is that one cannot check whether the Banach $\mathcal{M}$-bimodule $\mathcal{M} \widehat{\otimes}_\sigma \mathcal{M}$ is normal or not. But if one assumes that the normal projective tensor product is associative, then it is easy to check that $\mathcal{M} \widehat{\otimes}_\sigma \mathcal{M}$ is a normal bimodule.

Theorem 3.2. *Let $\mathcal{M}, \mathcal{N}$ be two unital dual operator algebras. We suppose that $\mathcal{M}$ has a normal virtual h-diagonal $u \in \mathcal{M} \otimes_{\sigma h} \mathcal{M}$. Then, for any $\varepsilon \in (0, 1)$, for any $\mu > 0$, there exists $\delta > 0$ such that: for every unital w^*-continuous linear map*

$L : \mathcal{M} \to \mathcal{N}$ satisfying $\|L\|_{\mathrm{cb}} \le \mu$ and $\|L^\vee\|_{\mathrm{cb}} \le \delta$, there is a unital w^*-continuous completely bounded algebra homomorphism $\pi : \mathcal{M} \to \mathcal{N}$ such that $\|L - \pi\|_{\mathrm{cb}} \le \varepsilon$.

Proof. Let $\varepsilon \in (0, 1)$, $\mu > 0$ and let L be a unital w^*-continuous linear map from $\mathcal{M}$ into $\mathcal{N}$ such that $\|L\|_{\mathrm{cb}} \le \mu$. The trilinear map

$$(x, y, z) \in \mathcal{M} \times \mathcal{M} \times \mathcal{M} \mapsto L(x)L^\vee(y, z) \in \mathcal{N} \tag{3-1}$$

is separately w^*-continuous and completely bounded. By the universal property of the normal Haagerup tensor product, it extends to a w^*-continuous completely bounded linear map

$$\Lambda_L : \mathcal{M} \otimes_{\sigma h} \mathcal{M} \otimes_{\sigma h} \mathcal{M} \to \mathcal{N}$$

such that

$$\Lambda_L(x \otimes y \otimes z) = L(x)L^\vee(y, z)$$

and $\|\Lambda_L\|_{\mathrm{cb}} \le \|L\|_{\mathrm{cb}}\|L^\vee\|_{\mathrm{cb}}$. By definition and associativity of the normal Haagerup tensor product, the linear map

$$m \in \mathcal{M} \mapsto u \otimes m \in (\mathcal{M} \otimes_{\sigma h} \mathcal{M}) \otimes_{\sigma h} \mathcal{M} = \mathcal{M} \otimes_{\sigma h} \mathcal{M} \otimes_{\sigma h} \mathcal{M}$$

is w^*-continuous; see (2-1). We can define $R : \mathcal{M} \to \mathcal{N}$ by

$$R(m) = \Lambda_L(u \otimes m)$$

which is w^*-continuous and

$$\|R\|_{\mathrm{cb}} \le \|u\| \, \|L\|_{\mathrm{cb}}\|L^\vee\|_{\mathrm{cb}}. \tag{3-2}$$

As $u \in \mathcal{M} \otimes_{\sigma h} \mathcal{M}$, there is a net $(u_t)_t$ in $\mathcal{M} \otimes \mathcal{M}$ converging to u in the w^*-topology of $\mathcal{M} \otimes_{\sigma h} \mathcal{M}$. For any t, there are finite families $(a_k^t)_k$, $(b_k^t)_k$ of elements in $\mathcal{M}$ such that

$$u_t = \sum_k a_k^t \otimes b_k^t.$$

Now fixing $m \in \mathcal{M}$, once again by definition and associativity of the normal Haagerup tensor product, the linear map

$$v \in \mathcal{M} \otimes_{\sigma h} \mathcal{M} \mapsto v \otimes m \in (\mathcal{M} \otimes_{\sigma h} \mathcal{M}) \otimes_{\sigma h} \mathcal{M} = \mathcal{M} \otimes_{\sigma h} \mathcal{M} \otimes_{\sigma h} \mathcal{M}$$

is w^*-continuous as well. Hence, using the w^*-continuity of Λ_L, we obtain

$$R(m) = w^* - \lim_t \Lambda_L(u_t \otimes m) = w^* - \lim_t \sum_k L(a_k^t)L^\vee(b_k^t, m).$$

From this point, we just need to check that the computations of [Johnson 1988, Theorem 3.1] remain valid with matrix coefficients. Fix $n \in \mathbb{N}$, let x, y be in the unit ball of $\mathbb{M}_n(\mathcal{M})$ (in the following computation, I_n denotes the identity matrix in $\mathbb{M}_n$, and the other subscripts n denote the n-th ampliation of a linear or bilinear map), then as in [Johnson 1988], we have

$$
\begin{aligned}
(L+R)_n^{\vee}(x,y)= &\\
&L_n^{\vee}(x,y)-w^*-\lim_t\sum_k L_n(I_n\otimes a_k^t b_k^t)L_n^{\vee}(x,y)\\
&-R_n(x)R_n(y)\\
&+w^*-\lim_t\sum_k L_n^{\vee}(x,I_n\otimes a_k^t)L_n^{\vee}(I_n\otimes b_k^t,y)\\
&+w^*-\lim_t\sum_k L_n^{\vee}(I_n\otimes a_k^t,I_n\otimes b_k^t)L_n^{\vee}(x,y)\\
&+w^*-\lim_t\sum_k L_n(I_n\otimes a_k^t)L_n((I_n\otimes b_k^t)xy)-L_n(x(I_n\otimes a_k^t))L_n((I_n\otimes b_k^t)y)\\
&-w^*-\lim_t\sum_k(L_n(I_n\otimes a_k^t)L_n((I_n\otimes b_k^t)x)-L_n(x(I_n\otimes a_k^t))L_n(I_n\otimes b_k^t))L_n(y).
\end{aligned}
$$

To evaluate the norm of $(L+R)_n^{\vee}$, we treat each line of the right-hand side successively. As u is a normal virtual h-diagonal, $w^*-\lim_t\sum_k a_k^t b_k^t=1$. But L is unital and w^*-continuous, so

$$w^*-\lim_t\sum_k L_n(I_n\otimes a_k^t b_k^t)=1$$

and the first line of the right-hand side is 0. Clearly, the norm of the term in the second line is bounded by $\|R\|_{\mathrm{cb}}^2$. Now let us show that the norm of the term in the third line is bounded by $\|u\|\,\|L^{\vee}\|_{\mathrm{cb}}^2$. The quadrilinear map

$$(x,y,z,t)\in\mathcal{M}\times\mathcal{M}\times\mathcal{M}\times\mathcal{M}\mapsto L^{\vee}(x,y)L^{\vee}(z,t)\in\mathcal{N} \tag{3-3}$$

is separately w^*-continuous and completely bounded. By the universal property of the normal Haagerup tensor product, it extends to a w^*-continuous completely bounded linear map $\Gamma_L:\mathcal{M}\otimes_{\sigma h}\mathcal{M}\otimes_{\sigma h}\mathcal{M}\otimes_{\sigma h}\mathcal{M}\to\mathcal{N}$ such that

$$\Gamma_L(x\otimes y\otimes z\otimes t)=L^{\vee}(x,y)L^{\vee}(z,t)$$

and $\|\Gamma_L\|_{\mathrm{cb}}\le\|L^{\vee}\|_{\mathrm{cb}}\|L^{\vee}\|_{\mathrm{cb}}$. The bilinear map

$$B:(x,y)\in\mathcal{M}\times\mathcal{M}\mapsto x\otimes u\otimes y\in\mathcal{M}\otimes_{\sigma h}\mathcal{M}\otimes_{\sigma h}\mathcal{M}\otimes_{\sigma h}\mathcal{M}$$

is separately w^*-continuous and $\|B\|_{\mathrm{cb}}\le\|u\|$. The bilinear map $\Gamma_L\circ B:\mathcal{M}\times\mathcal{M}\to\mathcal{N}$ is also separately w^*-continuous and

$$\|\Gamma_L\circ B\|_{\mathrm{cb}}\le\|\Gamma_L\|_{\mathrm{cb}}\,\|B\|_{\mathrm{cb}}\le\|u\|\,\|L^{\vee}\|_{\mathrm{cb}}^2.$$

We claim that the term of the third line is the n-th ampliation of the bilinear map $\Gamma_L\circ B$ applied to x, y in the unit ball of $\mathbb{M}_n(\mathcal{M})$ (and this gives the desired estimate).

Note first that

$$B_n(x,y)=\Big[\sum_l x_{il}\otimes u\otimes y_{lj}\Big]_{i,j}=w^*-\lim_t\sum_k\Big[\sum_l x_{il}\otimes a_k^t\otimes b_k^t\otimes y_{lj}\Big]_{i,j}$$

and also that $(L^\vee)_n=(L_n)^\vee$ and

$$(L^\vee)_n(x,I_n\otimes a_k^t)=\Big[\sum_l L^\vee(x_{il},(I_n\otimes a_k^t)_{lj})\Big]_{i,j}=\Big[L^\vee(x_{ij},a_k^t)\Big]_{i,j}$$

and similarly

$$(L^\vee)_n(I_n\otimes b_k^t,y)=\big[L^\vee(b_k^t,y_{ij})\big]_{i,j}.$$

Using these computations, we can prove our claim:

$$\begin{aligned}(\Gamma_L\circ B)_n(x,y)&=(\Gamma_L)_n(B_n(x,y))\\&=(\Gamma_L)_n\Big(w^*-\lim_t\sum_k\Big[\sum_l x_{il}\otimes a_k^t\otimes b_k^t\otimes y_{lj}\Big]_{i,j}\Big)\\&=w^*-\lim_t\sum_k\Big[\sum_l\Gamma_L(x_{il}\otimes a_k^t\otimes b_k^t\otimes y_{lj})\Big]_{i,j}\\&=w^*-\lim_t\sum_k\Big[\sum_l L^\vee(x_{il},a_k^t)L^\vee(b_k^t,y_{lj})\Big]_{i,j}\\&=w^*-\lim_t\sum_k\big([L^\vee(x_{ij},a_k^t)]_{i,j}\cdot[L^\vee(b_k^t,y_{ij})]_{i,j}\big)\\&=w^*-\lim_t\sum_k L_n^\vee(x,I_n\otimes a_k^t)L_n^\vee(I_n\otimes b_k^t,y).\end{aligned}$$

Consequently, we can estimate the norm of the term of the third line

$$\Big\|w^*-\lim_t\sum_k L_n^\vee(x,I_n\otimes a_k^t)L_n^\vee(I_n\otimes b_k^t,y)\Big\|\le\|\Gamma_L\circ B\|_{\mathrm{cb}}\le\|u\|\,\|L^\vee\|_{\mathrm{cb}}^2.$$

In the same manner, one can prove that the norm of the term in the fourth line is bounded by $\|u\|\,\|L^\vee\|_{\mathrm{cb}}^2$. For the term in the fifth line, note that its (i,j)-entry is

$$w^*-\lim_t\sum_k\sum_{p=1}^n\big(L(a_k^t)L(b_k^tx_{ip}y_{pj})-L(x_{ip}a_k^t)L(b_k^ty_{pj})\big)\in\mathcal{N}.$$

But u is a normal virtual h-diagonal, so for any i and p,

$$w^*-\lim_t(x_{ip}\cdot u_t-u_t\cdot x_{ip})=0,$$

hence for any i, j, p,

$$w^*-\lim_t\Big(\sum_k x_{ip}\cdot a_k^t\otimes b_k^t\cdot y_{pj}-\sum_k a_k^t\otimes b_k^t\cdot x_{ip}y_{pj}\Big)=0.$$

The bilinear map

$$(x, y) \in \mathcal{M} \times \mathcal{M} \mapsto L(x)L(y) \in \mathcal{N}$$

extends to a w^*-continuous map, consequently the term in the fifth line is 0. Analogously, the term in the sixth line is also 0. Finally we obtain

$$\|(L+R)^\vee\|_{\mathrm{cb}} \le 2\|u\|\|L^\vee\|_{\mathrm{cb}}^2 + \|R\|_{\mathrm{cb}}^2 \le (2\|u\| + \|u\|^2\|L\|_{\mathrm{cb}}^2)\|L^\vee\|_{\mathrm{cb}}^2. \tag{3-4}$$

Now we are in position to follow the induction of [Johnson 1988] with cb-norms instead of norms (for the reader's convenience, we reproduce it here). The important point is that each L^q (and thus each R^q) defined below is w^*-continuous. Define

$$\delta = \frac{\varepsilon}{4\|u\| + 8\mu^2\|u\|^2}. \tag{3-5}$$

Suppose that $\|L^\vee\|_{\mathrm{cb}} \le \delta$. Inductively, we define a sequence of linear maps from $\mathcal{M}$ into $\mathcal{N}$ by $L_0 = L$ and $R_0 = R$, and for $q \ge 0$,

$$L^{q+1} = L^q + R^q \quad \text{and} \quad R^{q+1}(\,\cdot\,) = \Lambda_{L^{q+1}}(u \otimes \,\cdot\,).$$

We also define $\mu_q = (2 - 2^{-q})\mu$ and $\delta_q = 2^{-q}\delta$. By induction, we prove that $\|(L^q)^\vee\|_{\mathrm{cb}} \le \delta_q$ and $\|L^q\|_{\mathrm{cb}} \le \mu_q$, for all q. It is obvious for $q = 0$. Then using the inequality (3-4) above, we have

$$\|(L^{q+1})^\vee\|_{\mathrm{cb}} \le (2\|u\| + \|u\|^2\mu_q^2)\delta_q^2 \le \delta_{q+1},$$

and using (3-2) to majorize the cb-norm of R^q, we obtain

$$\|L^{q+1}\|_{\mathrm{cb}} \le \mu_q + \|u\|\mu_q\delta_q \le \mu_{q+1},$$

(the last inequality coming from the fact that $\|u\|\delta \le 4^{-1}$). Consequently,

$$\|R^q\|_{\mathrm{cb}} \le \|u\|\|L^q\|_{\mathrm{cb}}\,\|(L^q)^\vee\|_{\mathrm{cb}} \le 2\|u\|\mu\delta_q,$$

so $\sum_{q\ge 0} R^q$ converges in cb-norm. We can define

$$\pi = L + \sum_{q\ge 0} R^q,$$

in other words $\pi = \lim_q L^q$, so π is w^*-continuous. Hence $\pi^\vee = \lim_q (L^q)^\vee$, but we proved that $\|(L^q)^\vee\|_{\mathrm{cb}} \le \delta_q$, so π is multiplicative. Moreover,

$$\|\pi - L\|_{\mathrm{cb}} = \|\sum_{q\ge 0} R^q\|_{\mathrm{cb}} \le 4\|u\|\mu\delta < \varepsilon. \qquad \square$$

Remark 3.3. One important point which does not appear in the statement of the previous theorem is that δ is an explicit function of μ, ε and $\|u\|$; see (3-5).

4. Neighboring representations

We now show that two representations of a dual operator algebra, admitting a normal virtual h-diagonal, which are close enough in cb-norm are necessarily similar. Apparently, this phenomena is well known to Banach algebraists (see, e.g., Chapter 8 of [Runde 2002]). We give here a quick proof for dual operator algebras. This proposition will enable us to perform the third step mentioned in the Introduction. If $S \in \mathbb{B}(H)$ is an invertible operator, we denote by Ad_S the similarity implemented by S.

Proposition 4.1. *Let $\mathcal{M}$ be a unital dual operator algebra. We suppose that $\mathcal{M}$ has a normal virtual h-diagonal $u \in \mathcal{M} \otimes_{\sigma h} \mathcal{M}$. Let π_1 and π_2 be two unital w^*-continuous completely bounded representations on the same Hilbert space K. If $\|\pi_1 - \pi_2\|_{\mathrm{cb}} < \|u\|^{-1} \max\{\|\pi_1\|_{\mathrm{cb}}^{-1}, \|\pi_2\|_{\mathrm{cb}}^{-1}\}$, then there exists an invertible operator S in the w^*-closed algebra generated by $\pi_1(\mathcal{M}) \cup \pi_2(\mathcal{M})$ such that $\pi_1 = \mathrm{Ad}_S \circ \pi_2$.*

Proof. Let π_1, π_2 be as above. For two completely bounded w^*-continuous linear maps $F, G : \mathcal{M} \to \mathbb{B}(K)$, we denote (with notation of Sections 2B and 2C)

$$\Psi_{F,G} = \mathrm{m}_{\sigma h} \circ (F \otimes_{\sigma h} G),$$

which is a completely bounded w^*-continuous linear map defined on $\mathcal{M} \otimes_{\sigma h} \mathcal{M}$. Now, define

$$S = \Psi_{\pi_1,\pi_2}(u) \in \mathbb{B}(K).$$

As $u \in \mathcal{M} \otimes_{\sigma h} \mathcal{M}$, there is a net $(u_t)_t$ in $\mathcal{M} \otimes \mathcal{M}$ converging to u in the w^*-topology of $\mathcal{M} \otimes_{\sigma h} \mathcal{M}$. For any t, there are finite families $(a_k^t)_k$, $(b_k^t)_k$ of elements in $\mathcal{M}$ such that

$$u_t = \sum_k a_k^t \otimes b_k^t.$$

Hence, $S = w^* - \lim_t \sum_k \pi_1(a_k^t)\pi_2(b_k^t)$. Let $m \in \mathcal{M}$, then

$$\begin{aligned} \pi_1(m)S &= \pi_1(m).w^* - \lim_t \sum_k \pi_1(a_k^t)\pi_2(b_k^t) \\ &= w^* - \lim_t \sum_k \pi_1(m a_k^t)\pi_2(b_k^t) \\ &= w^* - \lim_t \Psi_{\pi_1,\pi_2}(m \cdot u_t) \\ &= \Psi_{\pi_1,\pi_2}(m \cdot u). \end{aligned}$$

Analogously, we can show that

$$S\pi_2(m) = \Psi_{\pi_1,\pi_2}(u \cdot m).$$

But u is a normal virtual h-diagonal, so $m \cdot u = u \cdot m$, hence

$$\pi_1(m)S = S\pi_2(m).$$

To conclude, we just need to prove that S is invertible. Without loss of generality we can assume that $\|\pi_1 - \pi_2\|_{\mathrm{cb}} < \|u\|^{-1}\|\pi_1\|_{\mathrm{cb}}^{-1}$. As above, we have $\Psi_{\pi_1,\pi_1}(u) = w^* - \lim_t \sum_k \pi_1(a_k^t)\pi_1(b_k^t)$. Using the condition (C2) defining a normal virtual h-diagonal, we obtain

$$\begin{aligned}\Psi_{\pi_1,\pi_1}(u) =& w^* - \lim_t \sum_k \pi_1(a_k^t b_k^t) = \pi_1\Big(w^* - \lim_t \sum_k a_k^t b_k^t\Big)\\ =&\pi_1(\mathrm{m}_{\sigma h}(u)) = \pi_1(1) = I_K.\end{aligned}$$

Consequently,

$$\begin{aligned}\|S - I_K\| &= \|\Psi_{\pi_1,\pi_2}(u) - \Psi_{\pi_1,\pi_1}(u)\| = \|\Psi_{\pi_1,\pi_2-\pi_1}(u)\|\\ &\leq \|u\|\|\pi_1 - \pi_2\|_{\mathrm{cb}}\|\pi_1\|_{\mathrm{cb}} < 1.\end{aligned}$$

□

5. Proof of the main theorems

We start this section with a very simple lemma that we will use repeatedly in the proof of the next theorem; we just sketch the proof. Recall that $T^\vee$ denotes the bilinear map from $\mathcal{M} \times \mathcal{M}$ into $\mathcal{N}$ defined by $T^\vee(x, y) = T(xy) - T(x)T(y)$. Also in this section, we denote by $\mathrm{id}_{\mathcal{A}}$ the identity representation of a concretely represented operator algebra $\mathcal{A}$.

Lemma 5.1. *Let $\mathcal{A}, \mathcal{B} \subset \mathbb{B}(H)$ be two operator algebras and $T : \mathcal{A} \to \mathcal{B}$ be a completely bounded linear map. Then*:

(i) $\|T^\vee\|_{\mathrm{cb}} \leq (2 + \|T\|_{\mathrm{cb}})\|T - \mathrm{id}_{\mathcal{A}}\|_{\mathrm{cb}}$.

(ii) *If $\|T - \mathrm{id}_{\mathcal{A}}\|_{\mathrm{cb}} < 1$, then T is injective and has closed range. Moreover*, *if there exists $\alpha \in [0, 1)$ such that for any y in the unit ball of $\mathcal{B}$*, *there is x in $\mathcal{A}$ satisfying $\|T(x) - y\| \leq \alpha$, then T is bijective and*

$$\|T^{-1}\|_{\mathrm{cb}} \leq \frac{1}{1 - \|T - \mathrm{id}_{\mathcal{A}}\|_{\mathrm{cb}}}.$$

Proof. Let x, y be in the unit ball of $\mathbb{M}_n(\mathcal{M})$, then (i) follows from the decomposition

$$\begin{aligned}(T^\vee)_n(x, y) &= T_n(xy) - T_n(x)T_n(y)\\ &= T_n(xy) - xy + xy - xT_n(y) + xT_n(y) - T_n(x)T_n(y).\end{aligned}$$

The first assertion of (ii) follows from

$$\|T_n(x)\| \geq \big|\|T_n(x) - x\| - \|x\|\big| \geq (1 - \|T - \mathrm{id}_{\mathcal{A}}\|_{\mathrm{cb}})\|x\|.$$

The surjectivity of T is proved by induction. Let y be in the unit ball of $\mathcal{N}$, then for any integer j, we can find $t_1, \dots, t_j$ in the range of T such that

$$\|y - (t_1 + t_2 + \cdots + t_j)\| \leq \alpha^j.$$

As $\alpha < 1$, we conclude that y belongs to the closure of the range of T. □

Note that, in the following theorem, $\mathcal{M}$ is just assumed to be a dual operator algebra, but we require a near cb-inclusion of $\mathcal{M}$ into $\mathcal{N}$ (see Section 2A).

Theorem 5.2. *Let $\mathcal{M}, \mathcal{N} \subset \mathbb{B}(H)$ be two unital w^*-closed operator algebras. We suppose that $\mathcal{N}$ is an injective von Neumann algebra. If $\mathcal{N} \subset^1 \mathcal{M}$ and $\mathcal{M} \subseteq^{\gamma}_{\mathrm{cb}} \mathcal{N}$, with $\gamma < \frac{1}{164}$, then there exists an invertible operator S in the w^*-closed algebra generated by $\mathcal{M} \cup \mathcal{N}$ such that $S\mathcal{M}S^{-1} = \mathcal{N}$.*

Proof. Since $\mathcal{N}$ is injective, there is a completely contractive projection P from $\mathbb{B}(H)$ onto $\mathcal{N}$. Denote $T = P_{|\mathcal{M}}$. Let x be in the unit ball of $\mathbb{M}_n(\mathcal{M})$, then there is y in $\mathbb{M}_n(\mathcal{N})$ such that $\|x - y\| \leq \gamma$.

$$\|T_n(x) - x\| = \|T_n(x - y) - (x - y)\|_{\mathrm{cb}} \leq 2\gamma,$$

hence

$$\|T - \mathrm{id}_{\mathcal{M}}\|_{\mathrm{cb}} \leq 2\gamma < 1.$$

Let us prove that T is surjective. Since $\mathcal{N} \subset^1 \mathcal{M}$, there is $\gamma' < 1$ such that $\mathcal{N} \subseteq^{\gamma'} \mathcal{M}$. Let y be in the unit ball of $\mathcal{N}$, then there exists x in $\mathcal{M}$ such that $\|y - x\| \leq \gamma'$, therefore from Lemma 5.1(ii), T is a linear cb-isomorphism and

$$\|T^{-1}\|_{\mathrm{cb}} \leq \frac{1}{1-2\gamma}. \tag{5-1}$$

The problem is that T is not necessarily w^*-continuous, so we are going to consider the normal of T^{-1} (see [Tomiyama 1959], we denote with an exponent n the normal part of a linear map defined on $\mathcal{N}$). Note first that

$$\|T^{-1} - \mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}} \leq \|T^{-1}\|_{\mathrm{cb}} \|T - \mathrm{id}_{\mathcal{M}}\|_{\mathrm{cb}} \leq \frac{2\gamma}{1-2\gamma}. \tag{5-2}$$

Let $V = (T^{-1})^{\mathrm{n}} : \mathcal{N} \to \mathcal{M}$ be the normal part of T^{-1}. Using Lemma 5.1(ii) again, let us show that V is a completely bounded w^*-continuous linear isomorphism from $\mathcal{N}$ onto $\mathcal{M}$. As taking the normal part is a completely contractive operation, we have

$$\|V - \mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}} = \|(T^{-1} - \mathrm{id}_{\mathcal{N}})^{\mathrm{n}}\|_{\mathrm{cb}} \leq \|T^{-1} - \mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}} \leq \frac{2\gamma}{1-2\gamma}, \tag{5-3}$$

thus V is an injective map and has closed range. Now let y be in the unit ball of $\mathcal{M}$, and pick x in $\mathcal{N}$ such that $\|x - y\| \leq \gamma$. Thus $\|x\| \leq 1 + \gamma$ and

$$\begin{aligned}\|V(x) - y\| &\leq \|V(x) - x\| + \|x - y\| \\ &\leq \frac{2\gamma}{1-2\gamma}(1+\gamma) + \gamma \\ &\leq \frac{5\gamma}{1-2\gamma} < 1,\end{aligned}$$

so V is surjective.

In order to apply Theorem 3.2, we need to unitize V. From equation (5-3), $\|V(1)-1\| \le (2\gamma)/(1-2\gamma) < 1$, hence $V(1)$ is invertible in $\mathcal{M}$ and

$$\|V(1)^{-1}\| \le \frac{1}{1-\|V(1)-1\|} \le \frac{1-2\gamma}{1-4\gamma}. \tag{5-4}$$

Denote $L = V(1)^{-1}V$, so L is a unital w^*-continuous completely bounded isomorphism from $\mathcal{N}$ onto $\mathcal{M}$ and from (5-1) we have

$$\|L\|_{\text{cb}} \le \|V(1)^{-1}\|\|V\|_{\text{cb}} \le \|V(1)^{-1}\|\|T^{-1}\|_{\text{cb}} \le \frac{1}{1-4\gamma}. \tag{5-5}$$

Let us compute the norm of $L^\vee$, the defect of multiplicativity of L (see Section 3 for notation). Fix n, let x be in unit ball of $\mathbb{M}_n(\mathcal{N})$, then from (5-3) and (5-4) we obtain

$$\begin{aligned}\|L_n(x)-x\| &\le \|I_n \otimes V(1)^{-1}(V_n(x)-x)\| + \|I_n \otimes V(1)^{-1}x - x\|\\ &\le \|V(1)^{-1}\|\|V-\mathrm{id}_{\mathcal{N}}\|_{\text{cb}} + \|V(1)^{-1}\|\|V(1)-1\|\\ &\le \frac{4\gamma}{1-4\gamma},\end{aligned}$$

which means that

$$\|L-\mathrm{id}_{\mathcal{N}}\|_{\text{cb}} \le \frac{4\gamma}{1-4\gamma}. \tag{5-6}$$

Therefore, by Lemma 5.1(i) and equation (5-5) we obtain

$$\|L^\vee\|_{\text{cb}} \le (2+\|L\|_{\text{cb}})\|L-\mathrm{id}_{\mathcal{N}}\|_{\text{cb}} \le \frac{12\gamma}{(1-4\gamma)^2}.$$

We want to apply Theorem 3.2 to L. Put

$$\mu = \frac{1}{1-4\gamma} \quad \text{and} \quad \delta = \frac{12\gamma}{(1-4\gamma)^2}.$$

As $\mathcal{N}$ is an injective von Neumann algebra, we can find a normal virtual h-diagonal u of norm 1 [Effros 1988; Effros and Kishimoto 1987], and thus (see (3-5)) let

$$\varepsilon = \delta(4\|u\| + 8\mu^2\|u\|^2) = \frac{12\gamma}{(1-4\gamma)^2}\left(4 + \frac{8}{(1-4\gamma)^2}\right).$$

We can then apply Theorem 3.2 to L and find a unital w^*-continuous completely bounded homomorphism $\pi : \mathcal{N} \to \mathcal{M}$ such that

$$\|L-\pi\|_{\text{cb}} \le \varepsilon.$$

Consequently, from (5-6),

$$\begin{aligned}\|\pi-\mathrm{id}_{\mathcal{N}}\|_{\text{cb}} &\le \|\pi-L\|_{\text{cb}} + \|L-\mathrm{id}_{\mathcal{N}}\|_{\text{cb}}\\ &\le \varepsilon + \frac{4\gamma}{1-4\gamma},\end{aligned}$$

and this last quantity is strictly smaller than 1, because $\gamma < \frac{1}{164}$. Therefore, we can apply Proposition 4.1 to π and $\mathrm{id}_{\mathcal{N}}$ and find an invertible operator S in the w^*-closed algebra generated by $\mathcal{M} \cup \mathcal{N}$ such that

$$\mathrm{Ad}_S \circ \pi = \mathrm{id}_{\mathcal{N}}$$

(in particular π is injective and has closed range). To achieve the proof, it is sufficient to prove that the range of π is $\mathcal{M}$. Let y be in the unit ball of $\mathcal{M}$, then

$$\|\pi(L^{-1}(y)) - y\| \leq \|\pi - L\|_{\mathrm{cb}} \|L^{-1}\|_{\mathrm{cb}},$$

so by Lemma 5.1(ii), we just need to check that this last quantity is strictly smaller than 1. From (5-6)

$$\|L^{-1}\|_{\mathrm{cb}} \leq \frac{1}{1 - \|L - \mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}}} \leq \frac{1-4\gamma}{1-8\gamma},$$

it follows that

$$\|\pi - L\|_{\mathrm{cb}} \, \|L^{-1}\|_{\mathrm{cb}} \leq \frac{1-4\gamma}{1-8\gamma} \varepsilon$$

which is strictly smaller than 1, because $\gamma < \frac{1}{164}$. □

At this point, we want to get rid of the near cb-inclusion hypothesis appearing in the previous theorem. The task is to find conditions of "automatic near cb-inclusion" on the algebra $\mathcal{M}$. More explicitly, under which conditions does a near inclusion $\mathcal{M} \subseteq^{\gamma} \mathcal{N}$ automatically imply a near cb-inclusion? For C^*-algebras, Christensen isolated property D_k which ensures such an "automatic near cb-inclusion" result. Recall that a C^*-algebra $\mathcal{A}$ has property D_k if for any unital $*$-representation (π, K) one has

$$\forall x \in \mathbb{B}(K), \quad \mathrm{d}(x, \pi(\mathcal{A})') \leq k \|\delta(x)_{|\pi(\mathcal{A})}\|,$$

where d denotes the usual distance between subsets and $\delta(x)$ denotes the inner derivation implemented by x on $\mathbb{B}(K)$. It is well known that amenable C^*-algebras (or injective von Neumann algebras) have D_1, the next easy lemma is the nonselfadjoint analog of this fact (it also works for amenable Banach algebras).

Lemma 5.3. *Let $\mathcal{M}$ be a unital w^*-closed operator admitting a normal virtual diagonal $u \in \mathcal{M} \widehat{\otimes}_\sigma \mathcal{M}$. Then, for any unital w^*-continuous contractive representation (π, K) of $\mathcal{M}$ which satisfies $\pi(\mathcal{M}) = \overline{\pi(\mathcal{M})}^{w^*}$, we have*

$$\forall x \in \mathbb{B}(K), \quad \mathrm{d}(x, \pi(\mathcal{M})') \leq \|u\| \, \|\delta(x)_{|\pi(\mathcal{M})}\|. \tag{5-7}$$

Proof. Let us denote $\mathcal{N} = \pi(\mathcal{M}) \subset \mathbb{B}(K)$ and $v = \pi \widehat{\otimes}_\sigma \pi(u) \in \mathcal{N} \widehat{\otimes}_\sigma \mathcal{N}$, hence $\|v\| \leq \|u\|$. Since π has w^*-closed range, v is a normal virtual diagonal for the dual operator algebra $\mathcal{N}$. Note that $\mathbb{B}(K)$ is obviously a normal dual Banach $\mathcal{N}$-bimodule (in the sense of [Runde 2002, Definition 4.4.6]). Now, let x be in $\mathbb{B}(K)$ and consider the w^*-continuous bounded derivation $D = \delta(x)_{|\mathcal{N}} : \mathcal{N} \to \mathbb{B}(K)$. Adapting the proof of Johnson's theorem on characterization of amenability by virtual diagonals, we know that there is $\varphi \in \mathbb{B}(K)$ such that $D = \delta(\varphi)_{|\mathcal{N}}$. Actually $\varphi = D\widehat{\otimes}_\sigma \mathrm{id}_{\mathcal{N}}(v)$, so $\|\varphi\| \le \|D\|\|v\|$. As $D = \delta(x)_{|\mathcal{N}} = \delta(\varphi)_{|\mathcal{N}}$, we have $x - \varphi \in \mathcal{N}'$. Therefore,

$$\mathrm{d}(x, \pi(\mathcal{M})') = \mathrm{d}(\varphi, \mathcal{N}') \le \|\varphi\| \le \|D\|\|v\| \le \|u\|\|\delta(x)_{|\pi(\mathcal{M})}\|,$$

which ends the proof. □

Lemma 5.4. *Let $\mathcal{M} \subset \mathbb{B}(H)$ be a unital w^*-closed operator algebra admitting a normal virtual diagonal $u \in \mathcal{M}\widehat{\otimes}_\sigma\mathcal{M}$. Let $\mathcal{N}$ be an injective von Neumann subalgebra of $\mathbb{B}(H)$. Then, for any $\gamma > 0$, $\mathcal{M} \subseteq^\gamma \mathcal{N}$ implies $\mathcal{M} \subseteq_{\mathrm{cb}}^{4\|u\|\gamma} \mathcal{N}$.*

Proof. This follows from the previous lemma and the first lines of the proof of Theorem 3.1 in [Christensen 1980], with $D = \mathbb{M}_n$ (for arbitrary n), with $k = \|u\|$ (by (5-7)) and the $\frac{3}{2}$ must replaced by 1 because $\mathcal{N}$ is injective, so we get $4\|u\|\gamma$ instead of $6k\gamma$. □

Now, using the previous lemma and Theorem 5.2 above, we can prove Theorem 2.

This question of "automatic near cb-inclusion" can be thought of as an analog of the "automatic complete boundedness" question for homomorphisms (or equivalently Kadison's similarity problem). For this problem, Pisier defined the notion of length for operator algebras (see [Pisier 1998; 2000; 2001a; 2001b; 2007]). The connection between this notion of length and property D_k is now well known for C^*-algebras, see [Christensen et al. 2010a]. As we are working with dual operator algebras, C. Le Merdy's notion of length (or degree) denoted d_* in [Le Merdy 2000] is more appropriate (we call this quantity *normal length* in the following corollary).

Corollary 5.5. *Let $\mathcal{M}, \mathcal{N} \subset \mathbb{B}(H)$ be two unital w^*-closed operator algebras. Suppose that $\mathcal{M}$ has finite normal length at most d_* with constant at most $C > 0$. We suppose that $\mathcal{N}$ is an injective von Neumann algebra. If $\mathcal{N} \subset^1 \mathcal{M}$ and $\mathcal{M} \subseteq^\gamma \mathcal{N}$, with $\gamma < (1 + 1/(164C))^{1/d_*} - 1$, then there exists an invertible operator S in the w^*-closed algebra generated by $\mathcal{M} \cup \mathcal{N}$ such that $S\mathcal{M}S^{-1} = \mathcal{N}$ and consequently, $d_*(\mathcal{M}) \le 2$.*

Proof. If $\mathcal{M} \subseteq^\gamma \mathcal{N}$, then $\mathcal{M} \subseteq_{\mathrm{cb}}^{C((1+\gamma)^{d_*}-1)} \mathcal{N}$ as in Proposition 2.10 in [Christensen et al. 2010a]. The result follows from the similarity degree characterization of injectivity for von Neumann algebras in [Pisier 2006]. □

Remark 5.6. As explained in the Introduction, the main benefit of Theorem 2 (compared to Theorem 1) is that we can get rid of the selfadjointness hypothesis on one of the algebras. It would be very interesting to improve our theorem to both algebras being nonselfadjoint. More precisely, let $\mathcal{M}, \mathcal{N} \subset \mathbb{B}(H)$ be two unital w^*-closed operator algebras, suppose that $\mathcal{M}$ has a normal virtual diagonal u and that $\mathcal{N}$ is the range of a completely bounded projection P. Does there exist a continuous function $f : [1, \infty)^2 \to [0, \infty)$ with $f(1, 1) = 0$ such that if $d_{\mathrm{cb}}(\mathcal{M}, \mathcal{N}) < f(\|P\|_{\mathrm{cb}}, \|u\|)$, then $\mathcal{M}$ and $\mathcal{N}$ are similar? In our proof of Theorem 2, the only characterization of injectivity of a von Neumann algebra that we use is that of being the range of completely contractive projection. This is one advantage of our proof, because if one wants to positively answer the preceding question, the only obstruction in our proof is to find a Tomiyama type decomposition (into normal and singular parts) for nonselfadjoint dual operator algebras admitting a normal virtual diagonal.

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