Pacific Journal of Mathematics

SCALAR CURVATURE AND SINGULAR METRICS

YUGUANG SHI AND LUEN-FAI TAM

Volume 293 No. 2 April 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 293, No. 2, 2018
dx.doi.org/10.2140/pjm.2018.293.427

SCALAR CURVATURE AND SINGULAR METRICS

YUGUANG SHI AND LUEN-FAI TAM

Let M^n, $n \geq 3$, be a compact differentiable manifold with nonpositive Yamabe invariant $\sigma(M)$. Suppose g_0 is a continuous metric with volume $V(M, g_0) = 1$, smooth outside a compact set Σ, and is in $W^{1,p}_{\mathrm{loc}}$ for some $p > n$. Suppose the scalar curvature of g_0 is at least $\sigma(M)$ outside Σ. We prove that g_0 is Einstein outside Σ if the codimension of Σ is at least 2. If in addition, g_0 is Lipschitz then g_0 is smooth and Einstein after a change of the smooth structure. If Σ is a compact embedded hypersurface, g_0 is smooth up to Σ from two sides of Σ, and if the difference of the mean curvatures along Σ at two sides of Σ has a fixed appropriate sign, then g_0 is also Einstein outside Σ. For manifolds with dimension between 3 and 7, without a spin assumption we obtain a positive mass theorem on an asymptotically flat manifold for metrics with a compact singular set of codimension at least 2.

1. Introduction

There are two celebrated results on manifolds with nonnegative scalar curvature. The first result is on compact manifolds. It was proved by Schoen and Yau [1979a; 1979c] that any smooth metric on a torus T^n, $n \leq 7$, with nonnegative scalar curvature must be flat. Later, the result was proved to be true for all n by Gromov and Lawson [1983]. The second result is the positive mass theorem on noncompact manifolds. Schoen and Yau [1979b; 1981; Schoen 1989] proved that the Arnowitt–Deser–Misner (ADM) mass of each end of an n-dimensional asymptotically flat (AF) manifold with $3 \leq n \leq 7$ with nonnegative scalar curvature is nonnegative and if the ADM mass of an end is zero, then the manifold is isometric to the Euclidean space. Under the additional assumption that the manifold is spin, the same result is still true and was proved by Witten [1981]; see also [Parker and Taubes 1982; Bartnik 1986]. In the two results the metrics are assumed to be smooth.

There are many results on positive mass theorem for nonsmooth metrics. Miao [2002] and the authors [Shi and Tam 2002] studied and proved positive mass

Shi was partially supported by NSFC 11671015. Tam was partially supported by Hong Kong RGC General Research Fund #CUHK 14305114.
MSC2010: primary 53C20; secondary 83C99.
Keywords: Yamabe invariants, positive mass theorems, singular metrics.

theorems for metrics with corners. The metrics are smooth away from a compact hypersurface, which are Lipschitz and satisfy certain conditions on the mean curvatures of the hypersurface. The result was used to prove the positivity of the Brown–York quasilocal mass [Shi and Tam 2002]. Lee [2013] considered a positive mass theorem for metrics with bounded C^2 norm and are smooth away from a singular set with codimension greater than $n/2$, where n is the dimension of the manifold. On the other hand, McFeron and Székelyhidi [2012] were able to prove Miao's result using Ricci flow and Ricci–DeTurck flow, which was studied in detail by Simon [2002]. There is a positive mass theorem for spin manifolds or manifolds with dimension n less than eight obtained by Grant and Tassotti [2014] under the assumptions that the metric is continuous and in Sobolev space $W^{2,n/2}_{\mathrm{loc}}$. More recently, Lee and LeFloch [2015] were able to prove for spin manifolds, under rather general conditions, a positive mass theorem for metrics which may be singular. Their theorem can be applied to all previous results for nonsmooth metrics under the additional assumption that the manifold is spin.

Motivated by these studies of singular metrics on AF manifolds, we want to understand singular metrics on compact manifolds. One of the questions is to see if there are nonflat metrics with nonnegative scalar curvature on T^n which may be singular somewhere. Another question can be described as follows. It is now well known that in every conformal class of smooth metrics on a compact manifold without boundary there is a metric with constant scalar curvature; see [Yamabe 1960; Trudinger 1968; Aubin 1976a; 1976b; Schoen 1984]. One motivation for the result is to obtain Einstein metric. It is well known that if a smooth metric on a compact manifold attains the Yamabe invariant and if the invariant is nonpositive, then the metric is Einstein. See [Schoen 1989, pp. 126–127]. In this work, we will study the question whether this last result is still true for nonsmooth metrics.

Let us recall the definition of Yamabe invariant, which is called σ-invariant in [Schoen 1989]. Let $\mathcal{C}$ be a conformal class of smooth Riemannian metrics g on a smooth compact manifold M^n; then the *Yamabe constant of* $\mathcal{C}$ is defined as

$$Y(\mathcal{C}) = \inf_{g\in\mathcal{C}} \frac{\int_M \mathcal{S}_g \, dv_g}{(V(M,g))^{1-2/n}},$$

where $\mathcal{S}_g$ is the scalar curvature and $V(M,g)$ is the volume of M with respect to g. The *Yamabe invariant* is defined as

$$\sigma(M) = \sup\nolimits_{\mathcal{C}} Y(\mathcal{C}).$$

The supremum is taken among all conformal classes of smooth metrics. It is finite; see [Aubin 1976a]. If g attains $\sigma(M) > 0$, then in general it is still unclear whether g is Einstein or not; see [Macbeth 2017].

To answer the question on Einstein metrics, let M^n be a compact smooth manifold without boundary and let g_0 be a continuous Riemannian metric on M with $V(M, g_0) = 1$ such that it is smooth outside a compact set Σ. The first case is that Σ has codimension at least 2 and g_0 is in $W^{1,p}_{\text{loc}}$ for some $p > n$ (see Sections 3 and 5 for more precise definitions).

Theorem 1.1. *Let (M^n, g_0) be as above. Suppose $\sigma(M) \leq 0$ and suppose the scalar curvature of g_0 outside Σ is at least $\sigma(M)$. Then g_0 is Einstein outside Σ. If in addition g_0 is Lipschitz, then after changing the smooth structure, g_0 is smooth and Einstein.*

In the case that Σ is a compact embedded hypersurface, as in [Miao 2002] we assume that near Σ, $g_0 = dt^2 + g_\pm(z, t)$, $z \in \Sigma$, $t \in (-\epsilon, \epsilon)$ such that (t, z) are smooth coordinates and $g_-(\,\cdot\,, 0) = g_+(\,\cdot\,, 0)$, where g_+, g_- are defined on the neighborhood of Σ where $t > 0$ and $t < 0$ respectively and are smooth up to Σ. Moreover, with respect to the unit normal $\frac{\partial}{\partial t}$ the mean curvature H_+ of Σ with respect to g_+ and the mean curvature H_- of Σ with respect to g_- satisfies $H_- \geq H_+$. Under these assumptions, we have:

Theorem 1.2. *Let (M^n, g_0) be as above with $V(m, g_0) = 1$. Suppose $\sigma(M) \leq 0$ and suppose the scalar curvature of g_0 outside Σ is at least $\sigma(M)$. Then g_0 is Einstein outside Σ. Moreover, $H_+ = H_-$.*

Note that it is easy to construct examples so that the theorem is not true if the assumption $H_- \geq H_+$ is removed.

In the process of proving the theorems, one also obtains the following: In the case that M^n is T^n, under the regularity assumptions in Theorem 1.1 or Theorem 1.2 and if g_0 has nonnegative scalar curvature outside Σ, then g_0 is flat outside Σ.

The method of proof of the above results can also be adapted to AF manifolds. We want to discuss the positive mass theorem with singular metric on an AF manifold with dimension $3 \leq n \leq 7$ without assuming that the manifold is spin. We will prove the following:

Theorem 1.3. *Let (M^n, g_0) be an AF manifold with $3 \leq n \leq 7$, where g_0 is a continuous metric on M with regularity assumptions as in Theorem 1.1. Suppose g_0 has nonnegative scalar curvature outside Σ. Then the ADM mass of each end is nonnegative. Moreover, if the ADM mass of one of the ends is zero, then M is diffeomorphic to $\mathbb{R}^n$ and is flat outside Σ.*

We should mention that all the results mentioned above for nonsmooth metrics, all the metrics are assumed to be continuous. On the other hand, one can construct an example of AF metric with a cone singularity and nonnegative scalar curvature and with negative ADM mass; see Section 2. One can also construct examples of metrics on compact manifolds with a cone singularity so that Theorem 1.1 is not true. In these examples, the metrics are not continuous.

The structure of the paper is as follows. In Section 2, we construct examples which are related to results in later sections; in Section 3 we obtain some estimates for the Ricci–DeTurck flow; in Section 4 we use the Ricci–DeTurck flow to approximate singular metrics; in Sections 5 and 6 we prove Theorems 1.1 and 1.2; in Section 7 we prove Theorem 1.3. In this work, the dimension of any manifold is assumed to be at least three. We will also use the Einstein summation convention.

2. Examples of metrics with cone singularities

In previous results on positive mass theorems on AF manifolds with singular metrics mentioned in Section 1, the metrics are all assumed to be continuous. To understand this condition on continuity and to motivate our study, in this section, we construct some examples with cone singularities which are related to the study in the later sections.

The following lemma is standard. See [Petersen 1998].

Lemma 2.1. *Consider the metric $g = dr^2 + \phi^2(r)h_0$ on $(0, r_0) \times \mathbb{S}^{n-1}$, where h_0 is the standard metric of $\mathbb{S}^{n-1}$, $n \geq 3$, and ϕ is a smooth positive function on $(0, r_0)$. Then the scalar curvature of g is given by*

$$\mathcal{S} = (n-1)\left[-\frac{2\phi''}{\phi} + (n-2)\frac{1-(\phi')^2}{\phi^2}\right].$$

Suppose $\phi = \alpha r^\beta$, with $\alpha, \beta > 0$. Then $\mathcal{S} > 0$ if $\alpha < 1$, $\beta = 1$ or if $0 < \beta \leq 2/n$. In both cases, the metric is not continuous up to $r = 0$. If $\alpha > 1$, $\beta = 1$, then $\mathcal{S} < 0$ for r small enough.

We can construct asymptotically flat manifolds with nonnegative scalar curvature defined on $\mathbb{R}^3 \setminus \{0\}$ such that the metric behaves like $dr^2 + (\alpha r)^2 h_0$ near the origin for some $0 < \alpha < 1$ with positive mass.

Proposition 2.2. *Let $0 < \epsilon < \frac{1}{2}$ and let $\eta(x) = \eta(r)$, with $r = |x|$, be a smooth function on $\mathbb{R}^3 \setminus \{0\}$ such that*

$$\begin{cases} \eta(r) = -\epsilon(1-\epsilon)r^{-\epsilon-2} & \text{if } 0 < r \leq 1, \\ \eta(r) < 0 & \text{if } 1 \leq r \leq 2, \\ \eta(r) = 0 & \text{if } r \geq 2. \end{cases}$$

Let ϕ be the function defined on $\mathbb{R}^3 \setminus \{0\}$ with

$$\phi(r) = \int_1^r \frac{1}{s^2}\left(\int_0^s t^2 \eta(t)\,dt\right) ds.$$

Then there are constants $a, b > 0$ *such that if*

$$u = \phi + b + \frac{a}{2} + 1$$

then $u > 0$. *Moreover, if* $g = u^4 g_e$, *where* g_e *is the standard Euclidean metric, then near infinity,*

$$g = \left(1 + \frac{a}{r}\right)^4 g_e,$$

and near $r = 0$,

$$g = d\rho^2 + \big((1-2\epsilon)^2\rho^2 + O(\rho^{2+\delta})\big)h_0$$

for some $\delta > 0$, *where*

$$\rho = \int_0^r u^2(t)\,dt.$$

The metric g *has nonnegative scalar curvature and has zero scalar curvature outside a compact set. Moreover, the end near infinity is asymptotically flat in the sense of Definition 7.1 in Section 7, and has positive mass* $2a$.

Proof. Let Δ_0 be the Euclidean Laplacian. Then one can check that

$$\Delta_0 \phi = \eta \le 0.$$

For $0 < r \le 1$,

$$\phi(r) = r^{-\epsilon} - 1.$$

For $r \ge 2$, let

$$a = -\int_0^r s^2 \eta(s)\,ds > 0,$$

and

$$b = -\int_1^2 \frac{1}{s^2}\left(\int_0^s \tau^2\eta(\tau)d\tau\right)ds > 0.$$

Then

$$\begin{aligned}
\phi(r) &= -b + \int_2^r \frac{1}{s^2}\left(\int_0^s t^2\eta(t)\,dt\right)ds \\
&= -b - a\int_2^r \frac{1}{s^2}\,ds \\
&= -b - \frac{a}{2} + \frac{a}{r}.
\end{aligned}$$

Hence if $u = \phi + b + a/2 + 1$, then $\Delta_0 u = \eta \le 0$. Since $u \to \infty$ as $r \to 0$ and $u \to 1$ as $r \to \infty$, $u > 0$ by the strong maximum principle. The metric

$$g = u^4 g_e$$

is defined on $\mathbb{R}^3 \setminus \{0\}$, has nonnegative scalar curvature and has zero scalar curvature near infinity. g is also asymptotically flat. Near $r = 0$,

$$u = b + \frac{a}{2} + r^{-\epsilon}.$$

Since $0 < \epsilon < \frac{1}{2}$, we let

$$\rho = \int_0^r u^2(t)\,dt = \frac{1}{(1-2\epsilon)} r^{1-2\epsilon} + O(r^{1-\epsilon}).$$

So

$$\rho^2 = \frac{1}{(1-2\epsilon)^2} r^{2-4\epsilon} + O(r^{2-3\epsilon}).$$

Hence near $r = 0$,

$$\begin{aligned} g &= d\rho^2 + u^4 r^2 h_0 \\ &= d\rho^2 + (r^{2-4\epsilon} + O(r^{2-3\epsilon}))h_0 \\ &= d\rho^2 + ((1-2\epsilon)^2\rho^2 + O(r^{2-3\epsilon}))h_0 \\ &= d\rho^2 + (\alpha^2\rho^2 + O(r^{2-3\epsilon}))h_0, \end{aligned}$$

where $\alpha = 1 - 2\epsilon$. Note that $r^{2-3\epsilon} = O(\rho^{2+\delta})$ for some $\delta > 0$. □

The following example is the type of singularity which is called zero area singularity in [Bray and Jauregui 2013].

Proposition 2.3. *Let $m > 0$ and let $\phi = 1 - 2m/r$. Then the metric*

$$g = \phi^4 g_e$$

is asymptotically flat defined on $r > 2m$ in $\mathbb{R}^3$, with zero scalar curvature and with negative mass $-m$. Moreover, near $r = 2m$,

$$g = d\rho^2 + c\rho^{4/3}(1 + O(\rho^{2/3}))h_0$$

for some $c > 0$, where

$$\rho = \int_0^{r-2m} \phi^2(t+2m)\,dt.$$

Hence near $\rho = 0$ the metric is asymptotically of the form as in Lemma 2.1 with $\beta = \frac{2}{3}$.

Proof. We only need to consider g near $r = 2m$. The rest is well known. Let $t = r - 2m$, $r > 2m$. Then

$$\tilde{\phi}(t) = \phi(t+2m) = \frac{t}{t+2m} = \frac{t}{2m}\left(1 - \frac{t}{2m} + \frac{t^2}{4m^2} + O(t^3)\right)$$

and

$$\rho = \int_0^t \tilde{\phi}^2(s)\, ds = \int_0^t \frac{s^2}{(s+2m)^2}\, ds.$$

Note that as $r \to 2m$, $\rho \to 0$. In terms of ρ, near $\rho = 0$,

$$g = d\rho^2 + \phi^4 r^2 h_0.$$

Near $\rho = 0$,

$$\begin{aligned}\phi^4 r^2 &= \frac{t^4}{(t+2m)^4}(t+2m)^2 \\ &= c\rho^{4/3}(1 + O(\rho^{2/3}))\end{aligned}$$

for some $c > 0$. □

We can also construct a conical metric on $T^3 \setminus \{\text{a point}\}$, with nonnegative scalar curvature and with positive scalar curvature somewhere.

First, we have

Proposition 2.4. *Let $m > 0$. There is a metric g on $\mathbf{R}^3 \setminus B(2m)$ such that*

(i) *the scalar curvature R is nonnegative and $R > 0$ somewhere*;

(ii) *there exist r_0 and r_1 with $r_1 > r_0 > 2m$ such that $g = (1 - 2m/r)^4 g_e$ for any $r \in (2m, r_0)$ and $g = g_e$ for any $r \geq r_1$, where g_e is the Euclidean metric.*

Proof. Let $r_1 > r_0 > 2m$ to be chosen later. Let $\eta(r)$ be a smooth nonincreasing function with

$$\eta(r) = \begin{cases} 2m, & 2m \leq r \leq r_0, \\ 0, & r \geq r_1. \end{cases} \tag{2-1}$$

For any $\rho \geq 2m$, let

$$y(\rho) = \int_{2m}^{\rho} \frac{\eta(r)}{r^2}\, dr.$$

By choosing suitable r_0, r_1, we may get $y(\rho) = 1$ for any $\rho \geq r_1$; then we see that

$$y(r) = \begin{cases} 1 - 2m/r, & 2m \leq r \leq r_0, \\ 1, & r \geq r_1. \end{cases} \tag{2-2}$$

We claim that

$$\Delta_0 y \leq 0 \quad \text{on } \mathbb{R}^3 \setminus B_{2m};$$

here Δ_0 is the standard Laplace operator on $\mathbb{R}^3$. By a direct computation, we see that

$$\Delta_0 y = y'' + \frac{2}{r} y' = r^{-2}(r^2 y')' = r^{-2}\eta' \leq 0. \tag{2-3}$$

For any $x \in \mathbb{R}^3 \setminus B_{2m}$, let $u(x) = y(|x|)$; then $g = u^4(dr^2 + r^2 h_0)$ is the required metric. □

Suppose $T^3(r)$ is a flat torus, by taking r large enough we may glue $(B_r \setminus B_{2m}, g)$ with $T^3(r) \setminus B_r$ directly. As in Proposition 2.3, near $r = 2m$, the metric can be considered as a metric with cone singularity. The question is whether we have a metric on n-torus which has a cone singularity of the form $dr^2 + \alpha^2 r^2 h_0$ with $0 < \alpha < 1$ and with nonnegative scalar curvature. This will be answered in Section 4. The problem can be reduced to the study of singular metrics on T^n with nonnegative scalar curvature.

3. Gradient estimates for solutions to the h-flow

We want to use the Ricci–DeTurck flow to deform a singular metric to a smooth one. We need some basic facts about the flow.

Let (M^n, h) be a complete manifold without boundary. We assume that the curvature of h and its covariant derivatives are bounded:

$$|\widetilde{\nabla}^{(i)}\widetilde{\mathrm{Rm}}| \le k_i \tag{3-1}$$

for all $3 \ge i \ge 0$. Here $\widetilde{\nabla}$ is the covariant derivative with respect to h and $\widetilde{\mathrm{Rm}}$ is the curvature tensor of h. A smooth family of metrics $g(t)$ on $M \times (0, T]$, $T > 0$, is said to be a solution to the h-flow if $g(t)$ satisfies

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij} = g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta g_{ij} - g^{\alpha\beta} g_{ip} h^{pq}\widetilde{\mathrm{Rm}}_{j\alpha q\beta} - g^{\alpha\beta} g_{jp} h^{pq}\widetilde{\mathrm{Rm}}_{i\alpha q\beta} \\ + \tfrac{1}{2} g^{\alpha\beta} g^{pq}\big(\widetilde{\nabla}_i g_{p\alpha}\cdot\widetilde{\nabla}_j g_{q\beta} + 2\widetilde{\nabla}_\alpha g_{jp}\cdot\widetilde{\nabla}_q g_{i\beta} - 2\widetilde{\nabla}_\alpha g_{jp}\cdot\widetilde{\nabla}_\beta g_{iq} \\ - 2\widetilde{\nabla}_j g_{\alpha p}\cdot\widetilde{\nabla}_\beta g_{iq} - 2\widetilde{\nabla}_i g_{\alpha p}\cdot\widetilde{\nabla}_\beta g_{jq}\big).\end{aligned} \tag{3-2}$$

The h-flow is closely related to the Ricci flow

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Ric}(g).$$

Suppose g_0 is a smooth metric with bounded curvature; then the solution to the h-flow with $h = g_0$ such that $g(0) = g_0$ is the solution to the usual Ricci–DeTurck flow. Using the solution to the Ricci–DeTurck flow, one can obtain a solution to the Ricci flow through a smooth family of diffeomorphisms. Hence h-flow can be considered as a generalization of Ricci flow with initial data which may not be smooth.

Let

$$\square = \frac{\partial}{\partial t} - g^{ij}\widetilde{\nabla}_i\widetilde{\nabla}_j. \tag{3-3}$$

For a constant $\delta > 1$, h is said to be δ close to a metric g if

$$\delta^{-1} h \le g \le \delta h.$$

Theorem 3.1 [Simon 2002]. *There exists $\epsilon = \epsilon(n) > 0$ depending only on n such that if (M^n, g_0) is an n-dimensional compact or noncompact manifold without boundary with continuous Riemannian metric g_0 which is $(1+\epsilon(n))$ close to a smooth complete Riemannian metric h with curvature bounded by k_0, then the h-flow (3-2) has a smooth solution on $M \times (0, T]$ for some $T > 0$ with T depending only on n, k_0 such that $g(t) \to g_0$ as $t \to 0$ uniformly on compact sets and such that*

$$\sup_{x\in M} |\widetilde{\nabla}^i g(t)|^2 \le \frac{C_i}{t^i}$$

for all i, where C_i depends only on $n, k_0, \dots, k_i$ where k_j is the bound of $|\widetilde{\nabla}^j \mathrm{Rm}(h)|$. Moreover, h is $(1+2\epsilon)$ close to $g(t)$ for all t. Here and in the following $|\cdot|$ is the norm with respect to h.

In the case that g_0 is smooth, and if $|\widetilde{\nabla} g_0|$ is bounded, then it is also proved in [Simon 2002] that

$$|\widetilde{\nabla} g(t)| \le C, \quad |\widetilde{\nabla}^2 g(t)| \le C t^{-1/2}.$$

We want to obtain estimates in the case that $g_0 \in W^{1,p}_{\mathrm{loc}}$ in the sense that $|\widetilde{\nabla} g_0|$ is in L^p_{loc}, for $p > n$. We have the following:

Lemma 3.2. *Fix $p \ge 2$. There is $b = b(n, p) > 0$ depending only on n, p, with $e^b \le 1+\epsilon(n)$, where $\epsilon(n)$ is the constant in Theorem 3.1, such that if g_0 is a smooth metric which is e^b close to h, where h is smooth and satisfies (3-1) for $0 \le i \le 2$, then the solution $g(t)$ of the h-flow with initial metric g_0 on $M \times [0, T]$ described in Theorem 3.1 satisfies the following estimates. There is a constant $C > 0$ depending only n, p, h such that for any $x_0 \in M$ with injectivity radius $\iota(x_0)$ with respect to h,*

$$|\widetilde{\nabla} g(t, x_0)|^2 \le \frac{CD}{t^{n/(2p)}}$$

for $T > t > 0$, where D is a constant depending only n, the lower bound of $\iota(x_0)$ and the L^{2p} norm of $|\widetilde{\nabla} g_0|$ in $B(x_0, \iota(x_0))$, which is the geodesic ball with respect to h.

Proof. Suppose g_0 is $e^b < 1+\epsilon(n)$ close to h; then for any $\lambda > 0$, λg_0 is also e^b close to λh. Moreover, if $g(t)$ is the solution to the h-flow, then $\lambda g\left(\frac{1}{\lambda}t\right)$ is a solution to the λh-flow. Hence by scaling, we may assume that $k_0 + k_1 + k_2 \le 1$. The solution $g(t)$ constructed in [Simon 2002] is e^{2b} close to h. Moreover, we may assume that $T \le 1$.

Denote $\iota(x_0)$ by ι_0 and we may assume that $\iota_0 \le 1$. In the following c_i will denote a constant depending only on n. Let $m \ge 2$ be an integer, which will be chosen depending only on n, p. Let $b = 1/(2m)$. First choose m so that $e^b \le 1+\epsilon(n)$. Let $f_1 = |\widetilde{\nabla} g|$ and $\psi = \left(a + \sum_{i=1}^n \lambda_i^m\right) f_1^2$ with $a > 0$, where λ_i are the eigenvalues of $g(t)$ with respect to h. By choosing a depending only on n and m large enough depending only on n, as in [Shi 1989; Simon 2002] (see also [Huang and Tam 2015, (5.8)]), we have

$$\Box\psi \le c_1 - c_2 m^2 f_1^4. \tag{3-4}$$

Let x^i be normal coordinates in $B(x_0, \iota_0)$. Since $k_0 + k_1 + k_2 \le 1$, by [Hamilton 1995, Corollary 4.11] on $B(x_0, \iota_0)$ we have

$$\begin{cases} \frac{1}{2}|\xi|^2 \le h_{ij}\xi^i\xi^j \le 2|\xi|^2 & \text{for } \xi \in \mathbb{R}^n, \\ |D_x^\beta h_{ij}| \le c_3 & \text{for all } i, j, \end{cases} \tag{3-5}$$

where

$$h_{ij} = h\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

and $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index with $|\beta| \le 2$ and

$$D_{x^k} = \frac{\partial}{\partial x^k}.$$

Let η be a smooth function on $[0, 1]$ such that $0 \le \eta \le 1$, $\eta(s) = 0$ for $s \ge \frac{3}{4}$, $\eta(s) = 1$ for $0 \le s \le \frac{1}{2}$. Still denote $\eta(|x|/\iota_0)$ by $\eta(x)$. Then $|\widetilde{\nabla}\eta| \le c_4\iota_0^{-1}$. We have

$$\begin{aligned}
\frac{d}{dt}&\int_{B(x_0,\iota_0)} \eta^2\psi^p \, dv_h \\
&= p\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1}\psi_t \, dv_h \\
&\le p\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1} g^{ij}\widetilde{\nabla}_i\widetilde{\nabla}_j\psi \, dv_h + p\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1}(c_1 - c_2m^2f_1^4)\, dv_h \\
&\le -p(p-1)c_5\int_{B(x_0,\iota_0)} \eta^2\psi^{p-2}|\widetilde{\nabla}\psi|^2 \, dv_h + pc_6\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1} f_1|\widetilde{\nabla}\psi| \, dv_h \\
&\quad + pc_7\iota_0^{-1}\int_{B(x_0,\iota_0)} \eta\eta'\psi^{p-1}|\widetilde{\nabla}\psi| \, dv_h + p\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1}(c_1 - c_2m^2f_1^4)\, dv_h \\
&\le \frac{c_6^2}{2c_5(p-1)}\int_{B(x_0,\iota_0)} f_1^2\eta^2\psi^p \, dv_h + \frac{c_7^2}{2c_5(p-1)\iota_0^2}\int_{B(x_0,\iota_0)} (\eta')^2\psi^p \, dv_h \\
&\quad + p\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1}(c_1 - c_2m^2f_1^4)\, dv_h \\
&\le \frac{c_8 p}{p-1}\int_{B(x_0,\iota_0)} f_1^4\eta^2\psi^{p-1} \, dv_h + \frac{c_7^2}{2c_5(p-1)\iota_0^2}\int_{B(x_0,\iota_0)} (\eta')^2\psi^p \, dv_h \\
&\quad + p\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1}(c_1 - c_2m^2f_1^4)\, dv_h,
\end{aligned}$$

where we have used the fact that $\psi \le c f_1^2$ for some constant c depending only on n by the fact that $2bm = 1$ so that $\lambda_i^m \le 1$ for all i. We have also used the fact that

$$c_6 \int_{B(x_0,\iota_0)} \eta^2 \psi^{p-1} f_1 |\widetilde{\nabla}\psi|\, dv_h \le \tfrac{1}{2} c_5(p-1) \int_{B(x_0,\iota_0)} \eta^2 \psi^{p-2} |\widetilde{\nabla}\psi|^2\, dv_h + \frac{c_6^2}{2c_5(p-1)} \int_{B(x_0,\iota_0)} f_1^2 \eta^2 \psi^p\, dv_h$$

and

$$c_7 \iota_0^{-1} \int_{B(x_0,\iota_0)} \eta\eta' \psi^{p-1} |\widetilde{\nabla}\psi|\, dv_h \le \tfrac{1}{2} c_5(p-1) \int_{B(x_0,\iota_0)} \eta^2 \psi^{p-2} |\widetilde{\nabla}\psi|^2\, dv_h + \frac{c_7^2}{2c_5(p-1)\iota_0^2} \int_{B(x_0,\iota_0)} (\eta')^2 \psi^p\, dv_h.$$

Hence by choosing m large enough depending only on n, p and if $b = 1/(2m)$, we have

$$\frac{d}{dt} \int_{B(x_0,\iota_0)} \eta^2 \psi^p\, dv_h \le c_9 p \iota_0^{-2} \biggl(\int_{B(x_0,\iota_0)} (\eta')^2 \psi^p\, dv_h + \int_{B(x_0,\iota_0)} \eta^2 \psi^{p-1}\, dv_h \biggr).$$

By replacing η by η^q for $q \ge 1$, we may assume that $|\eta'| \le C\eta^{1-1/q}$, where C depends only on q. Let $q = 2p$, say; then we have

$$\begin{aligned}
&\frac{d}{dt} \int_{B(x_0,\iota_0)} \eta^2 \psi^p\, dv_h \\
&\le C_1 \iota_0^{-2} \biggl(\int_{B(x_0,\iota_0)} (\eta^2)^{1-\frac{1}{2p}} \psi^p\, dv_h + \int_{B(x_0,\iota_0)} \eta^2 \psi^{p-1}\, dv_h \biggr) \\
&\le C_1 \iota_0^{-2} \biggl[\biggl(\int_{B(x_0,\iota_0)} \eta^2 \psi^p\, dv_h \biggr)^{1-\frac{1}{2p}} \biggl(\int_{B(x_0,\iota_0)} \psi^p\, dv_h \biggr)^{\frac{1}{2p}} + \biggl(\int_{B(x_0,\iota_0)} \eta^2 \psi^p\, dv_h \biggr)^{1-\frac{1}{p}} \biggr] \\
&\le C_2 \iota_0^{-2} \biggl[\biggl(\int_{B(x_0,\iota_0)} \eta^2 \psi^p\, dv_h \biggr)^{1-\frac{1}{2p}} t^{-1/2} + \biggl(\int_{B(x_0,\iota_0)} \eta^2 \psi^p\, dv_h \biggr)^{1-\frac{1}{p}} \biggr].
\end{aligned}$$

Here and below upper case C_i denote a positive constant depending only on n, p and h. Here we have used the estimates in Theorem 3.1. Let

$$F = \int_{B(x_0,\iota_0)} \eta^2 \psi^p\, dv_h + 1.$$

Then we have

$$\frac{d}{dt} F \le C_3 \iota_0^{-2} F^{1-\frac{1}{2p}} t^{-\frac{1}{2}}.$$

Let $I = \int_{B(x_0,\iota_0)} |\widetilde{\nabla} g_0|^{2p}\, dv_h$. We conclude that

$$F(t) \le C_4 (I + \iota_0^{-4p}),$$

or

$$\int_{B\left(x_0,\frac{1}{2}\iota_0\right)} \psi^p \, dv_h \le C_5(I+\iota_0^{-4p}).$$

Hence $0 < t_0 < T$, by the mean value equality [Lieberman 1996, Theorem 7.21] applied to (3-4) to $B(x_0, r) \times (t_0 - r^2, t_0)$ with $r = \frac{1}{2}\sqrt{t_0}$, we have

$$\psi^p(x_0, t_0) \le C_6 r^{-n}(I + \iota_0^{-2p} + 1).$$

From this the result follows. □

Assume $2p > n$ and let $\delta = n/(2p)$. Let b as in Lemma 3.2. Assume h satisfies (3-1), for $0 \le i \le 2$.

Lemma 3.3. *Let $x_0 \in M$ and let $r_0 > 0$. Let*

$$I := \int_{B(x_0,r_0)} |\widetilde{\nabla} g_0|^{2p} \, dv_h.$$

Let ι be the infimum of the injectivity radii $\iota(x)$, $x \in B(x_0, r_0)$. Then there is a constant C depending only on n, p, h, r_0, the lower bound of ι and the upper bound of I such that

$$|\widetilde{\nabla}^2 g(x_0, t)|^2 \le C t^{-1-\delta}.$$

Proof. In the following, C_i will denote a constant depending only on the quantities mentioned in the lemma. By Lemma 3.2, we have

$$\sup_{x\in B\left(x_0,\frac{r_0}{2}\right)} |\widetilde{\nabla} g(x, t)|^2 \le C_1 t^{-\delta}. \tag{3-6}$$

Let $f_i = |\widetilde{\nabla}^i g|$. As in [Shi 1989; Simon 2002] (see also [Huang and Tam 2015, (5.11)]), one can find $a > 0$ depending only on the quantities mentioned in the lemma such that if $\psi = (at^{-\delta} + f_1^2) f_2^2$, then

$$\Box\psi \le -\tfrac{1}{8} f_2^4 + C_2 t^{-4\delta} \tag{3-7}$$

on $B(x_0, r_0/2) \times (0, T]$. We may assume that $\iota(x_0) \le r_0/2$. Let η be a cutoff function such that $(\eta')^2 + |\eta''| \le c\eta$ for some absolute constant as in the proof of Lemma 3.3, let $F = t^{1+2\delta}\eta\psi$. Since g is smooth up to $t = 0$, and $f_1^2 \le C_1 t^{-\delta}$, we have $F(\,\cdot\,, 0) = 0$. If F has a positive maximum, then there is $x_1 \in B(x_0, \iota)$ and $T \ge t_1 > 0$ such that

$$F(x_1, t_1) = \sup_{B(x_0,\iota)\times[0,T]} F.$$

Hence at (x_1, t_1), we have

$$\eta\widetilde{\nabla}_i\psi + \psi\widetilde{\nabla}_i\eta = 0$$

and

$$
\begin{aligned}
0 &\le \Box F \\
&= t_1^{1+2\delta}(\eta\Box\psi + \psi\Box\eta - 2g^{ij}\widetilde{\nabla}_i\psi\widetilde{\nabla}_j\eta) + (1+2\delta)t_1^{-1}F \\
&\le t_1^{1+2\delta}\big[\eta\big(-\tfrac{1}{8}f_2^4 + C_2t^{-4\delta}\big) - \psi g^{ij}\widetilde{\nabla}_i\widetilde{\nabla}_j\eta + 2g^{ij}\eta^{-1}\psi\widetilde{\nabla}_i\eta\widetilde{\nabla}_j\eta\big] + (1+2\delta)t_1^{-1}F \\
&\le t_1^{1+2\delta}\big[\eta\big(-\tfrac{1}{8}f_2^4 + C_2t^{-4\delta}\big) + C_3\psi\big] + (1+2\delta)t_1^{-1}F.
\end{aligned}
$$

Multiply the inequality by $t_1^{1+2\delta}\eta(at^{-\delta}+f_1^2)^2 = F\psi^{-1}(at^{-\delta}+f_1^2)$, we have

$$
\begin{aligned}
0 &\le -\tfrac{1}{8}F^2 + C_3t_1^{1+\delta}(at^{-\delta}+f_1^2)F + (1+2\delta)t^{2\delta}(at^{-\delta}+f_1^2)^2F \\
&\le -\tfrac{1}{8}F^2 + C_4F.
\end{aligned}
$$

Hence $F \le 8C_4$. From this it is easy to see that the result follows. $\square$

4. Approximation of singular metrics

Let $(M^n, \mathfrak{b})$ be a smooth complete Riemannian manifold of dimension n without boundary. Let g_0 be a continuous Riemannian metric on M satisfying the following:

(a1) There is a compact subset Σ such that g_0 is smooth on $M \setminus \Sigma$.

(a2) The metric g_0 is in $W^{1,p}_{\mathrm{loc}}$ for some $p \ge 1$ in the sense that g_0 has weak derivative and $|g_0|_{\mathfrak{b}}$, $|{}^{\mathfrak{b}}\nabla g_0|_{\mathfrak{b}} \in L^p_{\mathrm{loc}}$ with respect to the metric $\mathfrak{b}$.

We want to approximate g_0 by smooth metrics with uniform bound on the $W^{1,p}$ norm locally. As in [Lee 2013], cover Σ by finitely many precompact coordinate patches $U_1, \dots, U_N$ and cover M with $U_1, \dots, U_N$ and U_{N+1} such that U_{N+1} is an open set with $U_{N+1} \cap \Sigma = \varnothing$. We may assume that there is a partition of unity ψ_k with $\mathrm{supp}(\psi_k) \subset U_k$. Since g_0 is continuous, we may assume that g_0, $\mathfrak{b}$ and the Euclidean metric are equivalent in each U_k, $1 \le k \le N$. For any $a > 0$, let $\Sigma(a) = \{x \in M \mid d_{\mathfrak{b}}(x, \Sigma) < a\}$. By [Lee 2013, Lemma 3.1], for each $1 \le k \le N$, there is a smooth function $\epsilon \ge \rho_k \ge 0$ in U_k such that for $\epsilon > 0$ small enough

$$
\begin{cases}
\rho_k = \epsilon, & \Sigma(\epsilon) \cap U_k; \\
\rho_k = 0, & U_k \setminus \Sigma(2\epsilon); \\
|\partial\rho_k| \le C; & \\
|\partial^2\rho_k| \le C\epsilon^{-1} &
\end{cases}
\tag{4-1}
$$

for some C independent of ϵ and k. Here $\partial\rho_k$ and $\partial^2\rho_k$ are derivatives with respect to the Euclidean metric. Let $g_0^k = \psi_k g_0$ and for $1 \le k \le N$, let

$$
(g^k_{\epsilon,0})_{ij}(x) = \int_{\mathbb{R}^n} g^k_{0,ij}(x - \lambda\rho_k(x)y)\varphi(y)\,dy. \tag{4-2}
$$

Here φ is a nonnegative smooth function in $\mathbb{R}^n$ with support in $B(1)$ and integral equal to 1. $\lambda > 0$ is a constant independent of ϵ and k, to be determined. Finally, define

$$g_{\epsilon,0} = \sum_{k=1}^{N} g_{\epsilon,0}^{k} + \psi_{N+1} g_0. \tag{4-3}$$

Lemma 4.1. *For $\epsilon > 0$ small enough, $g_{\epsilon,0}$ is a smooth metric such that $g_{\epsilon,0}$ converges to g_0 in C^0 norm as $\epsilon \to 0$, and $g_{\epsilon,0} = g_0$ outside $\Sigma(2\epsilon)$. Moreover, there is a constant C independent of ϵ such that*

$$\int_{\Sigma(1)} |{}^{\mathfrak{b}}\nabla g_{\epsilon,0}|_{\mathfrak{b}}^{p}\, dv_{\mathfrak{b}} \le C.$$

Proof. It is easy to see that $g_{\epsilon,0}$ is smooth and converges to g_0 uniformly as $\epsilon \to 0$. In order to estimate the $W^{1,p}_{\mathrm{loc}}$ norm of $g_{\epsilon,0}$, it is sufficient to estimate the norm in each U_k, $1 \le k \le N$. Moreover, we may assume that $\mathfrak{b}$ is the Euclidean metric. So it is sufficient to prove the following: For fixed k, $1 \le k \le N$, and for any $u \in W^{1,p}_{\mathrm{loc}}$ if

$$v(x) = \int_{\mathbb{R}^n} u(x - \lambda\rho_k(x)y)\varphi(y)\, dy,$$

then the $W^{1,p}$ norm of v in $\Sigma(1)$ can be estimated in terms of the $W^{1,p}$ norm of u in $\Sigma(2)$, say. For fixed y with $|y| \le 1$, let $z = x - \lambda\rho_k(x)y$. Then

$$\frac{\partial z^i}{\partial x^j} = \delta_{ij} - y^i \lambda \frac{\partial \rho_k}{\partial x^i}.$$

By (4-1), we can choose $\lambda > 0$ small enough independent of ϵ and k so that

$$2 \ge \det\left(\delta_{ij} - \lambda y^i \frac{\partial \rho_k}{\partial x^i}\right) \ge \tfrac{1}{2},$$

and so that $z = z(x)$ is a diffeomorphism with the Jacobian being bounded above and below by some constants independent of ϵ, k. Hence

$$\begin{aligned}
\left(\int_{\Sigma(1)\cap U_k} |v|^p(x)\, dx\right)^{\frac{1}{p}} &\le \left[\int_{\Sigma(1)\cap U_k} \left(\int_{\mathbb{R}^n} |u(x - \lambda\rho_k(x)y)|\varphi(y)\, dy\right)^p dx\right]^{\frac{1}{p}} \\
&\le \int_{\mathbb{R}^n} \varphi(y)\left(\int_{\Sigma(1)\cap U_k} |u(x - \lambda\rho_k(x)y)|^p\, dx\right)^{\frac{1}{p}} dy \\
&= \int_{B(1)} \varphi(y)\left(\int_{\Sigma(1)\cap U_k} |u(x - \lambda\rho_k(x)y)|^p\, dx\right)^{\frac{1}{p}} dy \\
&\le C_1\left(\int_{\Sigma(2)} |u(z)|^p\, dz\right)^{\frac{1}{p}}
\end{aligned}$$

for some constant C_1 independent of ϵ, k provided ϵ is small enough, where we have used Minkowski's integral inequality [Stein 1970, Section A.1]. Now, if $x \notin \Sigma(2\epsilon)$, then $v(x) = u(x)$ and if $x \in \Sigma(\epsilon)$, then $v(x)$ is the standard mollification. If $x \in \Sigma(2\epsilon) \setminus \Sigma(\epsilon)$, then

$$|\partial v|(x) \le \int_{\mathbb{R}^n} |\partial u|(x - \lambda\rho_k(x)y)\lambda|\partial\rho_k(x)|\varphi(y)\,dy.$$

Since $|\partial\rho_k|$ is bounded by (4-1), we can prove as before that

$$\left(\int_{\Sigma(1)\cap U_k} |\partial v|^p(x)\,dx\right)^{\frac{1}{p}} \le C_2\left(\int_{\Sigma(2)} |\partial u|^p(z)\,dz\right)^{\frac{1}{p}}$$

for some constant C_2 independent of ϵ, k provided ϵ is small enough. □

In addition to (a1) and (a2), assume

(a3) The scalar curvature $\mathcal{S}_{g_0}$ of g_0 satisfies $\mathcal{S}_{g_0} \ge \sigma$ in $M \setminus \Sigma$, where σ is a constant.

We want to modify $g_{\epsilon,0}$ to obtain a smooth metric with scalar curvature bounded below by σ. We first consider the case that M is compact. Let $\epsilon_0 > 0$ be small enough so that for all $\epsilon_0 \ge \epsilon > 0$,

$$(1+\epsilon(n))^{-1} g_{\epsilon_0,0} \le g_{\epsilon,0} \le (1+\epsilon(n)) g_{\epsilon_0,0},$$

where $\epsilon(n) > 0$ is the constant depending only on n in Theorem 3.1. Hence if we let $h = g_{\epsilon_0,0}$, then the h-flow has solution $g_\epsilon(t)$ on $M \times [0, T]$ for some $T > 0$ independent of ϵ, with initial data $g_{\epsilon,0}$ in the sense that $\lim_{t\to 0} g_\epsilon(x,t) = g_{\epsilon,0}(x)$ uniformly in M; see Theorem 3.1. The curvature and all the covariant derivatives of curvature of h are bounded because M is compact.

By [Simon 2002] and Lemmas 3.2, 3.3 and 4.1 we have the following:

Lemma 4.2. *Let M be compact and suppose g_0 satisfies (a1)–(a3). Suppose $p > n$. Let $\delta = n/p < 1$. Then*

$$|{}^h\nabla g_\epsilon(t)|_h^2 \le Ct^{-\delta} \quad \textit{and} \quad |{}^h\nabla^2 g_\epsilon(t)|^2 \le Ct^{-1-\delta}$$

for some constant C independent of ϵ, t. Moreover, $g_\epsilon(t)$ subconverges to the solution $g(t)$ of the h-flow with initial data g_0 in C^∞ norm in compact sets of $M \times (0, T]$ and in compact sets of $M \setminus \Sigma \times [0, T]$.

For $\epsilon > 0$ small enough, let

$$W^k = (g_\epsilon(t))^{pq}\left(\Gamma^k_{pq}(g_\epsilon(t)) - \Gamma^k_{pq}(h)\right), \tag{4-4}$$

and let Φ_t be the diffeomorphism given by

$$\frac{\partial}{\partial t}\Phi_t(x) = -W(\Phi_t(x), t), \quad \Phi_0(x) = x. \tag{4-5}$$

Let $\tilde{g}_\epsilon(t) = \Phi_t^* g_\epsilon(t)$. Then $\tilde{g}_\epsilon(t)$ satisfies the Ricci flow equation with initial data $g_{\epsilon,0}$. Note that W and Φ_t depend also on ϵ. Recall the Ricci flow equation is

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}. \tag{4-6}$$

Lemma 4.3. *With the same assumptions and notation as in Lemma 4.2, for ϵ small enough, $|W|_h \le Ct^{-\frac{1}{2}\delta}$, $|\mathrm{Rm}(\tilde{g}_\epsilon(t))| \le Ct^{-\frac{1}{2}(1+\delta)}$ and*

$$C^{-1}h \le g_\epsilon(t) \le Ch$$

for some C, independent of ϵ, t.

Proof. The bound of W is given by Lemma 4.2. Since the bound of curvature is unchanged under diffeomorphism, $|\mathrm{Rm}(\tilde{g}_\epsilon(t))| \le Ct^{-\frac{1}{2}(1+\delta)}$ by Lemma 4.2. From this we conclude from the Ricci flow equation that $\tilde{g}_\epsilon(t)$ is uniformly equivalent to $g_{0,\epsilon}$ which is uniformly equivalent to h. □

Lemma 4.4. *Let $\mathcal{S}(t)$ be the scalar curvature of $g(t)$. Then there is a constant $C > 0$ independent of t, ϵ such that*

$$\exp(-Ct^{\frac{1}{2}(1-\delta)}) \int_M (\mathcal{S}(t) - \sigma)_- \, dv_{g(t)}$$

is nonincreasing in $(0, T]$, where $f_- = \max\{-f, 0\}$ is the negative part of f.

Proof. As in [McFeron and Székelyhidi 2012], fix $\theta > 0$, and for $\epsilon > 0$, let

$$v = \big((\mathcal{S}_\epsilon(t) - \sigma)^2 + \theta\big)^{1/2} - (\mathcal{S}_\epsilon(t) - \sigma),$$

where $\mathcal{S}_\epsilon(t)$ is the scalar curvature of $\tilde{g}_\epsilon(t)$. Let Δ and ∇ be the Laplacian and covariant derivative with respect to $\tilde{g}_\epsilon(t)$. Using the evolution equation of the scalar curvature in Ricci flow, we have

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta\Big)v &= \bigg(\frac{\mathcal{S}_\epsilon(t) - \sigma}{\big((\mathcal{S}_\epsilon(t)-\sigma)^2+\theta\big)^{1/2}} - 1\bigg)\Big(\frac{\partial}{\partial t} - \Delta\Big)\mathcal{S}_\epsilon(t) - \frac{\theta|\nabla\mathcal{S}_\epsilon|^2}{\big((\mathcal{S}_\epsilon(t)-\sigma)^2+\theta\big)^{1/2}} \\
&= \bigg(\frac{\mathcal{S}_\epsilon(t) - \sigma}{\big((\mathcal{S}_\epsilon(t)-\sigma)^2+\theta\big)^{1/2}} - 1\bigg)\cdot 2|\nabla \mathrm{Ric}(t)|^2 - \frac{\theta|\nabla\mathcal{S}_\epsilon(t)|^2}{\big((\mathcal{S}_\epsilon(t)-\sigma)^2+\theta\big)^{3/2}} \\
&\le 0,
\end{aligned}$$

where $\mathrm{Ric}(t)$ is the Ricci tensor of $\tilde{g}_\epsilon(t)$. Using Lemma 4.3 we have

$$\begin{aligned}
\frac{d}{dt}\int_M v\, dv_{\tilde{g}_\epsilon(t)} &= \int_M \frac{\partial}{\partial t} v\, dv_{\tilde{g}_\epsilon(t)} - \int_M \mathcal{S}_\epsilon(t) v\, dv_{\tilde{g}_\epsilon(t)} \\
&\le \int_M \Delta v\, dv_{\tilde{g}_\epsilon(t)} + C_1 t^{-\frac{1}{2}(1+\delta)} \int_M v\, dv_{\tilde{g}_\epsilon(t)} \\
&= C_1 t^{-\frac{1}{2}(1+\delta)} \int_M v\, dv_{\tilde{g}_\epsilon(t)}
\end{aligned} \tag{4-7}$$

for some constant C_1 independent of t, ϵ. From this and letting $\theta \to 0$, we conclude that for some constant C independent of t and ϵ,

$$\exp(-Ct^{\frac{1}{2}(1-\delta)}) \int_M (\mathcal{S}_\epsilon(t)-\sigma)_- \, dv_{\tilde{g}_\epsilon(t)}$$

is nonincreasing in $(0, T]$. Noting that $\tilde{g}_\epsilon(t) = \Phi_t^*(g_\epsilon(t))$, by Lemma 4.2 let $\epsilon \to 0$, the result follows. □

We first consider the case that the codimension of Σ is at least 2 in the following sense:

(a4) The volume $V(\Sigma(\epsilon), g_0)$ with respect to g_0 of the ϵ-neighborhood $\Sigma(\epsilon)$ of Σ is bounded by $C\epsilon^2$ for some constant C independent of ϵ. Here

$$\Sigma(\epsilon) = \{x \in M \mid d_{g_0}(x, \Sigma) < \epsilon\}.$$

Lemma 4.5. *With the same assumptions and notation as in Lemma 4.2, suppose (a4) is true. Then $S(t) \geq \sigma$ for all $t > 0$.*

Proof. By Lemma 4.4, it is sufficient to prove that

$$\lim_{t\to 0} \int_M (\mathcal{S}(t)-\sigma)_- \, dv_{g(t)} = 0. \tag{4-8}$$

For any $\epsilon > 0$, let Φ_t be the diffeomorphisms as before so that $\tilde{g}_\epsilon(t) = \Phi_t^*(g_\epsilon(t))$ is the solution to the Ricci flow. For any $\theta > 0$, let v as in the proof of Lemma 4.4. Let

$$\beta = \frac{1}{\epsilon}\left(\epsilon - \sum_{k=1}^N \psi_k \rho_k\right).$$

We may modify ρ_k so that if ϵ is small enough then β is a smooth function on M such that $\beta = 0$ in $\Sigma(2\epsilon)$, $\beta = 1$ outside $\Sigma(4\epsilon)$, $0 \leq \beta \leq 1$, $|{}^h\nabla\beta| \leq C\epsilon^{-1}$, and $|{}^h\nabla^2\beta| \leq C\epsilon^{-2}$ for some constant C independent of ϵ, t. Let

$$\tilde{\beta}(t, x) = \beta(\Phi_t(x)).$$

Then

$$\begin{aligned}
\frac{d}{dt}\int_M \tilde{\beta}^2 v \, dv_{\tilde{g}_\epsilon(t)} &= \int_M v\frac{\partial}{\partial t}(\tilde{\beta}^2)\, dv_{\tilde{g}_\epsilon(t)} + \int_M \tilde{\beta}^2 \frac{\partial}{\partial t} v \, dv_{\tilde{g}_\epsilon(t)} - \int_M \mathcal{S}_\epsilon(t)\tilde{\beta}^2 v \, dv_{\tilde{g}_\epsilon(t)} \\
&\leq \int_M v\frac{\partial}{\partial t}(\tilde{\beta})^2\, dv_{\tilde{g}_\epsilon(t)} + \int_M \tilde{\beta}^2 \Delta_{\tilde{g}_\epsilon(t)} v \, dv_{\tilde{g}_\epsilon(t)} \\
&\qquad + C_1 t^{-\frac{1}{2}(1+\delta)} \int_M \tilde{\beta}^2 v \, dv_{\tilde{g}_\epsilon(t)} \\
&= I + II + C_1 t^{-\frac{1}{2}(1+\delta)} \int_M \tilde{\beta}^2 v \, dv_{\tilde{g}_\epsilon(t)}.
\end{aligned}$$

for some constant $C_1 > 0$ independent of t, ϵ, θ by Lemma 4.3. Let $w(y) = v(\Phi_t^{-1}(y))$. Since in local coordinates,

$$\Delta_{g_\epsilon(t)} f = g_\epsilon^{ij}(\partial_i\partial_j f - \Gamma_{ij}^k \partial_k)$$

with $|\Gamma_{ij}^k| \le Ct^{-\delta/2}$ for some constant C independent of ϵ, t by Lemma 4.2, we have $|w| \le Ct^{-\frac{1}{2}(1+\delta)}$ for some constant C independent of ϵ, t, θ. Using also (a4) and Lemma 4.1, we have

$$\begin{aligned}
II &= \int_M \beta^2 \Delta_{g_\epsilon(t)} w \, dv_{g_\epsilon(t)} \\
&= \int_M w \Delta_{g_\epsilon(t)}(\beta^2)\, dv_{g_\epsilon(t)} \\
&\le C_2 \int_{\Sigma(4\epsilon)\setminus\Sigma(2\epsilon)} w|\epsilon^{-2} + \epsilon^{-1} t^{-\delta/2}\beta| \, dv_{g_\epsilon(t)} \\
&\le C_3\Big(t^{-\frac{1}{2}(1+\delta)} + \epsilon^{-1} t^{-\frac{\delta}{2}-\frac{1}{4}(1+\delta)} \int_{\Sigma(4\epsilon)} \beta w^{1/2}\, dv_{g_\epsilon(t)}\Big) \\
&\le C_4\Big[t^{-\frac{1}{2}(1+\delta)} + t^{-\frac{1}{4}(1+3\delta)}\Big(\int_M \tilde\beta^2 v\, dv_{\tilde g_\epsilon(t)}\Big)^{\frac{1}{2}}\Big]
\end{aligned}$$

for some constants $C_2 - C_4$ independent of ϵ, t, θ, where we have used Lemma 4.2, the fact that $\beta = 1$ outside $\Sigma(4\epsilon)$, the Hölder inequality and the fact that $V(\Sigma(4\epsilon)) = O(\epsilon^2)$. To estimate I, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde\beta &= (d\tilde\beta)\Big(\frac{\partial}{\partial t}\Big) \\
&= d\beta \circ d\Phi_t\Big(\frac{\partial}{\partial t}\Big) \\
&= d\beta(W).
\end{aligned}$$

Hence by Lemma 4.2, we have

$$\Big|\frac{\partial}{\partial t}\tilde\beta\Big|(x) \le C_5|{}^h\nabla\beta|(\Phi_t(x))| \le C_6\epsilon^{-1}t^{-\delta/2}$$

for some constants C_5, C_6 independent of ϵ, t, θ. Hence if w is as above, then

$$\begin{aligned}
I &\le C_6\epsilon^{-1}t^{-\delta/2}\int_{\Sigma(4\epsilon)} \beta w(y)\, dv_{g_\epsilon(t)} \\
&\le C_7 t^{-\frac{1}{4}(1+3\delta)}\Big(\int_{\Sigma(4\epsilon)} \tilde\beta^2 v\, dv_{\tilde g_\epsilon(t)}\Big)^{\frac{1}{2}}
\end{aligned}$$

for some constant C_7 independent of ϵ, t, θ. To summarize, if we let

$$F = \int_M \tilde\beta^2 v\, dv_{\tilde g_\epsilon(t)},$$

then

$$\frac{dF}{dt} \le C_8(t^{-\frac{1}{2}(1+\delta)} + t^{-\frac{1}{2}(1+\delta)}F + t^{-\frac{1}{4}(1+3\delta)}F^{1/2})$$
$$\le C_8(t^{-\frac{1}{2}(1+\delta)} + t^{-\delta} + 2t^{-\frac{1}{2}(1+\delta)}F)$$

for some constant C_8 independent of ϵ, t, θ. Integrate from 0 to t, and let $\theta \to 0$. Since $g_{\epsilon,0} = g_0$ outside $\Sigma(2\epsilon)$, $\Phi_0 = \mathrm{id}$, and $\beta = 0$ on $\Sigma(2\epsilon)$, and $\mathcal{S}_{g_0} \ge \sigma$ outside Σ, there exist constants $C_9 - C_{10}$ independent of ϵ, t such that

$$\exp(-C_9 t^{\frac{1}{2}(1-\delta)}) \int_M \tilde{\beta}^2 (\mathcal{S}_\epsilon(t) - \sigma)_- \, dv_{\tilde{g}_\epsilon(t)} \le C_{10}(t^{\frac{1}{2}(1-\delta)} + t^{1-\delta})$$

because $0 < \delta < 1$. Letting $\epsilon \to 0$, we see that (4-8) is true and the proof of the lemma is completed. □

By Lemmas 4.2 and 4.5, using $g(t)$ we have:

Corollary 4.6. *Let $(M^n, \mathfrak{b})$ be a smooth compact manifold and let g_0 be a continuous Riemannian metric satisfying the following*:

(a) *There is a compact set Σ such that g_0 is smooth on $M \setminus \Sigma$ with scalar curvature bounded below by σ.*

(b) *The metric g_0 is in $W^{1,p}_{\mathrm{loc}}$ for some $p > n$.*

(c) $V(\Sigma(\epsilon), g_0) = O(\epsilon^2)$ *as* $\epsilon \to 0$, *where* $\Sigma(\epsilon) = \{x \in M \mid d_{\mathfrak{b}}(x, \Sigma) < \epsilon\}$.

Then there exists a sequence of smooth metrics g_i satisfying the following: (*i*) *as i tends to infinity g_i converges to g_0 uniformly in M, and converges to g_0 in C^∞ norm on any compact subset of $M \setminus \Sigma$*; (*ii*) *the scalar curvature $\mathcal{S}_i$ of g_i satisfies $\mathcal{S}_i \ge \sigma$.*

Remark 4.7. If the codimension of Σ is only assumed to be larger than 1, then the conclusions of Lemma 4.5 and Corollary 4.6 are still true under some additional assumptions on the second derivatives of g_0.

Next let us consider the case that Σ is an embedded hypersurface. Let (M^n, g_0) be a Riemannian metric satisfying the following:

(b1) Σ is a compact embedded orientable hypersurface, and g_0 is smooth on $M \setminus \Sigma$ with scalar curvature $\mathcal{S}_{g_0} \ge \sigma$.

(b2) There is a neighborhood U of Σ and a smooth function t defined near U such that U is diffeomorphic to $\{-a < t < a\} \times \Sigma$ for some $a > 0$ with $\Sigma = \{t = 0\}$. Moreover, $g_0 = dt^2 + g_\pm(z, t)$, $z \in \Sigma$, such that (t, z) are smooth coordinates and $g_-(\cdot, 0) = g_+(\cdot, 0)$, where g_+ is defined and smooth on $t \ge 0$, g_- is defined and smooth on $t \le 0$.

(b3) Let $U_+ = \{t > 0\}$, $U_- = \{t < 0\}$. With respect to the unit normal $\frac{\partial}{\partial t}$ the mean curvature H_+ of Σ with respect to g_+ and the mean curvature H_- of Σ with respect to g_- satisfy $H_- \ge H_+$.

By [Miao 2002, Proposition 3.1], letting $\epsilon > 0$ be small enough, one can find a smooth metric $g_{\epsilon,0}$ such that (i) $g_{\epsilon,0} = g_0$ outside $U(\epsilon) = \{-\epsilon < t < \epsilon\}$; (ii) $g_{0,\epsilon}$ converges uniformly to g_0; (iii) $|{}^h\nabla g_{0,\epsilon}|_h \le C$ with respect to some fixed background smooth metric h; (iv) there exists a constant $c > 0$ independent of ϵ such that the scalar curvature $\mathcal{S}_{g_{0,\epsilon}}$ satisfies

$$\begin{cases} \mathcal{S}_{g_{0,\epsilon}} = \mathcal{S}_{g_0} & \text{outside } U(\epsilon), \\ |\mathcal{S}_{g_{0,\epsilon}}| \le c & \text{in } \dfrac{\epsilon^2}{100} < |t| \le \epsilon, \\ \mathcal{S}_{g_{0,\epsilon}}(z,t) \ge -c + (H_-(z) - H_+(z))\epsilon^{-2}\phi\Big(\dfrac{100t}{\epsilon^2}\Big) & \text{in } -\dfrac{\epsilon^2}{100} < t \le \dfrac{\epsilon^2}{100}, \\ |\mathcal{S}_{g_{0,\epsilon}}| \le c\epsilon^{-2} \end{cases} \tag{4-9}$$

for $z \in \Sigma$. Here $\phi \ge 0$ is a smooth function in $\mathbb{R}$ with compact support in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ such that

$$\int_{\mathbb{R}} \phi(s)\, ds = 1.$$

Using arguments similar to those before using h-flow, we can conclude:

Corollary 4.8. *Let M^n be a compact smooth manifold and let g_0 be a Riemannian metric satisfying (b1)–(b3) such that the scalar curvature of g_0 on $M \setminus \Sigma$ is at least σ. Then there exists a sequence of smooth metrics g_i such that as i tends to infinity g_i converges to g_0 uniformly in M, and converges to g_0 in C^∞ norm on any compact subset of $M \setminus \Sigma$. Moreover, $\mathcal{S}_{g_i} \ge \sigma$.*

Proof. As before, choose $h = g_{0,\epsilon_0}$ for ϵ_0 small enough, one can solve the h-flow with initial data $g_{0,\epsilon}$. Let $g_\epsilon(t)$ be the solution and let $\mathcal{S}_\epsilon(t)$ be its scalar curvature. From the proof of Lemma 4.4, one can conclude that

$$\begin{aligned} \exp(-C_3 t^{1/2}) \int_M (\mathcal{S}_\epsilon(t) - \sigma)_-\, dv_{g_\epsilon(t)} &\le \int_M (\mathcal{S}_{g_{0,\epsilon}} - \sigma)_-\, dv_{g_{0,\epsilon}} \\ &= \int_{U(\epsilon)} (\mathcal{S}_{g_{0,\epsilon}} - \sigma)_-\, dv_{g_{0,\epsilon}} \\ &\le C_1 \epsilon \end{aligned}$$

for some $C_1, C_3 > 0$ independent of ϵ, t. Here we used the fact that $H_- - H_+ \ge 0$. Let $\epsilon \to 0$, we conclude that the solution $g(t)$ of the h-flow with initial value g_0 has scalar curvature no less than σ. The result follows as before. ☐

Remark 4.9. By [Miao 2002], suppose Σ is a compact orientable hypersurface, and a neighborhood of Σ is of the disjoint union of U_1, U_2 and Σ. Assume g_0 is smooth up Σ from each side U_i of Σ and such that the mean curvatures H_1, H_2 with respect to unit normals in the two sides of Σ satisfying $H_1 + H_2 \ge 0$, where unit normals are chosen to be outward pointing in each side. Then one can find a smooth structure so that (b2) and (b3) are true.

We give some applications.

Corollary 4.10. *Let (M^n, g) be a compact manifold such that M^n is the topological n-torus, g is smooth except at a point, where it has a cone singularity of the form*

$$g = dr^2 + \alpha^2 r^2 h_0$$

with $0 < \alpha \leq 1$ and where h_0 is the standard metric on $\mathbb{S}^{n-1}$. Suppose the scalar curvature of g is nonnegative; then g must be flat and $\alpha = 1$.

Proof. For r small, the mean curvature of the level set $\{r\} \times \mathbb{S}^{n-1}$ with respect to the normal ∂_r is $H = (n-1)/r$. Consider the Euclidean ball $B(\alpha r)$ of radius αr with center at the origin. Then metric of the boundary is $(\alpha r)^2 h_0$. Moreover, the mean curvature is $H_0 = (n-1)/(\alpha r)$. Since $\alpha \leq 1$, $H_0 \geq H$. By gluing $B(\alpha r)$ along with M along $\{r\} \times \mathbb{S}^{n-1}$, we obtain a metric with corner so that (b1)–(b3) are true by changing the smooth structure if necessary. Still denote this metric by g. By Corollary 4.8, there exist smooth metrics g_i on the new manifold with nonnegative scalar curvature such that $g_i \to g$ in C^∞ away from the singular part. By [Schoen and Yau 1979a; 1979c; Gromov and Lawson 1983], g_i is flat. Hence g must be flat away from the singular part. Let $r \to 0$, we conclude that the original metric g is flat, and we must have $\alpha = 1$. □

Similarly, one can prove the following:

Corollary 4.11. *Let (M^n, g) be a compact manifold such that M^n is the topological n-torus and g is smooth away from some compact set with codimension at least 2. Moreover, assume g is in $W^{1,p}_{\text{loc}}$ for some $p > n$. Suppose the scalar curvature of g is nonnegative; then g must be flat.*

Remark 4.12. Suppose M is asymptotically flat with nonnegative scalar curvature and with some cone singularities as in Corollary 4.10; then we still have positive mass for each end by [Miao 2002]. The proof is similar. Compare this result with the example in Proposition 2.3.

Let us consider the case that M^n is noncompact. Let g_0 be a continuous Riemannian metric on M which is smooth outside a compact set Σ. Suppose there is a family of smooth complete metrics $g_{\epsilon,0}$ on M such that $g_{\epsilon,0}$ converges uniformly to g_0 and converges smoothly on compact sets of $M \setminus \Sigma$. Assume $g_{\epsilon,0}$ has bounded curvature for all ϵ. As before, we can find $\epsilon_0 > 0$ such that if $h = g_{\epsilon_0,0}$ then there are solutions $g_\epsilon(t)$ to the h-flow with initial data $g_{\epsilon,0}$, and solution to the h-flow with initial data g_0 on some fixed interval $[0, T]$, $T > 0$. As in [Simon 2002], using [Shi 1989], we may assume that all the derivatives of the curvature of h are bounded. Moreover, $g_\epsilon(t)$ converges uniformly on compact sets of $M \times (0, T]$ and

$M \setminus \Sigma \times [0, T]$. Suppose the scalar curvature of g_0 satisfies $\mathcal{S}_{g_0} \geq \sigma$. We want to find conditions so that the scalar curvature of $g(t)$ is also bounded below by σ.

Lemma 4.13. *With the above assumptions and notation, suppose*

(i) $g_{\epsilon,0} = g_0$ *outside* $\Sigma(2\epsilon)$;

(ii) $|{}^h\nabla g_\epsilon(t)| \leq Ct^{-\frac{\delta}{2}}$ *and* $|{}^h\nabla^2 g_\epsilon(t)| \leq Ct^{-\frac{1}{2}(1+\delta)}$ *for some C independent of* ϵ, t;

(iii) *there is an* $R_0 > 0$ *and a* $C > 0$ *independent of* ϵ, t *such that*

$$\int_{M \setminus B(o,R_0)} |\mathcal{S}_\epsilon(t) - \sigma| \, dv_h \leq C,$$

where $B(o, R_0)$ *is the geodesic ball with respect to h and* $\mathcal{S}_\epsilon(t)$ *is the scalar curvature of* $g_\epsilon(t)$;

(iv) $V(\Sigma(2\epsilon), g_0) = O(\epsilon^2)$.

Then the scalar curvature $\mathcal{S}(t)$ *of* $g(t)$ *satisfies* $\mathcal{S}(t) \geq \sigma$ *for all* $t > 0$.

Proof. By [Shi 1989; Tam 2010], we can find a smooth function ρ such that

$$C_1^{-1}(r(x) + 1) \leq \rho(x) \leq C_1(1 + r(x))$$

for some constant $C_1 > 0$ where $r(x)$ is the distance function to a fixed point o with respect to h. Moreover, the gradient and Hessian of ρ with respect to h are uniformly bounded. (Hence the constants in the lemma may depend also on the choice of o.)

Let $0 \leq \eta \leq 1$ be a smooth function on $\mathbb{R}$ such that $\eta = 1$ on $[0, 1]$ and $\eta = 0$ on $[2, \infty)$. We proceed as in the proofs of Lemmas 4.4 and 4.5. For $R \gg 1$, denote $\eta(\rho(x)/R)$ still by $\eta(x)$. Let $\tilde{g}_\epsilon$ be the Ricci flow corresponding to the $g_\epsilon(t)$ and let $\mathcal{S}_\epsilon(t)$ be its scalar curvature. Let $\theta > 0$ and let v be as in the proof of Lemma 4.4. We have

$$\begin{aligned}
\frac{d}{dt} \int_M \eta v \, dv_{\tilde{g}_\epsilon(t)} &\\
&\leq C_2 \left(t^{-\frac{1}{2}(1+\delta)} \int_M \eta v \, dv_{\tilde{g}_\epsilon(t)} + \int_M v |\Delta \eta| \, dv_{\tilde{g}_\epsilon(t)} \right) \\
&\leq C_3 \left(t^{-\frac{1}{2}(1+\delta)} \int_M \eta v \, dv_{\tilde{g}_\epsilon(t)} + t^{-\delta/2} R^{-1} \int_{M \setminus B(o, 2C_1 R)} (|\mathcal{S}_\epsilon(t) - \sigma| + \theta) \, dv_{\tilde{g}_\epsilon(t)} \right)
\end{aligned}$$

for some positive constants C_2, C_3 independent of t, ϵ, θ. Hence

$$\frac{d}{dt} \left(\exp(-C_4 t^{\frac{1}{2}(1+\delta)}) \int_M \eta v \, dv_{\tilde{g}_\epsilon(t)} \right) \leq C_5 t^{-\delta/2} R^{-1} \int_{M \setminus B(o, 2C_1 R)} (|\mathcal{S}_\epsilon(t) - \sigma| + \theta) \, dv_{\tilde{g}_\epsilon(t)}$$

for some positive constants C_4, C_5 independent of t, ϵ, θ. Integrating from $0 < t_1 < t_2$, let $\theta \to 0$ and then let $R \to \infty$. Using condition (iii), we conclude that

$$\exp(-C_4 t^{\frac{1}{2}(1+\delta)}) \int_M (\mathcal{S}_\epsilon(t) - \sigma)_- \, dv_{\tilde{g}_\epsilon(t)}$$

is nonincreasing in t. Let $\epsilon \to 0$, we conclude that

$$\exp(-C_4 t^{\frac{1}{2}(1+\delta)}) \int_M (\mathcal{S}(t) - \sigma)_- \, dv_{g_\epsilon(t)}$$

is nonincreasing in t.

Next we proceed as in the proof of Lemma 4.5. But we need the cutoff function η. For $\epsilon > 0$ and $\theta > 0$ as in the proof of Lemma 4.5, let β, $\tilde{\beta}$ as in that proof, we have for $R \gg 1$,

$$\begin{aligned} \text{(4-10)} \quad \frac{d}{dt} F \, dv &\le C_6\left(t^{-\frac{1}{2}(1+\delta)} + t^{-\delta} + t^{-\frac{1}{2}(1+\delta)} F + \int_M |\Delta \eta| v \tilde{\beta}^2 \, dv_{\tilde{g}_\epsilon(t)} \right) \\ &\le C_7\bigg(t^{-\frac{1}{2}(1+\delta)} + t^{-\delta} + t^{-\frac{1}{2}(1+\delta)} F \\ &\qquad\qquad + \frac{1}{R} \int_{M \setminus B(o, 2C_1 R)} (|\mathcal{S}_\epsilon(t) - \sigma| + \theta) \, dv_{\tilde{g}_\epsilon(t)} \bigg) \end{aligned}$$

for some constants C_6, C_7 independent of ϵ, t, θ where

$$F = \int_M \eta \tilde{\beta}^2 v \, dv_{\tilde{g}_\epsilon(t)}.$$

Integrate from 0 to t and let $\theta \to 0$. We have

$$\begin{aligned} &\int_M \eta \tilde{\beta}^2 (\mathcal{S}_\epsilon(t) - \sigma)_- \, dv_{\tilde{g}_\epsilon(t)} \\ &\qquad \le C_8\left(t^{1-\delta} + t^{\frac{1}{2}(1-\delta)} + \frac{1}{R} \int_0^t \int_{M \setminus B(o, 2C_1 R)} (|\mathcal{S}_\epsilon(s) - \sigma| \, dv_{\tilde{g}_\epsilon(s)}) \, ds \right) \end{aligned}$$

for some constant C_8 independent of ϵ, t. Here we have used the fact that $g_{\epsilon,0} = g_0$ outside $\Sigma(2\epsilon)$ and the fact that $\mathcal{S}_{g_0} \ge \sigma$. Let $R \to \infty$, using (iii), and finally let $\epsilon \to 0$, we conclude that

$$\int_M (\mathcal{S}(t) - \sigma)_- \, dv_{g(t)} \le C_8 (t^{1-\delta} + t^{\frac{1}{2}(1-\delta)}).$$

Since

$$\exp(-C_4 t^{\frac{1}{2}(1+\delta)}) \int_M (\mathcal{S}(t) - \sigma)_- \, dv_{g_\epsilon(t)}$$

is nonincreasing in t, we conclude that the lemma is true. $\square$

5. Singular metrics realizing the nonpositive Yamabe invariant

In this section, we will apply the results in previous sections to study singular metrics on compact manifolds. Let M^n be a compact smooth manifold without boundary. Then as in the Introduction, we may define the *Yamabe invariant* $\sigma(M)$. It is well known that if $\sigma(M) \le 0$ and if g is a smooth metric which realizes $\sigma(M)$, then g is Einstein; see [Schoen 1989, pp. 126–127] for example. If $\sigma(M) > 0$, the situation is more complicated; for some recent results see [Macbeth 2017].

In this section we want to discuss the following question:

Suppose g is a continuous Riemannian metric on M which is smooth outside some compact set Σ such that the volume of g is normalized to be 1. *Suppose the scalar curvature of g satisfies $\mathcal{S}_g \ge \sigma(M)$ away from Σ. What can we say about g?*

In the case that Σ has codimension at least 2, we have the following:

Theorem 5.1. *Let M^n be a smooth compact manifold such that $\sigma(M) \le 0$. Suppose g_0 is a Riemannian metric with $V(M, g_0) = 1$ satisfying the following*:

(i) *There is a compact subset Σ such that g_0 is smooth on $M \setminus \Sigma$ with scalar curvature $\mathcal{S}_{g_0} \ge \sigma(M)$ away from Σ.*

(ii) *The metric g_0 is in $W^{1,q}_{\mathrm{loc}}$ for some $q > n$ in the sense that g_0 has weak derivative and $|g_0|_{\mathfrak{b}}, |{}^{\mathfrak{b}}\nabla g_0|_{\mathfrak{b}} \in L^q_{\mathrm{loc}}$ with respect to a smooth background metric $\mathfrak{b}$.*

(iii) *The volume $V(\Sigma(\epsilon), g_0)$ with respect to g_0 of the ϵ-neighborhood $\Sigma(\epsilon)$ of Σ is bounded by $C\epsilon^2$ for some constant C independent of ϵ. Here*

$$\Sigma(\epsilon) = \{x \in M \mid d_{g_0}(x, \Sigma) < \epsilon\}.$$

Then g_0 is Einstein on $M \setminus \Sigma$.

To prove the theorem, let (M^n, g_0) be as in the theorem. Let

$$\mathring{\mathrm{Ric}}(g_0) = \mathrm{Ric}(g_0) - \frac{\mathcal{S}_0}{n} g_0$$

be the traceless part of $\mathrm{Ric}(g_0)$ where $\mathcal{S}_0 = \mathcal{S}_{g_0}$ is the scalar curvature of g_0. Let $x_0 \in M \setminus \Sigma$. We want to prove that $\mathring{\mathrm{Ric}}(x_0) = 0$. Suppose $\mathring{\mathrm{Ric}}(g_0)(x_0) \ne 0$, then there is $r > 0$ such that $B_{x_0}(4r; g_0) \cap \Sigma = \varnothing$ and there is $c > 0$, $|\mathring{\mathrm{Ric}}(g_0)|(x_0) \ge 2c$ in $B_{x_0}(3r)$. By Corollary 4.6, we can find smooth metrics g_i such that (i) g_i converges uniformly to g_0 and converges in C^∞ norm on any compact sets in $M \setminus \Sigma$; (ii) $V(M, g_i) = 1$; (iii) the scalar curvature $\mathcal{S}_i$ of g_i satisfies $\mathcal{S}_i \ge \sigma - \delta_i$ for all i with $\delta_i \downarrow 0$. Hence we may assume that

$$|\mathring{\mathrm{Ric}}(g_i)|(x) \ge c \tag{5-1}$$

in $B_{x_0}(2r; g_i)$ for all i, and $B_{x_0}(r; g_i) \subset B_{x_0}(2r; g)$, $B_{x_0}(2r; g_i) \subset B_{x_0}(3r; g)$. We may also assume that the distance function $r_i(x)$ from x_0 with respect to g_i are smooth in $B_{x_0}(3r; g)$, provided $r > 0$ is small enough, independent of i.

Let ϕ be a smooth function on $[0, \infty)$ with $\phi \geq 0$, $\phi = 1$ on $[0, 1]$ and $\phi = 0$ on $[2, \infty)$ and such that $|\phi'|^2 \leq C\phi$, with C being an absolute constant. Let

$$h_i(x) = \phi\left(\frac{r_i(x)}{r}\right)\mathring{\mathrm{Ric}}(g_i)(x).$$

For $|\tau| > 0$, let $G_{i;\tau} = g_i + \tau h_i$. Then there is $\tau_0 > 0$ such that $G_{i;\tau}$ are smooth metrics for all i and for all $0 < |\tau| \leq \tau_0$.

In the following, $E_k = E_k(x, \tau)$ $(k = 1, 2)$ will denote a quantity such that $|E_k| \leq C|\tau|^k$ for some C independent of x, i and τ.

Lemma 5.2. *We have*

$$dv_{G_{i;t}} = dv_{g_i}(1 + E_2)$$

and

$$V(M, G_{i;t}) = 1 + E_2;$$

here dv_g denotes the volume element of metric g.

Proof. Since $g_i \to g$ uniformly on compact sets of $M \setminus \Sigma$ in C^∞ norm and since h_i is traceless, the results follow. □

We have the following general fact [Brendle and Marques 2011, Proposition 4]:

Lemma 5.3. *Let (Ω^n, g) be a smooth Riemannian manifold. Let $\bar{g} = g + h$ with $|h|_g \leq \frac{1}{2}$. Then the scalar curvatures are related as*

$$\mathcal{S}_{\bar{g}} - \mathcal{S}_g = \mathrm{div}_g(\mathrm{div}_g(h)) - \Delta_g \operatorname{tr}_g h - \langle h, \mathrm{Ric}(g)\rangle_g + F,$$

where

$$|F| \leq C\big(|\nabla h|^2 + |h|_g|\nabla^2 h|_g + |\mathrm{Ric}(g)|\,|h|_g^2\big)$$

for some constant C depending only on n. Here ∇ is the covariant derivative with respect to g.

Lemma 5.4. *Let $\mathcal{S}_i$ be the scalar curvature of g_i and $\mathcal{S}_{i;\tau}$ be the scalar curvature of $G_{i;\tau}$. Then*

$$\mathcal{S}_{i;\tau} = \mathcal{S}_i + \tau\, \mathrm{div}_{g_i}(\mathrm{div}_{g_i} h_i) - \tau\langle h_i, \mathrm{Ric}(g_i)\rangle_{g_i} + E_2(\tau).$$

Note that $\mathcal{S}_{i;\tau} = \mathcal{S}_i$ outside $B_{x_0}(2r, g_i)$ and is bounded below by a constant independent of i, τ.

Proof. This follows from Lemma 5.3, the fact that h_i is traceless, $h_i = 0$ outside $B_{x_0}(2r, g_i)$, the fact that $g_i \to g$ in C^∞ outside Σ and the fact that $\mathcal{S}_i \geq \sigma - \delta_i$. □

In the following, let

$$a = \frac{4(n-1)}{n-2}, \quad p = \frac{2n}{n-2}. \tag{5-2}$$

By the resolution of the Yamabe conjecture [Yamabe 1960; Trudinger 1968; Aubin 1976b; Schoen 1984], for each i, τ, we can find a smooth positive solution $u_{i;\tau}$ of

$$-a\Delta_{G_{i;\tau}} u_{i;\tau} + \mathcal{S}_{i;\tau} u_{i;\tau} = \lambda_{i;\tau} V_{i;\tau}^{-2/n} u_{i;\tau}^{p-1} \tag{5-3}$$

with $\lambda_{i;\tau} = Y(\mathcal{C}_{i,\tau})$ which is less than or equal to σ (in particular, it is nonpositive), where $\mathcal{C}_{i,\tau}$ is the class of smooth metrics conformal to $G_{i;\tau}$. Moreover, $u_{i;\tau}$ is normalized by

$$\int_M u_{i;\tau}^p \, dv_{G_{i;\tau}} = 1,$$

and $V_{i,\tau} = V(M, G_{i;\tau})$.

Lemma 5.5. *There is* $0 < \tau_1 \leq \tau_0$ *independent of* i *such that if* $0 > \tau \geq -\tau_1$, *then*

$$\frac{a}{2}\int_M |{}^{(i;\tau)}\nabla u_{i;\tau}|^2_{G_{i;\tau}} \, dv_{G_{i;\tau}} - \lambda_{i;\tau} V_{i;\tau}^{-2/n} + \sigma \leq -C|\tau| \int_{B_{x_0}(2r, g_i)} \phi u_{i;\tau}^2 \, dv_{g_i} + C'\delta_i + E_2(\tau)$$

for some positive constants C, C' *independent of* i *and* τ. *Here* ${}^{(i;\tau)}\nabla$ *is the covariant derivative with respect to* $G_{i;\tau}$.

Proof. For simplicity of notation, in the following we denote ${}^{(i;\tau)}\nabla$ by ∇, $G_{i;\tau}$ by G; g_i by g; $u_{i;\tau}$ by u; $\lambda_{i;\tau}$ by λ; $\mathcal{S}_{i;\tau}$ by $\mathcal{S}_G$; $\mathcal{S}_i$ by $\mathcal{S}_g$; and $V_{i;\tau}$ by V.

Multiply (5-3) by u and integrating by parts, using the fact that

$$\int_M u^p \, dv_G = 1,$$

we have

$$\begin{aligned} a\int_M |\nabla u|^2_G \, dv_G - \lambda V^{-2/n} &= -\int_M \mathcal{S}_G u^2 \, dv_G \\ &\leq -\int_M \mathcal{S}_G u^2 \, dv_g + E_2(\tau)\int_M u^2 \, dv_g \end{aligned} \tag{5-4}$$

by Lemmas 5.2 and 5.4 and the fact that g_i converges in C^∞ norm in $B_{x_0}(3r, g_0) \supset B_{x_0}(g_i, 2r)$. On the other hand, by Lemma 5.4, for any $0 < \epsilon < 1$,

$$
\begin{aligned}
\text{(5-5)}\quad & -\int_M \mathcal{S}_G u^2\, dv_g \\
&\le -\int_M \mathcal{S}_g u^2\, dv_g - \tau \int_M \big(\mathrm{div}_g(\mathrm{div}_g\, h) - \langle h, \mathrm{Ric}(g)\rangle_g\big) u^2\, dv_g \\
&\qquad + E_2(\tau) \int_{B_{x_0}(2r;g)} u^2\, dv_g \\
&\le -\int_M \mathcal{S}_g u^2\, dv_g + C_1|\tau| \int_M u|{}^g\nabla u|_g \big(|\phi'||\mathring{\mathrm{Ric}}(g)|_g + \phi|{}^g\nabla \mathcal{S}_0|_g\big)\, dv_g \\
&\qquad - |\tau| \int_M \phi |\mathring{\mathrm{Ric}}(g)|^2 u^2\, dv_g + E_2(\tau) \int_{B_{x_0}(2r;g)} u^2\, dv_g \\
&\le (-\sigma + \delta) \int_M u^2\, dv_g + (C_2 + \epsilon^{-1})|\tau| \int_M |{}^g\nabla u|_g^2\, dv_g \\
&\qquad - C_3|\tau| \int_M \phi |\mathring{\mathrm{Ric}}(g)|^2 u^2\, dv_g + (E_2(\tau) + C_2\epsilon|\tau|) \int_{B_{x_0}(2r;g)} \phi u^2\, dv_g \\
&\le (-\sigma + \delta) \int_M u^2\, dv_g + (C_2 + \epsilon^{-1})|\tau| \int_M |{}^g\nabla u|_g^2\, dv_g \\
&\qquad + (E_1(\tau) + C_2\epsilon - C_3 c)|\tau| \int_{B_{x_0}(2r;g)} \phi u^2\, dv_g
\end{aligned}
$$

for some constants $C_1, C_2, C_3 > 0$ independent of i, τ. Here we have used the fact that $|\phi'|^2 \le C\phi$ and the fact that $\mathcal{S}_g \ge \sigma - \delta_i$ which is negative, where we denote δ_i by δ. Choose $\epsilon > 0$ so that $C_2\epsilon = \frac{1}{2}C_3 c$. Then the result follows if $\tau_1 > 0$ is small enough and independent of i, by (5-4), (5-5), the Hölder inequality, the fact that g, G are uniformly equivalent, and the fact that

$$\int_M u^p\, dv_G = 1, \quad V(M, g) = 1,$$

and

$$V(M, G) = 1 + E_2(\tau). \qquad \square$$

Let $0 > \tau_k > -\tau_1$, $\tau_k \to 0$. Since $\delta_i \to 0$, for each k we can find i_k such that $\delta_{i_k} \le \tau_k^2$, $i_k \to \infty$. Let us denote $G_{i_k;\tau_k}$ by G_k, and $u_{i_k;\tau_k}$ by u_k. We want to prove the following:

Lemma 5.6. *There is a constant $C > 0$ such that for all k,*

$$\inf_{B_{x_0}(3r, g_0)} u_k \ge C.$$

Proof of Theorem 5.1. Suppose the lemma is true then we will have a contradiction. In fact, if we denote δ_{i_k} by δ_k, since $V(M, G_k) = 1 + E_2(\tau_k)$, $\lambda \le \sigma$, by Lemma 5.5, we have

$$\begin{aligned}\frac{a}{2}\int_M |{}^{G_k}\nabla u_k|^2_{G_k}\, dv_{G_k} &\le -C_1|\tau_k| \int_{B_{x_0}(2r, g_{i_k})} \phi u_k^2\, dv_{g_{i_k}} + C_2\delta_k + C_2\tau_k^2 \\ &\le -C_1|\tau_k| \int_{B_{x_0}(2r, g_{i_k})} \phi u_k^2\, dv_{g_{i_k}} + (C_2+1)\tau_k^2\end{aligned}$$

for some positive constants C_1, C_2 independent of k. By Lemma 5.6, this is impossible if k is large enough. Hence $\mathring{\mathrm{Ric}}(g_0)(x_0)$ must be zero. Theorem 5.1 then follows. □

It remains to prove Lemma 5.6. Consider the equation

$$-a\Delta u + \mathcal{S}u = \lambda u^{p-1}. \tag{5-6}$$

Lemma 5.7. *Let (M^n, g) be a smooth metric with scalar curvature $\mathcal{S} \ge -s_0$, with $s_0 \ge 0$. Let $u > 0$ be a solution of (5-6) with $\|u\|_p = 1$ and with $\lambda \le 0$. Then for any $q > p$,*

$$\|u\|_q \le C(s_0, V(M; g), n, q).$$

Proof. See [Trudinger 1968]; see also [Lee and Parker 1987, Proposition 4.4]. □

Lemma 5.8. *Using the notation of Lemma 5.6,*

(i) *for any $q > p$, there is a constant C independent of k such that*

$$\|u_k\|_{q,g_0} \le C;$$

(ii) *u_k subconverges in C^2 norm with respect to g_0 in any compact set $K \subset M \setminus \Sigma$;*

(iii) $\lim_{k\to\infty} \int_M \|{}^{g_0}\nabla u_k\|^2_{g_0}\, dv_{g_0} = 0$;

(iv) $\lim_{k\to\infty} \lambda_k = \sigma$, *where $\lambda_k = \lambda_{i_k;\tau_k}$ as in (5-3).*

Proof. Since $\mathcal{S}_{i_k;\tau_k} \ge \sigma - \delta_k$ and $\delta_k \to 0$, (i) follows from Lemma 5.7 and the fact that $C^{-1}g_0 \le G_k \le Cg_0$ for some $C > 0$ for all k.

To prove (ii), for any compact set $K \subset M \setminus \Sigma$, there is an open set $K \Subset U \subset M \setminus \Sigma$ such that G_k converges in C^∞ norm to g_0 on U. By Lemma 5.5, we conclude that $0 \le -\lambda_k \le C$ for some constant independent of k. Then by (i), and [Lee and Parker 1987, Theorem 2.4], we conclude that for any $U' \Subset U$,

$$\|u_k\|_{L_2^q(U')} \le C_1$$

for some constant C independent of k. We then use the Sobolev embedding theorem to conclude that the C^α norm of u_k are uniformly bounded in $U' \Subset U$. From this the result follows by Schauder estimates.

Parts (iii) and (iv) follow from Lemma 5.5. □

Corollary 5.9. *After passing to a subsequence, u_k converges in C^2 norm locally in $M \setminus \Sigma$ to a function $\mathfrak{u}$. Moreover, $\mathfrak{u} = 1$ in $M \setminus \Sigma$ and*

$$\mathcal{S}_{g_0} = \sigma.$$

In particular Lemma 5.6 is true.

Proof. By Lemma 5.8, after passing to a subsequence, u_k converges in C^2 norm locally in $M \setminus \Sigma$ to a function $\mathfrak{u}$. Moreover, $\mathfrak{u}$ is constant in each component of $M \setminus \Sigma$. We claim that there is $C_1 > 0$ such that $0 \le u_k \le C_1$ for all k.

Since the scalar curvature $\mathcal{S}_{G_k} \ge -s_0$ for some $s_0 > 0$ independent of k and since $\lambda_k \le 0$, we have

$$-a\Delta_{G_k} u_k - s_0 u_k \le -a\Delta_{G_k} u_k + \mathcal{S}_{G_k} u_k \le 0.$$

Moreover, $\int_M u_k^p \, dv_{G_k} = 1$ and G_k is equivalent to g_0 uniformly in k, the claim follows from mean value inequality [Gilbarg and Trudinger 1983, Theorem 8.17].

Since $u_k \to \mathfrak{u}$ almost everywhere, and G_k converges uniformly to g_0, we have

$$\int_M \mathfrak{u}^p \, dv_{g_0} = 1.$$

In particular, $\mathfrak{u} > 0$ somewhere.

Next we want to prove that $\mathfrak{u}$ is constant on M. By Lemma 5.8, there is a constant C_2 independent of k such that

$$\int_M (|{}^{g_0}\nabla u_k|_{g_0}^2 + u_k^2) \, dv_{g_0} \le C_2.$$

Passing to a subsequence, we may assume that u_k converges weakly in $W^{1,2}(M, g_0)$ to v say. We claim that v is constant. In fact, for any $\ell \ge 1$, the sequence $u_{\ell+k}$, $k \ge 1$, also weakly converges to v. Then we can find convex combinations of $u_{\ell+k}$ which converge to v strongly in $W^{1,2}(M, g_0)$. Namely, for any $\epsilon > 0$, there exists $\alpha_1, \ldots, \alpha_m$ with $\alpha_k \ge 0$, $\sum_{k=1}^m \alpha_k = 1$ such that if $w = \sum_{k=1}^m \alpha_k u_{\ell+k}$, then

$$\|w - v\|_{W^{1,2}(M, g_0)} \le \epsilon.$$

On the other hand, by Lemma 5.8, if ℓ is large enough, then

$$\begin{aligned}\left(\int_M |{}^{g_0}\nabla w|_{g_0}^2 \, dv_{g_0}\right)^{\frac{1}{2}} &\le \left(\int_M \left(\sum_k \alpha_k |{}^{g_0}\nabla u_{\ell+k}|_{g_0}\right)^2 dv_{g_0}\right)^{\frac{1}{2}} \\ &\le \sum_k \alpha_k \left(\int_M |{}^{g_0}\nabla u_{l+k}|_{g_0}^2 \, dv_{g_0}\right)^{\frac{1}{2}} \\ &\le \epsilon.\end{aligned}$$

Hence

$$\int_M |{}^{g_0}\nabla v|^2\, dv_{g_0} \le (2\epsilon)^2.$$

This implies ${}^{g_0}\nabla v = 0$, a.e. Since $v \in W^{1,2}(M, g_0)$, we conclude that $v = c$ is a constant as claimed.

On the other hand, for any smooth function ϕ on M

$$\begin{aligned}\lim_{k\to\infty}\int_M (\langle {}^{g_0}\nabla\phi, {}^{g_0}\nabla u_k\rangle_{g_0} + \phi u_k)\, dv_{g_0} &= \int_M (\langle {}^{g_0}\nabla\phi, {}^{g_0}\nabla v\rangle_{g_0} + \phi v)\, dv_{g_0} \\ &= \int_M \phi v\, dv_{g_0}.\end{aligned}$$

Also by Lemma 5.8 again, and the fact that u_k are uniformly bounded and $u_k \to \mathfrak{u}$ a.e., we have

$$\lim_{k\to\infty}\int_M (\langle {}^{g_0}\nabla\phi, {}^{g_0}\nabla u_k\rangle_{g_0} + \phi u_k)\, dv_{g_0} = \int_M \phi\mathfrak{u}\, dv_{g_0}.$$

So

$$\int_M \phi\mathfrak{u}\, dv_{g_0} = \int_M \phi v\, dv_{g_0}.$$

Hence $\mathfrak{u} = v$ is a constant. Since $\int_M \mathfrak{u}^p\, dv_{g_0} = 1$ so $\mathfrak{u} = 1$. Since $\mathfrak{u}$ satisfies

$$-a\Delta_{g_0}\mathfrak{u} + \mathcal{S}_{g_0}\mathfrak{u} = \sigma\mathfrak{u}^p,$$

the last assertion follows. □

This completes the proof of Theorem 5.1. Next we want to discuss the case that Σ has codimension one. We have the following:

Theorem 5.10. *Let M^n be a smooth compact manifold such that $\sigma(M) \le 0$. Suppose g_0 is a Riemannian metric with $V(M, g_0) = 1$ satisfying (b1)–(b3) in Section 4. Then g_0 is Einstein on $M \setminus \Sigma$ and $\mathcal{S}_{g_0} = \sigma(M)$. Moreover, $H_- = H_+$.*

Proof. Let $g_i = g_{\epsilon_i,0}$ be the smooth approximation of g_0 by [Miao 2002] as given in Section 4. The fact that g_0 is Einstein outside Σ can be proved similarly as above using Corollary 4.8. It remains to prove that $H_- = H_+$. Let $\epsilon_i \to 0$ and let u_i be the positive solution of

$$-a\Delta_i u_i + \mathcal{S}_i u_i = \lambda_i u_i^{p-1}$$

normalized as

$$\int_M u_i^p\, dv_i = 1.$$

Here Δ_i is the Laplacian of g_i etc. Also $\lambda_i \le \sigma$, where $\sigma := \sigma(M)$. Suppose $H_-(z) > H_+(z)$ somewhere; then one can easily check that there is a positive constant b such that for i large enough,

$$\int_M \mathcal{S}_i \, dv_i \geq \sigma + b. \tag{5-7}$$

As before, passing to a subsequence if necessary, $u_i \to 1$ outside Σ and uniform in C^∞ norm in any compact set of $M \setminus \Sigma$. Moreover, u_i are uniformly bounded, and $\lambda_i \to \sigma$. Since $\mathcal{S}_i$ be bounded below by $-s_0$, for some $s_0 \geq 0$ and u_i is bounded from below, we have

$$\begin{aligned}\sigma &= \lim_{i\to\infty} \lambda_i \int_M u_i^{p-1} \, dv_i \\ &= \lim_{i\to\infty} \int_M \mathcal{S}_i u_i \, dv_i \\ &\geq \lim_{i\to\infty} \int_M \mathcal{S}_i (u_i - 1) \, dv_i + \sigma + b,\end{aligned}$$

where we have used the fact that $V(M, g_{0,\epsilon_i}) \to V(M, g_0) = 1$ and (5-7). We claim

$$\lim_{i\to\infty} \int_M \mathcal{S}_i (u_i - 1) \, dv_i = 0.$$

If the claim is true, then we have a contradiction because $b > 0$. To prove the claim, note that on $|t| \leq a$, the original metric g_0 is of the form

$$g_0(z,t) = dt^2 + g_{ij}(z,t) \, dz^i \, dz^j.$$

We assume that $g_{ij}(z,t)$ (which will be denoted by $h^t_{ij}(z)$) is uniformly equivalent to $g_{ij}(z,0)$ (which will be denoted by $h_{ij}(z)$). For any $z \in \Sigma$ and for any $1 \geq t \geq 0$,

$$|u_i(z,a) - u_i(z,t)| \leq \int_0^a \left| \frac{\partial u_i(z,s)}{\partial s} \right| ds \leq \int_0^1 |^{g_0}\nabla u_i|(z,s) \, ds.$$

By the properties of $g_{0,\epsilon}$,

$$\int_{\epsilon_i^2/100 \leq |t| \leq \epsilon_i} |\mathcal{S}_i (u_i - 1)| \, dv_i = o(1) \tag{5-8}$$

because u_i are uniformly bounded. So

$$\begin{aligned}\int_{|t|\leq\epsilon_i^2/100} &\mathcal{S}_i(z,t)(u_i(z,t) - 1) \, dv_{g_i} \\ &= \int_{|t|\leq\epsilon_i^2/100} \mathcal{S}_i(z,t)(u_i(z,1) - 1) \, dv_{g_i} \\ &\quad + \int_{|t|\leq\epsilon_i^2/100} \mathcal{S}_i(z,t)(u_i(z,t) - u_i(z,1)) \, dv_{g_i} \\ &= I + II.\end{aligned} \tag{5-9}$$

Since $u_i(z,1) \to 1$ uniformly on $z \in \Sigma$, and $\int_M |\mathcal{S}_i|\, dv_{g_i}$ is bounded, we conclude that

$$I = o(1) \tag{5-10}$$

as $i \to \infty$. On the other hand,

$$\begin{aligned}
|II| &\le \int_{|t| \le \epsilon_i^2/100} |\mathcal{S}_i(z,t)(u_i(z,t) - u_i(z,1))|\, dv_{g_i} \\
&\le c \int_{z \in \Sigma} \left(\int_{-\epsilon_i^2/100}^{\epsilon_i^2/100} \epsilon_i^{-2} \int_0^1 |\nabla u_i(z,s)|\, ds \right) dt\, dv_h \\
&\le c \int_{z \in \Sigma} \left(\int_0^a |\nabla u_i(z,s)|\, ds \right) dt\, dv_h \\
&\le c \int_M |\nabla u_i|\, dv_{g_i} \\
&= o(1)
\end{aligned} \tag{5-11}$$

by the Schwartz inequality and Lemma 5.8. The claim follows from (5-8)–(5-11). □

6. Singular Einstein metrics

In the conclusions of Theorem 5.1, one obtains metrics which are smooth and Einstein outside some singular sets. In this section, we want to prove that under certain conditions, one may introduce a smooth structure so that the Einstein metric is actually smooth. More precisely, we have the following:

Theorem 6.1. *Let M^n, $n \ge 3$, be a smooth manifold and g be a Riemannian metric on M satisfying the following conditions: There is a compact set Σ in M such that*

(i) *g is Lipschitz and g is smooth on $M \setminus \Sigma$;*

(ii) *$g = \lambda \operatorname{Ric}$ on $M \setminus \Sigma$ for some constant λ;*

(iii) *the codimension of Σ is larger than 1 in the sense that $V(\Sigma(\epsilon), g) = O(\epsilon^{1+\theta})$ for some $\theta > 0$, where $\Sigma(\epsilon) = \{x \in M \mid d(x,\Sigma) < \epsilon\}$.*

Then for any open set U containing Σ, there is a smooth structure on M which is the same as the original smooth structure on $M \setminus U$ so that g is a smooth Einstein metric on M.

We want to construct the required smooth structure using harmonic coordinates. First recall the following.

Lemma 6.2. *Let $B(1)$ be the unit ball in $\mathbb{R}^n$ with center at the origin. Let (a_{ij}) be a symmetric matrix such that*

$$\lambda |\xi|^2 \le a^{ij} \xi^i \xi^j \le \Lambda |\xi|^2,$$

for some $\Lambda > \lambda > 0$ for all $\xi \in \mathbb{R}^n$ and where a^{ij} is Lipschitz with Lipschitz constant L. Let $f \in L^\infty(B(1))$. Then the boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial x^i}\left(a^{ij}\dfrac{\partial u}{\partial x^j}\right) = f & \text{in } B(1), \\ u = 0 & \text{on } \partial B(1) \end{cases}$$

has a unique solution in $W^{2,p}(B(1))$ for any $p > 1$ with $u \in W_0^{1,p}(B(1))$. Moreover, we have

$$\|u\|_{2,p} \le C(\|u\|_p + \|f\|_p)$$

for some constant C depending only on $p, n, \lambda, \Lambda, L$. Here $\|u\|_{2,p}$ is the $W^{2,p}$ norm on $B(1)$ and $\|u\|_p$ is the L^p norm in $B(1)$.

Proof. The results follow from [Gilbarg and Trudinger 1983, Theorem 9.15, Corollary 9.13]. By taking $p > n$ and the Sobolev embedding theorem, u is continuous up to the boundary and $u = 0$ at the boundary. □

With the same assumptions and notation as in Theorem 6.1, let $q \in \Sigma$. Let $U_\delta = \{(x^1, \dots, x^n) \mid |x| < \delta\}$ be a smooth local coordinate neighborhood with q being at the origin such that g_{ij} is equivalent to the Euclidean metric and g_{ij} is Lipschitz with Lipschitz constant L

Lemma 6.3. *With the above assumptions and notation, there is $\delta > \epsilon > 0$ and functions $u^1, \dots, u^n$ on $U_\epsilon = \{(x^1, \dots, x^n) \mid |x| < \epsilon\}$ such that the mapping $(x^1, \dots, x^n) \to (u^1, \dots, u^n)$ is a local $C^{1,\alpha}$ diffeomorphism at the origin for some $0 < \alpha < 1$, $u^i \in W^{2,p}(U_\epsilon)$ for all $p > 1$ and u^i is harmonic with respect to g for $1 \le i \le n$. Moreover, u^i is smooth outside Σ.*

Proof. Let $\delta > \epsilon > 0$ to be chosen later. Fix ℓ, let $f = \Delta_g x^\ell$ which is bounded by the assumption on g_{ij}. Let $\lambda, \Lambda > 0$ be such that

$$\lambda|\xi|^2 \le g^{ij}\xi^i\xi^j \le \Lambda|\xi|^2 \tag{6-1}$$

in U_δ.

Let $y = \epsilon^{-1}x$. Consider the following boundary value problem on $B(1)$ in the y-space

$$\begin{cases} \dfrac{\partial}{\partial y^i}\left(\sqrt{g}g^{ij}\dfrac{\partial v}{\partial y^j}\right) = \epsilon^2\sqrt{g}f & \text{in } B(1), \\ v = 0 & \text{on } \partial B(1). \end{cases} \tag{6-2}$$

By Lemma 6.2, the boundary value problem has a solution v satisfying the conclusions in that lemma. Here we have used the fact that g_{ij} has Lipschitz constant bounded by ϵL and still satisfies (6-1) as functions of y. In particular, we have

$$\|v\|_{2,p;y} \le C_1(\|v\|_{p;y} + \epsilon^2).$$

Here and below, C_i will denote positive constants independent of ϵ. Let $p > n$ be fixed; then one can see that there is $1 > \alpha > 0$ such that $v \in C^{1,\alpha}(B(1))$ in the y-space and

(6-3) $$\|v\|_{C^{1,\alpha}(B(1))} \le C_2(\|v\|_{p;y} + \epsilon^2)$$

for some positive constants $C_2 - C_4$ independent of ϵ.

Let $w(x) = v(\epsilon^{-1}x)$ with $x \in B(\epsilon)$ in the x-space. Then w satisfies

$$\begin{cases} \dfrac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\dfrac{\partial w}{\partial x^j}\right) = \sqrt{g}f & \text{in } B(\epsilon), \\ w = 0 & \text{on } \partial B(\epsilon) \end{cases}$$

in the x-space. Moreover, $w \in W^{2,p}(B(\epsilon))$. Let $u^\ell = w - x^\ell$. Then u^ℓ is harmonic, namely, u^ℓ satisfies

$$\begin{cases} \dfrac{1}{\sqrt{g}}\dfrac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\dfrac{\partial u^\ell}{\partial x^j}\right) = 0 & \text{in } B(\epsilon), \\ u^\ell = x^\ell & \text{on } \partial B(\epsilon). \end{cases}$$

By the maximum principle, we conclude that $|u^\ell| \le \epsilon$ and so $|w| \le 2\epsilon$, and moreover, we have

(6-4) $$\sup_{B(\epsilon)} |\partial_x w| = \epsilon^{-1} \sup_{B(1)} |\partial_y v| \le C_2\epsilon^{-1}(\|v\|_{p;y} + \epsilon^2).$$

To estimate the right-hand side, multiply (6-2) by v and integrating by parts, using the Poincaré inequality, we have

$$\int_{B(1)} v^2\,dy \le C_3\epsilon^2 \int_{B(1)} |v|\,dy$$

and so

$$\begin{aligned} \|v\|_{p;y} &\le \left(\sup_{B(1)} |v|\right)^{1-2/p}\left(\int_{B(1)} v^2\,dy\right)^{1/p} \\ &\le C_4\epsilon^{1-2/p}\cdot\epsilon^{4/p} \\ &= C_4\epsilon^{1+2/p}, \end{aligned}$$

where we have used the Hölder inequality and the fact that $|v| = |w| \le 2\epsilon$. By (6-4) we conclude that

$$\sup_{B(\epsilon)} |\partial_x w| \le C_5\epsilon^{2/p}.$$

Hence

$$\frac{\partial u^\ell}{\partial x^i} = \delta_i^\ell + O(\epsilon^{2/p}).$$

From this and the fact that g is smooth outside Σ it is easy to see that the lemma is true, provided ϵ is small enough. $\square$

Proof of Theorem 6.1. Let U be any open set containing Σ. For any $q \in \Sigma$, by Lemma 6.3, we can find smooth coordinates neighborhood $V_q \Subset U$ around q and $C^{1,\alpha}$ functions $u^1, \dots, u^n$ on V_q near q which are in $W^{2,p}(V_q)$ as functions of x. Moreover, $(x^1, \dots, x^n) \to (u^1, \dots, u^n)$ is a C^1 diffeomorphism from V_q to its image $\widetilde{V}_q$ in the u-space. Let

$$h_{ab} = g\left(\frac{\partial}{\partial u^a}, \frac{\partial}{\partial u^b}\right) = \frac{\partial x^i}{\partial u^a}\frac{\partial x^j}{\partial u^b} g_{ij}, \tag{6-5}$$

where

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

Let

$$R_{ab} = \mathrm{Ric}\left(\frac{\partial}{\partial u^a}, \frac{\partial}{\partial u^b}\right).$$

Since each u^a is harmonic, and $R_{ab} = \lambda h_{ab}$ by assumption, away from Σ for all a, b we have

$$h^{cd} h_{ab,cd} = -2\lambda h_{ab} + \partial h^{-1} * \partial h + h^{-1} * h^{-1} * \partial h * \partial h := Q(h, \partial h), \tag{6-6}$$

where $(h^{cd}) = (h_{cd})^{-1}$,

$$h_{ab,c} = \frac{\partial}{\partial u^c} h_{ab}$$

etc., and $\partial h^{-1} * \partial h$ denotes a sum of finite terms of the form

$$\left(\frac{\partial}{\partial u^c} h^{ab}\right)\left(\frac{\partial}{\partial u^f} h_{de}\right)$$

etc. By (6-5),

$$h_{ab,c} = 2\frac{\partial^2 x^i}{\partial u^a \partial u^c}\frac{\partial x^j}{\partial u^b} g_{ij} + \frac{\partial x^i}{\partial u^a}\frac{\partial x^j}{\partial u^b}\frac{\partial x^k}{\partial u^c}\frac{\partial}{\partial x^k} g_{ij}. \tag{6-7}$$

We may assume that $\widetilde{V}_q$ contains the origin which is the coordinates of q. Then by shrinking $\widetilde{V}_q$ if necessary, by Lemma 6.3, h_{ab} is bounded and $h_{ab,c}$ is in L^p for all $p > 1$ for all a, b, c as functions of u. In particular, h_{ab} is in $W^{1,p}(\widetilde{V}_q)$ for all $p > 1$. Moreover, (h^{ab}) is uniformly elliptic. Since h^{ab} is only in C^α with $0 < \alpha < 1$, we cannot apply the standard L^p estimate as in [Gilbarg and Trudinger 1983, Theorem 9.19]. Hence, we want to prove that h_{ab} is in $W^{2,p}(B(\delta))$ for all a, b for all $p > n$ and for some $\delta > 0$ in the u-space, where $B(\delta) = \{u \mid |u| < \delta\}$. Suppose this is true; then $h_{ab} \in C^{0,1}_{\mathrm{loc}}(B(\delta))$ and $\partial h \in W^{1,p}_{\mathrm{loc}}(B(\delta))$. This implies $Q(h, \partial h)$ in (6-6) is in $W^{1,p/2}_{\mathrm{loc}}(B(\delta))$. Since this is true for all $p > n$, by [Gilbarg and Trudinger 1983, Theorem 9.19], we conclude that h_{ab} is in $W^{3,p}(B(\delta))$. Continuing in this way, we conclude that $h_{ab} \in W^{k,p}_{\mathrm{loc}}(B(\delta))$ for all $k \ge 1$ and $p > n$ by a bootstrap argument. Hence h_{ab} is smooth near the origin.

It remains to prove that $h_{ab} \in W^{2,p}(B(\delta))$ for all $p > n$ for all a, b for some $\delta > 0$. Fix a, b and let $w = \phi h_{ab}$ where ϕ is a smooth cutoff function in $B(2\delta)$ such that $\phi = 1$ in $B(\delta)$, $\phi = 0$ outside $B\big(\frac{3}{2}\delta\big)$, where $\delta > 0$ is small enough so that $B(2\delta) \Subset \widetilde{V}_q$. Then away from Σ, w satisfies

$$h^{cd} w_{cd} = Q_1(h, \partial h, \phi, \partial\phi, \partial^2\phi). \tag{6-8}$$

Since Q_1 is in $L^p(B(2\delta))$ by Lemma 6.3 and (h^{cd}) is continuous and is uniformly elliptic, by [Gilbarg and Trudinger 1983, Theorem 9.15] for any $p > n$ there is $v \in W^{2,p}(B(2\delta)) \cap W_0^{1,p}(B(2\delta))$ such that

$$h^{cd} v_{cd} = Q_1(h, \partial h, \phi, \partial\phi, \partial^2\phi).$$

Since $h^{cd} \in W^{1,p}(B(2\delta))$ for all p, for any smooth function η with compact support in $B(2\delta)$, we have

$$\int_{B(2\delta)} \left(h^{cd} \frac{\partial \eta}{\partial u^a} \frac{\partial v}{\partial u^b} + \eta s^d \frac{\partial v}{\partial u^d} \right) du = -\int_{B(2\delta))} \eta Q_1 \, du. \tag{6-9}$$

where $s^d = \frac{\partial}{\partial u^c} h^{cd}$. We want to prove that w also satisfies this relation.

To prove the claim, note that if we consider $\Sigma \cap \widetilde{V}_q$ then the codimension of Σ in the u-space is at least $1 + \theta$ for some $\theta > 0$ because h_{ab} and the Euclidean metric are uniformly equivalent. As in [Lee 2013], for $\epsilon > 0$ small enough, we can find a smooth function $0 \le \xi_\epsilon \le 1$ in $\widetilde{V}_q$ such that $\xi_\epsilon = 1$ outside $\Sigma_{2\epsilon}$ and is zero in $\Sigma_\epsilon \cap \widetilde{V}_q$ where $\Sigma_\epsilon = \{u \in \widetilde{V}_q \mid d(u, \Sigma) < \epsilon\}$ where the distance is the Euclidean distance. Moreover, $|\partial \xi_\epsilon| \le C_1 \epsilon^{-1}$. Here and below C_i denotes a positive constant independent of ϵ. Now let η be a smooth function with compact support in $B(2\delta)$. Multiply (6-8) by $\eta\xi_\epsilon$ and integrating by parts, we have

$$-\int_{B(2\delta)} \eta \xi_\epsilon Q_1 \, du = \int_{B(2\delta)} \left[h^{cd} \left(\xi_\epsilon \frac{\partial \eta}{\partial u^a} + \eta \frac{\partial \xi_\epsilon}{\partial u^a} \right) \frac{\partial w}{\partial u^b} + \eta \xi_\epsilon s^d \frac{\partial w}{\partial u^d} \right] du.$$

Since $w, h^{cd} \in L^{1,p}(B(2\delta))$ for all $p > 1$, we have

$$\int_{B(2\delta)} |\eta(\xi_\epsilon - 1) Q_1| \, du \le \left(\int_{B(2\delta)} |\eta(\xi_\epsilon - 1) Q_1|^2 \, du \right)^{1/2} V(\Sigma_{2\epsilon})^{1/2} \to 0$$

as $\epsilon \to 0$. Similarly, one can prove that

$$\int_{B(2\delta)} \left| h^{cd} (\xi_\epsilon - 1) \frac{\partial \eta}{\partial u^a} \frac{\partial w}{\partial u^b} + \eta (\xi_\epsilon - 1) s^d \frac{\partial w}{\partial u^d} \right| du \to 0$$

as $\epsilon \to 0$. On the other hand,

$$\begin{aligned}\int_{B(2\delta)} \left| h^{cd} \eta \frac{\partial \xi_\epsilon}{\partial u^a} \frac{\partial w}{\partial u^b} \right| du &\le C_2 \epsilon^{-1} \int_{\Sigma_{2\epsilon}} |\partial w|\, du \\ &\le C_3 \epsilon^{-1} \left(\int_{\Sigma_{2\epsilon}} |\partial w|^p\, du \right)^{\frac{1}{p}} (V(\Sigma(2\epsilon)))^{1-\frac{1}{p}} \\ &\le C_4 \epsilon^{-1+(1+\theta)(1-1/p)} \left(\int_{\Sigma_{2\epsilon}} |\partial w|^p\, du \right)^{\frac{1}{p}} \\ &\to 0\end{aligned}$$

as $\epsilon \to 0$ provided p is large enough. Hence we have

$$\int_{B(2\delta)} \left(h^{cd} \frac{\partial \eta}{\partial u^a} \frac{\partial w}{\partial u^b} + \eta s^d \frac{\partial w}{\partial u^d} \right) du = -\int_{B(2\delta))} \eta Q_1\, du \tag{6-10}$$

for all smooth functions η with compact support $B(2\delta)$.

Let $\zeta = v - w$; then $v - w \in W_0^{1,p}$ for all $p > 1$ and

$$\int_{B(2\delta)} \left(h^{cd} \frac{\partial \eta}{\partial u^a} \frac{\partial \zeta}{\partial u^b} + \eta s^d \frac{\partial \zeta}{\partial u^d} \right) du = 0 \tag{6-11}$$

for all smooth functions η with compact support in $B(2\delta)$. Using the fact that $s^d \in L^p(B(2\delta))$ we can proceed as in the proof of [Gilbarg and Trudinger 1983, Theorem 8.1] to conclude that $\zeta \equiv 0$, because $s^q \in L^p(B(2\delta))$ for all $p > 1$.

To summarize we have proved that $h_{ab} \in W^{2,p}(B(2\delta))$ for all $p > n$ and h_{ab} is smooth in u for all a, b.

We can cover Σ by such harmonic coordinate neighborhoods V_q so that the components of g are smooth with respect to these coordinates. By [Taylor 2006, Theorem 2.1] one can conclude that the theorem is true. □

Corollary 6.4. *Suppose (M^n, g_0) is as in Theorem 5.1. If in addition, g_0 is Lipschitz, then there is a smooth structure on M such that g_0 is smooth and Einstein.*

7. A positive mass theorem with singular set

In this section, we will use the results in Sections 3 and 4 to study positive mass theorems on asymptotically flat manifolds with singular metrics. We want to discuss the theorem without assuming that the manifold is spin. There are different definitions for asymptotically flat manifold. For our purpose, we use the following:

Definition 7.1. An n-dimensional Riemannian manifold (M^n, g), where g is continuous, is said to be asymptotically flat (AF) if there is a compact subset K such that g is smooth on $M \setminus K$, and $M \setminus K$ has finitely many components E_k, $1 \le k \le l$, each E_k is called an end of M, such that each E_k is diffeomorphic to $\mathbb{R}^n \setminus B(R_k)$ for some Euclidean ball $B(R_k)$, and the following are true:

(i) In the standard coordinates x^i of $\mathbb{R}^n$,

$$g_{ij} = \delta_{ij} + \sigma_{ij}$$

with

$$\sup_{E_k}\left\{\sum_{s=0}^{2} |x|^{\tau+s}|\partial^s\sigma_{ij}| + [|x|^{\alpha+2+\tau}\partial\partial\sigma_{ij}]_\alpha\right\} < \infty,$$

for some $0 < \alpha \le 1$, $\tau > (n-2)/2$, where ∂f and $\partial^2 f$ are the gradient and Hessian of f with respect to the Euclidean metric, and $[f]_\alpha$ is the α-Hölder norm of f.

(ii) The scalar curvature $\mathcal{S}$ satisfies the decay condition

$$|\mathcal{S}|(x) \le C(1 + d(x))^{-q}$$

for some $q > n$. Here $d(x)$ is the distance function from a fixed point in M.

The coordinate chart satisfying (i) is said to be *admissible*.

Without loss of generality, we assume that $q \le n+2$. This assumption will be used in (7-3).

In the following, for a function f defined near infinity or $\mathbb{R}^n$, and for $k \ge 0$, $f = O_k(r^{-\tau})$ refers to $\sum_{i=0}^{k} r^i|\partial^i f| = O(r^{-\tau})$ as $r \to \infty$, where $r = |x|$.

Definition 7.2. The Arnowitt–Deser–Misner (ADM) mass (see [Arnowitt et al. 1961]) of an end E of an AF manifold M is defined as

$$\mathfrak{m}_{\mathrm{ADM}}(E) = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (g_{ij,i} - g_{ii,j})\nu^j \, d\Sigma_r^0 \tag{7-1}$$

in an admissible coordinate chart where S_r is the Euclidean sphere, ω_{n-1} is the volume of the $(n-1)$-dimensional unit sphere, $d\Sigma_r^0$ is the volume element induced by the Euclidean metric, ν is the outward unit normal of S_r in $\mathbb{R}^n$ and the derivative is the ordinary partial derivative.

By [Bartnik 1986], $\mathfrak{m}_{\mathrm{ADM}}(E)$ is well-defined, i.e., it is independent of the choice of admissible charts.

For smooth metrics, without assuming the manifold is spin, we have the following positive mass theorem by Schoen and Yau [1979b; 1981; Schoen 1989]:

Theorem 7.3. *Let (M^n, g), $3 \le n \le 7$, be an AF manifold with nonnegative scalar curvature $\mathcal{S} \ge 0$. Then the ADM mass of each end is nonnegative. Moreover, if the ADM mass of one of the ends is zero, then (M^n, g) is isometric to $\mathbb{R}^n$ with the standard metric.*

We want to prove the following positive mass theorem for metrics which are smooth outside a compact set of codimension at least 2. More precisely, we want to prove the following:

Theorem 7.4. *Let (M^n, g_0) be an AF manifold with $3 \le n \le 7$, g_0 being a continuous metric on M such that*

(i) *g_0 is smooth outside a compact set Σ with codimension at least 2 as in (a4) in Section 4*,

(ii) *the scalar curvature $\mathcal{S}$ of g_0 is nonnegative outside Σ*,

(iii) *$g_0 \in W^{1,p}_{\mathrm{loc}}$ for some $p > n$ as in (a2) in Section 4*,

(iv) *on each end E, in an admissible coordinate chart*,

$$g_{ij} = \delta_{ij} + \sigma_{ij}$$

with $\sigma_{ij} = O_5(r^{-\tau})$ with $\tau > (n-2)/2$.

Then the ADM mass of each end is nonnegative. Moreover, if the mass of one of the ends is zero, then M is diffeomorphic to $\mathbb{R}^n$, and g_0 is flat outside Σ.

Remark 7.5. (a) The assumption of continuity of the metric cannot be removed. See the construction in Proposition 2.3.

(b) The case that the singular set is an embedded hypersurface has been studied in [Miao 2002; Shi and Tam 2002]; see also [McFeron and Székelyhidi 2012].

(c) In the case that the singular set has codimension larger than 1, for spin manifolds, positive mass theorems have been obtained under rather general assumptions in [Lee and LeFloch 2015]. Without the spin condition, there are also results for metrics with bounded C^2 norm and with singular set having codimension at least $n/2$ [Lee 2013].

We proceed as in [McFeron and Székelyhidi 2012]. As in Section 4, let $\epsilon > 0$, $\epsilon \to 0$. We can construct a family of metrics $g_{\epsilon,0}$ such that

(i) $g_{\epsilon,0} \to g_0$ uniformly,

(ii) $g_{\epsilon,0} = g_0$ outside $\Sigma(2\epsilon)$,

(iii) the $W^{1,p}$ norm of $g_{\epsilon,0}$ in a fixed precompact open set containing Σ is bounded by a constant independent of ϵ.

As in Section 4, we can choose $\epsilon_0 > 0$ small enough and let $h = g_{\epsilon_0,0}$. Then there is a $T > 0$ independent of ϵ such that if $0 < \epsilon \le \epsilon_0$, then there is a smooth solution $g_\epsilon(t)$ on $M \times [0, T]$ to the h-flow with initial data $g_{\epsilon,0}$. There is also a smooth solution $g(t)$ on $M \times (0, T]$ to the h-flow such that $g(t) \to g_0$ uniformly on compact sets as $t \to 0$. Moreover, Lemma 4.2 is still true with M being noncompact in this case because M is AF.

Let $\tilde{g}_\epsilon(t)$ be the corresponding solution to the Ricci flow with $\tilde{g}_\epsilon(t) = \Phi_t^*(g_\epsilon(t))$ as in the compact case in Section 4. Then we have the following:

Lemma 7.6. (i) *The metrics $g_\epsilon(t), \tilde{g}_\epsilon(t), g(t)$ are AF in the sense of Definition 7.1.*

(ii) *For each end E of M, $\mathfrak{m}(E)(\epsilon, t) = \mathfrak{m}(E)(\epsilon, 0) = \mathfrak{m}(E)$, where $\mathfrak{m}(E)(\epsilon, t)$ is the mass with respect to $g_\epsilon(t)$ or $\tilde{g}_\epsilon(t)$, and $\mathfrak{m}(E)(\epsilon, 0)$ is the mass with respect to $g_{\epsilon,0}$ or g_0.*

Proof. (i) First note that $C_1^{-1}h \le g_\epsilon(t) \le C_1 h$ for some constant $C_1 > 0$ independent of ϵ, t. On the other hand, by Lemma 4.2 applied to the noncompact case, we conclude that the curvature of $\tilde{g}_\epsilon(t)$ is bounded by $Ct^{-\frac{1}{2}(1+\delta)}$ for some $0 < \delta < 1$ where C, δ are independent of ϵ, t. Hence we also have $C_1^{-1} g_{\epsilon,0} \le g_\epsilon(t) \le C_1 g_{\epsilon,0}$ and $C_1^{-1}h \le \tilde{g}_\epsilon(t) \le C_1 h$, with possible larger C_1.

Using the fact that $\sigma_{ij} = O_5(r^{-\tau})$, we can proceed with some modifications as in [Dai and Ma 2007; McFeron and Székelyhidi 2012] to show that outside a fixed compact set, for $0 \le l \le 3$,

$$|{}^h\nabla^l g_\epsilon(x,t)| \le C_2 d^{-l-\tau}(x)$$

for some constant C_2 independent of ϵ, t, x, where $d(x)$ is the distance function from a fixed point with respect to h. Here we use the fact that $\sigma_{ij} = O_5(r^{-\tau})$. The proof is similar to the proof for the decay rate of scalar curvature. So we only carry out the proof for this case in more detail.

We want to prove that there is a constant $C_3 > 0$ independent of ϵ, t and a compact set K such that if $\tilde{\mathcal{S}}_\epsilon(t)$ is the scalar curvature of $\tilde{g}_\epsilon(t)$, then

$$\sup_{M\setminus K} d^q(x)|\tilde{\mathcal{S}}_\epsilon(x,t)| \le C_3. \tag{7-2}$$

We will prove this on each end. Fix ϵ. Denote the scalar curvature of $g_\epsilon(t)$ simply by $\mathcal{S}$ and curvature by Rm etc. Let E be an end which is diffeomorphic to $\mathbb{R}^n \setminus B(R)$, say. By [Simon 2002], by choosing R large enough so that $g_{\epsilon,0} = h = g_0$ outside $B(R/2)$ and g_0 is smooth there, we may assume that $|\mathrm{Rm}(g_\epsilon(t))| \le C_4$ for some constant C_4 independent of ϵ, t outside $B(R/2)$. Here we have used the fact that $g_\epsilon(t), \tilde{g}_\epsilon(t)$ are uniformly equivalent.

Let g_e be the standard Euclidean metric and let $0 \le \phi \le 1$ be a fixed smooth function on $\mathbb{R}^n$ such that $\phi = 1$ in $B(R)$ and $\phi = 0$ outside $B(2R)$. Consider the metric $\phi g_e + (1-\phi) g_\epsilon(t)$. Still denote its curvature by Rm etc.

Let ρ be a fixed function $\rho \ge 1$, $\rho = 1$ in $B(R)$, $\rho(x) = |x|$ outside $B(2R)$. Hence the gradient and the Hessian of ρ with respect to $g_\epsilon(t)$ are bounded by a constant independent of ϵ, t. We have

$$\frac{\partial}{\partial t}\mathcal{S}^2 \le \Delta \mathcal{S}^2 + C_5$$

in $B(2R)$ and

$$\frac{\partial}{\partial t}\mathcal{S}^2 = \Delta\mathcal{S}^2 + 2\mathcal{S}|\mathrm{Ric}|^2 - 2|\nabla\mathcal{S}|^2$$

outside $B(2R)$.

Let $F = \rho^{2q}\mathcal{S}^2$; then outside $B(2R)$,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)F &= \rho^{2q}(2\mathcal{S}|\mathrm{Ric}|^2 - 2|\nabla\mathcal{S}|^2) - 2\langle\nabla\rho^{2q}, \nabla\mathcal{S}^2\rangle + F\Delta\rho^{2q}\\ &\le C_6\rho^{q-4-2\tau}\rho^q\mathcal{S} - 4q\rho^{-1}\langle\nabla\rho, \nabla F\rangle + C_6F\\ &\le C_7 - 4q\rho^{-1}\langle\nabla\rho, \nabla F\rangle + C_7F\end{aligned}\tag{7-3}$$

for some constants C_6, C_7 independent of ϵ, t since $q-4-2\tau < q-(n+2) \le 0$. The inequality is still true in $B(2R)$ because in $B(R)$, $\nabla\rho = 0$ and in $B(2R)\setminus B(R)$, $|\nabla\rho|$ and $|\nabla\mathcal{S}|$ are uniformly bounded. Hence if $\tilde{F} = e^{-C_7t}F - C_7t$, then

$$\left(\frac{\partial}{\partial t} - \Delta\right)\tilde{F} \le -4q\rho^{-1}\langle\nabla\rho, \nabla\tilde{F}\rangle.\tag{7-4}$$

Let $A > 0$ to be chosen later. Let $\eta = \exp(2At + \rho)$. Then

$$\left(\frac{\partial}{\partial t} - \Delta\right)\eta \ge 2A\eta - C\eta$$

for some constant C independent of ϵ, t. Choose $A = C$; then we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\eta \ge A\eta.$$

Let $\kappa > 0$ be any positive number; then

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\tilde{F} - \kappa\eta) \le -4q\rho^{-1}\langle\nabla\rho, \nabla\tilde{F}\rangle - \kappa A\eta.$$

Since $\tilde{F}$ has at most polynomial growth, if $\tilde{F} - \kappa\eta$ has a positive maximum, then the maximum will be attained at some point (x_0, t_0). Suppose $t_0 > 0$; then at (x_0, t_0),

$$\nabla\tilde{F} = \kappa\nabla\eta.$$

Hence at (x_0, t_0),

$$\begin{aligned}0 &\le \left(\frac{\partial}{\partial t} - \Delta\right)(\tilde{F} - \kappa\eta)\\ &\le -4q\rho^{-1}\langle\nabla\rho, \nabla\tilde{F}\rangle - \kappa A\eta\\ &= -4q\rho^{-1}\kappa\langle\nabla\rho, \nabla\eta\rangle - \kappa A\eta\\ &\le -\kappa A\eta,\end{aligned}$$

which is impossible. Hence either $\tilde{F} - \kappa\eta \le 0$, or

$$\tilde{F} - \kappa\eta \le \sup_{\mathbb{R}^n}\left(\rho^{2q}(x)\mathcal{S}^2(0)\right),$$

where $\mathcal{S}(0)$ is the scalar curvature of $\phi g_e + (1 - \phi g_0)$. Let $\kappa \to 0$, we conclude the (7-2) is true.

(ii) Since $g_{\epsilon,0} = g_0$ outside a compact set, $\mathfrak{m}(E) = \mathfrak{m}(E)(\epsilon, 0)$. On the other hand, by the fact that $\tilde{g}_\epsilon(t)$ and $\tilde{g}(t)$ are given by a diffeomorphism and by (i) and [Bartnik 1986], the mass of E is the same whether it is computed with respect to $\tilde{g}_\epsilon(t)$ or $g_\epsilon(t)$.

The fact that $\mathfrak{m}(E)(\epsilon, t) = \mathfrak{m}(E)(\epsilon, 0)$ follows from [Dai and Ma 2007]. □

Proof of Theorem 7.4. By Lemmas 4.1 and 4.13, we conclude that $g(t)$ is AF and with nonnegative scalar curvature for $t > 0$. Let E be an end. Using the notation in Lemma 7.6, by the lemma and [McFeron and Székelyhidi 2012, Theorem 14] (see also [Jauregui 2014]), the mass $\mathfrak{m}(E)(t)$ of E with respect to $g(t)$ satisfies

$$\begin{aligned}\mathfrak{m}(E) &= \liminf_{\epsilon\to 0} \mathfrak{m}(E)(\epsilon, 0)\\ &= \liminf_{\epsilon\to 0} \mathfrak{m}(E)(\epsilon, t)\\ &\geq \mathfrak{m}(E)(t).\end{aligned}$$

By Theorem 7.3, $\mathfrak{m}(E)(t) \geq 0$, we have $\mathfrak{m}(E) \geq 0$. If $\mathfrak{m}(E) = 0$, then $\mathfrak{m}(E)(t) = 0$ and $(M^n, g(t))$ is isometric to the Euclidean space. Since $g(t)$ converges to g_0 in C^∞ as $t \to 0$ away from Σ, g_0 is flat outside Σ. □

Acknowledgments

The authors thank Xue Hu and Richard Schoen for useful discussions and also thank the referee for some suggested revisions.

References

[Arnowitt et al. 1961] R. Arnowitt, S. Deser, and C. W. Misner, "Coordinate invariance and energy expressions in general relativity", *Phys. Rev.* (2) **122**:3 (1961), 997–1006. MR Zbl

[Aubin 1976a] T. Aubin, "Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire", *J. Math. Pures Appl.* (9) **55**:3 (1976), 269–296. MR Zbl

[Aubin 1976b] T. Aubin, "The scalar curvature", pp. 5–18 in *Differential geometry and relativity*, edited by M. Cahen and M. Flato, Math. Phys. Appl. Math. **3**, Reidel, Dordrecht, 1976. MR Zbl

[Bartnik 1986] R. Bartnik, "The mass of an asymptotically flat manifold", *Comm. Pure Appl. Math.* **39**:5 (1986), 661–693. MR Zbl

[Bray and Jauregui 2013] H. L. Bray and J. L. Jauregui, "A geometric theory of zero area singularities in general relativity", *Asian J. Math.* **17**:3 (2013), 525–559. MR Zbl

[Brendle and Marques 2011] S. Brendle and F. C. Marques, "Scalar curvature rigidity of geodesic balls in S^n", *J. Differential Geom.* **88**:3 (2011), 379–394. MR Zbl

[Dai and Ma 2007] X. Dai and L. Ma, "Mass under the Ricci flow", *Comm. Math. Phys.* **274**:1 (2007), 65–80. MR Zbl

[Gilbarg and Trudinger 1983] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Math. Wissenschaften **224**, Springer, 1983. MR Zbl

[Grant and Tassotti 2014] J. D. E. Grant and N. Tassotti, "A positive mass theorem for low-regularity Riemannian metrics", preprint, 2014. arXiv

[Gromov and Lawson 1983] M. Gromov and H. B. Lawson, Jr., "Positive scalar curvature and the Dirac operator on complete Riemannian manifolds", *Inst. Hautes Études Sci. Publ. Math.* **58** (1983), 83–196. MR Zbl

[Hamilton 1995] R. S. Hamilton, "A compactness property for solutions of the Ricci flow", *Amer. J. Math.* **117**:3 (1995), 545–572. MR Zbl

[Huang and Tam 2015] S. Huang and L.-F. Tam, "Kähler–Ricci flow with unbounded curvature", 2015. To appear in *Amer. J. Math.* arXiv

[Jauregui 2014] J. L. Jauregui, "On the lower semicontinuity of the ADM mass", 2014. To appear in *Comm. Anal. Geom.* arXiv

[Lee 2013] D. A. Lee, "A positive mass theorem for Lipschitz metrics with small singular sets", *Proc. Amer. Math. Soc.* **141**:11 (2013), 3997–4004. MR Zbl

[Lee and LeFloch 2015] D. A. Lee and P. G. LeFloch, "The positive mass theorem for manifolds with distributional curvature", *Comm. Math. Phys.* **339**:1 (2015), 99–120. MR Zbl

[Lee and Parker 1987] J. M. Lee and T. H. Parker, "The Yamabe problem", *Bull. Amer. Math. Soc.* (*N.S.*) **17**:1 (1987), 37–91. MR Zbl

[Lieberman 1996] G. M. Lieberman, *Second order parabolic differential equations*, World Sci., River Edge, NJ, 1996. MR Zbl

[Macbeth 2017] H. R. Macbeth, "Conformal classes realizing the Yamabe invariant", *Int. Math. Res. Not.* **2017** (online publication July 2017).

[McFeron and Székelyhidi 2012] D. McFeron and G. Székelyhidi, "On the positive mass theorem for manifolds with corners", *Comm. Math. Phys.* **313**:2 (2012), 425–443. MR Zbl

[Miao 2002] P. Miao, "Positive mass theorem on manifolds admitting corners along a hypersurface", *Adv. Theor. Math. Phys.* **6**:6 (2002), 1163–1182. MR

[Parker and Taubes 1982] T. Parker and C. H. Taubes, "On Witten's proof of the positive energy theorem", *Comm. Math. Phys.* **84**:2 (1982), 223–238. MR Zbl

[Petersen 1998] P. Petersen, *Riemannian geometry*, Graduate Texts in Mathematics **171**, Springer, 1998. MR Zbl

[Schoen 1984] R. Schoen, "Conformal deformation of a Riemannian metric to constant scalar curvature", *J. Differential Geom.* **20**:2 (1984), 479–495. MR Zbl

[Schoen 1989] R. Schoen, "Variational theory for the total scalar curvature functional for Riemannian metrics and related topics", pp. 120–154 in *Topics in calculus of variations* (Montecatini Terme, 1987), edited by M. Giaquinta, Lecture Notes in Math. **1365**, Springer, 1989. MR Zbl

[Schoen and Yau 1979a] R. Schoen and S. T. Yau, "Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature", *Ann. of Math.* (2) **110**:1 (1979), 127–142. MR Zbl

[Schoen and Yau 1979b] R. Schoen and S. T. Yau, "On the proof of the positive mass conjecture in general relativity", *Comm. Math. Phys.* **65**:1 (1979), 45–76. MR Zbl

[Schoen and Yau 1979c] R. Schoen and S. T. Yau, "On the structure of manifolds with positive scalar curvature", *Manuscripta Math.* **28**:1-3 (1979), 159–183. MR Zbl

[Schoen and Yau 1981] R. Schoen and S. T. Yau, "The energy and the linear momentum of space-times in general relativity", *Comm. Math. Phys.* **79**:1 (1981), 47–51. MR Zbl

[Shi 1989] W.-X. Shi, "Deforming the metric on complete Riemannian manifolds", *J. Differential Geom.* **30**:1 (1989), 223–301. MR Zbl

[Shi and Tam 2002] Y. Shi and L.-F. Tam, "Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature", *J. Differential Geom.* **62**:1 (2002), 79–125. MR Zbl

[Simon 2002] M. Simon, "Deformation of C^0 Riemannian metrics in the direction of their Ricci curvature", *Comm. Anal. Geom.* **10**:5 (2002), 1033–1074. MR Zbl

[Stein 1970] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series **30**, Princeton Univ. Press, 1970. MR Zbl

[Tam 2010] L.-F. Tam, "Exhaustion functions on complete manifolds", pp. 211–215 in *Recent advances in geometric analysis* (Taipei, 2007), edited by Y.-I. Lee et al., Adv. Lect. Math. **11**, Int. Press, Somerville, MA, 2010. MR Zbl

[Taylor 2006] M. Taylor, "Existence and regularity of isometries", *Trans. Amer. Math. Soc.* **358**:6 (2006), 2415–2423. MR Zbl

[Trudinger 1968] N. S. Trudinger, "Remarks concerning the conformal deformation of Riemannian structures on compact manifolds", *Ann. Scuola Norm. Sup. Pisa* (3) **22** (1968), 265–274. MR Zbl

[Witten 1981] E. Witten, "A new proof of the positive energy theorem", *Comm. Math. Phys.* **80**:3 (1981), 381–402. MR Zbl

[Yamabe 1960] H. Yamabe, "On a deformation of Riemannian structures on compact manifolds", *Osaka Math. J.* **12** (1960), 21–37. MR Zbl

Received December 1, 2016. Revised May 4, 2017.

YUGUANG SHI
KEY LABORATORY OF PURE AND APPLIED MATHEMATICS
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING
CHINA
ygshi@math.pku.edu.cn

LUEN-FAI TAM
INSTITUTE OF MATHEMATICAL SCIENCES AND DEPARTMENT OF MATHEMATICS
CHINESE UNIVERSITY OF HONG KONG
SHATIN
CHINA
lftam@math.cuhk.edu.hk

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2018 is US $475/year for the electronic version, and $640/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 293 No. 2 April 2018

Sums of CR functions from competing CR structures 257
DAVID E. BARRETT and DUSTY E. GRUNDMEIER

On Tate duality and a projective scalar property for symmetric algebras 277
FLORIAN EISELE, MICHAEL GELINE, RADHA KESSAR and MARKUS LINCKELMANN

Coaction functors, II 301
S. KALISZEWSKI, MAGNUS B. LANDSTAD and JOHN QUIGG

Construction of a Rapoport–Zink space for GU(1, 1) in the ramified 2-adic case 341
DANIEL KIRCH

Group and round quadratic forms 391
JAMES O'SHEA

Dual operator algebras close to injective von Neumann algebras 407
JEAN ROYDOR

Scalar curvature and singular metrics 427
YUGUANG SHI and LUEN-FAI TAM

On the differentiability issue of the drift-diffusion equation with nonlocal Lévy-type diffusion 471
LIUTANG XUE and ZHUAN YE