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SUMS OF CR FUNCTIONS FROM COMPETING CR STRUCTURES

DAVID E. BARRETT AND DUSTY E. GRUNDMEIER

We characterize sums of CR functions from competing CR structures in two scenarios. In one scenario the structures are conjugate and we are adding to the theory of pluriharmonic boundary values. In the second scenario the structures are related by projective duality considerations. In both cases we provide explicit vector field-based characterizations for two-dimensional circular domains satisfying natural convexity conditions.

1. Introduction

The Dirichlet problem for pluriharmonic functions is a natural problem in several complex variables with a long history going back at least to Amoroso [1912], Severi [1931], Wirtinger [1927], and others. It was known early on that the problem is not solvable for general boundary data, so we may try to characterize the admissible boundary values with a system of tangential partial differential operators. This was first done for the ball by Bedford [1974]; see Section 2.1 for details. More precisely, given a bounded domain Ω with smooth boundary S, we seek a system $\mathcal{L}$ of partial differential operators tangential to S such that a function $u \in \mathcal{C}^\infty(S, \mathbb{C})$ satisfies $\mathcal{L}u = 0$ if and only if there exists $U \in \mathcal{C}^\infty(\overline{\Omega})$ such that $U|_S = u$ and $\partial\bar{\partial}U = 0$. The problem may also be considered locally.

While natural in its own right, this problem also arises in less direct fashion in many areas of complex analysis and geometry. For instance, this problem plays a fundamental role in Graham's work [1983] on the Bergman Laplacian, Lee's work [1988] on pseudo-Einstein structures, and Case, Chanillo, and Yang's work [Case et al. 2016] on CR Paneitz operators. From another point of view, the existence of nontrivial restrictions on pluriharmonic boundary values points to the need to look elsewhere (such as to the Monge–Ampère equations studied in [Bedford and Taylor 1976]) for Dirichlet problems solvable for general boundary data.

The pluriharmonic boundary value problem is closely related to the problem of characterizing sums of CR functions from different, competing CR structures;

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indeed, when the competing CR structures are conjugate then these problems coincide (in simply connected settings); see Propositions 3 and 4 below. Another natural construction leading to competing CR structures arises from the study of projective duality (see Section 3 or [Barrett 2016] for precise definitions).

In each of these two scenarios, we precisely characterize sums of CR functions from the two competing CR structures in the setting of two-dimensional circular domains satisfying appropriate convexity conditions. For conjugate structures we assume strong pseudoconvexity; our result appears as Theorem A below. In the projective duality scenario we assume strong convexity (the correct assumption without the circularity assumption would be strong $\mathbb{C}$-convexity, but these notions coincide in the circular case; see Section 3.1), and the main result appears as Theorem B below (with an expanded version appearing later in Section 3.2). Our techniques for these two related problems are interconnected to a surprising extent, and the reader will notice that the projective dual scenario actually turns out to have more structure and symmetry.

Theorem A. *Let $S \subset \mathbb{C}^2$ be a strongly pseudoconvex circular hypersurface. Then there exist nowhere-vanishing tangential vector fields X, Y on S satisfying the following conditions*:

(1-1a) *If u is a smooth function on a relatively open subset of S, then u is CR if and only if $Xu = 0$.*

(1-1b) *If u is a smooth function on a relatively open subset of S, then u is CR if and only if $Y\bar{u} = 0$.*

(1-1c) *If S is compact, then a smooth function u on S is a pluriharmonic boundary value (in the sense of Proposition 3 below) if and only if $XXYu = 0$.*

(1-1d) *A smooth function u on a relatively open subset of S is a pluriharmonic boundary value (in the sense of Proposition 4 below) if and only if $XXYu = 0 = \overline{XXY}u$.*

Theorem B. *Let $S \subset \mathbb{C}^2$ be a strongly convex circular hypersurface. Then there exist nowhere-vanishing tangential vector fields X, T on S satisfying the following conditions*:

(1-2a) *If u is a smooth function on a relatively open subset of S, then u is CR if and only if $Xu = 0$.*

(1-2b) *If u is a smooth function on a relatively open subset of S, then u is dual-CR if and only if $Tu = 0$.*

(1-2c) *If S is compact, then a smooth function u on S is the sum of a CR function and a dual-CR function if and only if $XXTu = 0$.*

(1-2d) *If S is simply connected (but not necessarily compact), then a smooth function u on S is the sum of a CR function and a dual-CR function if and only if $XXTu = 0 = TTXu$.*

This paper is organized as follows. In Section 2 we focus on the case of conjugate CR structures (the pluriharmonic case). In Section 3 we study the competing CR structures coming from projective duality. In Section 4 we prove Theorem B, while Theorem A is proved in Section 5. The final Section 6 includes a discussion of uniqueness issues.

2. Conjugate structures

2.1. ***Results on the ball.*** Early work focused on the case of the ball B^n in $\mathbb{C}^n$. In particular, Nirenberg observed that there is no second-order system of differential operators tangent to S^3 that exactly characterize pluriharmonic functions (see Section 6.2 for more details). Third-order characterizations were developed by Bedford in the global case and Audibert in the local case (which requires stronger conditions). To state these results, we define the tangential operators

$$L_{kl} = z_k \frac{\partial}{\partial \bar{z}_l} - z_l \frac{\partial}{\partial \bar{z}_k}, \qquad \overline{L_{kl}} = \bar{z}_k \frac{\partial}{\partial z_l} - \bar{z}_l \frac{\partial}{\partial z_k} \quad \text{for } 1 \le k, l \le n. \tag{2-1}$$

Theorem 1 [Bedford 1974]. *Let u be smooth on S^{2n-1}. Then*

$$\overline{L_{kl}}\,\overline{L_{kl}}\,L_{kl} u = 0$$

for $1 \le k, l \le n$ if and only if u extends to a pluriharmonic function on B^n.

Theorem 2 [Audibert 1977]. *Let S be a relatively open subset of S^{2n-1}, and let u be smooth on S. Then*

$$L_{jk} L_{lm} \overline{L_{rs}} u = 0 = \overline{L_{jk}}\,\overline{L_{lm}}\,L_{rs} u$$

for $1 \le j, k, l, m, r, s \le n$ if and only if u extends to a pluriharmonic function on a one-sided neighborhood of S.

For a treatment of both of these results along with further details and examples, see §18.3 of [Rudin 1980].

2.2. ***Other results.*** Laville [1977; 1984] also gave a fourth order operator to solve the global problem. Bedford and Federbush [1974] solved the local problem in the more general setting where $b\Omega$ has nonzero Levi form at some point. Later Bedford [1980] used the induced boundary complex $(\partial\bar{\partial})_b$ to solve the local problem in certain settings. In Lee's work [1988] on pseudo-Einstein structures, he gives a characterization for abstract CR manifolds using third order pseudohermitian covariant derivatives. Case, Chanillo, and Yang [Case et al. 2016] study when the kernel of the CR Paneitz operator characterizes CR-pluriharmonic functions.

2.3. ***Relation to decomposition on the boundary.*** Outside the proof of Theorem 30 below, all forms, functions, and submanifolds will be assumed $\mathcal{C}^\infty$-smooth.

Proposition 3. *Let $S \subset \mathbb{C}^n$ be a compact, connected and simply connected real hypersurface, and let Ω be the bounded domain with boundary S. Then for $u : S \to \mathbb{C}$ the following conditions are equivalent*:

(2-2a) *u extends to a (smooth) function U on $\overline{\Omega}$ that is pluriharmonic on Ω;*

(2-2b) *u is the sum of a CR function and a conjugate-CR function.*

Proof. In the proof that (2-2a) implies (2-2b), the CR term is the restriction to S of an antiderivative for ∂U on a simply connected one-sided neighborhood of S, and the conjugate-CR term is the restriction to S of an antiderivative for $\bar\partial U$ on a one-sided neighborhood of S (adjusting one term by a constant as needed).

To see that (2-2b) implies (2-2a), we use the global CR extension result [Hörmander 1990, Theorem 2.3.2] to extend the terms to holomorphic and conjugate-holomorphic functions, respectively; U is then the sum of the extensions. □

Proposition 4. *Let $S \subset \mathbb{C}^n$ be a simply connected, strongly pseudoconvex real hypersurface. Then for $u : S \to \mathbb{C}$ the following conditions are equivalent*:

(2-3a) *there is an open subset W of $\mathbb{C}^n$ with $S \subset bW$ (with W lying locally on the pseudoconvex side of S) so that u extends to a (smooth) function U on $W \cup S$ that is pluriharmonic on W;*

(2-3b) *u is the sum of a CR function and a conjugate-CR function.*

Proof. The proof follows the proof of Proposition 3 above, replacing the global CR extension result by the Hans Lewy local CR extension result as stated in [Boggess 1991, Section 14.1, Theorem 1]. □

3. Projective dual structures

3.1. ***Projective dual hypersurfaces.*** Let $S \subset \mathbb{C}^n$ be an oriented real hypersurface with defining function ρ. Then S is said to be *strongly $\mathbb{C}$-convex* if S is locally equivalent via a projective transformation (that is, via an automorphism of projective space) to a strongly convex hypersurface; this condition is equivalent to either of the following two equivalent conditions:

(3-1a) the second fundamental form for S is positive definite on the maximal complex subspace $H_z S$ of each $T_z S$;

(3-1b) the complex tangent (affine) hyperplanes for S lie to one side (the "concave side") of S near the point of tangency with minimal order of contact.

Theorem 5. *When S is compact and strongly $\mathbb{C}$-convex the complex tangent hyperplanes for S are in fact disjoint from the domain bounded by S.*

Proof. [Andersson et al. 2004, §2.5]. □

We note that strongly $\mathbb{C}$-convex hypersurfaces are also strongly pseudoconvex.

A circular hypersurface (that is, a hypersurface invariant under rotations $z \mapsto e^{i\theta} z$) is strongly $\mathbb{C}$-convex if and only if it is strongly convex [Černe 2002, Proposition 3.7].

The proper general context for the notion of strong $\mathbb{C}$-convexity is in the study of real hypersurfaces in complex projective space $\mathbb{CP}^n$ (see for example [Barrett 2016] and [Andersson et al. 2004]).

We specialize now to the two-dimensional case.

Lemma 6. *Let $S \subset \mathbb{C}^2$ be a compact strongly $\mathbb{C}$-convex hypersurface enclosing the origin. Then there is a uniquely determined map*

$$\mathfrak{D} : S \to \mathbb{C}^2 \setminus \{0\}, \quad z \mapsto w(z) = (w_1(z), w_2(z))$$

satisfying

(3-2a) $z_1 w_1 + z_2 w_2 = 1$ *on S;*

(3-2b) *the vector field*

$$Y \stackrel{\text{def}}{=} w_2 \frac{\partial}{\partial z_1} - w_1 \frac{\partial}{\partial z_2}$$

is tangent to S. Moreover, Y annihilates conjugate-CR functions on any relatively open subset of S.

Proof. It is easy to check that (3-2a) and (3-2b) force

$$w_1(z) = \frac{\frac{\partial \rho}{\partial z_1}}{z_1 \frac{\partial \rho}{\partial z_1} + z_2 \frac{\partial \rho}{\partial z_2}}, \quad w_2(z) = \frac{\frac{\partial \rho}{\partial z_2}}{z_1 \frac{\partial \rho}{\partial z_1} + z_2 \frac{\partial \rho}{\partial z_2}}.$$

establishing uniqueness. Existence follows provided that the denominators do not vanish, but the vanishing of the denominators occurs precisely when the complex tangent line for S at z passes through the origin, and Theorem 5 above guarantees that this does not occur under the given hypotheses. □

Remark 7. It is clear from the proof that the conclusions of Lemma 6 also hold under the assumption that S is a (not necessarily compact) hypersurface satisfying

(3-3) no complex tangent line for S passes through the origin.

Remark 8. Any tangential vector field annihilating conjugate-CR functions will be a scalar multiple of Y.

Remark 9. The complex line tangent to S at z is given by

$$\{\zeta \in \mathbb{C}^2 : w_1(z)\zeta_1 + w_2(z)\zeta_2 = 1\}. \tag{3-4}$$

Remark 10. The maximal complex subspace $H_z S$ of each $T_z S$ is annihilated by the form $w_1\,dz_1 + w_2\,dz_2$.

Proposition 11. *For S strongly $\mathbb{C}$-convex satisfying (3-3), the map $\mathfrak{D}$ is a local diffeomorphism onto an immersed strongly $\mathbb{C}$-convex hypersurface S^*, with each maximal complex subspace $H_z S$ of $T_z S$ mapped (not $\mathbb{C}$-linearly) by $\mathfrak{D}'_z$ onto the corresponding maximal complex subspace of $H_{w(z)} S^*$. For S strongly $\mathbb{C}$-convex and compact, S^* is an embedded strongly $\mathbb{C}$-convex hypersurface and $\mathfrak{D}$ is a diffeomorphism.*

Proof. [Barrett 2016, §6], [Andersson et al. 2004, §2.5]. □

For S strongly $\mathbb{C}$-convex satisfying (3-3) we may extend $\mathfrak{D}$ to a smooth map on an open set in $\mathbb{C}^2$; the extended map $\mathfrak{D}^\star$ will be a local diffeomorphism in some neighborhood U of S. We may then define vector fields $\partial/\partial w_1, \partial/\partial w_2, \partial/\partial \overline{w}_1, \partial/\partial \overline{w}_2$ on U by applying $((\mathfrak{D}^\star)^{-1})'$ to the corresponding vector fields on $\mathfrak{D}^\star(U)$; these newly defined vector fields will depend on the choice of the extension $\mathfrak{D}^\star$.

Lemma 12. *The nonvanishing vector field*

$$V \overset{\text{def}}{=} z_2 \frac{\partial}{\partial w_1} - z_1 \frac{\partial}{\partial w_2}$$

is tangent to S and is independent of the choice of the extension $\mathfrak{D}^\star$.

Proof. From (3-2a) we have

$$0 = d(z_1 w_1 + z_2 w_2) = z_1\,dw_1 + z_2\,dw_2 + w_1\,dz_1 + w_2\,dz_2 \quad \text{on } T_z S.$$

From Remark 10 we deduce that the null space in $T_z\mathbb{C}^2$ of $z_1\,dw_1 + z_2\,dw_2$ is precisely the maximal complex subspace $H_z S$ of $T_z S$ (and moreover the null space in $(T_z\mathbb{C}^2) \otimes \mathbb{C}$ of $z_1\,dw_1 + z_2\,dw_2$ is precisely $(H_z S) \otimes \mathbb{C}$). If we apply $z_1\,dw_1 + z_2\,dw_2$ to V we obtain

$$z_1 \cdot V w_1 + z_2 \cdot V w_2 = z_1 \cdot z_2 - z_2 \cdot z_1 = 0$$

showing that V takes values in $(H_z S) \otimes \mathbb{C}$ and is thus tangential.

If an alternate tangential vector field $\tilde{V}$ is constructed with the use of an alternate extension $\widetilde{\mathfrak{D}}^\star$ of $\mathfrak{D}$, then

$$\tilde{V} w_j = \pm z_{3-j} = V w_j, \quad \tilde{V} \overline{w}_j = 0 = V \overline{w}_j$$

along S, so $\tilde{V} = V$ along S. □

Definition 13. A function u on a relatively open subset of S will be called *dual-CR* if $\overline{V} u = 0$.

Example 14. If S is the unit sphere in $\mathbb{C}^2$, then $w(z) = \bar{z}$ and the set of dual-CR functions on S coincides with the set of conjugate-CR functions on S.

The set of dual-CR functions will only rarely coincide with the set of conjugate-CR functions as we see from the following two related results.

Theorem 15. *If S is a compact strongly $\mathbb{C}$-convex hypersurface in $\mathbb{C}^2$, then the set of dual-CR functions on S will coincide with the set of conjugate-CR functions on S if and only if S is a complex affine image of the unit sphere.*

Theorem 16. *If S is a strongly $\mathbb{C}$-convex hypersurface in $\mathbb{C}^2$, then the set of dual-CR functions on S will coincide with the set of conjugate-CR functions on S if and only if S is locally the image of a relatively open subset of the unit sphere by a projective transformation.*

For proofs of these results see [Jensen 1983], [Detraz and Trépreau 1990], and [Bolt 2008].

Remark 17. The constructions of the vector fields Y and V transform naturally under complex affine mappings of S. The construction of the dual-CR structure transforms naturally under projective transformation of S. (See for example [Barrett 2016, §6].)

Lemma 18. *Relations of the form*

$$V = \chi Y + \sigma \bar{Y}, \quad Y = \kappa V + \xi \bar{V}$$

hold along S with σ and ξ nowhere vanishing.

Proof. This follows from the following facts:

- V, $\bar{V}$, Y and $\bar{Y}$ all take values in the two-dimensional space $(H_z S) \otimes \mathbb{C}$;
- V and $\bar{V}$ are $\mathbb{C}$-linearly independent, as are Y and $\bar{Y}$;
- the map $\mathfrak{D}'_z : (H_z S) \otimes \mathbb{C} \to (H_z S^*) \otimes \mathbb{C}$ is not $\mathbb{C}$-linear (see Proposition 11). □

Lemma 19. *If f_1, f_2 are CR functions and g_1, g_2 are dual-CR functions on a connected relatively open subset W of S with $f_1 + g_1 = f_2 + g_2$, then $g_2 - g_1 = f_1 - f_2$ is constant.*

Proof. From Lemma 18 we deduce that the directional derivatives of $g_2 - g_1 = f_1 - f_2$ vanish in every direction belonging to the maximal complex subspace of TS. Applying one Lie bracket we find that in fact all directional derivatives along S of $g_2 - g_1 = f_1 - f_2$ vanish. □

Corollary 20. *If W is a simply connected relatively open subset of S and u is a function on W that is locally decomposable as the sum of a CR function and a dual-CR function, then u is decomposable on all of W as the sum of a CR function and a dual-CR function.*

3.2. ***Circular hypersurfaces in*** $\mathbb{C}^2$**.** We begin the section with an expanded restatement of the main theorem in the projective setting.

Theorem B [expanded statement]. *Let $S \subset \mathbb{C}^2$ be a strongly ($\mathbb{C}$-)convex circular hypersurface. Then there exist scalar functions ϕ and ψ on S so that the vector fields*

$$X = V + \phi\overline{V}, \tag{3-5a}$$

$$T = Y + \psi\overline{Y} \tag{3-5b}$$

satisfy the following conditions.

(3-6a) *If u is a smooth function on a relatively open subset of S, then u is CR if and only if $Xu = 0$; equivalently, X is a nonvanishing scalar multiple $\alpha\overline{Y}$ of $\overline{Y}$.*

(3-6b) *If u is a smooth function on a relatively open subset of S, then u is dual-CR if and only if $Tu = 0$; equivalently, T is a nonvanishing scalar multiple $\beta\overline{V}$ of $\overline{V}$.*

(3-6c) *If S is compact, then a smooth function u on S is the sum of a CR function and a dual-CR function if and only if $XXTu = 0$.*

(3-6d) *If S is simply connected (but not necessarily compact), then a smooth function u on S is the sum of a CR function and a dual-CR function if and only if $XXTu = 0 = TTXu$.*

As we shall see the vector field X in Theorem B will also work as the vector field X in Theorem A.

Example 21 (cf. [Audibert 1977]). The function z_1/w_2 satisfies $XXT(z_1/w_2) = 0$ but is not globally defined. Since $TTX(z_1/w_2) = 2 \neq 0$, this function is not locally the sum of a CR function and a dual-CR function.

Conditions (3-5a), (3-6a) and (3-6b) uniquely determine X and T. See Section 6.1 for some discussion of what can happen without condition (3-5a).

4. Proof of Theorem B

To prove Theorem B we start by consulting Lemma 18 and note that (3-5a), (3-6a) and (3-6b) will hold if we set

$$\alpha = 1/\bar{\xi}, \quad \beta = 1/\bar{\sigma}, \quad \phi = \bar{\kappa}/\bar{\xi}, \quad \psi = \bar{\chi}/\bar{\sigma};$$

it remains to check (3-6c) and (3-6d).

We note for future reference and the reader's convenience that

$$
\begin{array}{llll}
Xw_1 = z_2, & Xw_2 = -z_1, & \bar{Y}w_1 = \bar{\xi}z_2, & \bar{Y}w_2 = -\bar{\xi}z_1, \\
X\overline{w}_1 = \phi\bar{z}_2, & X\overline{w}_2 = -\phi\bar{z}_1, & Xz_1 = \bar{Y}z_1 = 0, & Xz_2 = \bar{Y}z_2 = 0, \\
X\bar{z}_1 = \alpha\overline{w}_2, & X\bar{z}_2 = -\alpha\overline{w}_1, & Tz_1 = w_2, & Tz_2 = -w_1, \\
\bar{V}z_1 = \bar{\sigma}w_2, & \bar{V}z_2 = -\bar{\sigma}w_1, & T\bar{z}_1 = \psi\overline{w}_2, & T\bar{z}_2 = -\psi\overline{w}_1, \\
Tw_1 = \bar{V}w_1 = 0, & Tw_2 = \bar{V}w_2 = 0, & T\overline{w}_1 = \beta z_2, & T\overline{w}_2 = -\beta z_1.
\end{array}
\tag{4-1}
$$

Lemma 22.

$$
\begin{aligned}
[Y, \bar{Y}] &= \bar{\xi}\Big(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}\Big) - \xi\Big(\bar{z}_1\frac{\partial}{\partial \bar{z}_1} + \bar{z}_2\frac{\partial}{\partial \bar{z}_2}\Big)\\
[V, \bar{V}] &= \bar{\sigma}\Big(w_1\frac{\partial}{\partial w_1} + w_2\frac{\partial}{\partial w_2}\Big) - \sigma\Big(\overline{w}_1\frac{\partial}{\partial \overline{w}_1} + \overline{w}_2\frac{\partial}{\partial \overline{w}_2}\Big).
\end{aligned}
$$

Proof. The first statement follows from

$$
[Y, \bar{Y}] = (Y\overline{w}_2)\frac{\partial}{\partial \bar{z}_1} - (Y\overline{w}_1)\frac{\partial}{\partial \bar{z}_2} - (\bar{Y}w_2)\frac{\partial}{\partial z_1} + (\bar{Y}w_1)\frac{\partial}{\partial z_2}
$$

along with (4-1).

The proof of the second statement is similar. □

We note that the assumption that S is circular has not been used so far in this section. We now bring it into play by introducing the real tangential vector field

$$
R \overset{\text{def}}{=} i\Big(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - \bar{z}_1\frac{\partial}{\partial \bar{z}_1} - \bar{z}_2\frac{\partial}{\partial \bar{z}_2}\Big)
$$

generating the rotations of $z \mapsto e^{i\theta}z$ of S.

Lemma 23. *The following equalities hold.*

(4-2a) $\bar{\xi} = \xi$.

(4-2b) $\bar{\sigma} = \sigma$.

(4-2c) $\bar{\alpha} = \alpha$.

(4-2d) $\bar{\beta} = \beta$.

(4-2e) $R = -i\Big(w_1\frac{\partial}{\partial w_1} + w_2\frac{\partial}{\partial w_2} - \overline{w}_1\frac{\partial}{\partial \overline{w}_1} - \overline{w}_2\frac{\partial}{\partial \overline{w}_2}\Big)$.

(4-2f) $[Y, \bar{Y}] = -i\xi R$.

(4-2g) $[V, \bar{V}] = i\sigma R$.

(4-2h) $[X, Y] = iR - (Y\alpha)\bar{Y}$.

Proof. We start by considering the tangential vector field

$$
[Y, \bar{Y}] + i\xi R = (\bar{\xi} - \xi)\Big(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}\Big);
$$

if (4-2a) fails, then $z_1\,\partial/\partial z_1 + z_2\,\partial/\partial z_2$ is a nonvanishing holomorphic tangential vector field on some nonempty relatively open subset of S, contradicting the strong pseudoconvexity of S.

To prove (4-2e) we first note from Lemma 6 that $w(e^{i\theta}z) = e^{-i\theta}w(z)$; differentiation with respect to θ yields (4-2e).

The proof of (4-2a) now may be adapted to prove (4-2b). (4-2c) and (4-2d) follow immediately.

Using Lemma 22 in combination with (4-2a) and (4-2b) we obtain (4-2f) and (4-2g).

From (3-6a) and (4-2f) we obtain (4-5b). □

Lemma 24. $$[X, T] = iR.$$

Proof. On the one hand,

$$\begin{aligned}[X, T] = [V + \phi\bar{V}, \beta\bar{V}] &= ((V + \phi\bar{V})\beta - \beta(\bar{V}\phi))\bar{V} + i\beta\sigma R \\ &= ((V + \phi\bar{V})\beta - \beta(\bar{V}\phi))\bar{V} + iR.\end{aligned}$$

On the other hand,

$$\begin{aligned}[X, T] = [\alpha\bar{Y}, Y + \psi\bar{Y}] &= (\alpha(\bar{Y}\psi) - (Y + \psi\bar{Y})\alpha)\bar{Y} + i\alpha\xi R \\ &= (\alpha(\bar{Y}\psi) - (Y + \psi\bar{Y})\alpha)\bar{Y} + iR.\end{aligned}$$

Since $\bar{V}$ and $\bar{Y}$ are linearly independent, it follows that $[X, T] = iR$. □

Lemma 25. *The following equalities hold.*

(4-3a) $[R, Y] = -2iY$.

(4-3b) $[R, \bar{Y}] = 2i\bar{Y}$.

(4-3c) $[R, V] = 2iV$.

(4-3d) $[R, \bar{V}] = -2i\bar{V}$.

(4-3e) $[R, X] = 2iX$.

(4-3f) $[R, \bar{X}] = -2i\bar{X}$.

(4-3g) $[R, T] = -2iT$.

(4-3h) $[R, \bar{T}] = 2i\bar{T}$.

(4-3i) $R\alpha = 0$.

(4-3j) $R\beta = 0$.

Proof. (4-3a), (4-3b), (4-3c) and (4-3d) follow from direct calculation.

For (4-3g) first note that writing $T = \beta\bar{V}$ and using (4-3d) we see that $[R, T]$ is a scalar multiple of T. Then writing

$$[R, T] = [R, Y + \psi\bar{Y}] = -2iY + (\text{multiple of } \bar{Y}),$$

we conclude using (3-5a) that $[R, T] = -2iT$. The proof of (4-3e) is similar, and (4-3f) and (4-3h) follow by conjugation.

Using (3-6a) along with (4-3b) and (4-3e) we obtain (4-3i); (4-3j) is proved similarly. □

Lemma 26. *$XXf = 0$ if and only if $f = f_1 w_1 + f_2 w_2$ with f_1, f_2 CR.*

Proof. From (3-6a) and (4-1) it is clear that $XX(f_1 w_1 + f_2 w_2) = 0$ if f_1 and f_2 are CR.

For the other direction, suppose that $XXf = 0$. Then if we set

$$f_1 \stackrel{\text{def}}{=} z_1 f + w_2 X f, \qquad f_2 \stackrel{\text{def}}{=} z_2 f - w_1 X f,$$

it is clear that $f = f_1 w_1 + f_2 w_2$; with the use of (3-6a) and (4-1) it is also easy to check that f_1 and f_2 are CR. □

Lemma 27. *Suppose that $XXTu = 0$ so that by Lemma 26 we may write $Tu = f_1 w_1 + f_2 w_2$ with f_1, f_2 CR. Then*

$$TTXu = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}. \tag{4-4}$$

In particular, $TTXu$ is CR.

The nontangential derivatives appearing in (4-4) may be interpreted using the Hans Lewy local CR extension result mentioned in the proof of Proposition 4, or else by rewriting them in terms of tangential derivatives (as in the last step of the proof below).

Proof.
$$\begin{aligned}
TTXu &= TXTu + T[T, X]u \\
&= TX(f_1 w_1 + f_2 w_2) - iTRu && \text{(Lemma 24)} \\
&= T(f_1 z_2 - f_2 z_1) - iRTu - i[T, R]u && \text{(3-6a), (4-1)} \\
&= T(f_1 z_2 - f_2 z_1) - iR(f_1 w_1 + f_2 w_2) + 2Tu && \text{(4-3g)} \\
&= (Tf_1)z_2 - f_1 w_1 - (Tf_2)z_1 - f_2 w_2 \\
&\quad - i(Rf_1)w_1 - f_2 w_2 - i(Rf_2)w_2 - f_2 w_2 \\
&\quad + 2(f_1 w_1 + f_2 w_2) && \text{(4-1), (4-2e)} \\
&= (z_2 T - i w_1 R) f_2 - (z_1 T + i w_2 R) f_2 \\
&= (z_2 Y - i w_1 R) f_2 - (z_1 Y + i w_2 R) f_2 \\
&= \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}.
\end{aligned}$$
□

Lemma 28. *The following statements hold.*

(4-5a) *The operator XT maps CR functions to CR functions.*

(4-5b) *The operator XY maps CR functions to CR functions.*

(4-5c) *The operator TX maps dual-CR functions to dual-CR functions.*

(4-5d) *The operator $\overline{XY}$ maps conjugate-CR functions to conjugate-CR functions.*

Proof. To prove (4-5a) and (4-5b) note that for u CR we have $XTu = XYu = -z_1\,\partial u/\partial z_1 - z_2\,\partial u/\partial z_2$, which is also CR. The other proofs are similar. □

Proof of (3-6d). To get the required lower bound on the null spaces, it will suffice to show that XXT and TTX annihilate CR functions and dual-CR functions. This follows from (3-6a) and (3-6b) along with (4-5a) and (4-5c).

For the other direction, if $XXTu = 0 = TTXu$, then from Lemma 27 we have a closed 1-form $\omega \overset{\text{def}}{=} f_2\,dz_1 - f_1\,dz_2$ on S where f_1 and f_2 are CR functions satisfying $Tu = f_1 w_1 + f_2 w_2$. Since S is simply connected we may write $\omega = df$ with f CR. Then from (3-5a) we have

$$Tf = Yf = w_2 f_2 + w_1 f_1 = Tu.$$

Thus u is the sum of the CR function f and the dual-CR function $u - f$. □

To set up the proof of the global result (3-6c) we introduce the form

$$\nu \overset{\text{def}}{=} (z_2\,dz_1 - z_1\,dz_2) \wedge dw_1 \wedge dw_2 \tag{4-6}$$

and the $\mathbb{C}$-bilinear pairing

$$\langle\langle \mu, \eta \rangle\rangle \overset{\text{def}}{=} \int_S \mu\eta \cdot \nu \tag{4-7}$$

between functions on S (but see Technical Remark 32 below).

Lemma 29. $\langle\langle T\gamma, \eta\rangle\rangle = -\langle\langle \gamma, T\eta\rangle\rangle$.

Proof. In the sequence of equalities below we will use

- the definition (4-7) of the pairing $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$,
- the Leibniz rule $\iota_T(\varphi_1 \wedge \varphi_2) = (\iota_T\varphi_1) \wedge \varphi_2 + (-1)^{\deg\varphi_1}\varphi_1 \wedge (\iota_T\varphi_2)$ for the interior product ι_T,
- the fact that S is integral for 4-forms,
- Stokes' theorem,
- the rules (4-1),
- the relation (3-2a).

$$\begin{aligned}
\langle\langle T\gamma, \eta\rangle\rangle + \langle\langle \gamma, T\eta\rangle\rangle &= \int_S T(\gamma\eta)\cdot\nu \\
&= \int_S \iota_T d(\gamma\eta)\cdot\nu \\
&= \int_S d(\gamma\eta)\cdot\iota_T\nu \\
&= \int_S d(\gamma\eta\cdot\iota_T\nu) - \int_S \gamma\eta\cdot d(\iota_T\nu) \\
&= 0 - \int_S \gamma\eta\cdot d\big(\iota_T((z_2\,dz_1 - z_1\,dz_2)\wedge dw_1\wedge dw_2)\big) \\
&= -\int_S \gamma\eta\cdot d((z_2\cdot Tz_1 - z_1\cdot Tz_2)\cdot dw_1\wedge dw_2) \\
&\qquad + \int_S \gamma\eta\cdot d((z_2\,dz_1 - z_1\,dz_2)\cdot Tw_1\wedge dw_2) \\
&\qquad - \int_S \gamma\eta\cdot d((z_2\,dz_1 - z_1\,dz_2)\wedge dw_1\cdot Tw_2) \\
&= -\int_S \gamma\eta\cdot d((z_2w_2 + z_1w_1)\,dw_1\wedge dw_2) + 0 - 0 \\
&= -\int_S \gamma\eta\cdot d(dw_1\wedge dw_2) \\
&= 0. \qquad \square
\end{aligned}$$

Theorem 30. *Let μ be a CR function on a compact strongly $\mathbb{C}$-convex hypersurface S. Then $\mu = 0$ if and only if $\langle\langle\mu, \eta\rangle\rangle = 0$ for all dual-CR η on S.*

Proof. [Barrett 2016, (4.3d) from Theorem 3]. (Note also definition enclosing [Barrett 2016, (4.2)].) □

Proof of (3-6c). Assume that $XXTu = 0$. Noting that S is simply connected, from (3-6d) it suffices to prove that $TTXu = 0$. From Lemma 27 we know that $TTXu$ is CR. By Theorem 30 it will suffice to show that

$$\langle\langle TTXu, \eta\rangle\rangle = 0$$

for dual-CR η. But from Lemma 29 we have, as required,

$$\langle\langle TTXu, \eta\rangle\rangle = -\langle\langle TXu, T\eta\rangle\rangle = 0. \qquad \square$$

Remark 31. From symmetry of formulas in Lemmas 6 and 12 we have

$$X_{S^*} = \mathcal{D}_* T_S, \quad T_{S^*} = \mathcal{D}_* X_S, \quad S^{**} = S.$$

These facts serve to explain why the formulas throughout this section appear in dual pairs.

Technical Remark 32. In [Barrett 2016] the pairing (4-7) applies not to functions μ, ν but rather to forms $\mu(z)\,(dz_1 \wedge dz_2)^{2/3}$, $\mu(w)\,(dw_1 \wedge dw_2)^{2/3}$; the additional notation is important there for keeping track of invariance properties under projective transformation but is not needed here.

Note also that (4-7) coincides (up to a constant) with the pairing (3.1.8) in [Andersson et al. 2004] with $s = w_1\,dz_1 + w_2\,dz_2$.

5. Proof of Theorem A

For the reader's convenience we restate the main theorem in the conjugate setting.

Theorem A. *Let $S \subset \mathbb{C}^2$ be a strongly pseudoconvex circular hypersurface. Then there exist nowhere-vanishing tangential vector fields X, Y on S satisfying the following conditions*:

(5-1a) *If u is a smooth function on a relatively open subset of S, then u is CR if and only if $Xu = 0$.*

(5-1b) *If u is a smooth function on a relatively open subset of S, then u is CR if and only if $Y\bar{u} = 0$.*

(5-1c) *If S is compact, then a smooth function u on S is a pluriharmonic boundary value* (*in the sense of Proposition 3 below*) *if and only if $XXYu = 0$.*

(5-1d) *A smooth function u on a relatively open subset of S is a pluriharmonic boundary value* (*in the sense of Proposition 4 below*) *if and only if*

$$XXYu = 0 = \overline{XXY}u.$$

It is not possible in general to have $Y = \bar{X}$.

Lemma 33. *Suppose that $XXYu = 0$ so that by Lemma 26 we may write*

$$Yu = f_1 w_1 + f_2 w_2$$

with f_1, f_2 CR. Then

$$\overline{XXY}u = \alpha\left(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}\right). \tag{5-2}$$

In particular, $\alpha^{-1}\,\overline{XXY}u$ is CR.

Proof. We have

$$
\begin{aligned}
\overline{XXY}u &= \overline{XYX}u + \bar{X}[\bar{X},\bar{Y}]u \\
&= \overline{XY}(\alpha(f_1w_1+f_2w_2)) + \bar{X}(-iR-(\bar{Y}\alpha)Y)u \qquad \text{(3-6a), (4-2c), (4-5b)} \\
&= \bar{X}(\alpha\bar{Y}(f_1w_1+f_2w_2)) - i\bar{X}Ru \\
&= \bar{X}(f_1z_2-f_2z_1) - iR\bar{X}u - i[\bar{X},R]u \qquad \text{(3-6a), (4-1)} \\
&= \bar{X}(f_1z_2-f_2z_1) - iR(\alpha(f_1w_1+f_2w_2)) + 2\bar{X}u \qquad \text{(3-6a), (4-3f)} \\
&= (\bar{X}f_1)\cdot z_2 - f_1\cdot\alpha w_1 - (\bar{X}f_2)\cdot z_1 - f_2\cdot\alpha w_2 \\
&\quad - i\alpha((Rf_1)\cdot w_1 - f_1\cdot(iw_1) + (Rf_2)\cdot w_2 - f_2\cdot(iw_2)) \qquad \text{(4-1),(4-3i),} \\
&\quad + 2\alpha(f_1w_1+f_2w_2) \qquad \text{(4-2e), (3-6a)} \\
&= (\bar{X}f_1)\cdot z_2 - (\bar{X}f_2)\cdot z_1 - i\alpha((Rf_1)\cdot w_1 + (Rf_2)\cdot w_2) \\
&= \alpha\big((z_2Y - iw_1R)f_1 - (z_1Y + iw_2R)f_2\big) \\
&= \alpha\Big(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}\Big). \qquad \square
\end{aligned}
$$

Proof of (1-1d). To get the required lower bound on the null spaces, it will suffice to show that XXY and $\overline{XXY}$ annihilate CR functions and conjugate-CR functions. This follows from (1-1a) along with (4-5b) and (4-5d).

For the other direction, if $XXYu = 0 = \overline{XXY}u$, then from Lemma 27 we have a closed 1-form $\tilde{\omega} \overset{\text{def}}{=} f_2\,dz_1 - f_1\,dz_2$ on the open subset of S where f_1 and f_2 are CR functions satisfying $Yu = f_1w_1 + f_2w_2$. Restricting our attention to a simply connected subset, we may write $\omega = df$ with f CR. Then we have

$$Yf = w_2f_2 + w_1f_1 = Yu.$$

Thus u is the sum of the CR function f and the conjugate-CR function $u - f$.

The general case follows by localization. □

Lemma 34. $\operatorname{div} Y \overset{\text{def}}{=} \partial w_2/\partial z_1 - \partial w_1/\partial z_2$ *and* $\operatorname{div} \bar{Y} \overset{\text{def}}{=} \partial \bar{w}_2/\partial \bar{z}_1 - \overline{\partial \bar{w}_1/\partial \bar{z}_2}$ *vanish on* S.

Proof. Since S is circular, any defining function ρ for S will satisfy

$$\operatorname{Im}\Big(z_1\frac{\partial\rho}{\partial z_1} + z_2\frac{\partial\rho}{\partial z_2}\Big) = -\frac{R\rho}{2} = 0.$$

Adjusting our choice of defining function we may arrange that

$$z_1\frac{\partial\rho}{\partial z_1} + z_2\frac{\partial\rho}{\partial z_2} \equiv 1$$

in some neighborhood of S. Then from the proof of Lemma 6 we have

$$\frac{\partial w_2}{\partial z_1} - \frac{\partial w_1}{\partial z_2} = \frac{\partial^2 \rho}{\partial z_1 \partial z_2} - \frac{\partial^2 \rho}{\partial z_2 \partial z_1} = 0. \tag{5-3}$$

The remaining statement follows by conjugation. □

Lemma 35.
$$\int_S (X\gamma)\eta \, \frac{dS}{\alpha} = -\int_S \gamma(X\eta) \, \frac{dS}{\alpha}$$

Proof.
$$\begin{aligned}
\int_S (X\gamma)\eta \, \frac{dS}{\alpha} &= \int_S (\bar{Y}\gamma)\eta \, dS && \text{(3-6a)}\\
&= -\int_S \gamma(\bar{Y}\eta) \, dS && \text{(Lemma 34)}\\
&= -\int_S \gamma(X\eta) \, \frac{dS}{\alpha} && \text{(3-6a)}
\end{aligned}$$

(The integration by parts above may be justified by applying the divergence theorem on a tubular neighborhood of S and passing to a limit.) □

Proof of (1-1c). Assume that $XXYu = 0$. Noting that S is simply connected, from (1-1d) it suffices to prove that $\overline{XXY}u = 0$. From Lemma 27 we know that $\alpha^{-1}\overline{XXY}u$ is CR. The desired conclusion now follows from

$$\begin{aligned}
\int_S |\overline{XXY}u|^2 \, \frac{dS}{\alpha^2} &= \int_S \alpha^{-1}\overline{XXY}u \cdot XXY\bar{u} \, \frac{dS}{\alpha}\\
&= -\int_S X(\alpha^{-1}\overline{XXY}u) \cdot XY\bar{u} \, \frac{dS}{\alpha} && \text{(Lemma 35)}\\
&= -\int_S 0 \cdot XY\bar{u} \, \frac{dS}{\alpha} && \text{(Lemma 33)}\\
&= 0.
\end{aligned}$$ □

6. Further comments

6.1. *Remarks on uniqueness.*

Proposition 36. *Suppose that in the setting of Theorem B we have vector fields $\tilde{X}$, $\tilde{T}$ satisfying (suitably modified) (3-6a) and (3-6b). Then $\tilde{X}\tilde{X}\tilde{T}$ annihilates CR functions and dual-CR functions if and only if there are CR functions f_1, f_2 and f_3 so that $f_1 w_1 + f_2 w_2$ and f_3 are nonvanishing and*

$$\tilde{X} = f_3(f_1 w_1 + f_2 w_2)^2 X, \quad \tilde{T} = \frac{1}{f_1 w_1 + f_2 w_2} T.$$

Proof. From (3-6a) and (3-6b) we have $\tilde{X} = \gamma X$, $\tilde{T} = \eta T$ with nonvanishing scalar functions γ and η.

Suppose that $\tilde{X}\tilde{X}\tilde{T}$ annihilates CR functions and dual-CR functions. By routine computation we have

$$\tilde{X}\tilde{X}\tilde{T} = \gamma^2\eta XXT + \gamma\big((2\gamma(X\eta) + \eta(X\gamma))XT + X(\gamma(X\eta))T\big).$$

The operator $(2\gamma(X\eta)+\eta(X\gamma))XT + X(\gamma(X\eta))T$ must in particular annihilate CR functions. But if f is CR, then using Lemma 24 we have

$$\big((2\gamma(X\eta)+\eta(X\gamma))XT+X(\gamma(X\eta))T\big)f=\big(i(2\gamma(X\eta)+\eta(X\gamma))R+X(\gamma(X\eta))T\big)f$$

Since R and T are $\mathbb{C}$-linearly independent and f is arbitrary it follows that

$$X(\gamma\eta^2) = 2\gamma(X\eta) + \eta(X\gamma) = 0, \quad X(\gamma(X\eta)) = 0.$$

We set $f_3 = \gamma\eta^2$, which is CR and nonvanishing. Then the second equation above yields

$$-f_3 \cdot XX(\eta^{-1}) = X(f_3\eta^{-2}(X\eta)) = X(\gamma(X\eta)) = 0,$$

and hence $XX(\eta^{-1}) = 0$. From Lemma 26 we have $\eta = 1/(f_1w_1 + f_2w_2)$ with f_1 and f_2 CR. The result now follows.

The converse statement follows by reversing steps. □

Proposition 37. *Suppose that in the setting of Theorem A we have vector fields $\tilde{X}$, $\tilde{T}$ satisfying (suitably modified) (1-1a) and (1-1b). Then $\tilde{X}\tilde{X}\tilde{Y}$ annihilates CR functions and conjugate-CR functions if and only if there are CR functions f_1, f_2 and f_3 so that $f_1w_1 + f_2w_2$ and f_3 are nonvanishing and*

$$\tilde{X} = f_3(f_1w_1 + f_2w_2)^2 X, \quad \tilde{Y} = \frac{1}{f_1w_1+f_2w_2}Y.$$

The proof is similar to that of Proposition 36, using (4-2h) in place of Lemma 26.

6.2. *Nirenberg-type result.*

Proposition 38. *Given a point p on a strongly pseudoconvex hypersurface $S \subset \mathbb{C}^2$, any 2-jet at p of a $\mathbb{C}$-valued function on S is the 2-jet of the restriction to S of a pluriharmonic function on $\mathbb{C}^2$.*

Proof. After performing a standard local biholomorphic change of coordinates we may reduce to the case where $p = 0$ and S is described near 0 by an equation of the form

$$y_2 = z_1\bar{z}_1 + O(\|(z_1, x_2)\|)^3.$$

The projection $(z_1, x_2 + iy_2) \mapsto (z_1, x_2)$ induces a bijection between 2-jets at 0 along S and 2-jets at 0 along $\mathbb{C} \times \mathbb{R}$. It suffices now to note that the 2-jet

$$A + Bz_1 + C\bar{z}_1 + Dx_2 + Ez_1^2 + F\bar{z}_1^2 + Gz_1\bar{z}_1 + Hz_1x_2 + I\bar{z}_1x_2 + Jx_2^2$$

is induced by the pluriharmonic polynomial

$$A + Bz_1 + C\bar{z}_1 + \frac{D-iG}{2}z_2 + \frac{D+iG}{2}\bar{z}_2 + Ez_1^2 + F\bar{z}_1^2 + Hz_1z_2 + I\bar{z}_1\bar{z}_2 + J\bar{z}_2^2. \quad \square$$

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ON TATE DUALITY AND A PROJECTIVE SCALAR PROPERTY FOR SYMMETRIC ALGEBRAS

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We identify a class of symmetric algebras over a complete discrete valuation ring $\mathcal{O}$ of characteristic zero to which the characterisation of Knörr lattices in terms of stable endomorphism rings in the case of finite group algebras can be extended. This class includes finite group algebras, their blocks and source algebras and Hopf orders. We also show that certain arithmetic properties of finite group representations extend to this class of algebras. Our results are based on an explicit description of Tate duality for lattices over symmetric $\mathcal{O}$-algebras whose extension to the quotient field of $\mathcal{O}$ is separable.

1. Introduction

Let p be a prime. Let $\mathcal{O}$ be a complete discrete valuation ring with maximal ideal $J(\mathcal{O}) = \pi\mathcal{O}$ for some $\pi \in \mathcal{O}$, residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p, and field of fractions K of characteristic zero. An $\mathcal{O}$-algebra A is *symmetric* if A is isomorphic to its $\mathcal{O}$-dual A^* as an A-A-bimodule; this implies that A is free of finite rank over $\mathcal{O}$. The image s of 1_A under a bimodule isomorphism $A \cong A^*$ is called a *symmetrising form* for A; it has the property that $s(ab) = s(ba)$ for all $a, b \in A$ and that the bimodule isomorphism $A \cong A^*$ sends $a \in A$ to the map $s_a \in A^*$ defined by $s_a(b) = s(ab)$ for all $a, b \in A$. Since the automorphism group of A as an A-A-bimodule is canonically isomorphic to $Z(A)^\times$, any other symmetrising form of A is of the form s_z for some $z \in Z(A)^\times$. If X is an $\mathcal{O}$-basis of A, then any symmetrising form s of A determines a dual basis $X^\vee = \{x^\vee \mid x \in X\}$ satisfying $s(xx^\vee) = 1$ for $x \in X$ and $s(xy^\vee) = 0$ for $x, y \in X$, $x \neq y$. We denote by $\mathrm{Tr}_1^A : A \to Z(A)$ the $Z(A)$-linear map defined by $\mathrm{Tr}_1^A(a) = \sum_{x \in X} xax^\vee$ for all $a \in A$. This map depends on the choice of s but not on the choice of the basis X. We set $z_A = \mathrm{Tr}_1^A(1_A)$ and call z_A the *relative projective element of A in $Z(A)$ with respect to s*. This is also called the *central Casimir element* in [Broué 2009]. If $z \in Z(A)^\times$ and $s' = s_z$, then the dual basis of X with respect to s' is equal to $X^\vee z^{-1}$,

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where $X^\vee$ is the dual basis of X with respect to s, and hence the relatively projective element in $Z(A)$ with respect to s' is equal to $z'_A = z_A z^{-1}$. If we do not specify a symmetric form of a symmetric algebra A, then the relative projective elements form a $Z(A)^\times$-orbit in $Z(A)$. See Broué [2009] for more details.

The purpose of this paper is to examine situations in which some relative projective element is a scalar multiple of the identity.

Definition 1.1. A symmetric $\mathcal{O}$-algebra A is said to have the *projective scalar property* if there exists a symmetrising form s of A such that the corresponding relative projective element z_A is of the form $z_A = \lambda 1_A$ for some $\lambda \in \mathcal{O}$.

Throughout the paper we will be working with a symmetric $\mathcal{O}$-algebra A such that the K-algebra $K \otimes_{\mathcal{O}} A$ is separable. Since K has characteristic zero, $K \otimes_{\mathcal{O}} A$ is separable if and only if it is semisimple. This in turn is equivalent to the condition that the relative projective element with respect to some, and hence any, symmetrising form on A is invertible in $Z(K \otimes_{\mathcal{O}} A)$, see [Broué 2009, Proposition 3.6]. In particular, when A has the projective scalar property, the separability of $K \otimes_{\mathcal{O}} A$ is equivalent to the property that the relative projective elements of A are nonzero.

Matrix algebras, finite group algebras, blocks and source algebras of finite group algebras, as well as Hopf algebras whose extension to K is semisimple have the projective scalar property (see Examples 5.1, 5.2, and 5.3), but Iwahori–Hecke algebras and rings of generalised characters do not typically have this property (see Examples 5.4, 5.5, and 5.6). The projective scalar property is invariant under taking direct factors and tensor products but not under direct products, and is not invariant under Morita equivalences (see Example 5.1).

Our motivation for studying algebras with the projective scalar property comes from a characterisation of Knörr lattices for a finite group algebra in terms of the relatively $\mathcal{O}$-stable module category of the algebra. Recall that an A-lattice is a left unital A-module which is free of finite rank as an $\mathcal{O}$-module. An indecomposable A-lattice U is called a *Knörr lattice* if the linear trace form tr_U on $\mathrm{End}_{\mathcal{O}}(U)$ satisfies $\mathrm{tr}_U(\alpha)\mathcal{O} \subseteq \mathrm{rank}_{\mathcal{O}}(U)\mathcal{O}$ for every $\alpha \in \mathrm{End}_A(U)$, with equality precisely when α is an automorphism.

Now for two finitely generated A-modules U and V, we denote by $\underline{\mathrm{Hom}}_A(U, V)$ the homomorphism space in the $\mathcal{O}$-stable category $\underline{\mathrm{mod}}(A)$ of finitely generated A-modules; that is, $\underline{\mathrm{Hom}}_A(U, V)$ is the quotient of $\mathrm{Hom}_A(U, V)$ by the subspace $\mathrm{Hom}_A^{\mathrm{pr}}(U, V)$ of A-homomorphisms $U \to V$ which factor through a relatively $\mathcal{O}$-projective A-module. We write

$$\mathrm{End}_A^{\mathrm{pr}}(U) = \mathrm{Hom}_A^{\mathrm{pr}}(U, U) \quad \text{and} \quad \underline{\mathrm{End}}_A(U) = \underline{\mathrm{Hom}}_A(U, U).$$

For an A-lattice U, let $a(U)$ denote the smallest nonnegative integer such that $\pi^{a(U)}$ annihilates $\underline{\mathrm{End}}_A(U)$. In [Carlson and Jones 1989], the element $\pi^{a(U)}$ is referred to as the *exponent* of U. If U is indecomposable nonprojective, U is said to have the *stable exponent property* if the socle of $\underline{\mathrm{End}}_A(U)$ as a (left or right) module over itself is equal to $\pi^{a(U)-1}\underline{\mathrm{End}}_A(U)$.

Carlson and Jones [1989], and independently Thévenaz [1988] and Knörr [1987] proved that for G a finite group, an absolutely indecomposable nonprojective $\mathcal{O}G$-lattice is a Knörr lattice if and only if it has the stable exponent property. The projective scalar property guarantees such an equivalence:

Theorem 1.2. *Let A be a symmetric $\mathcal{O}$-algebra such that $K \otimes_{\mathcal{O}} A$ is separable. Suppose that A has the projective scalar property. Then an indecomposable nonprojective A-lattice U is a Knörr lattice if and only if U is absolutely indecomposable and has the stable exponent property.*

The converse to this theorem is false. In Example 5.8, we shall see a symmetric algebra without the projective scalar property for which the Knörr lattices coincide with those having the stable exponent property. Thus, the equivalence between the Knörr and stable exponent properties does not provide a characterisation of the projective scalar property. Also, in Example 5.7, we shall see both Knörr lattices which do not have the stable exponent property, as well as lattices with the stable exponent property which are not Knörr.

Example 5.7 will, in addition, show that the property of being a Knörr lattice is not invariant under Morita equivalences. However, it is easy to see that the stable exponent property is invariant under such equivalences. Thus, two subclasses can be identified within a given Morita equivalence class of symmetric algebras, namely, those for which the above two types of lattices coincide, and those with the projective scalar property.

The basic ingredient for the proof of Theorem 1.2 is a description of Tate duality for lattices over symmetric $\mathcal{O}$-algebras with separable coefficient extensions which makes the role of the relative projective element explicit. Note that $\underline{\mathrm{Hom}}_A(U, V)$ is a torsion $\mathcal{O}$-module for any A-lattices U and V when $K \otimes_{\mathcal{O}} A$ is separable. This follows from the Gaschütz–Ikeda lemma (cf. [Geck and Pfeiffer 2000, Lemma 7.1.11]), which is a special case of Higman's criterion for modules over symmetric algebras in Broué [2009].

Theorem 1.3. *Let A be a symmetric $\mathcal{O}$-algebra with symmetrising form s such that $K \otimes_{\mathcal{O}} A$ is separable. Set $z = z_A$. Let U and V be A-lattices. The map sending $(\alpha, \beta) \in \mathrm{Hom}_A(U, V) \times \mathrm{Hom}_A(V, U)$ to $\mathrm{tr}_{K\otimes_{\mathcal{O}}U}(z^{-1}\beta \circ \alpha) \in K$ induces a nondegenerate pairing*

$$\underline{\mathrm{Hom}}_A(U, V) \times \underline{\mathrm{Hom}}_A(V, U) \to K/\mathcal{O}.$$

Here, $\mathrm{tr}_{K\otimes_{\mathcal{O}}U}(z^{-1}\beta \circ \alpha)$ is the trace of the K-linear endomorphism of $K \otimes_{\mathcal{O}} U$ obtained from extending the endomorphism $\beta\circ\alpha$ of U linearly to $K\otimes_{\mathcal{O}}U$, composed with the endomorphism given by multiplication on $K \otimes_{\mathcal{O}} U$ with the inverse z^{-1} of z in $Z(K \otimes_{\mathcal{O}} A)$. If A has the projective scalar property, then the Tate duality pairing admits the following description (which is well known in this form for finite group algebras, see [Brown 1982, Theorem 7.4]).

Corollary 1.4. *Let A be a symmetric $\mathcal{O}$-algebra such that $K \otimes_{\mathcal{O}} A$ is separable. Suppose that $z_A = \pi^n 1_A$ for some choice of a symmetrising form of A and some positive integer n. Let U and V be A-lattices. The map sending $(\alpha, \beta) \in \mathrm{Hom}_A(U, V) \times \mathrm{Hom}_A(V, U)$ to $\mathrm{tr}_U(\beta \circ \alpha)$ induces a nondegenerate pairing*

$$\underline{\mathrm{Hom}}_A(U, V) \times \underline{\mathrm{Hom}}_A(V, U) \to \mathcal{O}/\pi^n\mathcal{O}.$$

Remark 1.5. Theorem 1.3, applied to $U = V$, shows that if U is an indecomposable nonprojective lattice for a symmetric $\mathcal{O}$-algebra A such that $K \otimes_{\mathcal{O}} A$ is separable, then the socle of $\underline{\mathrm{End}}_A(U)$ as a module over itself is simple, since it is dual to $\underline{\mathrm{End}}_A(U)/J(\underline{\mathrm{End}}_A(U)) \cong k$. This fact is well known (see Roggenkamp [1977]) and this is the key step in the existence proof of almost split sequences of A-modules. Applying Theorem 1.3 to Heller translates of V yields nondegenerate pairings

$$\widehat{\mathrm{Ext}}_A^n(U, V) \times \widehat{\mathrm{Ext}}_A^{-n}(V, U) \to K/\mathcal{O}$$

for any integer n. Applied to $U = V = A$ as a module over $A \otimes_{\mathcal{O}} A^{\mathrm{op}}$ this yields nondegenerate pairings in Tate–Hochschild cohomology;

$$\widehat{HH}^n(A) \times \widehat{HH}^{-n}(A) \to K/\mathcal{O}.$$

Theorem 1.2 is a special case of the following consequence of Theorem 1.3 which gives a characterisation of absolutely indecomposable modules with the stable exponent property for symmetric $\mathcal{O}$-algebras. Denote by ν a π-adic valuation on K.

Theorem 1.6. *Let A be a symmetric $\mathcal{O}$-algebra with symmetrising form s such that $K \otimes_{\mathcal{O}} A$ is separable. Denote by z the associated relatively projective element of A in $Z(A)$. Let U be an indecomposable nonprojective A-lattice. The following are equivalent*:

(i) *For any $\alpha \in \mathrm{End}_A(U)$ we have $\nu(\mathrm{tr}_{K \otimes_{\mathcal{O}} U}(z^{-1}\alpha)) \geq \nu(\mathrm{tr}_{K \otimes_{\mathcal{O}} U}(z^{-1}\mathrm{Id}_U))$, with equality if and only if α is an automorphism of U.*

(ii) *The A-lattice U is absolutely indecomposable and has the stable exponent property.*

Symmetric $\mathcal{O}$-algebras with split semisimple coefficient extensions to K having the projective scalar property can be characterised as follows.

Theorem 1.7. *Let A be a symmetric $\mathcal{O}$-algebra such that $K \otimes_{\mathcal{O}} A$ is split semisimple. Denote by $\rho : A \to \mathcal{O}$ the regular character of A. The following are equivalent*:

(i) *The algebra A has the projective scalar property.*

(ii) *There exists a nonnegative integer n such that $\pi^{-n}\rho$ is a symmetrising form of A.*

(iii) *There exists a nonnegative integer n such that for any A-lattice U we have*

$$\mathrm{tr}_U(\mathrm{End}_A(U)) = \pi^{n-a(U)}\mathcal{O}.$$

Moreover, if these three equivalent statements hold, then the integers n in (ii) *and* (iii) *coincide, and $\pi^n 1_A$ is a relative projective element with respect to some symmetrising form of A.*

We also have a characterisation, in terms of the decomposition matrix, of symmetric $\mathcal{O}$-algebras A such that some algebra in the Morita or derived equivalence class of A has the scalar projective property. Recall that if B is a split finite-dimensional algebra over a field F then the set of characters of simple A-modules is a linearly independent subset of the F-vector space of functions from B to F (see, for instance, [Nagao and Tsushima 1989, Chapter 3, Theorem 3.13]), and hence may be identified with a set of representatives of the isomorphism classes of simple B-modules.

Theorem 1.8. *Let A be a symmetric $\mathcal{O}$-algebra such that $K\otimes_{\mathcal{O}} A$ is split semisimple and $k\otimes_{\mathcal{O}} A$ is split. Denote by $\mathrm{Irr}_K(A)$ the set of characters of simple $K\otimes_{\mathcal{O}} A$-modules and by $\mathrm{Irr}_k(A)$ the set of characters of simple $k\otimes_{\mathcal{O}} A$-modules. For $\chi\in\mathrm{Irr}_K(A)$ and $\varphi\in\mathrm{Irr}_k(A)$, denote by $d_{\chi,\varphi}$ the multiplicity of S as a composition factor of $k\otimes_{\mathcal{O}} V$, where V is an A-lattice such that $K\otimes_{\mathcal{O}} V$ has character χ, and S is a simple $k\otimes_{\mathcal{O}} A$-module with character φ. The following are equivalent*:

(i) *There exists an algebra Morita equivalent to A with the projective scalar property.*

(ii) *There exists an algebra derived equivalent to A with the projective scalar property.*

(iii) *There exist a nonnegative integer n and positive integers m_φ, where $\varphi\in\mathrm{Irr}_k(A)$, such that setting*

$$a_\chi := \sum_{\varphi\in\mathrm{Irr}_k(A)} m_\varphi d_{\chi,\varphi},$$

where $\chi\in\mathrm{Irr}_K(A)$, the form $\pi^{-n}\sum_{\chi\in\mathrm{Irr}_K(A)} a_\chi\chi$ is a symmetrising form for A.

(iv) *There exist a nonnegative integer n and integers m_φ, where $\varphi\in\mathrm{Irr}_k(A)$, such that setting $a_\chi := \sum_{\varphi\in\mathrm{Irr}_k(A)} m_\varphi d_{\chi,\varphi}$, where $\chi\in\mathrm{Irr}_K(A)$, the form $\pi^{-n}\sum_{\chi\in\mathrm{Irr}_K(A)} a_\chi\chi$ is a symmetrising form for A.*

We point out that certain arithmetic features of finite group representations carry over to algebras with the projective scalar property. Recall that the degree of an ordinary irreducible character of a finite group G divides the order of G and that if

U is a projective $\mathcal{O}G$-lattice, then the p-part of $|G|$ divides the p-part of the $\mathcal{O}$-rank of U.

Proposition 1.9. *Let A be a symmetric $\mathcal{O}$-algebra such that $K \otimes_{\mathcal{O}} A$ is split semisimple. Assume that A has the projective scalar property and let $\pi^n 1_A$ be a relative projective element with respect to some symmetrising form on A.*

(i) *If U is a Knörr A-lattice, then the p-part of the $\mathcal{O}$-rank of U divides π^n in $\mathcal{O}$.*

(ii) *If U is a projective A-lattice, then the p-part of the $\mathcal{O}$-rank of U is divisible in $\mathcal{O}$ by π^n.*

Theorem 1.7 and Proposition 1.9 combine to give the following generalisation of the Brauer–Nesbitt theorem of classical modular representation theory.

Proposition 1.10. *Under the assumptions of Proposition 1.9, suppose that U is a Knörr lattice. Then U is projective if and only if the p-part of the $\mathcal{O}$-rank of U is equal to π^n.*

Remark 1.11. Note that if $A = \mathcal{O}G$, then $|G| \cdot 1_{\mathcal{O}G}$ is the relative projective element with respect to the standard symmetrising form, see [Broué 2009, Examples and Remarks after Proposition 3.3]. Moreover, an absolutely irreducible $\mathcal{O}G$-lattice is a Knörr $\mathcal{O}G$-lattice. Hence, letting p vary across all primes in (i), one sees that the above does generalise the corresponding results for group algebras. A related global divisibility criterion for irreducible lattices of symmetric algebras has been given by Jacoby and Lorenz [2017, Corollary 6] in the context of Kaplansky's sixth conjecture.

For U an A-lattice, define the height of U to be the number $h(U)$ such that

$$\mathrm{rank}(U)_p = p^{m+h(U)},$$

where m is defined by

$$p^m = \min_V \{\mathrm{rank}(V)_p\}$$

as V ranges over all irreducible A-lattices. Note that $h(U)$ is a nonnegative integer.

It is well known that a Morita equivalence between blocks of finite group algebras or between a block algebra and the corresponding source algebra preserves the height of corresponding irreducible characters, see [Broué 1990; 1994]. The following theorem generalises this to algebras with the projective scalar property and to Knörr lattices.

Theorem 1.12. *Let A be a symmetric $\mathcal{O}$-algebra such that $K \otimes_{\mathcal{O}} A$ is split semisimple. Let A' be an $\mathcal{O}$-algebra Morita equivalent to A, and suppose that both A and A' have the projective scalar property. Let U be a Knörr A-lattice and let U' be an A'-lattice corresponding to U through a Morita equivalence between A and A'. Then U' is a Knörr A'-lattice and $h(U) = h(U')$.*

Finally we point out that although the stable exponent property does not apply to projective lattices, we can, following Knörr [1989, Lemma 1.9], characterise projective Knörr lattices in the presence of the projective scalar property.

Proposition 1.13. *Let A be as in the previous theorem. Assume that U is an A-lattice which is both projective and Knörr. Then $U/\pi U$ is a simple $A/\pi A$-module. In particular, $K \otimes_{\mathcal{O}} U$ is an irreducible $K \otimes_{\mathcal{O}} A$-module.*

Section 2 contains the proof of Theorems 1.3, 1.6 and 1.2. We prove Theorems 1.7 and 1.8 in Section 3. This section also contains a characterisation of the projective scalar property in terms of rational centres. Section 4 discusses arithmetic properties of Knörr lattices in the presence of the projective scalar property, including the proof of Proposition 1.9 and Theorem 1.12. Section 5 contains various examples.

2. Tate duality for symmetric algebras

The proof of Theorem 1.3 is an adaptation of ideas from Thévenaz [1988, Section 1]. We keep the notation used in Theorem 1.3. For simplicity, we write in this section $KA = K \otimes_{\mathcal{O}} A$, $KU = K \otimes_{\mathcal{O}} U$, and $KV = K \otimes_{\mathcal{O}} V$. We write $K\mathrm{Hom}_{\mathcal{O}}(U, V) = K \otimes_{\mathcal{O}} \mathrm{Hom}_{\mathcal{O}}(U, V)$ and identify this space with $\mathrm{Hom}_K(KU, KV)$ whenever convenient. Similarly, we write $K\mathrm{Hom}_A(U, V) = K \otimes_{\mathcal{O}} \mathrm{Hom}_A(U, V)$ and identify this space with $\mathrm{Hom}_{KA}(KU, KV)$. Let X, $X^\vee$ be a pair of $\mathcal{O}$-bases of A dual to each other with respect to the symmetrising form s; in particular, the relative projective element with respect to s is

$$z_A = \sum_{x \in X} x x^\vee = \sum_{x \in X} x^\vee x,$$

where $x^\vee$ denotes the unique element in $X^\vee$ satisfying $s(xx^\vee) = 1$, for $x \in X$. We denote by

$$\mathrm{Tr}_1^A : K\mathrm{Hom}_{\mathcal{O}}(U, V) \to K\mathrm{Hom}_A(U, V)$$

the K-linear map which sends $\alpha \in \mathrm{Hom}_{\mathcal{O}}(U, V)$ to $\sum_{x \in X} x\alpha x^\vee$. Here $x\alpha x^\vee \in \mathrm{Hom}_{\mathcal{O}}(U, V)$ is defined by $(x\alpha x^\vee)(u) = x\alpha(x^\vee u)$ for $u \in U$ and $x \in X$. Clearly, Tr_1^A restricts to a map $\mathrm{Hom}_{\mathcal{O}}(U, V) \to \mathrm{Hom}_A(U, V)$. By Higman's criterion for symmetric algebras (cf. [Broué 2009]), we have $\mathrm{Tr}_1^A(\mathrm{Hom}_{\mathcal{O}}(U, V)) = \mathrm{Hom}_A^{\mathrm{pr}}(U, V)$. Denote by

$$\varphi : K\mathrm{Hom}_{\mathcal{O}}(U, V) \times K\mathrm{Hom}_{\mathcal{O}}(V, U) \to K$$

the K-linear map sending $(\alpha, \beta) \in \mathrm{Hom}_{\mathcal{O}}(U, V) \times \mathrm{Hom}_{\mathcal{O}}(V, U)$ to $\mathrm{tr}_U(\beta \circ \alpha)$, and denote by

$$\varphi_A : K\mathrm{Hom}_A(U, V) \times K\mathrm{Hom}_A(V, U) \to K$$

the map sending $(\alpha, \beta) \in \mathrm{Hom}_A(U, V) \times \mathrm{Hom}_A(V, U)$ to $\mathrm{tr}_{KU}(z_A^{-1}\beta \circ \alpha)$, where α and β are extended linearly to maps between KU and KV. The following fact generalises [Thévenaz 1988, Proposition 1.1].

Proposition 2.1. *Using the same notation as above, for* $\alpha \in \mathrm{Hom}_{\mathcal{O}}(U, V)$ *and* $\beta \in \mathrm{Hom}_A(V, U)$ *we have*

$$\varphi_A(\mathrm{Tr}_1^A(\alpha), \beta) = \varphi(\alpha, \beta).$$

Similarly, for $\gamma \in \mathrm{Hom}_A(U, V)$ *and* $\delta \in \mathrm{Hom}_{\mathcal{O}}(V, U)$ *we have*

$$\varphi_A(\gamma, \mathrm{Tr}_1^A(\delta)) = \varphi(\gamma, \delta).$$

In particular, φ_A *is nondegenerate.*

Proof. We regard $\mathrm{Hom}_{\mathcal{O}}(U, V)$ and $\mathrm{Hom}_{\mathcal{O}}(V, U)$ as A-A-bimodules in the canonical way. If $\mu \in \mathrm{Hom}_{\mathcal{O}}(U, V)$ and $\nu \in \mathrm{Hom}_{\mathcal{O}}(V, U)$, then for any $a \in A$, we have $\nu \circ a\mu = \nu a \circ \mu$. If $\epsilon \in \mathrm{End}_{\mathcal{O}}(U)$ and $a \in A$, then $\mathrm{tr}_U(\epsilon a) = \mathrm{tr}_U(a\epsilon)$. Thus we have

$$\begin{aligned}\varphi_A(\mathrm{Tr}_1^A(\alpha), \beta) &= \mathrm{tr}_{KU}\Big(z_A^{-1}\sum_{x\in X}\beta \circ x\alpha x^\vee\Big) = \mathrm{tr}_{KU}\Big(z_A^{-1}\sum_{x\in X}x^\vee\beta \circ x\alpha\Big)\\ &= \mathrm{tr}_{KU}\Big(z_A^{-1}\sum_{x\in X}x^\vee\beta x \circ \alpha\Big) = \mathrm{tr}_{KU}\Big(z_A^{-1}\sum_{x\in X}x^\vee x\beta \circ \alpha\Big)\\ &= \mathrm{tr}_{KU}\big(z_A^{-1}z_A\beta \circ \alpha\big) = \varphi(\alpha, \beta).\end{aligned}$$

This shows the first equality, and the proof of the second is analogous. Clearly φ is nondegenerate, and hence so is φ_A. □

Proof of Theorem 1.3. For E an $\mathcal{O}$-submodule of $\mathrm{Hom}_{KA}(KU, KV)$ denote by $E^\perp$ the $\mathcal{O}$-submodule in $\mathrm{Hom}_{KA}(KV, KU)$ consisting of all $\beta \in \mathrm{Hom}_{KA}(KV, KU)$ such that $\varphi_A(\epsilon, \beta) \in \mathcal{O}$ for all $\epsilon \in E$. By the previous proposition, φ_A is nondegenerate, and hence if E is a lattice in $\mathrm{Hom}_{KA}(KU, KV)$, then $E^\perp$ is a lattice in $\mathrm{Hom}_{KA}(KV, KU)$, and we have $(E^\perp)^\perp = E$. We need to show that $(\mathrm{Hom}_A^{\mathrm{pr}}(U, V))^\perp = \mathrm{Hom}_A(V, U)$. Let $\beta \in \mathrm{Hom}_{KA}(KU, KV)$. We have $\beta \in (\mathrm{Hom}_A^{\mathrm{pr}}(U, V))^\perp$ if and only if $\varphi_A(\mathrm{Tr}_1^A(\alpha), \beta) \in \mathcal{O}$ for all $\alpha \in \mathrm{Hom}_{\mathcal{O}}(U, V)$. By Proposition 2.1, this is equivalent to $\mathrm{tr}_{KU}(\beta \circ \alpha) \in \mathcal{O}$ for all $\alpha \in \mathrm{Hom}_{\mathcal{O}}(U, V)$. This, in turn, is the case if and only if β belongs to the subspace $\mathrm{Hom}_A(U, V)$ of $\mathrm{Hom}_{KA}(KU, KV)$. (To see this, choose a basis of U, a basis of V, and let α range over the maps sending exactly one basis element in U to a basis element in V and all other basis elements of U to 0). □

Proof of Corollary 1.4. We have $z_A = \pi^n 1_A$. The nondegenerate pairing

$$\underline{\mathrm{Hom}}_A(U, V) \times \underline{\mathrm{Hom}}_A(V, U) \to K/\mathcal{O}$$

from Theorem 1.3 has image contained in the submodule $\pi^{-n}\mathcal{O}/\mathcal{O}$ of $K/\mathcal{O}$. Multiplication by π^n yields an isomorphism $\pi^{-n}\mathcal{O}/\mathcal{O} \cong \mathcal{O}/\pi^n\mathcal{O}$. Thus Corollary 1.4 follows from Theorem 1.3. □

In order to prove Theorem 1.6, we need the following generalisation of [Carlson and Jones 1989, Proposition 4.2].

Proposition 2.2. *Let A be a symmetric $\mathcal{O}$-algebra with symmetrising form s such that $K \otimes_{\mathcal{O}} A$ is separable. Set $z = z_A$. Let U be an A-lattice and let a be the smallest nonnegative integer such that π^a annihilates $\underline{\mathrm{End}}_A(U)$. Then*

$$\pi^a \mathrm{tr}_{KU}(z^{-1}\mathrm{End}_A(U)) = \mathcal{O}.$$

Proof. Let $\alpha \in \mathrm{End}_A(U)$. By the assumptions we have $\pi^a\alpha \in \mathrm{End}_A^{\mathrm{pr}}(U)$. Applying Theorem 1.3 with $U = V$ and $\beta = \mathrm{Id}_U$ implies that $\pi^a \mathrm{tr}_{KU}(z^{-1}\alpha) \in \mathcal{O}$. Thus $\pi^a \mathrm{tr}_{KU}(z^{-1}\mathrm{End}_A(U)) \subseteq \mathcal{O}$. For the reverse inclusion, consider first the case that U is nonprojective. Then $a \geq 1$, and $\pi^{a-1}\mathrm{Id}_U$ is not contained in $\mathrm{End}_A^{\mathrm{pr}}(U)$; equivalently, its image in $\underline{\mathrm{End}}_A(U)$ is nonzero. Again by Theorem 1.3, there exists $\alpha \in \mathrm{End}_A(U)$ such that $\pi^{a-1}\mathrm{tr}_U(z^{-1}\alpha) \notin \mathcal{O}$. Thus $\pi^a \mathrm{tr}_{KU}(z^{-1}\mathrm{End}_A(U))$ is not contained in $\pi\mathcal{O}$, whence the equality in this case. Suppose U is projective, so $a = 0$. Let $\alpha \in \mathrm{End}_{\mathcal{O}}(U)$ be such that $\mathrm{tr}_U(\alpha) = 1$ and set $\beta = \mathrm{Tr}_1^A(\alpha) \in \mathrm{End}_A(U)$. By Proposition 2.1, we have

$$\mathrm{tr}_U(z^{-1}\beta) = \varphi_A(\mathrm{Tr}_1^A(\alpha), \mathrm{Id}_U) = \varphi(\alpha, \mathrm{Id}_U) = \mathrm{tr}_U(\alpha) = 1.$$

The result follows. □

Proof of Theorem 1.6. Let a be the smallest positive integer such that π^a annihilates $\underline{\mathrm{End}}_A(U)$. The algebra $\underline{\mathrm{End}}_A(U)$ is local, as U is indecomposable nonprojective. The duality in Theorem 1.3 implies that $\mathrm{soc}(\underline{\mathrm{End}}_A(U))$ is simple.

Suppose that (i) holds. We show first that U is absolutely indecomposable. The inequality in (i) applied to the endomorphism α given by multiplication with z shows that

$$\nu(\mathrm{rank}_{\mathcal{O}}(U)) = \nu(\mathrm{tr}_U(\mathrm{Id}_U)) \geq \nu(\mathrm{tr}_{KU}(z^{-1}\mathrm{Id}_U)),$$

so in particular, $\mathrm{tr}_{KU}(z^{-1}\mathrm{Id}_U)$ is nonzero. The inequality in (i) applied to an arbitrary $\alpha \in \mathrm{End}_A(U)$ implies that the scalar τ defined by

$$\tau = \mathrm{tr}_{KU}(z^{-1}\alpha)\mathrm{tr}_{KU}(z^{-1}\mathrm{Id}_U)^{-1}$$

belongs to $\mathcal{O}$. One then has

$$\mathrm{tr}_{KU}(z^{-1}(\alpha - \tau\mathrm{Id}_U)) = 0.$$

Thus (i) implies that $\alpha - \tau\mathrm{Id}_U$ is not an automorphism, and is hence in $J(\mathrm{End}_A(U))$. It follows that $\mathrm{End}_A(U) = \mathcal{O} \cdot \mathrm{Id}_U + J(\mathrm{End}_A(U))$, and hence U is absolutely indecomposable.

We show next that U has the stable exponent property. Since the socle of $\underline{\mathrm{End}}_A(U)$ is simple, we have

$$\mathrm{soc}(\underline{\mathrm{End}}_A(U)) \subseteq \pi^{a-1}\underline{\mathrm{End}}_A(U),$$

and it thus suffices to show that $\pi^{a-1}\underline{\mathrm{End}}_A(U)$ is a semisimple $\underline{\mathrm{End}}_A(U)$-module. That is, it suffices to show that $\pi^{a-1}\underline{\mathrm{End}}_A(U)$ is annihilated by $J(\underline{\mathrm{End}}_A(U))$. Let

$\alpha \in J(\mathrm{End}_A(U))$. The assumptions in (i) together with Proposition 2.2 imply that $\pi^a \mathrm{tr}_{KU}(z^{-1}\alpha) \in \pi\mathcal{O}$, hence $\pi^{a-1}\mathrm{tr}_{KU}(z^{-1}\alpha) \in \mathcal{O}$. By Theorem 1.3, this is equivalent to $\pi^{a-1}\alpha \in \mathrm{End}_A^{\mathrm{pr}}(U)$, or equivalently, to $\pi^{a-1}\underline{\alpha} = 0$. This shows that (i) implies (ii).

Suppose conversely that (ii) holds. In particular, the socle of $\underline{\mathrm{End}}_A(U)$ is simple and equal to $\pi^{a-1}\underline{\mathrm{End}}_A(U)$. Let $\alpha \in J(\mathrm{End}_A(U))$. The image $\underline{\alpha}$ in $\underline{\mathrm{End}}_A(U)$ is contained in $J(\underline{\mathrm{End}}_A(U))$, and hence $\underline{\alpha}$ annihilates $\pi^{a-1}\underline{\mathrm{End}}_A(U)$. Thus $\pi^{a-1}\underline{\alpha} = 0$. Theorem 1.3 implies that $\pi^{a-1}\mathrm{tr}_{KU}(z^{-1}\alpha) \in \mathcal{O}$, hence $\pi^a \mathrm{tr}_{KU}(z^{-1}\alpha) \in \pi\mathcal{O}$.

By Proposition 2.2, there exists $\alpha \in \mathrm{End}_A(U)$ such that $\pi^a \mathrm{tr}_{KU}(z^{-1}\alpha) = 1$. By the previous argument, this forces $\alpha \notin J(\mathrm{End}_A(U))$. Since U is absolutely indecomposable, it follows that $\mathrm{End}_A(U)$ is split local, and hence we have $\alpha = \lambda \mathrm{Id}_U + \rho$ for some $\lambda \in \mathcal{O}^\times$ and some $\rho \in J(\mathrm{End}_A(U))$. Since $\pi^a \mathrm{tr}_{KU}(z^{-1}\rho) \in \pi\mathcal{O}$, it follows that

$$\pi^a \mathrm{tr}_{KU}(z^{-1}\lambda \mathrm{Id}_U) \in \mathcal{O}^\times.$$

Then in fact $\pi^a \mathrm{tr}_{KU}(z^{-1}\lambda \mathrm{Id}_U) \in \mathcal{O}^\times$ for any $\lambda \in \mathcal{O}^\times$, and hence $\pi^a \mathrm{tr}_{KU}(z^{-1}\alpha) \in \mathcal{O}^\times$ for any automorphism α of U. This shows that (ii) implies (i). □

Proof of Theorem 1.2. Let n be the positive integer such that $z_A = \pi^n 1_A$, for some choice of a symmetrising form. Theorem 1.6(i) is then equivalent to stating that U is a Knörr lattice. Thus Theorem 1.2 follows from Theorem 1.6. □

3. Characterisations of the projective scalar property

Throughout this section, A will denote an $\mathcal{O}$-order such that $K \otimes_{\mathcal{O}} A$ is separable. We identify A with its canonical image in $KA = K \otimes_{\mathcal{O}} A$. Denote by $\mathrm{Irr}_K(A)$ the set of the characters of the simple KA-modules. For $\chi \in \mathrm{Irr}_K(A)$ denote by $e(\chi)$ the unique primitive idempotent in $Z(KA)$ satisfying $\chi(e(\chi)) \neq 0$. We will use this notation for other orders as well.

Proof of Theorem 1.7. Suppose that $K \otimes_{\mathcal{O}} A$ is split semisimple. Proposition 2.2 shows that (i) implies (iii).

By the assumptions, $KAe(\chi)$ is a matrix algebra over K of dimension $\chi(1)^2$. In particular, $KAe(\chi)$ is symmetric with symmetrising form χ, and we have $Z(KA) = \prod_{\chi \in \mathrm{Irr}_K(A)} Ke(\chi)$. Fix a symmetrising form s of A. Then s extends to a symmetrising form of KA, still denoted s, and we have

$$s = \sum_{\chi \in \mathrm{Irr}_K(A)} \sigma_\chi \cdot \chi,$$

for some $\sigma_\chi \in K$. The relative projective element of the matrix algebra $KAe(\chi)$ with respect to χ is $\chi(1) \cdot e(\chi)$, and hence the relative projective element of A with respect to s is

$$z_A = \sum_{\chi \in \mathrm{Irr}_K(A)} \sigma_\chi^{-1} \cdot \chi(1) \cdot e(\chi).$$

Suppose that (ii) holds; that is, we may assume that s satisfies

$$s = \pi^{-n}\rho = \sum_{\chi \in \mathrm{Irr}_K(A)} \pi^{-n} \cdot \chi(1) \cdot \chi.$$

Comparing coefficients in the two expressions for s now gives $\sigma_\chi = \pi^{-n} \cdot \chi(1)$. Plugging this into the expression for z_A yields $z_A = \pi^n \cdot 1_A$. So (ii) implies (i).

Suppose finally that (iii) holds. We need to show that (ii) holds. For U an A-lattice, write as before $KU = K \otimes_{\mathcal{O}} U$, and denote by $a(U)$ the smallest nonnegative integer such that $\pi^{a(U)}$ annihilates $\underline{\mathrm{End}}_A(U)$. By the assumptions in (iii) and by Proposition 2.2, there is a nonnegative integer n such that

$$\pi^n \cdot \mathrm{tr}_{KU}(z_A^{-1} \cdot \mathrm{End}_A(U)) = \mathrm{tr}_U(\mathrm{End}_A(U)) = \pi^{n-a(U)}\mathcal{O}$$

for any A-lattice U. We apply this first to $U = A$. Since A is projective as a left A-module, we have $a(A) = 0$, and hence

$$\pi^n \cdot \mathrm{tr}_{KA}(z_A^{-1} \cdot \mathrm{End}_A(A)) = \mathrm{tr}_A(\mathrm{End}_A(A)) = \pi^n\mathcal{O}.$$

Any A-endomorphism is given by right multiplication with an element a in A. By elementary linear algebra, the trace of this endomorphism is equal to the trace of the linear endomorphism given by left multiplication with a, and hence this trace is equal to $\rho(a)$. Thus $\mathrm{tr}_A(\mathrm{End}_A(A)) = \rho(A) = \pi^n\mathcal{O}$, which implies that $\pi^{-n}\rho$ sends A to $\mathcal{O}$. Thus we have

$$\pi^{-n}\rho = s_w,$$

for some $w \in Z(A)$. In order to show that $\pi^{-n}\rho$ is a symmetrising form on A we need to show that $w \in Z(A)^\times$. Writing

$$w = \sum_{\chi \in \mathrm{Irr}_K(A)} \omega_\chi e(\chi)$$

with coefficients $\omega_\chi \in \mathcal{O}$, we need to show that $\omega_\chi \in \mathcal{O}^\times$.

In terms of the coefficients σ_χ already introduced in the expression for s, we have

$$s_w = \sum_{\chi \in \mathrm{Irr}_K(A)} \sigma_\chi \omega_\chi \chi.$$

Comparing coefficients with $\pi^{-n}\rho$ yields therefore

$$\sigma_\chi \omega_\chi = \pi^{-n}\chi(1),$$

for all $\chi \in \mathrm{Irr}_K(A)$, and hence

$$\nu(\sigma_\chi \omega_\chi) = \nu(\pi^{-n}\chi(1)).$$

Let $\chi \in \mathrm{Irr}_K(A)$, and let V be an A-lattice such that $KV = K \otimes_{\mathcal{O}} V$ has character χ. Using that $\mathrm{End}_A(V) = \mathcal{O} \cdot \mathrm{Id}_V$, we get from the above that

$$\nu(\pi^n \cdot \mathrm{tr}_{KV}(z_A^{-1})) = \nu(\mathrm{Id}_V) = \nu(\chi(1)).$$

By the above formula for z_A, we have $z_A^{-1} = \sum_{\chi \in \mathrm{Irr}_K(A)} \sigma_\chi \cdot \chi(1)^{-1} \cdot e(\chi)$, and hence $\mathrm{tr}_{KV}(z_A^{-1}) = \sigma_\chi$. Thus

$$\nu(\pi^n \sigma_\chi) = \nu(\chi(1)).$$

Combining the previous statements yields

$$\nu(\sigma_\chi \omega_\chi) = \nu(\pi^{-n}\chi(1)) = \nu(\sigma_\chi)$$

and hence ω_χ is invertible in $\mathcal{O}$. This shows that (iii) implies (ii). The last statement in Theorem 1.7 on the integer n is obvious from the proofs of the implications. □

Remark 3.1. The coefficients σ_χ^{-1} in the above proof are called *Schur elements* in [Geck and Pfeiffer 2000, §7.2].

Next, we prove Theorem 1.8. As in the theorem, let $\mathrm{Irr}_k(A)$ denote the set of characters afforded by the simple $k \otimes_{\mathcal{O}} A$-modules, and for $\chi \in \mathrm{Irr}_K(A)$ and $\varphi \in \mathrm{Irr}_k(A)$, denote by $d_{\chi,\varphi}$ the multiplicity of S as a composition factor of $k \otimes_{\mathcal{O}} V$, where V is an A-lattice such that $K \otimes_{\mathcal{O}} V$ has character χ, and S is a simple $k \otimes_{\mathcal{O}} A$-module with character φ. We adopt the analogous notation for other orders.

Lemma 3.2. *Let A' be an $\mathcal{O}$-order which is derived equivalent to A. Then we have* $|\mathrm{Irr}_k(A)| = |\mathrm{Irr}_k(A')|$ *and* $|\mathrm{Irr}_K(A)| = |\mathrm{Irr}_K(A')|$. *Further, there exists a bijection* $\chi \to \chi'$ *from* $\mathrm{Irr}_K(A)$ *to* $\mathrm{Irr}_K(A')$, *signs* $\epsilon_\chi \in \{\pm 1\}$, $\chi \in \mathrm{Irr}_K(A)$, *and integers* $u_{\varphi,\psi}$, $\varphi \in \mathrm{Irr}_k(A)$, $\psi \in \mathrm{Irr}_k(A')$ *such that*:

(i) *For* $\chi \in \mathrm{Irr}_K(A)$, $\psi \in \mathrm{Irr}_k(A')$,

$$d_{\chi',\psi} = \epsilon_\chi \sum_{\varphi \in \mathrm{Irr}_k(A)} d_{\chi,\varphi} u_{\varphi,\psi}.$$

(ii) *The form* $s = \sum_{\chi \in \mathrm{Irr}_K(A)} \sigma_\chi \chi$, $\sigma_\chi \in K$ *is a symmetrising form of A if and only if the form* $s' = \sum_{\chi \in \mathrm{Irr}_K(A)} \epsilon_\chi \sigma_\chi \chi'$ *is a symmetrising form of A'.*

If A and A' are Morita equivalent, then in addition there is a bijection $\varphi \to \varphi'$ from $\mathrm{Irr}_k(A)$ *to* $\mathrm{Irr}_k(A')$ *such that* $d_{\chi',\varphi'} = d_{\chi,\varphi}$ *and* $\epsilon_\chi = 1$ *for all* $\chi \in \mathrm{Irr}_K(A)$, $\varphi \in \mathrm{Irr}_k(A)$.

Proof. The first statement follows from [Zimmermann 2014, Theorem 6.8.8]. The transfer of symmetrising forms as in (ii) is proved in [Eisele 2012, Theorem 4.7]. □

Proof of Theorem 1.8. Suppose that the $\mathcal{O}$-order A' is Morita equivalent to A and let $\chi \to \chi'$, and let $\varphi \to \varphi'$ be the bijections of Lemma 3.2. Denoting by n_φ the k-dimension of the simple A'-module labelled by φ' ($\varphi \in \mathrm{Irr}_k(A)$), we have that $\chi'(1) = \sum_{\varphi\in\mathrm{Irr}_k(A)} n_\varphi \cdot d_{\chi,\varphi}$ for all $\chi \in \mathrm{Irr}_K(A)$. The equivalence between (i) and (iii) is now immediate from Lemma 3.2 and the equivalence between (i) and (ii) of Theorem 1.7. We now prove that (iv) implies (iii). Let n and m_φ, $\varphi \in \mathrm{Irr}_k(A)$, be integers such that $\pi^{-n}\sum_{\chi\in\mathrm{Irr}_K(A)} a_\chi\chi$ is a symmetrising form of A, where $a_\chi = \sum_{\varphi\in\mathrm{Irr}_k(A)} m_\varphi d_{\chi,\varphi}$, $\chi \in \mathrm{Irr}_K(A)$. Let X be an $\mathcal{O}$-basis of A. Choose a positive integer t such that $\pi^{-n}\cdot p^t \cdot d_{\chi,\varphi}\chi(x) \in \pi\mathcal{O}$ and $m'_\varphi := m_\varphi + p^t > 0$ for all $\chi \in \mathrm{Irr}_K(A)$, $\varphi \in \mathrm{Irr}_k(A)$ and $x \in X$. Set $s' = \pi^{-n}\sum_{\chi\in\mathrm{Irr}_K(A)} a'_\chi \cdot \chi$, where $a'_\chi = \sum_{\varphi\in\mathrm{Irr}_k(A)} m'_\varphi \cdot d_{\chi,\varphi}$, $\chi \in \mathrm{Irr}_K(A)$. Then for all $a \in A$, $s'(a) - s(a) \in \pi\mathcal{O}$. Hence by considering the determinant of the Gram matrices of the bilinear forms associated to s and s', it follows that s' is also a symmetrising form of A. This proves that (iii) holds. Since (i) clearly implies (ii) and (iii) implies (iv), in order to complete the proof, it suffices to show that (ii) implies (iv). Suppose that A' has the projective scalar property and that A' and A are derived equivalent. Then (iii) holds for A', say for the integers m_ψ, $\psi \in \mathrm{Irr}_k(A')$. Then by Lemma 3.2, we have that (iv) holds for A with the integers $n_\varphi = \sum_{\psi\in\mathrm{Irr}_k(A')} m_\psi u_{\varphi,\psi}$, $\varphi \in \mathrm{Irr}_k(A)$. $\square$

For the rest of this section we will expand on the question of the extent to which the characterisations of the projective scalar property up to Morita equivalence given in Theorem 1.8(iii) and (iv) are constructive. The point here is that the set of symmetrising forms for an order A is actually a $Z(A)^\times$-orbit, and $Z(A)$ is an $\mathcal{O}$-order for a (potentially) quite large ring $\mathcal{O}$. But in fact, as we will see, the criterion can be reduced to linear algebra over $\mathbb{Q}$.

The following proposition shows that the projective scalar property is essentially independent of the choice of the ring $\mathcal{O}$. This is particularly interesting to note since we often make the assumption that K is a splitting field.

Proposition 3.3. *Let A be an $\mathcal{O}$-order and let $\mathcal{E} \supseteq \mathcal{O}$ be a discrete valuation ring containing $\mathcal{O}$ such that $J(\mathcal{E}) \cap \mathcal{O} = J(\mathcal{O})$. Then the $\mathcal{O}$-order A has the projective scalar property if and only if the $\mathcal{E}$-order $\mathcal{E} \otimes_{\mathcal{O}} A$ has the projective scalar property.*

Proof. By the characterisation in Theorem 1.7, A having the projective scalar property is equivalent to some multiple of the regular trace being a symmetrising form for A. But the regular trace on A and the regular trace of $\mathcal{E} \otimes_{\mathcal{O}} A$ have the same Gram matrix (when the same basis is chosen for both of them), and invertibility of a multiple of said Gram matrix over $\mathcal{O}$ is equivalent to invertibility over $\mathcal{E}$, provided of course that we multiplied by an element of $\mathcal{O}$.

So the only thing that still requires proof is that if τ is a generator of $J(\mathcal{E})$, then the integer m such that $\tau^{-m}\cdot\rho$ is a symmetrising form for $\mathcal{E} \otimes_{\mathcal{O}} A$, satisfies $\tau^m\mathcal{E} = \pi^n\mathcal{E}$ for some $n \in \mathbb{Z}_{\geq 0}$ (since this means that $\pi^{-n}\cdot\rho$ is a symmetrising form for A). But by Theorem 1.7 we have $\tau^m \mathcal{E} = \mathrm{tr}_{\mathcal{E}\otimes_{\mathcal{O}} A}(\mathrm{End}_{\mathcal{E}\otimes_{\mathcal{O}} A}(\mathcal{E}\otimes_{\mathcal{O}} A)) = \mathcal{E}\otimes_{\mathcal{O}} \mathrm{tr}_A(\mathrm{End}_A(A))$, and $\mathrm{tr}_A(\mathrm{End}_A(A))$ is certainly of the form $\pi^n\mathcal{O}$ for some nonnegative integer n. □

Definition 3.4. Let A be an $\mathcal{O}$-algebra which is free of finite rank as an $\mathcal{O}$-module such that KA is split semisimple. Fix an isomorphism

$$\varphi : Z(KA) \xrightarrow{\sim} K \times \cdots \times K.$$

We define the *rational centre* $Z^{\mathrm{rat}}(KA)$ of KA to be the $\mathbb{Q}$-algebra

$$\varphi^{-1}(\mathbb{Q} \times \cdots \times \mathbb{Q}).$$

We define the *rational centre of* A, denoted by $Z^{\mathrm{rat}}(A)$, as the intersection of A with $Z^{\mathrm{rat}}(KA)$.

We say that A is *rationally symmetric* if there is an element

$$\tilde{\sigma} = \sum_{\chi\in\mathrm{Irr}_K(A)} \tilde{\sigma}_\chi e_\chi \in Z^{\mathrm{rat}}(A)$$

and an $n \in \mathbb{Z}$ such that

$$\pi^{-n} \cdot \sum_{\chi\in\mathrm{Irr}_K(A)} \tilde{\sigma}_\chi \cdot \chi$$

is a symmetrising form for A.

We should note that $\sigma_\chi = \pi^{-n} \cdot \tilde{\sigma}_\chi$ with σ_χ defined as earlier. Therefore rational symmetry is not the same as asking that the σ_χ be rational. Not even the projective scalar property implies rationality of the σ_χ.

The rational centre of A is a $\mathbb{Z}_{(p)}$-order, and the projective scalar property implies rational symmetry. We should remark that, if $\mathcal{O}$ is ramified over $\mathbb{Z}_p$, then rational symmetry is not necessarily preserved under direct sums. Neither is the projective scalar property, or even the property of being Morita-equivalent to an order which satisfies the projective scalar property. This is due to the possibility that the rational symmetrising forms involve different powers of π, whose quotient may have a nonintegral p-valuation (using the convention $\nu(p) = 1$).

Remark 3.5. An element $\tilde{\sigma}$ (together with an $n \in \mathbb{Z}$) as above and the central projective element z_A are related by the formula

$$z_A = \pi^n \cdot \tilde{\sigma}^{-1} \cdot \sum_{\chi\in\mathrm{Irr}_K(A)} \chi(1) \cdot e_\chi.$$

In particular, $\tilde{\sigma}$ can be chosen in $Z^{\mathrm{rat}}(A)$ if and only if $z_A \in K^\times \cdot Z^{\mathrm{rat}}(A)$. Now we can reinterpret the projective scalar property and rational symmetry in the following way: we consider the orbit $Z(A)^\times \cdot z_A$. If it intersects nontrivially with $K^\times \cdot Z^{\mathrm{rat}}(A)$, then A is rationally symmetric, and if it intersects nontrivially with $K^\times \cdot 1_A$, then A has the projective scalar property.

In view of everything we have seen so far, the following is fairly straightforward.

Proposition 3.6. *Assume that A is rationally symmetric, and $\tilde{\sigma} \in Z^{\mathrm{rat}}(A)$ is as before. Then A has the projective scalar property if and only if*

$$(1)\quad \Big\langle \sum_{\chi\in\mathrm{Irr}_K(A)} d_{\chi,\varphi}\cdot\chi \mid \varphi\in\mathrm{Irr}_k(A)\Big\rangle_{\mathbb{Q}} \cap \Big\{\sum_{\chi\in\mathrm{Irr}_K(A)} \tilde{\sigma}_\chi\cdot\frac{\chi(z)}{\chi(1)}\cdot\chi \;\Big|\; z\in Z^{\mathrm{rat}}(A)\Big\}$$

properly contains

$$(2)\quad \Big\langle \sum_{\chi\in\mathrm{Irr}_K(A)} d_{\chi,\varphi}\cdot\chi \mid \varphi\in\mathrm{Irr}_k(A)\Big\rangle_{\mathbb{Q}} \cap \Big\{\sum_{\chi\in\mathrm{Irr}_K(A)} \tilde{\sigma}_\chi\cdot\frac{\chi(z)}{\chi(1)}\cdot\chi \;\Big|\; z\in I\Big\}$$

for all maximal ideals I in $Z^{\mathrm{rat}}(A)$.

Note that the right-hand side in both (1) and (2) is the intersection of a $\mathbb{Q}$-vector space and a $\mathbb{Z}_{(p)}$-lattice, which can be computed by means of linear algebra.

We conclude this section with an example of a symmetric algebra which is not rationally symmetric, to show that the two notions are not equivalent.

Example 3.7. Assume that k has characteristic two and $\mathcal{O}$ is unramified, i.e., $\pi = p = 2$. Let $x \in \mathcal{O}^\times$ be an arbitrary unit in $\mathcal{O}$. We consider the order $A = \langle\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle_{\mathcal{O}}$ in the commutative split-semisimple K-algebra $K \times K \times K \times K$, where

$$(3)\quad \begin{aligned} \lambda_1 &= (1\ \ 1\ \ 1\ \ 1), \\ \lambda_2 &= (0\ \ 2\ \ 0\ \ 2x), \\ \lambda_3 &= (0\ \ 0\ \ 2\ \ 2x), \\ \lambda_4 &= (0\ \ 0\ \ 0\ \ 4x). \end{aligned}$$

We claim that the map

$$(4)\quad s : K\times K\times K\times K \longrightarrow K : (a_1, a_2, a_3, a_4) \mapsto \frac{2-x^{-1}}{4}a_1 + \frac{1}{4}a_2 + \frac{1}{4}a_3 + \frac{x^{-1}}{4}a_4$$

defines a symmetrising form for A. The Gram matrix of s with respect to the basis $(\lambda_1, \ldots, \lambda_4)$ is

$$(5)\quad (s(\lambda_i\cdot\lambda_j))_{i,j} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & x & 2x \\ 1 & x & 1+x & 2x \\ 1 & 2x & 2x & 4x \end{pmatrix}.$$

The determinant of this matrix is congruent to 1 mod $2\mathcal{O}$, which implies that it is invertible over $\mathcal{O}$, which in turn implies that A is a self-dual lattice with respect to s. So, clearly, A is a symmetric $\mathcal{O}$-order. However, if $x + 2\mathcal{O} \neq 1 + 2\mathcal{O}$, then A is not rationally symmetric. To see this we consider the family of forms

$$s_u : K \times K \times K \times K \longrightarrow K : (a_1, a_2, a_3, a_4) \mapsto \tfrac{1}{4} \cdot \sum_{i=1}^{4} u_i \cdot a_i, \tag{6}$$

where $u \in (K^\times)^4$. By definition, the order A is rationally symmetric if and only if s_u is a symmetrising form for A for some $u \in (\mathbb{Q}^\times)^4$. We know that the symmetrising forms for A are exactly the forms $s(z \cdot -)$ with $z \in Z(A)^\times$ and s as in (4). The form $s(z \cdot -)$ is equal to $s_{z \cdot v}$ with $v = (2 - x^{-1}, 1, 1, x^{-1})$. Since z is a unit each z_i lies in $\mathcal{O}^\times$, and so do all v_i. So if A is symmetric with respect to s_u, then each u_i needs to lie in $\mathcal{O}^\times$. Moreover, A being symmetric with respect to s_u would necessitate A being integral with respect to s_u, which in particular would require $s_u(\lambda_2) = 2^{-1} \cdot u_2 + 2^{-1} \cdot u_4 \cdot x \in \mathcal{O}$. That is, $-u_2/u_4 + 2\mathcal{O} = x + 2\mathcal{O}$, which can only hold true for rational u_i if $x + 2\mathcal{O}$ lies in the prime field of k, which means $x + 2\mathcal{O} = 1 + 2\mathcal{O}$ (since we asked that x be a unit, the case $x + 2\mathcal{O} = 0 + 2\mathcal{O}$ is impossible).

4. Heights and degrees of Knörr lattices

Proof of Proposition 1.9. Let U be a Knörr lattice. Then, $\mathrm{tr}_U(\mathrm{End}_A(U)) = \mathrm{rank}(U)\mathcal{O}$. By Theorem 1.7(iii), we have that $\mathrm{tr}_U(\mathrm{End}_A(U)) = \pi^{n-a(U)}\mathcal{O}$. By Theorem 1.7(ii), we have

$$\frac{\mathrm{rank}(A)}{\pi^n} = \frac{\rho(1_A)}{\pi^n} \in \mathcal{O}.$$

It then follows that

$$\mathrm{rank}(A)\mathcal{O} \subseteq \pi^n\mathcal{O} \subseteq \pi^{n-a(U)}\mathcal{O} = \mathrm{rank}(U)\mathcal{O}.$$

This proves (i). Now suppose that U is a projective lattice. Then, by Theorem 1.7(iii) we have that $\mathrm{tr}_U(\mathrm{End}_A(U)) = \pi^n\mathcal{O}$. On the other hand, $\mathrm{rank}(U)\mathcal{O} \subseteq \mathrm{tr}_U(\mathrm{End}_A(U))$. This proves (ii). □

The next lemma is needed to prove Theorem 1.12.

Lemma 4.1. *Let A be a symmetric $\mathcal{O}$-algebra such that $K \otimes_{\mathcal{O}} A$ is split semisimple. Assume that A has the projective scalar property and let $\pi^n 1_A$ be a relative projective element with respect to some symmetrising form on A. Let $a_0 = \max_V\{a(V)\}$ as V ranges over all A-lattices. There exists $\chi \in \mathrm{Irr}_K(A)$ such that $\chi(1)\mathcal{O} = \pi^{n-a_0}\mathcal{O}$.*

Proof. By Theorem 1.7(ii), $a_0 \leq n$ and we have that $\chi(1)\mathcal{O} \subseteq \pi^{n-a_0}\mathcal{O}$ for all $\chi \in \mathrm{Irr}_K(A)$. Let U be an A-lattice and $\alpha \in \mathrm{End}_A(U)$ be such that $\mathrm{tr}_U(\alpha)\mathcal{O} = \pi^{n-a_0}\mathcal{O}$. Let $f \in \mathcal{O}[x]$ be the characteristic polynomial of α and let $g \in \mathcal{O}[x]$ be an irreducible monic factor of f. Let $\overline{K}$ be an algebraic closure of K, let λ_i, where $i \in \mathcal{I}$, be the roots of g in $\overline{K}$ and for $i \in \mathcal{I}$ let W_i be the generalised λ_i-eigenspace of α in $\overline{K} \otimes_{\mathcal{O}} U$.

Set $\lambda_g := \sum_{i\in\mathcal{I}} \lambda_i \in \mathcal{O}$ and $W_g := \oplus_{i\in\mathcal{I}} W_i$. We have $\dim(W_i) = \dim(W_j) =: d_g$ for all $i, j \in \mathcal{I}$. So,

$$\mathrm{tr}_{W_g}(\alpha) = \lambda_g d_g.$$

Since $\mathrm{tr}_U(\alpha)$ is the sum of $\mathrm{tr}_{W_g}(\alpha)$ as g runs through the irreducible factors of f and since $\lambda_g \in \mathcal{O}$, replacing g by some other irreducible factor of f if necessary, we may assume that

$$\pi^{n-a_0}\mathcal{O} = \mathrm{tr}_U(\alpha)\mathcal{O} \subseteq d_g\mathcal{O}.$$

Now $\alpha \in \mathrm{End}_A(U)$, hence U_i is a $\bar{K} \otimes_{\mathcal{O}} A$-submodule of $\bar{K} \otimes_{\mathcal{O}} A$. In particular, d_g is the dimension of a $\bar{K} \otimes_{\mathcal{O}} A$-module. Since $K \otimes_{\mathcal{O}} A$ is split it follows that there exists some $\chi \in I$ such that

$$\pi^{n-a_0}\mathcal{O} = \mathrm{tr}_U(\alpha)\mathcal{O} \subseteq d_g\mathcal{O} \subseteq \chi(1)\mathcal{O} \subseteq \pi^{n-a_0}\mathcal{O}.$$

Hence, $\chi(1)\mathcal{O} = \pi^{n-a_0}\mathcal{O}$ as desired. □

Proof of Theorem 1.12 . The fact that U' is a Knörr A'-lattice is a consequence of Theorem 1.2. Let $a_0 = \max_V\{a(V)\}$ as V ranges over all A-lattices. Then a_0 also equals $\max_{V'}\{a(V')\}$ as V' ranges over all A'-lattices. Further, $a(U) = a(U')$. Let $\pi^e\mathcal{O} = p\mathcal{O}$ and let $\pi^n 1_A$ be a relative projective element of A. For any A-lattice V, $\mathrm{rank}(V)\mathcal{O} \subseteq \mathrm{tr}_V(\mathrm{End}_A(V))$, hence by Lemma 4.1 and Theorem 1.7,

$$p^{(n-a_0)/e} = \min_V\{\mathrm{rank}(V)_p\}$$

as V ranges over all A-lattices. Since U is a Knörr A-lattice and using again Theorem 1.7, it follows that

$$\pi^{n-a(U)}\mathcal{O} = p^{(n-a_0)/e+h(U)}\mathcal{O}$$

and hence

$$h(U) = \frac{a_0 - a(U)}{e}.$$

Applying the same argument to A' and U' gives the desired result. □

Proof of Proposition 1.13. Let u be an element of $U \setminus \pi U$. Let $\varphi : U \to U$ be an $\mathcal{O}$-linear projection onto $\mathcal{O}u$, and let $\mathrm{Tr}_1^A(\varphi)$ be the corresponding A-endomorphism of U. A calculation similar to that in Proposition 2.1 and using the assumption that $\pi^n 1_A$ is a relative projective element as in Theorem 1.7, shows that

$$\mathrm{tr}_U(\mathrm{Tr}_1^A(\varphi)) = \pi^n.$$

Now because U is projective, we have $a(U) = 0$. It follows that

$$\mathrm{tr}_U(\mathrm{End}_A(U)) = \pi^n\mathcal{O}.$$

Because U is a Knörr lattice, we can conclude that $\mathrm{Tr}_1^A(\varphi)$ is an invertible element of $\mathrm{End}_A(U)$. In particular, it is surjective. However, the image of $\mathrm{Tr}_1^A(\varphi)$ is contained in the A-lattice Au. We thus have $Au = U$. The result follows because u was an arbitrary element of $U \setminus \pi U$. □

5. Examples

Example 5.1. If $A = \mathrm{Mat}_n(\mathcal{O})$ for some positive integer n or if $A = \mathcal{O}G$ for some finite group G, then A has the scalar projective property, see [Broué 2009, Examples and Remarks after Proposition 3.3]. If an $\mathcal{O}$-algebra A has the projective scalar property, and if B is a direct factor of A, then B has the projective scalar property. This is immediate from the fact that the relative projective element with respect to a symmetrising form on A is independent of the choice of an $\mathcal{O}$-basis. If $\mathcal{O}$-algebras A and B have the projective scalar property, then so does $A \otimes_{\mathcal{O}} B$. However, the projective scalar property is not preserved under taking direct products, whilst the property of being symmetric is. For instance if $p = 2$, then by Proposition 1.9, $\mathcal{O} \times \mathrm{Mat}_2(\mathcal{O})$ does not have the projective scalar property. Further, $\mathcal{O} \times \mathrm{Mat}_2(\mathcal{O})$ is Morita equivalent to $\mathcal{O} \times \mathcal{O}$ from which we see that the scalar projective property is not invariant under Morita equivalence.

Example 5.2. Source algebras of blocks of finite groups have the projective scalar property. More precisely, if A is a source algebra of a block of a finite group algebra with defect group P, and k is a splitting field for the underlying finite group and its subgroups, then there is a symmetrising form on A such that the relative projective element of A is equal to $|P| \cdot 1$. To see this, let G be a finite group, B a block algebra of $\mathcal{O}G$, P a defect group of B, and i a source idempotent of B; that is, i is a primitive idempotent in B^P satisfying $\mathrm{Br}_P(i) \neq 0$, where $\mathrm{Br}_P : (\mathcal{O}G)^P \to kC_G(P)$ is the Brauer homomorphism. Assume that k is a splitting field for G and all of its subgroups. The source algebra $A = i\mathcal{O}Gi$ is again symmetric, and any symmetrising form on $\mathcal{O}G$ restricts to a symmetrising form on A. Denote by $s : \mathcal{O}G \to \mathcal{O}$ the canonical symmetrising form, sending 1_G to $1_{\mathcal{O}}$ and $x \in G \setminus \{1_G\}$ to zero. With respect to this form, the relative trace $\mathrm{Tr}_1^{\mathcal{O}G}$ on $\mathcal{O}G$ is equal to the relative trace map Tr_1^G, sending $a \in \mathcal{O}G$ to $\sum_{x \in G} xax^{-1}$. The relative trace map Tr_1^A with respect to the symmetrising form s restricted to A satisfies $\mathrm{Tr}_1^A(a) = \mathrm{Tr}_1^G(a)i$. In particular, we have $\mathrm{Tr}_1^A(i) = \mathrm{Tr}_1^G(i)i$. As a consequence of [Picaronny and Puig 1987] or [Thévenaz 1988, 9.3], the element $u = \mathrm{Tr}_P^G(i)$ is invertible in $Z(B)$. Moreover, we have $\mathrm{Tr}_1^G(i) = |P|\mathrm{Tr}_P^G(i) = |P|u$. Denote by t the symmetrising form given by $t(a) = s(ua)$. The relative trace map on A with respect to the form t sends the unit element i of A to $|P|uu^{-1}i = |P|i$ as required.

Example 5.3. If A is a Hopf algebra over $\mathcal{O}$ such that $K \otimes_{\mathcal{O}} A$ is semisimple, then A has the projective scalar property. This is well known to Hopf algebra experts — we just sketch the trail of ideas. By [Larson and Radford 1988a, Theorem 3.3] and [Larson and Radford 1988b, Theorem 4], the antipode of $K \otimes_{\mathcal{O}} A$ and of $K \otimes_{\mathcal{O}} A^* = (K \otimes_{\mathcal{O}} A)^*$ has order 2. Hence the same is true for the antipode of A and A^*. By the main theorem of [Larson and Sweedler 1969], A has a nonsingular left integral, say, λ. Then λ is also a nonsingular left integral for $K \otimes A$. Hence by

Propositions 3 and 4 of the same paper, $\epsilon(\lambda) \neq 0$ and A is unimodular. Since the antipode of A^* also has order 2, by the second corollary to [Larson and Sweedler 1969, Proposition 8], applied with the roles of A and A^* reversed, we have that if $\Lambda \in A^*$ is a nonsingular integral (Λ exists by the main theorem of [Larson and Sweedler 1969] applied to A^*), then Λ is a symmetrising form on A. Further, by [Lorenz 2011, Section 5.3], the corresponding projective element is a scalar.

Example 5.4. This example shows that very few local commutative symmetric $\mathcal{O}$-algebras of $\mathcal{O}$-rank 2 have the projective scalar property. Let A be an indecomposable $\mathcal{O}$-algebra such that $K \otimes_{\mathcal{O}} A = K \times K$; in particular, A is commutative. Then there is a unique positive integer m such that

$$A = \{(\alpha, \beta) \in \mathcal{O} \times \mathcal{O} \mid \beta - \alpha \in \pi^m \mathcal{O}\} = \{(\alpha, \alpha + \beta) \mid \alpha \in \mathcal{O},\ \beta \in \pi^m \mathcal{O}\}.$$

The algebra A is local commutative and symmetric, with symmetrising form s sending $(\alpha, \alpha + \beta) \in A$ to $\pi^{-m}\beta$. We are going to show that A has the projective scalar property if and only if $p = 2$ and $2 \in \pi^m \mathcal{O}$.

The $\mathcal{O}$-basis $X = \{(1, 1), (0, \pi^m)\}$ of A has, with respect to s, the dual basis $\{(-\pi^m, 0), (1, 1)\}$. Thus the relative projective element with respect to the symmetrising form s is $z_A = (-\pi^m, \pi^m)$. We have $A^\times = \{(\alpha, \alpha + \pi^m \gamma) \mid \alpha \in \mathcal{O}^\times, \gamma \in \mathcal{O}\}$. Thus the $A^\times$-orbit of z_A is

$$\{-\pi^m \alpha, \pi^m \alpha + \pi^{2m} \gamma \mid \alpha \in \mathcal{O}^\times,\ \gamma \in \mathcal{O}\}.$$

An element in this set is a scalar if and only if $\pi^m \gamma = -2\alpha$. For p odd, this is impossible as the right side is invertible in $\mathcal{O}$ whereas the left side has a positive valuation of at least m. This shows that for p odd, A does not have the projective scalar property. For $p = 2$, the algebra A has the scalar property if and only if π^m divides 2 in $\mathcal{O}$.

Note that since A is local, any $\mathcal{O}$-algebra Morita equivalent to A is a matrix algebra over A. Hence if A does not have the projective scalar property, then neither does any algebra Morita equivalent to A.

Example 5.5. Let (W, S) be a finite Coxeter group with length function ℓ and $q \in \mathcal{O}^\times$. Let $\mathcal{H} = \mathcal{H}_q(W, S)$ be the associated Iwahori–Hecke algebra over $\mathcal{O}$ with parameter q. That is, $\mathcal{H}$ has an $\mathcal{O}$-basis $\{T_w\}_{w \in W}$, with multiplication given by $T_w T_y = T_{wy}$ if $w, y \in W$ such that $\ell(wy) = \ell(w) + \ell(y)$, and $(T_s)^2 = qT_1 + (1-q)T_s$ for $s \in S$. By [Geck and Pfeiffer 2000, Proposition 8.1.1], the algebra $\mathcal{H}$ is symmetric, with a symmetrising form sending T_1 to 1 and T_w to 0 for $w \in W \setminus \{1\}$. The dual basis of $\{T_w\}_{w \in W}$ with respect to this form is $\{q^{-\ell(w)} T_{w^{-1}}\}_{w \in W}$, and hence the associated relative projective element is

$$z_{\mathcal{H}} = \sum_{w \in W} q^{-\ell(w)} T_w T_{w^{-1}}.$$

Whether $\mathcal{H}$ has the projective scalar property seems to be difficult to read off this expression. If $p = 2$ and $W = S_2 = S = \{1, s\}$, and if q is an odd integer, then the map sending T_1 to $(1, 0)$ and T_s to $(1, 1-q)$ is an injective algebra homomorphism from $\mathcal{H}$ to $\mathcal{O} \times \mathcal{O}$. The previous example shows that $\mathcal{H}$ has the scalar property if and only if $q \equiv 3 \mod 4$.

Example 5.6. Let G be a finite group and assume that $\mathcal{O}$ contains the values of all irreducible characters of G. Let

$$A = \mathcal{O}[\mathrm{Irr}(G)] = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Irr}(G)].$$

The irreducible characters of G form an $\mathcal{O}$-basis for A. For K-valued functions α and β on G, define the usual

$$[\alpha, \beta] = \frac{1}{|G|} \sum_{g \in G} \alpha(g)\beta(g^{-1}) \in K.$$

For $\chi \in \mathrm{Irr}(G)$, let $\overline{\chi}$ denote the character of the contragredient representation, so $\overline{\chi}(g) = \chi(g^{-1})$ for all $g \in G$. Finally, let 1_G denote the trivial character of G.

For $\chi, \psi \in \mathrm{Irr}(G)$, the identity $[\chi\overline{\psi}, 1_G] = [\chi, \psi] = \delta_{\chi,\psi}$ implies that the $\mathcal{O}$-linear function $s : A \to \mathcal{O}$ given by

$$s(\alpha) = \text{coefficient of } 1_G \text{ in } \alpha$$

is a symmetrising form on A. The same identity makes it clear that the basis of A dual to $\mathrm{Irr}(G)$ with respect to s is given by $\chi^\vee = \overline{\chi}$. The corresponding relative projective element, $z = \sum_{\chi \in \mathrm{Irr}(G)} \chi\overline{\chi}$, coincides with the function on G sending g to $|C_G(g)|$. Clearly, z is a scalar multiple of 1_G if and only if G is abelian. In this case, we have $z = |G| \cdot 1_G$. However, to see exactly when A has the projective scalar property, it is necessary to consider the action of $A^\times$ on z. Let u be an invertible element of A. Then u is a function from G to $\mathcal{O}^\times$. Assume that $uz = \lambda \cdot 1_G$ for some element $\lambda \in \mathcal{O}$. We must then have

$$u(g) = \frac{\lambda}{|C_G(g)|} \in \mathcal{O}^\times \tag{7}$$

for all $g \in G$. Thus, $|C_G(g)|_p$ is independent of g. We deduce that every element of G must centralise a Sylow p-subgroup. So, let P be a Sylow p-subgroup. By Sylow's theorem, every element of G is conjugate to an element of $C_G(P)$. A well known application of Burnside's counting lemma allows us to conclude that $C_G(P) = G$. Thus, P is abelian, and $G \cong P \times H$ for some group H of p' order. Conversely, we claim that if $G = P \times H$, with P an abelian p-group and H a group of p'-order, then A has the projective scalar property. All that remains to do is to verify that the function $u(g) = 1/(|C_G(g)|_{p'})$ for $g \in G$ actually lies in A, assuming $G = P \times H$ as above. So let $\chi \in \mathrm{Irr}(G)$. We must show that $[\chi, u] \in \mathcal{O}$. We can write $\chi = \theta \otimes \psi$ for irreducible characters θ of P and ψ of H. One verifies

$$[\chi, u] = \begin{cases} \frac{1}{|H|} \sum_{h \in H} \frac{\psi(h)}{|C_H(h)|} & \text{if } \theta = 1_P, \\ 0 & \text{if } \theta \neq 1_P. \end{cases}$$

In both cases, we have $[\chi, u] \in \mathcal{O}$.

Finally, we remark that if $\mathcal{O}$ is a Dedekind domain in which no prime dividing the order of G is invertible, then A has the projective scalar property if and only if G is abelian.

Example 5.7. The Knörr property is not preserved by Morita equivalences in general. The idea is that all absolutely indecomposable A-lattices of p'-rank are Knörr, but among those of rank divisible by p, only the absolutely irreducible lattices tend to have the property. Indeed, the proof of [Knörr 1989, Corollary 1.6] does not require the $\mathcal{O}$-algebra to be a group ring (nor even a symmetric algebra). Thus, any Morita equivalence that sends a lattice of p'-rank which is indecomposable but not irreducible to a lattice of rank divisible by p is likely to give an example.

Specifically, let $p = 2$ and assume that $\mathcal{O}$ is unramified and k is algebraically closed. Let A be the principal block algebra of $\mathcal{O}A_5$, where A_5 is the alternating group of degree 5. Then $|\mathrm{Irr}_K(A)| = 4$, $|\mathrm{Irr}_k(A)| = 3$, and the decomposition matrix of A with respect to some ordering of $\mathrm{Irr}_K(A)$ is

(8)

	φ_1	φ_2	φ_3
χ_1	1	0	0
χ_2	1	0	1
χ_3	1	1	0
χ_4	1	1	1

where φ_1 corresponds to a one-dimensional kA-module, and φ_2 and φ_3 correspond to simple kA-modules of dimension 2.

For each i, where $1 \le i \le 3$, let P_i denote a projective indecomposable A-module such that $P_i/\mathrm{rad}(P_i)$ is isomorphic to a simple kA-module corresponding to φ_i. Let e be an idempotent in A such that $Ae \cong P_1 + 2P_2 + P_3$ as left A-modules. Then A and eAe are Morita equivalent via the functor sending an A-module M to the eAe-module eM and an A-module homomorphism $\alpha : M \to N$ to the eAe-module homomorphism $e \cdot \alpha : eM \to eN$ defined through restriction to eM. The simple eAe-modules corresponding to φ_1 and φ_3 have dimension 1 whereas the simple eAe-module corresponding to φ_2 has dimension 2.

From the decomposition matrix above, one sees that the character afforded by KP_1 has two irreducible constituents, one of degree 5 and the other of degree 3. It follows from [Knörr 1989, Lemma 1.9] that P_1 is not a Knörr A-lattice. However, the rank of the eAe-lattice eP_1 is 7. Thus eP_1 is a Knörr eAe-lattice.

To obtain an example in which neither lattice is projective, it is enough to inflate the P_i above to lattices for the group $A_5 \times C_2$, where C_2 is a cyclic group of order 2.

Notice also that although eP_1 is Knörr, it does not have the stable exponent property. This is the case for both the A_5 and $A_5 \times C_2$ situations. Next, we produce a lattice with the stable exponent property which is not Knörr.

First, we have $\mathbb{Q}(\sqrt{5}) \subseteq K$, so KA is split semisimple. Let M be the unique quotient lattice of P_1 such that KM has character $\chi_1+\chi_2+\chi_3$. Since M has rank 7, M is a Knörr A-lattice. Because A has the projective scalar property, M and hence eM also have the stable exponent property. We shall show that eM is not Knörr.

Let L be the unique $\mathcal{O}$-free quotient of M affording the character χ_3 and let $\alpha : M \to L$ be the projection map. Since α is surjective, and L and M are not projective, $\alpha \notin \mathrm{Hom}_A^{\mathrm{pr}}(M, L)$. Thus, by Corollary 1.4, there exists $\beta \in \mathrm{Hom}_A(L, M)$ such that $\mathrm{tr}_M(\beta \circ \alpha) \notin 4\mathcal{O}$ (since $4 \cdot 1_A$ is a projective scalar element of A).

Let $\tau = \beta\alpha$ and denote also by τ the K-linear extension of τ to KM. For each i, $1 \le i \le 4$, let e_i be the primitive central idempotent of KA corresponding to χ_i. Since $\tau(KM)$ is contained in $e_3(KM)$, we have that $(e_2+e_4)(KM)$ is contained in the kernel of τ. On the other hand, $1-e = (e_2+e_4)(1-e)$. Thus, $\mathrm{tr}_{KM}(\tau) = \mathrm{tr}_{e(KM)}(\tau)$. It follows that

$$\mathrm{tr}_{eM}(e \cdot \tau) = \mathrm{tr}_{KM}(\tau) = \mathrm{tr}_M(\tau) \notin 4\mathcal{O}.$$

Since eM has rank 6, we have that $\nu_2(\mathrm{tr}_{eM}(e\cdot\tau)) \le \nu_2(\mathrm{rank}_{\mathcal{O}}(eM))$. Since τ is not invertible, neither is $e \cdot \tau$, hence eM is not a Knörr eAe-lattice.

Example 5.8. Let $\mathcal{O} = \mathbb{Z}_3$, and consider the $\mathcal{O}$-order $A = \mathcal{O}S_3$, that is, the group ring of the symmetric group on three points. The decomposition matrix of A is

(9)

	$\varphi_{(3)}$	$\varphi_{(2,1)}$
$\chi_{(3)}$	1	0
$\chi_{(2,1)}$	1	1
$\chi_{(1^3)}$	0	1

Here we use the standard indexing of ordinary and modular irreducible characters of symmetric groups via partitions. Let $e_{(3)}$, $e_{(2,1)}$ and $e_{(1^3)}$ denote the primitive idempotents in $Z(KA)$. The inertial index of this block is 2, and, according to [Bessenrodt 1982], that means that this block has six isomorphism types of indecomposable lattices (one can also show this in an elementary way). It is also easy to enumerate those isomorphism types: there are two indecomposable projective lattices, which are nonirreducible. Then there is a unique lattice with character $\chi_{(3)}$ and a unique lattice with character $\chi_{(1^3)}$. Moreover there is a lattice with character $\chi_{(2,1)}$ whose top has Brauer character $\varphi_{(3)}$ and there is a lattice with character $\chi_{(2,1)}$ whose top has Brauer character $\varphi_{(2,1)}$ (those two lattices are the projective lattices over the order $Ae_{(2,1)}$). As there are but six lattices in total we know that there can be no further indecomposable lattices. In particular, all indecomposable lattices are either projective or absolutely irreducible. This implies that each algebra in the Morita equivalence class of A has the property that Knörr-lattices and absolutely indecomposable nonprojective lattices with the stable exponent property coincide. Any algebra in the Morita equivalence class of A which does not possess the projective scalar property will therefore provide a counterexample to the converse of Theorem 1.2.

Choose B in the Morita equivalence class of A such that the Morita equivalence sends the simple module with character $\varphi_{(3)}$ to a one-dimensional module and the simple module with character $\varphi_{(2,1)}$ to a two-dimensional module. Note that

$$\tfrac{1}{3}\cdot\big(\chi_{(3)}(-)+2\cdot\chi_{(2,1)}(-)+\chi_{(1^3)}(-)\big) \tag{10}$$

is a symmetrising form for A, and therefore also for B (with the characters replaced by the corresponding characters of B). It follows that

$$z_B=3\cdot\big(e_{(3)}+\tfrac{3}{2}\cdot e_{(2,1)}+2\cdot e_{(1^3)}\big) \tag{11}$$

and this element is determined uniquely up to multiplications by units in $Z(A)=Z(B)$. But multiplication by units cannot turn the above element into a scalar, since it will leave the 3-valuation of the coefficients of the idempotents $e_{(3)}$, $e_{(2,1)}$ and $e_{(1^3)}$ invariant. Hence B does not possess the projective scalar property.

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COACTION FUNCTORS, II

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In their study of the application of crossed-product functors to the Baum–Connes conjecture, Buss, Echterhoff, and Willett introduced various properties that crossed-product functors may have. Here we introduce and study analogues of some of these properties for coaction functors, making sure that the properties are preserved when the coaction functors are composed with the full crossed product to make a crossed-product functor. The new properties for coaction functors studied here are functoriality for generalized homomorphisms and the correspondence property. We also study the connections with the ideal property. The study of functoriality for generalized homomorphisms requires a detailed development of the Fischer construction of maximalization of coactions with regard to possibly degenerate homomorphisms into multiplier algebras. We verify that all "KLQ" functors arising from large ideals of the Fourier–Stieltjes algebra $B(G)$ have all the properties we study, and at the opposite extreme we give an example of a coaction functor having none of the properties.

1. Introduction

As part of their study of the Baum–Connes conjecture, [Baum et al. 2016] considered *exotic crossed products* between the full and reduced crossed products of a C^*-dynamical system, and a crucial feature was that the construction be *functorial* for equivariant homomorphisms. In [Kaliszewski et al. 2016a], we introduced a two-step construction of crossed-product functors: first form the full crossed product, then apply a *coaction functor*. Although this recipe does not give all crossed-product functors, there is some evidence that it might produce the functors that are most important for the program of [Baum et al. 2016].

In [Baum et al. 2016], the applications to the Baum–Connes conjecture lead to the desire that the crossed-product functors be *exact* and *Morita compatible*, and it was proved that there is a smallest (for a suitable partial ordering) crossed product with these properties. The idea is that every family of crossed-product functors has a greatest lower bound, and that exactness and Morita compatibility are preserved

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by greatest lower bounds. In [Kaliszewski et al. 2016a] we proved analogues of these facts for coaction functors.

In further study of the application of crossed-product functors to the Baum–Connes conjecture, Buss et al. [2014] studied various other properties that crossed-product functors may have. This motivated us to investigate in the current paper the analogous properties of coaction functors.

There is a subtlety regarding the appropriate choices of categories. To study short exact sequences, the morphisms should be homomorphisms between the C^*-algebras themselves, and we call the resulting categories *classical*. On the other hand, some of the properties considered in [Buss et al. 2014] (hereafter cited as [BEW]) require homomorphisms into multiplier algebras. Most of the literature on noncommutative C^*-crossed-product duality uses *nondegenerate categories*, where the morphisms are nondegenerate homomorphisms into multiplier algebras; the nondegeneracy guarantees that the maps can be composed. On the other hand, for some of the properties studied in [BEW] it is actually important to allow *possibly degenerate* homomorphisms into multiplier algebras. Of course this is problematic in terms of composing morphisms, but nevertheless Buss et al. introduced a reasonable notation of *functoriality for generalized homomorphisms*, involving such possibly degenerate homomorphisms. In this paper we chose to develop the theory along three parallel tracks: first we prove what we can in the context of generalized homomorphisms, then we specialize to the classical and the nondegenerate categories. However, our main interest is in the classical categories, and for much of this paper the classical case will be our default, with occasional mention of nondegenerate categories.

Nondegenerate equivariant categories have been well studied, but (perhaps unexpectedly) the classical counterparts have not, especially in noncommutative crossed-product duality. In [Kaliszewski et al. 2016a], we began to fill in some of these gaps in the theory of classical categories, and here we will continue this, to prepare the way for our study of analogues for coaction functors of some of the properties introduced in [BEW]. In [Kaliszewski et al. 2016a], we gave a brief indication of how maximalization of coactions is a functor on the classical category of coactions, which we make more precise in Section 3.

We begin Section 2 by recording a few of our conventions for coactions and actions. We also discuss the distinction between *nondegenerate* and *classical* categories of C^*-algebras with extra structure. For the study of exactness of coaction functors, the classical categories are appropriate, so we focus upon them in this paper. Coaction functors involve maximalization of coactions, and we outline Fischer's construction of maximalization as a composition of three simpler functors. We finish Section 2 with a short discussion of coaction functors, taken from [Kaliszewski et al. 2016a; 2016b]. In particular, we recall a few properties that coaction functors may have: *exactness*, *Morita compatibility*, and the *ideal property*. The first of these occupies a central position in the application of coaction functors to the crossed-product functors of [Baum et al. 2016], while the second and third are analogues of properties of action-crossed-product functors discussed in [BEW]. In Proposition 2.3, we record a more precise statement of a result in [Kaliszewski et al. 2016a] regarding greatest lower bounds of exact or Morita compatible coaction functors. The whole point of coaction functors is that they give a large (albeit not exhaustive) source of crossed-product functors in the sense of [Baum et al. 2016]. There are numerous open problems regarding the relationship between these two types of functors, and in Section 2 we mention one of these, involving greatest lower bounds. We also recall another type of coaction functor: *decreasing*, which include those coaction functors arising from *large ideals* of the Fourier–Stieltjes algebra $B(G)$; the associated crossed-product functors for actions have been referred to as "KLQ functors" [Buss et al. 2014; 2016] or "KLQ crossed products" [Baum et al. 2016].

In Section 3, we discuss how to maximalize possibly degenerate equivariant homomorphisms into multiplier algebras, with an eye toward developing an analogue for coaction functors of the *functoriality for generalized homomorphisms* discussed in [BEW]. This requires consideration of generalized homomorphisms for each of the three steps in the Fischer construction. As a side benefit, we close Section 3 by remarking how Theorem 3.9 gives a more precise justification than the one in [Kaliszewski et al. 2016a, Section 3] that maximalization is a functor on the classical category of coactions.

In Section 4, we introduce an analogue for coaction functors of the property called *functoriality for generalized homomorphisms* in [BEW]. Here the term "generalized homomorphism" refers to a possibly degenerate homomorphism $\phi : A \to M(B)$; these are somewhat delicate, and some care must be exercised in dealing with them. We prove some analogues for coaction functors of results of [BEW]; for example, coaction functors that are functorial for generalized homomorphisms in the sense of Definition 4.1 satisfy a limited version of the usual composability aspect of actual functors, and every functor arising from a large ideal of $B(G)$ has this generalized functoriality property. We also give a further discussion of the ideal property, in particular proving that it is implied by functoriality for generalized homomorphisms. This is weaker than the corresponding result of [BEW], namely that for crossed-product functors these two properties are equivalent. We also prove that both the ideal property and functoriality for generalized homomorphisms are inherited by greatest lower bounds.

In Section 5, we introduce the *correspondence property* for coaction functors, which is an analogue of the *correspondence crossed-product functors* of [BEW]. This is much stronger than Morita compatibility, and we need to do a bit of work to develop it. As a side benefit of this work, we prove that if a coaction functor is Morita compatible then the associated crossed-product functor for actions is strongly Morita compatible in the sense of [BEW], and we also prove a technical lemma showing that, in the presence of the ideal property, the test for Morita compatibility can be relaxed somewhat. We prove that a coaction functor has the correspondence property if and only if it is both Morita compatible and functorial for generalized homomorphisms, which is an analogue of a similar equivalence for crossed-product functors in [BEW]. It follows that if a coaction functor has the correspondence property then the associated crossed-product functor for actions is a correspondence crossed-product functor in the sense of [BEW]. Among the consequences, we deduce that every coaction functor arising from a large ideal of $B(G)$ has the correspondence property, and that the correspondence property is inherited by greatest lower bounds, so that in particular there is a smallest coaction functor with the correspondence property. Also, a result of [BEW] showing that the output of a correspondence crossed-product functor carries a quotient of the dual coaction on the full crossed product strengthens our belief that the most important crossed-product functors are those arising from coaction functors.

2. Preliminaries

Throughout, G will be a locally compact group, A, B, C, D will be C^*-algebras, actions of G are denoted by letters such as α, β, γ, and coactions of G by letters such as δ, ϵ, ζ. Throughout, we assume that G is second countable, so that the Hilbert space $L^2(G)$ will be separable; second countability of G is needed for the use of Fischer's result, and in that proof separability of $L^2(G)$ is essential. We refer to [Echterhoff et al. 2004; 2006, Appendix A] for conventions regarding actions and coactions, and to [Echterhoff et al. 2006, Chapters 1–2] for C^*-correspondences[1] and imprimitivity bimodules.

We write $A \rtimes_\alpha G$ for the crossed product of an action (A, α), and (i_A, i_G) for the universal covariant homomorphism from (A, G) to the multiplier algebra $M(A \rtimes_\alpha G)$, occasionally writing i_G^α to avoid ambiguity. We write $\hat{\alpha}$ for the dual coaction.

We write $A \rtimes_\delta G$ for the crossed product of a coaction (A, δ), and (j_A, j_G) for the universal covariant homomorphism from $(A, C_0(G))$ to $M(A \rtimes_\delta G)$, occasionally writing j_G^δ to avoid ambiguity. We write $\hat{\delta}$ for the dual action.

Given a coaction (A, δ), we find it convenient to use the associated $B(G)$-module structure given by

$$f \cdot a = (\mathrm{id} \otimes f) \circ \delta(a) \quad \text{for } f \in B(G), a \in A,$$

and in [Kaliszewski et al. 2016a, Appendix A] we recorded a few properties. We will need the following mild strengthening of [Kaliszewski et al. 2016a, Proposition A.1]:

[1] These are called *right-Hilbert bimodules* in [Echterhoff et al. 2006].

Proposition 2.1. *Let (A,δ) and (B,ϵ) be coactions of G, and let $\phi : A \to M(B)$ be a homomorphism. Then ϕ is $\delta - \epsilon$ equivariant if and only if it is a module map, that is,*

$$\phi(f \cdot a) = f \cdot \phi(a) \quad \text{for all } f \in B(G),\ a \in A.$$

Proof. As we mentioned in [Kaliszewski et al. 2016b, proof of Lemma 3.17], the argument of [Kaliszewski et al. 2016a, Proposition A.1] carries over, with the minor adjustment that in the expression "$(\mathrm{id} \otimes f)((\phi \otimes \mathrm{id}) \circ \delta(a))$" there, the map $\phi \otimes \mathrm{id}$ must be replaced by the canonical extension

$$\overline{\phi \otimes \mathrm{id}} : \widetilde{M}(A \otimes C^*(G)) \to M(B \otimes C^*(G)),$$

which exists by [Echterhoff et al. 2006, Proposition A.6], and where we recall the notation

$$\widetilde{M}(A \otimes C^*(G)) \\ = \{m \in M(A \otimes C^*(G)) : m(1 \otimes C^*(G)) \cup (1 \otimes C^*(G))m \subset A \otimes C^*(G)\}. \quad \square$$

Classical and nondegenerate categories. In all of our categories, the objects will be C^*-algebras, usually equipped with some extra structure, and the morphisms will be homomorphisms that preserve this extra structure in some sense. We consider two main types of homomorphisms: *nondegenerate* homomorphisms $\phi : A \to M(B)$, and what we call *classical* homomorphisms $\phi : A \to B$, and these give rise to what we call *nondegenerate* and *classical* categories, respectively. We are concerned mainly with the classical case, but occasionally we will refer to the nondegenerate case, and sometimes we will develop the two in parallel. We also need to consider what Buss, Echterhoff, and Willett call *generalized homomorphisms* $\phi : A \to M(B)$, which are allowed to be degenerate. Perhaps surprisingly, in the noncommutative crossed-product duality literature, the nondegenerate categories are used almost exclusively; here we will devote more attention to developing the tools we need for the classical categories.

Warning: in this paper we will slightly modify some of the notation from [Kaliszewski et al. 2016a]: given a coaction (A,δ), recall from [Echterhoff et al. 2004] that δ is called *maximal* if the canonical map $\Phi : A \rtimes_\delta G \rtimes_{\widehat{\delta}} G \to A \otimes \mathcal{K}(L^2(G))$ is an isomorphism. Recall also that an arbitrary (A,δ) has a *maximalization*, which is a maximal coaction (A^m,δ^m) and a $\delta^m - \delta$ equivariant surjection, which we will write as $\psi_A : A^m \to A$, rather than q^m_A, having the property that

$$\psi_A \rtimes G : A^m \rtimes_{\delta^m} G \to A \rtimes_\delta G$$

is an isomorphism. On the nondegenerate category of coactions, Fischer proves that ψ_A gives a natural transformation from maximalization to the identity functor; in [Kaliszewski et al. 2016a] we stated this for the classical category, and we will make this more precise in Theorem 3.9.

On the other hand, we will use the same notation as in [Kaliszewski et al. 2016a] for the surjections $\Lambda_A : A \to A^n$ giving a natural transformation from the identity functor to the normalization functor $(A, \delta) \mapsto (A^n, \delta^n)$ (for both the classical and the nondegenerate categories).

Given a coaction (A, δ), we call a C^*-subalgebra B of $M(A)$ *strongly δ-invariant* if

$$\overline{\text{span}}\{\delta(B)(1 \otimes C^*(G))\} = B \otimes C^*(G),$$

in which case, by [Quigg 1994, Lemma 1.6], δ restricts to a coaction δ_B on B. If I is a strongly δ-invariant ideal of A, then by [Nilsen 1999, Propositions 2.1 and 2.2, Theorem 2.3] (see also [Landstad et al. 1987, Proposition 4.8]), $I \rtimes_{\delta_I} G$ can be naturally identified with an ideal of $A \rtimes_\delta G$, and δ descends to a coaction δ^I on A/I in such a manner that

$$0 \to I \rtimes_{\delta_I} G \to A \rtimes_\delta G \to (A/I) \rtimes_{\delta^I} G \to 0$$

is a short exact sequence in the classical category of C^*-algebras.

Remark 2.2. Given a coaction (A, δ) and an ideal I of A, the existence of a coaction δ^I on the quotient A/I such that the quotient map $A \to A/I$ is $\delta - \delta^I$ equivariant is a weaker condition than the above strong invariance, and when it is satisfied we say that δ descends to a coaction on A/I.

The Fischer construction. For convenient reference we record the following rough outline of Fischer's construction of the maximalization of a coaction (A, δ) [Fischer 2004, Section 6] (see also [Kaliszewski et al. 2016c; 2017]). First of all, letting $\mathcal{K}$ denote the algebra of compact operators on a separable infinite-dimensional Hilbert space, a *$\mathcal{K}$-algebra* is a pair (A, ι), where A is a C^*-algebra and $\iota : \mathcal{K} \to M(A)$ is a nondegenerate homomorphism. Given a $\mathcal{K}$-algebra (A, ι), the *A-relative commutant of $\mathcal{K}$* is

$$C(A, \iota) := \{m \in M(A) : m\iota(k) = \iota(k)m \in A \quad \text{for all } k \in \mathcal{K}\}.$$

The *canonical isomorphism* $\theta_A : C(A, \iota) \otimes \mathcal{K} \xrightarrow{\simeq} A$ is determined by

$$\theta_A(a \otimes k) = a\iota(k)$$

for $a \in C(A, \iota)$, $k \in \mathcal{K}$ (see [Fischer 2004, Remark 3.1; Kaliszewski et al. 2016c, Proposition 3.4]). If $(B, \jmath)$ is another $\mathcal{K}$-algebra and $\phi : A \to M(B)$ is a nondegenerate homomorphism such that $\phi \circ \iota = \jmath$, then there is a unique nondegenerate homomorphism $C(\phi) : C(A, \iota) \to M(C(B, \jmath))$ making the diagram

$$\begin{tikzcd}
A \arrow[r, "\phi"] & M(B) \\
C(A, \iota) \otimes \mathcal{K} \arrow[u, "\theta_A"] \arrow[r, "C(\phi)\otimes\text{id}"'] & M(C(B, \jmath) \otimes \mathcal{K}) \arrow[u, "\theta_B"']
\end{tikzcd}$$

commute.

A $\mathcal{K}$*-coaction* is a triple (A,δ,ι), where (A,δ) is a coaction and (A,ι) is a $\mathcal{K}$-algebra such that $\delta\circ\iota=\iota\otimes 1$. If (A,δ,ι) is a $\mathcal{K}$-coaction, then the relative commutant $C(A,\iota)$ is strongly δ-invariant, and the restricted coaction $C(\delta)=\delta_{C(A,\iota)}$ is maximal if δ is, and θ_A is $(C(\delta)\otimes_* \mathrm{id})-\delta$ equivariant [Kaliszewski et al. 2017, Lemma 3.2].

An *equivariant action* is a triple (A,α,μ), where (A,α) is an action of G and $\mu : C_0(G)\to M(A)$ is a nondegenerate $\mathrm{rt}-\alpha$ equivariant homomorphism, and where, in turn, rt is the action of G on $C_0(G)$ given by $\mathrm{rt}_s(f)(t)=f(ts)$.

A *cocycle* for a coaction (A,δ) is a unitary element $U\in M(A\otimes C^*(G))$ such that

$$(\mathrm{id}\otimes\delta_G)(U)=(U\otimes 1)(\delta\otimes\mathrm{id})(U)\quad\text{and}\quad \mathrm{Ad}\,U\circ\delta(A)(1\otimes C^*(G))\subset A\otimes C^*(G).$$

Then $\mathrm{Ad}\,U\circ\delta$ is a coaction on A, and is Morita equivalent to δ, and hence is maximal if and only if δ is. If U is a δ-cocycle, (B,ϵ) is another coaction, and $\phi: A\to M(B)$ is a nondegenerate $\delta-\epsilon$ equivariant homomorphism, then $(\phi\otimes\mathrm{id})(U)$ is an ϵ-cocycle and ϕ is $\mathrm{Ad}\,U\circ\delta-\mathrm{Ad}(\phi\otimes\mathrm{id})(U)\circ\epsilon$ equivariant.

Given an equivariant action (A,α,μ), the unitary element

$$V_A:=((i_A\circ\mu)\otimes\mathrm{id})(w_G)$$

is an $\hat{\alpha}$-cocycle, and we write $\tilde{\alpha}=\mathrm{Ad}\,V_A\circ\hat{\alpha}$. Then $(A\rtimes_\alpha G,\tilde{\alpha},\mu\rtimes G)$ is a maximal $\mathcal{K}$-coaction [Kaliszewski et al. 2017, Lemma 3.1].

Now, if (A,δ) is a coaction, then $(A\rtimes_\delta G,\hat{\delta},j_G)$ is an equivariant action, so

$$(A\rtimes_\delta G\rtimes_{\hat{\delta}} G,\tilde{\hat{\delta}},j_G\rtimes G)$$

is a $\mathcal{K}$-coaction, and hence

$$(A^m,\delta^m):=(C(A\rtimes_\delta G\rtimes_{\hat{\delta}} G,j_G\rtimes G),C(\tilde{\hat{\delta}}))$$

is a maximal coaction. Letting

$$\Phi_A : A\rtimes_\delta G\rtimes_{\hat{\delta}} G\to A\otimes\mathcal{K}$$

be the *canonical surjection*, which is $\tilde{\hat{\delta}}-(\delta\otimes_*\mathrm{id})$ equivariant, Fischer proves that there is a unique $\delta^m-\delta$ equivariant surjective homomorphism $\psi_A : A^m\to A$ such that the diagram

$$\begin{tikzcd}
& A\rtimes_\delta G\times_{\hat{\delta}} G \arrow[dr, "\Phi_A"] & \\
A^m\otimes\mathcal{K} \arrow[ur, "\theta_{A\rtimes_\delta G\rtimes_{\hat{\delta}} G}"] \arrow[rr, "\psi_A\otimes\mathrm{id}"'] & & A\otimes\mathcal{K}
\end{tikzcd}$$

commutes, and moreover $\psi_A : (A^m,\delta^m)\to(A,\delta)$ is a maximalization of (A,δ). Fischer goes on to prove that maximalization is a functor on the nondegenerate category of coactions, by showing that if $\phi : A \to M(B)$ is a nondegenerate $\delta - \epsilon$ equivariant homomorphism then there is a unique homomorphism

$$\phi^m : A^m \to M(B^m)$$

making the diagram

$$\begin{tikzcd}
 & A \rtimes_\delta G \rtimes_{\hat{\delta}} G \arrow[dr, "\Phi_A"] \arrow[dd, "\phi \rtimes G \rtimes G"] & \\
A^m \otimes \mathcal{K} \arrow[ur, "\theta_{A \rtimes_\delta G \rtimes_{\hat{\delta}} G}", "\simeq"'] \arrow[rr, "\psi_A \otimes \mathrm{id}"'] \arrow[dd, "\phi^m \otimes \mathrm{id}"'] & & A \otimes \mathcal{K} \arrow[dd, "\phi \otimes \mathrm{id}"] \\
 & M(B \rtimes_\epsilon G \rtimes_{\hat{\epsilon}} G) \arrow[dr, "\Phi_B"] & \\
M(B^m \otimes \mathcal{K}) \arrow[ur, "\simeq", "\theta_{B \rtimes_\epsilon G \rtimes_{\hat{\epsilon}} G}"'] \arrow[rr, "\psi_B \otimes \mathrm{id}"'] & & M(B \otimes \mathcal{K})
\end{tikzcd}$$

commute. Consequently, the diagram

$$\begin{tikzcd}
A^m \arrow[r, "\phi^m"] \arrow[d, "\psi_A"'] & M(B^m) \arrow[d, "\psi_B"] \\
A \arrow[r, "\phi"'] & M(B)
\end{tikzcd}$$

also commutes, and ϕ^m is nondegenerate and $\delta^m - \epsilon^m$ equivariant.

Coaction functors. A functor $\tau : (A, \delta) \mapsto (A^\tau, \delta^\tau)$, $\phi \mapsto \phi^\tau$ on the classical category of coactions is a *coaction functor* if it fits into a commutative diagram

$$\begin{tikzcd}
 & (A^m, \delta^m) \arrow[dl, "\psi_A"'] \arrow[dr, "q_A^\tau"] & \\
(A, \delta) \arrow[dr, "\Lambda_A"'] & & (A^\tau, \delta^\tau) \arrow[dl, "\Lambda_A^\tau"] \\
 & (A^n, \delta^n) &
\end{tikzcd} \tag{2-1}$$

of surjective natural transformations. In [Kaliszewski et al. 2016a, Lemma 4.3], we proved that the existence of the natural transformation Λ^τ is automatic, provided we insist that $\ker q_A^\tau \subset \ker \Lambda_A \circ \psi_A$.

We observed in [Kaliszewski et al. 2016a, Example 4.2] that maximalization, normalization, and the identity functor are all coaction functors.

Given two coaction functors τ and σ, we say σ is *smaller* than τ, written $\sigma \le \tau$, if there is a natural transformation $\Gamma^{\tau,\sigma}$ fitting into commutative diagrams

$$\begin{tikzcd}
& (A^m, \delta^m) \arrow[dl, "q_A^\tau"'] \arrow[dr, "q_A^\sigma"] & \\
(A^\tau, \delta^\tau) \arrow[rr, "\Gamma_A^{\tau,\sigma}"] \arrow[dr, "\Lambda_A^\tau"'] & & (A^\sigma, \delta^\sigma) \arrow[dl, "\Lambda_A^\sigma"] \\
& (A^n, \delta^n) &
\end{tikzcd}$$

in other words, $\ker q_A^\tau \subset \ker q_A^\sigma$. In [Kaliszewski et al. 2016a, Theorem 4.9], we proved that every nonempty family $\mathcal{T}$ of coaction functors has a greatest lower bound $\operatorname{glb}\mathcal{T}$, characterized by

$$\ker q^{\operatorname{glb}\mathcal{T}} = \overline{\operatorname*{span}_{\tau\in\mathcal{T}}} \ker q^\tau.$$

A coaction functor τ is *exact* [Kaliszewski et al. 2016a, Definition 4.10] if for every short exact sequence

$$0 \to (I, \gamma) \xrightarrow{\phi} (A, \delta) \xrightarrow{\psi} (B, \epsilon) \to 0$$

in the classical category of coactions the image

$$0 \to (I^\tau, \gamma^\tau) \xrightarrow{\phi^\tau} (A^\tau, \delta^\tau) \xrightarrow{\psi^\tau} (B^\tau, \epsilon^\tau) \to 0$$

under τ is also exact. Maximalization is exact, see [Kaliszewski et al. 2016a, Theorem 4.11].

A coaction functor τ is *Morita compatible* (as defined in [Kaliszewski et al. 2016a, Definition 4.16]) if for every $(A, \delta) - (B, \epsilon)$ imprimitivity-bimodule coaction (X, ζ), with associated $(A^m, \delta^m) - (B^m, \epsilon^m)$ imprimitivity-bimodule coaction (X^m, ζ^m), the Rieffel correspondence of ideals satisfies

$$\ker q_A^\tau = X^m\text{-}\operatorname{Ind} \ker q_B^\tau,$$

equivalently there are an $A^\tau - B^\tau$ imprimitivity bimodule X^τ and a surjective $q_A^\tau - q_B^\tau$ compatible imprimitivity-bimodule homomorphism $q_X^\tau : X^m \to X^\tau$ [Kaliszewski et al. 2016a, Lemma 4.19]. Trivially, maximalization is Morita compatible, and routine linking-algebra techniques show that the identity functor is Morita compatible [Kaliszewski et al. 2016a, Lemma 4.21]. In [Kaliszewski et al. 2016a, Theorem 4.22], we proved that the greatest lower bound of the family of all exact and Morita compatible coaction functors is itself exact and Morita compatible. It is easy to check that the arguments can be used to prove the following more precise statement:

Proposition 2.3. *Let $\mathcal{T}$ be a nonempty family of coaction functors. If every functor in $\mathcal{T}$ is exact, then so is* $\operatorname{glb}\mathcal{T}$, *and if every functor in $\mathcal{T}$ is Morita compatible then so is* $\operatorname{glb}\mathcal{T}$.

In particular, there are both a smallest exact coaction functor and a smallest Morita compatible coaction functor.

Every coaction functor τ determines a crossed-product functor CP^τ on actions by composing with the full-crossed-product functor $(A,\alpha) \mapsto (A \rtimes_\alpha G, \hat{\alpha})$. If τ is exact or Morita compatible then so is CP^τ, and if $\tau \le \sigma$ then $\mathrm{CP}^\tau \le \mathrm{CP}^\sigma$. However, if $\mathcal{T}$ is a nonempty family of coaction functors, and $\mathcal{S} = \{\mathrm{CP}^\tau : \tau \in \mathcal{T}\}$ is the associated family of crossed-product functors, with respective greatest lower bounds $\operatorname{glb}\mathcal{S}$ and $\operatorname{glb}\mathcal{T}$, then

$$\mathrm{CP}^{\operatorname{glb}\mathcal{T}} \le \operatorname{glb}\mathcal{S},$$

but we do not know whether this is always an equality. In particular (see [Kaliszewski et al. 2016a, Question 4.25]), we do not know whether the smallest exact and Morita compatible crossed-product functor is naturally isomorphic to the composition with the full crossed product of the smallest exact and Morita compatible coaction functor.

A coaction functor τ is *decreasing* if there is a natural transformation Q^τ fitting into the embellishment

$$\begin{tikzcd}
& (A^m,\delta^m) \arrow[dl, "\psi_A"'] \arrow[dr, "q^\tau_A"] & \\
(A,\delta) \arrow[rr, "Q^\tau_A"] \arrow[dr, "\Lambda_A"'] & & (A^\tau,\delta^\tau) \arrow[dl, "\Lambda^\tau_A"] \\
& (A^n,\delta^n) &
\end{tikzcd}$$

of the diagram (2-1), equivalently $\tau \le \mathrm{id}$ (the identity functor). This property tends to simplify considerations of various properties of coaction functors, mainly by replacing q^τ by Q^τ. For example, a decreasing coaction functor τ is Morita compatible if and only if whenever (X,ζ) is an $(A,\delta)-(B,\epsilon)$ imprimitivity-bimodule coaction, there are an $A^\tau - B^\tau$ imprimitivity bimodule X^τ and a $Q^\tau_A - Q^\tau_B$ compatible imprimitivity-bimodule homomorphism $Q^\tau_X : X \to X^\tau$ [Kaliszewski et al. 2016a, Proposition 5.5].

The most studied decreasing coaction functors are those determined by *large ideals* of the Fourier–Stieltjes algebra $B(G)$, i.e., nonzero G-invariant weak* closed ideals E of $B(G)$. The preannihilator ${}^\perp E$ is an ideal of $C^*(G)$, and, denoting the quotient map by

$$q_E : C^*(G) \to C^*_E(G) := C^*(G)/{}^\perp E,$$

for any coaction (A,δ) we let

$$A^E = A/\ker((\mathrm{id}\otimes q_E)\circ\delta).$$

Then δ descends to a coaction δ^E on the quotient A^E, and the assignments $(A,\delta)\mapsto(A^E,\delta^E)$ determine a decreasing coaction functor τ_E. We write

$$Q^E = Q^{\tau_E} : A \to A^E.$$

The maximalization functor is not decreasing, so is not of the form τ_E for any large ideal E. Moreover, [Kaliszewski et al. 2016b, Example 3.16] gives an example of a decreasing coaction functor τ such that for every large ideal E the restrictions of τ and τ_E to the subcategory of maximal coactions are not naturally isomorphic; in particular, τ is not itself of the form τ_E.

We call the large ideal E *exact* if the coaction functor τ_E is exact. It is quite frustrating that so far we have few exact large ideals; for arbitrary G we only know of one exact large ideal, namely $B(G)$, and $\tau_{B(G)}$ is the identity functor. If the group G is exact, then it seems plausible — although we have not checked this — that $B_r(G)$ is also an exact large ideal, and would obviously be the smallest one. The frustrating thing is that for arbitrary G we do not know whether there is a smallest exact large ideal E. On the other hand, for every large ideal E the coaction functor τ_E is Morita compatible [Kaliszewski et al. 2016a, Proposition 6.10]. We do not know whether the intersection of all exact large ideals is exact; the best we can say for now is that the set of all exact large ideals is closed under finite intersections [Kaliszewski et al. 2016b, Theorem 3.2]. In a similar vein, if $\mathcal{F}$ is a collection of large ideals, with intersection F, we do not know whether τ_F is the greatest lower bound of $\{\tau_E : E \in \mathcal{F}\}$.

A coaction functor τ has the *ideal property* [Kaliszewski et al. 2016b, Definition 3.10] if for every coaction (A,δ) and every strongly δ-invariant ideal I of A, letting $\iota : I \hookrightarrow A$ denote the inclusion map, the induced map $\iota^\tau : I^\tau \to A^\tau$ is injective. For every large ideal E, the coaction τ_E has the ideal property [Kaliszewski et al. 2016b, Lemma 3.11]. We do not know an example of a decreasing coaction functor that is Morita compatible and does not have the ideal property (see [Kaliszewski et al. 2016b, Remark 3.12]).

3. Maximalization of degenerate homomorphisms

Our main objects of study are coaction functors, which involve maximalization of coactions. We will need to maximalize possibly degenerate homomorphisms. Maximalization can be characterized by a universal property (see [Fischer 2004, Lemma 6.2] for nondegenerate morphisms, and [Kaliszewski et al. 2016a] for the classical case), but this does not seem well-suited to handling possibly degenerate homomorphisms. Instead, we rely upon the Fischer construction, which involves three steps: first form the crossed product by the coaction, then the crossed product by the dual action, and finally *destabilize*, which roughly means extract A from $A \otimes \mathcal{K}$.

Our strategy for maximalizing possibly degenerate homomorphisms is to do it for each of the three steps in the Fischer construction, then combine. The steps are Lemmas 3.1, 3.7, and 3.8, which will be combined in Theorem 3.9.

Lemma 3.1. *Let (A,δ) and (B,ϵ) be coactions, and let $\phi : A \to M(B)$ be a possibly degenerate $\delta - \epsilon$ equivariant homomorphism. Then there is a unique homomorphism*

$$\phi \rtimes G : A \rtimes_\delta G \to M(B \rtimes_\epsilon G)$$

such that

$$(\phi \rtimes G)\bigl(j_A(a)j_G^\delta(g)\bigr) = j_B \circ \phi(a) j_G^\epsilon(g) \quad \text{for all } a \in A,\ g \in C_c(G) \subset C^*(G). \tag{3-1}$$

Moreover, $\phi \rtimes G$ is nondegenerate if ϕ is, and is $\hat\delta - \hat\epsilon$ equivariant, and if $\phi(A) \subset B$ then

$$(\phi \rtimes G)(A \rtimes_\delta G) \subset B \rtimes_\epsilon G.$$

Finally, given a third action (C,γ) and a possibly degenerate $\epsilon - \gamma$ equivariant homomorphism $\psi : B \to M(C)$, if either $\phi(A) \subset B$ or ψ is nondegenerate then

$$(\psi \rtimes G) \circ (\phi \rtimes G) = (\psi \circ \phi) \rtimes G.$$

Proof. The first part is [Echterhoff et al. 2006, Lemma A.46], and the other statements follow from direct calculation. □

For the next step, we need some ancillary lemmas. Lemmas 3.2–3.4 are completely routine — we record them for convenient reference. Lemmas 3.5–3.6 are included to prepare for Lemma 3.7.

Lemma 3.2. *Let B be a C^*-algebra, and let D and E be C^*-subalgebras of $M(B)$. Suppose that*

$$\overline{\text{span}}\{ED\} = D,$$

so that also $\overline{\text{span}}\{DE\} = D$. Then there is a unique homomorphism $\rho : E \to M(D)$ such that

$$\rho(m)d = md \quad \text{for all } m \in E,\ d \in D,$$

and moreover ρ is nondegenerate.

Lemma 3.3. *Let D, B, F be C^*-algebras, with $D \subset M(B)$, and let $\nu : F \to M(B)$ be a nondegenerate homomorphism. Suppose that $\overline{\text{span}}\{\nu(F)D\} = D$. Let $E = \nu(F)$. Let $\rho : E \to M(D)$ be the homomorphism from Lemma 3.2. Then*

$$\tau := \rho \circ \nu : F \to M(D)$$

is the unique nondegenerate homomorphism satisfying

$$\nu(f)d = \tau(f)d \quad \text{for all } f \in F,\ d \in D. \tag{3-2}$$

Lemma 3.4. *Keep the notation from Lemma 3.3, and let C be another C^*-algebra. Let $w \in M(F \otimes C)$. Define*

$$U = (\nu \otimes \mathrm{id})(w) \in M(E \otimes C) \subset M(B \otimes C),$$
$$W = (\tau \otimes \mathrm{id})(w) \in M(D \otimes C).$$

Then

$$W = (\rho \otimes \mathrm{id})(U),$$

and

$$Wm = Um \quad \textit{for all } m \in \widetilde{M}(D \otimes C).$$

Let D, B, and C be C^*-algebras, with $D \subset M(B)$. Let $\sigma : D \hookrightarrow M(B)$ be the inclusion map. Then, by [Echterhoff et al. 2006, Proposition A.6], $\sigma \otimes \mathrm{id} : D \otimes C \hookrightarrow M(B \otimes C)$ extends canonically to an injective homomorphism,

$$\overline{\sigma \otimes \mathrm{id}} : \widetilde{M}(D \otimes C) \to M(B \otimes C),$$

that is continuous from the C-strict topology to the strict topology, and we frequently identify $\widetilde{M}(D \otimes C)$ with its image in $M(B \otimes C)$.

Lemma 3.5. *Keep the notation from the Lemmas 3.2–3.4, and let $F = C_0(G)$, $C = C^*(G)$, and $w = w_G$. Also let ϵ be a coaction of G on B. Suppose that D is strongly ϵ-invariant, and let $\zeta = \epsilon_D$. Suppose that $U := (\nu \otimes \mathrm{id})(w_G)$ is an ϵ-cocycle, and $W := (\tau \otimes \mathrm{id})(w_G)$ is a ζ-cocycle. Define*

$$\tilde{\epsilon} := \operatorname{Ad} U \circ \epsilon \quad \textit{and} \quad \tilde{\zeta} := \operatorname{Ad} W \circ \zeta.$$

Then D is also strongly $\tilde{\epsilon}$-invariant, and $\tilde{\zeta} = \tilde{\epsilon}_D$.

Proof. For $d \in D$, we have

$$\begin{aligned}
\tilde{\epsilon}(d) &= \operatorname{Ad} U \circ \epsilon(d) \\
&= \operatorname{Ad} U \circ \zeta(d) && (\text{since } \zeta = \epsilon_B) \\
&= \operatorname{Ad} W \circ \zeta(d) && (\text{by Lemma 3.4}) \\
&= \tilde{\zeta}(d).
\end{aligned}$$

Since $\tilde{\zeta}$ is a coaction of G on D, we conclude that D is strongly $\tilde{\epsilon}$-invariant. □

Lemma 3.6. *Let (A, δ) and (B, ϵ) be coactions, and let $\phi : A \to M(B)$ be a possibly degenerate $\delta - \epsilon$ equivariant homomorphism. Let $\mu : C_0(G) \to M(A)$ and $\nu : C_0(G) \to M(B)$ be nondegenerate homomorphisms, and assume that*

$$\phi\big(a\mu(f)\big) = \phi(a)\nu(f) \quad \textit{for all } a \in A,\ f \in C_0(G).$$

Define

$$V = (\mu \otimes \mathrm{id})(w_G) \in M(A \otimes C^*(G)) \quad \textit{and} \quad U = (\nu \otimes \mathrm{id})(w_G) \in M(B \otimes C^*(G)).$$

Suppose that V is a δ-cocycle and U is an ϵ-cocycle. Define

$$\tilde{\delta} = \operatorname{Ad} V \circ \delta \quad \textit{and} \quad \tilde{\epsilon} = \operatorname{Ad} U \circ \epsilon.$$

Then ϕ is also $\tilde{\delta} - \tilde{\epsilon}$ equivariant.

Proof. Define $D = \phi(A)$. Then there is a unique coaction ζ of G on D such that the surjection $\phi : A \to D$ is $\delta - \zeta$ equivariant. It follows that D is strongly ϵ-invariant. Moreover, $\zeta = \epsilon_D$, since for all $d \in D$ we can choose $a \in A$ such that $d = \phi(a)$, and then, regarding $\widetilde{M}(D \otimes C^*(G))$ as a subset of $M(B \otimes C^*(G))$,

$$\begin{aligned} \zeta(d) &= \zeta \circ \phi(d) = (\phi \otimes \mathrm{id}) \circ \delta(a) \\ &= \epsilon \circ \phi(a) = \epsilon(d). \end{aligned}$$

The canonical extension $\bar{\phi} : M(A) \to M(D)$ takes μ to the unique nondegenerate homomorphism $\tau : C_0(G) \to M(D)$ satisfying (3-2) with $F = C_0(G)$, and the unitary

$$W := (\phi \otimes \mathrm{id})(V) = (\tau \otimes \mathrm{id})(w_G)$$

is a ζ-cocycle. The hypotheses imply that $\nu(C_0(G))D = D$. Thus we can apply Lemma 3.5: the right-front rectangle (involving D and $M(B)$) of the diagram

$$\begin{tikzcd}
A \arrow[rr, "\phi"] \arrow[dr, "\phi"] \arrow[dd, "\tilde{\delta}"'] & & M(B) \arrow[dd, "\tilde{\epsilon}"] \\
& D \arrow[ur, hook'] \arrow[dd, "\tilde{\zeta}", near start] & \\
\widetilde{M}(A \otimes C^*(G)) \arrow[rr, "\overline{\phi\otimes\mathrm{id}}", near start] \arrow[dr, "\phi\otimes\mathrm{id}"'] & & M(B \otimes C^*(G)) \\
& \widetilde{M}(D \otimes C^*(G)) \arrow[ur, hook'] &
\end{tikzcd}$$

commutes, and the left-front rectangle (involving A and D) commutes by naturality of cocycles, and therefore the rear rectangle (involving A and $M(B)$) commutes, giving $\tilde{\delta} - \tilde{\epsilon}$ equivariance of ϕ. $\square$

We are now ready for the second step of the Fischer construction for possibly degenerate homomorphisms:

Lemma 3.7. *Let (A, α, μ) and (B, β, ν) be equivariant actions, and $\phi : A \to M(B)$ be a possibly degenerate $\alpha - \beta$ equivariant homomorphism such that*

$$\phi(a\mu(f)) = \phi(a)\nu(f) \quad \textit{for all } \ a \in A, \ f \in C_0(G).$$

Then there is a unique (possibly degenerate) homomorphism

$$\phi \rtimes G : A \rtimes_\alpha G \to M(B \rtimes_\beta G)$$

such that

$$(\phi \rtimes G)\big(i_A(a)i_G^\alpha(c)\big) = i_B \circ \phi(a)i_G^\beta(c) \quad \text{for all } a \in A,\ c \in C^*(G). \tag{3-3}$$

Moreover, $\phi \rtimes G$ is nondegenerate if ϕ is, and is $\tilde{\alpha} - \tilde{\beta}$ equivariant, and

$$(\phi \rtimes G)\big(c(\mu \rtimes G)(k)\big) = (\phi \rtimes G)(c)(\nu \rtimes G)(k) \quad \text{for all } c \in A \rtimes_\alpha G,\ k \in \mathcal{K}. \tag{3-4}$$

Also, if $\phi(A) \subset B$ then

$$(\phi \rtimes G)(A \rtimes_\alpha G) \subset B \rtimes_\beta G.$$

Finally, given a third action (C, γ) and a possibly degenerate $\beta - \gamma$ equivariant homomorphism $\psi : B \to M(C)$, if either $\phi(A) \subset B$ or ψ is nondegenerate then

$$(\psi \rtimes G) \circ (\phi \rtimes G) = (\psi \circ \phi) \rtimes G.$$

Proof. The first statement, up to and including (3-3), is [Echterhoff et al. 2006, Remark A.8(4)], the preservation of nondegeneracy is well known, and the last part, starting with "Also", follows from direct calculation. We must verify the $\tilde{\alpha} - \tilde{\beta}$ equivariance and (3-4). We first claim that for all $c \in A \rtimes_\alpha G,\ d \in C^*(G),\ a \in A$, and $f \in C_0(G)$ we have

$$(\phi \rtimes G)\big(c\, i_G^\alpha(d)\big) = (\phi \rtimes G)(c)i_B^\beta(d) \tag{3-5}$$

$$(\phi \rtimes G)\big(c\, i_A(a)\big) = (\phi \rtimes G)(c)i_B \circ \phi(a) \tag{3-6}$$

$$(\phi \rtimes G)\big(c\, i_A \circ \mu(f)\big) = (\phi \rtimes G)(c)i_B \circ \nu(f). \tag{3-7}$$

Equations (3-5) and (3-6) follow by first replacing c by appropriately chosen generators, and to see (3-7) we use nondegeneracy of i_A and the Cohen factorization theorem to write

$$c = c'\, i_A(b) \quad \text{for } c' \in A \rtimes_\alpha G, b \in A,$$

and then compute

$$\begin{aligned}
(\phi \rtimes G)(c\, i_A \circ \mu(f)) &= (\phi \rtimes G)(c'\, i_A(b)i_A \circ \mu(f)) \\
&= (\phi \rtimes G)(c' i_A(b\mu(f))) \\
&= (\phi \rtimes G)(c')i_B \circ \phi(b\mu(f)) \\
&= (\phi \rtimes G)(c')i_B(\phi(b)\nu(f)) \\
&= (\phi \rtimes G)(c')i_B(\phi(b))i_B(\nu(f)) \\
&= (\phi \rtimes G)(c' i_A(b))i_B(\nu(f)) \\
&= (\phi \rtimes G)(c)i_B \circ \nu(f).
\end{aligned}$$

Combining (3-7) with the other hypotheses, we can apply Lemma 3.6 to conclude that $\phi \rtimes G$ is $\tilde{\alpha} - \tilde{\beta}$ equivariant.

For (3-4), it suffices to consider a generator

$$k = i_{C_0(G)}(f) i_G^{\mathrm{rt}}(d) \quad \text{for } f \in C_0(G), d \in C^*(G),$$

and then compute

$$\begin{aligned}
(\phi \rtimes G)\big(c(\mu \rtimes G)(k)\big) &= (\phi \rtimes G)\big(c i_A \circ \mu(f) i_G^{\alpha}(d)\big) \\
&= (\phi \rtimes G)\big(c i_A \circ \mu(f)\big) i_B^{\beta}(d) \qquad \text{(by (3-5))} \\
&= (\phi \rtimes G)(c) i_B \circ \nu(f) i_B^{\beta}(d) \qquad \text{(by (3-7))} \\
&= (\phi \rtimes G)(c)(\nu \rtimes G)(k).
\end{aligned}$$

□

Finally, we are ready for the third step of the Fischer construction for possibly degenerate homomorphisms:

Lemma 3.8. *Let (A, δ, ι) and $(B, \epsilon, \jmath)$ be $\mathcal{K}$-coactions, and let $\phi : A \to M(B)$ be a possibly degenerate $\delta - \epsilon$ equivariant homomorphism such that*

$$\phi(a\iota(k)) = \phi(a)\jmath(k) \quad \text{for all } a \in A,\ k \in \mathcal{K}.$$

Then there is a unique (possibly degenerate) homomorphism,

$$C(\phi) : C(A, \iota) \to M(C(B, \jmath)),$$

making the diagram

$$\begin{array}{ccc}
C(A,\iota) \otimes \mathcal{K} & \xrightarrow[\simeq]{\theta_A} & A \\
{\scriptstyle C(\phi)\otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \phi} \\
M(C(B,\jmath) \otimes \mathcal{K}) & \xrightarrow[\theta_B]{\simeq} & M(B)
\end{array} \tag{3-8}$$

commute. Moreover, $C(\phi)$ is nondegenerate if ϕ is, and is $C(\delta) - C(\epsilon)$ equivariant. Also, if $\phi(A) \subset B$ then $C(\phi)(C(A, \iota)) \subset C(B, \jmath)$. Finally, given a third $\mathcal{K}$-coaction (C, ζ, ω) and a possibly degenerate $\epsilon - \zeta$ equivariant homomorphism $\psi : B \to M(C)$ satisfying $\psi(b\jmath(k)) = \psi(b)\omega(k)$ for all $b \in B$ and $k \in \mathcal{K}$, if either $\phi(A) \subset B$ or ψ is nondegenerate then

$$C(\psi) \circ C(\phi) = C(\psi \circ \phi). \tag{3-9}$$

Proof. By [Deaconu et al. 2012, Lemma A.5], ϕ extends uniquely to a homomorphism

$$\bar{\phi} : M_{\mathcal{K}}(A) \to M(B)$$

that is continuous from the $\mathcal{K}$-strict topology to the strict topology. Since $C(A, \iota) \subset M_{\mathcal{K}}(A)$, we can define

$$C(\phi) = \bar{\phi}|_{C(A,\iota)}.$$

We will show that the diagram (3-8) commutes, and then the uniqueness will be obvious. For $m \in C(A,\iota)$ and $k \in \mathcal{K}$ we have

$$\begin{aligned}\theta_B \circ (C(\phi)\otimes \mathrm{id})(m\otimes k) &= \theta_B\big(\bar{\phi}(m)\otimes k\big)\\ &= \bar{\phi}(m)\jmath(k)\\ &\overset{*}{=} \phi(m\iota(k))\\ &= \phi\circ\theta_A(m\otimes k),\end{aligned}$$

where the equality at $*$ follows from $\mathcal{K}$-strict to strict continuity. The preservation of nondegeneracy is proven in [Kaliszewski et al. 2016c, Theorem 4.4], and follows from a routine approximate-identity argument.

For the equivariance, let $f \in B(G)$, $m \in C(A,\iota)$, and $k \in \mathcal{K}$. Since $C(A,\iota)$ is a $B(G)$-submodule of $M(A)$, we can compute as follows:

$$\begin{aligned}C(\phi)(f\cdot m)\jmath(k) &= \bar{\phi}(f\cdot m)\jmath(k) && (\text{since } C(\phi)=\bar{\phi}|_{C(A,\iota)})\\ &= \phi((f\cdot m)\iota(k)) && (\text{by [Deaconu et al. 2012, Lemma A.5]})\\ &= \phi(f\cdot(m\iota(k))) && (\text{since } \delta\circ\iota=\iota\otimes 1)\\ &= f\cdot\phi(m\iota(k)) && (\text{by Proposition 2.1})\\ &= f\cdot(\bar{\phi}(m)\jmath(k))\\ &= f\cdot(\bar{\phi}(m))\jmath(k)\\ &= f\cdot(C(\phi)(m))\jmath(k).\end{aligned}$$

Thus $C(\phi)(f\cdot m) = f\cdot C(\phi)(m)$ since $\jmath : \mathcal{K}\to M(B)$ is nondegenerate, and hence ϕ is equivariant by Proposition 2.1.

Now suppose that $\phi(A)\subset B$. Then for all $m \in C(A,\iota)$ and $k\in\mathcal{K}$ we have

$$\begin{aligned}C(\phi)(m)\jmath(k) &= \bar{\phi}(m)\jmath(k)\\ &= \phi(m\iota(k)) = \phi(\iota(k)m)\\ &= \jmath(k)\bar{\phi}(m) = \jmath(k)C(\phi)(m),\end{aligned}$$

which is an element of B since $m\iota(k)\in A$.

The final statement, regarding composition, seems to not be recorded in the literature, so we give the proof here. First suppose that $\phi(A)\subset B$. Then by [Deaconu et al. 2012, Lemma A.5] the extension $\bar{\phi}$ maps $M_{\mathcal{K}}(A)$ into $M_{\mathcal{K}}(B)$ and is continuous for the $\mathcal{K}$-strict topologies. Also, $\bar{\psi} : M_{\mathcal{K}}(B)\to M(C)$ is continuous from the $\mathcal{K}$-strict topology to the strict topology. Let $\{a_i\}$ be a net in A converging $\mathcal{K}$-strictly to $m\in M_{\mathcal{K}}(A)$. Then $\phi(a_i)\to\bar{\phi}(m)$ $\mathcal{K}$-strictly in $M_{\mathcal{K}}(B)$, and so

$$\psi(\phi(a_i))\to\bar{\psi}(\bar{\phi}(m))\quad\text{strictly in } M(C).$$

On the other hand, the composition

$$\bar{\psi} \circ \bar{\phi} : M_{\mathcal{K}}(A) \to M(C)$$

is continuous from the $\mathcal{K}$-strict topology to the strict topology, so

$$\overline{\psi \circ \phi}(a_i) \to \overline{\psi \circ \phi}(m).$$

Since $\psi(\phi(a_i)) = (\psi \circ \phi)(a_i)$ for all i, we conclude that

$$\bar{\psi} \circ \bar{\phi}(m) = \overline{\psi \circ \phi}(m).$$

Since $C(\phi)$ and $C(\psi)$ are the restrictions to the relative commutants $C(A, \iota)$ and $C(B, \jmath)$, respectively, we get $C(\psi \circ \phi) = C(\psi) \circ C(\phi)$.

For the other case, where ψ is nondegenerate, we use the canonical extension of ψ to $M(B)$ to compose, getting a $\delta - \zeta$ equivariant homomorphism $\psi \circ \phi : A \to M(C)$ such that

$$(\psi \circ \phi)\big(a\iota(k)\big) = (\psi \circ \phi)(a)\omega(k) \quad \text{for all } \ a \in A,\ k \in \mathcal{K},$$

so that $C(\psi \circ \phi)$ makes sense. Since $C(\phi)$ is computed by restricting the canonical extension $\bar{\phi} : M_{\mathcal{K}}(A) \to M(B)$, and similarly for $C(\psi \circ \phi)$, and since we can compute the extension of ψ on all of $M(B)$, (3-9) follows. $\square$

We are now ready to maximalize possibly degenerate homomorphisms:

Theorem 3.9. *Let (A, δ) and (B, ϵ) be coactions, and let $\phi : A \to M(B)$ be a possibly degenerate $\delta - \epsilon$ equivariant homomorphism. Then there is a unique (possibly degenerate) homomorphism $\phi^m : A^m \to M(B^m)$ making the diagram*

(3-10)
$$\begin{tikzcd}
& A \rtimes_\delta G \rtimes_{\hat{\delta}} G \arrow[dr, "\Phi_A"] \arrow[dd, "\phi \rtimes G \rtimes G"] & \\
A^m \otimes \mathcal{K} \arrow[ur, "\theta_{A \rtimes_\delta G \rtimes_{\hat{\delta}} G}", "\simeq"'] \arrow[rr, "\psi_A \otimes \mathrm{id}"', crossing over] \arrow[dd, dashed, "\phi^m \otimes \mathrm{id}"'] & & A \otimes \mathcal{K} \arrow[dd, "\phi \otimes \mathrm{id}"] \\
& M(B \rtimes_\epsilon G \rtimes_{\hat{\epsilon}} G) \arrow[dr, "\Phi_B"] & \\
M(B^m \otimes \mathcal{K}) \arrow[ur, "\simeq", "\theta_{B \rtimes_\epsilon G \rtimes_{\hat{\epsilon}} G}"'] \arrow[rr, "\psi_B \otimes \mathrm{id}"'] & & M(B \otimes \mathcal{K})
\end{tikzcd}$$

commute, where $\psi_A : (A^m, \delta^m) \to (A, \delta)$ is the maximalization (and similarly for ψ_B). Moreover, ϕ^m is nondegenerate if ϕ is, the diagram

(3-11)
$$\begin{tikzcd}
A^m \arrow[r, "\phi^m"] \arrow[d, "\psi_A"'] & M(B^m) \arrow[d, "\psi_B"] \\
A \arrow[r, "\phi"'] & M(B)
\end{tikzcd}$$

also commutes, and ϕ^m *is* $\delta^m - \epsilon^m$ *equivariant. Further, if* $\phi(A) \subset B$ *then* $\phi^m(A^m) \subset B^m$. *Finally, given a third coaction* (C, ζ) *and a possibly degenerate* $\epsilon - \zeta$ *equivariant homomorphism* $\pi : B \to M(C)$, *if either* $\phi(A) \subset B$ *or* π *is nondegenerate then*

$$(\pi \circ \phi)^m = \pi^m \circ \phi^m.$$

Proof. The right-rear rectangle in the diagram (3-10) (involving $A \rtimes G \rtimes G$ and $A \otimes \mathcal{K}$) commutes by direct computation.

Now, $(A \rtimes_\delta G, \hat{\delta}, j_G^\delta)$ and $(B \rtimes_\epsilon G, \hat{\epsilon}, j_G^\epsilon)$ are equivariant actions. By Lemma 3.1, the homomorphism

$$\phi \rtimes G : A \rtimes_\delta G \to M(B \rtimes_\epsilon G)$$

is $\hat{\delta} - \hat{\epsilon}$ equivariant and satisfies

$$(\phi \times G)\big(c j_G^\delta(f)\big) = (\phi \rtimes G)(c) j_G^\epsilon(f) \quad \text{for all } \ c \in A \rtimes_\delta G,\ f \in C_0(G).$$

Thus, by Lemma 3.7 the homomorphism

$$\phi \rtimes G \rtimes G : A \rtimes_\delta G \rtimes_{\hat{\delta}} G \to M(B \rtimes_\epsilon G \rtimes_{\hat{\epsilon}} G)$$

is $\tilde{\delta} - \tilde{\epsilon}$ equivariant and satisfies

$$(\phi \rtimes G \rtimes G)\big(c(j_G^\delta \rtimes G)(k)\big) = (\phi \rtimes G \rtimes G)(c)(j_G^\epsilon \rtimes G)(k)$$

for all $c \in A \rtimes_\delta G \rtimes_{\hat{\delta}} G$ and $k \in \mathcal{K}$. Furthermore, $(A \rtimes_\delta G \rtimes_{\hat{\delta}} G, \tilde{\delta}, j_G^\delta \rtimes G)$ and $(B \rtimes_\epsilon G \rtimes_{\hat{\epsilon}} G, \tilde{\epsilon}, j_G^\epsilon \rtimes G)$ are $\mathcal{K}$-coactions. Thus, by Lemma 3.8 the homomorphism

$$C(\phi \rtimes G \rtimes G) : C(A \rtimes_\delta G \rtimes_{\hat{\delta}} G, j_G^\delta \rtimes G) \to M\big(C(B \rtimes_\epsilon G \rtimes_{\hat{\epsilon}} G, j_G^\epsilon \rtimes G)\big)$$

makes the diagram

$$\begin{tikzcd}
C(A \rtimes_\delta G \rtimes_{\hat{\delta}} G, j_G^\delta \rtimes G) \otimes \mathcal{K} \arrow[r, "\theta_{A \rtimes_\delta G \rtimes_{\hat{\delta}} G}", "\simeq"'] \arrow[d, "C(\phi \rtimes G \rtimes G) \otimes \mathrm{id}"'] & A \rtimes_\delta G \rtimes_{\hat{\delta}} G \arrow[d, "\phi \rtimes G \rtimes G"] \\
M\big(C(B \rtimes_\epsilon G \rtimes_{\hat{\epsilon}} G, j_G^\epsilon \rtimes G) \otimes \mathcal{K}\big) \arrow[r, "\simeq", "\theta_{B \rtimes_\epsilon G \rtimes_{\hat{\epsilon}} G}"'] & M(B \rtimes_\epsilon G \rtimes_{\hat{\epsilon}} G)
\end{tikzcd}$$

commute. Since

$$A^m = C(A \rtimes_\delta G \rtimes_{\hat{\delta}} G, i_{A \rtimes_\delta G} \circ j_G^\delta),$$

by Lemma 3.8 we can define

$$\phi^m = C(\phi \rtimes G \rtimes G),$$

which is then the unique homomorphism making the left-rear rectangle in the diagram (3-10) (involving $A^m \otimes \mathcal{K}$ and $A \rtimes G \rtimes G$) commute. The preservation of nondegeneracy follows immediately from the corresponding properties of the functors whose composition is $\phi \mapsto \phi^m$. Then the front rectangle (involving $A^m \otimes \mathcal{K}$ and $A \otimes \mathcal{K}$) commutes, and hence so does the diagram (3-11). Moreover, since $\delta^m = C(\delta)$ and $\epsilon^m = C(\epsilon)$, by Lemma 3.8 again we see that ϕ^m is $\delta^m - \epsilon^m$ equivariant.

For the final statement, involving composition, suppose that we have C, ζ, and π. We consider the two cases separately: first of all, assume that $\phi(A) \subset B$. Then from Lemma 3.1 we conclude that the equivariant actions

$$(A \rtimes_\delta G, \hat{\delta}, j_G^\delta),$$
$$(B \rtimes_\epsilon G, \hat{\epsilon}, j_G^\epsilon),$$
$$(C \rtimes_\zeta G, \hat{\zeta}, j_G^\zeta)$$

and the homomorphisms

$$\phi \rtimes G : A \rtimes_\delta G \to B \rtimes_\epsilon G,$$
$$\pi \rtimes G : B \rtimes_\epsilon G \to M(C \rtimes_\zeta G)$$

satisfy the hypotheses of Lemma 3.7. Thus, Lemma 3.7 now tells us that the $\mathcal{K}$-coactions

$$(A \rtimes_\delta G \rtimes_{\hat{\delta}} G, \tilde{\delta}, j_G^\delta \rtimes G),$$
$$(B \rtimes_\epsilon G \rtimes_{\hat{\epsilon}} G, \tilde{\epsilon}, j_G^\epsilon \rtimes G),$$
$$(C \rtimes_\zeta G \rtimes_{\hat{\zeta}} G, \tilde{\zeta}, j_G^\zeta \rtimes G)$$

and the homomorphisms

$$\phi \rtimes G \rtimes G : A \rtimes_\delta G \rtimes_{\hat{\delta}} G \to B \rtimes_\epsilon G \rtimes_{\hat{\epsilon}} G,$$
$$\pi \rtimes G \rtimes G : B \rtimes_\epsilon G \rtimes_{\hat{\epsilon}} G \to M(C \rtimes_\zeta G \rtimes_{\hat{\zeta}} G)$$

satisfy the hypotheses of Lemma 3.8, and hence, by construction of the maximalizations $\delta^m, \epsilon^m, \zeta^m$ of δ, ϵ, ζ, we get

$$\pi^m \circ \phi^m = (\pi \circ \phi)^m.$$

On the other hand, if we assume that π is nondegenerate instead of $\phi(A) \subset B$, the argument proceeds similarly, except we keep tacitly using the canonical extension to multiplier algebras of any homomorphism constructed from π. □

Remark 3.10. Theorem 3.9 gives a precise justification that the assignments

$$(A, \delta) \mapsto (A^m, \delta^m),$$
$$\phi \mapsto \phi^m$$

define a functor on the classical category of coactions.

4. Generalized homomorphisms

Definition 4.1. We say that a coaction functor τ is *functorial for generalized homomorphisms* if whenever (A, δ) and (B, ϵ) are coactions and $\phi : A \to M(B)$ is a possibly degenerate $\delta - \epsilon$ equivariant homomorphism there is a (necessarily unique) possibly degenerate homomorphism ϕ^τ making the following diagram commute:

$$\begin{tikzcd} A^m \arrow[r, "\phi^m"] \arrow[d, "q_A^\tau"'] & M(B^m) \arrow[d, "q_B^\tau"] \\ A^\tau \arrow[r, dashed, "\phi^\tau"'] & M(B^\tau) \end{tikzcd} \tag{4-1}$$

Note that the existence of the homomorphism ϕ^m is guaranteed by Theorem 3.9. If ϕ^τ is only presumed to exist when ϕ is nondegenerate, then we say that τ is *functorial for nondegenerate homomorphisms*. Note that if τ is functorial for generalized homomorphisms, it automatically sends nondegenerate homomorphisms to nondegenerate homomorphisms. This follows immediately from the corresponding property for the maximalization functor $A \mapsto A^m$.

Remark 4.2. Let τ be a coaction functor, and let CP^τ be the associated crossed-product functor for actions, given by full crossed product followed by τ. If τ is functorial for generalized homomorphisms, then CP^τ is also functorial for generalized homomorphisms in the sense of Buss et al. — see the paragraph following Definition 3.1 in [BEW].

Thus, a coaction functor τ is functorial for generalized homomorphisms if and only if for every possibly degenerate $\delta-\epsilon$ equivariant homomorphism $\phi : A \to M(B)$ we have

$$\ker q_A^\tau \subset \ker q_B^\tau \circ \phi^m,$$

and similarly for nondegenerate functoriality.

Example 4.3. The maximalization functor is functorial for generalized homomorphisms, by Theorem 3.9. Thus the identity functor id is functorial for generalized homomorphisms, since we can take $q_A^{\mathrm{id}} = \psi_A$ and $\phi^{\mathrm{id}} = \phi$.

Remark 4.4. Suppose that τ is functorial for generalized homomorphisms, and that $\phi : A \to B$ is $\delta - \epsilon$ equivariant. Then the map ϕ^τ vouchsafed by Definition 4.1 agrees with the one that we get by the assumption that τ is a coaction functor. In particular, if $\iota : A \hookrightarrow M(A)$ is the canonical embedding then ι^τ coincides with the canonical embedding $A^\tau \hookrightarrow M(A^\tau)$.

Lemma 4.5. *Let τ be a coaction functor that is functorial for generalized homomorphisms, let (A, δ), (B, ϵ), and (C, ζ) be coactions, and let $\phi : A \to M(B)$*

and $\psi : B \to M(C)$ be possibly degenerate equivariant homomorphisms. If either $\phi(A) \subset B$ or ψ is nondegenerate, then $(\psi \circ \phi)^\tau = \psi^\tau \circ \phi^\tau$.

Proof. First assume that $\phi(A) \subset B$. Then $\psi \circ \phi : A \to M(C)$ is $\delta - \zeta$ equivariant. Consider the following diagram:

$$\begin{tikzcd}
A^m \arrow[rr, "\phi^m"] \arrow[dr, "(\psi\circ\phi)^m"'] \arrow[dd, "q_A^\tau"'] & & B^m \arrow[dl, "\psi^m"] \arrow[dd, "q_B^\tau"] \\
& M(C^m) \arrow[dd, "q_C^\tau"] & \\
A^\tau \arrow[rr, "\phi^\tau"] \arrow[dr, "(\psi\circ\phi)^\tau"'] & & B^\tau \arrow[dl, "\psi^\tau"] \\
& M(C^\tau) &
\end{tikzcd}$$

The top triangle commutes by Theorem 3.9. The rear, right-front, and left-front rectangles commute since τ is functorial for generalized homomorphisms. Since the left vertical arrow q_A^τ is surjective, it follows that the bottom triangle commutes, as desired.

On the other hand, assume that ψ is nondegenerate. Then again we have a $\delta - \zeta$ equivariant homomorphism $\psi \circ \phi$ (extending ψ canonically to $M(B)$), the above diagram becomes

$$\begin{tikzcd}
A^m \arrow[rr, "\phi^m"] \arrow[dr, "(\psi\circ\phi)^m"'] \arrow[dd, "q_A^\tau"'] & & M(B^m) \arrow[dl, "\psi^m"] \arrow[dd, "q_B^\tau"] \\
& M(C^m) \arrow[dd, "q_C^\tau"] & \\
A^\tau \arrow[rr, "\phi^\tau"] \arrow[dr, "(\psi\circ\phi)^\tau"'] & & M(B^\tau) \arrow[dl, "\psi^\tau"] \\
& M(C^\tau) &
\end{tikzcd}$$

and the argument proceeds as in the first part. $\square$

Essentially the same techniques as in the above proof can be used to verify the following:

Lemma 4.6. *Let τ be a coaction functor that is functorial for nondegenerate homomorphisms, let (A, δ), (B, ϵ), and (C, ζ) be coactions, and let $\phi : A \to M(B)$ and $\psi : B \to M(C)$ be possibly degenerate equivariant homomorphisms. If ψ is nondegenerate, and if either $\phi(A) \subset B$ or ϕ is nondegenerate, then $(\psi \circ \phi)^\tau = \psi^\tau \circ \phi^\tau$. In particular, every coaction functor that is functorial for nondegenerate* *homomorphisms in the sense of Definition 4.1 is also a functor on the nondegenerate category of coactions.*

As usual, things are simpler for decreasing coaction functors:

Lemma 4.7. *A decreasing coaction functor τ is functorial for generalized homomorphisms if and only if whenever (A, δ) and (B, ϵ) are coactions and $\phi : A \to M(B)$ is a possibly degenerate $\delta - \epsilon$ equivariant homomorphism there is a (necessarily unique) possibly degenerate homomorphism ϕ^τ making the diagram*

(4-2)
$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "Q_A^\tau"'] & M(B) \arrow[d, "Q_B^\tau"] \\ A^\tau \arrow[r, dashed, "\phi^\tau"'] & M(B^\tau) \end{tikzcd}$$

commute. If ϕ^τ is only presumed to exist when ϕ is nondegenerate, then τ is functorial for nondegenerate homomorphisms.

Proof. The above diagram fits into a bigger one:

(4-3)
$$\begin{tikzcd} A^m \arrow[rr, "\psi_A"] \arrow[dr, "q_A^\tau"] \arrow[dd, "\phi^m"'] & & A \arrow[dl, "Q_A^\tau"'] \arrow[dd, "\phi"] \\ & A^\tau \arrow[dd, dashed, "\phi^\tau"] & \\ M(B^m) \arrow[rr, "\psi_B"] \arrow[dr, "q_B^\tau"'] & & M(B) \arrow[dl, "Q_B^\tau"] \\ & M(B^\tau) & \end{tikzcd}$$

The top and bottom triangles commute since τ is a decreasing coaction functor. The rear rectangle commutes since the identity functor is functorial for generalized homomorphisms. If there is a homomorphism ϕ^τ making the left-front rectangle commute, then the right-front rectangle also commutes since ψ_A is surjective. Conversely, if there is a homomorphism ϕ^τ making the diagram (4-2) commute, then the right-front rectangle in the diagram (4-3) commutes, and hence so does the left-front rectangle. □

Thus, a decreasing coaction functor τ is functorial for generalized homomorphisms if and only if for every possibly degenerate $\delta - \epsilon$ equivariant homomorphism $\phi : A \to M(B)$ we have

$$\ker Q_A^\tau \subset \ker Q_B^\tau \circ \phi.$$

Example 4.8. We apply Lemma 4.7 to show that for every large ideal E of $B(G)$, the coaction functor τ_E is functorial for generalized homomorphisms. Let $\phi : A \to M(B)$ be a $\delta - \epsilon$ equivariant homomorphism, and let

$$a \in \ker Q_A^E = \{b \in A : E \cdot a = \{0\}\}.$$

Then for all $f \in E$ we have

$$\begin{aligned} f \cdot \phi(a) &= \phi(f \cdot a) \qquad \text{(by equivariance)} \\ &= 0, \end{aligned}$$

so $a \in \ker Q_B^E \circ \phi$. In particular, the identity functor and the normalization functor are functorial for generalized homomorphisms. For the identity functor this fact was already noted in Example 4.3.

The ideal property. A coaction functor τ has the *ideal property* [Kaliszewski et al. 2016b, Definition 3.10] if for every coaction (A, δ) and every strongly invariant ideal I of A, letting $\iota : I \hookrightarrow A$ denote the inclusion map, the induced map

$$\iota^\tau : I^\tau \to A^\tau$$

is injective.

Example 4.9. The identity functor trivially has the ideal property.

Example 4.10. Every exact coaction functor has the ideal property, and hence by [Kaliszewski et al. 2016a, Theorem 4.11] maximalization has the ideal property. However, normalization has the ideal property, but is not exact unless G is, since by [Kaliszewski et al. 2016a, Proposition 4.24] the composition of an exact coaction functor with the full-crossed-product functor is an exact crossed-product functor, and the composition of normalization with the full-crossed-product functor is the reduced crossed product, which is not an exact crossed-product functor unless G is an exact group.

Remark 4.11. If a coaction functor τ has the ideal property, then the associated crossed-product functor for actions has the ideal property in the sense of [BEW, Definition 3.2], since the full-crossed-product functor is exact [Green 1978, Proposition 12]. For crossed-product functors, [BEW, Lemma 3.3] includes the fact that functoriality for generalized homomorphisms and the ideal property are equivalent. In the following proposition we show that part of this carries over to coaction functors. However, our naive attempts to adapt the argument from [BEW] to show that the ideal property implies functoriality for generalized homomorphisms seem to require that if $\phi : A \to M(B)$ is a $\delta - \epsilon$ equivariant homomorphism then there is a strongly ϵ-invariant C^*-subalgebra E of $M(B)$ containing both B and $\phi(A)$, which we have unfortunately been unable to prove.

Proposition 4.12. *If a coaction functor τ is functorial for nondegenerate homomorphisms, in particular if τ is functorial for generalized homomorphisms, then τ has the ideal property.*

Proof. We adapt the proof from [BEW]: let (A, δ) be a coaction and let I be a strongly δ-invariant ideal of A. Let $\phi : I \hookrightarrow A$ be the inclusion map, let $\psi : A \to M(I)$ be the canonical map, and let $\iota : I \hookrightarrow M(I)$ be the canonical embedding. Note that ι and ψ are nondegenerate equivariant homomorphisms, and ϕ is a classical equivariant homomorphism. We have $\psi \circ \phi = \iota$, so by Lemma 4.6 we also have $\psi^\tau \circ \phi^\tau = \iota^\tau$. Since ι^τ is the canonical embedding $I^\tau \hookrightarrow M(I^\tau)$, we conclude that ϕ^τ is injective. □

Remark 4.13. By combining Example 4.8 with Proposition 4.12, we recover [Kaliszewski et al. 2016b, Lemma 3.11]: for every large ideal E of $B(G)$ the coaction functor τ_E has the ideal property. In particular, the identity functor and the normalization functor have the ideal property (and for the identity functor we already noted this in Example 4.9).

Example 4.14. We adapt the techniques of [Kaliszewski et al. 2016b, Example 3.16] (which was in turn adapted from the techniques of [Buss et al. 2014, Section 2.5 and Example 3.5]) to show that if G is nonamenable then there is a decreasing coaction functor for G that does not have the ideal property, and hence is not exact, and also, by Proposition 4.12, is not functorial for nondegenerate homomorphisms, and a fortiori is not functorial for generalized homomorphisms. Let

$$\mathcal{R} = \big\{\big(C[0,1) \otimes C^*(G), \mathrm{id} \otimes \delta_G\big)\big\},$$

and for every coaction (A, δ) let $\mathcal{R}_{(A,\delta)}$ be the collection of all triples (B, ϵ, ϕ), where either $(B, \epsilon) \in \mathcal{R}$ and $\phi : A \to B$ is a $\delta - \epsilon$ equivariant homomorphism or $(B, \epsilon) = (A^n, \delta^n)$ and $\phi : A \to A^n$ is the normalization map. Then let

$$\Big(\bigoplus\nolimits_{(B,\epsilon,\phi)\in\mathcal{R}_{A,\delta}} (B, \epsilon), \bigoplus\nolimits_{(B,\epsilon,\phi)\in\mathcal{R}_{A,\delta}} \epsilon\Big)$$

be the direct-sum coaction. Define a nondegenerate $\delta - \bigoplus_{(B,\epsilon,\phi)\in\mathcal{R}_{A,\delta}} \epsilon$ equivariant homomorphism

$$Q_A^{\mathcal{R}} = \bigoplus\nolimits_{(B,\epsilon,\phi)\in\mathcal{R}_{A,\delta}} \phi : A \to M\Big(\bigoplus\nolimits_{(B,\epsilon,\phi)\in\mathcal{R}_{A,\delta}} B\Big),$$

and let $A^{\mathcal{R}} = Q_A^{\mathcal{R}}(A)$. Then there is a unique coaction $\delta^{\mathcal{R}}$ of G on $A^{\mathcal{R}}$ such that $Q_A^{\mathcal{R}}$ is $\delta - \delta^{\mathcal{R}}$ equivariant. Moreover, for every morphism $\phi : (A, \delta) \to (B, \epsilon)$ in the classical category of coactions there is a unique homomorphism $\phi^{\mathcal{R}}$ making the diagram

$$\begin{tikzcd}
(A, \delta) \arrow[r, "\phi"] \arrow[d, "Q_A^{\mathcal{R}}"'] & (B, \epsilon) \arrow[d, "Q_B^{\mathcal{R}}"] \\
(A^{\mathcal{R}}, \delta^{\mathcal{R}}) \arrow[r, dashed, "\phi^{\mathcal{R}}"'] & (B^{\mathcal{R}}, \epsilon^{\mathcal{R}})
\end{tikzcd}$$

commute, giving a decreasing coaction functor $\tau^{\mathcal{R}}$ with $(A^{\tau_{\mathcal{R}}}, \delta^{\tau_{\mathcal{R}}}) = (A^{\mathcal{R}}, \delta^{\mathcal{R}})$ and $\phi^{\tau_{\mathcal{R}}} = \phi^{\mathcal{R}}$.

We will show that (assuming that G is nonamenable) the coaction functor $\tau_{\mathcal{R}}$ does not have the ideal property. Consider the coaction

$$(A, \delta) = \bigl(C[0, 1] \otimes C^*(G), \mathrm{id} \otimes \delta_G\bigr).$$

Then

$$I := C[0, 1) \otimes C^*(G)$$

is a strongly invariant ideal of A, because δ restricts on I to the coaction

$$\delta_I := \mathrm{id}_{C[0,1)} \otimes \delta_G.$$

To see that $Q_I^{\mathcal{R}}$ is faithful, note that $\mathcal{R}_{(I,\delta_I)}$ contains the triple $(I, \delta_I, \mathrm{id})$. On the other hand, to see that $Q_A^{\mathcal{R}}$ is not faithful on I, note that, since I has no nonzero projections, there is no nonzero homomorphism from $C[0, 1]$ to I, and hence no nonzero homomorphism from $A = C[0, 1] \otimes C^*(G)$ to I, and so the only morphism in $\mathcal{R}_{(A,\delta)}$ is the normalization map

$$\mathrm{id} \otimes \lambda : C[0, 1] \otimes C^*(G) \to C[0, 1] \otimes C_r^*(G),$$

which is not faithful on I because G is nonamenable.

Proposition 4.15. *Let $\mathcal{T}$ be a nonempty family of coaction functors. If every functor in $\mathcal{T}$ is functorial for generalized homomorphisms, then so is* $\mathrm{glb}\,\mathcal{T}$.

Proof. Let $\phi : A \to M(B)$ be a $\delta - \epsilon$ equivariant homomorphism. We must show

$$\ker q_A^\sigma \subset \ker(q_B^\sigma \circ \phi^m),$$

equivalently

$$\phi^m(\ker q_A^\sigma)B^m \subset \ker q_B^\sigma. \tag{4-4}$$

For each $\tau \in \mathcal{T}$ we have

$$\phi^m(\ker q_A^\tau)B^m \subset \ker q_B^\tau \subset \ker q_B^\sigma,$$

so by linearity

$$\phi^m\Bigl(\operatorname*{span}_{\tau\in\mathcal{T}} \ker q_A^\tau\Bigr)B^m = \operatorname*{span}_{\tau\in\mathcal{T}} \phi^m(\ker q_A^\tau)B^m \subset \ker q_B^\sigma,$$

and hence by density and continuity

$$\phi^m\Bigl(\operatorname*{\overline{span}}_{\tau\in\mathcal{T}} \ker q_A^\tau\Bigr)B^m \subset \ker q_B^\sigma.$$

By definition of greatest lower bound, we have verified (4-4). □

Proposition 4.16. *Let $\mathcal{T}$ be a nonempty family of coaction functors. If every functor in $\mathcal{T}$ has the ideal property, then so does* $\mathrm{glb}\,\mathcal{T}$.

Proof. Let (A,δ) be a coaction, let I be a strongly invariant ideal of A, and let $\iota : I \hookrightarrow A$ denote the inclusion map. We must show that the induced map

$$\iota^\sigma : I^\sigma \to A^\sigma$$

is injective, equivalently

$$\iota^m(\ker q_I^\sigma) = \iota^m(I^m) \cap \ker q_A^\sigma. \tag{4-5}$$

We know that for every $\tau \in \mathcal{T}$ the map

$$\iota^\tau : I^\tau \to A^\tau$$

is injective. The computation justifying (4-5) is the same as part of the proof of [Kaliszewski et al. 2016a, Theorem 4.22]:

$$\begin{aligned}
&\iota^m(\ker q_I^\sigma)\\
&\quad = \iota^m\Bigl(\overline{\operatorname{span}}_{\tau\in\mathcal{T}} \ker q_I^\tau\Bigr)\\
&\quad = \overline{\operatorname{span}}_{\tau\in\mathcal{T}} \iota^m(\ker q_I^\tau)\\
&\quad = \overline{\operatorname{span}}_{\tau\in\mathcal{T}}\bigl(\iota^m(I^m)\cap \ker q_A^\tau\bigr) && \text{(since } \tau \text{ has the ideal property)}\\
&\quad = \iota^m(I^m) \cap \overline{\operatorname{span}}_{\tau\in\mathcal{T}} \ker q_A^\tau && \text{(since all spaces involved are ideals in } C^*\text{-algebras)}\\
&\quad = \iota^m(I^m)\cap \ker q_A^\sigma. && \square
\end{aligned}$$

This might be an appropriate place to record a similar fact for decreasing coaction functors:

Proposition 4.17. *The greatest lower bound of any family of decreasing coaction functors is itself decreasing.*

Proof. We first point out a routine fact: if σ and τ are coaction functors, and if $\sigma \le \tau$ and τ is decreasing, then σ is decreasing. To see this, let (A,δ) be a coaction. Since $\sigma \le \tau$,

$$\ker q_A^\tau \subset \ker q_A^\sigma.$$

Since τ is decreasing,

$$\ker \psi_A \subset \ker q_A^\tau.$$

Thus $\ker \psi_A \subset \ker q_A^\sigma$, so σ is decreasing.

Now let σ be the greatest lower bound of $\mathcal{T}$. For every $\tau \in \mathcal{T}$ we have $\sigma \le \tau$ and τ is decreasing, so σ is decreasing. $\square$

5. Correspondence property

Given C^*-algebras A and B, recall that an $A-B$ *correspondence* is a Hilbert B-module X equipped with a homomorphism $\varphi_A : A \to \mathcal{L}(X)$, inducing a left A-module structure via $ax = \varphi_A(a)x$. We sometimes write $X = {}_AX_B$ to emphasize A and B. If $A = B$ we call X an *A-correspondence*.

The closed span of the inner product, written $\operatorname{span}\{\langle X, X\rangle_B\}$, is an ideal of B, and X is *full* if this ideal is dense. By the Cohen–Hewitt factorization theorem, the set $AX = \{ax : a \in A, x \in X\}$ is an $A-B$ subcorrespondence, and X is *nondegenerate* if $AX = X$.

If $\phi : A \to M(B)$ is a homomorphism, the associated *standard $A-B$ correspondence*, denoted by ${}_AB_B$, has left-module homomorphism $\varphi_A = \phi$.

If X is an $A-B$ correspondence and Y is a $C-D$ correspondence, a *correspondence homomorphism* from X to Y is a triple (π, ψ, ρ), where $\pi : A \to C$ and $\rho : B \to D$ are homomorphisms and $\psi : X \to Y$ is a linear map such that $\psi(ax) = \pi(a)\psi(x)$, $\psi(xb) = \psi(x)\rho(b)$, and $\langle\psi(x), \psi(y)\rangle_D = \rho(\langle x, y\rangle_B)$ (and recall that the second property, involving xb, is automatic). If π and ρ are understood we sometimes write ψ for the correspondence homomorphism. If π, ψ, and ρ are all bijections then ψ is a *correspondence isomorphism*, and we write $X \simeq Y$. If $A = C$, $B = D$, $\pi = \mathrm{id}_A$, and $\rho = \mathrm{id}_B$, we call ψ an *$A-B$ correspondence homomorphism*, and an *$A-B$ correspondence isomorphism* is an $A-B$ correspondence homomorphism that is also a correspondence isomorphism.

An *$A-B$ Hilbert bimodule* is an $A-B$ correspondence X equipped with a left A-valued inner product ${}_A\langle\cdot,\cdot\rangle$ that is compatible with the B-valued one. X is *left-full* if $\overline{\operatorname{span}}\{{}_A\langle X, X\rangle\} = A$; to avoid ambiguity we sometimes say X is *right-full* if $\overline{\operatorname{span}}\{\langle X, X\rangle_B\} = B$. If X is both left- and right-full, it is an *$A-B$ imprimitivity bimodule*. We write X^* for the *reverse* $B-A$ Hilbert bimodule.[2] The *linking algebra* of an $A-B$ Hilbert bimodule X is $L(X) = \left(\begin{smallmatrix} A & X \\ X^* & B \end{smallmatrix}\right)$, but we frequently just write $\left(\begin{smallmatrix} A & X \\ * & B \end{smallmatrix}\right)$ because the lower-left corner takes care of itself. The linking algebra of the reverse bimodule is $L(X^*) = \left(\begin{smallmatrix} B & X^* \\ X & B \end{smallmatrix}\right)$. The linking algebra of an $A-B$ correspondence X is defined as the linking algebra of the associated (left-full) $\mathcal{K}(X) - B$ Hilbert bimodule.

Recall from [Echterhoff et al. 2006, Definition 1.7] that if X is an $A-B$ correspondence and I is an ideal of B, then XI is an $A-B$ subcorrespondence of X, and the ideal

$$X\text{-Ind}\, I = X\text{-Ind}_B^A\, I := \{a \in A : aX \subset XI\}$$

of A is said to be *induced from I via X*. If $X \simeq Y$ as $A-B$ correspondences, then $X\text{-Ind}\, I = Y\text{-Ind}\, I$ for every ideal I of B.

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[2] Although the notation $\widetilde{X}$ is perhaps more common, it would conflict with another usage of $\widetilde{\ }$ we will need later.

The quotient X/XI becomes an $(A/X\text{-Ind}\, I)-(B/I)$ correspondence.

Let $J=\overline{\operatorname{span}}\{\langle X, X\rangle_B\}$. Then X is a nondegenerate right J-module and J is an ideal of B, so

$$XI=(XJ)I=X(JI)=X(JI).$$

Thus $X\text{-Ind}\, I=X\text{-Ind}(JI)$. Moreover, X may also be regarded as an $A-J$ correspondence, and the quotient X/XI may also be regarded as an $(A/X\text{-Ind}_J^A(JI))-(J/(JI))$ correspondence.

If I and J are ideals of B, and we regard J as a $J-B$ correspondence with the given algebraic operations, then

$$J\text{-Ind}_B^J\, I=\{a\in J : aJ\subset JI\}=JI.$$

On the other hand, regarding B as a $J-B$ correspondence with the given algebraic operations, then, since $BI=I$, we nevertheless still get the same result:

$$B\text{-Ind}_B^J\, I=\{a\in J : aB\subset I\}=J\cap I=JI.$$

Given a homomorphism $\phi : A\to M(B)$ and an ideal I of B, and regarding B as the associated standard $A-B$ correspondence (with left-module multiplication given by $a\cdot b=\phi(a)b$ for $a\in A$ and $b\in B$), then

$$B\text{-Ind}_B^A\, I=\{a\in A : \phi(a)B\subset I\}$$

is sometimes denoted by $\phi^*(I)$.

Regarding A as a standard $A-A$ correspondence, for every ideal I of A we have $A\text{-Ind}_A^A\, I=I$.

If X is an $A-B$ correspondence and Y is a $B-C$ correspondence, we write $X\otimes_B Y$ for the balanced tensor product, which is an $A-C$ correspondence. Letting $K=\mathcal{K}(X)$, X becomes a left-full $K-B$ Hilbert bimodule, and

$${}_AX_B\simeq({}_AK_K)\otimes_K({}_KY_B).$$

Letting $J=\overline{\operatorname{span}}\{\langle X, X\rangle_B\}$, X becomes a full $A-J$ correspondence, and

$${}_AX_B\simeq({}_AX_J)\otimes_J({}_JB_B).$$

By Rieffel's induction in stages theorem, if X is an $A-B$ correspondence, Y is a $B-C$ correspondence, and I is an ideal of C, then

$$(X\otimes_B Y)\text{-Ind}_C^A\, I=X\text{-Ind}_B^A\, Y\text{-Ind}_C^B\, I.$$

If X is an $A-B$ imprimitivity bimodule then

$$X^*\otimes_A X\simeq {}_BB_B,$$

so if I is an ideal of B, then

$$X^*\text{-Ind}_A^B\, X\text{-Ind}_B^A\, I=I.$$

Given actions α and β of G on A and B, respectively, and an $\alpha-\beta$ compatible action γ on X, we say (X,γ) is an $(A,\alpha)-(B,\beta)$ *correspondence action*. The crossed product $X\rtimes_\gamma G$ is an $(A\rtimes_\alpha G)-(B\rtimes_\beta G)$ correspondence, and we let $i_X: X\to M(X\rtimes_\gamma G)$ denote the canonical i_A-i_B compatible correspondence homomorphism. Writing $\gamma^{(1)}$ for the induced action of G on $\mathcal{K}(X)$, there is a canonical isomorphism

$$\mathcal{K}(X\rtimes_\gamma G)\simeq\mathcal{K}(X)\rtimes_{\gamma^{(1)}} G,$$

and, blurring the distinction between these two isomorphic algebras, the left-module homomorphism of the crossed-product correspondence is given by

$$\varphi_{A\rtimes_\alpha G}=\varphi_A\rtimes G: A\rtimes_\alpha G\to M(\mathcal{K}(X)\rtimes_{\gamma^{(1)}} G).$$

In particular, if X is a left-full $A-B$ Hilbert bimodule, then $X\rtimes_\gamma G$ is a left-full $(A\rtimes_\alpha G)-(B\rtimes_\beta G)$ bimodule, and is moreover an imprimitivity bimodule if X is.

Let (X,γ) be an $(A,\alpha)-(B,\beta)$ correspondence action, and let $J=\overline{\operatorname{span}}\{\langle X,X\rangle_B\}$. Then J is a β-invariant ideal of B, and we write η for the action on J gotten by restricting β. As in [Echterhoff et al. 2006, Proposition 3.2],[3]

$$\overline{\operatorname{span}}\langle X\rtimes_\gamma G, X\rtimes_\gamma G\rangle_{B\rtimes_\beta G}=J\rtimes_\eta G,$$

where the latter is identified with an ideal of $B\rtimes_\beta G$ in the canonical way.

If (X,γ) is an $(A,\alpha)-(B,\beta)$ Hilbert bimodule action (so that ${}_A\langle\gamma_s(x),\gamma_s(y)\rangle=\alpha_s({}_A\langle x,y\rangle)$ also), there are a canonical $\beta-\alpha$ compatible action γ^* on X^* and a canonical isomorphism

$$(X\rtimes_\gamma G)^*\simeq X^*\rtimes_{\gamma^*} G.$$

Dually, given coactions δ and ϵ of G on A and B, respectively, and a $\delta-\epsilon$ compatible coaction ζ on X, we say (X,ζ) is an $(A,\delta)-(B,\epsilon)$ *correspondence coaction*. The crossed product $X\rtimes_\zeta G$ is an $(A\rtimes_\delta G)-(B\rtimes_\epsilon G)$ correspondence, and we let $j_X: X\to M(X\rtimes_\zeta G)$ denote the canonical j_A-j_B compatible correspondence homomorphism. Writing $\zeta^{(1)}$ for the induced coaction of G on $\mathcal{K}(X)$, there is a canonical isomorphism

$$\mathcal{K}(X\rtimes_\zeta G)\simeq\mathcal{K}(X)\rtimes_{\zeta^{(1)}} G,$$

and, blurring the distinction between these two isomorphic algebras, the left-module homomorphism of the crossed-product correspondence is given by

$$\varphi_{A\rtimes_\delta G}=\varphi_A\rtimes G: A\rtimes_\delta G\to M(\mathcal{K}(X)\rtimes_{\zeta^{(1)}} G).$$

In particular, if X is a left-full $A-B$ Hilbert bimodule, then $X\rtimes_\zeta G$ is a left-full $(A\rtimes_\delta G)-(B\rtimes_\epsilon G)$ bimodule, and is moreover an imprimitivity bimodule if X is.

Suppose that (X,ζ) is an $(A,\delta)-(B,\epsilon)$ correspondence coaction, and let $J=\overline{\operatorname{span}}\{\langle X,X\rangle_B\}$. Then J is a strongly ϵ-invariant ideal of B [Echterhoff et al.

[3]The theory of [Echterhoff et al. 2006] uses reduced crossed products, but for the results of concern to us here the same techniques handle the case of full crossed products.

2006, Lemma 2.32], and we write η for the coaction on J gotten by restricting ϵ. As in [Echterhoff et al. 2006, Proposition 3.9],

$$\overline{\operatorname{span}}\langle X \rtimes_\zeta G, X \rtimes_\zeta G\rangle_{B\rtimes_\epsilon G} = J \rtimes_\eta G,$$

where the latter is identified with an ideal of $B \rtimes_\epsilon G$ in the canonical way.

If (X, ζ) is an $(A, \delta) - (B, \epsilon)$ Hilbert-bimodule coaction (so that

$${}_{M(A\otimes C^*(G))}\langle \zeta(x), \zeta(y)\rangle = \delta({}_A\langle x, y\rangle)$$

also), there are a canonical $\epsilon - \delta$ compatible coaction ζ^* on X^* and a canonical isomorphism

$$(X \rtimes_\zeta G)^* \simeq X^* \rtimes_{\zeta^*} G.$$

If (X, γ) is an $(A, \alpha) - (B, \beta)$ correspondence action, the *dual coaction* $\hat{\gamma}$ on $X\rtimes_\gamma G$ is $\hat{\alpha}-\hat{\beta}$ compatible, and dually if (X, ζ) is an $(A, \delta)-(B, \epsilon)$ correspondence coaction, the *dual action* $\hat{\zeta}$ on $X \rtimes_\zeta G$ is $\hat{\delta} - \hat{\epsilon}$ compatible. Moreover, if (X, γ) is an $(A, \alpha) - (B, \beta)$ Hilbert-bimodule action, the isomorphism $(X \rtimes_\gamma G)^* \simeq X^* \rtimes_{\gamma^*} G$ is $\hat{\gamma}^* - \widehat{\gamma^*}$ equivariant, and dually if (X, ζ) is an $(A, \delta) - (B, \epsilon)$ Hilbert bimodule coaction, the isomorphism $(X \rtimes_\zeta G)^* \simeq X^* \rtimes_{\zeta^*} G$ is $\hat{\zeta}^* - \widehat{\zeta^*}$ equivariant.

Given equivariant actions (A, α, μ) and (B, β, ν), and an $(A, \alpha) - (B, \beta)$ correspondence action (X, γ), by [Kaliszewski et al. 2017, Lemma 6.1], there is an $\tilde{\alpha} - \tilde{\beta}$ compatible coaction[4] $\tilde{\gamma}$ on $X \rtimes_\gamma G$ given by

$$\tilde{\gamma}(y) = V_A\hat{\gamma}(y)V_B^*.$$

Moreover, if (X, γ) is a Hilbert bimodule action, the isomorphism $(X \rtimes_\gamma G)^* \simeq X^* \rtimes_{\gamma^*} G$ is $\tilde{\gamma}^* - \widetilde{\gamma^*}$ equivariant.[5]

Given $\mathcal{K}$-algebras (A, ι) and $(B, \jmath)$, and an $A - B$ correspondence X, Theorem 6.4 of [Kaliszewski et al. 2016c] and its proof construct a $C(A, \iota) - C(B, \jmath)$ correspondence $C(X, \iota, \jmath)$ given by

$$C(X, \iota, \jmath) = \{x \in M(X) : \iota(k)\cdot x = x\cdot \jmath(k) \in X \text{ for all } k \in \mathcal{K}\}.$$

Writing $\kappa : \mathcal{K} \to M(\mathcal{K}(X))$ for the induced nondegenerate homomorphism, there is a canonical isomorphism

$$\mathcal{K}\big(C(X, \iota, \jmath)\big) \simeq C\big(\mathcal{K}(X), \kappa\big),$$

and, blurring the distinction between these two isomorphic algebras, the left-module homomorphism of the relative-commutant correspondence is given by

$$\varphi_{C(A,\iota)} = C(\varphi_A) : C(A, \iota) \to M(C(\mathcal{K}(X), \kappa)).$$

[4] Recall from Section 2 the definition of $\tilde{\alpha}$. We define $\tilde{\beta}$ similarly.

[5] Here is where the notation * for the reverse bimodule is important.

In particular, if X is a left-full $A-B$ Hilbert bimodule, then $C(X, \iota, \jmath)$ is a left-full $C(A, \iota)-C(B, \jmath)$ bimodule, and is moreover an imprimitivity bimodule if X is.

Given $\mathcal{K}$-coactions (A, δ, ι) and $(B, \epsilon, \jmath)$, and an $(A, \delta)-(B, \epsilon)$ correspondence coaction (X, ζ), by [Kaliszewski et al. 2017, Lemma 6.3] there is a $C(\delta)-C(\epsilon)$ compatible coaction $C(\zeta)$ on $C(X, \iota, \jmath)$ given by the restriction of the canonical extension to $M(X)$ of ζ. As before, let $J = \overline{\operatorname{span}}\{\langle X, X\rangle_B\}$, and let $\eta = \epsilon_J$ be the restricted coaction. Letting $\rho : B \to M(J)$ be the canonical homomorphism, which is nondegenerate, we can define a nondegenerate homomorphism

$$\omega = \rho \circ \jmath : \mathcal{K} \to M(J),$$

and (J, η, ω) is a $\mathcal{K}$-coaction. It is not hard to verify that

$$\overline{\operatorname{span}}\{\langle C(X, \iota, \jmath), C(X, \iota, \jmath)\rangle_{C(B,\jmath)}\} = C(J, \omega),$$

which we identify with an ideal of $C(B, \jmath)$.

If (A, δ, ι) and $(B, \epsilon, \jmath)$ are $\mathcal{K}$-coactions and X is an $(A, \delta)-(B, \epsilon)$ Hilbert bimodule coaction, there is an isomorphism

$$C(X, \iota, \jmath)^* \simeq C(X^*, \jmath, \iota)$$

of $C(B, \jmath)-C(A, \iota)$ Hilbert bimodules, and moreover this isomorphism is $C(\zeta)^*-C(\zeta^*)$ equivariant.

Recall that the maximalization of a coaction (A, δ) is the coaction

$$(A^m, \delta^m) = \big(C(A \rtimes_\delta G \rtimes_{\hat{\delta}} G, j_G^\delta \rtimes G), C(\tilde{\delta})\big),$$

where

$$\tilde{\delta} = \tilde{\hat{\hat{\delta}}} = \operatorname{Ad} V_{A\rtimes_\delta G} \circ \hat{\hat{\delta}}.$$

Definition 5.1. Given coactions (A, δ) and (B, ϵ), the *maximalization* of an $(A, \delta)-(B, \epsilon)$ correspondence coaction (X, ζ) is the $(A^m, \delta^m)-(B^m, \epsilon^m)$ correspondence coaction

$$(X^m, \zeta^m) := \big(C(X \rtimes_\zeta G \rtimes_{\hat{\zeta}} G, j_G^\delta \rtimes G, j_G^\epsilon \rtimes G), C(\tilde{\zeta})\big),$$

where

$$\tilde{\zeta}(y) = \tilde{\hat{\hat{\zeta}}}(y) = V_{A\rtimes_\delta G}\, \hat{\hat{\zeta}}(y) V_{B\rtimes_\epsilon G}$$

for $y \in X^m$.

There is a canonical isomorphism

$$\big(\mathcal{K}(X^m), (\zeta^m)^{(1)}\big) \simeq \big(\mathcal{K}(X)^m, (\zeta^{(1)})^m\big). \tag{5-1}$$

Blurring the distinction between these two isomorphic algebras, the left-module homomorphism of the $A^m - B^m$ correspondence X^m is given by

$$\varphi_{A^m} = \varphi_A^m : A^m \to M(\mathcal{K}(X)^m) = M(\mathcal{K}(X^m)).$$

In particular, if X is a left-full $A-B$ Hilbert bimodule, then X^m is a left-full A^m-B^m Hilbert bimodule, and is moreover an imprimitivity bimodule if X is.

Letting $J=\overline{\text{span}}\{\langle X, X\rangle_B\}$ with coaction $\eta=\epsilon_J$ as before, it follows from the above properties of the functors in the factorization of the Fischer construction that

$$\overline{\text{span}}\{\langle X^m, X^m\rangle_{B^m}\}=J^m,$$

which we identify with an ideal of B^m.

If (X,ζ) is an $(A,\delta)-(B,\epsilon)$ Hilbert bimodule coaction, then it follows from the properties of the steps in the Fischer construction that there is a canonical isomorphism

$$(X^{m*},\zeta^{m*})\simeq(X^{*m},\zeta^{*m}).$$

Let τ be a coaction functor, and let (X,ζ) be a Hilbert (B,ϵ)-module coaction (equivalently, a $(\mathbb{C},\delta_{\text{triv}})-(B,\epsilon)$ correspondence coaction, where δ_{triv} is the trivial coaction on $\mathbb{C}$). Then $X^m\ker q_B^\tau$ is a Hilbert B^m-submodule of X^m. We define

$$X^\tau=X^m/X^m\ker q_B^\tau,$$

which is a Hilbert B^τ-module, and we further write

$$q_X^\tau: X^m\to X^\tau$$

for the quotient map, which is a surjective homomorphism of the Hilbert B^m-module X^m onto the Hilbert B^τ-module X^τ. It follows quickly from the definitions that there is a (necessarily unique) Hilbert-module homomorphism ζ^τ making the diagram

$$\begin{tikzcd}
X^m \arrow[r, "\zeta^m"] \arrow[d, "q_X^\tau"'] & \widetilde{M}(X^m\otimes C^*(G)) \arrow[d, "q_X^\tau\otimes\text{id}"] \\
X^\tau \arrow[r, dashed, "\zeta^\tau"'] & \widetilde{M}(X^\tau\otimes C^*(G))
\end{tikzcd}$$

commute, and that ζ^τ is moreover a coaction on the Hilbert B^τ-module X^τ. Let

$$(q_X^\tau)^{(1)}:\mathcal{K}(X^m)\to\mathcal{K}(X^\tau)$$

be the induced surjection, which is equivariant for the induced coactions $(\zeta^m)^{(1)}$ on $\mathcal{K}(X^m)$ and $(\zeta^\tau)^{(1)}$ on $\mathcal{K}(X^\tau)$.

Recall from [Kaliszewski et al. 2016a, Definition 4.16] that we call a coaction functor τ Morita compatible if whenever (X,ζ) is an $(A,\delta)-(B,\epsilon)$ imprimitivity-bimodule coaction we have

$$\ker q_A^\tau=X^m\text{-Ind}\ker q_B^\tau.$$

Remark 5.2. Lemma 4.19 of [Kaliszewski et al. 2016a] says that a coaction functor τ is Morita compatible if and only if for every $(A,\delta)-(B,\epsilon)$ imprimitivity-bimodule coaction (X,ζ) the maximalization X^m descends to an $A^\tau - B^\tau$ imprimitivity bimodule X^τ. Thus, if CP^τ is the crossed-product functor given by τ composed with the full crossed product, then Morita compatibility of τ implies that CP^τ is *strongly Morita compatible* in the sense of [BEW, Definition 4.7].

Example 5.3. The maximalization functor, and also the functors τ_E for large ideals E of $B(G)$, are Morita compatible, by [Kaliszewski et al. 2016a, Lemma 4.15, Remark 4.18, and Proposition 6.10].

Remark 5.4. Proposition 5.5 of [Kaliszewski et al. 2016a] can be equivalently stated as follows: a decreasing coaction functor τ is Morita compatible if and only if whenever (X,ζ) is an $(A,\delta)-(B,\epsilon)$ imprimitivity-bimodule coaction we have

$$\ker Q_A^\tau = X\text{-}\mathrm{Ind}_B^A \ker Q_B^\tau.$$

Remark 5.5. Let (A,δ) be a coaction, and let I be a strongly δ-invariant ideal of A. The diagram

$$\begin{tikzcd} I^m \arrow[r, "\iota^m"] \arrow[d, "q_I^\tau"'] & A^m \arrow[d, "q_A^\tau"] \\ I^\tau \arrow[r, "\iota^\tau"'] & A^\tau \end{tikzcd} \tag{5-2}$$

commutes because τ is a coaction functor. The top arrow is always injective, so we can identify I^m with the ideal $\iota^m(I^m)$ of A^m. Thus we always have

$$\ker q_I^\tau \subset \ker(q_A^\tau \circ \iota^m) = I^m \cap \ker q_A^\tau,$$

and since $\ker q_I^\tau \subset I^m$ we have $\ker q_I^\tau \subset \ker q_A^\tau$. The ideal property for τ means that the bottom arrow is injective, equivalently

$$\ker q_I^\tau = I^m \cap \ker q_A^\tau, \tag{5-3}$$

in which case the quotient map q_I^τ may be regarded as the restriction of q_A^τ to the ideal I^m.

Lemma 5.6. *Let τ be a coaction functor that has the ideal property. Then τ is Morita compatible if and only if for every left-full $(A,\delta)-(B,\epsilon)$ Hilbert-bimodule coaction (X,ζ) we have*

$$\ker q_A^\tau = X^m\text{-}\mathrm{Ind}_{B^m}^{A^m} \ker q_B^\tau. \tag{5-4}$$

Proof. The condition involving (5-4) of course implies Morita compatibility, so suppose that τ is Morita compatible and (X,ζ) is a left-full $(A,\delta)-(B,\epsilon)$ Hilbert-bimodule coaction.

As before, let $J=\overline{\text{span}}\{\langle X, X\rangle_B\}$ with the restricted coaction $\eta=\epsilon_J$. Then (X,ζ) is an $(A,\delta)-(J,\eta)$ imprimitivity-bimodule coaction, so by Morita compatibility we have

$$\ker q_A^\tau = X^m\text{-Ind}_{J^m}^{A^m}\ker q_J^\tau. \tag{5-5}$$

Identify J^m with an ideal of B^m in the usual way. Regarding B^m as a standard J^m-B^m correspondence, we have

$$\ker q_J^\tau = J^m\cap\ker q_B^\tau = B^m\text{-Ind}_{B^m}^{J^m}\ker q_B^\tau. \tag{5-6}$$

Thus by induction in stages we can combine (5-5) and (5-6) to conclude that

$$\ker q_A^\tau = X^m\text{-Ind}_{B^m}^{A^m}\ker q_B^\tau. \qquad\square$$

Definition 5.7. We say that a coaction functor τ has the *correspondence property* if for every $(A,\delta)-(B,\epsilon)$ correspondence coaction (X,ζ) we have

$$\ker q_A^\tau \subset X^m\text{-Ind}_{B^m}^{A^m}\ker q_B^\tau.$$

Note that we have a commutative diagram

$$\begin{tikzcd} A^m \arrow[r, "\varphi_{A^m}"] \arrow[d] & \mathcal{L}(X^m) \arrow[d, "q_X^\tau"] \\ A^m/X^m\text{-Ind}\ker q_B^\tau \arrow[r] & \mathcal{L}(X^\tau) \end{tikzcd}$$

with

$$X^m\text{-Ind}\ker q_B^\tau = \ker(q_X^\tau\circ\varphi_{A^m}).$$

The composition $q_X^\tau\circ\varphi_{A^m}$ gives X^τ a left A^m-module multiplication, and τ has the correspondence property if and only if this left A^m-module multiplication on X^τ factors through a left A^τ-module multiplication, making (X^τ,ζ^τ) into a $(A^\tau,\delta^\tau)-(B^\tau,\epsilon^\tau)$ correspondence coaction.

Example 5.8. Trivially the maximalization functor has the correspondence property.

Theorem 5.9. *A coaction functor τ has the correspondence property if and only if it is Morita compatible and functorial for generalized homomorphisms.*

Proof. First assume that τ has the correspondence property. For the Morita compatibility, let (X,ζ) be an $(A,\delta)-(B,\epsilon)$ imprimitivity bimodule coaction. We must show that

$$\ker q_A^\tau = X^m - \text{Ind}\ker q_B^\tau. \tag{5-7}$$

By the correspondence property the left side is contained in the right side. Since

(X^*, ζ^*) is a $(B, \epsilon) - (A, \delta)$ imprimitivity bimodule coaction, we also have

$$\ker q_B^\tau \subset X^{*m}\text{-Ind} \ker q_A^\tau.$$

By induction in stages and the properties of reverse bimodules,

$$\ker q_A^\tau \subset X^m\text{-Ind} \ker q_B^\tau \subset X^m\text{-Ind}\, X^{*m}\text{-Ind} \ker q_A^\tau = \ker q_A^\tau,$$

so we must have equality throughout, and in particular (5-7) holds.

For the functoriality, let $\phi : A \to M(B)$ be a $\delta - \epsilon$ equivariant homomorphism. Then (B, ϵ) is a standard $(A, \delta) - (B, \epsilon)$ correspondence coaction. By assumption, we have $\ker q_A^\tau \subset B^m\text{-Ind} \ker q_B^\tau$. Since

$$B^m\text{-Ind} \ker q_B^\tau = \{a \in A^m : \phi^m(a) B^m \subset \ker q_B^\tau\} = \ker(q_B^\tau \circ \phi^m),$$

τ is functorial for generalized homomorphisms.

Conversely, assume that τ is Morita compatible and functorial for generalized homomorphisms. Let (X, ζ) be an $(A, \delta) - (B, \epsilon)$ correspondence coaction. We need to show that

$$\ker q_A^\tau \subset X^m\text{-Ind}_{B^m}^{A^m} \ker q_B^\tau. \tag{5-8}$$

Let $K = \mathcal{K}(X)$, with induced coaction μ. Let $\varphi_A : A \to M(K)$ be the left-module homomorphism, which is $\delta - \mu$ equivariant. We use the associated $\delta^m - \mu^m$ equivariant homomorphism $\varphi_A^m : A^m \to M(K^m)$ to regard (K^m, ζ^m) as a standard $(A^m, \delta^m) - (K^m, \mu^m)$ correspondence coaction. By functoriality for generalized homomorphisms we have

$$\ker q_A^\tau \subset K^m\text{-Ind}_{K^m}^{A^m} \ker q_K^\tau. \tag{5-9}$$

Note that (X, ζ) may be regarded as a left-full $(K, \mu) - (B, \epsilon)$ Hilbert-bimodule coaction. Since τ is functorial for generalized homomorphisms, by Proposition 4.12 it has the ideal property, so, since τ is also assumed to be Morita compatible, by Lemma 5.6 we have

$$\ker q_K^\tau = X^m\text{-Ind}_{B^m}^{K^m} \ker q_B^\tau. \tag{5-10}$$

By induction in stages we can combine (5-9) and (5-10) to deduce (5-8). □

Remark 5.10. Although we do not need it in the current paper, it is natural to wonder whether a coaction functor with the correspondence property will automatically be functorial under composition of correspondences. More precisely, let τ be a coaction functor with the correspondence property, and let (X, ζ) and (Y, η) be $(A, \delta) - (B, \epsilon)$ and $(B, \epsilon) - (C, \nu)$ correspondence coactions, respectively. Then the balanced tensor product $(X \otimes_B Y, \zeta \sharp \eta)$ is an $(A, \delta) - (C, \nu)$ correspondence coaction (see [Echterhoff et al. 2006, Proposition 2.13]). The assumption that τ has the correspondence property implies that there are $(A^\tau,\delta^\tau)-(B^\tau,\epsilon^\tau)$, $(B^\tau,\epsilon^\tau)-(C^\tau,\nu^\tau)$, and $(A^\tau,\delta^\tau)-(C^\tau,\nu^\tau)$ correspondence coactions (X^τ,ζ^τ), (Y^τ,η^τ), and $((X\otimes_B Y)^\tau,(\zeta\sharp\eta)^\tau)$, respectively. The functoriality property we are wondering about here is whether there is a natural isomorphism

$$\big((X\otimes_B Y)^\tau,(\zeta\sharp\eta)^\tau\big)\simeq(X^\tau\otimes_{B^\tau}Y^\tau,\zeta^\tau\sharp\eta^\tau)$$

of $(A^\tau,\delta^\tau)-(C^\tau,\nu^\tau)$ correspondence coactions. It seems plausible that this could be checked via a tedious diagram chase, or via linking algebras.

Example 5.11. Combining Example 4.8, Example 5.3, and Theorem 5.9, we see that τ_E has the correspondence property for every large ideal E of $B(G)$.

Remark 5.12. Theorem 5.9 is similar to the equivalence (2)$\Longleftrightarrow$(3) in [BEW, Theorem 4.9], except that, as we mentioned in Remark 4.11, we have not been able to prove that for coaction functors the ideal property is equivalent to functoriality for generalized homomorphisms.

Remark 5.13. [BEW, Theorem 5.6] shows that every correspondence crossed-product functor produces C^*-algebras carrying a quotient of the dual coaction on the full crossed product. This reinforces our belief in the importance of studying crossed-product functors arising from coaction functors composed with the full cross product.

Corollary 5.14. *Let $\mathcal{T}$ be a nonempty family of coaction functors. If every functor in $\mathcal{T}$ has the correspondence property, then so does* $\operatorname{glb}\mathcal{T}$. *In particular, there is a smallest coaction functor with the correspondence property.*

Not surprisingly, the correspondence property is simpler for decreasing functors:

Lemma 5.15. *A decreasing coaction functor τ has the correspondence property if and only if for every $(A,\delta)-(B,\epsilon)$ correspondence coaction (X,ζ) we have*

$$\ker Q_A^\tau\subset X\text{-}\mathrm{Ind}_B^A\ker Q_B^\tau.$$

Proof. We must show that the stated condition involving Q_A^τ holds if and only if $\ker q_A^\tau\subset X^m\text{-}\mathrm{Ind}_{B^m}^{A^m}\ker q_B^\tau$. Let

$$I=\ker\psi_A,\qquad J=\ker\psi_B,\qquad K=\ker q_A^\tau,\qquad L=\ker q_B^\tau.$$

Then $I\subset K\cap X^m\text{-}\mathrm{Ind}\,J$, $I\subset K$, and $J\subset L$, and we can identify A with A^m/I, $\ker Q_A^\tau$ with K/I, X with X^m/X^mJ, B with B^m/J and $\ker Q_B^\tau$ with L/J, so the desired equivalence follows from the general Lemma 5.16 below, which is probably folklore. □

Lemma 5.16. *Let X be an $A-B$ correspondence, let $I\subset K$ be ideals of A, and let $J\subset L$ be ideals of B. Suppose that $I\subset X\text{-}\mathrm{Ind}\,J$, so that X/XJ is an $(A/I)-(B/J)$ correspondence. Then $K\subset X\text{-}\mathrm{Ind}\,L$ if and only if $K/I\subset(X/XJ)\text{-}\mathrm{Ind}\,L/J$.*

Proof. Let

$$\phi : A \to A/I, \quad \psi : X \to X/XJ, \quad \rho : B \to B/J$$

be the quotient maps. First assume that $K \subset X\text{-Ind}\, L$. Then

$$\begin{aligned}(K/I)(X/XJ) &= \phi(K)\psi(X)\\ &= \psi(KX) \subset \psi(XL)\\ &= \psi(X)\rho(L) = (X/XJ)(L/J),\end{aligned}$$

so $K/I \subset (X/XJ)\text{-Ind}\, L/J$.

Conversely, assume that $K/I \subset (X/XJ)\text{-Ind}\, L/J$. Then

$$\begin{aligned}KX &\subset \psi^{-1}(\psi(KX)) = \psi^{-1}\big(\phi(K)\psi(X)\big)\\ &\subset \psi^{-1}\big(\psi(X)\rho(L)\big) \overset{*}{=} \psi^{-1}(\psi(XL)) = XL,\end{aligned}$$

where the equality at * holds since ψ is a surjective homomorphism of correspondences and XL is a closed subcorrespondence containing $\ker \psi = KJ$. □

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CONSTRUCTION OF A RAPOPORT–ZINK SPACE FOR GU(1, 1) IN THE RAMIFIED 2-ADIC CASE

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Let $F|\mathbb{Q}_2$ be a finite extension. In this paper, we construct an RZ-space $\mathcal{N}_E$ for split GU(1, 1) over a ramified quadratic extension $E|F$. For this, we first introduce the naive moduli problem $\mathcal{N}_E^{\text{naive}}$ and then define $\mathcal{N}_E \subseteq \mathcal{N}_E^{\text{naive}}$ as a canonical closed formal subscheme, using the so-called straightening condition. We establish an isomorphism between $\mathcal{N}_E$ and the Drinfeld moduli problem, proving the 2-adic analogue of a theorem of Kudla and Rapoport. The formulation of the straightening condition uses the existence of certain polarizations on the points of the moduli space $\mathcal{N}_E^{\text{naive}}$. We show the existence of these polarizations in a more general setting over any quadratic extension $E|F$, where $F|\mathbb{Q}_p$ is a finite extension for any prime p.

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1. Introduction

Rapoport–Zink spaces (RZ-spaces for short) are moduli spaces of p-divisible groups endowed with additional structure. Rapoport and Zink [1996] studied two major classes of RZ-spaces, called EL type and PEL type. The abbreviations EL and PEL indicate, in analogy to the case of Shimura varieties, whether the extra structure comes in the form of Endomorphisms and Level structure or in the form of Polarizations, Endomorphisms and Level structure. Rapoport and Zink [1996] developed a theory of these spaces, including important theorems about the existence of local models and nonarchimedean uniformization of Shimura varieties, for the EL type and for the PEL type whenever $p \neq 2$.

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The blanket assumption $p \neq 2$ made by Rapoport and Zink in the PEL case is by no means of a cosmetic nature, but originates with various serious difficulties that arise for $p = 2$. However, we recall that one can still use their definition in that case to obtain "naive" moduli spaces that still satisfy basic properties like being representable by a formal scheme.

In this paper, we construct the 2-adic Rapoport–Zink space $\mathcal{N}_E$ corresponding to the group of unitary similitudes of size 2 relative to any (wildly) ramified quadratic extension $E|F$, where $F|\mathbb{Q}_2$ is a finite extension. It is given as the closed formal subscheme of the corresponding naive RZ-space $\mathcal{N}_E^{\mathrm{naive}}$ described by the so-called "straightening condition", which is defined below. The main result of this paper is a natural isomorphism $\eta : \mathcal{M}_{Dr} \xrightarrow{\sim} \mathcal{N}_E$, where $\mathcal{M}_{Dr}$ is Deligne's formal model of the Drinfeld upper half-plane (cf. [Boutot and Carayol 1991]). This result is in analogy with Kudla and Rapoport's construction [2014] of a corresponding isomorphism for $p \neq 2$ and also for $p = 2$ when $E|F$ is an unramified extension. The formal scheme $\mathcal{M}_{Dr}$ solves a certain moduli problem of p-divisible groups and, in this way, it carries the structure of an RZ-space of EL type. In particular, $\mathcal{M}_{Dr}$ is defined even for $p = 2$.

As in [Kudla and Rapoport 2014], there are natural group actions by $\mathrm{SL}_2(F)$ and the split $\mathrm{SU}_2(F)$ on the spaces $\mathcal{M}_{Dr}$ and $\mathcal{N}_E$, respectively. The isomorphism η is hence a geometric realization of the exceptional isomorphism of these groups. As a consequence, one cannot expect a similar result in higher dimensions. Of course, the existence of "good" RZ-spaces is still expected, but a general definition will probably need a different approach.

The study of residue characteristic 2 is interesting and important for the following reasons: First of all, from the general philosophy of RZ-spaces and, more generally, of local Shimura varieties [Rapoport and Viehmann 2014], it follows that there should be a uniform approach for all primes p. In this sense, the present paper is in the same spirit as the recent constructions of RZ-spaces of Hodge type of W. Kim [2013], Howard and Pappas [2017] and Bültel and Pappas [2017]. Second, Rapoport–Zink spaces have been used to determine the arithmetic intersection numbers of special cycles on Shimura varieties [Kudla et al. 2006]; in this kind of problem, it is necessary to deal with all places, even those of residue characteristic 2. Finally, studying the cases of residue characteristic 2 also throws light on the cases previously known. In the specific case at hand, the methods we develop also give a simplification of the proof for $p \neq 2$ of Kudla and Rapoport [2014]; see Remark 5.3 (2).

We will now explain the results of this paper in greater detail. Let F be a finite extension of $\mathbb{Q}_2$ and $E|F$ a ramified quadratic extension. Following [Jacobowitz 1962], we consider the following dichotomy for this extension (see Section 2):

(R-P) There is a uniformizer $\pi_0 \in F$ such that $E = F[\Pi]$ with $\Pi^2 + \pi_0 = 0$. Then the rings of integers O_F of F and O_E of E satisfy $O_E = O_F[\Pi]$.

(R-U) $E|F$ is given by an Eisenstein equation of the form $\Pi^2 - t\Pi + \pi_0 = 0$. Here, π_0 is again a uniformizer in F and $t \in O_F$ satisfies $\pi_0|t|2$. We still have $O_E = O_F[\Pi]$. Note that in this case $E|F$ is generated by a square root of the unit $1 - 4\pi_0/t^2$ in F.

An example of an extension of type R-P is $\mathbb{Q}_2(\sqrt{-2})|\mathbb{Q}_2$, whereas $\mathbb{Q}_2(\sqrt{-1})|\mathbb{Q}_2$ is of type R-U. Note that for $p > 2$, any ramified quadratic extension over $\mathbb{Q}_p$ is of the form R-P.

Our results in the cases R-P and R-U are similar, but different. We first describe the results in the case R-P. Let $E|F$ be of type R-P.

We first define a naive moduli problem $\mathcal{N}_E^{\text{naive}}$, which merely copies the definition from $p \neq 2$ (cf. [Kudla and Rapoport 2014]). Let $\breve{F}$ be the completion of the maximal unramified extension of F and $\breve{O}_F$ its ring of integers. Then $\mathcal{N}_E^{\text{naive}}$ is a set-valued functor on $\mathrm{Nilp}_{\breve{O}_F}$, the category of $\breve{O}_F$-schemes where π_0 is locally nilpotent. For $S \in \mathrm{Nilp}_{\breve{O}_F}$, the set $\mathcal{N}_E^{\text{naive}}(S)$ is the set of equivalence classes of tuples $(X, \iota, \lambda, \varrho)$. Here, X/S is a formal O_F-module of height 4 and dimension 2, equipped with an action $\iota : O_E \to \mathrm{End}(X)$. This action satisfies the Kottwitz condition of signature $(1, 1)$, i.e., for any $\alpha \in O_E$, the characteristic polynomial of $\iota(\alpha)$ on $\mathrm{Lie}\, X$ is given by

$$\mathrm{char}(\mathrm{Lie}\, X, T \mid \iota(\alpha)) = (T - \alpha)(T - \bar{\alpha}).$$

Here, $\alpha \mapsto \bar{\alpha}$ denotes the Galois conjugation of $E|F$. The right side of this equation is a polynomial with coefficients in $\mathcal{O}_S$ via the structure map $O_F \hookrightarrow \breve{O}_F \to \mathcal{O}_S$. The third entry λ is a principal polarization $\lambda : X \to X^\vee$ such that the induced Rosati involution satisfies $\iota(\alpha)^* = \iota(\bar{\alpha})$ for all $\alpha \in O_E$. (Here, $X^\vee$ is the dual of X as a formal O_F-module.) Finally, ϱ is a quasi-isogeny of height 0 (and compatible with all previous data) to a fixed framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\bar{k} = \breve{O}_F/\pi_0$. This framing object is unique up to isogeny under the condition that

$$\{\varphi \in \mathrm{End}^0(\mathbb{X}, \iota_{\mathbb{X}}) \mid \varphi^*(\lambda_{\mathbb{X}}) = \lambda_{\mathbb{X}}\} \simeq \mathrm{U}(C, h),$$

for a split $E|F$-hermitian vector space (C, h) of dimension 2; see Proposition 3.2.

Recall that this is exactly the definition used in [Kudla and Rapoport 2014] for the ramified case with $p > 2$. There, $\mathcal{N}_E = \mathcal{N}_E^{\text{naive}}$ and we have natural isomorphism

$$\eta : \mathcal{M}_{Dr} \xrightarrow{\sim} \mathcal{N}_E,$$

where $\mathcal{M}_{Dr}$ is the Drinfeld moduli problem mentioned above.

However, for $p = 2$, it turns out that the definition of $\mathcal{N}_E^{\text{naive}}$ is not the "correct" one in the sense that it is not isomorphic to the Drinfeld moduli problem. Hence this naive definition of the moduli space is not in line with the results from [Kudla and Rapoport 2014] and the general philosophy of (conjectural) local Shimura varieties (see [Rapoport and Viehmann 2014]). In order to remedy this, we will describe a new condition on $\mathcal{N}_E^{\mathrm{naive}}$, which we call the *straightening condition*, and show that this cuts out a closed formal subscheme $\mathcal{N}_E \subseteq \mathcal{N}_E^{\mathrm{naive}}$ that is naturally isomorphic to $\mathcal{M}_{Dr}$. Interestingly, the straightening condition is not trivial on the rigid-analytic generic fiber of $\mathcal{N}_E^{\mathrm{naive}}$ (as originally assumed by the author), but it cuts out an (admissible) open and closed subspace; see Remark 3.13.

We would like to explicate the defect of the naive moduli space. For this, let us recall the definition of $\mathcal{M}_{Dr}$. It is a functor on $\mathrm{Nilp}_{\breve{O}_F}$, mapping a scheme S to the set $\mathcal{M}_{Dr}(S)$ of equivalence classes of tuples (X, ι_B, ϱ). Again, X/S is a formal O_F-module of height 4 and dimension 2. Let B be the quaternion division algebra over F and O_B its ring of integers. Then ι_B is an action of O_B on X, satisfying the *special* condition of Drinfeld (see [Boutot and Carayol 1991] or Section 3C below). The last entry ϱ is an O_B-linear quasi-isogeny of height 0 to a fixed framing object $(\mathbb{X}, \iota_{\mathbb{X},B})$ over $\bar{k}$. This framing object is unique up to isogeny (cf. [Boutot and Carayol 1991, II. Proposition 5.2]).

Fix an embedding $O_E \hookrightarrow O_B$ and consider the involution $b \mapsto b^* = \Pi b' \Pi^{-1}$ on B, where $b \mapsto b'$ is the standard involution. By work of Drinfeld (see Proposition 3.14 below), there exists a principal polarization $\lambda_{\mathbb{X}}$ on the framing object $(\mathbb{X}, \iota_{\mathbb{X},B})$ of $\mathcal{M}_{Dr}$ such that the induced Rosati involution satisfies $\iota_{\mathbb{X},B}(b)^* = \iota_{\mathbb{X},B}(b^*)$ for all $b \in O_B$. This polarization is unique up to a scalar in $O_F^\times$. Furthermore, for any $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$, the pullback $\lambda = \varrho^*(\lambda_{\mathbb{X}})$ is a principal polarization on X.

We now set

$$\eta(X, \iota_B, \varrho) = (X, \iota_B|_{O_E}, \lambda, \varrho).$$

By Lemma 3.15, this defines a closed embedding $\eta : \mathcal{M}_{Dr} \hookrightarrow \mathcal{N}_E^{\mathrm{naive}}$. But η is far from being an isomorphism, as the following proposition shows:

Proposition 1.1. *The induced map $\eta(\bar{k}) : \mathcal{M}_{Dr}(\bar{k}) \to \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ is not surjective.*

Let us sketch the proof here. Using Dieudonné theory, we can write $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ naturally as a union

$$\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) = \bigcup_{\Lambda \subseteq C} \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}),$$

where the union runs over all O_E-lattices Λ in the hermitian vector space (C, h) that are Π^{-1}-modular, i.e., the dual $\Lambda^\sharp$ of Λ with respect to h is given by $\Lambda = \Pi^{-1}\Lambda^\sharp$ (see Lemma 3.7). By [Jacobowitz 1962], there exist different types (i.e., $\mathrm{U}(C, h)$-orbits) of such lattices $\Lambda \subseteq C$ that are parametrized by their norm ideal $\mathrm{Nm}(\Lambda) = \langle \{h(x, x) | x \in \Lambda\} \rangle \subseteq F$. In the case at hand, $\mathrm{Nm}(\Lambda)$ can be any ideal with $2O_F \subseteq \mathrm{Nm}(\Lambda) \subseteq O_F$. It is easily checked (see Section 2) that the norm ideal of Λ is minimal, that is $\mathrm{Nm}(\Lambda) = 2O_F$, if and only if Λ admits a basis consisting of isotropic vectors, and hence we call these lattices *hyperbolic*. Now, the image under η of $\mathcal{M}_{Dr}(\bar{k})$ is the union of all lines $\mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k})$ where $\Lambda \subseteq C$ is hyperbolic. This is a consequence of Remark 3.12 and Theorem 3.16 below.

On the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of $\mathcal{N}_E^{\text{naive}}$, there exists a principal polarization $\widetilde{\lambda}_{\mathbb{X}}$ such that the induced Rosati involution is the identity on O_E. This polarization is unique up to a scalar in $O_E^{\times}$ (see Theorem 5.2 (1)). On C, the polarization $\widetilde{\lambda}_{\mathbb{X}}$ induces an E-linear alternating form b, such that $\det b$ and $\det h$ differ only by a unit (for a fixed basis of C). After possibly rescaling b by a unit in $O_E^{\times}$, a Π^{-1}-modular lattice $\Lambda \subseteq C$ is hyperbolic if and only if $b(x, y) + h(x, y) \in 2O_F$ for all $x, y \in \Lambda$. This enables us to describe the "hyperbolic" points of $\mathcal{N}_E^{\text{naive}}$ (i.e., those that lie on a projective line corresponding to a hyperbolic lattice $\Lambda \subseteq C$) in terms of polarizations.

We now formulate the closed condition that characterizes $\mathcal{N}_E$ as a closed formal subscheme of $\mathcal{N}_E^{\text{naive}}$. For a suitable choice of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ and $\widetilde{\lambda}_{\mathbb{X}}$, we may assume that $\frac{1}{2}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}})$ is a polarization on $\mathbb{X}$. The following definition is a reformulation of Definition 3.11.

Definition 1.2. Let $S \in \mathrm{Nilp}_{\breve{O}_F}$. An object $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\text{naive}}(S)$ satisfies the *straightening* condition if $\lambda_1 = \frac{1}{2}(\lambda + \widetilde{\lambda})$ is a polarization on X. Here, $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$.

We remark that $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$ is a polarization on X. This is a consequence of Theorem 5.2, which states the existence of certain polarizations on points of a larger moduli space $\mathcal{M}_E$ containing $\mathcal{N}_E^{\text{naive}}$; see below.

For $S \in \mathrm{Nilp}_{\breve{O}_F}$, let $\mathcal{N}_E(S) \subseteq \mathcal{N}_E^{\text{naive}}(S)$ be the subset of all tuples $(X, \iota, \lambda, \varrho)$ that satisfy the straightening condition. By [Rapoport and Zink 1996, Proposition 2.9], this defines a closed formal subscheme $\mathcal{N}_E \subseteq \mathcal{N}_E^{\text{naive}}$. An application of Drinfeld's proposition (Proposition 3.14; see also [Boutot and Carayol 1991]) shows that the image of $\mathcal{M}_{Dr}$ under η lies in $\mathcal{N}_E$. The main theorem in the R-P case can now be stated as follows; see Theorem 3.16.

Theorem 1.3. *$\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E$ is an isomorphism of formal schemes.*

This concludes our discussion of the R-P case. From now on, we assume that $E|F$ is of type R-U.

In the case R-U, we have to make some adaptations for $\mathcal{N}_E^{\text{naive}}$. For $S \in \mathrm{Nilp}_{\breve{O}_F}$, let $\mathcal{N}_E^{\text{naive}}(S)$ be the set of equivalence classes of tuples $(X, \iota, \lambda, \varrho)$ with (X, ι) as in the R-P case. But now, the polarization $\lambda : X \to X^\vee$ is supposed to have kernel $\ker \lambda = X[\Pi]$ (in contrast to the R-P case, where λ is a principal polarization). As before, the Rosati involution of λ induces the conjugation on O_E. There exists a framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\operatorname{Spec} \bar{k}$ for $\mathcal{N}_E^{\text{naive}}$, which is unique up to isogeny under the condition that

$$\{\varphi \in \mathrm{End}^0(\mathbb{X}, \iota_{\mathbb{X}}) \mid \varphi^*(\lambda_{\mathbb{X}}) = \lambda_{\mathbb{X}}\} \simeq \mathrm{U}(C, h),$$

where (C, h) is a split $E|F$-hermitian vector space of dimension 2 (see Proposition 4.1). Finally, ϱ is a quasi-isogeny of height 0 from X to $\mathbb{X}$, respecting all structure.

Fix an embedding $E \hookrightarrow B$. Using some subtle choices of elements in B (these are described in Lemma 2.3 (2)) and by Drinfeld's proposition, we can construct a polarization λ as above for any $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$. This induces a closed embedding

$$\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E^{\text{naive}}, \quad (X, \iota_B, \varrho) \mapsto (X, \iota_B|_{O_E}, \lambda, \varrho).$$

We can write $\mathcal{N}_E^{\text{naive}}(\bar{k})$ as a union of projective lines,

$$\mathcal{N}_E^{\text{naive}}(\bar{k}) = \bigcup_{\Lambda \subseteq C} \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}),$$

where the union now runs over all self-dual O_E-lattices $\Lambda \subseteq (C, h)$ with $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. As in the R-P case, these lattices $\Lambda \subseteq C$ are classified up to isomorphism by their norm ideal $\mathrm{Nm}(\Lambda)$. Since Λ is self-dual with respect to h, the norm ideal can be any ideal satisfying $t O_F \subseteq \mathrm{Nm}(\Lambda) \subseteq O_F$. We call Λ *hyperbolic* when the norm ideal is minimal, i.e., $\mathrm{Nm}(\Lambda) = t O_F$. Equivalently, the lattice Λ has a basis consisting of isotropic vectors. Recall that here t is the element showing up in the Eisenstein equation for the R-U extension $E|F$ and that $\pi_0 | t | 2$. Hence there exists at least one type of self-dual lattices $\Lambda \subseteq C$ with $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. In the case R-U, it may happen that $|t| = |\pi_0|$, in which case all lattices Λ in the description of $\mathcal{N}_E^{\text{naive}}(\bar{k})$ are hyperbolic.

The image of $\mathcal{M}_{Dr}(\bar{k})$ under η in $\mathcal{N}_E^{\text{naive}}(\bar{k})$ is the union of all projective lines corresponding to hyperbolic lattices. Unless $|t| = |\pi_0|$, it follows that $\eta(\bar{k})$ is not surjective and thus η cannot be an isomorphism. For the case $|t| = |\pi_0|$, we will show that η is an isomorphism on reduced loci $(\mathcal{M}_{Dr})_{\text{red}} \xrightarrow{\sim} (\mathcal{N}_E^{\text{naive}})_{\text{red}}$ (see Remark 4.11), but η is not an isomorphism of formal schemes. This follows from the nonflatness of the deformation ring for certain points of $\mathcal{N}_E^{\text{naive}}$; see Section 4D.

On the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of $\mathcal{N}_E^{\text{naive}}$, there exists a polarization $\tilde{\lambda}_{\mathbb{X}}$ such that $\ker \tilde{\lambda}_{\mathbb{X}} = \mathbb{X}[\Pi]$ and such that the Rosati involution induces the identity on O_E. After a suitable choice of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ and $\tilde{\lambda}_{\mathbb{X}}$, we may assume that $\frac{1}{t}(\lambda_{\mathbb{X}} + \tilde{\lambda}_{\mathbb{X}})$ is a polarization on $\mathbb{X}$. The straightening condition for the R-U case is given as follows (see Definition 4.10).

Definition 1.4. Let $S \in \mathrm{Nilp}_{\breve{O}_F}$. An object $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\text{naive}}(S)$ satisfies the *straightening* condition if $\lambda_1 = \frac{1}{t}(\lambda + \tilde{\lambda})$ is a polarization on X. Here, $\tilde{\lambda} = \varrho^*(\tilde{\lambda}_{\mathbb{X}})$.

Note that $\tilde{\lambda} = \varrho^*(\tilde{\lambda}_{\mathbb{X}})$ is a polarization on X by Theorem 5.2.

The straightening condition defines a closed formal subscheme $\mathcal{N}_E \subseteq \mathcal{N}_E^{\text{naive}}$ that contains the image of $\mathcal{M}_{Dr}$ under η. The main theorem in the R-U case can now be stated as follows; compare Theorem 4.14.

Theorem 1.5. *$\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E$ is an isomorphism of formal schemes.*

When formulating the straightening condition in the R-U and the R-P case, we mentioned that $\tilde{\lambda} = \varrho^*(\tilde{\lambda}_{\mathbb{X}})$ is a polarization for any $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\text{naive}}(S)$. This fact is a corollary of Theorem 5.2, which states the existence of this polarization in the following more general setting.

Let $F|\mathbb{Q}_p$ be a finite extension for any prime p and $E|F$ an arbitrary quadratic extension. We consider the following moduli space $\mathcal{M}_E$ of EL type. For $S \in \mathrm{Nilp}_{\breve{O}_F}$, the set $\mathcal{M}_E(S)$ consists of equivalence classes of tuples (X, ι_E, ϱ), where X is a formal O_F-module of height 4 and dimension 2 and ι_E is an O_E-action on X satisfying the Kottwitz condition of signature $(1, 1)$ as above. The entry ϱ is an O_E-linear quasi-isogeny of height 0 to a supersingular framing object $(\mathbb{X}, \iota_{\mathbb{X},E})$.

The points of $\mathcal{M}_E$ are equipped with polarizations in the following natural way; see Theorem 5.2.

Theorem 1.6. (1) *There exists a principal polarization $\tilde{\lambda}_{\mathbb{X}}$ on $(\mathbb{X}, \iota_{\mathbb{X},E})$ such that the Rosati involution induces the identity on O_E, i.e., $\iota(\alpha)^* = \iota(\alpha)$ for all $\alpha \in O_E$. This polarization is unique up to a scalar in $O_E^\times$.*

(2) *Fix $\tilde{\lambda}_{\mathbb{X}}$ as in part* (1). *For any $S \in \mathrm{Nilp}_{\breve{O}_F}$ and $(X, \iota_E, \varrho) \in \mathcal{M}_E(S)$, there exists a unique principal polarization $\tilde{\lambda}$ on X such that the Rosati involution induces the identity on O_E and such that $\tilde{\lambda} = \varrho^*(\tilde{\lambda}_{\mathbb{X}})$.*

If $p = 2$ and $E|F$ is ramified of R-P or R-U type, then there is a canonical closed embedding $\mathcal{N}_E \hookrightarrow \mathcal{M}_E$ that forgets about the polarization λ. In this way, it follows that $\tilde{\lambda}$ is a polarization for any $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\text{naive}}(S)$.

The statement of Theorem 1.6 can also be expressed in terms of an isomorphism of moduli spaces $\mathcal{M}_{E,\text{pol}} \xrightarrow{\sim} \mathcal{M}_E$. Here $\mathcal{M}_{E,\text{pol}}$ is a moduli space of PEL type, defined by mapping $S \in \mathrm{Nilp}_{\breve{O}_F}$ to the set of tuples $(X, \iota, \tilde{\lambda}, \varrho)$ where $(X, \iota, \varrho) \in \mathcal{M}_E(S)$ and $\tilde{\lambda}$ is a polarization as in the theorem.

We now briefly describe the contents of the subsequent sections of this paper. In Section 2, we recall some facts about the quadratic extensions of F, the quaternion algebra $B|F$ and hermitian forms. In Sections 3 and 4, we define the moduli spaces $\mathcal{N}_E^{\text{naive}}$, introduce the straightening condition describing $\mathcal{N}_E \subseteq \mathcal{N}_E^{\text{naive}}$ and prove our main theorem in both the cases R-P and R-U. Although the techniques are quite similar in both cases, we decided to treat these cases separately, since the results in both cases differ in important details. Finally, in Section 5 we prove Theorem 1.6 on the existence of the polarizations $\tilde{\lambda}$.

2. Preliminaries on quaternion algebras and hermitian forms

Let $F|\mathbb{Q}_2$ be a finite extension. In this section we will recall some facts about the quadratic extensions of F, the quaternion division algebra $B|F$ and certain hermitian forms. For more information on quaternion algebras, see for example the book by Vignéras [1980]. A systematic classification of hermitian forms over local fields has been done by Jacobowitz [1962].

Let $E|F$ be a quadratic field extension and denote by O_F, resp. O_E, the rings of integers. There are three mutually exclusive possibilities for $E|F$:

- $E|F$ is unramified. Then $E = F[\delta]$ for δ a square root of a unit in F. We can choose δ such that $\delta^2 = 1+4u$ for some $u \in O_F^\times$. In this case, $O_E = O_F[(1+\delta)/2]$. The element $\gamma = (1+\delta)/2$ satisfies the Eisenstein equation $\gamma^2 - \gamma - u = 0$. In the following we will write $F^{(2)}$ instead of E and $O_F^{(2)}$ instead of O_E when talking about the unramified extension of F.

- $E|F$ is ramified and E is generated by the square root of a uniformizer in F. That is, $E = F[\Pi]$ and Π is given by the Eisenstein equation $\Pi^2 + \pi_0 = 0$ for a uniformizing element $\pi_0 \in O_F$. We also have $O_E = O_F[\Pi]$. Following Jacobowitz, we will say $E|F$ is of type R-P (which stands for "ramified-prime").

- Finally, $E|F$ can be given by an Eisenstein equation of the form $\Pi^2 - t\Pi + \pi_0 = 0$ for a uniformizer π_0 and $t \in O_F$ such that $\pi_0|t|2$. Then $E|F$ is ramified and $O_E = O_F[\Pi]$. Here, E is generated by the square root of a unit in F. Indeed, for $\vartheta = 1 - 2\Pi/t$ we have $\vartheta^2 = 1 - 4\pi_0/t^2 \in O_F^\times$. Thus $E|F$ is said to be of type R-U (for "ramified-unit").

We will use this notation throughout the paper.

Remark 2.1. The isomorphism classes of quadratic extensions of F correspond to nontrivial equivalence classes of $F^\times/(F^\times)^2$. We have $F^\times/(F^\times)^2 \simeq \mathrm{H}^1(G_F, \mathbb{Z}/2\mathbb{Z})$ for the absolute Galois group G_F of F and $\dim \mathrm{H}^1(G_F, \mathbb{Z}/2\mathbb{Z}) = 2 + d$, where $d = [F : \mathbb{Q}_2]$ is the degree of F over $\mathbb{Q}_2$ (see, for example, [Neukirch et al. 2008, Corollary 7.3.9]).

A representative of an equivalence class in $F^\times/F^{\times 2}$ can be chosen to be either a prime or a unit, and exactly half of the classes are represented by prime elements, the others being represented by units. It follows that there are, up to isomorphism, 2^{1+d} different extensions $E|F$ of type R-P and $2^{1+d} - 2$ extensions of type R-U. (We have to exclude the trivial element $1 \in F^\times/F^{\times 2}$ and one unit element corresponding to the unramified extension.)

Lemma 2.2. *The inverse different of $E|F$ is given by $\mathfrak{D}_{E|F}^{-1} = \frac{1}{2\Pi} O_E$ in the case R-P and by $\mathfrak{D}_{E|F}^{-1} = \frac{1}{t} O_E$ in the case R-U.*

Proof. The inverse different is defined as

$$\mathfrak{D}_{E|F}^{-1} = \{\alpha \in E \mid \mathrm{Tr}_{E|F}(\alpha O_E) \subseteq O_F\}.$$

It is enough to check the condition on the trace for the elements 1 and $\Pi \in O_E$. If we write $\alpha = \alpha_1 + \Pi\alpha_2$ with $\alpha_1, \alpha_2 \in F$, we get

$$\begin{aligned}\mathrm{Tr}_{E|F}(\alpha \cdot 1) &= \alpha + \bar{\alpha} = 2\alpha_1 + \alpha_2(\Pi + \overline{\Pi}),\\ \mathrm{Tr}_{E|F}(\alpha \cdot \Pi) &= \alpha\Pi + \overline{\alpha\Pi} = \alpha_1(\Pi + \overline{\Pi}) + \alpha_2(\Pi^2 + \overline{\Pi}^2).\end{aligned}$$

In the case R-P we have $\Pi + \overline{\Pi} = 0$ and $\Pi^2 + \overline{\Pi}^2 = 2\pi_0$, while in the case R-U, $\Pi + \overline{\Pi} = t$ and $\Pi^2 + \overline{\Pi}^2 = t^2 - 2\pi_0$. It is now easy to deduce that the inverse different is of the claimed form. □

Over F, there exists up to isomorphism exactly one quaternion division algebra B, with unique maximal order O_B. For every quadratic extension $E|F$, there exists an embedding $E \hookrightarrow B$ and this induces an embedding $O_E \hookrightarrow O_B$. If $E|F$ is ramified, a basis for O_E as O_F-module is given by $(1, \Pi)$. We would like to extend this to an O_F-basis of O_B.

Lemma 2.3. (1) *If $E|F$ is of type R-P, there exists an embedding $F^{(2)} \hookrightarrow B$ such that $\delta\Pi = -\Pi\delta$. An O_F-basis of O_B is then given by* $(1, \gamma, \Pi, \gamma \cdot \Pi)$, *where* $\gamma = (1+\delta)/2$.

(2) *If $E|F$ is of type R-U, there exists an embedding $E_1 \hookrightarrow B$, where $E_1|F$ is of type R-P with uniformizer Π_1 such that $\vartheta\Pi_1 = -\Pi_1\vartheta$. The tuple* $(1, \vartheta, \Pi_1, \vartheta\Pi_1)$ *is an F-basis of B.*

Furthermore, there is also an embedding $\widetilde{E} \hookrightarrow B$ with $\widetilde{E}|F$ of type R-U with elements $\widetilde{\Pi}$ and $\tilde{\vartheta}$ as above, such that $\vartheta\tilde{\vartheta} = -\tilde{\vartheta}\vartheta$ and $\tilde{\vartheta}^2 = 1 + (t^2/\pi_0) \cdot u$ for some unit $u \in F$. In terms of this embedding, an O_F-basis of O_B is given by $(1, \Pi, \widetilde{\Pi}, \Pi \cdot \widetilde{\Pi}/\pi_0)$. *Also,*

$$\frac{\Pi \cdot \widetilde{\Pi}}{\pi_0} = \gamma \tag{2-1}$$

for some embedding $F^{(2)} \hookrightarrow B$ of the unramified extension and $\gamma^2 - \gamma - u = 0$. Hence, $O_B = O_F[\Pi, \gamma]$ as O_F-algebra.

Proof. (1) This is [Vignéras 1980, II. Corollary 1.7].

(2) By [Vignéras 1980, I. Corollary 2.4], it suffices to find a uniformizer $\Pi_1^2 \in F^\times \setminus \mathrm{Nm}_{E|F}(E^\times)$ in order to prove the first part. But $\mathrm{Nm}_{E|F}(E^\times) \subseteq F^\times$ is a subgroup of order 2 and $F^{\times 2} \subseteq \mathrm{Nm}_{E|F}(E^\times)$. On the other hand, the residue classes of uniformizing elements in $F^\times/F^{\times 2}$ generate the whole group. Thus they cannot all be contained in $\mathrm{Nm}_{E|F}(E^\times)$.

For the second part, choose a unit $\delta \in F^{(2)}$ with $\delta^2 = 1 + 4u \in F^\times \setminus F^{\times 2}$ for some $u \in O_F^\times$ and set $\gamma = (1+\delta)/2$. Let $\widetilde{E}|F$ be of type R-U, generated by $\tilde{\vartheta}$ with $\tilde{\vartheta}^2 = 1 + (t^2/\pi_0) \cdot u$. We have to show that $\tilde{\vartheta}^2$ is not contained in $\mathrm{Nm}_{E|F}(E^\times)$.

Assume it is a norm, so $\tilde{\vartheta}^2 = \mathrm{Nm}_{E|F}(b)$ for a unit $b \in E^\times$. Then b is of the form $b = 1 + x \cdot (t/\Pi)$ for some $x \in O_E$. Indeed, let ℓ be the Π-adic valuation of $b-1$, i.e., $b = 1 + x \cdot \Pi^\ell$ and $x \in O_E^\times$. We have

$$1 + (t^2/\pi_0) \cdot u = \mathrm{Nm}_{E|F}(b) = 1 + \mathrm{Tr}_{E|F}(x\Pi^\ell) + \mathrm{Nm}_{E|F}(x\Pi^\ell). \tag{2-2}$$

Let v be the π_0-adic valuation on F; then $v(\mathrm{Nm}_{E|F}(x\Pi^\ell)) = \ell$ and $v(\mathrm{Tr}_{E|F}(x\Pi^\ell)) \geq v(t) + \lfloor \frac{\ell}{2} \rfloor$, by Lemma 2.2. On the left-hand side, we have $v((t^2/\pi_0) \cdot u) = 2v(t) - 1$. Comparing the valuations on both sides of (2-2), the assumption $\ell < 2v(t) - 1$ now quickly leads to a contradiction.

Hence $\ell \geq 2v(t) - 1$ and $b = 1 + x \cdot (t/\Pi)$ for some $x \in O_E$. Again,

$$1 + (t^2/\pi_0) \cdot u = \mathrm{Nm}_{E|F}(b) = 1 + \mathrm{Tr}_{E|F}(xt/\Pi) + \mathrm{Nm}_{E|F}(xt/\Pi).$$

An easy calculation shows that the residue $\bar{x} \in k = O_E/\Pi = O_F/\pi_0$ of x satisfies $u = x + x^2$. But this equation has no solution in k, since a solution of $\gamma^2 - \gamma - u = 0$ generates the unramified quadratic extension of F. It follows that $\tilde{\vartheta}^2$ cannot be a norm.

Using again [Vignéras 1980, I. Corollary 2.4], we find an embedding $\widetilde{E} \hookrightarrow B$ such that $\vartheta\tilde{\vartheta} = -\tilde{\vartheta}\vartheta$.

We have $\Pi = t(1+\vartheta)/2$ and $\widetilde{\Pi} = \pi_0(1+\tilde{\vartheta})/t$; thus

$$\frac{\Pi \cdot \widetilde{\Pi}}{\pi_0} = \frac{(1+\vartheta)\cdot(1+\tilde{\vartheta})}{2} = \frac{1+\vartheta+\tilde{\vartheta}+\vartheta\cdot\tilde{\vartheta}}{2},$$

and

$$\begin{aligned}(\vartheta + \tilde{\vartheta} + \vartheta \cdot \tilde{\vartheta})^2 &= \vartheta^2 + \tilde{\vartheta}^2 - \vartheta^2 \cdot \tilde{\vartheta}^2 \\ &= (1 - 4\pi_0/t^2) + (1 + t^2u/\pi_0) - (1 - 4\pi_0/t^2)(1 + t^2u/\pi_0) \\ &= 1 + 4u.\end{aligned}$$

Hence $\gamma \mapsto \Pi \cdot \widetilde{\Pi}/\pi_0$ induces an embedding $F^{(2)} \hookrightarrow B$.

It remains to prove that the tuple $u = (1, \Pi, \widetilde{\Pi}, \Pi \cdot \widetilde{\Pi}/\pi_0)$ is a basis of O_B as O_F-module. By [Vignéras 1980, I. Corollary 4.8], it suffices to check that the discriminant

$$\mathrm{disc}(u) = \det(\mathrm{Trd}(u_i u_j)) \cdot O_F$$

is equal to $\mathrm{disc}(O_B)$. An easy calculation shows $\det(\mathrm{Trd}(u_i u_j)) \cdot O_F = \pi_0 O_F$ and then the assertion follows from [Vignéras 1980, V, II. Corollary 1.7]. □

For the remainder of this section, we will consider lattices Λ in a 2-dimensional E-vector space C with a split $E|F$-hermitian[1] form h. Recall from [Jacobowitz 1962] that, up to isomorphism, there are two different $E|F$-hermitian vector

[1] Here and in the following, sesquilinear forms will be linear from the left and semilinear from the right.

spaces (C, h) of fixed dimension n, parametrized by the discriminant $\operatorname{disc}(C, h) \in F^\times/\operatorname{Nm}_{E|F}(E^\times)$. A hermitian space (C, h) is called *split* whenever $\operatorname{disc}(C, h) = 1$. In our case, where (C, h) is split of dimension 2, we can find a basis (e_1, e_2) of C with $h(e_i, e_i) = 0$ and $h(e_1, e_2) = 1$.

Denote by $\Lambda^\sharp$ the dual of a lattice $\Lambda \subseteq C$ with respect to h. The lattice Λ is called Π^i*-modular* if $\Lambda = \Pi^i \Lambda^\sharp$ (resp. *unimodular* or *self-dual* when $i = 0$). In contrast to the p-adic case with $p > 2$, there exist Π^i-modular lattices of more than one type in our case (cf. [Jacobowitz 1962]):

Proposition 2.4. *Define the norm ideal* $\operatorname{Nm}(\Lambda)$ *of* Λ *by*

$$\operatorname{Nm}(\Lambda) = \langle \{h(x, x) | x \in \Lambda\} \rangle \subseteq F. \tag{2-3}$$

Any Π^i*-modular lattice* $\Lambda \subseteq C$ *is determined up to the action of* $\mathrm{U}(C, h)$ *by the ideal* $\operatorname{Nm}(\Lambda) = \pi_0^\ell O_F \subseteq F$. *For* $i = 0$ *or* 1, *the exponent* ℓ *can be any integer such that*

$$\begin{aligned} |2| \le |\pi_0|^\ell &\le |1| && \textit{for } E|F \textit{ R-P}, \textit{ unimodular } \Lambda,\\ |2\pi_0| \le |\pi_0|^\ell &\le |\pi_0| && \textit{for } E|F \textit{ R-P}, \Pi\textit{-modular } \Lambda,\\ |t| \le |\pi_0|^\ell &\le |1| && \textit{for } E|F \textit{ R-U}, \textit{ unimodular } \Lambda,\\ |t| \le |\pi_0|^\ell &\le |\pi_0| && \textit{for } E|F \textit{ R-U}, \Pi\textit{-modular } \Lambda, \end{aligned}$$

where $|\cdot|$ *is the* (*normalized*) *absolute value on* F. *Two* Π^i*-modular lattices* Λ *and* Λ' *are isomorphic if and only if* $\operatorname{Nm}(\Lambda) = \operatorname{Nm}(\Lambda')$. □

For any other i, the possible values of ℓ for a given Π^i-modular lattice Λ are easily obtained by shifting. In fact, we can choose an integer j such that $\Pi^j \Lambda$ is either unimodular or Π-modular. Then $\operatorname{Nm}(\Lambda) = \pi_0^{-j} \operatorname{Nm}(\Pi^j \Lambda)$ and we can apply the proposition above.

Since (C, h) is split, any Π^i-modular lattice Λ contains an *isotropic* vector v (i.e., with $h(v, v) = 0$). After rescaling with a suitable power of Π, we can extend v to a basis of Λ. Hence there always exists a basis (e_1, e_2) of Λ such that h is represented by a matrix of the form

$$H_\Lambda = \begin{pmatrix} x & \overline{\Pi}^i \\ \Pi^i & \end{pmatrix}, \quad x \in F. \tag{2-4}$$

If $x = 0$ in this representation, then $\operatorname{Nm}(\Lambda) = \pi_0^\ell O_F$ is as small as possible, or in other words, the absolute value of $|\pi_0|^\ell$ is minimal. On the other hand, whenever $|\pi_0|^\ell$ takes the minimal absolute value for a given Π^i-modular lattice Λ, there exists a basis (e_1, e_2) of Λ such that h is represented by H_Λ with $x = 0$. Indeed, this follows because the ideal $\operatorname{Nm}(\Lambda)$ already determines Λ up to isomorphism. In this case (when $x = 0$), we call Λ a *hyperbolic* lattice. By the arguments above, a

Π^i-modular lattice is thus hyperbolic if and only if its norm is minimal. In all other cases, where Λ is Π^i-modular but not hyperbolic, we have $\mathrm{Nm}(\Lambda) = x O_F$.

For further reference, we explicitly write down the norm of a hyperbolic lattice for the cases that we need later. For other values of i, the norm can easily be deduced from this by shifting (see also [Jacobowitz 1962, Table 9.1]).

Lemma 2.5. *A Π^i-modular lattice Λ is hyperbolic if and only if*

$$\mathrm{Nm}(\Lambda) = 2O_F \quad \textit{for } E|F \textit{ R-P}, \ i = 0 \textit{ or } -1,$$
$$\mathrm{Nm}(\Lambda) = tO_F \quad \textit{for } E|F \textit{ R-U}, \ i = 0 \textit{ or } 1.$$

The norm ideal of Λ is minimal among all norm ideals for Π^i-modular lattices in C. □

In the following, we will only consider the cases $i = 0$ or -1 for $E|F$ R-P and the cases $i = 0$ or 1 for $E|F$ R-U, since these are the cases we will need later. We want to study the following question:

Question 2.6. Assume $E|F$ is R-P. Fix a Π^{-1}-modular lattice $\Lambda_{-1} \subseteq C$ (not necessarily hyperbolic). How many unimodular lattices $\Lambda_0 \subseteq \Lambda_{-1}$ are there and what norms $\mathrm{Nm}(\Lambda_0)$ can appear? Dually, for a fixed unimodular lattice $\Lambda_0 \subseteq C$, how many Π^{-1}-modular lattices Λ_{-1} with $\Lambda_0 \subseteq \Lambda_{-1}$ exist and what are their norms?

We can ask the same question for $E|F$ R-U and unimodular, resp. Π-modular, lattices.

Of course, such an inclusion is always of index 1. The inclusions $\Lambda_0 \subseteq \Lambda_{-1}$ of index 1 correspond to lines in $\Lambda_{-1}/\Pi\Lambda_{-1}$. Denote by q the number of elements in the common residue field of O_F and O_E. Then there exist at most $q+1$ such Π-modular lattices Λ_0 for a given Λ_{-1}. The same bound holds in the dual case, i.e., there are at most $q+1$ Π^{-1}-modular lattices containing a given unimodular lattice Λ_0. Propositions 2.7 and 2.8 below provide an exhaustive answer to Question 2.6. Since the proofs consist of a lengthy but simple case-by-case analysis, we will leave them to the interested reader.

Proposition 2.7. *Let $E|F$ be of type R-P.*

(1) *Let $\Lambda_{-1} \subseteq C$ be a Π^{-1}-modular hyperbolic lattice. There are $q+1$ hyperbolic unimodular lattices contained in Λ_{-1}.*

(2) *Let $\Lambda_{-1} \subseteq C$ be a Π^{-1}-modular nonhyperbolic lattice. Let $\mathrm{Nm}(\Lambda_{-1}) = \pi_0^{\ell} O_F$. Then Λ_{-1} contains one unimodular lattice Λ_0 with $\mathrm{Nm}(\Lambda_0) = \pi_0^{\ell+1} O_F$ and q unimodular lattices of norm $\pi_0^{\ell} O_F$.*

(3) *Let $\Lambda_0 \subseteq C$ be a unimodular hyperbolic lattice. There are two hyperbolic Π^{-1}-modular lattices $\Lambda_{-1} \supseteq \Lambda_0$ and $q-1$ nonhyperbolic Π^{-1}-modular lattices $\Lambda_{-1} \supseteq \Lambda_0$ with $\mathrm{Nm}(\Lambda_{-1}) = 2/\pi_0 O_F$.*

(4) *Let $\Lambda_0 \subseteq C$ be unimodular nonhyperbolic. Let* $\mathrm{Nm}(\Lambda_0) = \pi_0^\ell O_F$. *There exists one Π^{-1}-modular lattice $\Lambda_{-1} \supseteq \Lambda_0$ with* $\mathrm{Nm}(\Lambda_{-1}) = \pi_0^\ell O_F$ *and, unless $\ell = 0$, there are q nonhyperbolic Π^{-1}-modular lattices $\Lambda_{-1} \supseteq \Lambda_0$ with* $\mathrm{Nm}(\Lambda_{-1}) = \pi_0^{\ell-1} O_F$.

Note that the total number of unimodular, resp. Π^{-1}-modular, lattices found for $\Lambda = \Lambda_{-1}$, resp. Λ_0, is $q+1$ except in the case of Proposition 2.7 (4) when $\ell = 0$. In that particular case, there is just one Π^{-1}-modular lattice contained in Λ_0. The same phenomenon also appears in the case R-U; see part (2) of the following proposition.

Proposition 2.8. *Let $E|F$ be of type R-U.*

(1) *Let $\Lambda_0 \subseteq C$ be a unimodular hyperbolic lattice. There are $q+1$ hyperbolic Π-modular lattices $\Lambda_1 \subseteq \Lambda_0$.*

(2) *Let $\Lambda_0 \subseteq C$ be unimodular nonhyperbolic with* $\mathrm{Nm}(\Lambda_0) = \pi_0^\ell O_F$. *There is one Π-modular lattice $\Lambda_1 \subseteq \Lambda_0$ with norm ideal* $\mathrm{Nm}(\Lambda_1) = \pi_0^{\ell+1} O_F$ *and if $\ell \neq 0$, there are also q nonhyperbolic Π-modular lattices $\Lambda_1 \subseteq \Lambda_0$ with* $\mathrm{Nm}(\Lambda_1) = \pi_0^\ell O_F$.

(3) *Let $\Lambda_1 \subseteq C$ be a Π-modular hyperbolic lattice. There are two unimodular hyperbolic lattices containing Λ_1 and $q-1$ unimodular lattices Λ_0 with $\Lambda_1 \subseteq \Lambda_0$ and* $\mathrm{Nm}(\Lambda_0) = t/\pi_0 O_F$.

(4) *Let $\Lambda_1 \subseteq C$ be a Π-modular nonhyperbolic lattice and let* $\mathrm{Nm}(\Lambda_1) = \pi_0^\ell O_F$. *The lattice Λ_1 is contained in q unimodular lattices of norm $\pi_0^{\ell-1} O_F$ and in one unimodular lattice Λ_0 with* $\mathrm{Nm}(\Lambda_0) = \pi_0^\ell O_F$.

If $E|F$ is a quadratic extension of type R-U such that $|t| = |\pi_0|$, there exist only hyperbolic Π-modular lattices in C and hence case (4) of Proposition 2.8 does not appear.

3. The moduli problem in the case R-P

Throughout this section, $E|F$ is a quadratic extension of type R-P, i.e., there exist uniformizing elements $\pi_0 \in F$ and $\Pi \in E$ such that $\Pi^2 + \pi_0 = 0$. Then $O_E = O_F[\Pi]$ for the rings of integers O_F and O_E of F and E, respectively. Let k be the common residue field with q elements, $\bar{k}$ an algebraic closure, and $\breve{F}$ the completion of the maximal unramified extension of F, with ring of integers $\breve{O}_F = W_{O_F}(\bar{k})$. Let σ be the lift of the Frobenius in $\mathrm{Gal}(\bar{k}|k)$ to $\mathrm{Gal}(\breve{O}_F|O_F)$.

3A. ***The definition of the naive moduli problem $\mathcal{N}_E^{\mathrm{naive}}$.*** We first construct a functor $\mathcal{N}_E^{\mathrm{naive}}$ on $\mathrm{Nilp}_{\breve{O}_F}$, the category of $\breve{O}_F$-schemes S such that $\pi_0 \mathcal{O}_S$ is locally nilpotent. We consider tuples (X, ι, λ), where

- X is a formal O_F-module over S of dimension 2 and height 4.

- $\iota : O_E \to \mathrm{End}(X)$ is an action of O_E satisfying the *Kottwitz condition*: The characteristic polynomial of $\iota(\alpha)$ on $\mathrm{Lie}\, X$ for any $\alpha \in O_E$ is
$$\mathrm{char}(\mathrm{Lie}\, X, T \mid \iota(\alpha)) = (T - \alpha)(T - \bar{\alpha}).$$
Here $\alpha \mapsto \bar{\alpha}$ is the nontrivial Galois automorphism and the right-hand side is a polynomial with coefficients in $\mathcal{O}_S$ via the composition $O_F[T] \hookrightarrow \breve{O}_F[T] \to \mathcal{O}_S[T]$.
- $\lambda : X \to X^\vee$ is a principal polarization on X such that the Rosati involution satisfies $\iota(\alpha)^* = \iota(\bar{\alpha})$ for $\alpha \in O_E$.

Definition 3.1. A *quasi-isogeny* (resp. an *isomorphism*) $\varphi : (X, \iota, \lambda) \to (X', \iota', \lambda')$ of two such tuples (X, ι, λ) and (X', ι', λ') over S is an O_E-linear quasi-isogeny of height 0 (resp. an O_E-linear isomorphism) $\varphi : X \to X'$ such that $\lambda = \varphi^*(\lambda')$.

Denote the group of quasi-isogenies $\varphi : (X, \iota, \lambda) \to (X, \iota, \lambda)$ by $\mathrm{QIsog}(X, \iota, \lambda)$.

For $S = \mathrm{Spec}\, \bar{k}$ we have the following proposition:

Proposition 3.2. *Up to isogeny, there exists precisely one tuple* $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ *over* $\mathrm{Spec}\, \bar{k}$ *such that the group* $\mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ *contains* $\mathrm{SU}(C, h)$ *as a closed subgroup. Here* $\mathrm{SU}(C, h)$ *is the special unitary group for a* 2*-dimensional* E*-vector space* C *with split* $E|F$*-hermitian form* h*.*

Remark 3.3. If $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ is as in the proposition, we always have $\mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) \cong \mathrm{U}(C, h)$. This follows directly from the proof and gives a more natural way to describe the framing object. However, we will need the slightly stronger statement of the proposition later, in Lemma 3.15.

Proof of Proposition 3.2. We first show uniqueness. Let $(X, \iota, \lambda)/\mathrm{Spec}\, \bar{k}$ be such a tuple. Its (relative) rational Dieudonné module N_X is a 4-dimensional vector space over $\breve{F}$ with an action of E and an alternating form $\langle\ ,\ \rangle$ such that for all $x, y \in N_X$,

$$\langle x, \Pi y \rangle = -\langle \Pi x, y \rangle. \tag{3-1}$$

The space N_X has the structure of a 2-dimensional vector space over $\breve{E} = E \otimes_F \breve{F}$ and we can define an $\breve{E}|\breve{F}$-hermitian form on it via

$$h(x, y) = \langle \Pi x, y \rangle + \Pi \langle x, y \rangle. \tag{3-2}$$

The alternating form can be recovered from h by

$$\langle x, y \rangle = \mathrm{Tr}_{\breve{E}|\breve{F}}\Big(\frac{1}{2\Pi} \cdot h(x.y) \Big). \tag{3-3}$$

Furthermore we have on N_X a σ-linear operator $\boldsymbol{F}$, the Frobenius, and a σ^{-1}-linear operator $\boldsymbol{V}$, the Verschiebung, that satisfy $\boldsymbol{V}\boldsymbol{F} = \boldsymbol{F}\boldsymbol{V} = \pi_0$. Recall that σ is the lift of the Frobenius on $\breve{O}_F$. Since $\langle\,,\rangle$ comes from a polarization, we have

$$\langle \boldsymbol{F}x, y\rangle = \langle x, \boldsymbol{V}y\rangle^\sigma$$

and

$$h(\boldsymbol{F}x, y) = h(x, \boldsymbol{V}y)^\sigma$$

for all $x, y \in N_X$. Let us consider the σ-linear operator $\tau = \Pi \boldsymbol{V}^{-1}$. Its slopes are all zero, since N_X is isotypical of slope $\frac{1}{2}$. (This follows from the condition on $\mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$.) We set $C = N_X^\tau$. This is a 2-dimensional vector space over E and $N_X = C \otimes_E \breve{E}$. Now h induces an $E|F$-hermitian form on C since

$$h(\tau x, \tau y) = h(-\boldsymbol{F}\Pi^{-1}x, \Pi \boldsymbol{V}^{-1}y) = -h(\Pi^{-1}x, \Pi y)^\sigma = h(x, y)^\sigma.$$

A priori, there are up to isomorphism two possibilities for (C, h), either h is split on C or nonsplit. But automorphisms of (C, h) correspond to elements of $\mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. The unitary groups of (C, h) for h split and h nonsplit are not isomorphic and they cannot contain each other as a closed subgroup. Hence the condition on $\mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ implies that h is split.

Assume now we have two different objects (X, ι, λ) and (X', ι', λ') as in the proposition. These give us isomorphic vector spaces (C, h) and (C', h') and an isomorphism between these extends to an isomorphism between N_X and N'_X (respecting all rational structure) which corresponds to a quasi-isogeny between (X, ι, λ) and (X', ι', λ').

The existence of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ now follows from the fact that a 2-dimensional E-vector space (C, h) with split $E|F$-hermitian form contains a unimodular lattice Λ. Indeed, this gives us a lattice $M = \Lambda \otimes_{O_E} \breve{O}_E \subseteq C \otimes_E \breve{E}$. We extend h to $N = C \otimes_E \breve{E}$ and define the $\breve{F}$-linear alternating form $\langle\,,\rangle$ as in (3-3). Now M is unimodular with respect to $\langle\,,\rangle$, because $\frac{1}{2\Pi}\breve{O}_E$ is the inverse different of $\breve{E}|\breve{F}$ (see Lemma 2.2). We choose the operators $\boldsymbol{F}$ and $\boldsymbol{V}$ on M such that $\boldsymbol{F}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{F} = \pi_0$ and $\Lambda = M^\tau$ for $\tau = \Pi \boldsymbol{V}^{-1}$. This makes M a (relative) Dieudonné module and we define $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ as the corresponding formal O_F-module. □

We fix such a framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\operatorname{Spec} \bar{k}$.

Definition 3.4. For arbitrary $S \in \mathrm{Nilp}_{\breve{O}_F}$, let $\bar{S} = S \times_{\operatorname{Spf} \breve{O}_F} \operatorname{Spec} \bar{k}$. Define $\mathcal{N}_E^{\mathrm{naive}}(S)$ as the set of equivalence classes of tuples $(X, \iota, \lambda, \varrho)$ over S, where (X, ι, λ) as above and

$$\varrho : X \times_S \bar{S} \to \mathbb{X} \times_{\operatorname{Spec} \bar{k}} \bar{S}$$

is a quasi-isogeny between the tuple (X, ι, λ) and the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ (after base change to $\bar{S}$). Two objects $(X, \iota, \lambda, \varrho)$ and $(X', \iota', \lambda', \varrho')$ are equivalent if and only if there exists an isomorphism $\varphi : (X, \iota, \lambda) \to (X', \iota', \lambda')$ such that $\varrho = \varrho' \circ (\varphi \times_S \bar{S})$.

Remark 3.5. (1) The morphism ϱ is a quasi-isogeny in the sense of Definition 3.1, i.e., we have $\lambda = \varrho^*(\lambda_{\mathbb{X}})$. Similarly, we have $\lambda = \varphi^*(\lambda')$ for the isomorphism φ. We obtain an equivalent definition of $\mathcal{N}_E^{\text{naive}}$ if we replace strict equality by the condition that, locally on S, λ and $\varrho^*(\lambda_{\mathbb{X}})$ (resp. $\varphi^*(\lambda')$) only differ by a scalar in $O_F^\times$. This variant is used in the definition of RZ-spaces of PEL type for $p > 2$ in [Rapoport and Zink 1996]. In this paper we will use the version with strict equality, since it simplifies the formulation of the straightening condition; see Definition 3.11 below.

(2) $\mathcal{N}_E^{\text{naive}}$ is pro-representable by a formal scheme, formally locally of finite type over Spf $\breve{O}_F$. This follows from [Rapoport and Zink 1996, Theorem 3.25].

As a next step, we use Dieudonné theory in order to get a better understanding of the special fiber of $\mathcal{N}_E^{\text{naive}}$. Let $N = N_{\mathbb{X}}$ be the rational Dieudonné module of the base point $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of $\mathcal{N}_E^{\text{naive}}$. This is a 4-dimensional vector space over $\breve{F}$, equipped with an E-action, an alternating form $\langle\, ,\rangle$ and two operators $\boldsymbol{V}$ and $\boldsymbol{F}$. As in the proof of Proposition 3.2, the form $\langle\, ,\rangle$ satisfies condition (3-1):

$$\langle x, \Pi y\rangle = -\langle \Pi x, y\rangle. \tag{3-4}$$

A point $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\text{naive}}(\bar{k})$ corresponds to an $\breve{O}_F$-lattice $M_X \subseteq N$. It is stable under the actions of the operators $\boldsymbol{V}$ and $\boldsymbol{F}$ and of the ring O_E. Furthermore M_X is unimodular under $\langle\, ,\rangle$, i.e., $M_X = M_X^\vee$, where

$$M_X^\vee = \{x \in N \mid \langle x, y\rangle \in \breve{O}_F \text{ for all } y \in M_X\}.$$

We can regard N as a 2-dimensional vector space over $\breve{E}$ with the $\breve{E}|\breve{F}$-hermitian form h defined by

$$h(x, y) = \langle \Pi x, y\rangle + \Pi\langle x, y\rangle. \tag{3-5}$$

Let $\breve{O}_E = O_E \otimes_{O_F} \breve{O}_F$. Then $M_X \subseteq N$ is an $\breve{O}_E$-lattice and we have

$$M_X = M_X^\vee = M_X^\sharp,$$

where $M_X^\sharp$ is the dual lattice of M_X with respect to h. The latter equality follows from the formula

$$\langle x, y\rangle = \operatorname{Tr}_{\breve{E}|\breve{F}}\Big(\frac{1}{2\Pi} \cdot h(x.y)\Big) \tag{3-6}$$

and the fact that the inverse different of $E|F$ is $\mathfrak{D}_{E|F}^{-1} = \frac{1}{2\Pi} O_E$ (see Lemma 2.2). We can thus write the set $\mathcal{N}_E^{\text{naive}}(\bar{k})$ as

$$\mathcal{N}_E^{\text{naive}}(\bar{k}) = \{\breve{O}_E\text{-lattices } M \subseteq N_{\mathbb{X}} \mid M^\sharp = M, \pi_0 M \subseteq \boldsymbol{V} M \subseteq M\}. \tag{3-7}$$

Let $\tau = \Pi \boldsymbol{V}^{-1}$. This is a σ-linear operator on N with all slopes zero. The elements invariant under τ form a 2-dimensional E-vector space $C = N^\tau$. The hermitian form

h is invariant under τ, hence it induces a split hermitian form on C which we denote again by h. With the same proof as in [Kudla and Rapoport 2014, Lemma 3.2], we have:

Lemma 3.6. *Let $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$. Then*:

(1) *$M + \tau(M)$ is τ-stable.*

(2) *Either M is τ-stable and $\Lambda_0 = M^\tau \subseteq C$ is unimodular ($\Lambda_0^\sharp = \Lambda_0$) or M is not τ-stable and then $\Lambda_{-1} = (M + \tau(M))^\tau \subseteq C$ is Π^{-1}-modular ($\Lambda_{-1}^\sharp = \Pi\Lambda_{-1}$).*

Under the identification $N = C \otimes_E \breve{E}$, we get $M = \Lambda_0 \otimes_{O_E} \breve{O}_E$ for a τ-stable Dieudonné lattice M. If M is not τ-stable, we have $M + \tau M = \Lambda_{-1} \otimes_{O_E} \breve{O}_E$ and $M \subseteq \Lambda_{-1} \otimes_{O_E} \breve{O}_E$ is a sublattice of index 1. The next lemma is the analogue of [Kudla and Rapoport 2014, Lemma 3.3].

Lemma 3.7. (1) *Fix a Π^{-1}-modular lattice $\Lambda_{-1} \subseteq C$. There is an injective map*

$$i_{\Lambda_{-1}} : \mathbb{P}(\Lambda_{-1}/\Pi\Lambda_{-1})(\bar{k}) \hookrightarrow \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$$

mapping a line $\ell \subseteq (\Lambda_{-1}/\Pi\Lambda_{-1}) \otimes \bar{k}$ to its preimage in $\Lambda_{-1} \otimes \breve{O}_E$. Identify $\mathbb{P}(\Lambda_{-1}/\Pi\Lambda_{-1})(\bar{k})$ with its image in $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$. Then $\mathbb{P}(\Lambda_{-1}/\Pi\Lambda_{-1})(k) \subseteq \mathbb{P}(\Lambda_{-1}/\Pi\Lambda_{-1})(\bar{k})$ is the set of τ-invariant Dieudonné lattices $M \subseteq \Lambda_{-1} \otimes \breve{O}_E$.

(2) *The set $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ is a union*

$$\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) = \bigcup_{\Lambda_{-1} \subseteq C} \mathbb{P}(\Lambda_{-1}/\Pi\Lambda_{-1})(\bar{k}), \tag{3-8}$$

ranging over all Π^{-1}-modular lattices $\Lambda_{-1} \subseteq C$. The projective lines corresponding to the lattices Λ_{-1} and Λ'_{-1} intersect in $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ if and only if $\Lambda_0 = \Lambda_{-1} \cap \Lambda'_{-1}$ is unimodular. In this case, their intersection consists of the point $M = \Lambda_0 \otimes \breve{O}_E \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$.

Proof. We only have to prove that the map $i_{\Lambda_{-1}}$ is well-defined. Denote by M the preimage of $\ell \subseteq (\Lambda_{-1}/\Pi\Lambda_{-1}) \otimes \bar{k}$ in $\Lambda_{-1} \otimes \breve{O}_E$. We need to show that M is an element in $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ under the identification of (3-7). It is clearly a sublattice of index 1 in $\Lambda_{-1} \otimes \breve{O}_E$, stable under the actions of $\boldsymbol{F}$, $\boldsymbol{V}$ and O_E.

Let $e_1 \in \Lambda_{-1} \otimes \breve{O}_E$ such that $e_1 \otimes \bar{k}$ generates ℓ. We can extend this to a basis (e_1, e_2) of Λ_{-1} and with respect to this basis, h is represented by a matrix of the form

$$\begin{pmatrix} x & -\Pi^{-1} \\ \Pi^{-1} & y \end{pmatrix},$$

with $x, y \in \Pi^{-1}\breve{O}_E \cap \breve{O}_F = \breve{O}_F$. The lattice $M \subseteq \Lambda_{-1} \otimes \breve{O}_E$ is generated by e_1 and Πe_2. With respect to this new basis, h is now given by the matrix

$$\begin{pmatrix} x & 1 \\ 1 & \pi_0 y \end{pmatrix}.$$

Since all entries of the matrix are integral, we have $M \subseteq M^\sharp$. But this already implies $M^\sharp = M$, because they both have index 1 in $\Lambda_{-1} \otimes \breve{O}_E$. Thus $M \in \mathcal{N}_E^{\text{naive}}(\bar{k})$ and $i_{\Lambda_{-1}}$ is well-defined. □

Remark 3.8. (1) Recall from Proposition 2.4 that the isomorphism type of a Π^i-modular lattice $\Lambda \subseteq C$ only depends on its norm ideal $\mathrm{Nm}(\Lambda) = \langle\{h(x,x) | x \in \Lambda\}\rangle = \pi_0^\ell O_F \subseteq F$. In the case that $\Lambda = \Lambda_0$ or Λ_{-1} is unimodular or Π^{-1}-modular, ℓ can be any integer such that $|1| \geq |\pi_0|^\ell \geq |2|$. In particular, there are always at least two possible values for ℓ. Recall from Lemma 2.5 that Λ is *hyperbolic* if and only if $\mathrm{Nm}(\Lambda) = 2O_F$.

(2) The intersection behavior of the projective lines in $\mathcal{N}_E^{\text{naive}}(\bar{k})$ can be deduced from Proposition 2.7. In particular, for a given unimodular lattice $\Lambda_0 \subseteq C$ with $\mathrm{Nm}(\Lambda_0) \subseteq \pi_0 O_F$, there are $q+1$ lines intersecting in $M = \Lambda_0 \otimes \breve{O}_E$. If $\mathrm{Nm}(\Lambda_0) = O_F$, the lattice $M = \Lambda_0 \otimes \breve{O}_E$ is only contained in one projective line. On the other hand, a projective line $\mathbb{P}(\Lambda_{-1}/\Pi\Lambda_{-1})(\bar{k}) \subseteq \mathcal{N}_E^{\text{naive}}(\bar{k})$ contains $q+1$ points corresponding to unimodular lattices in C. By Lemma 3.7 (1), these are exactly the k-rational points of $\mathbb{P}(\Lambda_{-1}/\Pi\Lambda_{-1})$.

(3) If we restrict the union at the right-hand side of (3-8) to hyperbolic Π^{-1}-modular lattices $\Lambda_{-1} \subseteq C$ (i.e., $\mathrm{Nm}(\Lambda_{-1}) = 2O_F$; see Lemma 2.5), we obtain a canonical subset $\mathcal{N}_E(\bar{k}) \subseteq \mathcal{N}_E^{\text{naive}}(\bar{k})$ and there is a description of $\mathcal{N}_E$ as a pro-representable functor on $\mathrm{Nilp}_{\breve{O}_F}$ (see below). We will see later (Theorem 3.16) that $\mathcal{N}_E$ is isomorphic to the Drinfeld moduli space $\mathcal{M}_{Dr}$, described in [Boutot and Carayol 1991, I.3]. In particular, the underlying topological space of $\mathcal{N}_E$ is connected. (The induced topology on the projective lines is the Zariski topology; see Proposition 3.9.) Moreover, each projective line in $\mathcal{N}_E(\bar{k})$ has $q+1$ intersection points and there are two projective lines intersecting in each such point (see also Proposition 2.7).

We fix such an intersection point $P \in \mathcal{N}_E(\bar{k})$. Now going back to $\mathcal{N}_E^{\text{naive}}(\bar{k})$, there are $q-1$ additional lines going through $P \in \mathcal{N}_E^{\text{naive}}(\bar{k})$ that correspond to nonhyperbolic lattices in C (see Proposition 2.7). Each of these additional lines contains P as its only "hyperbolic" intersection point, all other intersection points on this line and the line itself correspond to unimodular, resp. Π^{-1}-modular, lattices $\Lambda \subseteq C$ of norm $\mathrm{Nm}(\Lambda) = (2/\pi_0)O_F$ (whereas all hyperbolic lattices occurring have the norm ideal $2O_F$; see Lemma 2.5). Assume $\mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}) \subseteq \mathcal{N}_E^{\text{naive}}(\bar{k})$ is such a line and let $P' \in \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k})$ be an intersection point, where $P \neq P'$. There are again q more lines going through P' (always $q+1$ in total) that correspond to lattices with norm ideal $\mathrm{Nm}(\Lambda) = (2/\pi_0^2)O_F$, and these lines again have more intersection points and so on. This goes on until we reach lines $\mathbb{P}(\Lambda'/\Pi\Lambda')(\bar{k})$ with $\mathrm{Nm}(\Lambda') = O_F$. Each of these lines contains q points that correspond to unimodular lattices $\Lambda_0 \subseteq C$ with $\mathrm{Nm}(\Lambda_0) = O_F$. Such a lattice is only contained

(a) $e = 1,\ f = 1$ (b) $e = 2,\ f = 1$

Figure 1. The reduced locus of $\mathcal{N}_E^{\mathrm{naive}}$ for $E|F$ of type R-P where $F|\mathbb{Q}_2$ has ramification index e and inertia degree f. Solid lines are given by subschemes $\mathcal{N}_{E,\Lambda}$ for hyperbolic lattices Λ.

in one Π^{-1}-modular lattice (see part (4) of Proposition 2.7). Hence, these points are only contained in one projective line, namely $\mathbb{P}(\Lambda'/\Pi\Lambda')(\bar{k})$.

In other words, each intersection point $P \in \mathcal{N}_E(\bar{k})$ has a "tail", consisting of finitely many projective lines, which is the connected component of P in $(\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) \setminus \mathcal{N}_E(\bar{k})) \cup \{P\}$. Figure 1 shows a drawing of $(\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}}$ for the cases $F = \mathbb{Q}_2$ (on the left-hand side) and $F|\mathbb{Q}_2$ a ramified quadratic extension (on the right-hand side). The "tails" are indicated by dashed lines.

Fix a Π^{-1}-modular lattice $\Lambda = \Lambda_{-1} \subseteq C$. Let X_Λ^+ be the formal O_F-module over $\operatorname{Spec} \bar{k}$ associated to the Dieudonné lattice $M = \Lambda \otimes \breve{O}_E \subseteq N$. It comes with a canonical quasi-isogeny

$$\varrho_\Lambda^+ : \mathbb{X} \to X_\Lambda^+$$

of F-height 1. We define a subfunctor $\mathcal{N}_{E,\Lambda} \subseteq \mathcal{N}_E^{\mathrm{naive}}$ by mapping $S \in \mathrm{Nilp}_{\breve{O}_F}$ to

$$\mathcal{N}_{E,\Lambda}(S) = \{(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S) \mid (\varrho_\Lambda^+ \times S) \circ \varrho \text{ is an isogeny}\}. \tag{3-9}$$

Note that the condition of (3-9) is closed; cf. [Rapoport and Zink 1996, Proposition 2.9]. Hence $\mathcal{N}_{E,\Lambda}$ is representable by a closed formal subscheme of $\mathcal{N}_E^{\mathrm{naive}}$. On geometric points, we have a bijection

$$\mathcal{N}_{E,\Lambda}(\bar{k}) \xrightarrow{\sim} \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}), \tag{3-10}$$

as a consequence of Lemma 3.7 (1).

Proposition 3.9. *The reduced locus of $\mathcal{N}_E^{\mathrm{naive}}$ is given by*

$$(\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}} = \bigcup_{\Lambda \subseteq C} \mathcal{N}_{E,\Lambda},$$

where Λ runs over all Π^{-1}-modular lattices in C. For each Λ, there is an isomorphism of reduced schemes

$$\mathcal{N}_{E,\Lambda} \xrightarrow{\sim} \mathbb{P}(\Lambda/\Pi\Lambda),$$

inducing the map (3-10) on $\bar{k}$-valued points.

Proof. The embedding

$$\bigcup_{\Lambda\subseteq C} (\mathcal{N}_{E,\Lambda})_{\mathrm{red}} \hookrightarrow (\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}} \tag{3-11}$$

is closed, because each embedding $\mathcal{N}_{E,\Lambda} \subseteq \mathcal{N}_E^{\mathrm{naive}}$ is closed and, locally on $(\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}}$, the left-hand side is always only a finite union of $(\mathcal{N}_{E,\Lambda})_{\mathrm{red}}$. It follows already that (3-11) is an isomorphism, since it is a bijection on $\bar{k}$-valued points (see (3-8) and (3-10)) and $(\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}}$ is reduced by definition and locally of finite type over $\operatorname{Spec}\bar{k}$ by Remark 3.5 (2).

For the second part of the proposition, we follow the proof presented in [Kudla and Rapoport 2014, Lemma 4.2]. Fix a Π^{-1}-modular lattice $\Lambda \subseteq C$ and let $M = \Lambda \otimes \breve{O}_E \subseteq N$, as above. Now X_Λ^+ is the formal O_F-module associated to M, but we also get a formal O_F-module X_Λ^- associated to the dual $M^\sharp = \Pi M$ of M. This comes with a natural isogeny

$$\mathrm{nat}_\Lambda : X_\Lambda^- \to X_\Lambda^+$$

and a quasi-isogeny $\varrho_\Lambda^- : X_\Lambda^- \to \mathbb{X}$ of F-height 1. For $(X,\iota,\lambda,\varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$ where $S \in \mathrm{Nilp}_{\breve{O}_F}$, we consider the composition

$$\varrho_{\Lambda,X}^- = \varrho^{-1} \circ (\varrho_\Lambda^- \times S) : (X_\Lambda^- \times S) \to X.$$

By [Kudla and Rapoport 2014, Lemma 4.2], this composition is an isogeny if and only if $(\varrho_\Lambda^+ \times S) \circ \varrho$ is an isogeny, or, in other words, if and only if $(X,\iota,\lambda,\varrho) \in \mathcal{N}_{E,\Lambda}(S)$. Let $\mathbb{D}_{X_\Lambda^-}(S)$ be the (relative) Grothendieck–Messing crystal of X_Λ^- evaluated at S (cf. [Ahsendorf et al. 2016, Definition 3.24] or [Ahsendorf 2011, Section 5.2]). This is a locally free $\mathcal{O}_S$-module of rank 4, isomorphic to $\Lambda/\pi_0\Lambda \otimes_{O_F} \mathcal{O}_S$. The kernel of $\mathbb{D}(\mathrm{nat}_\Lambda)(S)$ is given by $(\Lambda/\Pi\Lambda) \otimes_{O_F} \mathcal{O}_S$, locally a direct summand of rank 2 of $\mathbb{D}_{X_\Lambda^-}(S)$. For any $(X,\iota,\lambda,\varrho) \in \mathcal{N}_{E,\Lambda}(S)$, the kernel of $\varrho_{\Lambda,X}^-$ is contained in $\ker(\mathrm{nat}_\Lambda)$. It follows from [Vollaard and Wedhorn 2011, Corollary 4.7] (see also [Kudla and Rapoport 2014, Proposition 4.6]) that $\ker \mathbb{D}(\varrho_{\Lambda,X}^-)(S)$ is locally a direct summand of rank 1 of $(\Lambda/\Pi\Lambda) \otimes_{O_F} \mathcal{O}_S$. This induces a map

$$\mathcal{N}_{E,\Lambda}(S) \to \mathbb{P}(\Lambda/\Pi\Lambda)(S),$$

functorial in S, and the arguments of [Vollaard and Wedhorn 2011, Section 4.7] show that it is an isomorphism. (One easily checks that their results indeed carry over to the relative setting over O_F.) $\square$

3B. *Construction of the closed formal subscheme $\mathcal{N}_E \subseteq \mathcal{N}_E^{\mathbf{naive}}$.* We now use a result from Section 5. By Theorem 5.2 and Remark 5.1 (2), there exists a principal polarization $\widetilde{\lambda}_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}^\vee$ on $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$, unique up to a scalar in $O_E^\times$, such that the induced Rosati involution is the identity on O_E. Furthermore, for any $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$, the pullback $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$ is a principal polarization on X.

The next proposition is crucial for the construction of $\mathcal{N}_E$. Recall the notion of a *hyperbolic* lattice from Proposition 2.4 and the subsequent discussion.

Proposition 3.10. *It is possible to choose* $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ *and* $\widetilde{\lambda}_{\mathbb{X}}$ *such that*

$$\lambda_{\mathbb{X},1} = \tfrac{1}{2}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}}) \in \mathrm{Hom}(\mathbb{X}, \mathbb{X}^\vee).$$

Fix such a choice and let $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$. *Then,* $\frac{1}{2}(\lambda + \widetilde{\lambda}) \in \mathrm{Hom}(X, X^\vee)$ *if and only if* $(X, \iota, \lambda, \varrho) \in \mathcal{N}_{E,\Lambda}(\bar{k})$ *for some hyperbolic lattice* $\Lambda \subseteq C$.

Proof. The polarization $\widetilde{\lambda}_{\mathbb{X}}$ on $\mathbb{X}$ induces an alternating form $(\,,\,)$ on the rational Dieudonné module $N = M_{\mathbb{X}} \otimes_{\breve{O}_F} \breve{F}$. For all $x, y \in N$, the form $(\,,\,)$ satisfies

$$(\boldsymbol{F}x, y) = (x, \boldsymbol{V}y)^\sigma,$$
$$(\Pi x, y) = (x, \Pi y).$$

It induces an $\breve{E}$-alternating form b on N via

$$b(x, y) = \delta((\Pi x, y) + \Pi(x, y)),$$

where $\delta \in \breve{O}_F$ is a unit generating the unramified quadratic extension of F, chosen such that $\delta^\sigma = -\delta$ and $(1+\delta)/2 \in \breve{O}_F$; see page 348. On the other hand, we can describe $(\,,\,)$ in terms of b,

$$(x, y) = \mathrm{Tr}_{\breve{E}|\breve{F}}\Big(\frac{1}{2\Pi\delta} \cdot b(x, y)\Big). \tag{3-12}$$

The form b is invariant under $\tau = \Pi \boldsymbol{V}^{-1}$, since

$$b(\tau x, \tau y) = b(-\boldsymbol{F}\Pi^{-1}x, \Pi \boldsymbol{V}^{-1} y) = b(\Pi^{-1}x, \Pi y)^\sigma = b(x, y)^\sigma.$$

Hence b defines an E-linear alternating form on $C = N^\tau$, which we again denote by b. Denote by $\langle\,,\,\rangle$ the alternating form on $M_{\mathbb{X}}$ induced by the polarization $\lambda_{\mathbb{X}}$ and let h be the corresponding hermitian form; see (3-2). On $N_{\mathbb{X}}$, we define the alternating form $\langle\,,\,\rangle_1$ by

$$\langle x, y\rangle_1 = \tfrac{1}{2}(\langle x, y\rangle + (x, y)).$$

This form is integral on $M_{\mathbb{X}}$ if and only if $\lambda_{\mathbb{X},1} = \frac{1}{2}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}})$ is a polarization on $\mathbb{X}$.

We choose $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ such that it corresponds to a unimodular hyperbolic lattice $\Lambda_0 \subseteq (C, h)$ under the identifications of (3-7) and Lemma 3.6. There exists a basis

(e_1, e_2) of Λ_0 such that

$$h \mathrel{\widehat{=}} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad b \mathrel{\widehat{=}} \begin{pmatrix} & u \\ -u & \end{pmatrix} \tag{3-13}$$

for some $u \in E^\times$. Since $\tilde{\lambda}_{\mathbb{X}}$ is principal, the alternating form b is perfect on Λ_0, thus $u \in O_E^\times$. After rescaling $\tilde{\lambda}_{\mathbb{X}}$, we may assume that $u = 1$. We now have

$$\tfrac{1}{2}(h(x, y) + b(x, y)) \in O_E,$$

for all $x, y \in \Lambda_0$. Thus $\frac{1}{2}(h + b)$ is integral on $M_{\mathbb{X}} = \Lambda_0 \otimes_{O_E} \breve{O}_E$. This implies that

$$\begin{aligned} \langle x, y\rangle_1 &= \tfrac{1}{2}(\langle x, y\rangle + (x, y)) = \tfrac{1}{2}\operatorname{Tr}_{\breve{E}|\breve{F}}\Big(\frac{1}{2\Pi}\cdot h(x.y) + \frac{1}{2\Pi\delta}\cdot b(x, y)\Big) \\ &= \operatorname{Tr}_{\breve{E}|\breve{F}}\Big(\frac{1}{4\Pi}(h(x, y) + b(x, y))\Big) + \operatorname{Tr}_{\breve{E}|\breve{F}}\Big(\frac{1-\delta}{4\Pi\delta}\cdot b(x, y)\Big) \in \breve{O}_F \end{aligned}$$

for all $x, y \in M_{\mathbb{X}}$. Indeed, in the definition of b, the unit δ has been chosen such that $(1+\delta)/2 \in \breve{O}_F$, so the second summand is in $\breve{O}_F$. The first summand is integral, since $\frac{1}{2}(h + b)$ is integral. It follows that $\lambda_{\mathbb{X},1} = \frac{1}{2}(\lambda_{\mathbb{X}} + \tilde{\lambda}_{\mathbb{X}})$ is a polarization on $\mathbb{X}$.

Let $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\text{naive}}(\bar{k})$ and assume that $\lambda_1 = \frac{1}{2}(\lambda + \tilde{\lambda}) = \varrho^*(\lambda_{\mathbb{X},1})$ is a polarization on X. Then $\langle\, ,\rangle_1$ is integral on the Dieudonné module $M \subseteq N$ of X. By the above calculation, this is equivalent to $\frac{1}{2}(h + b)$ being integral on M. In particular, this implies that

$$h(x, x) = h(x, x) + b(x, x) \in 2\breve{O}_F,$$

for all $x \in M$. Let $\Lambda = (M + \tau(M))^\tau$. Then $h(x, x) \in 2O_F$ for all $x \in \Lambda$; hence $\operatorname{Nm}(\Lambda) \subseteq 2O_F$. By Lemma 2.5 and the bound of norm ideals, we have $\operatorname{Nm}(\Lambda) = 2O_F$ and Λ is a hyperbolic lattice. It follows that $(X, \iota, \lambda, \varrho) \in \mathcal{N}_{E,\Lambda'}(\bar{k})$ for some hyperbolic Π^{-1}-modular lattice $\Lambda' \subseteq C$. Indeed, if $M^\tau \subsetneq \Lambda$ then Λ is Π^{-1}-modular and $\Lambda' = \Lambda$. If $M^\tau = \Lambda$ then it is contained in some Π^{-1}-modular hyperbolic lattice Λ' by Proposition 2.7.

Conversely, assume that $(X, \iota, \lambda, \varrho) \in \mathcal{N}_{E,\Lambda}(\bar{k})$ for some hyperbolic lattice $\Lambda \subseteq C$. It suffices to show that $\frac{1}{2}(h + b)$ is integral on Λ. Indeed, it follows that $\frac{1}{2}(h + b)$ is integral on the Dieudonné module M. Thus $\langle\, ,\rangle_1$ is integral on M and this is equivalent to $\lambda_1 = \frac{1}{2}(\lambda + \tilde{\lambda}) \in \operatorname{Hom}(X, X^\vee)$.

Let $\Lambda' \subseteq C$ be the Π^{-1}-modular lattice generated by e_1 and $\Pi^{-1}e_2$, where (e_1, e_2) is the basis of the lattice Λ_0 corresponding to the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. By (3-13), h and b have the following form with respect to the basis $(e_1, \Pi^{-1}e_2)$,

$$h \mathrel{\widehat{=}} \begin{pmatrix} & -\Pi^{-1} \\ \Pi^{-1} & \end{pmatrix}, \quad b \mathrel{\widehat{=}} \begin{pmatrix} & \Pi^{-1} \\ -\Pi^{-1} & \end{pmatrix}.$$

In particular, Λ' is hyperbolic and $\frac{1}{2}(h + b)$ is integral on Λ'. By Proposition 2.4, there exists an automorphism $g \in \operatorname{SU}(C, h)$ mapping Λ onto Λ'. Since $\det g = 1$, the alternating form b is invariant under g. It follows that $\frac{1}{2}(h+b)$ is also integral on Λ. □

From now on, we assume $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ and $\widetilde{\lambda}_{\mathbb{X}}$ chosen in a way such that

$$\lambda_{\mathbb{X},1} = \tfrac{1}{2}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}}) \in \operatorname{Hom}(\mathbb{X}, \mathbb{X}^{\vee}).$$

Note that this determines the polarization $\widetilde{\lambda}_{\mathbb{X}}$ up to a scalar in $1+2O_E$. If we replace $\widetilde{\lambda}_{\mathbb{X}}$ by $\widetilde{\lambda}'_{\mathbb{X}} = \widetilde{\lambda}_{\mathbb{X}} \circ \iota_{\mathbb{X}}(1+2u)$ for some $u \in O_E$, then $\lambda'_{\mathbb{X},1} = \lambda_{\mathbb{X},1} + \widetilde{\lambda}_{\mathbb{X}} \circ \iota_{\mathbb{X}}(u)$.

We can now formulate the straightening condition.

Definition 3.11. Let $S \in \operatorname{Nilp}_{\breve{O}_F}$. An object $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\text{naive}}(S)$ satisfies the *straightening condition* if

$$\lambda_1 \in \operatorname{Hom}(X, X^{\vee}), \tag{3-14}$$

where $\lambda_1 = \frac{1}{2}(\lambda + \widetilde{\lambda}) = \varrho^*(\lambda_{\mathbb{X},1})$.

This definition is clearly independent of the choice of the polarization $\widetilde{\lambda}_{\mathbb{X}}$. We define $\mathcal{N}_E$ as the functor that maps $S \in \operatorname{Nilp}_{\breve{O}_F}$ to the set of all tuples $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\text{naive}}(S)$ that satisfy the straightening condition. By [Rapoport and Zink 1996, Proposition 2.9], $\mathcal{N}_E$ is representable by a closed formal subscheme of $\mathcal{N}_E^{\text{naive}}$.

Remark 3.12. The reduced locus of $\mathcal{N}_E$ can be written as

$$(\mathcal{N}_E)_{\text{red}} = \bigcup_{\Lambda \subseteq C} \mathcal{N}_{E,\Lambda} \simeq \bigcup_{\Lambda \subseteq C} \mathbb{P}(\Lambda/\Pi\Lambda),$$

where we take the unions over all *hyperbolic* Π^{-1}-modular lattices $\Lambda \subseteq C$. By Proposition 2.7 and Lemma 3.7, each projective line contains $q+1$ points corresponding to unimodular lattices and there are two lines intersecting in each such point. Recall from Remark 3.8 (1) that there exist nonhyperbolic Π^{-1}-modular lattices $\Lambda \subseteq C$; thus we have $\mathcal{N}_E(\bar{k}) \neq \mathcal{N}_E^{\text{naive}}(\bar{k})$, and in particular $(\mathcal{N}_E)_{\text{red}} \neq (\mathcal{N}_E^{\text{naive}})_{\text{red}}$.

Remark 3.13. As has been pointed out to the author by A. Genestier, the straightening condition is not trivial on the rigid-analytic generic fiber of $\mathcal{N}_E^{\text{naive}}$. However, we can show that it is open and closed. Since a proper study of the generic fiber would go beyond the scope of this paper, we restrain ourselves to indications rather than complete proofs.

Let C be an algebraically closed extension of F and $\mathcal{O}_C$ its ring of integers. Take a point $x = (X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\text{naive}}(\mathcal{O}_C)$ and consider its 2-adic Tate module $T_2(x)$. It is a free O_E-module of rank 2 and λ endows $T_2(x)$ with a perfect (nonsplit) hermitian form h. If $x \in \mathcal{N}_E(\mathcal{O}_C)$, then the straightening condition implies that $(T_2(x), h)$ is a lattice with minimal norm[2] $\operatorname{Nm}(T_2(x))$ in the vector space $V_2(x) = T_2(x) \otimes_{O_E} E$ (see Proposition 2.4 and [Jacobowitz 1962]). But $V_2(x)$ also contains

[2] Calling this lattice “hyperbolic” doesn’t make much sense here since it is anisotropic.

self-dual lattices with nonminimal norm ideal. Let $\Lambda \subseteq V_2(x)$ be such a lattice with $\mathrm{Nm}(\Lambda) \neq \mathrm{Nm}(T_2(x))$. Let Λ' be the intersection of $T_2(x)$ and Λ in $V_2(x)$. The inclusions $\Lambda' \hookrightarrow \Lambda$ and $\Lambda' \hookrightarrow T_2(x)$ define canonically a formal O_F-module Y with $T_2(Y) = \Lambda'$ and a quasi-isogeny $\varphi : X \to Y$. By inheriting all data, Y becomes a point in $\mathcal{N}_E^{\mathrm{naive}}(\mathcal{O}_C)$ that does not satisfy the straightening condition.

To see that the straightening condition is open and closed on the generic fiber, consider the universal formal O_F-module $\mathcal{X} = (\mathcal{X}, \iota_{\mathcal{X}}, \lambda_{\mathcal{X}})$ over $\mathcal{N}_E^{\mathrm{naive}}$ and let $T_2(\mathcal{X})$ be its Tate module. Then $T_2(\mathcal{X})$ is a locally constant sheaf over $\mathcal{N}_E^{\mathrm{naive,rig}}$ with respect to the étale topology. The polarization $\lambda_{\mathcal{X}}$ defines a hermitian form h on $T_2(\mathcal{X})$. Since $T_2(\mathcal{X})$ is a locally constant sheaf, the norm ideal $\mathrm{Nm}(T_2(\mathcal{X}))$ with respect to h (see Proposition 2.4) is locally constant as well. Hence the locus where $\mathrm{Nm}(T_2(\mathcal{X}))$ is minimal is open and closed in $\mathcal{N}_E^{\mathrm{naive,rig}}$. But this is exactly $\mathcal{N}_E^{\mathrm{rig}} \subseteq \mathcal{N}_E^{\mathrm{naive,rig}}$.

3C. ***The isomorphism to the Drinfeld moduli problem.*** We now recall the Drinfeld moduli problem $\mathcal{M}_{Dr}$ on $\mathrm{Nilp}_{\breve{O}_F}$. Let B be the quaternion division algebra over F and O_B its ring of integers. Let $S \in \mathrm{Nilp}_{\breve{O}_F}$. Then $\mathcal{M}_{Dr}(S)$ is the set of equivalence classes of objects (X, ι_B, ϱ), where

- X is a formal O_F-module over S of dimension 2 and height 4;
- $\iota_B : O_B \to \mathrm{End}(X)$ is an action of O_B on X satisfying the *special* condition, i.e., $\mathrm{Lie}\, X$ is, locally on S, a free $(\mathcal{O}_S \otimes_{O_F} O_F^{(2)})$-module of rank 1, where $O_F^{(2)} \subseteq O_B$ is any embedding of the unramified quadratic extension of O_F into O_B (cf. [Boutot and Carayol 1991]);
- $\varrho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\, \bar{k}} \bar{S}$ is an O_B-linear quasi-isogeny of height 0 to a fixed framing object $(\mathbb{X}, \iota_{\mathbb{X}}) \in \mathcal{M}_{Dr}(\bar{k})$.

Such a framing object exists and is unique up to isogeny. By a proposition of Drinfeld, cf. [Boutot and Carayol 1991, p. 138], there always exist polarizations on these objects, as follows:

Proposition 3.14 [Drinfeld 1976]. *Let $\Pi \in O_B$ a uniformizer with $\Pi^2 \in O_F$ and let $b \mapsto b'$ be the standard involution of B. Then $b \mapsto b^* = \Pi b' \Pi^{-1}$ is another involution on B.*

(1) *There exists a principal polarization $\lambda_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}^\vee$ on $\mathbb{X}$ with associated Rosati involution $b \mapsto b^*$. It is unique up to a scalar in $O_F^\times$.*

(2) *Let $\lambda_{\mathbb{X}}$ be as in* (1)*. For $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$, there exists a unique principal polarization*

$$\lambda : X \to X^\vee$$

with Rosati involution $b \mapsto b^*$ such that $\varrho^*(\lambda_{\mathbb{X}}) = \lambda$ on $\bar{S}$.

We now relate $\mathcal{M}_{Dr}$ and $\mathcal{N}_E$. For this, we fix an embedding $E \hookrightarrow B$. Any choice of a uniformizer $\Pi \in O_E$ with $\Pi^2 \in O_F$ induces the same involution $b \mapsto b^* = \Pi b' \Pi^{-1}$ on B.

For the framing object $(\mathbb{X}, \iota_{\mathbb{X}})$ of $\mathcal{M}_{Dr}$, let $\lambda_{\mathbb{X}}$ be a polarization associated to this involution by Proposition 3.14 (1). Denote by $\iota_{\mathbb{X},E}$ the restriction of $\iota_{\mathbb{X}}$ to $O_E \subseteq O_B$. For any object $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$, let λ be the polarization with Rosati involution $b \mapsto b^*$ that satisfies $\varrho^*(\lambda_{\mathbb{X}}) = \lambda$; see Proposition 3.14 (2). Let ι_E be the restriction of ι_B to O_E.

Lemma 3.15. $(\mathbb{X}, \iota_{\mathbb{X},E}, \lambda_{\mathbb{X}})$ *is a framing object for* $\mathcal{N}_E^{\text{naive}}$. *Furthermore, the map*

$$(X, \iota_B, \varrho) \mapsto (X, \iota_E, \lambda, \varrho)$$

induces a closed immersion of formal schemes

$$\eta : \mathcal{M}_{Dr} \hookrightarrow \mathcal{N}_E^{\text{naive}}.$$

Proof. There are two things to check: that $\mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ contains $\mathrm{SU}(C, h)$ as a closed subgroup and that ι_E satisfies the Kottwitz condition. Indeed, once these two assertions hold, we can take $(\mathbb{X}, \iota_{\mathbb{X},E}, \lambda_{\mathbb{X}})$ as a framing object for $\mathcal{N}_E^{\text{naive}}$ and the morphism η is well-defined. For any $S \in \mathrm{Nilp}_{\breve{O}_F}$, the map $\eta(S)$ is injective, because (X, ι_B, ϱ) and $(X', \iota'_B, \varrho') \in \mathcal{M}_{Dr}(S)$ map to the same point in $\mathcal{N}_E^{\text{naive}}(S)$ under η if and only if the quasi-isogeny $\varrho' \circ \varrho$ on $\bar{S}$ lifts to an isomorphism on S, i.e., if and only if (X, ι_B, ϱ) and (X', ι'_B, ϱ') define the same point in $\mathcal{M}_{Dr}(S)$. The functor

$$F : S \mapsto \{(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\text{naive}}(S) \mid \iota \text{ extends to an } O_B\text{-action}\}$$

is pro-representable by a closed formal subscheme of $\mathcal{N}_E^{\text{naive}}$ by [Rapoport and Zink 1996, Proposition 2.9]. Now, the formal subscheme $\eta(\mathcal{M}_{Dr}) \subseteq F$ is given by the special condition. But the special condition is open and closed (see [Rapoport and Zink 2017, p. 7]), thus η is a closed embedding.

It remains to show the two assertions from the beginning of this proof. We first check the condition on $\mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. Let $G_{(\mathbb{X},\iota_{\mathbb{X}})}$ be the group of O_B-linear quasi-isogenies $\varphi : (\mathbb{X}, \iota_{\mathbb{X}}) \to (\mathbb{X}, \iota_{\mathbb{X}})$ of height 0 such that the induced homomorphism of Dieudonné modules has determinant 1. Then we have (noncanonical) isomorphisms $G_{(\mathbb{X},\iota_{\mathbb{X}})} \simeq \mathrm{SL}_{2,F}$ and $\mathrm{SL}_{2,F} \simeq \mathrm{SU}(C, h)$, since h is split. The uniqueness of the polarization $\lambda_{\mathbb{X}}$ (up to a scalar in $O_F^\times$) implies that $G_{(\mathbb{X},\iota_{\mathbb{X}})} \subseteq \mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. This is a closed embedding of linear algebraic groups over F, since a quasi-isogeny $\varphi \in \mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ lies in $G_{(\mathbb{X},\iota_{\mathbb{X}})}$ if and only if it is O_B-linear and has determinant 1, and these are closed conditions on $\mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$.

Finally, the special condition implies the Kottwitz condition for any element $b \in O_B$ (see [Rapoport and Zink 2017, Proposition 5.8]), i.e., the characteristic polynomial for the action of $\iota(b)$ on $\operatorname{Lie} X$ is

$$\operatorname{char}(\operatorname{Lie} X, T \mid \iota(b)) = (T-b)(T-b'),$$

where the right-hand side is a polynomial in $\mathcal{O}_S[T]$ via the structure homomorphism $O_F \hookrightarrow \breve{O}_F \to \mathcal{O}_S$. From this, the second assertion follows. $\square$

Let $O_F^{(2)} \subseteq O_B$ be an embedding such that conjugation with Π induces the nontrivial Galois action on $O_F^{(2)}$, as in Lemma 2.3 (1). Fix a generator $\gamma = (1+\delta)/2$ of $O_F^{(2)}$ with $\delta^2 \in O_F^\times$. On $(\mathbb{X}, \iota_{\mathbb{X}})$, the principal polarization $\tilde{\lambda}_{\mathbb{X}}$ given by

$$\tilde{\lambda}_{\mathbb{X}} = \lambda_{\mathbb{X}} \circ \iota_{\mathbb{X}}(\delta)$$

has a Rosati involution that induces the identity on O_E. For any $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$, we set $\tilde{\lambda} = \varrho^*(\tilde{\lambda}_{\mathbb{X}}) = \lambda \circ \iota_B(\delta)$. The tuple $(X, \iota_E, \lambda, \varrho) = \eta(X, \iota_B, \varrho)$ satisfies the straightening condition (3-14), since

$$\lambda_1 = \tfrac{1}{2}(\lambda + \tilde{\lambda}) = \lambda \circ \iota_B(\gamma) \in \operatorname{Hom}(X, X^\vee).$$

In particular, the tuple $(\mathbb{X}, \iota_{\mathbb{X},E}, \lambda_{\mathbb{X}})$ is a framing object of $\mathcal{N}_E$ and η induces a natural transformation

$$\eta : \mathcal{M}_{Dr} \hookrightarrow \mathcal{N}_E. \tag{3-15}$$

Note that this map does not depend on the above choices, as $\mathcal{N}_E$ is a closed formal subscheme of $\mathcal{N}_E^{\text{naive}}$.

Theorem 3.16. *$\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E$ is an isomorphism of formal schemes.*

We will first prove this on $\bar{k}$-valued points:

Lemma 3.17. *η induces a bijection $\eta(\bar{k}) : \mathcal{M}_{Dr}(\bar{k}) \to \mathcal{N}_E(\bar{k})$.*

Proof. We can identify the $\bar{k}$-valued points of $\mathcal{M}_{Dr}$ with a subset $\mathcal{M}_{Dr}(\bar{k}) \subseteq \mathcal{N}_E^{\text{naive}}(\bar{k})$. The rational Dieudonné module N of $\mathbb{X}$ is equipped with an action of B. Fix an embedding $F^{(2)} \hookrightarrow B$ as in Lemma 2.3 (1). This induces a $\mathbb{Z}/2$-grading $N = N_0 \oplus N_1$ of N, where

$$N_0 = \{x \in N \mid \iota(a)x = ax \text{ for all } a \in F^{(2)}\},$$
$$N_1 = \{x \in N \mid \iota(a)x = \sigma(a)x \text{ for all } a \in F^{(2)}\}$$

for a fixed embedding $F^{(2)} \hookrightarrow \breve{F}$. The operators $\boldsymbol{V}$ and $\boldsymbol{F}$ have degree 1 with respect to this decomposition. Recall that λ has Rosati involution $b \mapsto \Pi b' \Pi^{-1}$ on O_B which restricts to the identity on $O_F^{(2)}$. The subspaces N_0 and N_1 are therefore orthogonal with respect to $\langle\, , \rangle$.

Under the identification (3-7), a lattice $M \in \mathcal{M}_{Dr}(\bar{k})$ respects this decomposition, i.e., $M = M_0 \oplus M_1$ with $M_i = M \cap N_i$. Furthermore it satisfies the special condition

$$\dim M_0/\boldsymbol{V} M_1 = \dim M_1/\boldsymbol{V} M_0 = 1.$$

We already know that $\mathcal{M}_{Dr}(\bar{k}) \subseteq \mathcal{N}_E(\bar{k})$, so let us assume $M \in \mathcal{N}_E(\bar{k})$. We want to show that $M \in \mathcal{M}_{Dr}(\bar{k})$, i.e., that the lattice M is stable under the action of O_B on N and satisfies the special condition. It is stable under the O_B-action if and only if $M = M_0 \oplus M_1$ for $M_i = M \cap N_i$. Let $y \in M$ and $y = y_0 + y_1$ with $y_i \in N_i$. For any $x \in M$, we have

$$\langle x, y \rangle = \langle x, y_0 \rangle + \langle x, y_1 \rangle \in \breve{O}_F. \tag{3-16}$$

We can assume that $\lambda_{\mathbb{X},1} = \lambda_{\mathbb{X}} \circ \iota_B(\gamma)$ with $\gamma \in O_F^{(2)}$ under our fixed embedding $F^{(2)} \hookrightarrow B$. Recall that $\gamma^\sigma = 1 - \gamma$ from page 348. Let $\langle\, , \rangle_1$ be the alternating form on M induced by $\lambda_{\mathbb{X},1}$. Then,

$$\langle x, y \rangle_1 = \gamma \cdot \langle x, y_0 \rangle + (1-\gamma) \cdot \langle x, y_1 \rangle \in \breve{O}_F. \tag{3-17}$$

From (3-16) and (3-17), it follows that $\langle x, y_0 \rangle$ and $\langle x, y_1 \rangle$ lie in $\breve{O}_F$. Since $x \in M$ was arbitrary and $M = M^\vee$, this gives $y_0, y_1 \in M$. Hence M respects the decomposition of N and is stable under the action of O_B.

It remains to show that M satisfies the special condition: The alternating form $\langle\, , \rangle$ is perfect on M, thus the restrictions to M_0 and M_1 are perfect as well. If M is not special, we have $M_i = \boldsymbol{V} M_{i+1}$ for some $i \in \{0, 1\}$. But then, $\langle\, , \rangle$ cannot be perfect on M_i. In fact, for any $x, y \in M_{i+1}$,

$$\langle \boldsymbol{V}x, \boldsymbol{V}y \rangle^\sigma = \langle \boldsymbol{F}\boldsymbol{V}x, y \rangle = \pi_0 \cdot \langle x, y \rangle \in \pi_0 \breve{O}_F.$$

Thus M is indeed special, i.e., $M \in \mathcal{M}_{Dr}(\bar{k})$, and this finishes the proof of the lemma. □

Proof of Theorem 3.16. We already know that η is a closed embedding

$$\eta : \mathcal{M}_{Dr} \hookrightarrow \mathcal{N}_E.$$

Let $(\mathbb{X}, \iota_{\mathbb{X}})$ be the framing object of $\mathcal{M}_{Dr}$ and choose an embedding $O_F^{(2)} \subseteq O_B$ and a generator γ of $O_F^{(2)}$ as in Lemma 2.3 (1). We take $(\mathbb{X}, \iota_{\mathbb{X},E}, \lambda_{\mathbb{X}})$ as a framing object for $\mathcal{N}_E$ and set $\tilde{\lambda}_{\mathbb{X}} = \lambda_{\mathbb{X}} \circ \iota_{\mathbb{X}}(\delta)$.

Let $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(S)$ and $\tilde{\lambda} = \varrho^*(\tilde{\lambda}_{\mathbb{X}})$. We have

$$\varrho^{-1} \circ \iota_{\mathbb{X}}(\gamma) \circ \varrho = \varrho^{-1} \circ \lambda_{\mathbb{X}}^{-1} \circ \lambda_{\mathbb{X},1} \circ \varrho = \lambda^{-1} \circ \lambda_1 \in \operatorname{End}(X),$$

where $\lambda_{\mathbb{X},1} = \frac{1}{2}(\lambda_{\mathbb{X}} + \tilde{\lambda}_{\mathbb{X}})$ and $\lambda_1 = \frac{1}{2}(\lambda + \tilde{\lambda})$. Since $O_B = O_F[\Pi, \gamma]$, this induces an O_B-action ι_B on X and makes ϱ an O_B-linear quasi-isogeny. We have to check that (X, ι_B, ϱ) satisfies the special condition.

Recall that the special condition is open and closed (see [Rapoport and Zink 2017, p. 7]), so η is an open and closed embedding. Furthermore, $\eta(\bar{k})$ is bijective and the reduced loci $(\mathcal{M}_{Dr})_{\mathrm{red}}$ and $(\mathcal{N}_E)_{\mathrm{red}}$ are locally of finite type over $\operatorname{Spec} \bar{k}$. Hence η induces an isomorphism on reduced subschemes. But any open and closed embedding of formal schemes, that is, an isomorphism on the reduced subschemes, is already an isomorphism. □

4. The moduli problem in the case R-U

Let $E|F$ be a quadratic extension of type R-U, generated by a uniformizer Π satisfying an Eisenstein equation of the form $\Pi^2 - t\Pi + \pi_0 = 0$ where $t \in O_F$ and $\pi_0|t|2$. Let O_F and O_E be the rings of integers of F and E. We have $O_E = O_F[\Pi]$. As in the case R-P, let k be the common residue field, $\bar{k}$ an algebraic closure, $\breve{F}$ the completion of the maximal unramified extension with ring of integers $\breve{O}_F = W_{O_F}(\bar{k})$ and σ the lift of the Frobenius in $\mathrm{Gal}(\bar{k}|k)$ to $\mathrm{Gal}(\breve{O}_F|O_F)$.

4A. ***The naive moduli problem.*** Let $S \in \mathrm{Nilp}_{\breve{O}_F}$. Consider tuples (X, ι, λ), where

- X is a formal O_F-module over S of dimension 2 and height 4.
- $\iota : O_E \to \mathrm{End}(X)$ is an action of O_E on X satisfying the *Kottwitz condition*: The characteristic polynomial of $\iota(\alpha)$ for some $\alpha \in O_E$ is given by

$$\mathrm{char}(\mathrm{Lie}\, X, T \mid \iota(\alpha)) = (T - \alpha)(T - \bar{\alpha}).$$

  Here $\alpha \mapsto \bar{\alpha}$ is the Galois conjugation of $E|F$ and the right-hand side is a polynomial in $\mathcal{O}_S[T]$ via the structure morphism $O_F \hookrightarrow \breve{O}_F \to \mathcal{O}_S$.
- $\lambda : X \to X^\vee$ is a polarization on X with kernel $\ker \lambda = X[\Pi]$, where $X[\Pi]$ is the kernel of $\iota(\Pi)$. Further we demand that the Rosati involution of λ satisfies $\iota(\alpha)^* = \iota(\bar{\alpha})$ for all $\alpha \in O_E$.

We define quasi-isogenies $\varphi : (X, \iota, \lambda) \to (X', \iota', \lambda')$ and the group $\mathrm{QIsog}(X, \iota, \lambda)$ as in Definition 3.1.

Proposition 4.1. *Up to isogeny, there exists exactly one such tuple* $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ *over* $S = \mathrm{Spec}\, \bar{k}$ *under the condition that the group* $\mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ *contains a closed subgroup isomorphic to* $\mathrm{SU}(C, h)$ *for a* 2*-dimensional* E*-vector space* C *with split* $E|F$*-hermitian form* h.

Remark 4.2. As in the case R-P, we have $\mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) \cong \mathrm{U}(C, h)$ for $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ as in the proposition.

Proof of Proposition 4.1. We first show uniqueness of the object. Let $(X, \iota, \lambda)/\mathrm{Spec}\, \bar{k}$ be a tuple as in the proposition and consider its rational Dieudonné module N_X. This is a 4-dimensional vector space over $\breve{F}$ equipped with an action of E and an alternating form $\langle\, ,\, \rangle$ such that

$$\langle x, \Pi y\rangle = \langle \overline{\Pi} x, y\rangle \tag{4-1}$$

for all $x, y \in N_X$. Let $\breve{E} = \breve{F} \otimes_F E$. We can see N_X as 2-dimensional vector space over $\breve{E}$ with a hermitian form h given by

$$h(x, y) = \langle \Pi x, y \rangle - \overline{\Pi} \langle x, y \rangle. \tag{4-2}$$

Let $\boldsymbol{F}$ and $\boldsymbol{V}$ be the σ-linear Frobenius and the σ^{-1}-linear Verschiebung on N_X. We have $\boldsymbol{F}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{F} = \pi_0$ and, since $\langle\ ,\ \rangle$ comes from a polarization,

$$\langle \boldsymbol{F}x, y \rangle = \langle x, \boldsymbol{V}y \rangle^{\sigma}.$$

Consider the σ-linear operator $\tau = \Pi \boldsymbol{V}^{-1} = \boldsymbol{F}\overline{\Pi}^{-1}$. The hermitian form h is invariant under τ:

$$h(\tau x, \tau y) = h(\boldsymbol{F}\overline{\Pi}^{-1} x, \Pi \boldsymbol{V}^{-1} y) = h(\boldsymbol{F}x, \boldsymbol{V}^{-1} y) = h(x, y)^{\sigma}.$$

From the condition on $\mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ it follows that N_X is isotypical of slope $\frac{1}{2}$ and thus the slopes of τ are all zero. Let $C = N_X^{\tau}$. This is a 2-dimensional vector space over E with $N_X = C \otimes_E \breve{E}$ and h induces an $E|F$-hermitian form on C. A priori, there are two possibilities for (C, h), either h is split or nonsplit. The group $\mathrm{U}(C, h)$ of automorphisms is isomorphic to $\mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. But the unitary groups for h split and h nonsplit are not isomorphic and do not contain each other as a closed subgroup. Thus the condition on $\mathrm{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ implies that h is split.

Assume we are given two different objects (X, ι, λ) and (X', ι', λ') as in the proposition. Then there is an isomorphism between the spaces (C, h) and (C', h') extending to an isomorphism of N_X and $N_{X'}$ respecting all structure. This corresponds to a quasi-isogeny $\varphi : (X, \iota, \lambda) \to (X', \iota', \lambda')$.

Now we prove the existence of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. We start with a Π-modular lattice Λ in a 2-dimensional vector space (C, h) over E with split hermitian form. Then $M = \Lambda \otimes_{O_E} \breve{O}_E$ is an $\breve{O}_E$-lattice in $N = C \otimes_E \breve{E}$. The σ-linear operator $\tau = 1 \otimes \sigma$ on N has slopes are all 0. We can extend h to N such that

$$h(\tau x, \tau y) = h(x, y)^{\sigma},$$

for all $x, y \in N$. The operators $\boldsymbol{F}$ and $\boldsymbol{V}$ are given by $\tau = \Pi \boldsymbol{V}^{-1} = \boldsymbol{F}\overline{\Pi}^{-1}$. Finally, the alternating form $\langle\ ,\ \rangle$ is defined via

$$\langle x, y \rangle = \mathrm{Tr}_{\breve{E}|\breve{F}}\Big(\frac{1}{t\vartheta} \cdot h(x, y)\Big),$$

for $x, y \in N$. The lattice $M \subseteq N$ is the Dieudonné module of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. We leave it to the reader to check that this is indeed an object as considered above. $\square$

We fix such an object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ over $\operatorname{Spec} \bar{k}$ from the proposition. We define the functor $\mathcal{N}_E^{\mathrm{naive}}$ on $\mathrm{Nilp}_{\breve{O}_F}$ as in Definition 3.4.

Remark 4.3. $\mathcal{N}_E^{\mathrm{naive}}$ is pro-representable by a formal scheme, formally locally of finite type over $\operatorname{Spf}\breve{O}_F$; cf. [Rapoport and Zink 1996, Theorem 3.25].

We now study the $\bar{k}$-valued points of the space $\mathcal{N}_E^{\mathrm{naive}}$. Let $N = N_{\mathbb{X}}$ be the rational Dieudonné module of $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. This is a 4-dimensional vector space over $\breve{F}$, equipped with an action of E, with two operators $\boldsymbol{F}$ and $\boldsymbol{V}$ and an alternating form $\langle\,,\rangle$.

Let $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$. This corresponds to an $\breve{O}_F$-lattice $M = M_X \subseteq N$ which is stable under the actions of $\boldsymbol{F}$, $\boldsymbol{V}$ and O_E. The condition on the kernel of λ implies that $M = \Pi M^\vee$ for

$$M^\vee = \{x \in N \mid \langle x, y\rangle \in \breve{O}_F \text{ for all } y \in M\}.$$

The alternating form $\langle\,,\rangle$ induces an $\breve{E}|\breve{F}$-hermitian form h on N, seen as a 2-dimensional vector space over $\breve{E}$ (see (4-2)):

$$h(x, y) = \langle \Pi x, y\rangle - \overline{\Pi}\langle x, y\rangle.$$

We can recover the form $\langle\,,\rangle$ from h via

$$\langle x, y\rangle = \operatorname{Tr}_{\breve{E}|\breve{F}}\Big(\frac{1}{t\vartheta}\cdot h(x, y)\Big). \tag{4-3}$$

Since the inverse different of $E|F$ is $\mathfrak{D}_{E|F}^{-1} = \frac{1}{t}O_E$ (see Lemma 2.2), this implies that M is Π-modular with respect to h, as $\breve{O}_E$-lattice in N. We denote the dual of M with respect to h by $M^\sharp$. There is a natural bijection

$$\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) = \{\breve{O}_E\text{-lattices } M \subseteq N \mid M = \Pi M^\sharp,\ \pi_0 M \subseteq \boldsymbol{V}M \subseteq M\}. \tag{4-4}$$

Recall that $\tau = \Pi\boldsymbol{V}^{-1}$ is a σ-linear operator on N with slopes all 0. Further $C = N^\tau$ is a 2-dimensional E-vector space with hermitian form h.

Lemma 4.4. *Let $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$. Then*:

(1) *$M + \tau(M)$ is τ-stable.*

(2) *Either M is τ-stable and $\Lambda_1 = M^\tau \subseteq C$ is Π-modular with respect to h, or M is not τ-stable and then $\Lambda_0 = (M + \tau(M))^\tau \subseteq C$ is unimodular.*

The proof is the same as that of [Kudla and Rapoport 2014, Lemma 3.2]. We identify N with $C \otimes_E \breve{E}$. For any τ-stable lattice $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$, we have $M = \Lambda_1 \otimes_{O_E} \breve{O}_E$. If $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ is not τ-stable, there is an inclusion $M \subseteq \Lambda_0 \otimes_{O_E} \breve{O}_E$ of index 1. Recall from Proposition 2.4 that the isomorphism class of a Π-modular or unimodular lattice $\Lambda \subseteq C$ is determined by the norm ideal

$$\operatorname{Nm}(\Lambda) = \langle\{h(x, x) \mid x \in \Lambda\}\rangle.$$

There are always at least two types of unimodular lattices. However, not all of them appear in the description of $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$.

Lemma 4.5. (1) *Let* $\Lambda \subseteq C$ *be a unimodular lattice with* $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. *There is an injection*

$$i_\Lambda : \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}) \hookrightarrow \mathcal{N}_E^{\mathrm{naive}}(\bar{k}),$$

that maps a line $\ell \subseteq \Lambda/\Pi\Lambda \otimes_k \bar{k}$ *to its inverse image under the canonical projection*

$$\Lambda \otimes_{O_E} \breve{O}_E \to \Lambda/\Pi\Lambda \otimes_k \bar{k}.$$

The k*-valued points* $\mathbb{P}(\Lambda/\Pi\Lambda)(k) \subseteq \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k})$ *are mapped to* τ*-invariant Dieudonné modules* $M \subseteq \Lambda \otimes_{O_E} \breve{O}_E$ *under this embedding.*

(2) *Identify* $\mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k})$ *with its image under* i_Λ. *The set* $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ *can be written as*

$$\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) = \bigcup_{\Lambda \subseteq C} \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}),$$

where the union is taken over all lattices $\Lambda \subseteq C$ *with* $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$.

Proof. Let $\Lambda \subseteq C$ be a unimodular lattice. For any line $\ell \in \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k})$, denote its preimage in $\Lambda \otimes \breve{O}_E$ by M. The inclusion $M \subseteq \Lambda \otimes \breve{O}_E$ has index 1 and M is an $\breve{O}_E$-lattice with $\Pi(\Lambda \otimes \breve{O}_E) \subseteq M$. Furthermore $\Lambda \otimes \breve{O}_E$ is τ-invariant by construction, hence $\Pi(\Lambda \otimes \breve{O}_E) = \boldsymbol{V}(\Lambda \otimes \breve{O}_E) = \boldsymbol{F}(\Lambda \otimes \breve{O}_E)$. It follows that M is stable under the actions of $\boldsymbol{F}$ and $\boldsymbol{V}$. Thus $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ if and only if $M = \Pi M^\sharp$. The hermitian form h induces a symmetric form s on $\Lambda/\Pi\Lambda$. Now M is Π-modular if and only if it is the preimage of an isotropic line $\ell \subseteq \Lambda/\Pi\Lambda \otimes \bar{k}$. Note that s is also antisymmetric since we are in characteristic 2.

We first consider the case $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. We can find a basis of Λ such that h has the form

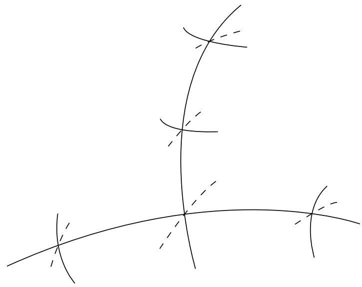
$$H_\Lambda = \begin{pmatrix} x & 1 \\ 1 & \end{pmatrix}, \quad x \in \pi_0 O_F;$$

see (2-4). It follows that the induced form s is even alternating (because $x \equiv 0 \bmod \pi_0$). Hence any line in $\Lambda/\Pi\Lambda \otimes \bar{k}$ is isotropic. This implies that i_Λ is well-defined, proving part (1) of the lemma.

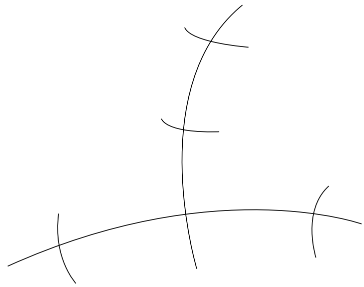
Now assume that $\mathrm{Nm}(\Lambda) = O_F$. There is a basis (e_1, e_2) of Λ such that h is represented by

$$H_\Lambda = \begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}.$$

The induced form s is given by the same matrix and $\ell = \bar{k} \cdot e_2$ is the only isotropic line in $\Lambda/\Pi\Lambda$. Since ℓ is already defined over k, the corresponding lattice $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ is of the form $M = \Lambda_1 \otimes \breve{O}_E$ for a Π-modular lattice $\Lambda_1 \subseteq \Lambda$. But, by Proposition 2.8, any Π-modular lattice in C is contained in a unimodular lattice Λ' with $\mathrm{Nm}(\Lambda') \subseteq \pi_0 O_F$.



(a) $e = 2,\ f = 1,\ v(t) = 2$



(b) $e = 2,\ f = 1,\ v(t) = 1$

Figure 2. The reduced locus of $\mathcal{N}_E^{\mathrm{naive}}$ for an R-U extension $E|F$ where e and f are the ramification index and the inertia degree of $F|\mathbb{Q}_2$ and $v(t)$ is the π_0-adic valuation of t. We always have $1 \le v(t) \le e$. The solid lines lie in $\mathcal{N}_E \subseteq \mathcal{N}_E^{\mathrm{naive}}$.

It follows that we can write $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ as a union

$$\mathcal{N}_E^{\mathrm{naive}}(\bar{k}) = \bigcup_{\Lambda \subseteq C} \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}),$$

where the union is taken over all unimodular lattices $\Lambda \subseteq C$ with $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. This shows the second part of the lemma. $\square$

Remark 4.6. We can use Proposition 2.8 to describe the intersection behavior of the projective lines in $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$. A τ-invariant point $M \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ corresponds to the Π-modular lattice $\Lambda_1 = M^\tau \subseteq C$. If $\mathrm{Nm}(\Lambda_1) \subseteq \pi_0^2 O_F$, there are $q+1$ lines going through M. If $\mathrm{Nm}(\Lambda_1) = \pi_0 O_F$, the point M is contained in one or two lines, depending on whether Λ_1 is hyperbolic or not; see parts (3) and (4) of Proposition 2.8. The former case (i.e., Λ_1 is hyperbolic) appears if and only if $\pi_0 O_F = \mathrm{Nm}(\Lambda_1) = t O_F$ (see Lemma 2.5). This happens only for a specific type of R-U extension $E|F$; see page 348. We refer to Remark 4.8, Remark 4.11 and Section 4D for a further discussion of this special case.

On the other hand, each projective line in $\mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ contains $q+1$ τ-invariant points. Such a τ-invariant point M is an intersection point of two or more projective lines if and only if $|t| = |\pi_0|$ or $\Lambda_1 = M^\tau \subseteq C$ has a norm ideal satisfying $\mathrm{Nm}(\Lambda_1) \subseteq \pi_0^2 O_F$.

Let $\Lambda \subseteq C$ as in Lemma 4.5. We denote by X_Λ^+ the formal O_F-module corresponding to the Dieudonné module $M = \Lambda \otimes \breve{O}_E$. There is a canonical quasi-isogeny

$$\varrho_\Lambda^+ : \mathbb{X} \to X_\Lambda^+$$

of F-height 1. For $S \in \mathrm{Nilp}_{\breve{O}_F}$, we define

$$\mathcal{N}_{E,\Lambda}(S) = \{(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S) \mid (\varrho_\Lambda^+ \times S) \circ \varrho \text{ is an isogeny}\}.$$

By [Rapoport and Zink 1996, Proposition 2.9], the functor $\mathcal{N}_{E,\Lambda}$ is representable by a closed formal subscheme of $\mathcal{N}_E^{\text{naive}}$. On geometric points, we have

$$\mathcal{N}_{E,\Lambda}(\bar{k}) \xrightarrow{\sim} \mathbb{P}(\Lambda/\Pi\Lambda)(\bar{k}), \tag{4-5}$$

as follows from Lemma 4.5 (1).

Proposition 4.7. *The reduced locus of $\mathcal{N}_E^{\text{naive}}$ is a union*

$$(\mathcal{N}_E^{\text{naive}})_{\text{red}} = \bigcup_{\Lambda \subseteq C} \mathcal{N}_{E,\Lambda},$$

where Λ runs over all unimodular lattices in C with $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. For each Λ, there exists an isomorphism

$$\mathcal{N}_{E,\Lambda} \xrightarrow{\sim} \mathbb{P}(\Lambda/\Pi\Lambda),$$

inducing the bijection (4-5) on $\bar{k}$-valued points.

The proof is analogous to that of Proposition 3.9.

Remark 4.8. Similar to Remark 3.8 (3), we let $(\mathcal{N}_E)_{\text{red}} \subseteq (\mathcal{N}_E^{\text{naive}})_{\text{red}}$ be the union of all projective lines $\mathcal{N}_{E,\Lambda}$ corresponding to *hyperbolic* unimodular lattices $\Lambda \subseteq C$. Later, we will define $\mathcal{N}_E$ as a functor on $\mathrm{Nilp}_{\breve{O}_F}$ and show that $\mathcal{N}_E \simeq \mathcal{M}_{Dr}$, where $\mathcal{M}_{Dr}$ is the Drinfeld moduli problem (see Theorem 4.14, a description of the formal scheme $\mathcal{M}_{Dr}$ can be found in [Boutot and Carayol 1991, I.3]). In particular, $(\mathcal{N}_E)_{\text{red}}$ is connected and each projective line in $(\mathcal{N}_E)_{\text{red}}$ has $q+1$ intersection points and there are two lines intersecting in each such point.

It might happen that $(\mathcal{N}_E)_{\text{red}} = (\mathcal{N}_E^{\text{naive}})_{\text{red}}$ (see, for example, Figure 2(b)) if there are no nonhyperbolic unimodular lattices $\Lambda \subseteq C$ with $\mathrm{Nm}(\Lambda) \subseteq \pi_0 O_F$. In fact, this is the case if and only if $|t| = |\pi_0|$; see Proposition 2.4 and Lemma 2.5. (Note however that we still have $\mathcal{N}_E \neq \mathcal{N}_E^{\text{naive}}$; see Remark 4.11 and Section 4D.)

Assume $|t| \neq |\pi_0|$ and let $P \in \mathcal{N}_E(\bar{k})$ be an intersection point. Then, as in the case where $E|F$ is of type R-P (compare Remark 3.8 (3)), the connected component of P in $((\mathcal{N}_E^{\text{naive}})_{\text{red}} \setminus (\mathcal{N}_E)_{\text{red}}) \cup \{P\}$ consists of a finite union of projective lines (corresponding to nonhyperbolic lattices, by definition of $(\mathcal{N}_E)_{\text{red}}$). In Figure 2(a), these components are indicated by dashed lines (they consist of just one projective line in that case).

4B. ***The straightening condition.*** As in the case R-P (see Section 3B) we use the results of Section 5 to define the straightening condition on $\mathcal{N}_E^{\text{naive}}$. By Theorem 5.2 and Remark 5.1 (2), there exists a principal polarization $\tilde{\lambda}_{\mathbb{X}}^0$ on the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ such that the Rosati involution is the identity on O_E. We set $\tilde{\lambda}_{\mathbb{X}} = \tilde{\lambda}_{\mathbb{X}}^0 \circ \iota_{\mathbb{X}}(\Pi)$, which is again a polarization on $\mathbb{X}$ with the Rosati involution inducing the identity on O_E, but with kernel $\ker \tilde{\lambda}_{\mathbb{X}} = \mathbb{X}[\Pi]$. This polarization is unique up to a scalar in $O_E^\times$, i.e., any two polarizations $\tilde{\lambda}_{\mathbb{X}}$ and $\tilde{\lambda}'_{\mathbb{X}}$ with these properties satisfy

$$\tilde{\lambda}'_{\mathbb{X}} = \tilde{\lambda}_{\mathbb{X}} \circ \iota(\alpha),$$

for some $\alpha \in O_E^\times$. For any $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$,

$$\tilde{\lambda} = \varrho^*(\tilde{\lambda}_{\mathbb{X}}) = \varrho^*(\tilde{\lambda}^0_{\mathbb{X}}) \circ \iota(\Pi)$$

is a polarization on X with kernel $\ker \tilde{\lambda} = X[\Pi]$; see Theorem 5.2 (2).

Recall that a unimodular or Π-modular lattice $\Lambda \subseteq C$ is called *hyperbolic* if there exists a basis (e_1, e_2) of Λ such that, with respect to this basis, h has the form

$$\begin{pmatrix} & \Pi^i \\ \overline{\Pi}^i & \end{pmatrix},$$

for $i = 0$ (resp. 1). By Lemma 2.5, this is the case if and only if $\mathrm{Nm}(\Lambda) = t\,O_F$.

Proposition 4.9. *For a suitable choice of* $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ *and* $\tilde{\lambda}_{\mathbb{X}}$, *the quasipolarization*

$$\lambda_{\mathbb{X},1} = \frac{1}{t}(\lambda_{\mathbb{X}} + \tilde{\lambda}_{\mathbb{X}})$$

is a polarization on $\mathbb{X}$. *Let* $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ *and* $\tilde{\lambda} = \varrho^*(\tilde{\lambda}_{\mathbb{X}})$. *Then* $\lambda_1 = \frac{1}{t}(\lambda + \tilde{\lambda})$ *is a polarization if and only if* $(X, \iota, \lambda, \varrho) \in \mathcal{N}_{E,\Lambda}(\bar{k})$ *for a hyperbolic unimodular lattice* $\Lambda \subseteq C$.

Proof. On the rational Dieudonné module $N = M_{\mathbb{X}} \otimes_{\breve{O}_F} \breve{F}$, denote by $\langle\,,\rangle$, $(\,,)$ and $\langle\,,\rangle_1$ the alternating forms induced by $\lambda_{\mathbb{X}}$, $\tilde{\lambda}_{\mathbb{X}}$ and $\lambda_{\mathbb{X},1}$, respectively. The form $\langle\,,\rangle_1$ is integral on $M_{\mathbb{X}}$ if and only if $\lambda_{\mathbb{X},1}$ is a polarization on $\mathbb{X}$. We have

$$\begin{aligned} (\boldsymbol{F}x, y) &= (x, \boldsymbol{V}y)^\sigma, \\ (\Pi x, y) &= (x, \Pi y), \\ \langle x, y\rangle_1 &= \frac{1}{t}(\langle x, y\rangle + (x, y)) \end{aligned}$$

for all $x, y \in N$. The form $(\,,)$ induces an $\breve{E}$-bilinear alternating form b on N by the formula

$$b(x, y) = c((\Pi x, y) - \overline{\Pi}(x, y)). \tag{4-6}$$

Here, c is a unit in $\breve{O}_E$ such that $c \cdot \sigma(c)^{-1} = \overline{\Pi}\Pi^{-1}$. Since

$$\frac{\overline{\Pi}}{\Pi} = \frac{t - \Pi}{\Pi} \in 1 + \frac{t}{\Pi}\breve{O}_E,$$

we can even choose $c \in 1 + t\Pi^{-1}\breve{O}_E$. The dual of M with respect to this form is again $M^\sharp = \Pi^{-1}M$, since

$$(x, y) = \mathrm{Tr}_{\breve{E}|\breve{F}}\Big(\frac{1}{t\vartheta c} \cdot b(x, y)\Big),$$

and the inverse different of $E|F$ is given by $\mathfrak{D}_{E|F}^{-1} = t^{-1}O_E$; see Lemma 2.2. Now b is invariant under the σ-linear operator $\tau = \Pi \boldsymbol{V}^{-1} = \boldsymbol{F}\overline{\Pi}^{-1}$, because

$$b(\tau x, \tau y) = b(\boldsymbol{F}\overline{\Pi}^{-1}x, \Pi \boldsymbol{V}^{-1}y) = \frac{c}{\sigma(c)} \cdot b(\overline{\Pi}^{-1}x, \Pi y)^{\sigma} = b(x, y)^{\sigma}.$$

Hence b defines an E-linear alternating form on C.

We choose the framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ such that $M_{\mathbb{X}}$ is τ-invariant (see Lemma 4.4) and such that $\Lambda_1 = M_{\mathbb{X}}^{\tau}$ is hyperbolic. We can find a basis (e_1, e_2) of Λ_1 such that

$$h \mathrel{\widehat{=}} \begin{pmatrix} & \Pi \\ \overline{\Pi} & \end{pmatrix}, \quad b \mathrel{\widehat{=}} \begin{pmatrix} & u \\ -u & \end{pmatrix}$$

for some $u \in E^{\times}$. Since $\widetilde{\lambda}_{\mathbb{X}}$ has the same kernel as $\lambda_{\mathbb{X}}$, we have $u = \overline{\Pi}u'$ for some unit $u' \in O_E^{\times}$. We can choose $\widetilde{\lambda}_{\mathbb{X}}$ such that $u' = 1$ and $u = \overline{\Pi}$. Now $\frac{1}{t}(h(x, y) + b(x, y))$ is integral for all $x, y \in \Lambda_1$. Hence $\frac{1}{t}(h(x, y) + b(x, y))$ is also integral for all $x, y \in M_{\mathbb{X}}$. For all $x, y \in M_{\mathbb{X}}$, we have

$$\begin{aligned} \langle x, y\rangle_1 &= \frac{1}{t}(\langle x, y\rangle + (x, y)) = \frac{1}{t}\operatorname{Tr}_{\breve{E}|\breve{F}}\left(\frac{1}{t\vartheta} \cdot h(x, y) + \frac{1}{t\vartheta c} \cdot b(x, y)\right) \\ &= \operatorname{Tr}_{\breve{E}|\breve{F}}\left(\frac{1}{t^2\vartheta} \cdot (h(x, y) + b(x, y))\right) + \operatorname{Tr}_{\breve{E}|\breve{F}}\left(\frac{1-c}{t^2\vartheta c} \cdot b(x, y)\right). \end{aligned}$$

The first summand is integral since $\frac{1}{t}(h(x, y) + b(x, y))$ is integral. The second summand is integral since $1 - c$ is divisible by $t\Pi^{-1}$ and $b(x, y)$ lies in $\Pi \breve{O}_E$. It follows that the second summand above is integral as well. Hence $\langle\, ,\rangle_1$ is integral on $M_{\mathbb{X}}$ and this implies that $\lambda_{\mathbb{X},1}$ is a polarization on $\mathbb{X}$.

Now let $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ and denote by $M \subseteq N$ its Dieudonné module. Assume that $\lambda_1 = t^{-1}(\lambda + \widetilde{\lambda})$ is a polarization on X. Then $\langle\, ,\rangle_1$ is integral on M. But this is equivalent to $t^{-1}(h(x, y) + b(x, y))$ being integral for all $x, y \in M$. For $x = y$, we have

$$h(x, x) = h(x, x) + b(x, x) \in t\breve{O}_F.$$

Let $\Lambda \subseteq C$ be the unimodular or Π-modular lattice given by $\Lambda = M^{\tau}$, resp. $\Lambda = (M + \tau(M))^{\tau}$; see Lemma 4.4. Then $h(x, x) \in tO_F$ for all $x \in \Lambda$. Thus $\operatorname{Nm}(\Lambda) \subseteq tO_F$ and, by minimality, this implies that $\operatorname{Nm}(\Lambda) = tO_F$ and Λ is hyperbolic (see Lemma 2.5). Hence, in either case, the point corresponding to $(X, \iota, \lambda, \varrho)$ lies in $\mathcal{N}_{E,\Lambda'}$ for a hyperbolic lattice Λ'.

Conversely, assume that $(X, \iota, \lambda, \varrho) \in \mathcal{N}_{E,\Lambda}(\bar{k})$ for some hyperbolic lattice $\Lambda \subseteq C$. We want to show that λ_1 is a polarization on X. This follows if $\langle\, ,\rangle_1$ is integral on M, or equivalently, if $t^{-1}(h(x, y) + b(x, y))$ is integral on M. For this, it is enough to show that $t^{-1}(h(x, y) + b(x, y))$ is integral on Λ. Let $\Lambda' \subseteq C$ be the unimodular lattice generated by $\overline{\Pi}^{-1}e_1$ and e_2, where (e_1, e_2) is the basis of the Π-modular lattice $\Lambda_1 = M_{\mathbb{X}}$. With respect to the basis $(\overline{\Pi}^{-1}e_1, e_2)$, we have

$$h \mathrel{\widehat{=}} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad b \mathrel{\widehat{=}} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

In particular, Λ' is a hyperbolic lattice and $t^{-1}(h+b)$ is integral on Λ'. By Proposition 2.4, there exists an element $g \in \mathrm{SU}(C,h)$ with $g\Lambda = \Lambda'$. Since $\det g = 1$, the alternating form b is invariant under g. Thus $t^{-1}(h+b)$ is also integral on Λ. □

From now on, we assume that $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ and $\tilde{\lambda}_{\mathbb{X}}$ are chosen in a way such that

$$\lambda_{\mathbb{X},1} = \frac{1}{t}(\lambda_{\mathbb{X}} + \tilde{\lambda}_{\mathbb{X}}) \in \mathrm{Hom}(\mathbb{X}, \mathbb{X}^\vee).$$

Definition 4.10. A tuple $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$ satisfies the *straightening condition* if

$$\lambda_1 = \frac{1}{t}(\lambda + \tilde{\lambda}) \in \mathrm{Hom}(X, X^\vee). \tag{4-7}$$

This condition is independent of the choice of $\tilde{\lambda}_{\mathbb{X}}$. In fact, we can only change $\tilde{\lambda}_{\mathbb{X}}$ by a scalar of the form $1 + t\Pi^{-1}u$, $u \in O_E$. But if $\tilde{\lambda}'_{\mathbb{X}} = \tilde{\lambda}_{\mathbb{X}} \circ \iota(1 + t\Pi^{-1}u)$, then $\lambda'_{\mathbb{X},1} = \lambda_{\mathbb{X},1} + \tilde{\lambda}_{\mathbb{X}} \circ \iota(\Pi^{-1}u) = \lambda_{\mathbb{X},1} + \tilde{\lambda}^0_{\mathbb{X}} \circ \iota(u)$ and $\lambda'_1 = \lambda_1 + \varrho^*(\tilde{\lambda}^0_{\mathbb{X}}) \circ \iota(u)$. Clearly, λ'_1 is a polarization if and only if λ_1 is one.

For $S \in \mathrm{Nilp}_{\breve{O}_F}$, let $\mathcal{N}_E(S)$ be the set of all tuples $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$ that satisfy the straightening condition. By [Rapoport and Zink 1996, Proposition 2.9], the functor $\mathcal{N}_E$ is representable by a closed formal subscheme of $\mathcal{N}_E^{\mathrm{naive}}$.

Remark 4.11. The reduced locus of $\mathcal{N}_E$ is given by

$$(\mathcal{N}_E)_{\mathrm{red}} = \bigcup_{\Lambda \subseteq C} \mathcal{N}_{E,\Lambda} \simeq \bigcup_{\Lambda \subseteq C} \mathbb{P}(\Lambda/\Pi\Lambda),$$

where the union goes over all *hyperbolic* unimodular lattices $\Lambda \subseteq C$. Note that, depending on the form of the R-U extension $E|F$, it may happen that all unimodular lattices are hyperbolic (when $|t| = |\pi_0|$) and in that case, we have $(\mathcal{N}_E)_{\mathrm{red}} = (\mathcal{N}_E^{\mathrm{naive}})_{\mathrm{red}}$. However, the equality does not extend to an isomorphism between $\mathcal{N}_E$ and $\mathcal{N}_E^{\mathrm{naive}}$. This will be discussed in Section 4D.

4C. ***The main theorem for the case R-U.*** As in the case R-P, we want to establish a connection to the Drinfeld moduli problem. Therefore, fix an embedding of E into the quaternion division algebra B. Let $(\mathbb{X}, \iota_{\mathbb{X}})$ be the framing object of the Drinfeld problem. We want to construct a polarization $\lambda_{\mathbb{X}}$ on $\mathbb{X}$ with $\ker \lambda_{\mathbb{X}} = \mathbb{X}[\Pi]$ and Rosati involution given by $b \mapsto \vartheta b' \vartheta^{-1}$ on B. Here $b \mapsto b'$ denotes the standard involution on B.

By Lemma 2.3 (2), there exists an embedding $E_1 \hookrightarrow B$ of a ramified quadratic extension $E_1|F$ of type R-P, such that $\Pi_1\vartheta = -\vartheta\Pi_1$ for a prime element $\Pi_1 \in E_1$. From Proposition 3.14 (1) we get a principal polarization $\lambda_{\mathbb{X}}^0$ on $\mathbb{X}$ with associated Rosati involution $b \mapsto \Pi_1 b' \Pi_1^{-1}$. If we assume fixed choices of E_1 and Π_1, this is unique up to a scalar in $O_F^\times$. We define

$$\lambda_{\mathbb{X}} = \lambda_{\mathbb{X}}^0 \circ \iota_{\mathbb{X}}(\Pi_1\vartheta).$$

Since $\lambda_{\mathbb{X}}^0$ is a principal polarization and $\Pi_1\vartheta$ and Π have the same valuation in O_B, we have $\ker \lambda_{\mathbb{X}} = \mathbb{X}[\Pi]$. The Rosati involution of $\lambda_{\mathbb{X}}$ is $b \mapsto \vartheta b' \vartheta^{-1}$. On the other hand, any polarization on $\mathbb{X}$ satisfying these two conditions can be constructed in this way (using the same choices for E_1 and Π_1). Hence:

Lemma 4.12. (1) *There exists a polarization* $\lambda_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}^\vee$, *unique up to a scalar in* $O_F^\times$, *with* $\ker \lambda_{\mathbb{X}} = \mathbb{X}[\Pi]$ *and associated Rosati involution* $b \mapsto \vartheta b' \vartheta^{-1}$.

(2) *Fix* $\lambda_{\mathbb{X}}$ *as in* (1) *and let* $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$. *There exists a unique polarization* λ *on* X *with* $\ker \lambda = X[\Pi]$ *and Rosati involution* $b \mapsto \vartheta b' \vartheta^{-1}$ *such that* $\varrho^*(\lambda_{\mathbb{X}}) = \lambda$ *on* $\bar{S} = S \times_{\operatorname{Spf} \breve{O}_F} \bar{k}$.

Note also that the involution $b \mapsto \vartheta b' \vartheta^{-1}$ does not depend on the choice of $\vartheta \in E$. We write $\iota_{\mathbb{X},E}$ for the restriction of $\iota_{\mathbb{X}}$ to $E \subseteq B$ and, in the same manner, we write ι_E for the restriction of ι_B to E for any $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$. Fix a polarization $\lambda_{\mathbb{X}}$ of $\mathbb{X}$ as in Lemma 4.12 (1). Accordingly for a tuple $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$, let λ be the polarization given by Lemma 4.12 (2).

Lemma 4.13. *The tuple* $(\mathbb{X}, \iota_{\mathbb{X},E}, \lambda_{\mathbb{X}})$ *is a framing object of* $\mathcal{N}_E^{\text{naive}}$. *Moreover, the map*

$$(X, \iota_B, \varrho) \mapsto (X, \iota_E, \lambda, \varrho)$$

induces a closed embedding of formal schemes

$$\eta : \mathcal{M}_{Dr} \hookrightarrow \mathcal{N}_E^{\text{naive}}.$$

Proof. We follow the same argument as in the proof of Lemma 3.15. Again it is enough to check that $\operatorname{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ contains $\operatorname{SU}(C, h)$ as a closed subgroup and that ι_E satisfies the Kottwitz condition.

By [Rapoport and Zink 2017, Proposition 5.8], the special condition on ι_B implies the Kottwitz condition for ι_E. It remains to show that $\operatorname{SU}(C, h) \subseteq \operatorname{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$. But the group $G_{(\mathbb{X},\iota_{\mathbb{X}})}$ of automorphisms of determinant 1 of $(\mathbb{X}, \iota_{\mathbb{X}})$ is isomorphic to $\operatorname{SL}_{2,F}$ and $G_{(\mathbb{X},\iota_{\mathbb{X}})} \subseteq \operatorname{QIsog}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ is a Zariski-closed subgroup by the same argument as in Lemma 3.15. Hence the statement follows from the exceptional isomorphism $\operatorname{SL}_{2,F} \simeq \operatorname{SU}(C, h)$. □

As a next step, we want to show that this already induces a closed embedding

$$\eta : \mathcal{M}_{Dr} \hookrightarrow \mathcal{N}_E. \tag{4-8}$$

Let $\widetilde{E} \hookrightarrow B$ an embedding of a ramified quadratic extension $\widetilde{E}|F$ of type R-U as in Lemma 2.3 (2). On the framing object $(\mathbb{X}, \iota_{\mathbb{X}})$ of $\mathcal{M}_{Dr}$, we define a polarization $\widetilde{\lambda}_{\mathbb{X}}$ via

$$\widetilde{\lambda}_{\mathbb{X}} = \lambda_{\mathbb{X}} \circ \iota_{\mathbb{X}}(\widetilde{\vartheta}),$$

where $\widetilde{\vartheta}$ is a unit in $\widetilde{E}$ of the form $\widetilde{\vartheta}^2 = 1 + (t^2/\pi_0) \cdot u$; see Lemma 2.3 (2). The Rosati involution of $\widetilde{\lambda}_{\mathbb{X}}$ induces the identity on O_E and we have

$$\begin{aligned}\lambda_{\mathbb{X},1} = \frac{1}{t}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}}) = \frac{1}{t} \cdot \lambda_{\mathbb{X}} \circ \iota_B(1 + \widetilde{\vartheta}) = \lambda_{\mathbb{X}} \circ \iota_B(\widetilde{\Pi}/\pi_0)\\ = \lambda_{\mathbb{X}} \circ \iota_B(\Pi^{-1}\gamma) \in \operatorname{Hom}(\mathbb{X}, \mathbb{X}^\vee),\end{aligned}$$

using the notation of Lemma 2.3 (2). For $(X, \iota_B, \varrho) \in \mathcal{M}_{Dr}(S)$, we set $\widetilde{\lambda} = \lambda \circ \iota_B(\widetilde{\vartheta})$. By the same calculation, we have $\lambda_1 = \frac{1}{t}(\lambda + \widetilde{\lambda}) \in \operatorname{Hom}(X, X^\vee)$. Thus the tuple $(X, \iota_E, \lambda, \varrho) = \eta(X, \iota_B, \varrho)$ satisfies the straightening condition. Hence we get a closed embedding of formal schemes $\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E$ which is independent of the choice of $\widetilde{E}$.

Theorem 4.14. *$\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E$ is an isomorphism of formal schemes.*

We first check this for $\bar{k}$-valued points:

Lemma 4.15. *η induces a bijection $\eta(\bar{k}) : \mathcal{M}_{Dr}(\bar{k}) \to \mathcal{N}_E(\bar{k})$.*

Proof. We only have to show surjectivity and we will use for this the Dieudonné theory description of $\mathcal{N}_E^{\text{naive}}(\bar{k})$; see (4-4). The rational Dieudonné module $N = N_{\mathbb{X}}$ of $\mathbb{X}$ now carries additionally an action of B. The embedding $F^{(2)} \hookrightarrow B$ given by

$$\gamma \mapsto \frac{\Pi \cdot \widetilde{\Pi}}{\pi_0}, \tag{4-9}$$

(see Lemma 2.3 (2)) induces a $\mathbb{Z}/2$-grading $N = N_0 \oplus N_1$. Here,

$$\begin{aligned}N_0 &= \{x \in N \mid \iota(a)x = ax \text{ for all } a \in F^{(2)}\},\\ N_1 &= \{x \in N \mid \iota(a)x = \sigma(a)x \text{ for all } a \in F^{(2)}\}\end{aligned}$$

for a fixed embedding $F^{(2)} \hookrightarrow \breve{F}$. The operators $\boldsymbol{F}$ and $\boldsymbol{V}$ have degree 1 with respect to this grading. The principal polarization

$$\lambda_{\mathbb{X},1} = \frac{1}{t}(\lambda_{\mathbb{X}} + \widetilde{\lambda}_{\mathbb{X}}) = \lambda_{\mathbb{X}} \circ \iota_{\mathbb{X}}(\Pi^{-1}\gamma)$$

induces an alternating form $\langle\, ,\rangle_1$ on N that satisfies

$$\langle x, y\rangle_1 = \langle x, \iota(\Pi^{-1}\gamma) \cdot y\rangle,$$

for all $x, y \in N$. Let $M \in \mathcal{N}_E(\bar{k}) \subseteq \mathcal{N}_E^{\text{naive}}(\bar{k})$ be an $\breve{O}_F$-lattice in N. We claim that $M \in \mathcal{M}_{Dr}(\bar{k})$. For this, it is necessary that M is stable under the action of $O_F^{(2)}$ (since $O_B = O_F[\Pi, \gamma] = O_F^{(2)}[\Pi]$; see Lemma 2.3 (2)) or equivalently, that M respects the grading of N, i.e., $M = M_0 \oplus M_1$ for $M_i = M \cap N_i$. Furthermore M has to satisfy the *special* condition:

$$\dim M_0/\boldsymbol{V} M_1 = \dim M_1/\boldsymbol{V} M_0 = 1.$$

We first show that $M = M_0 \oplus M_1$. Let $y = y_0 + y_1 \in M$ with $y_i \in N_i$. Since $M = \Pi M^\vee$, we have

$$\langle x, \iota(\Pi)^{-1} y\rangle = \langle x, \iota(\Pi)^{-1} y_0\rangle + \langle x, \iota(\Pi)^{-1} y_1\rangle \in \breve{O}_F,$$

for all $x \in M$. Together with

$$\begin{aligned}\langle x, y\rangle_1 = \langle x, y_0\rangle_1 + \langle x, y_1\rangle_1 &= \langle x, \iota(\widetilde{\Pi}/\pi_0) y_0\rangle + \langle x, \iota(\widetilde{\Pi}/\pi_0) y_1\rangle \\ &= \gamma \cdot \langle x, \iota(\Pi^{-1}) y_0\rangle + (1-\gamma) \cdot \langle x, \iota(\Pi^{-1}) y_1\rangle \in \breve{O}_F,\end{aligned}$$

this implies that $\langle x, \iota(\Pi^{-1}) y_0\rangle$ and $\langle x, \iota(\Pi^{-1}) y_1\rangle$ lie in $\breve{O}_F$ for all $x \in M$. Hence, $y_0, y_1 \in M$ and this means that M respects the grading. It follows that M is stable under the action of O_B.

In order to show that M is special, note that

$$\langle \boldsymbol{V} x, \boldsymbol{V} y\rangle_1^\sigma = \langle \boldsymbol{F}\boldsymbol{V} x, y\rangle_1 = \pi_0 \cdot \langle x, y\rangle_1 \in \pi_0 \breve{O}_F,$$

for all $x, y \in M$. The form $\langle\, ,\rangle_1$ comes from a principal polarization, so it induces a perfect form on M. Now it is enough to show that also the restrictions of $\langle\, ,\rangle_1$ to M_0 and M_1 are perfect. Indeed, if M was not special, we would have $M_i = \boldsymbol{V} M_{i+1}$ for some i and this would contradict $\langle\, ,\rangle_1$ being perfect on M_i. We prove that $\langle\, ,\rangle_1$ is perfect on M_i by showing $\langle M_0, M_1\rangle_1 \subseteq \pi_0 \breve{O}_F$.

Let $x \in M_0$ and $y \in M_1$. Then,

$$\begin{aligned}&\langle x, y\rangle_1 = (1-\gamma) \cdot \langle x, \iota(\Pi)^{-1} y\rangle, \\ &\langle x, y\rangle_1 = -\langle y, x\rangle_1 = -\gamma \cdot \langle y, \iota(\Pi)^{-1} x\rangle = \gamma \cdot \langle x, \iota(\overline{\Pi})^{-1} y\rangle.\end{aligned}$$

We take the difference of these two equations. From $\Pi \equiv \overline{\Pi} \bmod \pi_0$, it follows that $\langle x, \iota(\Pi)^{-1} y\rangle \equiv 0 \bmod \pi_0$ and thus also $\langle x, y\rangle_1 \equiv 0 \bmod \pi_0$. The form $\langle\, ,\rangle_1$ is hence perfect on M_0 and M_1 and the special condition follows. This finishes the proof of Lemma 4.15. ☐

Proof of Theorem 4.14. Let $(\mathbb{X}, \iota_{\mathbb{X}})$ be a framing object for $\mathcal{M}_{Dr}$ and let further

$$\eta(\mathbb{X}, \iota_{\mathbb{X}}) = (\mathbb{X}, \iota_{\mathbb{X},E}, \lambda_{\mathbb{X}})$$

be the corresponding framing object for $\mathcal{N}_E$. We fix an embedding $F^{(2)} \hookrightarrow B$ as in Lemma 2.3 (2). For $S \in \mathrm{Nilp}_{\breve{O}_F}$, let $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(S)$ and $\tilde{\lambda} = \varrho^*(\tilde{\lambda}_{\mathbb{X}})$. We have

$$\begin{aligned} \varrho^{-1} \circ \iota_{\mathbb{X}}(\gamma) \circ \varrho &= \varrho^{-1} \circ \iota_{\mathbb{X}}(\Pi) \circ \lambda_{\mathbb{X}}^{-1} \circ \lambda_{\mathbb{X},1} \circ \varrho \\ &= \iota(\Pi) \circ \lambda^{-1} \circ \lambda_1 \in \mathrm{End}(X) \end{aligned}$$

for $\lambda_1 = t^{-1}(\lambda + \tilde{\lambda})$, since $\ker \lambda = X[\Pi]$. But $O_B = O_F[\Pi, \gamma]$ (see Lemma 2.3 (2)), so this already induces an O_B-action ι_B on X. It remains to show that (X, ι_B, ϱ) satisfies the *special* condition (see the discussion before Proposition 3.14 for a definition).

The special condition is open and closed (see [Rapoport and Zink 2017, p. 7]) and η is bijective on $\bar{k}$-points. Hence η induces an isomorphism on reduced subschemes

$$(\eta)_{\mathrm{red}} : (\mathcal{M}_{Dr})_{\mathrm{red}} \xrightarrow{\sim} (\mathcal{N}_E)_{\mathrm{red}},$$

because $(\mathcal{M}_{Dr})_{\mathrm{red}}$ and $(\mathcal{N}_E)_{\mathrm{red}}$ are locally of finite type over $\mathrm{Spec}\,\bar{k}$. It follows that $\eta : \mathcal{M}_{Dr} \to \mathcal{N}_E$ is an isomorphism. □

4D. ***Deformation theory of intersection points.*** In this section, we will study the deformation rings of certain geometric points in $\mathcal{N}_E^{\mathrm{naive}}$ with the goal of proving that $\mathcal{N}_E \subseteq \mathcal{N}_E^{\mathrm{naive}}$ is a strict inclusion even in the case $|t| = |\pi_0|$. In contrast to the non-2-adic case, we are not able to use the theory of local models (see [Pappas et al. 2013] for a survey) since there is in general no normal form for the lattices $\Lambda \subseteq C$; see Proposition 2.4 and [Rapoport and Zink 1996, Theorem 3.16].[3] Thus we will take the more direct approach of studying the deformations of a fixed point $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ and using the theory of Grothendieck and Messing [Messing 1972].

Let $\Lambda \subseteq C$ be a Π-modular hyperbolic lattice. By Lemma 4.5, there is a unique point $x = (X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(\bar{k})$ with a τ-stable Dieudonné module $M \subseteq C \otimes_E \breve{E}$ and $M^\tau = \Lambda$. Since Λ is hyperbolic, x satisfies the straightening condition, i.e., $x \in \mathcal{N}_E(\bar{k})$. (In Figure 2, x would lie on the intersection of two solid lines.)

Let $\widehat{\mathcal{O}}_{\mathcal{N}_E^{\mathrm{naive}},x}$ be the formal completion of the local ring at x. It represents the following deformation functor Def_x. For an artinian $\breve{O}_F$-algebra R with residue field $\bar{k}$, we have

$$\mathrm{Def}_x(R) = \{(Y, \iota_Y, \lambda_Y)/R \mid Y_{\bar{k}} \cong X\},$$

where (Y, ι_Y, λ_Y) satisfies the usual conditions (see Section 4A) and the isomorphism $Y_{\bar{k}} \cong X$ is actually an isomorphism of tuples $(Y_{\bar{k}}, \iota_Y, \lambda_Y) \cong (X, \iota, \lambda)$ as in Definition 3.1.

[3] It is possible to define a local model for the nonnaive spaces $\mathcal{N}_E$ (also in the case R-P) and establish a local model diagram as in [Rapoport and Zink 1996, Definition 3.27]. The local model is then isomorphic to the local model of the Drinfeld moduli problem. This will be part of a future paper of the author.

Now assume the quotient map $R \to \bar{k}$ is an O_F-pd-thickening (see [Ahsendorf 2011]). For example, this is the case when $\mathfrak{m}^2 = 0$ for the maximal ideal $\mathfrak{m}$ of R. Then, by Grothendieck–Messing theory (see [Messing 1972] and [Ahsendorf 2011]), we get an explicit description of $\mathrm{Def}_x(R)$ in terms of liftings of the Hodge filtration:

The (relative) Dieudonné crystal $\mathbb{D}_X(R)$ of X evaluated at R is naturally isomorphic to the free R-module $\Lambda \otimes_{O_F} R$ and this isomorphism is equivariant under the action of O_E induced by ι and respects the perfect form $\Phi = \langle\, ,\rangle \circ (1, \Pi^{-1})$ induced by $\lambda \circ \iota(\Pi^{-1})$. The Hodge filtration of X is given by $\mathcal{F}_X = V \cdot \mathbb{D}_X(\bar{k}) \cong \Pi \cdot (\Lambda \otimes_{O_F} \bar{k}) \subseteq \Lambda \otimes_{O_F} \bar{k}$.

A point $Y \in \mathrm{Def}_x(R)$ now corresponds, via Grothendieck–Messing, to a direct summand $\mathcal{F}_Y \subseteq \Lambda \otimes_{O_F} R$ of rank 2 lifting $\mathcal{F}_X$, stable under the O_E-action and totally isotropic with respect to Φ. Furthermore, it has to satisfy the Kottwitz condition (see Section 4A): For the action of $\alpha \in O_E$ on $\mathrm{Lie}\, Y = (\Lambda \otimes_{O_F} R)/\mathcal{F}_Y$, we have

$$\mathrm{char}(\mathrm{Lie}\, Y, T \mid \iota(\alpha)) = (T - \alpha)(T - \bar{\alpha}).$$

Let us now fix an O_E-basis (e_1, e_2) of Λ and let us write everything in terms of the O_F-basis $(e_1, e_2, \Pi e_1, \Pi e_2)$. Since Λ is hyperbolic, we can fix (e_1, e_2) such that h is represented by the matrix

$$h \mathrel{\widehat{=}} \begin{pmatrix} & \Pi \\ \overline{\Pi} & \end{pmatrix},$$

and then

$$\Phi = \mathrm{Tr}_{E|F} \frac{1}{t\vartheta} h(\,\cdot\,, \Pi^{-1} \cdot\,) \mathrel{\widehat{=}} \left(\begin{array}{cc|cc} & t/\pi_0 & & 1 \\ & & -1 & \\ \hline & -1 + t^2/\pi_0 & & t \\ 1 & & & \end{array}\right).$$

An R-basis (v_1, v_2) of $\mathcal{F}_Y$ can now be chosen such that

$$(v_1 v_2) = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ 1 & \\ & 1 \end{pmatrix},$$

with $y_{ij} \in R$. As an easy calculation shows, the conditions on $\mathcal{F}_Y$ above are now equivalent to the following conditions on the y_{ij}:

$$\begin{gathered} y_{11} + y_{22} = t, \\ y_{11} y_{22} - y_{12} y_{21} = \pi_0, \\ t\left(\frac{t y_{22}}{\pi_0} + 2\right) = y_{11}\left(\frac{t y_{22}}{\pi_0} + 2\right) = y_{21}\left(\frac{t y_{22}}{\pi_0} + 2\right) = y_{12}\left(\frac{t y_{22}}{\pi_0} + 2\right) = 0. \end{gathered}$$

Let T be the closed subscheme of $\operatorname{Spec} O_F[y_{11}, y_{12}, y_{21}, y_{22}]$ given by these equations. Let T_y be the formal completion of the localization at the ideal generated by the y_{ij} and π_0. Then we have $\mathrm{Def}_x(R) \cong T_y(R)$ for any O_F-pd-thickening $R \to \bar{k}$. In particular, the first infinitesimal neighborhoods of Def_x and T_y coincide. The first infinitesimal neighborhood of T_y is given by $\operatorname{Spec} O_F[y_{ij}]/((y_{ij})^2, y_{11}+y_{22}-t, \pi_0)$, hence T_y has Krull dimension 3 and so has Def_x. However, $\mathcal{M}_{Dr}$ is regular of dimension 2; cf. [Boutot and Carayol 1991]. Thus:

Proposition 4.16. *$\mathcal{N}_E^{\mathrm{naive}} \neq \mathcal{M}_{Dr}$, even when $|t| = |\pi_0|$.*

Indeed, $\dim \widehat{\mathcal{O}}_{\mathcal{N}_E^{\mathrm{naive}},x} = \dim \mathrm{Def}_x = 3 > 2 = \dim \widehat{\mathcal{O}}_{\mathcal{N}_E,x}$.

5. A theorem on the existence of polarizations

In this section, we will prove the existence of the polarization $\tilde{\lambda}$ for any $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E^{\mathrm{naive}}(S)$ as claimed in the Sections 3B and 4B in both the cases R-P and R-U. In fact, we will show more generally that $\tilde{\lambda}$ exists even for the points of a larger moduli space $\mathcal{M}_E$ where we forget about the polarization λ.

We start with the definition of the moduli space $\mathcal{M}_E$. Let $F|\mathbb{Q}_p$ be a finite extension (not necessarily $p = 2$) and let $E|F$ be a quadratic extension (not necessarily ramified). We denote by O_F and O_E the rings of integers, by k the residue field of O_F and by $\bar{k}$ the algebraic closure of k. Furthermore, $\breve{F}$ is the completion of the maximal unramified extension of F and $\breve{O}_F$ its ring of integers. Let B be the quaternion division algebra over F and O_B the ring of integers.

If $E|F$ is unramified, we fix a common uniformizer $\pi_0 \in O_F \subseteq O_E$. If $E|F$ is ramified and $p > 2$, we choose a uniformizer $\Pi \in O_E$ such that $\pi_0 = \Pi^2 \in O_F$. If $E|F$ is ramified and $p = 2$, we use the notation of Section 2 for the cases R-P and R-U.

For $S \in \mathrm{Nilp}_{\breve{O}_F}$, let $\mathcal{M}_E(S)$ be the set of isomorphism classes of tuples (X, ι_E, ϱ) over S. Here, X is a formal O_F-module of dimension 2 and height 4 and ι_E is an action of O_E on X satisfying the Kottwitz condition for the signature $(1, 1)$, i.e., the characteristic polynomial for the action of $\iota_E(\alpha)$ on $\mathrm{Lie}(X)$ is

$$\operatorname{char}(\mathrm{Lie}\, X, T \mid \iota(\alpha)) = (T - \alpha)(T - \bar{\alpha}), \tag{5-1}$$

for any $\alpha \in O_E$, compare the definition of $\mathcal{N}_E^{\mathrm{naive}}$ in Sections 3 and 4. The last entry ϱ is an O_E-linear quasi-isogeny

$$\varrho : X \times_S \bar{S} \to \mathbb{X} \times_{\operatorname{Spec} \bar{k}} \bar{S},$$

of height 0 to the framing object $(\mathbb{X}, \iota_{\mathbb{X},E})$ defined over $\operatorname{Spec} \bar{k}$. The framing object for $\mathcal{M}_E$ is the Drinfeld framing object $(\mathbb{X}, \iota_{\mathbb{X},B})$ where we restrict the O_B-action to O_E for an arbitrary embedding $O_E \hookrightarrow O_B$. The special condition on $(\mathbb{X}, \iota_{\mathbb{X},B})$ implies the Kottwitz condition for any $\alpha \in O_E$ by [Rapoport and Zink 2017, Proposition 5.8].

Remark 5.1. (1) Up to isogeny, there is more than one pair (X, ι_E) over $\operatorname{Spec} \bar{k}$ satisfying the conditions above. Indeed, let N_X be the rational Dieudonné module of (X, ι_E). This is a 4-dimensional $\breve{F}$-vector space with an action of O_E. The Frobenius $\boldsymbol{F}$ on N_X commutes with the action of O_E. For a suitable choice of a basis of N_X, it may be of either of the following two forms,

$$\boldsymbol{F} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ \pi_0 & & & \\ & \pi_0 & & \end{pmatrix} \sigma \quad \text{or} \quad \boldsymbol{F} = \begin{pmatrix} \pi_0 & & & \\ & \pi_0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \sigma.$$

This follows from the classification of isocrystals; see, for example, [Rapoport and Zink 1996, p. 3]. In the left case, $\boldsymbol{F}$ is isoclinic of slope $1/2$ (the supersingular case), and in the right case, the slopes are 0 and 1. Our choice of the framing object above assures that we are in the supersingular case, since the framing object for the Drinfeld moduli problem can be written as a product of two formal O_F-modules of dimension 1 and height 2 (cf. [Boutot and Carayol 1991, p. 136–137]).

(2) Let $p = 2$ and $E|F$ ramified of type R-P or R-U. We can identify the framing objects $(\mathbb{X}, \iota_{\mathbb{X},E})$ for $\mathcal{N}_E^{\text{naive}}$, $\mathcal{M}_{Dr}$ and $\mathcal{M}_E$ by Proposition 3.14 and Lemma 4.13. In this way, we obtain a forgetful morphism $\mathcal{N}_E^{\text{naive}} \to \mathcal{M}_E$. This is a closed embedding, since the existence of a polarization λ for $(X, \iota_E, \varrho) \in \mathcal{M}_E(S)$ is a closed condition by [Rapoport and Zink 1996, Proposition 2.9].

By [Rapoport and Zink 1996, Theorem 3.25], $\mathcal{M}_E$ is pro-representable by a formal scheme over $\operatorname{Spf} \breve{O}_F$. We will prove the following theorem in this section.

Theorem 5.2. (1) *There exists a principal polarization $\widetilde{\lambda}_{\mathbb{X}}$ on $(\mathbb{X}, \iota_{\mathbb{X},E})$ such that the Rosati involution induces the identity on O_E, i.e., $\iota(\alpha)^* = \iota(\alpha)$ for all $\alpha \in O_E$. This polarization is unique up to a scalar in $O_E^\times$, that is, for any two polarizations $\widetilde{\lambda}_{\mathbb{X}}$ and $\widetilde{\lambda}'_{\mathbb{X}}$ of this form, there exists an element $\alpha \in O_E^\times$ such that $\widetilde{\lambda}'_{\mathbb{X}} = \widetilde{\lambda}_{\mathbb{X}} \circ \iota_{\mathbb{X},E}(\alpha)$.*

(2) *Fix $\widetilde{\lambda}_{\mathbb{X}}$ as in part* (1)*. For any $S \in \operatorname{Nilp}_{\breve{O}_F}$ and $(X, \iota_E, \varrho) \in \mathcal{M}_E(S)$, there exists a unique principal polarization $\widetilde{\lambda}$ on X such that the Rosati involution induces the identity on O_E and such that $\widetilde{\lambda} = \varrho^*(\widetilde{\lambda}_{\mathbb{X}})$.*

Remark 5.3. (1) We will see later that this theorem describes a natural isomorphism between $\mathcal{M}_E$ and another space $\mathcal{M}_{E,\text{pol}}$ which solves the moduli problem for tuples $(X, \iota_E, \widetilde{\lambda}, \varrho)$ where $\widetilde{\lambda}$ is a principal polarization with Rosati involution the identity on O_E. This is an RZ-space for the symplectic group $\mathrm{GSp}_2(E)$ and thus the theorem gives us another geometric realization of an exceptional isomorphism of reductive groups, in this case $\mathrm{GSp}_2(E) \cong \mathrm{GL}_2(E)$.

Since there is no such isomorphism in higher dimensions, the theorem does not generalize to these cases and a different approach is needed to formulate the straightening condition.

(2) With Theorem 5.2 established, one can give an easier proof of the isomorphism $\mathcal{N}_E \xrightarrow{\sim} \mathcal{M}_{Dr}$ for the cases where $E|F$ is unramified or $E|F$ is ramified and $p > 2$, which is the main theorem of [Kudla and Rapoport 2014]. Indeed, the main part of the proof in that paper consists of Propositions 2.1 and 3.1, which claim the existence of a certain principal polarization λ_X^0 for any point $(X, \iota, \lambda, \varrho) \in \mathcal{N}_E(S)$. But there is a canonical closed embedding $\mathcal{N}_E \hookrightarrow \mathcal{M}_E$ and under this embedding, λ_X^0 is just the polarization $\tilde{\lambda}$ of Theorem 5.2, for a suitable choice of $\tilde{\lambda}_{\mathbb{X}}$ on the framing object. More explicitly, using the notation on page 2 of [loc. cit.], we take $\tilde{\lambda}_{\mathbb{X}} = \lambda_{\mathbb{X}} \circ \iota_{\mathbb{X}}^{-1}(\Pi) = \lambda_{\mathbb{X}}^0 \circ \iota_{\mathbb{X}}(-\delta)$ in the unramified case and $\tilde{\lambda}_{\mathbb{X}} = \lambda_{\mathbb{X}} \circ \iota_{\mathbb{X}}(\zeta^{-1})$ in the ramified case.

We will split the proof of this theorem into several lemmata. As a first step, we use Dieudonné theory to prove the statement for all geometric points.

Lemma 5.4. *Part (1) of Theorem 5.2 holds. Furthermore, for a fixed polarization $\tilde{\lambda}_{\mathbb{X}}$ on $(\mathbb{X}, \iota_{\mathbb{X},E})$ and for any $(X, \iota_E, \varrho) \in \mathcal{M}_E(\bar{k})$, the pullback $\tilde{\lambda} = \varrho^*(\tilde{\lambda}_{\mathbb{X}})$ is a polarization on X.*

Proof. This follows almost immediately from the theory of affine Deligne–Lusztig varieties (see, for example, [Chen and Viehmann 2015]) since we are comparing the geometric points of RZ-spaces for the isomorphic groups $\mathrm{GL}_2(E)$ and $\mathrm{GSp}_2(E)$.

It is also possible to check this via a more direct computation using Dieudonné theory, as we will indicate briefly. Proceeding very similarly to Proposition 3.2 or Proposition 4.1 (cf. [Kudla and Rapoport 2014] in the unramified case), we can associate to $\mathbb{X}$ a lattice Λ in the 2-dimensional E-vector space C (the Frobenius invariant points of the (rational) Dieudonné module). The choice of a principal polarization on $\mathbb{X}$ with trivial Rosati involution corresponds now exactly to a choice of perfect alternating form on Λ. It immediately follows that such a polarization exists and that it is unique up to a scalar in $O_E^\times$.

For the second part, let $X \in \mathcal{M}_E(\bar{k})$ and $M \subseteq C \otimes_E \breve{E}$ be its Dieudonné module. Since ϱ has height 0, we have

$$[M : M \cap (\Lambda \otimes_E \breve{E})] = [(\Lambda \otimes_E \breve{E}) : M \cap (\Lambda \otimes_E \breve{E})],$$

and one easily checks that a perfect alternating form b on Λ is also perfect on M. □

In the following, we fix a polarization $\tilde{\lambda}_{\mathbb{X}}$ on $(\mathbb{X}, \iota_{\mathbb{X},E})$ as in Theorem 5.2 (1). Let $(X, \iota_E, \varrho) \in \mathcal{M}_E(S)$ for $S \in \mathrm{Nilp}_{\breve{O}_F}$ and consider the pullback $\tilde{\lambda} = \varrho^*(\tilde{\lambda}_{\mathbb{X}})$. In general, this is only a quasipolarization. It suffices to show that $\tilde{\lambda}$ is a polarization on X. Indeed, since ϱ is O_E-linear and of height 0, this is then automatically a principal polarization on X such that the Rosati involution is the identity on O_E.

Define a subfunctor $\mathcal{M}_{E,\mathrm{pol}} \subseteq \mathcal{M}_E$ by

$$\mathcal{M}_{E,\mathrm{pol}}(S) = \{(X, \iota_E, \varrho) \in \mathcal{M}_E(S) \mid \tilde{\lambda} = \varrho^*(\tilde{\lambda}_{\mathbb{X}}) \text{ is a polarization on } X\}.$$

This is a closed formal subscheme by [Rapoport and Zink 1996, Proposition 2.9]. Moreover, Lemma 5.4 shows that $\mathcal{M}_{E,\mathrm{pol}}(\bar{k}) = \mathcal{M}_E(\bar{k})$.

Remark 5.5. Equivalently, we can describe $\mathcal{M}_{E,\mathrm{pol}}$ as follows. For $S \in \mathrm{Nilp}_{\breve{O}_F}$, we define $\mathcal{M}_{E,\mathrm{pol}}(S)$ to be the set of equivalence classes of tuples $(X, \iota_E, \tilde{\lambda}, \varrho)$, where

- X is a formal O_F-module over S of height 4 and dimension 2,
- ι_E is an action of O_E on X that satisfies the Kottwitz condition in (5-1) and
- $\tilde{\lambda}$ is a principal polarization on X such that the Rosati involution induces the identity on O_E.
- Furthermore, we fix a framing object $(\mathbb{X}, \iota_{\mathbb{X},E}, \tilde{\lambda}_{\mathbb{X}})$ over $\operatorname{Spec} \bar{k}$, where $(\mathbb{X}, \iota_{\mathbb{X},E})$ is the framing object for $\mathcal{M}_E$ and $\tilde{\lambda}_{\mathbb{X}}$ is a polarization as in Theorem 5.2 (1). Then ϱ is an O_E-linear quasi-isogeny

$$\varrho : X \times_S \bar{S} \to \mathbb{X} \times_{\operatorname{Spec} \bar{k}} \bar{S},$$

of height 0 such that, locally on $\bar{S}$, the (quasi-)polarizations $\varrho^*(\tilde{\lambda}_{\mathbb{X}})$ and $\tilde{\lambda}$ on X only differ by a scalar in $O_E^{\times}$, i.e., there exists an element $\alpha \in O_E^{\times}$ such that $\varrho^*(\tilde{\lambda}_{\mathbb{X}}) = \tilde{\lambda} \circ \iota_E(\alpha)$. Two tuples $(X, \iota_E, \tilde{\lambda}, \varrho)$ and $(X', \iota'_E, \tilde{\lambda}', \varrho')$ are equivalent if there exists an O_E-linear isomorphism $\varphi : X \xrightarrow{\sim} X'$ such that $\varphi^*(\tilde{\lambda}')$ and $\tilde{\lambda}$ only differ by a scalar in $O_E^{\times}$.

In this way, we give a definition for $\mathcal{M}_{E,\mathrm{pol}}$ by introducing extra data on points of the moduli space $\mathcal{M}_E$, instead of extra conditions. It is now clear that $\mathcal{M}_{E,\mathrm{pol}}$ describes a moduli problem for p-divisible groups of PEL type. It is easily checked that the two descriptions of $\mathcal{M}_{E,\mathrm{pol}}$ give rise to the same moduli space.

Theorem 5.2 now holds if and only if $\mathcal{M}_{E,\mathrm{pol}} = \mathcal{M}_E$. This equality is a consequence of the following statement.

Lemma 5.6. *For any point $x = (X, \iota_E, \varrho) \in \mathcal{M}_{E,\mathrm{pol}}(\bar{k})$, the embedding $\mathcal{M}_{E,\mathrm{pol}} \hookrightarrow \mathcal{M}_E$ induces an isomorphism of completed local rings $\widehat{\mathcal{O}}_{\mathcal{M}_{E,\mathrm{pol}},x} \cong \widehat{\mathcal{O}}_{\mathcal{M}_E,x}$.*

For the proof of this lemma, we use the theory of local models; cf. [Rapoport and Zink 1996, Chapter 3]. We postpone the proof of this lemma to the end of this section and we first introduce the local models $\mathrm{M}_E^{\mathrm{loc}}$ and $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$ for $\mathcal{M}_E$ and $\mathcal{M}_{E,\mathrm{pol}}$.

Let C be a 4-dimensional F-vector space with an action of E and let $\Lambda \subseteq C$ be an O_F-lattice that is stable under the action of O_E. Furthermore, let $(\, , \,)$ be an F-bilinear alternating form on C with

$$(\alpha x, y) = (x, \alpha y) \tag{5-2}$$

for all $\alpha \in E$ and $x, y \in C$ and such that Λ is unimodular with respect to $(\,,\,)$. It is easily checked that $(\,,\,)$ is unique up to an isomorphism of C that commutes with the E-action and that maps Λ to itself.

For an O_F-algebra R, let $\mathrm{M}_E^{\mathrm{loc}}(R)$ be the set of all direct summands $\mathcal{F} \subseteq \Lambda \otimes_{O_F} R$ of rank 2 that are O_E-linear and satisfy the *Kottwitz condition*. That means, for all $\alpha \in O_E$, the action of α on the quotient $(\Lambda \otimes_{O_F} R)/\mathcal{F}$ has the characteristic polynomial

$$\mathrm{char}(\mathrm{Lie}\, X, T \mid \alpha) = (T - \alpha)(T - \bar{\alpha}).$$

The subset $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}(R) \subseteq \mathrm{M}_E^{\mathrm{loc}}(R)$ consists of all direct summands $\mathcal{F} \in \mathrm{M}_E^{\mathrm{loc}}(R)$ that are in addition totally isotropic with respect to $(\,,\,)$ on $\Lambda \otimes_{O_F} R$.

The functor $\mathrm{M}_E^{\mathrm{loc}}$ is representable by a closed subscheme of $\mathrm{Gr}(2, \Lambda)_{O_F}$, the Grassmannian of rank 2 direct summands of Λ, and $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$ is representable by a closed subscheme of $\mathrm{M}_E^{\mathrm{loc}}$. In particular, both $\mathrm{M}_E^{\mathrm{loc}}$ and $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$ are projective schemes over $\mathrm{Spec}\, O_F$.

These local models have already been studied by Deligne and Pappas. In particular, we have:

Proposition 5.7 [Deligne and Pappas 1994]. $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}} = \mathrm{M}_E^{\mathrm{loc}}$. *In other words, for an O_F-algebra R, any direct summand $\mathcal{F} \in \mathrm{M}_E^{\mathrm{loc}}(R)$ is totally isotropic with respect to* $(\,,\,)$.

The moduli spaces $\mathcal{M}_E$ and $\mathcal{M}_{E,\mathrm{pol}}$ are related to the local models $\mathrm{M}_E^{\mathrm{loc}}$ and $\mathrm{M}_{E,\mathrm{pol}}^{\mathrm{loc}}$ via local model diagrams; cf. [Rapoport and Zink 1996, Chapter 3]. Let $\mathcal{M}_E^{\mathrm{large}}$ be the functor that maps a scheme $S \in \mathrm{Nilp}_{\breve{O}_F}$ to the set of isomorphism classes of tuples $(X, \iota_E, \varrho; \gamma)$. Here,

$$(X, \iota_E, \varrho) \in \mathcal{M}_E(S),$$

and γ is an O_E-linear isomorphism

$$\gamma : \mathbb{D}_X(S) \xrightarrow{\sim} \Lambda \otimes_{O_F} \mathcal{O}_S.$$

On the left-hand side, $\mathbb{D}_X(S)$ denotes the (relative) Grothendieck–Messing crystal of X evaluated at S; cf. [Ahsendorf 2011, Section 5.2].

Let $\widehat{\mathrm{M}}_E^{\mathrm{loc}}$ be the π_0-adic completion of $\mathrm{M}_E^{\mathrm{loc}} \otimes_{O_F} \breve{O}_F$. Then there is a local model diagram:

$$\begin{tikzcd} & \mathcal{M}_E^{\mathrm{large}} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathcal{M}_E & & \widehat{\mathrm{M}}_E^{\mathrm{loc}} \end{tikzcd}$$

The morphism f on the left-hand side is the projection $(X, \iota_E, \varrho; \gamma) \mapsto (X, \iota_E, \varrho)$. The morphism g on the right-hand side maps $(X, \iota_E, \varrho; \gamma) \in \mathcal{M}_E^{\text{large}}(S)$ to

$$\mathcal{F} = \ker(\Lambda \otimes_{O_F} \mathcal{O}_S \xrightarrow{\gamma^{-1}} \mathbb{D}_X(S) \to \operatorname{Lie} X) \subseteq \Lambda \otimes_{O_F} \mathcal{O}_S.$$

By [Rapoport and Zink 1996, Theorem 3.11], the morphism f is smooth and surjective. The morphism g is formally smooth by Grothendieck–Messing theory; see [Messing 1972, V.1.6], resp. [Ahsendorf 2011, Chapter 5.2] for the relative setting (i.e., when $O_F \neq \mathbb{Z}_p$).

We also have a local model diagram for the space $\mathcal{M}_{E,\text{pol}}$. We define $\mathcal{M}_{E,\text{pol}}^{\text{large}}$ as the fiber product $\mathcal{M}_{E,\text{pol}}^{\text{large}} = \mathcal{M}_{E,\text{pol}} \times_{\mathcal{M}_E} \mathcal{M}_E^{\text{large}}$. Then $\mathcal{M}_{E,\text{pol}}^{\text{large}}$ is closed formal subscheme of $\mathcal{M}_E^{\text{large}}$ with the following moduli description. A point $(X, \iota_E, \varrho; \gamma) \in \mathcal{M}_E^{\text{large}}(S)$ lies in $\mathcal{M}_{E,\text{pol}}^{\text{large}}(S)$ if $\tilde{\lambda} = \varrho^*(\tilde{\lambda}_{\mathbb{X}})$ is a principal polarization on X. In that case, $\tilde{\lambda}$ induces an alternating form $(\ ,\ )^X$ on $\mathbb{D}_X(S)$ which, under the isomorphism γ, is equal to the form $(\ ,\ )$ on $\Lambda \otimes_{O_F} \mathcal{O}_S$, up to a unit in $O_E \otimes_{O_F} \mathcal{O}_S$.

The local model diagram for $\mathcal{M}_{E,\text{pol}}$ now looks as follows.

$$\begin{tikzcd}[column sep=small] & \mathcal{M}_{E,\text{pol}}^{\text{large}} \arrow[dl, "f_{\text{pol}}"'] \arrow[dr, "g_{\text{pol}}"] & \\ \mathcal{M}_{E,\text{pol}} & & \widehat{\mathrm{M}}_{E,\text{pol}}^{\text{loc}} \end{tikzcd} \tag{5-3}$$

Here, $\widehat{\mathrm{M}}_{E,\text{pol}}^{\text{loc}}$ is the π_0-adic completion of $\mathrm{M}_{E,\text{pol}}^{\text{loc}} \otimes_{O_F} \breve{O}_F$ and f_{pol} and g_{pol} are the restrictions of the morphisms f and g above. Again, g_{pol} is formally smooth by Grothendieck–Messing theory and f_{pol} is smooth and surjective by construction.

We can now finish the proof of Lemma 5.6.

Proof of Lemma 5.6. We have the following commutative diagram.

$$\begin{tikzcd} \mathcal{M}_{E,\text{pol}} \arrow[d, hook] & \mathcal{M}_{E,\text{pol}}^{\text{large}} \arrow[l, "f_{\text{pol}}"'] \arrow[r, "g_{\text{pol}}"] \arrow[d, hook] & \widehat{\mathrm{M}}_{E,\text{pol}}^{\text{loc}} \arrow[d, equal] \\ \mathcal{M}_E & \mathcal{M}_E^{\text{large}} \arrow[l, "f"'] \arrow[r, "g"] & \widehat{\mathrm{M}}_E^{\text{loc}} \end{tikzcd} \tag{5-4}$$

The equality on the right-hand side follows from Proposition 5.7. The other vertical arrows are closed embeddings.

Let $x \in \mathcal{M}_{E,\text{pol}}(\bar{k})$. By [Rapoport and Zink 1996, Proposition 3.33], there exists an étale neighborhood U of x in $\mathcal{M}_E$ and section $s : U \to \mathcal{M}_E^{\text{large}}$ such that $g \circ s$ is formally étale. Similarly, $U_{\text{pol}} = U \times_{\mathcal{M}_E} \mathcal{M}_{E,\text{pol}}$ and s_{pol} is the base change of s to U_{pol}. Then the composition $g_{\text{pol}} \circ s_{\text{pol}}$ is also formally étale. These formally étale maps induce isomorphisms of local rings

$$\widehat{\mathcal{O}}_{\mathcal{M}_E,x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\widehat{\mathrm{M}}_E^{\mathrm{loc}},x'} \quad \text{and} \quad \widehat{\mathcal{O}}_{\mathcal{M}_{E,\mathrm{pol}},x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\widehat{\mathrm{M}}_{E,\mathrm{pol}}^{\mathrm{loc}},x'}, \qquad x' = s(g(x)).$$

By Proposition 5.7, we have $\widehat{\mathcal{O}}_{\widehat{\mathrm{M}}_E^{\mathrm{loc}},x'} = \widehat{\mathcal{O}}_{\widehat{\mathrm{M}}_{E,\mathrm{pol}}^{\mathrm{loc}},x'}$ and since this identification commutes with $g \circ s$ (resp. $g_{\mathrm{pol}} \circ s_{\mathrm{pol}}$), we get the desired isomorphism $\widehat{\mathcal{O}}_{\mathcal{M}_{E,\mathrm{pol}},x} \cong \widehat{\mathcal{O}}_{\mathcal{M}_E,x}$. □

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GROUP AND ROUND QUADRATIC FORMS

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We offer some elementary characterisations of group and round quadratic forms. These characterisations are applied to establish new (and recover existing) characterisations of Pfister forms. We establish "going-up" results for group and anisotropic round forms with respect to iterated Laurent series field extensions, which contrast with the established results with respect to rational function field extensions. For forms of two-power dimension, we determine when there exists a field extension over which the form becomes an anisotropic group form that is not round.

1. Introduction

A quadratic form is round if its value set coincides with the multiplicative group of similarity factors associated with the form. Thus, round forms constitute a prominent subclass of group forms, forms whose value sets are multiplicative groups. As roundness is one of the fundamental properties of Pfister forms, the class of forms that occupy a central role in quadratic form theory, it is unsurprising that this notion has had a number of important consequences. However, while the structure and behaviour of round forms has received extensive treatment in the literature, this class of forms is still not fully understood and, as suggested in [Lam 2005], merits further study. The broader class of group forms is, comparatively, little understood.

Our opening results, which are invoked throughout this article, record elementary characterisations of the classes of group and round forms (see Proposition 2.2 to Corollary 2.5). In Section 2, we apply these results to obtain new characterisations of Pfister forms (see Theorem 2.7), in addition to reproving established ones (see Corollary 2.8), and to extend Elman's classification of odd-dimensional round forms in accordance with our broader definition of roundness.

The group and round properties of a form are intrinsically linked to its base field of definition, and thus are sensitive to scalar extension. Alpers [1991] remarks that while general "going-down" results exist with respect to roundness, with round forms over odd-degree extensions being round over their base fields for example, no general results are known in the "going-up" direction. We establish such results

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for group and anisotropic round forms with respect to iterated Laurent series fields (see Corollary 3.4), highlighting an interesting divergence in the behaviour of forms under extension to iterated Laurent series fields as opposed to rational function fields (see Remark 3.5).

Hsia and Johnson [1973a; 1973b] studied the problem of distinguishing between anisotropic group and round forms over local and global fields. In this spirit, we consider the following general question:

Question 1.1. For q an anisotropic form over F, does there exist an extension K/F such that q_K is an anisotropic group form that is not round?

Our characterisation of group forms allows for the construction of a generic field extension over which a form becomes an anisotropic group form. Thus, the adoption of Merkurjev's method of passing to iterated field extensions obtained by composing function fields of quadratic forms represents the natural approach to addressing the above question. While highlighting an obstruction to resolving Question 1.1 in general (see Proposition 3.6), we can employ this method to good effect in certain situations. In particular, Theorem 3.8 represents a complete answer to Question 1.1 with respect to forms of two-power dimension.

We let F denote a field of characteristic different from two, and recall that every nondegenerate quadratic form on a vector space over F can be diagonalised. We write $\langle a_1, \dots, a_n\rangle$ to denote the (n-dimensional) quadratic form on an n-dimensional F-vector space defined by $a_1, \dots, a_n \in F^\times$. We use the term "form" to refer to a nondegenerate quadratic form of positive dimension. If p and q are forms over F, we denote by $p \perp q$ their orthogonal sum and by $p \otimes q$ their tensor product. We use aq to denote $\langle a\rangle \otimes q$ for $a \in F^\times$. We write $p \simeq q$ to indicate that p and q are isometric, and say that p and q are *similar* if $p \simeq aq$ for some $a \in F^\times$. A form p is a *subform of* q if $q \simeq p \perp r$ for some form r, in which case we write $p \subseteq q$. For q a form over F and K/F a field extension, we will often employ the notation q_K when viewing q as a form over K via the canonical embedding. A form q represents $a \in F$ if there exists a vector v such that $q(v) = a$. We denote by $D_F(q)$ the set of values in $F^\times$ represented by q. A form over F is *isotropic* if it represents zero nontrivially, and *anisotropic* otherwise. Every form q has a decomposition $q \simeq q_{\text{an}} \perp i(q) \times \langle 1, -1\rangle$ where the anisotropic quadratic form q_{an} and the nonnegative integer $i(q)$ are uniquely determined. If a form q is isotropic over F, then $D_F(q) = F^\times$, as $((a+1)/2)^2 - ((a-1)/2)^2 = a$ for all $a \in F^\times$. A form q is *hyperbolic* if $i(q) = \frac{1}{2}\dim q$.

A form q is a *group form* over F if $D_F(q)$ is a subgroup of $F^\times$. The similarity factors of q constitute the group $G_F(q) = \{a \in F^\times \mid aq \simeq q\}$. Equivalently, $G_F(q) = \{a \in F^\times \mid \langle 1, -a\rangle \otimes q \text{ is hyperbolic}\}$. A group form q over F is said to be *round* if $D_F(q) = G_F(q)$. Equivalently, a form q is round over F if $D_F(q) \subseteq G_F(q)$, as if

$a \in D_F(q) \subseteq G_F(q)$, then $aq \simeq q$, whereby $1 \in D_F(q)$ and thus $G_F(q) \subseteq D_F(q)$. We use $H_F(q)$ to denote the set of products of two elements of $D_F(q)$. Per Lemma 2.1, we have that $H_F(q) = \{a \in F^\times \mid \langle 1, -a\rangle \otimes q$ is isotropic$\}$. For $n \in \mathbb{N}$, an *n-fold Pfister form* over F is a form isometric to $\langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$ for some $a_1, \ldots, a_n \in F^\times$ (the form $\langle 1\rangle$ is the 0-fold Pfister form). Isotropic Pfister forms are hyperbolic [Lam 2005, Theorem X.1.7]. Pfister forms are round (see [Lam 2005, Theorem X.1.8]). A form τ is a Pfister neighbour if τ is similar to a subform of a Pfister form π and $\dim \tau > \frac{1}{2} \dim \pi$.

We recall that every nonzero square class in $F((x))$, the Laurent series field in the variable x over F, can be represented by a or ax for some $a \in F^\times$, whereby every form over $F((x))$ can be written as $p \perp xq$ for p and q forms over F. We will often invoke the following folkloric result regarding isotropy over Laurent series fields.

Lemma 1.2. *Let p and q be forms over F. Considering $p \perp xq$ as a form over $F((x))$, we have that $i(p \perp xq) = i(p) + i(q)$.*

For a form q over F with $\dim q = n \geqslant 2$ and $q \not\simeq \langle 1, -1\rangle$, the *function field $F(q)$ of q* is the quotient field of the integral domain $F[X_1, \ldots, X_n]/(q(X_1, \ldots, X_n))$ (this is the function field of the affine quadric $q(X) = 0$ over F). To avoid case distinctions, we set $F(q) = F$ if $\dim q = 1$ or $q \simeq \langle 1, -1\rangle$. For q a form over F, we note the inclusion $F((x))(q) \subseteq F(q)((x))$, which we will apply in combination with Lemma 1.2. For all forms p over F and all extensions K/F such that q_K is isotropic, we have that $i(p_{F(q)}) \leqslant i(p_K)$ in accordance with Knebusch's specialisation results [1976, Proposition 3.1 and Theorem 3.3]. In particular, letting $i_1(q)$ denote $i(q_{F(q)})$, we have that $i_1(q) \leqslant i(q_K)$ for all extensions K/F such that q_K is isotropic. Invoking the Cassels–Pfister Subform Theorem [Lam 2005, Theorem X.4.5] of Wadsworth [1975, Theorem 2] and Knebusch [1976, Lemma 4.5], for p and q anisotropic forms over F of dimension at least two such that $p_{F(q)}$ is hyperbolic, one has that $aq \subseteq bp$ for all $a \in D_F(q)$ and $b \in D_F(p)$. For q an anisotropic form over F of dimension at least two, it is known that $q_{F(q)}$ is hyperbolic if and only if q is similar to a Pfister form over F by [Lam 2005, Theorem X.4.14], a result of Wadsworth [1975, Theorem 5] and Knebusch [1976, Theorem 5.8]. We will regularly invoke Hoffmann's separation theorem [1995, Theorem 1], which we recall below.

Theorem 1.3. *Let p and q be forms over F such that p is anisotropic. If*

$$\dim p \leqslant 2^n < \dim q$$

for some integer $n \geqslant 0$, then $p_{F(q)}$ is anisotropic.

In accordance with the above theorem and [Lam 2005, Exercise I.16], for q an anisotropic form over F, we note that $\dim q$ and $\dim q - i_1(q)$ belong to an interval of the form $[2^n, 2^{n+1}]$ for some $n \in \mathbb{N} \cup \{0\}$. We will also invoke the following isotropy criterion of Karpenko and Merkurjev [2003, Theorem 4.1].

Theorem 1.4. *For p and q anisotropic forms over F such that $p_{F(q)}$ is isotropic,*

(i) $\dim p - i_1(p) \geqslant \dim q - i_1(q)$;

(ii) $\dim p - i_1(p) = \dim q - i_1(q)$ *if and only if* $q_{F(p)}$ *is isotropic.*

We refer the reader to works by Vishik [2011] and Scully [2016a] for recent results in the spirit of Theorem 1.3 and Theorem 1.4.

2. Characterisations of group, round and Pfister forms

As above, the forms we consider are nondegenerate and of positive dimension over fields of characteristic different from two. In accordance with the associated definitions, we begin our study of the properties of a form q over F being group or round by considering the value set $D_F(q)$, the group of similarity factors $G_F(q)$, and the set of products of two elements of $D_F(q)$, usually denoted by $D_F(q)D_F(q)$.

Roussey [2005, Lemme 2.5.4] proved the following result in his thesis.

Lemma 2.1. *For p and q forms over F, we have that*

$$D_F(p)D_F(q) = \{a \in F^\times \mid p \perp -aq \text{ is isotropic}\}.$$

Proof. The statement clearly holds if either p or q is isotropic. Thus, assuming that p and q are anisotropic, we have that $p \perp -aq$ is isotropic if and only if there exist nonzero vectors v and w such that $p(v) - aq(w) = 0$. Thus, $p(v) = aq(w) \neq 0$, whereby $a = p(v)(q(w))^{-1}$. Hence,

$$a = p(v)\frac{1}{q(w)} = p(v)q\Big(\frac{w}{q(w)}\Big) \in D_F(p)D_F(q).$$

As $1 \in D_F(dq)$ for $d \in D_F(q)$, we have $p \perp -cdq$ is isotropic for $c \in D_F(p)$. □

We let $H_F(q) = \{a \in F^\times \mid \langle 1, -a\rangle \otimes q \text{ is isotropic}\}$, giving $H_F(q) = D_F(q)D_F(q)$ in accordance with Lemma 2.1. As with $D_F(q)$ and $G_F(q)$, we may restrict our attention to the square classes contained in $H_F(q)$, since $H_F(q) = (F^\times)^2 H_F(q)$. Clearly we have that $(F^\times)^2 \subseteq G_F(q) \subseteq H_F(q)$ for all forms q over F. Moreover, if $1 \in D_F(q)$, then

$$(F^\times)^2 \subseteq G_F(q) \subseteq D_F(q) \subseteq H_F(q).$$

If q is isotropic over F, then q is a group form over F, with $D_F(q) = F^\times = H_F(q)$ in this case. Our opening result records that Lemma 2.1 may be applied to characterise group forms.

Proposition 2.2. *A form q is a group form over F if and only if $H_F(q) \subseteq D_F(q)$.*

Proof. Letting $a \in D_F(q)$, there exists a nonzero vector v such that $q(v) = a$, whereby $q(a^{-1}v) = (a^{-1})^2 q(v) = a^{-1} \in D_F(q)$. Thus, q is a group form over F if and only if $D_F(q)D_F(q) \subseteq D_F(q)$, so the result follows by invoking Lemma 2.1. □

As group forms represent one, we thus obtain the following corollary.

Corollary 2.3. *A form q is a group form over F if and only if $H_F(q) = D_F(q)$.*

In accordance with our definition of roundness, if q is isotropic over F, then q is round over F if and only if q is hyperbolic or the nonzero form q_{an} is such that $D_F(q_{\mathrm{an}}) = F^\times = G_F(q_{\mathrm{an}})$. This observation follows from the fact that $G_F(q) = G_F(q_{\mathrm{an}})$, in accordance with Witt cancellation and the fact that $D_F(q) = F^\times$ for q isotropic over F. Per [Lam 2005, Example X.1.15(5)], the form $q \simeq \langle 1, -1, 1, 1\rangle$ over $\mathbb{F}_3$ is an example of an isotropic round form that is not hyperbolic.

Corollary 2.4. *A form q is round over F if and only if 1 is an element of $D_F(q)$ and $H_F(q) \subseteq G_F(q)$.*

Proof. If q is round over F, then $1 \in D_F(q) = G_F(q)$ and q is a group form over F. Invoking Proposition 2.2, it follows that $H_F(q) \subseteq D_F(q)$, whereby $H_F(q) \subseteq G_F(q)$.

Conversely, as $1 \in D_F(q)$, we recall that $G_F(q) \subseteq D_F(q) \subseteq H_F(q)$, whereby the equality $D_F(q) = G_F(q)$ follows from the assumption that $H_F(q) \subseteq G_F(q)$. □

We note that, for q a round form over F, the inclusion $H_F(q) \subseteq G_F(q)$ can also be derived from [Wadsworth and Shapiro 1977, Proposition 1 and Theorem 1].

Addressing the question of distinguishing between the classes of group and round forms over a given field, it is reasonable to restrict one's consideration to those forms that represent one, whereby the preceding characterisation may be simplified.

Corollary 2.5. *Let q be a form such that $1 \in D_F(q)$. The following are equivalent*:

(i) *q is round over F.*

(ii) *$H_F(q) \subseteq G_F(q)$.*

(iii) *$q \otimes \rho$ is anisotropic or hyperbolic for every 1-fold Pfister form ρ over F.*

(iv) *$q \otimes \beta$ is anisotropic or hyperbolic for every 2-dimensional form β over F.*

(v) *$q \otimes \pi$ is anisotropic or hyperbolic for every n-fold Pfister form π over F, for $n \in \mathbb{N}$.*

Proof. Statements (i), (ii) and (iii) are equivalent by Corollary 2.4. Statements (iii) and (iv) are equivalent, as scaling does not affect isotropy. Statement (v) clearly implies (iii). Assuming (i), it follows that $q \otimes \pi$ is round for every Pfister form π over F, by Witt's round form theorem [Lam 2005, Theorem X.1.14]. By repeatedly invoking statement (iii), we see that (v) follows. □

Next, note that scalar multiples of Pfister forms representing one are Pfister forms.

Lemma 2.6. *A form q over F is a Pfister form if and only if q is similar to a Pfister form and $1 \in D_F(q)$.*

Proof. To establish the right-to-left implication, we let $q \simeq a\pi$ for $a \in F^\times$ and π a Pfister form over F. As $1 \in D_F(q)$, it follows that $a \in D_F(\pi)$, whereby $q \simeq \pi$ as π is round. □

We can apply the above characterisations of round and group forms to obtain a new characterisation of Pfister forms.

Theorem 2.7. *Let q be an anisotropic form. The following are equivalent*:

(i) *q is a Pfister form over F.*

(ii) *q is a round form over $K = F((x))(q \otimes \langle 1, -x\rangle)$.*

(iii) *q is a group form over $K = F((x))(q \otimes \langle 1, -x\rangle)$.*

Proof. As Pfister forms are round, and round forms are group, it suffices to prove that (iii) implies (i).

The field K is the function field of $q\otimes\langle 1, -x\rangle$ over $F((x))$, which is an anisotropic form in accordance with Lemma 1.2. We will first show that

$$D_K(q) \cap F^\times = D_F(q).$$

Let $a \in F^\times$ be such that $q \perp \langle -a\rangle$ is anisotropic over F and suppose, for the sake of contradiction, that $q \perp \langle -a\rangle$ is isotropic over K. Invoking Theorem 1.4 (i),

$$\dim(q \perp \langle -a\rangle) - i_1(q \perp \langle -a\rangle) \geqslant \dim(q \otimes \langle 1, -x\rangle) - i_1(q \otimes \langle 1, -x\rangle).$$

In accordance with Theorem 1.3 and [Lam 2005, Exercise I.16], there exists $n \in \mathbb{N}$ such that

$$\dim(q \perp \langle -a\rangle) - i_1(q \perp \langle -a\rangle) = \dim(q \otimes \langle 1, -x\rangle) - i_1(q \otimes \langle 1, -x\rangle) = 2^n,$$

whereby $\dim q = 2^n$. Hence, $q \otimes \langle 1, -x\rangle$ is isotropic over $F((x))(q \perp \langle -a\rangle)$, in accordance with Theorem 1.4 (ii). Invoking [Izhboldin 2000, Lemma 5.4 (3)], it thus follows that q is isotropic over $F(q \perp \langle -a\rangle)$. However, as $\dim q = 2^n$, this contradicts Theorem 1.3, thereby establishing the claim.

We have that $1 \in D_K(q)$ by assumption, whereby $1 \in D_F(q)$ by the statement proven above. As $x \in H_K(q)$ by construction, it follows that $x \in D_K(q)$ in accordance with Proposition 2.2, whereby the form $q \perp \langle -x\rangle$ becomes isotropic over $F((x))(q \otimes \langle 1, -x\rangle)$. Arguing as above, it follows that, for some $n \in \mathbb{N}$,

$$\dim(q \perp \langle -x\rangle) - i_1(q \perp \langle -x\rangle) = \dim(q \otimes \langle 1, -x\rangle) - i_1(q \otimes \langle 1, -x\rangle) = 2^n,$$

and $\dim q = 2^n$. Hence, $i_1(q \otimes \langle 1, -x\rangle) = \dim q$, in accordance with this equality, whereby $q \otimes \langle 1, -x\rangle$ becomes hyperbolic over $F((x))(q \otimes \langle 1, -x\rangle)$. Invoking [O'Shea 2016, Proposition 3.2], it follows that q is hyperbolic over $F(q)$. Thus, q is similar to a Pfister form over F by [Lam 2005, Theorem X.4.14]. As $1 \in D_F(q)$, the result follows by invoking Lemma 2.6. □

We can invoke the above result to reprove the following characterisations of Pfister forms due to Pfister (see [Pfister 1965, Satz 5 and Theorem 2], [Scharlau 1985, Theorem 4.4 p.153] or [Elman et al. 2008, Theorem 23.2]).

Corollary 2.8. *Let q be an anisotropic form over F. The following are equivalent*:

(i) *q is a Pfister form over F.*

(ii) *q is round over K for every extension K/F.*

(iii) *q is group over K for every extension K/F.*

Similarly, we obtain the following corollary of Theorem 2.7.

Corollary 2.9. *For q an anisotropic form over F, let $K = F((x))(q \otimes \langle 1, -x \rangle)$.*

(i) *Then q is group over every extension of F if and only if q is group over K, and*

(ii) *q is round over every extension of F if and only if q is round over K.*

Remark 2.10. Per [Scharlau 1985, Theorem 4.4, p.153], Pfister established that, for q an anisotropic form of dimension n, the three statements in Corollary 2.8 are equivalent to each of the following statements:

(i) $q(x_1, \dots, x_n) \in G_K(q)$ for $K = F(x_1, \dots, x_n)$.

(ii) $q(x_1, \dots, x_n)q(x_{n+1}, \dots, x_{2n}) \in D_K(q)$ for $K = F(x_1, \dots, x_{2n})$.

Thus, for q an anisotropic form of dimension n, it follows that q is a Pfister form over F if and only if q is a round form over $F(x_1, \dots, x_n)$, and that q is a Pfister form over F if and only if q is a group form over $F(x_1, \dots, x_{2n})$.

In a similar spirit to the preceding results, we offer the following characterisation of scalar multiples of Pfister forms.

Proposition 2.11. *Letting q be an anisotropic form over F, the following are equivalent*:

(i) *q is similar to a Pfister form over F.*

(ii) *q_K is round for all extensions K/F such that $1 \in D_K(q)$.*

(iii) *q_K is round for all extensions K/F such that q_K is isotropic.*

Proof. Assuming (i), Lemma 2.6 implies that q_K is a Pfister form for K/F such that $1 \in D_K(q)$, whereby (ii) follows. As (ii) clearly implies (iii), it suffices to show that (iii) implies (i).

Letting $K = F(q)((x))$, we have that

$$D_K(q) = K^\times = G_K(q).$$

As $x \in G_K(q)$, the form $q \otimes \langle 1, -x \rangle$ is hyperbolic over $F(q)((x))$. Invoking Lemma 1.2, it follows that q is hyperbolic over $F(q)$. Thus, q is similar to a Pfister form over F by [Lam 2005, Theorem X.4.14]. □

We conclude this section by characterising the odd-dimensional round forms (see Elman's characterisation [1973] of odd-dimensional round forms in the situation where isotropic round forms are defined to be hyperbolic).

Proposition 2.12. *Let q be a form over F.*

(i) *If $D_F(q) = (F^\times)^2$, then q is round over F.*

(ii) *If $H_F(q) \neq (F^\times)^2$ and q is round over F, then q is even-dimensional.*

Proof. (i) If $D_F(q) = (F^\times)^2$, then $D_F(q) \subseteq G_F(q)$, whereby q is round over F.
(ii) Let $a \in H_F(q) \setminus (F^\times)^2$. As q is round over F, we have that $H_F(q) \subseteq G_F(q)$ by Corollary 2.4, whereby $q \perp -aq$ is hyperbolic over F. As a comparison of determinants yields the contradiction that $a \in (F^\times)^2$ for q odd-dimensional, the result follows. □

Adapting Elman's proof of [Elman 1973, Lemma], we obtain the following result as a corollary of Proposition 2.12.

Corollary 2.13. *Let q be an odd-dimensional form over F. If q is round, then*

$$q \simeq (2r+1) \times \langle 1 \rangle$$

for some $r \in \mathbb{N} \cup \{0\}$. Moreover, the following are equivalent:

(i) *$(2k+1) \times \langle 1 \rangle$ is round over F for some $k \in \mathbb{N}$,*

(ii) *$(2n+1) \times \langle 1 \rangle$ is round over F for every $n \in \mathbb{N} \cup \{0\}$,*

(iii) *F is Pythagorean.*

Corollary 2.14. *Let q be an odd-dimensional isotropic form over F. Then q is round over F if and only if F is quadratically closed.*

Proof. If q is round, then $D_F(q) = (F^\times)^2$ by Proposition 2.12. As q is isotropic, it follows that $D_F(q) = F^\times$, whereby F is quadratically closed.

If F is quadratically closed, then $q_{\mathrm{an}} \simeq \langle 1 \rangle$, whereby q is round. □

Corollary 2.15. *If q is an odd-dimensional anisotropic round form over F, then $q \otimes \beta$ is anisotropic over F for every anisotropic 2-dimensional form β over F.*

Proof. Let $\beta \simeq b\langle 1, -a \rangle$ be anisotropic over F for $a, b \in F^\times$. Suppose, for the sake of contradiction, that $q \otimes \beta$ is isotropic over F. Hence, $q \otimes \langle 1, -a \rangle$ is hyperbolic over F by Corollary 2.4. By repeatedly invoking [Elman and Lam 1973, Proposition 2.2], it follows that there exist binary forms $\beta_1, \dots, \beta_n$ over F such that $\beta_i \otimes \langle 1, -a \rangle$ is hyperbolic over F for $i = 1, \dots, n$ and such that

$$q \simeq \beta_1 \perp \cdots \perp \beta_n,$$

whereby q is even-dimensional, a contradiction. □

Remark 2.16. The preceding result can also be derived from the fact that F is Pythagorean and real, whereby its Witt ring is torsion free (see [Elman et al. 2008, Theorem 23.2]).

3. Group and round forms over field extensions

Alpers [1991] considers roundness with respect to algebraic extensions, establishing "going-down" and "going-up" results in certain situations. In particular, he remarks that a general going-down result holds for odd-degree extensions by Springer's theorem [Springer 1952] (see [Lam 2005, Theorem VII.2.7]). We generalise this remark below.

Proposition 3.1. *Let q be a form over F and let K be an extension of F.*

(i) *Suppose that $D_K(q) \cap F^\times \subseteq D_F(q)$. If q_K is a group form, then q is a group form over F.*

(ii) *Suppose that every anisotropic form over F of dimension at most* $\dim q + 1$ *is anisotropic over K. If q_K is a round form, then q is a round form over F.*

Proof. (i) As $D_K(q) \cap F^\times = D_F(q)$ follows from the assumption, if $D_K(q)$ is a group it readily follows that $D_F(q)$ is a group.

(ii) If q is anisotropic over F, the assumption on K implies that $D_K(q) \cap F^\times = D_F(q)$ and $G_K(q) \cap F^\times = G_F(q)$. Applying this argument to q_{an} in the case where q is isotropic over F, the result follows. □

Thus, as a consequence of the above, group and round forms satisfy going-down results with respect to purely transcendental extensions. Per Remark 2.10, going-up results do not hold for group or round forms with respect to rational function fields. However, we do have the following result with respect to Laurent series fields:

Proposition 3.2. *Let q be a form over F and let $K = F((x))$.*

(i) *q is a group form over F if and only if q is a group form over K.*

(ii) *If q is anisotropic, then q is round over F if and only if q is round over K.*

Proof. We remark that anisotropic forms over F are anisotropic over K.

(i) As isotropic forms are trivially group, we may assume, without loss of generality, that q is anisotropic over F. We consider the set $H_K(q)$, recalling that every nonzero square class in K can be represented by a or ax for some $a \in F^\times$. Invoking Lemma 1.2, it is apparent that $q \perp -axq$ is anisotropic over K for $a \in F^\times$. For $a \in F^\times$ such that $q \perp -aq$ is isotropic over K, it follows that $q \perp -aq$ is isotropic over F, whereby $q \perp \langle -a \rangle$ is isotropic over F by Proposition 2.2. Thus, as $q \perp \langle -a \rangle$ is isotropic over K, it follows from Proposition 2.2 that q is a group form over K.

For the converse statement, we may invoke Proposition 3.1 (i).

(ii) As $1 \in D_F(q)$ if and only if $1 \in D_K(q)$, we may argue as in the preceding proof of (i), with respect to Corollary 2.4 as opposed to Proposition 2.2, to establish the "only if" statement. The "if" statement can be established by invoking Proposition 3.1 (ii). □

Remark 3.3. We note the necessity of the restriction to anisotropic round forms in statement (ii) of the above result. If q is isotropic and round over $K = F((x))$, then it readily follows that q is isotropic and round over F, but the converse does not hold in general. In particular, the isotropic form $q \simeq \langle 1, -1, 1, 1\rangle$ is round over $\mathbb{F}_3$ but it is not round over $\mathbb{F}_3((x))$, as $x \notin D_K((q_K)_{\text{an}})$ in accordance with Lemma 1.2.

Iterating the above, we obtain the following result.

Corollary 3.4. *Let q be a form over F and let $K = F((x_1))\cdots((x_n))$ for $n \in \mathbb{N}$.*

(i) *q is a group form over F if and only if q is a group form over K.*

(ii) *If q is anisotropic, then q is round over F if and only if q is round over K.*

Remark 3.5. We note that the above result demonstrates a divergence in the behaviour, with respect to the properties of being group or round, of forms over F extended to iterated Laurent series fields as opposed to rational function fields. Corollary 3.4 contrasts with Pfister's result, per Remark 2.10, that an anisotropic form of dimension n over F is a round form over $F(x_1, \dots, x_n)$ if and only if it is a group form over $F(x_1, \dots, x_{2n})$ if and only if it is a Pfister form over F.

Motivated by the problem of distinguishing between anisotropic group and round forms, as studied over particular fields in [Hsia and Johnson 1973a; Hsia and Johnson 1973b], the rest of this section is devoted to addressing Question 1.1.

In accordance with Corollary 2.5, if q is an anisotropic group form over F, one can resolve Question 1.1 by determining whether there exists a Pfister form π over F such that $q \otimes \pi$ is isotropic but not hyperbolic. If such a form q is odd-dimensional, Question 1.1 further reduces to the problem of determining whether $H_F(q) \setminus (F^\times)^2$ is empty, in accordance with Proposition 2.12.

The natural approach towards answering Question 1.1 in the case where q is not a group form over F is to consider its extension to the generic extension K/F such that q_K is a group form. However, one encounters the following obstruction:

Proposition 3.6. *Let q be an anisotropic form over F. If there exists $a \in H_F(q)$ such that $i_1(q \perp \langle -a\rangle) > 1$, then there does not exist an extension K/F such that q_K is an anisotropic group form.*

Proof. Let K/F be an extension such that q_K is a group form. Since

$$a \in H_F(q) \subseteq H_K(q),$$

it follows that $a \in D_K(q)$ by Corollary 2.3, whereby $q \perp \langle -a\rangle$ is isotropic over K. Since $i_1(q \perp \langle -a\rangle) > 1$, it follows that $i((q \perp \langle -a\rangle)_K) > 1$, whereby q_K is isotropic (see [Lam 2005, Exercise I.16]). □

The following example illustrates that, provided that $\dim q \neq 2^n$ for $n \in \mathbb{N} \cup \{0\}$, there exist fields F, forms q over F and scalars $a \in F^\times$ that satisfy the hypotheses of Proposition 3.6.

Example 3.7. Let L a field and $a \in L^\times$ be such that $q \perp \langle -a \rangle$ is an anisotropic Pfister neighbour, where $\dim q \neq 2^n$ for $n \in \mathbb{N} \cup \{0\}$. Letting $F = L(q \perp -aq)$, we have that $a \in H_F(q)$ and that $q \perp \langle -a \rangle$ is anisotropic over F, by Theorem 1.3. As $q \perp \langle -a \rangle$ is a Pfister neighbour of dimension $\neq 2^n + 1$ for $n \in \mathbb{N} \cup \{0\}$, it follows that $i_1(q \perp \langle -a \rangle) > 1$.

In contrast with the above example, letting q be an arbitrary anisotropic form over F of dimension 2^n for some $n \in \mathbb{N}$, the proof of the following theorem demonstrates that there does exist an extension K/F such that q_K is an anisotropic group form. Moreover, when possible, we can find an extension K/F such that q_K is an anisotropic group form that is not round.

Theorem 3.8. *Let q be an anisotropic form over F of dimension 2^n for $n \in \mathbb{N}$.*

(i) *There exists an extension K/F such that q_K is an anisotropic group form.*

(ii) *If q is similar to a Pfister form over F, then q_K is round for every extension K/F such that q_K is a group form.*

(iii) *If q is not similar to a Pfister form over F, there exists an extension K/F such that q_K is an anisotropic group form that is not round.*

Proof. (i) In light of statement (iii), it suffices to prove this statement in the case where q is similar to a Pfister form over F. By Lemma 2.6, we have that q is a Pfister form if and only if it represents one. Thus, we may let $K = F$ in the case where $1 \in D_F(q)$. Otherwise, we may let $K = F(q \perp \langle -1 \rangle)$, as q_K is anisotropic by Theorem 1.3.

(ii) Let K/F be such that q_K is a group form. As $1 \in D_K(q)$, we have that q_K is a Pfister form by Lemma 2.6, whereby q_K is round.

(iii) If $1 \notin D_F(q)$, we may consider q as a form over $L = F(q \perp \langle -1 \rangle)$, whereby $1 \in D_L(q)$. In this case, q_L remains anisotropic by Theorem 1.3. As q is not similar to a Pfister form over F, it follows that q is not hyperbolic over $F(q)$ by [Lam 2005, Theorem X.4.14]. Since

$$i(q_{L(q)}) = i\big(q_{F(q)(q \perp \langle -1 \rangle)}\big),$$

we may invoke the Cassels–Pfister subform theorem [Lam 2005, Theorem X.4.5] to establish that q is not hyperbolic over $L(q)$, whereby it follows that q is not similar to a Pfister form over L by [Lam 2005, Theorem X.4.14]. Hence, we may assume, without loss of generality, that $1 \in D_F(q)$.

Let $K = F$ if q is a group form over F that is not round. Otherwise, let

$$L_0 = F((x))(q \otimes \langle 1, -x \rangle).$$

Since q is not similar to a Pfister form over F, it follows that q is not a group form over L_0 by Theorem 2.7. Hence, we have that $H_{L_0}(q) \setminus D_{L_0}(q)$ is a nonempty set by Corollary 2.3 (in particular, the proof of Theorem 2.7 establishes that $x \in H_{L_0}(q) \setminus D_{L_0}(q)$). Consider the set

$$Q(L_0) = \{q \perp \langle -a\rangle \mid a \in H_{L_0}(q) \setminus D_{L_0}(q)\},$$

which is a nonempty set of anisotropic forms over L_0. For $i \geqslant 0$, we inductively define L_{i+1} to be the compositum of all function fields of forms in $Q(L_i)$. For all L_i and $a \in H_{L_i}(q) \setminus D_{L_i}(q)$, we have that q is anisotropic over $L_i(q \perp \langle -a\rangle)$ by Theorem 1.3. Hence, letting $K = \bigcup_{i=0}^{\infty} L_i$, it follows that q_K is anisotropic. Moreover, as $H_K(q) = D_K(q)$ by construction, it follows that q_K is a group form by Corollary 2.3.

It remains to show that q_K is not round. By construction, we have that $q \perp -xq$ is isotropic over L_0, whereby $x \in H_K(q)$. Suppose, for the sake of contradiction, that $q \perp -xq$ is hyperbolic over K. Hence, for some $i \in \mathbb{N} \cup \{0\}$, there exists an extension L_i'/L_i and $a \in (L_i')^\times$ such that $((q \perp -xq)_{L_i'})_{\mathrm{an}}$ is hyperbolic over $L_i'(q \perp \langle -a\rangle)$. As a consequence of Elman and Lam's representation theorem [1972, Theorem 1.4], there exists a form p over L_i' such that $\dim p < \dim q$ and

$$((q \perp -xq)_{L_i'})_{\mathrm{an}} \simeq p \perp -xp$$

(see [Hoffmann 1996, Lemma 3.1]). Hence, invoking [Karpenko and Merkurjev 2003, Corollary 4.2], it follows that

$$\dim(p \perp -xp) - i\big((p \perp -xp)_{L_i'(q\perp\langle -a\rangle)}\big) \geqslant \dim(q \perp \langle -a\rangle) - i_1(q \perp \langle -a\rangle).$$

As $i_1(q \perp \langle -a\rangle) = 1$ by Theorem 1.3 and [Lam 2005, Exercise I.16], it follows that

$$\dim(p \perp -xp) - \frac{\dim(p \perp -xp)}{2} \geqslant \dim q,$$

in contradiction to the fact that $\dim p < \dim q$. Hence, having obtained our desired contradiction, we may conclude that $x \notin G_K(q)$, whereby q_K is not round by Corollary 2.5. □

Remark 3.9. Scully [2016b, Main Theorem] recently established that, for p and q anisotropic forms over F of dimension at least two with $2^i < \dim q \leqslant 2^{i+1}$, if $p_{F(q)}$ is hyperbolic, then $\dim p = 2^{i+1}k$ for some $k \in \mathbb{N}$. One may invoke this result to shorten the final component of the above proof.

Per Example 3.7, in order to answer Question 1.1 in the case where $\dim q \neq 2^n$ for $n \in \mathbb{N}$, we require some additional assumptions regarding the form q over F. Orderings are a useful tool in this regard, with their behaviour with respect to function-field extensions being governed by the following result due to Elman, Lam and Wadsworth [Elman et al. 1979, Theorem 3.5] and, independently, Knebusch [Gentile and Shapiro 1978, Lemma 10].

Theorem 3.10. *Let q be a form of dimension at least two over a real field F. An ordering P of F extends to $F(q)$ if and only if q is indefinite at P.*

Invoking Theorem 3.10, we can resolve Question 1.1 in the case where q is a positive-definite form over a real field.

Proposition 3.11. *Let F be a real field. Let q be a form over F that is positive definite with respect to some ordering of F. If q is not similar to a Pfister form over F, there exists an extension K/F such that q_K is an anisotropic group form that is not round.*

Proof. Let P be an ordering of F such that q is positive definite with respect to P. If $1 \notin D_F(q)$, we may consider q as a form over $L = F(q \perp \langle -1\rangle)$, whereby P is an ordering of L by Theorem 3.10 and $1 \in D_L(q)$. Per the proof of Theorem 3.8 (iii), q is not hyperbolic over $L(q)$, and thus is not similar to a Pfister form over L. Hence, we may assume, without loss of generality, that $1 \in D_F(q)$.

Let $K = F$ if q is a group form over F that is not round. Otherwise, let

$$L_0 = F((x))(q \otimes \langle 1, -x\rangle).$$

Since q is not similar to a Pfister form over F, it follows that q is not a group form over L_0 by Theorem 2.7. Hence, we have that $H_{L_0}(q) \setminus D_{L_0}(q)$ is a nonempty set by Corollary 2.3. Moreover, by Theorem 3.10, there exist orderings of L_0 such that q is positive definite.

Let L/L_0 be an extension such that q is positive definite with respect to an ordering P of L and $H_L(q) \setminus D_L(q)$ is not empty. Let $a \in H_L(q) \setminus D_L(q)$, whereby $a \in P$ and the form $q \perp \langle -a\rangle$ has signature $\dim q - 1$ with respect to P. As P extends to $L(q \perp \langle -a\rangle)$, by Theorem 3.10, it thus follows that $i_1(q \perp \langle -a\rangle) = 1$. Hence, q is anisotropic over $L(q \perp \langle -a\rangle)$ by Theorem 1.4 (i).

Equipped with the above, we may now proceed with the argument in the proof of Theorem 3.8 (iii) to establish the existence of an extension K/L_0 such that q_K is an anisotropic group form with $x \in D_K(q) \setminus G_K(q)$. □

As discussed in [O'Shea 2016], many properties of a form q over F are shared by its generic Pfister multiple $q \otimes \langle 1, x\rangle$ over $F((x))$. Invoking Proposition 2.2, we may show that this is also the case with respect to the group property.

Proposition 3.12. *A form q is a group form over F if and only if $q \otimes \langle 1, x\rangle$ is a group form over $K = F((x))$.*

Proof. Without loss of generality, we may assume that q is anisotropic over F.

Let q be a group form over F. As every nonzero square class in K can be represented by a or ax for some $a \in F^\times$, we first suppose that

$$q \otimes \langle 1, x\rangle \perp -ax(q \otimes \langle 1, x\rangle)$$

is isotropic over K for $a \in F^\times$. Then, because $x \in D_K(\langle 1, x\rangle)$, it follows that $q \otimes \langle 1, x\rangle \perp -a(q \otimes \langle 1, x\rangle)$ is isotropic over K in this case. Thus, supposing that $q \otimes \langle 1, x\rangle \perp -a(q \otimes \langle 1, x\rangle)$ is isotropic over K for $a \in F^\times$, it suffices to show that

$q \otimes \langle 1, x\rangle \perp \langle -a\rangle$ and $q \otimes \langle 1, x\rangle \perp \langle -ax\rangle$ are isotropic over K in accordance with Proposition 2.2. Invoking Lemma 1.2, it follows that $q \perp -aq$ is isotropic over F, whereby $q \perp \langle -a\rangle$ is isotropic over F by Proposition 2.2. Hence, it follows that $q \otimes \langle 1, x\rangle \perp \langle -a\rangle$ and $q \otimes \langle 1, x\rangle \perp \langle -ax\rangle$ are isotropic over K, as desired.

Conversely, let $q \otimes \langle 1, x\rangle$ be a group form over K. Letting $a \in F^\times$ be such that $q \perp -aq$ is isotropic over F, it follows that $q \otimes \langle 1, x\rangle \perp -a(q \otimes \langle 1, x\rangle)$ is isotropic over K, whereby $q \otimes \langle 1, x\rangle \perp -a$ is isotropic over K by Proposition 2.2. Hence, $q \perp \langle -a\rangle$ is isotropic over F by Lemma 1.2, whereby q is a group form over F by Proposition 2.2. □

Combining Proposition 3.12 with Proposition 3.2 (ii) and [O'Shea 2016, Proposition 3.11], we obtain the following corollary.

Corollary 3.13. *Let q be an anisotropic group form over F that is not round. Then $q \otimes \langle 1, x\rangle$ is an anisotropic group form over $K = F((x))$ that is not round.*

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DUAL OPERATOR ALGEBRAS CLOSE TO INJECTIVE VON NEUMANN ALGEBRAS

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We prove that if a nonselfadjoint dual operator algebra admitting a normal virtual diagonal and an injective von Neumann algebra are close enough for the Kadison–Kastler metric, then they are similar. The bound explicitly depends on the norm of the normal virtual diagonal. This is inspired by E. Christensen's work on perturbation of operator algebras and is related to a conjecture of G. Pisier on nonselfadjoint amenable operator algebras.

1. Introduction

The starting point of this paper is the conjunction of perturbation theory of operator algebras and a conjecture on amenable nonselfadjoint operator algebras. Let us first recall this conjecture and propose a dual version of it, then we will explain the connection with our main result.

A conjecture raised by G. Pisier asserts that a nonselfadjoint amenable operator algebra $\mathcal{A}$ should be similar to a nuclear C^*-algebra (i.e., there is an invertible operator S such that $S\mathcal{A}S^{-1}$ is a C^*-algebra). Recently, this conjecture has been proved for commutative amenable operator algebras in [Marcoux and Popov 2016]. It generalizes [Choi 2013; Willis 1995]; see also [Marcoux 2008] for more details around this conjecture. A nonseparable counter-example to Pisier's conjecture has been found [Choi et al. 2014] but the separable case remains open.

In his memoir, B.E. Johnson [1972] characterized amenability of Banach algebras by the existence of a virtual diagonal. Recall that injectivity for von Neumann algebras can be characterized by the existence of a normal virtual diagonal (in the sense of E.G. Effros [1988], see Section 2C below for details). Therefore, a dual version of Pisier's conjecture would be:

Conjecture. *A unital dual operator algebra admitting a normal virtual diagonal should be similar to an injective von Neumann algebra. In that case, it is expected that the similarity constant is controlled by a nondecreasing function of the norm of*

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the normal virtual diagonal. Note that one advantage of this conjecture is to avoid the separability question.

In 1972, R.V. Kadison and D. Kastler defined a metric d on the collection of all subspaces of the bounded operators on a fixed Hilbert space (see Section 2A). They conjectured [1972] that sufficiently close C^*-algebras are necessarily unitarily conjugated. A great amount of work around this conjecture has been done since then (see [Christensen et al. 2012] for a nice introduction on this topic). Notably, E. Christensen proved the conjecture for the class of type I von Neumann algebras [Christensen 1975] and for the class of injective von Neumann algebras [Christensen 1977]. Very recently, Kadison and Kastler's conjecture has been proved for the class of separable nuclear C^*-algebras in [Christensen et al. 2012] (see also [Christensen et al. 2010b]). The recent paper [Cameron et al. 2014] is an important breakthrough beyond amenability. Let us state Christensen's first result on perturbation of injective von Neumann algebras (this result has subsequently been improved in [Christensen 1980]):

Theorem 1 [Christensen 1977, Theorem 4.1]. *Let $\mathcal{M}, \mathcal{N}$ be two von Neumann subalgebras of a fixed $\mathbb{B}(H)$. We suppose that $\mathcal{M}$ has Schwartz's property (P) and $\mathcal{N}$ has the extension property. If $d(\mathcal{M},\mathcal{N}) < \frac{1}{169}$, then there is a unitary U in the von Neumann algebra generated by $\mathcal{M} \cup \mathcal{N}$ such that $U\mathcal{M}U^* = \mathcal{N}$. Moreover, $\|U - I_H\| \leq 19d(\mathcal{M},\mathcal{N})^{1/2}$.*

It is now well known, after the work of A. Connes [1976; 1978] and U. Haagerup [1985], that Schwartz's property (P), the extension property and injectivity (and thus the existence of a normal virtual diagonal) are equivalent conditions for von Neumann algebras.

The aforementioned conjecture leads to the following question: can we replace, in the preceding theorem, $\mathcal{M}$ by a unital nonselfadjoint dual operator algebra admitting a normal virtual diagonal? In other words, is the selfadjointness hypothesis on $\mathcal{M}$ necessary? Indeed, assume for a moment that our conjecture is true, then there would be an invertible S such that $S\mathcal{M}S^{-1}$ is an injective von Neumann algebra. Moreover,

$$d(\mathcal{M}, S\mathcal{M}S^{-1}) \leq 2(1 + \|S\|\|S^{-1}\|)\|S - I_H\|$$

(and this last quantity is controlled by a nondecreasing function of the norm of the normal virtual diagonal). Hence, if $d(\mathcal{M},\mathcal{N})$ is small enough such that the following strict inequality holds

$$d(\mathcal{N}, S\mathcal{M}S^{-1}) \leq d(\mathcal{M},\mathcal{N}) + 2(1 + \|S\|\|S^{-1}\|)\|S - I_H\| < \tfrac{1}{169},$$

then (from Theorem 1 above) the injective von Neumann algebras $\mathcal{N}$ and $S\mathcal{M}S^{-1}$ would be unitarily conjugated, so $\mathcal{M}$ and $\mathcal{N}$ would be similar. Therefore, it is not incongruous to try to replace $\mathcal{M}$ by a unital dual operator algebra admitting a normal virtual diagonal.

In this paper, we prove:

Theorem 2. *Let $\mathcal{M}, \mathcal{N} \subset \mathbb{B}(H)$ be two unital w^*-closed operator algebras. Suppose that $\mathcal{M}$ admits a normal virtual diagonal u and $\mathcal{N}$ is an injective von Neumann algebra. If $d(\mathcal{M}, \mathcal{N}) < 1/(656\|u\|)$, then there exists an invertible operator S in the w^*-closed algebra generated by $\mathcal{M} \cup \mathcal{N}$ such that $S\mathcal{M}S^{-1} = \mathcal{N}$. Moreover, $\|S - I_H\| \leq 656\|u\|d(\mathcal{M}, \mathcal{N})$.*

The proof of Theorem 2 is the consequence of Theorem 5.2 and Lemma 5.4. Note that von Neumann algebras enjoy a self-improvement phenomenon; if a von Neumann algebra admits a normal virtual diagonal then it admits a normal virtual diagonal of norm 1, see [Haagerup 1985; Effros 1988; Effros and Kishimoto 1987] (self-improvement phenomena are frequent for selfadjoint algebras, for instance nuclearity constant and exactness constant). This may explain why in Theorem 1 the bound is a universal constant, whereas in Theorem 2, the bound depends on the feature of the nonselfadjoint algebra involved. Moreover, from Theorem 7.4.18 (1) in [Blecher and Le Merdy 2004] and Remark 2.1 below, if a unital dual operator algebra admits a normal virtual diagonal of norm 1, then it is necessarily a von Neumann algebra (no similarity is needed in this extreme case). Hence, Theorem 1 corresponds exactly to the case $\|u\|$ equals 1 in Theorem 2 (as the unitary U is obtained by taking the polar decomposition of S, see Lemma 2.7 in [Christensen 1975]). Our bound in this special case is not as good as Christensen's one, but the important point is that we have been able to remove the selfadjointness hypothesis on $\mathcal{M}$. This is not a minor modification; knowing that nonselfadjoint algebras are less rigid than selfadjoint ones (no order structure for instance) and fewer tools are available (no continuous or Borel functional calculus), our proof requires new ingredients from operator space theory in particular the normal Haagerup tensor product of dual operator spaces.

Now let us sketch the main lines of our proof. There are three steps (as in Christensen's work [1977]):

Step 1. Find a linear isomorphism, between the two algebras, which is close to the identity representation.

Step 2. Find an algebra homomorphism close to the previous linear isomorphism.

Step 3. Prove this algebra homomorphism is similar to the identity representation.

For the first step, as $\mathcal{N}$ is injective, one just has to take the restriction to $\mathcal{M}$ of a completely contractive projection onto $\mathcal{N}$. This gives a linear isomorphism T from $\mathcal{M}$ onto $\mathcal{N}$ which is close to the identity representation of $\mathcal{M}$. But in order to apply certain averaging procedure for Step 2, we need a w^*-continuous linear isomorphism. For this, Christensen used Tomiyama's decomposition into normal and singular parts of bounded linear maps defined on von Neumann algebras. But when $\mathcal{M}$ is nonselfadjoint, such decomposition is not available. Hence, we have to consider the normal part of T^{-1}. This w^*-continuous linear isomorphism from $\mathcal{N}$ onto $\mathcal{M}$ is not necessarily completely positive, and moreover the target algebra $\mathcal{M}$ is not necessarily selfadjoint, thus we can not use Christensen's averaging trick [1977, Lemma 3.3] to accomplish the second step. The idea is to turn to Banach algebras results and operator spaces tools. More precisely, we will use a dual operator space version of a B.E. Johnson theorem [1988] on almost multiplicative maps. Indeed, the issue here is that we need to preserve the w^*-continuity, but we cannot use the normal projective tensor product of dual Banach spaces (as we could not check its associativity, see Section 3). This second step will force us to work with the normal Haagerup tensor product of dual operator spaces.

Finally, the third step, which is related to a more general problem on neighboring representations (already mentioned in [Kadison and Kastler 1972]), is done by an averaging technique. However, because of the second step, we have had to work in the operator space category and as a consequence we had to assume that the algebras $\mathbb{M}_n(\mathcal{M})$ nearly embed in $\mathbb{M}_n(\mathcal{N})$ uniformly in n (see the notion of near cb-inclusion defined in Section 2A). As an intermediate result, we prove a perturbation theorem with a near cb-inclusion assumption (see Theorem 5.2). Therefore, our final task is to notice that the existence of a normal virtual diagonal is an "automatic near cb-inclusion" condition (see Lemma 5.4).

To conclude this introduction, let us mention that an engaging objective would be to prove an analog of Theorem 2 when both algebras are nonselfadjoint (for details see Remark 5.6). We also should mention that after the writing and circulation of our paper, L. Dickson has obtained an improvement of our Theorem 2 (see [Dickson 2014, Theorem 6.1.1]). His result is interesting because he was able to get rid of the normal virtual diagonal hypothesis. His proof uses a variant of Johnson's result on almost multiplicative maps (like our proof) and also the characterization of injective von Neumann algebras as the w^*-closure of a net of finite-dimensional subalgebras. This is a strong approximation property characterization, but such a characterization is far from being available for nonselfadjoint operator algebras admitting normal virtual diagonal. Hence, unfortunately, we can not use Dickson's techniques for our perturbation problem (mentioned in Remark 5.6) when both algebras are nonselfadjoint.

2. Preliminaries

For background on completely bounded maps, operator space theory and nonselfadjoint algebra theory, the reader is referred to [Blecher and Le Merdy 2004; Effros and Ruan 2000; Paulsen 2002; Pisier 2003], especially Section 2.7 in [Blecher and Le Merdy 2004] for background on dual operator algebras.

2A. ***Perturbation theory.*** We first recall definitions and notations commonly used in perturbation theory of operator algebras (see, e.g., [Christensen et al. 2010a]). Let H be a Hilbert space, and $\mathbb{B}(H)$ be the von Neumann algebra of all bounded operators on H. Let $\mathcal{E}, \mathcal{F}$ be two subspaces of $\mathbb{B}(H)$. We denote by d the Kadison–Kastler metric, i.e., $d(\mathcal{E}, \mathcal{F})$ denotes the Hausdorff distance between the unit balls of $\mathcal{E}$ and $\mathcal{F}$. More explicitly,

$$d(\mathcal{E},\mathcal{F}) = \inf\big\{\gamma > 0 : \text{for all } x \in B_{\mathcal{E}},\ \text{there exists } x' \in B_{\mathcal{F}},\ \|x - x'\| < \gamma \text{ and} \\ \text{for all } y \in B_{\mathcal{F}},\ \text{there exists } y' \in B_{\mathcal{E}},\ \|y - y'\| < \gamma\big\},$$

where $B_{\mathcal{E}}$ (respectively, $B_{\mathcal{F}}$) denotes the unit ball of $\mathcal{E}$ (respectively, $\mathcal{F}$). Let $\gamma > 0$, then we write $\mathcal{E} \subseteq^{\gamma} \mathcal{F}$ if for any x in the unit ball of $\mathcal{E}$, there exists y in $\mathcal{F}$ such that

$$\|x - y\| \le \gamma.$$

We also write $\mathcal{E} \subset^{\gamma} \mathcal{F}$ if there exists $\gamma' < \gamma$ such that $\mathcal{E} \subseteq^{\gamma'} \mathcal{F}$. We will also need the notion of *near* cb-*inclusion*. As usual in operator space theory, $\mathbb{M}_n(\mathcal{E})$, the subspace of $n \times n$ matrices with coefficients in $\mathcal{E}$ is normed by the identification $\mathbb{M}_n(\mathcal{E}) \subset \mathbb{M}_n(\mathbb{B}(H)) = \mathbb{B}(\ell_n^2 \otimes H)$. We write

$$\mathcal{E} \subseteq^{\gamma}_{\text{cb}} \mathcal{F}$$

if $\mathbb{M}_n(\mathcal{E}) \subseteq^{\gamma} \mathbb{M}_n(\mathcal{F})$, for all n.

2B. ***The normal projective tensor product and normal Haagerup tensor product.*** For dual operator spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $(\mathcal{X} \otimes_h \mathcal{Y})^*_\sigma$ the space of all completely bounded bilinear forms which are separately w^*-continuous (see [Blecher and Le Merdy 2004, Paragraph 1.5.4] for the definition of completely bounded bilinear maps). The normal Haagerup tensor product, denoted $\otimes_{\sigma h}$, can be defined as

$$\mathcal{X} \otimes_{\sigma h} \mathcal{Y} = ((\mathcal{X} \otimes_h \mathcal{Y})^*_\sigma)^*, \tag{2-1}$$

see [Blecher and Le Merdy 2004, Paragraph 1.6.8]. The normal Haagerup tensor product is characterized by the following universal property: $\mathcal{X} \otimes \mathcal{Y}$ is w^*-dense in $\mathcal{X} \otimes_{\sigma h} \mathcal{Y}$ and, for any dual operator space $\mathbb{Z}$, for any w^*-continuous completely contractive bilinear map $B : \mathcal{X} \times \mathcal{Y} \to \mathbb{Z}$, there exists a (unique) w^*-continuous completely contractive linear map $\tilde{B} : \mathcal{X} \otimes_{\sigma h} \mathcal{Y} \to \mathbb{Z}$ such that $\tilde{B}(x \otimes y) = B(x, y)$, for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. We will also need the normal projective tensor product $\widehat{\otimes}_\sigma$ of dual Banach spaces. If $\mathcal{X}$ and $\mathcal{Y}$ are dual Banach spaces,

$$\mathcal{X} \widehat{\otimes}_\sigma \mathcal{Y} = ((\mathcal{X} \widehat{\otimes} \mathcal{Y})^*_\sigma)^*,$$

where $(\mathcal{X} \widehat{\otimes} \mathcal{Y})^*_\sigma$ denotes the space of all bounded bilinear forms on $\mathcal{X} \times \mathcal{Y}$ which are separately w^*-continuous. The normal projective tensor product enjoys a similar universal property to the normal Haagerup tensor product, but for separately w^*-continuous bounded bilinear maps instead of separately w^*-continuous completely bounded (for von Neumann algebras, the projective normal tensor product appeared for instance in [Effros 1988] under the name binormal projective tensor product). These two tensor products are "functorial" in the sense that, if $L_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$, are bounded (respectively, completely bounded) w^*-continuous linear maps between dual Banach spaces (dual operator spaces), then there is a unique bounded (completely bounded) w^*-continuous linear map

$$L_1 \widehat{\otimes}_\sigma L_2 : \mathcal{X}_1 \widehat{\otimes}_\sigma \mathcal{X}_2 \to \mathcal{Y}_1 \widehat{\otimes}_\sigma \mathcal{Y}_2$$

($L_1 \otimes_{\sigma h} L_2 : \mathcal{X}_1 \otimes_{\sigma h} \mathcal{X}_2 \to \mathcal{Y}_1 \otimes_{\sigma h} \mathcal{Y}_2$) extending $L_1 \otimes L_2$. Moreover,

$$\|L_1 \widehat{\otimes}_\sigma L_2\| \leq \|L_1\| \|L_2\|$$

($\|L_1 \otimes_{\sigma h} L_2\|_{\mathrm{cb}} \leq \|L_1\|_{\mathrm{cb}} \|L_2\|_{\mathrm{cb}}$).

The main difference between these two tensor products is that the normal Haagerup tensor product is associative (see Lemma 2.2 in [Blecher and Kashyap 2008]), whereas the normal projective tensor product does not seem to be associative in general (this difference will have an important consequence for us in Section 3).

2C. ***Normal virtual diagonals and normal virtual h-diagonals.*** Normal virtual diagonals appeared implicitly in [Haagerup 1985] and explicitly in [Effros 1988] (see p. 147 thereof). In this paper, we also need the notion of normal virtual h-diagonal (called reduced normal virtual diagonal in [Effros 1988], see also [Blecher and Le Merdy 2004, Paragraph 7.4.8] for more details). Let us just recall this notion. Replacing the normal Haagerup tensor product by the normal projective tensor product in the following, one can analogously obtain the definition of normal virtual diagonal. Let $\mathcal{M}$ be a unital dual operator algebra, and let us recall the $\mathcal{M}$-bimodule structure of $\mathcal{M} \otimes_{\sigma h} \mathcal{M}$. Letting $\psi \in (\mathcal{M} \otimes_h \mathcal{M})^*_\sigma$ and $a, b, c, d \in \mathcal{M}$,

$$\langle b \cdot \psi \cdot a, c \otimes d\rangle = \psi(ac, db).$$

Hence by duality, one can define actions of $\mathcal{M}$ on $\mathcal{M} \otimes_{\sigma h} \mathcal{M} = ((\mathcal{M} \otimes_h \mathcal{M})^*_\sigma)^*$. One can check that these actions are determined on the elementary tensors by

$$a \cdot (c \otimes d) \cdot b = ac \otimes db.$$

On a dual operator algebra, the multiplication is a separately w^*-continuous completely contractive bilinear map [Blecher and Le Merdy 2004, Proposition 2.7.4 (1)]. Consequently, it induces a w^*-continuous complete contraction,

$$\mathrm{m}_{\sigma h} : \mathcal{M} \otimes_{\sigma h} \mathcal{M} \to \mathcal{M}.$$

A *normal virtual h-diagonal* for $\mathcal{M}$ is an element $u \in \mathcal{M} \otimes_{\sigma h} \mathcal{M}$ satisfying

(C1) $m \cdot u = u \cdot m$ for any $m \in \mathcal{M}$,

(C2) $\mathrm{m}_{\sigma h}(u) = 1$.

Note that condition (C2) implies that the norm of a normal virtual h-diagonal is always greater than or equal to 1.

Remark 2.1. Note that the inclusion $(\mathcal{X} \otimes_h \mathcal{Y})^*_\sigma \subset (\mathcal{X} \widehat{\otimes} \mathcal{Y})^*_\sigma$ induces, by duality, a contraction from $\mathcal{M} \widehat{\otimes}_\sigma \mathcal{M}$ into $\mathcal{M} \otimes_{\sigma h} \mathcal{M}$ and this contraction sends normal virtual diagonals into normal virtual h-diagonals. Consequently, if $\mathcal{M}$ admits a normal virtual diagonal, it admits a normal virtual h-diagonal.

3. B.E. Johnson's theorem revisited

The aim of this section is to find a solution to the second step mentioned in the Introduction. Johnson [1988] proved that an approximately multiplicative map defined on an amenable Banach algebra is close to an actual algebra homomorphism. His result is the Banach algebraic version of an earlier result due to D. Kazhdan [1982] for amenable groups (see also [Burger et al. 2013]). If L is a linear map between operator algebras $\mathcal{M}$ and $\mathcal{N}$, we denote by $L^\vee : \mathcal{M} \times \mathcal{M} \to \mathcal{N}$ the bilinear map defined by

$$L^\vee(x, y) = L(xy) - L(x)L(y).$$

This enables us to measure the defect of multiplicativity of L.

In our present case, we have to take into account the dual operator space structure of our algebras. Starting from a w^*-continuous linear map from $\mathcal{M}$ into $\mathcal{N}$, we must obtain a w^*-continuous algebra homomorphism. This will force us to work in the category of operator spaces. The proof of Theorem 3.1 in [Johnson 1988] is by induction, the algebra homomorphism is the limit (in operator norm) of a sequence of linear maps with multiplicativity defect tending to zero. The problem is that these linear maps are defined using the w^*-topology of the target algebra (see equation $(*)$ in the proof of [Johnson 1988, Theorem 3.1]). Here, to justify the w^*-continuity of these linear maps, we must consider a trilinear map defined on $\mathcal{M} \times \mathcal{M} \times \mathcal{M}$; see (3-1) below. But the normal projective tensor product does not seem associative. To circumvent this difficulty, we will instead work with the normal Haagerup tensor product, which is associative [Blecher and Kashyap 2008, Lemma 2.2]. As a consequence, we have to control the cb-norm of the bilinear map $L^\vee$.

Remark 3.1. Actually, this difficulty concerning the associativity of the normal projective tensor product has already been encountered in disguise. The main issue in [Effros 1988] is that one cannot check whether the Banach $\mathcal{M}$-bimodule $\mathcal{M} \widehat{\otimes}_\sigma \mathcal{M}$ is normal or not. But if one assumes that the normal projective tensor product is associative, then it is easy to check that $\mathcal{M} \widehat{\otimes}_\sigma \mathcal{M}$ is a normal bimodule.

Theorem 3.2. *Let $\mathcal{M}, \mathcal{N}$ be two unital dual operator algebras. We suppose that $\mathcal{M}$ has a normal virtual h-diagonal $u \in \mathcal{M} \otimes_{\sigma h} \mathcal{M}$. Then, for any $\varepsilon \in (0, 1)$, for any $\mu > 0$, there exists $\delta > 0$ such that: for every unital w^*-continuous linear map*

$L:\mathcal{M}\to\mathcal{N}$ satisfying $\|L\|_{\mathrm{cb}}\le\mu$ and $\|L^\vee\|_{\mathrm{cb}}\le\delta$, there is a unital w^*-continuous completely bounded algebra homomorphism $\pi:\mathcal{M}\to\mathcal{N}$ such that $\|L-\pi\|_{\mathrm{cb}}\le\varepsilon$.

Proof. Let $\varepsilon\in(0,1)$, $\mu>0$ and let L be a unital w^*-continuous linear map from $\mathcal{M}$ into $\mathcal{N}$ such that $\|L\|_{\mathrm{cb}}\le\mu$. The trilinear map

$$(x,y,z)\in\mathcal{M}\times\mathcal{M}\times\mathcal{M}\mapsto L(x)L^\vee(y,z)\in\mathcal{N} \tag{3-1}$$

is separately w^*-continuous and completely bounded. By the universal property of the normal Haagerup tensor product, it extends to a w^*-continuous completely bounded linear map

$$\Lambda_L:\mathcal{M}\otimes_{\sigma h}\mathcal{M}\otimes_{\sigma h}\mathcal{M}\to\mathcal{N}$$

such that

$$\Lambda_L(x\otimes y\otimes z)=L(x)L^\vee(y,z)$$

and $\|\Lambda_L\|_{\mathrm{cb}}\le\|L\|_{\mathrm{cb}}\|L^\vee\|_{\mathrm{cb}}$. By definition and associativity of the normal Haagerup tensor product, the linear map

$$m\in\mathcal{M}\mapsto u\otimes m\in(\mathcal{M}\otimes_{\sigma h}\mathcal{M})\otimes_{\sigma h}\mathcal{M}=\mathcal{M}\otimes_{\sigma h}\mathcal{M}\otimes_{\sigma h}\mathcal{M}$$

is w^*-continuous; see (2-1). We can define $R:\mathcal{M}\to\mathcal{N}$ by

$$R(m)=\Lambda_L(u\otimes m)$$

which is w^*-continuous and

$$\|R\|_{\mathrm{cb}}\le\|u\|\,\|L\|_{\mathrm{cb}}\|L^\vee\|_{\mathrm{cb}}. \tag{3-2}$$

As $u\in\mathcal{M}\otimes_{\sigma h}\mathcal{M}$, there is a net $(u_t)_t$ in $\mathcal{M}\otimes\mathcal{M}$ converging to u in the w^*-topology of $\mathcal{M}\otimes_{\sigma h}\mathcal{M}$. For any t, there are finite families $(a_k^t)_k$, $(b_k^t)_k$ of elements in $\mathcal{M}$ such that

$$u_t=\sum_k a_k^t\otimes b_k^t.$$

Now fixing $m\in\mathcal{M}$, once again by definition and associativity of the normal Haagerup tensor product, the linear map

$$v\in\mathcal{M}\otimes_{\sigma h}\mathcal{M}\mapsto v\otimes m\in(\mathcal{M}\otimes_{\sigma h}\mathcal{M})\otimes_{\sigma h}\mathcal{M}=\mathcal{M}\otimes_{\sigma h}\mathcal{M}\otimes_{\sigma h}\mathcal{M}$$

is w^*-continuous as well. Hence, using the w^*-continuity of Λ_L, we obtain

$$R(m)=w^*-\lim_t\Lambda_L(u_t\otimes m)=w^*-\lim_t\sum_k L(a_k^t)L^\vee(b_k^t,m).$$

From this point, we just need to check that the computations of [Johnson 1988, Theorem 3.1] remain valid with matrix coefficients. Fix $n\in\mathbb{N}$, let x, y be in the unit ball of $\mathbb{M}_n(\mathcal{M})$ (in the following computation, I_n denotes the identity matrix in $\mathbb{M}_n$, and the other subscripts n denote the n-th ampliation of a linear or bilinear map), then as in [Johnson 1988], we have

$$\begin{aligned}
&(L+R)_n^{\vee}(x,y)=\\
&\qquad L_n^{\vee}(x,y)-w^*-\lim_t\sum_k L_n(I_n\otimes a_k^t b_k^t)L_n^{\vee}(x,y)\\
&\qquad -R_n(x)R_n(y)\\
&\qquad +w^*-\lim_t\sum_k L_n^{\vee}(x,I_n\otimes a_k^t)L_n^{\vee}(I_n\otimes b_k^t,y)\\
&\qquad +w^*-\lim_t\sum_k L_n^{\vee}(I_n\otimes a_k^t,I_n\otimes b_k^t)L_n^{\vee}(x,y)\\
&\qquad +w^*-\lim_t\sum_k L_n(I_n\otimes a_k^t)L_n((I_n\otimes b_k^t)xy)-L_n(x(I_n\otimes a_k^t))L_n((I_n\otimes b_k^t)y)\\
&\qquad -w^*-\lim_t\sum_k(L_n(I_n\otimes a_k^t)L_n((I_n\otimes b_k^t)x)-L_n(x(I_n\otimes a_k^t))L_n(I_n\otimes b_k^t))L_n(y).
\end{aligned}$$

To evaluate the norm of $(L+R)_n^{\vee}$, we treat each line of the right-hand side successively. As u is a normal virtual h-diagonal, $w^*-\lim_t\sum_k a_k^t b_k^t=1$. But L is unital and w^*-continuous, so

$$w^*-\lim_t\sum_k L_n(I_n\otimes a_k^t b_k^t)=1$$

and the first line of the right-hand side is 0. Clearly, the norm of the term in the second line is bounded by $\|R\|_{\mathrm{cb}}^2$. Now let us show that the norm of the term in the third line is bounded by $\|u\|\|L^{\vee}\|_{\mathrm{cb}}^2$. The quadrilinear map

$$(x,y,z,t)\in\mathcal{M}\times\mathcal{M}\times\mathcal{M}\times\mathcal{M}\mapsto L^{\vee}(x,y)L^{\vee}(z,t)\in\mathcal{N} \tag{3-3}$$

is separately w^*-continuous and completely bounded. By the universal property of the normal Haagerup tensor product, it extends to a w^*-continuous completely bounded linear map $\Gamma_L:\mathcal{M}\otimes_{\sigma h}\mathcal{M}\otimes_{\sigma h}\mathcal{M}\otimes_{\sigma h}\mathcal{M}\to\mathcal{N}$ such that

$$\Gamma_L(x\otimes y\otimes z\otimes t)=L^{\vee}(x,y)L^{\vee}(z,t)$$

and $\|\Gamma_L\|_{\mathrm{cb}}\le\|L^{\vee}\|_{\mathrm{cb}}\|L^{\vee}\|_{\mathrm{cb}}$. The bilinear map

$$B:(x,y)\in\mathcal{M}\times\mathcal{M}\mapsto x\otimes u\otimes y\in\mathcal{M}\otimes_{\sigma h}\mathcal{M}\otimes_{\sigma h}\mathcal{M}\otimes_{\sigma h}\mathcal{M}$$

is separately w^*-continuous and $\|B\|_{\mathrm{cb}}\le\|u\|$. The bilinear map $\Gamma_L\circ B:\mathcal{M}\times\mathcal{M}\to\mathcal{N}$ is also separately w^*-continuous and

$$\|\Gamma_L\circ B\|_{\mathrm{cb}}\le\|\Gamma_L\|_{\mathrm{cb}}\,\|B\|_{\mathrm{cb}}\le\|u\|\|L^{\vee}\|_{\mathrm{cb}}^2.$$

We claim that the term of the third line is the n-th ampliation of the bilinear map $\Gamma_L\circ B$ applied to x, y in the unit ball of $\mathbb{M}_n(\mathcal{M})$ (and this gives the desired estimate).

Note first that

$$B_n(x,y)=\Big[\sum_l x_{il}\otimes u\otimes y_{lj}\Big]_{i,j}=w^*-\lim_t\sum_k\Big[\sum_l x_{il}\otimes a_k^t\otimes b_k^t\otimes y_{lj}\Big]_{i,j}$$

and also that $(L^\vee)_n=(L_n)^\vee$ and

$$(L^\vee)_n(x,I_n\otimes a_k^t)=\Big[\sum_l L^\vee(x_{il},(I_n\otimes a_k^t)_{lj})\Big]_{i,j}=\Big[L^\vee(x_{ij},a_k^t)\Big]_{i,j}$$

and similarly

$$(L^\vee)_n(I_n\otimes b_k^t,y)=\big[L^\vee(b_k^t,y_{ij})\big]_{i,j}.$$

Using these computations, we can prove our claim:

$$\begin{aligned}(\Gamma_L\circ B)_n(x,y)&=(\Gamma_L)_n(B_n(x,y))\\&=(\Gamma_L)_n\Big(w^*-\lim_t\sum_k\Big[\sum_l x_{il}\otimes a_k^t\otimes b_k^t\otimes y_{lj}\Big]_{i,j}\Big)\\&=w^*-\lim_t\sum_k\Big[\sum_l\Gamma_L(x_{il}\otimes a_k^t\otimes b_k^t\otimes y_{lj})\Big]_{i,j}\\&=w^*-\lim_t\sum_k\Big[\sum_l L^\vee(x_{il},a_k^t)L^\vee(b_k^t,y_{lj})\Big]_{i,j}\\&=w^*-\lim_t\sum_k\big([L^\vee(x_{ij},a_k^t)]_{i,j}\cdot[L^\vee(b_k^t,y_{ij})]_{i,j}\big)\\&=w^*-\lim_t\sum_k L_n^\vee(x,I_n\otimes a_k^t)L_n^\vee(I_n\otimes b_k^t,y).\end{aligned}$$

Consequently, we can estimate the norm of the term of the third line

$$\Big\|w^*-\lim_t\sum_k L_n^\vee(x,I_n\otimes a_k^t)L_n^\vee(I_n\otimes b_k^t,y)\Big\|\le\|\Gamma_L\circ B\|_{\mathrm{cb}}\le\|u\|\,\|L^\vee\|_{\mathrm{cb}}^2.$$

In the same manner, one can prove that the norm of the term in the fourth line is bounded by $\|u\|\,\|L^\vee\|_{\mathrm{cb}}^2$. For the term in the fifth line, note that its (i,j)-entry is

$$w^*-\lim_t\sum_k\sum_{p=1}^n\big(L(a_k^t)L(b_k^tx_{ip}y_{pj})-L(x_{ip}a_k^t)L(b_k^ty_{pj})\big)\in\mathcal{N}.$$

But u is a normal virtual h-diagonal, so for any i and p,

$$w^*-\lim_t(x_{ip}\cdot u_t-u_t\cdot x_{ip})=0,$$

hence for any i, j, p,

$$w^*-\lim_t\Big(\sum_k x_{ip}\cdot a_k^t\otimes b_k^t\cdot y_{pj}-\sum_k a_k^t\otimes b_k^t\cdot x_{ip}y_{pj}\Big)=0.$$

The bilinear map

$$(x, y) \in \mathcal{M} \times \mathcal{M} \mapsto L(x)L(y) \in \mathcal{N}$$

extends to a w^*-continuous map, consequently the term in the fifth line is 0. Analogously, the term in the sixth line is also 0. Finally we obtain

$$\|(L+R)^\vee\|_{\text{cb}} \leq 2\|u\| \|L^\vee\|_{\text{cb}}^2 + \|R\|_{\text{cb}}^2 \leq (2\|u\| + \|u\|^2 \|L\|_{\text{cb}}^2) \|L^\vee\|_{\text{cb}}^2. \tag{3-4}$$

Now we are in position to follow the induction of [Johnson 1988] with cb-norms instead of norms (for the reader's convenience, we reproduce it here). The important point is that each L^q (and thus each R^q) defined below is w^*-continuous. Define

$$\delta = \frac{\varepsilon}{4\|u\| + 8\mu^2\|u\|^2}. \tag{3-5}$$

Suppose that $\|L^\vee\|_{\text{cb}} \leq \delta$. Inductively, we define a sequence of linear maps from $\mathcal{M}$ into $\mathcal{N}$ by $L_0 = L$ and $R_0 = R$, and for $q \geq 0$,

$$L^{q+1} = L^q + R^q \quad \text{and} \quad R^{q+1}(\,\cdot\,) = \Lambda_{L^{q+1}}(u \otimes \,\cdot\,).$$

We also define $\mu_q = (2 - 2^{-q})\mu$ and $\delta_q = 2^{-q}\delta$. By induction, we prove that $\|(L^q)^\vee\|_{\text{cb}} \leq \delta_q$ and $\|L^q\|_{\text{cb}} \leq \mu_q$, for all q. It is obvious for $q = 0$. Then using the inequality (3-4) above, we have

$$\|(L^{q+1})^\vee\|_{\text{cb}} \leq (2\|u\| + \|u\|^2\mu_q^2)\delta_q^2 \leq \delta_{q+1},$$

and using (3-2) to majorize the cb-norm of R^q, we obtain

$$\|L^{q+1}\|_{\text{cb}} \leq \mu_q + \|u\|\mu_q\delta_q \leq \mu_{q+1},$$

(the last inequality coming from the fact that $\|u\|\delta \leq 4^{-1}$). Consequently,

$$\|R^q\|_{\text{cb}} \leq \|u\| \|L^q\|_{\text{cb}} \|(L^q)^\vee\|_{\text{cb}} \leq 2\|u\|\mu\delta_q,$$

so $\sum_{q\geq 0} R^q$ converges in cb-norm. We can define

$$\pi = L + \sum_{q\geq 0} R^q,$$

in other words $\pi = \lim_q L^q$, so π is w^*-continuous. Hence $\pi^\vee = \lim_q (L^q)^\vee$, but we proved that $\|(L^q)^\vee\|_{\text{cb}} \leq \delta_q$, so π is multiplicative. Moreover,

$$\|\pi - L\|_{\text{cb}} = \Big\|\sum_{q\geq 0} R^q\Big\|_{\text{cb}} \leq 4\|u\|\mu\delta < \varepsilon. \qquad \square$$

Remark 3.3. One important point which does not appear in the statement of the previous theorem is that δ is an explicit function of μ, ε and $\|u\|$; see (3-5).

4. Neighboring representations

We now show that two representations of a dual operator algebra, admitting a normal virtual h-diagonal, which are close enough in cb-norm are necessarily similar. Apparently, this phenomena is well known to Banach algebraists (see, e.g., Chapter 8 of [Runde 2002]). We give here a quick proof for dual operator algebras. This proposition will enable us to perform the third step mentioned in the Introduction. If $S \in \mathbb{B}(H)$ is an invertible operator, we denote by Ad_S the similarity implemented by S.

Proposition 4.1. *Let $\mathcal{M}$ be a unital dual operator algebra. We suppose that $\mathcal{M}$ has a normal virtual h-diagonal $u \in \mathcal{M} \otimes_{\sigma h} \mathcal{M}$. Let π_1 and π_2 be two unital w^*-continuous completely bounded representations on the same Hilbert space K. If $\|\pi_1 - \pi_2\|_{\mathrm{cb}} < \|u\|^{-1} \max\{\|\pi_1\|_{\mathrm{cb}}^{-1}, \|\pi_2\|_{\mathrm{cb}}^{-1}\}$, then there exists an invertible operator S in the w^*-closed algebra generated by $\pi_1(\mathcal{M}) \cup \pi_2(\mathcal{M})$ such that $\pi_1 = \mathrm{Ad}_S \circ \pi_2$.*

Proof. Let π_1, π_2 be as above. For two completely bounded w^*-continuous linear maps $F, G : \mathcal{M} \to \mathbb{B}(K)$, we denote (with notation of Sections 2B and 2C)

$$\Psi_{F,G} = \mathrm{m}_{\sigma h} \circ (F \otimes_{\sigma h} G),$$

which is a completely bounded w^*-continuous linear map defined on $\mathcal{M} \otimes_{\sigma h} \mathcal{M}$. Now, define

$$S = \Psi_{\pi_1,\pi_2}(u) \in \mathbb{B}(K).$$

As $u \in \mathcal{M} \otimes_{\sigma h} \mathcal{M}$, there is a net $(u_t)_t$ in $\mathcal{M} \otimes \mathcal{M}$ converging to u in the w^*-topology of $\mathcal{M} \otimes_{\sigma h} \mathcal{M}$. For any t, there are finite families $(a_k^t)_k$, $(b_k^t)_k$ of elements in $\mathcal{M}$ such that

$$u_t = \sum_k a_k^t \otimes b_k^t.$$

Hence, $S = w^* - \lim_t \sum_k \pi_1(a_k^t)\pi_2(b_k^t)$. Let $m \in \mathcal{M}$, then

$$\begin{aligned} \pi_1(m)S &= \pi_1(m).w^* - \lim_t \sum_k \pi_1(a_k^t)\pi_2(b_k^t) \\ &= w^* - \lim_t \sum_k \pi_1(m a_k^t)\pi_2(b_k^t) \\ &= w^* - \lim_t \Psi_{\pi_1,\pi_2}(m \cdot u_t) \\ &= \Psi_{\pi_1,\pi_2}(m \cdot u). \end{aligned}$$

Analogously, we can show that

$$S\pi_2(m) = \Psi_{\pi_1,\pi_2}(u \cdot m).$$

But u is a normal virtual h-diagonal, so $m \cdot u = u \cdot m$, hence

$$\pi_1(m)S = S\pi_2(m).$$

To conclude, we just need to prove that S is invertible. Without loss of generality we can assume that $\|\pi_1 - \pi_2\|_{\mathrm{cb}} < \|u\|^{-1}\|\pi_1\|_{\mathrm{cb}}^{-1}$. As above, we have $\Psi_{\pi_1,\pi_1}(u) = w^* - \lim_t \sum_k \pi_1(a_k^t)\pi_1(b_k^t)$. Using the condition (C2) defining a normal virtual h-diagonal, we obtain

$$\begin{aligned}\Psi_{\pi_1,\pi_1}(u) =& w^* - \lim_t \sum_k \pi_1(a_k^t b_k^t) = \pi_1\Big(w^* - \lim_t \sum_k a_k^t b_k^t\Big)\\ =&\pi_1(\mathrm{m}_{\sigma h}(u)) = \pi_1(1) = I_K.\end{aligned}$$

Consequently,

$$\begin{aligned}\|S - I_K\| &= \|\Psi_{\pi_1,\pi_2}(u) - \Psi_{\pi_1,\pi_1}(u)\| = \|\Psi_{\pi_1,\pi_2-\pi_1}(u)\|\\ &\leq \|u\|\|\pi_1 - \pi_2\|_{\mathrm{cb}}\|\pi_1\|_{\mathrm{cb}} < 1.\end{aligned}$$

□

5. Proof of the main theorems

We start this section with a very simple lemma that we will use repeatedly in the proof of the next theorem; we just sketch the proof. Recall that $T^\vee$ denotes the bilinear map from $\mathcal{M} \times \mathcal{M}$ into $\mathcal{N}$ defined by $T^\vee(x, y) = T(xy) - T(x)T(y)$. Also in this section, we denote by $\mathrm{id}_{\mathcal{A}}$ the identity representation of a concretely represented operator algebra $\mathcal{A}$.

Lemma 5.1. *Let $\mathcal{A}, \mathcal{B} \subset \mathbb{B}(H)$ be two operator algebras and $T : \mathcal{A} \to \mathcal{B}$ be a completely bounded linear map. Then*:

(i) $\|T^\vee\|_{\mathrm{cb}} \leq (2 + \|T\|_{\mathrm{cb}})\|T - \mathrm{id}_{\mathcal{A}}\|_{\mathrm{cb}}$.

(ii) *If $\|T - \mathrm{id}_{\mathcal{A}}\|_{\mathrm{cb}} < 1$, then T is injective and has closed range. Moreover*, *if there exists $\alpha \in [0, 1)$ such that for any y in the unit ball of $\mathcal{B}$*, *there is x in $\mathcal{A}$ satisfying $\|T(x) - y\| \leq \alpha$, then T is bijective and*

$$\|T^{-1}\|_{\mathrm{cb}} \leq \frac{1}{1 - \|T - \mathrm{id}_{\mathcal{A}}\|_{\mathrm{cb}}}.$$

Proof. Let x, y be in the unit ball of $\mathbb{M}_n(\mathcal{M})$, then (i) follows from the decomposition

$$\begin{aligned}(T^\vee)_n(x, y) &= T_n(xy) - T_n(x)T_n(y)\\ &= T_n(xy) - xy + xy - xT_n(y) + xT_n(y) - T_n(x)T_n(y).\end{aligned}$$

The first assertion of (ii) follows from

$$\|T_n(x)\| \geq \big|\|T_n(x) - x\| - \|x\|\big| \geq (1 - \|T - \mathrm{id}_{\mathcal{A}}\|_{\mathrm{cb}})\|x\|.$$

The surjectivity of T is proved by induction. Let y be in the unit ball of $\mathcal{N}$, then for any integer j, we can find $t_1, \dots, t_j$ in the range of T such that

$$\|y - (t_1 + t_2 + \cdots + t_j)\| \leq \alpha^j.$$

As $\alpha < 1$, we conclude that y belongs to the closure of the range of T. □

Note that, in the following theorem, $\mathcal{M}$ is just assumed to be a dual operator algebra, but we require a near cb-inclusion of $\mathcal{M}$ into $\mathcal{N}$ (see Section 2A).

Theorem 5.2. *Let $\mathcal{M}, \mathcal{N} \subset \mathbb{B}(H)$ be two unital w^*-closed operator algebras. We suppose that $\mathcal{N}$ is an injective von Neumann algebra. If $\mathcal{N} \subset^1 \mathcal{M}$ and $\mathcal{M} \subseteq^{\gamma}_{\mathrm{cb}} \mathcal{N}$, with $\gamma < \frac{1}{164}$, then there exists an invertible operator S in the w^*-closed algebra generated by $\mathcal{M} \cup \mathcal{N}$ such that $S\mathcal{M}S^{-1} = \mathcal{N}$.*

Proof. Since $\mathcal{N}$ is injective, there is a completely contractive projection P from $\mathbb{B}(H)$ onto $\mathcal{N}$. Denote $T = P_{|\mathcal{M}}$. Let x be in the unit ball of $\mathbb{M}_n(\mathcal{M})$, then there is y in $\mathbb{M}_n(\mathcal{N})$ such that $\|x - y\| \leq \gamma$.

$$\|T_n(x) - x\| = \|T_n(x - y) - (x - y)\|_{\mathrm{cb}} \leq 2\gamma,$$

hence

$$\|T - \mathrm{id}_{\mathcal{M}}\|_{\mathrm{cb}} \leq 2\gamma < 1.$$

Let us prove that T is surjective. Since $\mathcal{N} \subset^1 \mathcal{M}$, there is $\gamma' < 1$ such that $\mathcal{N} \subseteq^{\gamma'} \mathcal{M}$. Let y be in the unit ball of $\mathcal{N}$, then there exists x in $\mathcal{M}$ such that $\|y - x\| \leq \gamma'$, therefore from Lemma 5.1(ii), T is a linear cb-isomorphism and

$$\|T^{-1}\|_{\mathrm{cb}} \leq \frac{1}{1-2\gamma}. \tag{5-1}$$

The problem is that T is not necessarily w^*-continuous, so we are going to consider the normal of T^{-1} (see [Tomiyama 1959], we denote with an exponent n the normal part of a linear map defined on $\mathcal{N}$). Note first that

$$\|T^{-1} - \mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}} \leq \|T^{-1}\|_{\mathrm{cb}} \|T - \mathrm{id}_{\mathcal{M}}\|_{\mathrm{cb}} \leq \frac{2\gamma}{1-2\gamma}. \tag{5-2}$$

Let $V = (T^{-1})^{\mathrm{n}} : \mathcal{N} \to \mathcal{M}$ be the normal part of T^{-1}. Using Lemma 5.1(ii) again, let us show that V is a completely bounded w^*-continuous linear isomorphism from $\mathcal{N}$ onto $\mathcal{M}$. As taking the normal part is a completely contractive operation, we have

$$\|V - \mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}} = \|(T^{-1} - \mathrm{id}_{\mathcal{N}})^{\mathrm{n}}\|_{\mathrm{cb}} \leq \|T^{-1} - \mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}} \leq \frac{2\gamma}{1-2\gamma}, \tag{5-3}$$

thus V is an injective map and has closed range. Now let y be in the unit ball of $\mathcal{M}$, and pick x in $\mathcal{N}$ such that $\|x - y\| \leq \gamma$. Thus $\|x\| \leq 1 + \gamma$ and

$$\begin{aligned} \|V(x) - y\| &\leq \|V(x) - x\| + \|x - y\| \\ &\leq \frac{2\gamma}{1-2\gamma}(1+\gamma) + \gamma \\ &\leq \frac{5\gamma}{1-2\gamma} < 1, \end{aligned}$$

so V is surjective.

In order to apply Theorem 3.2, we need to unitize V. From equation (5-3), $\|V(1)-1\| \leq (2\gamma)/(1-2\gamma) < 1$, hence $V(1)$ is invertible in $\mathcal{M}$ and

$$\|V(1)^{-1}\| \leq \frac{1}{1-\|V(1)-1\|} \leq \frac{1-2\gamma}{1-4\gamma}. \tag{5-4}$$

Denote $L = V(1)^{-1}V$, so L is a unital w^*-continuous completely bounded isomorphism from $\mathcal{N}$ onto $\mathcal{M}$ and from (5-1) we have

$$\|L\|_{\mathrm{cb}} \leq \|V(1)^{-1}\|\|V\|_{\mathrm{cb}} \leq \|V(1)^{-1}\|\|T^{-1}\|_{\mathrm{cb}} \leq \frac{1}{1-4\gamma}. \tag{5-5}$$

Let us compute the norm of $L^\vee$, the defect of multiplicativity of L (see Section 3 for notation). Fix n, let x be in unit ball of $\mathbb{M}_n(\mathcal{N})$, then from (5-3) and (5-4) we obtain

$$\begin{aligned}\|L_n(x)-x\| &\leq \|I_n \otimes V(1)^{-1}(V_n(x)-x)\| + \|I_n \otimes V(1)^{-1}x - x\| \\ &\leq \|V(1)^{-1}\|\|V-\mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}} + \|V(1)^{-1}\|\|V(1)-1\| \\ &\leq \frac{4\gamma}{1-4\gamma},\end{aligned}$$

which means that

$$\|L-\mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}} \leq \frac{4\gamma}{1-4\gamma}. \tag{5-6}$$

Therefore, by Lemma 5.1(i) and equation (5-5) we obtain

$$\|L^\vee\|_{\mathrm{cb}} \leq (2+\|L\|_{\mathrm{cb}})\|L-\mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}} \leq \frac{12\gamma}{(1-4\gamma)^2}.$$

We want to apply Theorem 3.2 to L. Put

$$\mu = \frac{1}{1-4\gamma} \quad \text{and} \quad \delta = \frac{12\gamma}{(1-4\gamma)^2}.$$

As $\mathcal{N}$ is an injective von Neumann algebra, we can find a normal virtual h-diagonal u of norm 1 [Effros 1988; Effros and Kishimoto 1987], and thus (see (3-5)) let

$$\varepsilon = \delta(4\|u\| + 8\mu^2\|u\|^2) = \frac{12\gamma}{(1-4\gamma)^2}\Big(4 + \frac{8}{(1-4\gamma)^2}\Big).$$

We can then apply Theorem 3.2 to L and find a unital w^*-continuous completely bounded homomorphism $\pi : \mathcal{N} \to \mathcal{M}$ such that

$$\|L-\pi\|_{\mathrm{cb}} \leq \varepsilon.$$

Consequently, from (5-6),

$$\begin{aligned}\|\pi-\mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}} &\leq \|\pi - L\|_{\mathrm{cb}} + \|L-\mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}} \\ &\leq \varepsilon + \frac{4\gamma}{1-4\gamma},\end{aligned}$$

and this last quantity is strictly smaller than 1, because $\gamma < \frac{1}{164}$. Therefore, we can apply Proposition 4.1 to π and $\mathrm{id}_{\mathcal{N}}$ and find an invertible operator S in the w^*-closed algebra generated by $\mathcal{M} \cup \mathcal{N}$ such that

$$\mathrm{Ad}_S \circ \pi = \mathrm{id}_{\mathcal{N}}$$

(in particular π is injective and has closed range). To achieve the proof, it is sufficient to prove that the range of π is $\mathcal{M}$. Let y be in the unit ball of $\mathcal{M}$, then

$$\|\pi(L^{-1}(y)) - y\| \leq \|\pi - L\|_{\mathrm{cb}} \|L^{-1}\|_{\mathrm{cb}},$$

so by Lemma 5.1(ii), we just need to check that this last quantity is strictly smaller than 1. From (5-6)

$$\|L^{-1}\|_{\mathrm{cb}} \leq \frac{1}{1 - \|L - \mathrm{id}_{\mathcal{N}}\|_{\mathrm{cb}}} \leq \frac{1-4\gamma}{1-8\gamma},$$

it follows that

$$\|\pi - L\|_{\mathrm{cb}} \, \|L^{-1}\|_{\mathrm{cb}} \leq \frac{1-4\gamma}{1-8\gamma}\varepsilon$$

which is strictly smaller than 1, because $\gamma < \frac{1}{164}$. □

At this point, we want to get rid of the near cb-inclusion hypothesis appearing in the previous theorem. The task is to find conditions of "automatic near cb-inclusion" on the algebra $\mathcal{M}$. More explicitly, under which conditions does a near inclusion $\mathcal{M} \subseteq^{\gamma} \mathcal{N}$ automatically imply a near cb-inclusion? For C^*-algebras, Christensen isolated property D_k which ensures such an "automatic near cb-inclusion" result. Recall that a C^*-algebra $\mathcal{A}$ has property D_k if for any unital $*$-representation (π, K) one has

$$\forall x \in \mathbb{B}(K), \quad \mathrm{d}(x, \pi(\mathcal{A})') \leq k \|\delta(x)_{|\pi(\mathcal{A})}\|,$$

where d denotes the usual distance between subsets and $\delta(x)$ denotes the inner derivation implemented by x on $\mathbb{B}(K)$. It is well known that amenable C^*-algebras (or injective von Neumann algebras) have D_1, the next easy lemma is the nonselfadjoint analog of this fact (it also works for amenable Banach algebras).

Lemma 5.3. *Let $\mathcal{M}$ be a unital w^*-closed operator admitting a normal virtual diagonal $u \in \mathcal{M}\widehat{\otimes}_\sigma\mathcal{M}$. Then, for any unital w^*-continuous contractive representation (π, K) of $\mathcal{M}$ which satisfies $\pi(\mathcal{M}) = \overline{\pi(\mathcal{M})}^{w^*}$, we have*

$$\forall x \in \mathbb{B}(K), \quad \mathrm{d}(x, \pi(\mathcal{M})') \leq \|u\| \, \|\delta(x)_{|\pi(\mathcal{M})}\|. \tag{5-7}$$

Proof. Let us denote $\mathcal{N} = \pi(\mathcal{M}) \subset \mathbb{B}(K)$ and $v = \pi\widehat{\otimes}_\sigma\pi(u) \in \mathcal{N}\widehat{\otimes}_\sigma\mathcal{N}$, hence $\|v\| \leq \|u\|$. Since π has w^*-closed range, v is a normal virtual diagonal for the dual operator algebra $\mathcal{N}$. Note that $\mathbb{B}(K)$ is obviously a normal dual Banach $\mathcal{N}$-bimodule (in the sense of [Runde 2002, Definition 4.4.6]). Now, let x be in $\mathbb{B}(K)$ and consider the w^*-continuous bounded derivation $D = \delta(x)_{|\mathcal{N}} : \mathcal{N} \to \mathbb{B}(K)$. Adapting the proof of Johnson's theorem on characterization of amenability by virtual diagonals, we know that there is $\varphi \in \mathbb{B}(K)$ such that $D = \delta(\varphi)_{|\mathcal{N}}$. Actually $\varphi = D\widehat{\otimes}_\sigma \mathrm{id}_{\mathcal{N}}(v)$, so $\|\varphi\| \le \|D\|\|v\|$. As $D = \delta(x)_{|\mathcal{N}} = \delta(\varphi)_{|\mathcal{N}}$, we have $x - \varphi \in \mathcal{N}'$. Therefore,

$$\mathrm{d}(x, \pi(\mathcal{M})') = \mathrm{d}(\varphi, \mathcal{N}') \le \|\varphi\| \le \|D\|\|v\| \le \|u\|\|\delta(x)_{|\pi(\mathcal{M})}\|,$$

which ends the proof. □

Lemma 5.4. *Let $\mathcal{M} \subset \mathbb{B}(H)$ be a unital w^*-closed operator algebra admitting a normal virtual diagonal $u \in \mathcal{M}\widehat{\otimes}_\sigma\mathcal{M}$. Let $\mathcal{N}$ be an injective von Neumann subalgebra of $\mathbb{B}(H)$. Then, for any $\gamma > 0$, $\mathcal{M} \subseteq^\gamma \mathcal{N}$ implies $\mathcal{M} \subseteq_{\mathrm{cb}}^{4\|u\|\gamma} \mathcal{N}$.*

Proof. This follows from the previous lemma and the first lines of the proof of Theorem 3.1 in [Christensen 1980], with $D = \mathbb{M}_n$ (for arbitrary n), with $k = \|u\|$ (by (5-7)) and the $\frac{3}{2}$ must replaced by 1 because $\mathcal{N}$ is injective, so we get $4\|u\|\gamma$ instead of $6k\gamma$. □

Now, using the previous lemma and Theorem 5.2 above, we can prove Theorem 2.

This question of "automatic near cb-inclusion" can be thought of as an analog of the "automatic complete boundedness" question for homomorphisms (or equivalently Kadison's similarity problem). For this problem, Pisier defined the notion of length for operator algebras (see [Pisier 1998; 2000; 2001a; 2001b; 2007]). The connection between this notion of length and property D_k is now well known for C^*-algebras, see [Christensen et al. 2010a]. As we are working with dual operator algebras, C. Le Merdy's notion of length (or degree) denoted d_* in [Le Merdy 2000] is more appropriate (we call this quantity *normal length* in the following corollary).

Corollary 5.5. *Let $\mathcal{M}, \mathcal{N} \subset \mathbb{B}(H)$ be two unital w^*-closed operator algebras. Suppose that $\mathcal{M}$ has finite normal length at most d_* with constant at most $C > 0$. We suppose that $\mathcal{N}$ is an injective von Neumann algebra. If $\mathcal{N} \subset^1 \mathcal{M}$ and $\mathcal{M} \subseteq^\gamma \mathcal{N}$, with $\gamma < (1 + 1/(164C))^{1/d_*} - 1$, then there exists an invertible operator S in the w^*-closed algebra generated by $\mathcal{M} \cup \mathcal{N}$ such that $S\mathcal{M}S^{-1} = \mathcal{N}$ and consequently, $d_*(\mathcal{M}) \le 2$.*

Proof. If $\mathcal{M} \subseteq^\gamma \mathcal{N}$, then $\mathcal{M} \subseteq_{\mathrm{cb}}^{C((1+\gamma)^{d_*}-1)} \mathcal{N}$ as in Proposition 2.10 in [Christensen et al. 2010a]. The result follows from the similarity degree characterization of injectivity for von Neumann algebras in [Pisier 2006]. □

Remark 5.6. As explained in the Introduction, the main benefit of Theorem 2 (compared to Theorem 1) is that we can get rid of the selfadjointness hypothesis on one of the algebras. It would be very interesting to improve our theorem to both algebras being nonselfadjoint. More precisely, let $\mathcal{M}, \mathcal{N} \subset \mathbb{B}(H)$ be two unital w^*-closed operator algebras, suppose that $\mathcal{M}$ has a normal virtual diagonal u and that $\mathcal{N}$ is the range of a completely bounded projection P. Does there exist a continuous function $f : [1, \infty)^2 \to [0, \infty)$ with $f(1, 1) = 0$ such that if $d_{\mathrm{cb}}(\mathcal{M}, \mathcal{N}) < f(\|P\|_{\mathrm{cb}}, \|u\|)$, then $\mathcal{M}$ and $\mathcal{N}$ are similar? In our proof of Theorem 2, the only characterization of injectivity of a von Neumann algebra that we use is that of being the range of completely contractive projection. This is one advantage of our proof, because if one wants to positively answer the preceding question, the only obstruction in our proof is to find a Tomiyama type decomposition (into normal and singular parts) for nonselfadjoint dual operator algebras admitting a normal virtual diagonal.

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SCALAR CURVATURE AND SINGULAR METRICS

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Let M^n, $n \geq 3$, be a compact differentiable manifold with nonpositive Yamabe invariant $\sigma(M)$. Suppose g_0 is a continuous metric with volume $V(M, g_0) = 1$, smooth outside a compact set Σ, and is in $W^{1,p}_{\text{loc}}$ for some $p > n$. Suppose the scalar curvature of g_0 is at least $\sigma(M)$ outside Σ. We prove that g_0 is Einstein outside Σ if the codimension of Σ is at least 2. If in addition, g_0 is Lipschitz then g_0 is smooth and Einstein after a change of the smooth structure. If Σ is a compact embedded hypersurface, g_0 is smooth up to Σ from two sides of Σ, and if the difference of the mean curvatures along Σ at two sides of Σ has a fixed appropriate sign, then g_0 is also Einstein outside Σ. For manifolds with dimension between 3 and 7, without a spin assumption we obtain a positive mass theorem on an asymptotically flat manifold for metrics with a compact singular set of codimension at least 2.

1. Introduction

There are two celebrated results on manifolds with nonnegative scalar curvature. The first result is on compact manifolds. It was proved by Schoen and Yau [1979a; 1979c] that any smooth metric on a torus T^n, $n \leq 7$, with nonnegative scalar curvature must be flat. Later, the result was proved to be true for all n by Gromov and Lawson [1983]. The second result is the positive mass theorem on noncompact manifolds. Schoen and Yau [1979b; 1981; Schoen 1989] proved that the Arnowitt–Deser–Misner (ADM) mass of each end of an n-dimensional asymptotically flat (AF) manifold with $3 \leq n \leq 7$ with nonnegative scalar curvature is nonnegative and if the ADM mass of an end is zero, then the manifold is isometric to the Euclidean space. Under the additional assumption that the manifold is spin, the same result is still true and was proved by Witten [1981]; see also [Parker and Taubes 1982; Bartnik 1986]. In the two results the metrics are assumed to be smooth.

There are many results on positive mass theorem for nonsmooth metrics. Miao [2002] and the authors [Shi and Tam 2002] studied and proved positive mass

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theorems for metrics with corners. The metrics are smooth away from a compact hypersurface, which are Lipschitz and satisfy certain conditions on the mean curvatures of the hypersurface. The result was used to prove the positivity of the Brown–York quasilocal mass [Shi and Tam 2002]. Lee [2013] considered a positive mass theorem for metrics with bounded C^2 norm and are smooth away from a singular set with codimension greater than $n/2$, where n is the dimension of the manifold. On the other hand, McFeron and Székelyhidi [2012] were able to prove Miao's result using Ricci flow and Ricci–DeTurck flow, which was studied in detail by Simon [2002]. There is a positive mass theorem for spin manifolds or manifolds with dimension n less than eight obtained by Grant and Tassotti [2014] under the assumptions that the metric is continuous and in Sobolev space $W^{2,n/2}_{\mathrm{loc}}$. More recently, Lee and LeFloch [2015] were able to prove for spin manifolds, under rather general conditions, a positive mass theorem for metrics which may be singular. Their theorem can be applied to all previous results for nonsmooth metrics under the additional assumption that the manifold is spin.

Motivated by these studies of singular metrics on AF manifolds, we want to understand singular metrics on compact manifolds. One of the questions is to see if there are nonflat metrics with nonnegative scalar curvature on T^n which may be singular somewhere. Another question can be described as follows. It is now well known that in every conformal class of smooth metrics on a compact manifold without boundary there is a metric with constant scalar curvature; see [Yamabe 1960; Trudinger 1968; Aubin 1976a; 1976b; Schoen 1984]. One motivation for the result is to obtain Einstein metric. It is well known that if a smooth metric on a compact manifold attains the Yamabe invariant and if the invariant is nonpositive, then the metric is Einstein. See [Schoen 1989, pp. 126–127]. In this work, we will study the question whether this last result is still true for nonsmooth metrics.

Let us recall the definition of Yamabe invariant, which is called σ-invariant in [Schoen 1989]. Let $\mathcal{C}$ be a conformal class of smooth Riemannian metrics g on a smooth compact manifold M^n; then the *Yamabe constant of* $\mathcal{C}$ is defined as

$$Y(\mathcal{C}) = \inf_{g\in\mathcal{C}} \frac{\int_M \mathcal{S}_g \, dv_g}{(V(M,g))^{1-2/n}},$$

where $\mathcal{S}_g$ is the scalar curvature and $V(M,g)$ is the volume of M with respect to g. The *Yamabe invariant* is defined as

$$\sigma(M) = \sup_{\mathcal{C}} Y(\mathcal{C}).$$

The supremum is taken among all conformal classes of smooth metrics. It is finite; see [Aubin 1976a]. If g attains $\sigma(M) > 0$, then in general it is still unclear whether g is Einstein or not; see [Macbeth 2017].

To answer the question on Einstein metrics, let M^n be a compact smooth manifold without boundary and let g_0 be a continuous Riemannian metric on M with $V(M, g_0) = 1$ such that it is smooth outside a compact set Σ. The first case is that Σ has codimension at least 2 and g_0 is in $W^{1,p}_{\mathrm{loc}}$ for some $p > n$ (see Sections 3 and 5 for more precise definitions).

Theorem 1.1. *Let (M^n, g_0) be as above. Suppose $\sigma(M) \le 0$ and suppose the scalar curvature of g_0 outside Σ is at least $\sigma(M)$. Then g_0 is Einstein outside Σ. If in addition g_0 is Lipschitz, then after changing the smooth structure, g_0 is smooth and Einstein.*

In the case that Σ is a compact embedded hypersurface, as in [Miao 2002] we assume that near Σ, $g_0 = dt^2 + g_\pm(z, t)$, $z \in \Sigma$, $t \in (-\epsilon, \epsilon)$ such that (t, z) are smooth coordinates and $g_-(\,\cdot\,, 0) = g_+(\,\cdot\,, 0)$, where g_+, g_- are defined on the neighborhood of Σ where $t > 0$ and $t < 0$ respectively and are smooth up to Σ. Moreover, with respect to the unit normal $\frac{\partial}{\partial t}$ the mean curvature H_+ of Σ with respect to g_+ and the mean curvature H_- of Σ with respect to g_- satisfies $H_- \ge H_+$. Under these assumptions, we have:

Theorem 1.2. *Let (M^n, g_0) be as above with $V(m, g_0) = 1$. Suppose $\sigma(M) \le 0$ and suppose the scalar curvature of g_0 outside Σ is at least $\sigma(M)$. Then g_0 is Einstein outside Σ. Moreover, $H_+ = H_-$.*

Note that it is easy to construct examples so that the theorem is not true if the assumption $H_- \ge H_+$ is removed.

In the process of proving the theorems, one also obtains the following: In the case that M^n is T^n, under the regularity assumptions in Theorem 1.1 or Theorem 1.2 and if g_0 has nonnegative scalar curvature outside Σ, then g_0 is flat outside Σ.

The method of proof of the above results can also be adapted to AF manifolds. We want to discuss the positive mass theorem with singular metric on an AF manifold with dimension $3 \le n \le 7$ without assuming that the manifold is spin. We will prove the following:

Theorem 1.3. *Let (M^n, g_0) be an AF manifold with $3 \le n \le 7$, where g_0 is a continuous metric on M with regularity assumptions as in Theorem 1.1. Suppose g_0 has nonnegative scalar curvature outside Σ. Then the ADM mass of each end is nonnegative. Moreover, if the ADM mass of one of the ends is zero, then M is diffeomorphic to $\mathbb{R}^n$ and is flat outside Σ.*

We should mention that all the results mentioned above for nonsmooth metrics, all the metrics are assumed to be continuous. On the other hand, one can construct an example of AF metric with a cone singularity and nonnegative scalar curvature and with negative ADM mass; see Section 2. One can also construct examples of metrics on compact manifolds with a cone singularity so that Theorem 1.1 is not true. In these examples, the metrics are not continuous.

The structure of the paper is as follows. In Section 2, we construct examples which are related to results in later sections; in Section 3 we obtain some estimates for the Ricci–DeTurck flow; in Section 4 we use the Ricci–DeTurck flow to approximate singular metrics; in Sections 5 and 6 we prove Theorems 1.1 and 1.2; in Section 7 we prove Theorem 1.3. In this work, the dimension of any manifold is assumed to be at least three. We will also use the Einstein summation convention.

2. Examples of metrics with cone singularities

In previous results on positive mass theorems on AF manifolds with singular metrics mentioned in Section 1, the metrics are all assumed to be continuous. To understand this condition on continuity and to motivate our study, in this section, we construct some examples with cone singularities which are related to the study in the later sections.

The following lemma is standard. See [Petersen 1998].

Lemma 2.1. *Consider the metric* $g = dr^2 + \phi^2(r)h_0$ *on* $(0, r_0) \times \mathbb{S}^{n-1}$, *where* h_0 *is the standard metric of* $\mathbb{S}^{n-1}$, $n \ge 3$, *and* ϕ *is a smooth positive function on* $(0, r_0)$. *Then the scalar curvature of* g *is given by*

$$\mathcal{S} = (n-1)\left[-\frac{2\phi''}{\phi} + (n-2)\frac{1-(\phi')^2}{\phi^2}\right].$$

Suppose $\phi = \alpha r^\beta$, *with* $\alpha, \beta > 0$. *Then* $\mathcal{S} > 0$ *if* $\alpha < 1$, $\beta = 1$ *or if* $0 < \beta \le 2/n$. *In both cases, the metric is not continuous up to* $r = 0$. *If* $\alpha > 1$, $\beta = 1$, *then* $\mathcal{S} < 0$ *for* r *small enough.*

We can construct asymptotically flat manifolds with nonnegative scalar curvature defined on $\mathbb{R}^3 \setminus \{0\}$ such that the metric behaves like $dr^2 + (\alpha r)^2 h_0$ near the origin for some $0 < \alpha < 1$ with positive mass.

Proposition 2.2. *Let* $0 < \epsilon < \frac{1}{2}$ *and let* $\eta(x) = \eta(r)$, *with* $r = |x|$, *be a smooth function on* $\mathbb{R}^3 \setminus \{0\}$ *such that*

$$\begin{cases} \eta(r) = -\epsilon(1-\epsilon)r^{-\epsilon-2} & \text{if } 0 < r \le 1,\\ \eta(r) < 0 & \text{if } 1 \le r \le 2,\\ \eta(r) = 0 & \text{if } r \ge 2. \end{cases}$$

Let ϕ *be the function defined on* $\mathbb{R}^3 \setminus \{0\}$ *with*

$$\phi(r) = \int_1^r \frac{1}{s^2}\left(\int_0^s t^2\eta(t)\,dt\right)ds.$$

Then there are constants $a, b > 0$ such that if

$$u = \phi + b + \frac{a}{2} + 1$$

then $u > 0$. Moreover, if $g = u^4 g_e$, where g_e is the standard Euclidean metric, then near infinity,

$$g = \left(1 + \frac{a}{r}\right)^4 g_e,$$

and near $r = 0$,

$$g = d\rho^2 + \big((1-2\epsilon)^2\rho^2 + O(\rho^{2+\delta})\big)h_0$$

for some $\delta > 0$, where

$$\rho = \int_0^r u^2(t)\,dt.$$

The metric g has nonnegative scalar curvature and has zero scalar curvature outside a compact set. Moreover, the end near infinity is asymptotically flat in the sense of Definition 7.1 in Section 7, and has positive mass $2a$.

Proof. Let Δ_0 be the Euclidean Laplacian. Then one can check that

$$\Delta_0 \phi = \eta \leq 0.$$

For $0 < r \leq 1$,

$$\phi(r) = r^{-\epsilon} - 1.$$

For $r \geq 2$, let

$$a = -\int_0^r s^2 \eta(s)\,ds > 0,$$

and

$$b = -\int_1^2 \frac{1}{s^2}\left(\int_0^s \tau^2 \eta(\tau) d\tau\right) ds > 0.$$

Then

$$\begin{aligned}
\phi(r) &= -b + \int_2^r \frac{1}{s^2}\left(\int_0^s t^2 \eta(t)\,dt\right) ds \\
&= -b - a\int_2^r \frac{1}{s^2}\,ds \\
&= -b - \frac{a}{2} + \frac{a}{r}.
\end{aligned}$$

Hence if $u = \phi + b + a/2 + 1$, then $\Delta_0 u = \eta \leq 0$. Since $u \to \infty$ as $r \to 0$ and $u \to 1$ as $r \to \infty$, $u > 0$ by the strong maximum principle. The metric

$$g = u^4 g_e$$

is defined on $\mathbb{R}^3 \setminus \{0\}$, has nonnegative scalar curvature and has zero scalar curvature near infinity. g is also asymptotically flat. Near $r = 0$,

$$u = b + \frac{a}{2} + r^{-\epsilon}.$$

Since $0 < \epsilon < \frac{1}{2}$, we let

$$\rho = \int_0^r u^2(t)\,dt = \frac{1}{(1-2\epsilon)} r^{1-2\epsilon} + O(r^{1-\epsilon}).$$

So

$$\rho^2 = \frac{1}{(1-2\epsilon)^2} r^{2-4\epsilon} + O(r^{2-3\epsilon}).$$

Hence near $r = 0$,

$$\begin{aligned} g &= d\rho^2 + u^4 r^2 h_0 \\ &= d\rho^2 + (r^{2-4\epsilon} + O(r^{2-3\epsilon}))h_0 \\ &= d\rho^2 + ((1-2\epsilon)^2\rho^2 + O(r^{2-3\epsilon}))h_0 \\ &= d\rho^2 + (\alpha^2\rho^2 + O(r^{2-3\epsilon}))h_0, \end{aligned}$$

where $\alpha = 1 - 2\epsilon$. Note that $r^{2-3\epsilon} = O(\rho^{2+\delta})$ for some $\delta > 0$. □

The following example is the type of singularity which is called zero area singularity in [Bray and Jauregui 2013].

Proposition 2.3. *Let $m > 0$ and let $\phi = 1 - 2m/r$. Then the metric*

$$g = \phi^4 g_e$$

is asymptotically flat defined on $r > 2m$ in $\mathbb{R}^3$, with zero scalar curvature and with negative mass $-m$. Moreover, near $r = 2m$,

$$g = d\rho^2 + c\rho^{4/3}(1 + O(\rho^{2/3}))h_0$$

for some $c > 0$, where

$$\rho = \int_0^{r-2m} \phi^2(t+2m)\,dt.$$

Hence near $\rho = 0$ the metric is asymptotically of the form as in Lemma 2.1 with $\beta = \frac{2}{3}$.

Proof. We only need to consider g near $r = 2m$. The rest is well known. Let $t = r - 2m$, $r > 2m$. Then

$$\tilde{\phi}(t) = \phi(t+2m) = \frac{t}{t+2m} = \frac{t}{2m}\left(1 - \frac{t}{2m} + \frac{t^2}{4m^2} + O(t^3)\right)$$

and

$$\rho = \int_0^t \tilde{\phi}^2(s)\,ds = \int_0^t \frac{s^2}{(s+2m)^2}\,ds.$$

Note that as $r \to 2m$, $\rho \to 0$. In terms of ρ, near $\rho = 0$,

$$g = d\rho^2 + \phi^4 r^2 h_0.$$

Near $\rho = 0$,

$$\begin{aligned}\phi^4 r^2 &= \frac{t^4}{(t+2m)^4}(t+2m)^2\\ &= c\rho^{4/3}(1 + O(\rho^{2/3}))\end{aligned}$$

for some $c > 0$. □

We can also construct a conical metric on $T^3 \setminus \{\text{a point}\}$, with nonnegative scalar curvature and with positive scalar curvature somewhere.

First, we have

Proposition 2.4. *Let $m > 0$. There is a metric g on $\mathbf{R}^3 \setminus B(2m)$ such that*

(i) *the scalar curvature R is nonnegative and $R > 0$ somewhere*;

(ii) *there exist r_0 and r_1 with $r_1 > r_0 > 2m$ such that $g = (1 - 2m/r)^4 g_e$ for any $r \in (2m, r_0)$ and $g = g_e$ for any $r \geq r_1$, where g_e is the Euclidean metric.*

Proof. Let $r_1 > r_0 > 2m$ to be chosen later. Let $\eta(r)$ be a smooth nonincreasing function with

$$\eta(r) = \begin{cases} 2m, & 2m \leq r \leq r_0, \\ 0, & r \geq r_1. \end{cases} \tag{2-1}$$

For any $\rho \geq 2m$, let

$$y(\rho) = \int_{2m}^{\rho} \frac{\eta(r)}{r^2}\,dr.$$

By choosing suitable r_0, r_1, we may get $y(\rho) = 1$ for any $\rho \geq r_1$; then we see that

$$y(r) = \begin{cases} 1 - 2m/r, & 2m \leq r \leq r_0, \\ 1, & r \geq r_1. \end{cases} \tag{2-2}$$

We claim that

$$\Delta_0 y \leq 0 \quad \text{on } \mathbb{R}^3 \setminus B_{2m};$$

here Δ_0 is the standard Laplace operator on $\mathbb{R}^3$. By a direct computation, we see that

$$\Delta_0 y = y'' + \frac{2}{r} y' = r^{-2}(r^2 y')' = r^{-2}\eta' \leq 0. \tag{2-3}$$

For any $x \in \mathbb{R}^3 \setminus B_{2m}$, let $u(x) = y(|x|)$; then $g = u^4(dr^2 + r^2 h_0)$ is the required metric. □

Suppose $T^3(r)$ is a flat torus, by taking r large enough we may glue $(B_r \setminus B_{2m}, g)$ with $T^3(r) \setminus B_r$ directly. As in Proposition 2.3, near $r = 2m$, the metric can be considered as a metric with cone singularity. The question is whether we have a metric on n-torus which has a cone singularity of the form $dr^2 + \alpha^2 r^2 h_0$ with $0 < \alpha < 1$ and with nonnegative scalar curvature. This will be answered in Section 4. The problem can be reduced to the study of singular metrics on T^n with nonnegative scalar curvature.

3. Gradient estimates for solutions to the *h*-flow

We want to use the Ricci–DeTurck flow to deform a singular metric to a smooth one. We need some basic facts about the flow.

Let (M^n, h) be a complete manifold without boundary. We assume that the curvature of h and its covariant derivatives are bounded:

$$|\widetilde{\nabla}^{(i)}\widetilde{\mathrm{Rm}}| \le k_i \tag{3-1}$$

for all $3 \ge i \ge 0$. Here $\widetilde{\nabla}$ is the covariant derivative with respect to h and $\widetilde{\mathrm{Rm}}$ is the curvature tensor of h. A smooth family of metrics $g(t)$ on $M \times (0, T]$, $T > 0$, is said to be a solution to the h-flow if $g(t)$ satisfies

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij} = g^{\alpha\beta}\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta g_{ij} - g^{\alpha\beta} g_{ip} h^{pq}\widetilde{\mathrm{Rm}}_{j\alpha q\beta} - g^{\alpha\beta} g_{jp} h^{pq}\widetilde{\mathrm{Rm}}_{i\alpha q\beta} \\ + \tfrac{1}{2} g^{\alpha\beta} g^{pq}\big(\widetilde{\nabla}_i g_{p\alpha}\cdot\widetilde{\nabla}_j g_{q\beta} + 2\widetilde{\nabla}_\alpha g_{jp}\cdot\widetilde{\nabla}_q g_{i\beta} - 2\widetilde{\nabla}_\alpha g_{jp}\cdot\widetilde{\nabla}_\beta g_{iq} \\ - 2\widetilde{\nabla}_j g_{\alpha p}\cdot\widetilde{\nabla}_\beta g_{iq} - 2\widetilde{\nabla}_i g_{\alpha p}\cdot\widetilde{\nabla}_\beta g_{jq}\big).\end{aligned} \tag{3-2}$$

The h-flow is closely related to the Ricci flow

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Ric}(g).$$

Suppose g_0 is a smooth metric with bounded curvature; then the solution to the h-flow with $h = g_0$ such that $g(0) = g_0$ is the solution to the usual Ricci–DeTurck flow. Using the solution to the Ricci–DeTurck flow, one can obtain a solution to the Ricci flow through a smooth family of diffeomorphisms. Hence h-flow can be considered as a generalization of Ricci flow with initial data which may not be smooth.

Let

$$\square = \frac{\partial}{\partial t} - g^{ij}\widetilde{\nabla}_i\widetilde{\nabla}_j. \tag{3-3}$$

For a constant $\delta > 1$, h is said to be δ close to a metric g if

$$\delta^{-1} h \le g \le \delta h.$$

Theorem 3.1 [Simon 2002]. *There exists $\epsilon = \epsilon(n) > 0$ depending only on n such that if (M^n, g_0) is an n-dimensional compact or noncompact manifold without boundary with continuous Riemannian metric g_0 which is $(1+\epsilon(n))$ close to a smooth complete Riemannian metric h with curvature bounded by k_0, then the h-flow (3-2) has a smooth solution on $M \times (0, T]$ for some $T > 0$ with T depending only on n, k_0 such that $g(t) \to g_0$ as $t \to 0$ uniformly on compact sets and such that*

$$\sup_{x \in M} |\widetilde{\nabla}^i g(t)|^2 \le \frac{C_i}{t^i}$$

for all i, where C_i depends only on $n, k_0, \dots, k_i$ where k_j is the bound of $|\widetilde{\nabla}^j \mathrm{Rm}(h)|$. Moreover, h is $(1+2\epsilon)$ close to $g(t)$ for all t. Here and in the following $|\cdot|$ is the norm with respect to h.

In the case that g_0 is smooth, and if $|\widetilde{\nabla} g_0|$ is bounded, then it is also proved in [Simon 2002] that

$$|\widetilde{\nabla} g(t)| \le C, \quad |\widetilde{\nabla}^2 g(t)| \le C t^{-1/2}.$$

We want to obtain estimates in the case that $g_0 \in W^{1,p}_{\mathrm{loc}}$ in the sense that $|\widetilde{\nabla} g_0|$ is in L^p_{loc}, for $p > n$. We have the following:

Lemma 3.2. *Fix $p \ge 2$. There is $b = b(n, p) > 0$ depending only on n, p, with $e^b \le 1+\epsilon(n)$, where $\epsilon(n)$ is the constant in Theorem 3.1, such that if g_0 is a smooth metric which is e^b close to h, where h is smooth and satisfies (3-1) for $0 \le i \le 2$, then the solution $g(t)$ of the h-flow with initial metric g_0 on $M \times [0, T]$ described in Theorem 3.1 satisfies the following estimates. There is a constant $C > 0$ depending only n, p, h such that for any $x_0 \in M$ with injectivity radius $\iota(x_0)$ with respect to h,*

$$|\widetilde{\nabla} g(t, x_0)|^2 \le \frac{CD}{t^{n/(2p)}}$$

for $T > t > 0$, where D is a constant depending only n, the lower bound of $\iota(x_0)$ and the L^{2p} norm of $|\widetilde{\nabla} g_0|$ in $B(x_0, \iota(x_0))$, which is the geodesic ball with respect to h.

Proof. Suppose g_0 is $e^b < 1+\epsilon(n)$ close to h; then for any $\lambda > 0$, λg_0 is also e^b close to λh. Moreover, if $g(t)$ is the solution to the h-flow, then $\lambda g\big(\frac{1}{\lambda} t\big)$ is a solution to the λh-flow. Hence by scaling, we may assume that $k_0 + k_1 + k_2 \le 1$. The solution $g(t)$ constructed in [Simon 2002] is e^{2b} close to h. Moreover, we may assume that $T \le 1$.

Denote $\iota(x_0)$ by ι_0 and we may assume that $\iota_0 \le 1$. In the following c_i will denote a constant depending only on n. Let $m \ge 2$ be an integer, which will be chosen depending only on n, p. Let $b = 1/(2m)$. First choose m so that $e^b \le 1+\epsilon(n)$. Let $f_1 = |\widetilde{\nabla} g|$ and $\psi = \big(a + \sum_{i=1}^n \lambda_i^m\big) f_1^2$ with $a > 0$, where λ_i are the eigenvalues of $g(t)$ with respect to h. By choosing a depending only on n and m large enough depending only on n, as in [Shi 1989; Simon 2002] (see also [Huang and Tam 2015, (5.8)]), we have

$$\Box\psi \le c_1 - c_2 m^2 f_1^4. \tag{3-4}$$

Let x^i be normal coordinates in $B(x_0, \iota_0)$. Since $k_0+k_1+k_2 \le 1$, by [Hamilton 1995, Corollary 4.11] on $B(x_0, \iota_0)$ we have

$$\begin{cases} \frac{1}{2}|\xi|^2 \le h_{ij}\xi^i\xi^j \le 2|\xi|^2 & \text{for } \xi \in \mathbb{R}^n, \\ |D_x^\beta h_{ij}| \le c_3 & \text{for all } i, j, \end{cases} \tag{3-5}$$

where

$$h_{ij} = h\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

and $\beta = (\beta_1, \dots, \beta_n)$ is a multi-index with $|\beta| \le 2$ and

$$D_{x^k} = \frac{\partial}{\partial x^k}.$$

Let η be a smooth function on $[0, 1]$ such that $0 \le \eta \le 1$, $\eta(s) = 0$ for $s \ge \frac{3}{4}$, $\eta(s) = 1$ for $0 \le s \le \frac{1}{2}$. Still denote $\eta(|x|/\iota_0)$ by $\eta(x)$. Then $|\widetilde{\nabla}\eta| \le c_4\iota_0^{-1}$. We have

$$\begin{aligned}
\frac{d}{dt}\int_{B(x_0,\iota_0)} \eta^2\psi^p\,dv_h
&= p\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1}\psi_t\,dv_h \\
&\le p\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1}g^{ij}\widetilde{\nabla}_i\widetilde{\nabla}_j\psi\,dv_h + p\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1}(c_1 - c_2m^2f_1^4)\,dv_h \\
&\le -p(p-1)c_5\int_{B(x_0,\iota_0)} \eta^2\psi^{p-2}|\widetilde{\nabla}\psi|^2\,dv_h + pc_6\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1}f_1|\widetilde{\nabla}\psi|\,dv_h \\
&\quad + pc_7\iota_0^{-1}\int_{B(x_0,\iota_0)} \eta\eta'\psi^{p-1}|\widetilde{\nabla}\psi|\,dv_h + p\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1}(c_1 - c_2m^2f_1^4)\,dv_h \\
&\le \frac{c_6^2}{2c_5(p-1)}\int_{B(x_0,\iota_0)} f_1^2\eta^2\psi^p\,dv_h + \frac{c_7^2}{2c_5(p-1)\iota_0^2}\int_{B(x_0,\iota_0)} (\eta')^2\psi^p\,dv_h \\
&\quad + p\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1}(c_1 - c_2m^2f_1^4)\,dv_h \\
&\le \frac{c_8 p}{p-1}\int_{B(x_0,\iota_0)} f_1^4\eta^2\psi^{p-1}\,dv_h + \frac{c_7^2}{2c_5(p-1)\iota_0^2}\int_{B(x_0,\iota_0)} (\eta')^2\psi^p\,dv_h \\
&\quad + p\int_{B(x_0,\iota_0)} \eta^2\psi^{p-1}(c_1 - c_2m^2f_1^4)\,dv_h,
\end{aligned}$$

where we have used the fact that $\psi \le c f_1^2$ for some constant c depending only on n by the fact that $2bm = 1$ so that $\lambda_i^m \le 1$ for all i. We have also used the fact that

$$c_6 \int_{B(x_0,\iota_0)} \eta^2 \psi^{p-1} f_1 |\widetilde{\nabla}\psi| \, dv_h \le \tfrac{1}{2} c_5(p-1) \int_{B(x_0,\iota_0)} \eta^2 \psi^{p-2} |\widetilde{\nabla}\psi|^2 \, dv_h + \frac{c_6^2}{2c_5(p-1)} \int_{B(x_0,\iota_0)} f_1^2 \eta^2 \psi^p \, dv_h$$

and

$$c_7 \iota_0^{-1} \int_{B(x_0,\iota_0)} \eta\eta' \psi^{p-1} |\widetilde{\nabla}\psi| \, dv_h \le \tfrac{1}{2} c_5(p-1) \int_{B(x_0,\iota_0)} \eta^2 \psi^{p-2} |\widetilde{\nabla}\psi|^2 \, dv_h + \frac{c_7^2}{2c_5(p-1)\iota_0^2} \int_{B(x_0,\iota_0)} (\eta')^2 \psi^p \, dv_h.$$

Hence by choosing m large enough depending only on n, p and if $b = 1/(2m)$, we have

$$\frac{d}{dt} \int_{B(x_0,\iota_0)} \eta^2 \psi^p \, dv_h \le c_9 p \iota_0^{-2} \biggl(\int_{B(x_0,\iota_0)} (\eta')^2 \psi^p \, dv_h + \int_{B(x_0,\iota_0)} \eta^2 \psi^{p-1} \, dv_h \biggr).$$

By replacing η by η^q for $q \ge 1$, we may assume that $|\eta'| \le C\eta^{1-1/q}$, where C depends only on q. Let $q = 2p$, say; then we have

$$\begin{aligned}
&\frac{d}{dt} \int_{B(x_0,\iota_0)} \eta^2 \psi^p \, dv_h \\
&\le C_1 \iota_0^{-2} \biggl(\int_{B(x_0,\iota_0)} (\eta^2)^{1-\frac{1}{2p}} \psi^p \, dv_h + \int_{B(x_0,\iota_0)} \eta^2 \psi^{p-1} \, dv_h \biggr) \\
&\le C_1 \iota_0^{-2} \biggl[\biggl(\int_{B(x_0,\iota_0)} \eta^2 \psi^p \, dv_h \biggr)^{1-\frac{1}{2p}} \biggl(\int_{B(x_0,\iota_0)} \psi^p \, dv_h \biggr)^{\frac{1}{2p}} + \biggl(\int_{B(x_0,\iota_0)} \eta^2 \psi^p \, dv_h \biggr)^{1-\frac{1}{p}} \biggr] \\
&\le C_2 \iota_0^{-2} \biggl[\biggl(\int_{B(x_0,\iota_0)} \eta^2 \psi^p \, dv_h \biggr)^{1-\frac{1}{2p}} t^{-1/2} + \biggl(\int_{B(x_0,\iota_0)} \eta^2 \psi^p \, dv_h \biggr)^{1-\frac{1}{p}} \biggr].
\end{aligned}$$

Here and below upper case C_i denote a positive constant depending only on n, p and h. Here we have used the estimates in Theorem 3.1. Let

$$F = \int_{B(x_0,\iota_0)} \eta^2 \psi^p \, dv_h + 1.$$

Then we have

$$\frac{d}{dt} F \le C_3 \iota_0^{-2} F^{1-\frac{1}{2p}} t^{-\frac{1}{2}}.$$

Let $I = \int_{B(x_0,\iota_0)} |\widetilde{\nabla} g_0|^{2p} \, dv_h$. We conclude that

$$F(t) \le C_4 (I + \iota_0^{-4p}),$$

or

$$\int_{B\left(x_0,\frac{1}{2}\iota_0\right)} \psi^p \, dv_h \le C_5(I + \iota_0^{-4p}).$$

Hence $0 < t_0 < T$, by the mean value equality [Lieberman 1996, Theorem 7.21] applied to (3-4) to $B(x_0, r) \times (t_0 - r^2, t_0)$ with $r = \frac{1}{2}\sqrt{t_0}$, we have

$$\psi^p(x_0, t_0) \le C_6 r^{-n}(I + \iota_0^{-2p} + 1).$$

From this the result follows. □

Assume $2p > n$ and let $\delta = n/(2p)$. Let b as in Lemma 3.2. Assume h satisfies (3-1), for $0 \le i \le 2$.

Lemma 3.3. *Let $x_0 \in M$ and let $r_0 > 0$. Let*

$$I := \int_{B(x_0,r_0)} |\widetilde{\nabla} g_0|^{2p} \, dv_h.$$

Let ι be the infimum of the injectivity radii $\iota(x)$, $x \in B(x_0, r_0)$. Then there is a constant C depending only on n, p, h, r_0, the lower bound of ι and the upper bound of I such that

$$|\widetilde{\nabla}^2 g(x_0, t)|^2 \le C t^{-1-\delta}.$$

Proof. In the following, C_i will denote a constant depending only on the quantities mentioned in the lemma. By Lemma 3.2, we have

$$\sup_{x\in B\left(x_0,\frac{r_0}{2}\right)} |\widetilde{\nabla} g(x, t)|^2 \le C_1 t^{-\delta}. \tag{3-6}$$

Let $f_i = |\widetilde{\nabla}^i g|$. As in [Shi 1989; Simon 2002] (see also [Huang and Tam 2015, (5.11)]), one can find $a > 0$ depending only on the quantities mentioned in the lemma such that if $\psi = (at^{-\delta} + f_1^2) f_2^2$, then

$$\Box\psi \le -\tfrac{1}{8} f_2^4 + C_2 t^{-4\delta} \tag{3-7}$$

on $B(x_0, r_0/2) \times (0, T]$. We may assume that $\iota(x_0) \le r_0/2$. Let η be a cutoff function such that $(\eta')^2 + |\eta''| \le c\eta$ for some absolute constant as in the proof of Lemma 3.3, let $F = t^{1+2\delta}\eta\psi$. Since g is smooth up to $t = 0$, and $f_1^2 \le C_1 t^{-\delta}$, we have $F(\,\cdot\,, 0) = 0$. If F has a positive maximum, then there is $x_1 \in B(x_0, \iota)$ and $T \ge t_1 > 0$ such that

$$F(x_1, t_1) = \sup_{B(x_0,\iota)\times[0,T]} F.$$

Hence at (x_1, t_1), we have

$$\eta\widetilde{\nabla}_i\psi + \psi\widetilde{\nabla}_i\eta = 0$$

and

$$
\begin{aligned}
0 &\le \Box F \\
&= t_1^{1+2\delta}(\eta\Box\psi + \psi\Box\eta - 2g^{ij}\widetilde{\nabla}_i\psi\widetilde{\nabla}_j\eta) + (1+2\delta)t_1^{-1}F \\
&\le t_1^{1+2\delta}\big[\eta\big(-\tfrac{1}{8}f_2^4 + C_2t^{-4\delta}\big) - \psi g^{ij}\widetilde{\nabla}_i\widetilde{\nabla}_j\eta + 2g^{ij}\eta^{-1}\psi\widetilde{\nabla}_i\eta\widetilde{\nabla}_j\eta\big] + (1+2\delta)t_1^{-1}F \\
&\le t_1^{1+2\delta}\big[\eta\big(-\tfrac{1}{8}f_2^4 + C_2t^{-4\delta}\big) + C_3\psi\big] + (1+2\delta)t_1^{-1}F.
\end{aligned}
$$

Multiply the inequality by $t_1^{1+2\delta}\eta(at^{-\delta}+f_1^2)^2 = F\psi^{-1}(at^{-\delta}+f_1^2)$, we have

$$
\begin{aligned}
0 &\le -\tfrac{1}{8}F^2 + C_3t_1^{1+\delta}(at^{-\delta}+f_1^2)F + (1+2\delta)t^{2\delta}(at^{-\delta}+f_1^2)^2F \\
&\le -\tfrac{1}{8}F^2 + C_4F.
\end{aligned}
$$

Hence $F \le 8C_4$. From this it is easy to see that the result follows. $\square$

4. Approximation of singular metrics

Let $(M^n, \mathfrak{b})$ be a smooth complete Riemannian manifold of dimension n without boundary. Let g_0 be a continuous Riemannian metric on M satisfying the following:

(a1) There is a compact subset Σ such that g_0 is smooth on $M \setminus \Sigma$.

(a2) The metric g_0 is in $W^{1,p}_{\mathrm{loc}}$ for some $p \ge 1$ in the sense that g_0 has weak derivative and $|g_0|_{\mathfrak{b}}$, $|{}^{\mathfrak{b}}\nabla g_0|_{\mathfrak{b}} \in L^p_{\mathrm{loc}}$ with respect to the metric $\mathfrak{b}$.

We want to approximate g_0 by smooth metrics with uniform bound on the $W^{1,p}$ norm locally. As in [Lee 2013], cover Σ by finitely many precompact coordinate patches $U_1, \dots, U_N$ and cover M with $U_1, \dots, U_N$ and U_{N+1} such that U_{N+1} is an open set with $U_{N+1} \cap \Sigma = \varnothing$. We may assume that there is a partition of unity ψ_k with $\mathrm{supp}(\psi_k) \subset U_k$. Since g_0 is continuous, we may assume that g_0, $\mathfrak{b}$ and the Euclidean metric are equivalent in each U_k, $1 \le k \le N$. For any $a > 0$, let $\Sigma(a) = \{x \in M \mid d_{\mathfrak{b}}(x, \Sigma) < a\}$. By [Lee 2013, Lemma 3.1], for each $1 \le k \le N$, there is a smooth function $\epsilon \ge \rho_k \ge 0$ in U_k such that for $\epsilon > 0$ small enough

$$
\begin{cases}
\rho_k = \epsilon, & \Sigma(\epsilon) \cap U_k; \\
\rho_k = 0, & U_k \setminus \Sigma(2\epsilon); \\
|\partial\rho_k| \le C; \\
|\partial^2\rho_k| \le C\epsilon^{-1}
\end{cases}
\tag{4-1}
$$

for some C independent of ϵ and k. Here $\partial\rho_k$ and $\partial^2\rho_k$ are derivatives with respect to the Euclidean metric. Let $g_0^k = \psi_k g_0$ and for $1 \le k \le N$, let

$$
(g^k_{\epsilon,0})_{ij}(x) = \int_{\mathbb{R}^n} g^k_{0,ij}(x - \lambda\rho_k(x)y)\varphi(y)\,dy. \tag{4-2}
$$

Here φ is a nonnegative smooth function in $\mathbb{R}^n$ with support in $B(1)$ and integral equal to 1. $\lambda > 0$ is a constant independent of ϵ and k, to be determined. Finally, define

$$g_{\epsilon,0} = \sum_{k=1}^{N} g^k_{\epsilon,0} + \psi_{N+1} g_0. \tag{4-3}$$

Lemma 4.1. *For $\epsilon > 0$ small enough, $g_{\epsilon,0}$ is a smooth metric such that $g_{\epsilon,0}$ converges to g_0 in C^0 norm as $\epsilon \to 0$, and $g_{\epsilon,0} = g_0$ outside $\Sigma(2\epsilon)$. Moreover, there is a constant C independent of ϵ such that*

$$\int_{\Sigma(1)} |{}^{\mathfrak{b}}\nabla g_{\epsilon,0}|^p_{\mathfrak{b}} \, dv_{\mathfrak{b}} \le C.$$

Proof. It is easy to see that $g_{\epsilon,0}$ is smooth and converges to g_0 uniformly as $\epsilon \to 0$. In order to estimate the $W^{1,p}_{\mathrm{loc}}$ norm of $g_{\epsilon,0}$, it is sufficient to estimate the norm in each U_k, $1 \le k \le N$. Moreover, we may assume that $\mathfrak{b}$ is the Euclidean metric. So it is sufficient to prove the following: For fixed k, $1 \le k \le N$, and for any $u \in W^{1,p}_{\mathrm{loc}}$ if

$$v(x) = \int_{\mathbb{R}^n} u(x - \lambda\rho_k(x)y)\varphi(y)\, dy,$$

then the $W^{1,p}$ norm of v in $\Sigma(1)$ can be estimated in terms of the $W^{1,p}$ norm of u in $\Sigma(2)$, say. For fixed y with $|y| \le 1$, let $z = x - \lambda\rho_k(x)y$. Then

$$\frac{\partial z^i}{\partial x^j} = \delta_{ij} - y^i \lambda \frac{\partial \rho_k}{\partial x^i}.$$

By (4-1), we can choose $\lambda > 0$ small enough independent of ϵ and k so that

$$2 \ge \det\left(\delta_{ij} - \lambda y^i \frac{\partial \rho_k}{\partial x^i}\right) \ge \tfrac{1}{2},$$

and so that $z = z(x)$ is a diffeomorphism with the Jacobian being bounded above and below by some constants independent of ϵ, k. Hence

$$\begin{aligned}
\left(\int_{\Sigma(1)\cap U_k} |v|^p(x)\, dx\right)^{\frac{1}{p}} &\le \left[\int_{\Sigma(1)\cap U_k}\left(\int_{\mathbb{R}^n} |u(x-\lambda\rho_k(x)y)|\varphi(y)\, dy\right)^p dx\right]^{\frac{1}{p}} \\
&\le \int_{\mathbb{R}^n} \varphi(y)\left(\int_{\Sigma(1)\cap U_k} |u(x-\lambda\rho_k(x)y)|^p\, dx\right)^{\frac{1}{p}} dy \\
&= \int_{B(1)} \varphi(y)\left(\int_{\Sigma(1)\cap U_k} |u(x-\lambda\rho_k(x)y)|^p\, dx\right)^{\frac{1}{p}} dy \\
&\le C_1\left(\int_{\Sigma(2)} |u(z)|^p\, dz\right)^{\frac{1}{p}}
\end{aligned}$$

for some constant C_1 independent of ϵ, k provided ϵ is small enough, where we have used Minkowski's integral inequality [Stein 1970, Section A.1]. Now, if $x \notin \Sigma(2\epsilon)$, then $v(x) = u(x)$ and if $x \in \Sigma(\epsilon)$, then $v(x)$ is the standard mollification. If $x \in \Sigma(2\epsilon) \setminus \Sigma(\epsilon)$, then

$$|\partial v|(x) \le \int_{\mathbb{R}^n} |\partial u|(x - \lambda\rho_k(x)y)\lambda|\partial\rho_k(x)|\varphi(y)\,dy.$$

Since $|\partial\rho_k|$ is bounded by (4-1), we can prove as before that

$$\left(\int_{\Sigma(1)\cap U_k} |\partial v|^p(x)\,dx\right)^{\frac{1}{p}} \le C_2\left(\int_{\Sigma(2)} |\partial u|^p(z)\,dz\right)^{\frac{1}{p}}$$

for some constant C_2 independent of ϵ, k provided ϵ is small enough. □

In addition to (a1) and (a2), assume

(a3) The scalar curvature $\mathcal{S}_{g_0}$ of g_0 satisfies $\mathcal{S}_{g_0} \ge \sigma$ in $M \setminus \Sigma$, where σ is a constant.

We want to modify $g_{\epsilon,0}$ to obtain a smooth metric with scalar curvature bounded below by σ. We first consider the case that M is compact. Let $\epsilon_0 > 0$ be small enough so that for all $\epsilon_0 \ge \epsilon > 0$,

$$(1+\epsilon(n))^{-1} g_{\epsilon_0,0} \le g_{\epsilon,0} \le (1+\epsilon(n)) g_{\epsilon_0,0},$$

where $\epsilon(n) > 0$ is the constant depending only on n in Theorem 3.1. Hence if we let $h = g_{\epsilon_0,0}$, then the h-flow has solution $g_\epsilon(t)$ on $M \times [0, T]$ for some $T > 0$ independent of ϵ, with initial data $g_{\epsilon,0}$ in the sense that $\lim_{t\to 0} g_\epsilon(x,t) = g_{\epsilon,0}(x)$ uniformly in M; see Theorem 3.1. The curvature and all the covariant derivatives of curvature of h are bounded because M is compact.

By [Simon 2002] and Lemmas 3.2, 3.3 and 4.1 we have the following:

Lemma 4.2. *Let M be compact and suppose g_0 satisfies (a1)–(a3). Suppose $p > n$. Let $\delta = n/p < 1$. Then*

$$|{}^h\nabla g_\epsilon(t)|_h^2 \le Ct^{-\delta} \quad \textit{and} \quad |{}^h\nabla^2 g_\epsilon(t)|^2 \le Ct^{-1-\delta}$$

for some constant C independent of ϵ, t. Moreover, $g_\epsilon(t)$ subconverges to the solution $g(t)$ of the h-flow with initial data g_0 in C^∞ norm in compact sets of $M \times (0, T]$ and in compact sets of $M \setminus \Sigma \times [0, T]$.

For $\epsilon > 0$ small enough, let

$$W^k = (g_\epsilon(t))^{pq}\big(\Gamma^k_{pq}(g_\epsilon(t)) - \Gamma^k_{pq}(h)\big), \tag{4-4}$$

and let Φ_t be the diffeomorphism given by

$$\frac{\partial}{\partial t}\Phi_t(x) = -W(\Phi_t(x), t), \quad \Phi_0(x) = x. \tag{4-5}$$

Let $\tilde{g}_\epsilon(t) = \Phi_t^* g_\epsilon(t)$. Then $\tilde{g}_\epsilon(t)$ satisfies the Ricci flow equation with initial data $g_{\epsilon,0}$. Note that W and Φ_t depend also on ϵ. Recall the Ricci flow equation is

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}. \tag{4-6}$$

Lemma 4.3. *With the same assumptions and notation as in Lemma 4.2, for ϵ small enough, $|W|_h \le Ct^{-\frac{1}{2}\delta}$, $|\mathrm{Rm}(\tilde{g}_\epsilon(t))| \le Ct^{-\frac{1}{2}(1+\delta)}$ and*

$$C^{-1}h \le g_\epsilon(t) \le Ch$$

for some C, independent of ϵ, t.

Proof. The bound of W is given by Lemma 4.2. Since the bound of curvature is unchanged under diffeomorphism, $|\mathrm{Rm}(\tilde{g}_\epsilon(t))| \le Ct^{-\frac{1}{2}(1+\delta)}$ by Lemma 4.2. From this we conclude from the Ricci flow equation that $\tilde{g}_\epsilon(t)$ is uniformly equivalent to $g_{0,\epsilon}$ which is uniformly equivalent to h. □

Lemma 4.4. *Let $\mathcal{S}(t)$ be the scalar curvature of $g(t)$. Then there is a constant $C > 0$ independent of t, ϵ such that*

$$\exp(-Ct^{\frac{1}{2}(1-\delta)}) \int_M (\mathcal{S}(t) - \sigma)_-\, dv_{g(t)}$$

is nonincreasing in $(0, T]$, where $f_- = \max\{-f, 0\}$ is the negative part of f.

Proof. As in [McFeron and Székelyhidi 2012], fix $\theta > 0$, and for $\epsilon > 0$, let

$$v = \big((\mathcal{S}_\epsilon(t) - \sigma)^2 + \theta\big)^{1/2} - (\mathcal{S}_\epsilon(t) - \sigma),$$

where $\mathcal{S}_\epsilon(t)$ is the scalar curvature of $\tilde{g}_\epsilon(t)$. Let Δ and ∇ be the Laplacian and covariant derivative with respect to $\tilde{g}_\epsilon(t)$. Using the evolution equation of the scalar curvature in Ricci flow, we have

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big)v &= \bigg(\frac{\mathcal{S}_\epsilon(t)-\sigma}{\big((\mathcal{S}_\epsilon(t)-\sigma)^2+\theta\big)^{1/2}} - 1\bigg)\Big(\frac{\partial}{\partial t} - \Delta\Big)\mathcal{S}_\epsilon(t) - \frac{\theta|\nabla\mathcal{S}_\epsilon|^2}{\big((\mathcal{S}_\epsilon(t)-\sigma)^2+\theta\big)^{1/2}} \\ &= \bigg(\frac{\mathcal{S}_\epsilon(t)-\sigma}{\big((\mathcal{S}_\epsilon(t)-\sigma)^2+\theta\big)^{1/2}} - 1\bigg)\cdot 2|\nabla \mathrm{Ric}(t)|^2 - \frac{\theta|\nabla\mathcal{S}_\epsilon(t)|^2}{\big((\mathcal{S}_\epsilon(t)-\sigma)^2+\theta\big)^{3/2}} \\ &\le 0,\end{aligned}$$

where $\mathrm{Ric}(t)$ is the Ricci tensor of $\tilde{g}_\epsilon(t)$. Using Lemma 4.3 we have

$$\begin{aligned}\frac{d}{dt}\int_M v\, dv_{\tilde{g}_\epsilon(t)} &= \int_M \frac{\partial}{\partial t} v\, dv_{\tilde{g}_\epsilon(t)} - \int_M \mathcal{S}_\epsilon(t) v\, dv_{\tilde{g}_\epsilon(t)} \\ &\le \int_M \Delta v\, dv_{\tilde{g}_\epsilon(t)} + C_1 t^{-\frac{1}{2}(1+\delta)} \int_M v\, dv_{\tilde{g}_\epsilon(t)} \\ &= C_1 t^{-\frac{1}{2}(1+\delta)} \int_M v\, dv_{\tilde{g}_\epsilon(t)}\end{aligned} \tag{4-7}$$

for some constant C_1 independent of t, ϵ. From this and letting $\theta \to 0$, we conclude that for some constant C independent of t and ϵ,

$$\exp(-Ct^{\frac{1}{2}(1-\delta)}) \int_M (\mathcal{S}_\epsilon(t) - \sigma)_- \, dv_{\tilde{g}_\epsilon(t)}$$

is nonincreasing in $(0, T]$. Noting that $\tilde{g}_\epsilon(t) = \Phi_t^*(g_\epsilon(t))$, by Lemma 4.2 let $\epsilon \to 0$, the result follows. □

We first consider the case that the codimension of Σ is at least 2 in the following sense:

(a4) The volume $V(\Sigma(\epsilon), g_0)$ with respect to g_0 of the ϵ-neighborhood $\Sigma(\epsilon)$ of Σ is bounded by $C\epsilon^2$ for some constant C independent of ϵ. Here

$$\Sigma(\epsilon) = \{x \in M \mid d_{g_0}(x, \Sigma) < \epsilon\}.$$

Lemma 4.5. *With the same assumptions and notation as in Lemma 4.2, suppose (a4) is true. Then $S(t) \geq \sigma$ for all $t > 0$.*

Proof. By Lemma 4.4, it is sufficient to prove that

$$\lim_{t \to 0} \int_M (\mathcal{S}(t) - \sigma)_- \, dv_{g(t)} = 0. \tag{4-8}$$

For any $\epsilon > 0$, let Φ_t be the diffeomorphisms as before so that $\tilde{g}_\epsilon(t) = \Phi_t^*(g_\epsilon(t))$ is the solution to the Ricci flow. For any $\theta > 0$, let v as in the proof of Lemma 4.4. Let

$$\beta = \frac{1}{\epsilon}\left(\epsilon - \sum_{k=1}^N \psi_k \rho_k\right).$$

We may modify ρ_k so that if ϵ is small enough then β is a smooth function on M such that $\beta = 0$ in $\Sigma(2\epsilon)$, $\beta = 1$ outside $\Sigma(4\epsilon)$, $0 \leq \beta \leq 1$, $|{}^h\nabla\beta| \leq C\epsilon^{-1}$, and $|{}^h\nabla^2\beta| \leq C\epsilon^{-2}$ for some constant C independent of ϵ, t. Let

$$\tilde{\beta}(t, x) = \beta(\Phi_t(x)).$$

Then

$$\begin{aligned}
\frac{d}{dt}\int_M \tilde{\beta}^2 v \, dv_{\tilde{g}_\epsilon(t)} &= \int_M v \frac{\partial}{\partial t}(\tilde{\beta}^2) \, dv_{\tilde{g}_\epsilon(t)} + \int_M \tilde{\beta}^2 \frac{\partial}{\partial t} v \, dv_{\tilde{g}_\epsilon(t)} - \int_M \mathcal{S}_\epsilon(t)\tilde{\beta}^2 v \, dv_{\tilde{g}_\epsilon(t)} \\
&\leq \int_M v \frac{\partial}{\partial t}(\tilde{\beta})^2 \, dv_{\tilde{g}_\epsilon(t)} + \int_M \tilde{\beta}^2 \Delta_{\tilde{g}_\epsilon(t)} v \, dv_{\tilde{g}_\epsilon(t)} \\
&\quad + C_1 t^{-\frac{1}{2}(1+\delta)} \int_M \tilde{\beta}^2 v \, dv_{\tilde{g}_\epsilon(t)} \\
&= I + II + C_1 t^{-\frac{1}{2}(1+\delta)} \int_M \tilde{\beta}^2 v \, dv_{\tilde{g}_\epsilon(t)}.
\end{aligned}$$

for some constant $C_1 > 0$ independent of t, ϵ, θ by Lemma 4.3. Let $w(y) = v(\Phi_t^{-1}(y))$. Since in local coordinates,

$$\Delta_{g_\epsilon(t)} f = g_\epsilon^{ij}(\partial_i \partial_j f - \Gamma_{ij}^k \partial_k)$$

with $|\Gamma_{ij}^k| \le C t^{-\delta/2}$ for some constant C independent of ϵ, t by Lemma 4.2, we have $|w| \le C t^{-\frac{1}{2}(1+\delta)}$ for some constant C independent of ϵ, t, θ. Using also (a4) and Lemma 4.1, we have

$$\begin{aligned} II &= \int_M \beta^2 \Delta_{g_\epsilon(t)} w \, dv_{g_\epsilon(t)} \\ &= \int_M w \Delta_{g_\epsilon(t)}(\beta^2)\, dv_{g_\epsilon(t)} \\ &\le C_2 \int_{\Sigma(4\epsilon)\setminus\Sigma(2\epsilon)} w|\epsilon^{-2} + \epsilon^{-1} t^{-\delta/2}\beta|\, dv_{g_\epsilon(t)} \\ &\le C_3 \left(t^{-\frac{1}{2}(1+\delta)} + \epsilon^{-1} t^{-\frac{\delta}{2}-\frac{1}{4}(1+\delta)} \int_{\Sigma(4\epsilon)} \beta w^{1/2}\, dv_{g_\epsilon(t)} \right) \\ &\le C_4 \left[t^{-\frac{1}{2}(1+\delta)} + t^{-\frac{1}{4}(1+3\delta)} \left(\int_M \tilde\beta^2 v \, dv_{\tilde g_\epsilon(t)} \right)^{\frac{1}{2}} \right] \end{aligned}$$

for some constants $C_2 - C_4$ independent of ϵ, t, θ, where we have used Lemma 4.2, the fact that $\beta = 1$ outside $\Sigma(4\epsilon)$, the Hölder inequality and the fact that $V(\Sigma(4\epsilon)) = O(\epsilon^2)$. To estimate I, we have

$$\begin{aligned} \frac{\partial}{\partial t}\tilde\beta &= (d\tilde\beta)\left(\frac{\partial}{\partial t}\right) \\ &= d\beta \circ d\Phi_t\left(\frac{\partial}{\partial t}\right) \\ &= d\beta(W). \end{aligned}$$

Hence by Lemma 4.2, we have

$$\left|\frac{\partial}{\partial t}\tilde\beta\right|(x) \le C_5 |{}^h\nabla\beta|(\Phi_t(x))| \le C_6 \epsilon^{-1} t^{-\delta/2}$$

for some constants C_5, C_6 independent of ϵ, t, θ. Hence if w is as above, then

$$\begin{aligned} I &\le C_6 \epsilon^{-1} t^{-\delta/2} \int_{\Sigma(4\epsilon)} \beta w(y)\, dv_{g_\epsilon(t)} \\ &\le C_7 t^{-\frac{1}{4}(1+3\delta)} \left(\int_{\Sigma(4\epsilon)} \tilde\beta^2 v\, dv_{\tilde g_\epsilon(t)} \right)^{\frac{1}{2}} \end{aligned}$$

for some constant C_7 independent of ϵ, t, θ. To summarize, if we let

$$F = \int_M \tilde\beta^2 v \, dv_{\tilde g_\epsilon(t)},$$

then

$$\begin{aligned}\frac{dF}{dt} &\le C_8(t^{-\frac{1}{2}(1+\delta)} + t^{-\frac{1}{2}(1+\delta)}F + t^{-\frac{1}{4}(1+3\delta)}F^{1/2})\\ &\le C_8(t^{-\frac{1}{2}(1+\delta)} + t^{-\delta} + 2t^{-\frac{1}{2}(1+\delta)}F)\end{aligned}$$

for some constant C_8 independent of ϵ, t, θ. Integrate from 0 to t, and let $\theta \to 0$. Since $g_{\epsilon,0} = g_0$ outside $\Sigma(2\epsilon)$, $\Phi_0 = \mathrm{id}$, and $\beta = 0$ on $\Sigma(2\epsilon)$, and $\mathcal{S}_{g_0} \ge \sigma$ outside Σ, there exist constants $C_9 - C_{10}$ independent of ϵ, t such that

$$\exp(-C_9 t^{\frac{1}{2}(1-\delta)}) \int_M \tilde{\beta}^2 (\mathcal{S}_\epsilon(t) - \sigma)_- \, dv_{\tilde{g}_\epsilon(t)} \le C_{10}(t^{\frac{1}{2}(1-\delta)} + t^{1-\delta})$$

because $0 < \delta < 1$. Letting $\epsilon \to 0$, we see that (4-8) is true and the proof of the lemma is completed. □

By Lemmas 4.2 and 4.5, using $g(t)$ we have:

Corollary 4.6. *Let $(M^n, \mathfrak{b})$ be a smooth compact manifold and let g_0 be a continuous Riemannian metric satisfying the following*:

(a) *There is a compact set Σ such that g_0 is smooth on $M \setminus \Sigma$ with scalar curvature bounded below by σ.*

(b) *The metric g_0 is in $W^{1,p}_{\mathrm{loc}}$ for some $p > n$.*

(c) $V(\Sigma(\epsilon), g_0) = O(\epsilon^2)$ *as* $\epsilon \to 0$, *where* $\Sigma(\epsilon) = \{x \in M \mid d_{\mathfrak{b}}(x, \Sigma) < \epsilon\}$.

Then there exists a sequence of smooth metrics g_i satisfying the following: (*i*) *as i tends to infinity g_i converges to g_0 uniformly in M, and converges to g_0 in C^∞ norm on any compact subset of $M \setminus \Sigma$*; (*ii*) *the scalar curvature $\mathcal{S}_i$ of g_i satisfies $\mathcal{S}_i \ge \sigma$.*

Remark 4.7. If the codimension of Σ is only assumed to be larger than 1, then the conclusions of Lemma 4.5 and Corollary 4.6 are still true under some additional assumptions on the second derivatives of g_0.

Next let us consider the case that Σ is an embedded hypersurface. Let (M^n, g_0) be a Riemannian metric satisfying the following:

(b1) Σ is a compact embedded orientable hypersurface, and g_0 is smooth on $M \setminus \Sigma$ with scalar curvature $\mathcal{S}_{g_0} \ge \sigma$.

(b2) There is a neighborhood U of Σ and a smooth function t defined near U such that U is diffeomorphic to $\{-a < t < a\} \times \Sigma$ for some $a > 0$ with $\Sigma = \{t = 0\}$. Moreover, $g_0 = dt^2 + g_\pm(z, t)$, $z \in \Sigma$, such that (t, z) are smooth coordinates and $g_-(\cdot, 0) = g_+(\cdot, 0)$, where g_+ is defined and smooth on $t \ge 0$, g_- is defined and smooth on $t \le 0$.

(b3) Let $U_+ = \{t > 0\}$, $U_- = \{t < 0\}$. With respect to the unit normal $\frac{\partial}{\partial t}$ the mean curvature H_+ of Σ with respect to g_+ and the mean curvature H_- of Σ with respect to g_- satisfy $H_- \ge H_+$.

By [Miao 2002, Proposition 3.1], letting $\epsilon > 0$ be small enough, one can find a smooth metric $g_{\epsilon,0}$ such that (i) $g_{\epsilon,0} = g_0$ outside $U(\epsilon) = \{-\epsilon < t < \epsilon\}$; (ii) $g_{0,\epsilon}$ converges uniformly to g_0; (iii) $|{}^h\nabla g_{0,\epsilon}|_h \le C$ with respect to some fixed background smooth metric h; (iv) there exists a constant $c > 0$ independent of ϵ such that the scalar curvature $\mathcal{S}_{g_{0,\epsilon}}$ satisfies

$$
\begin{cases}
\mathcal{S}_{g_{0,\epsilon}} = \mathcal{S}_{g_0} & \text{outside } U(\epsilon),\\
|\mathcal{S}_{g_{0,\epsilon}}| \le c & \text{in } \dfrac{\epsilon^2}{100} < |t| \le \epsilon,\\
\mathcal{S}_{g_{0,\epsilon}}(z,t) \ge -c + (H_-(z) - H_+(z))\epsilon^{-2}\phi\Big(\dfrac{100t}{\epsilon^2}\Big) & \text{in } -\dfrac{\epsilon^2}{100} < t \le \dfrac{\epsilon^2}{100},\\
|\mathcal{S}_{g_{0,\epsilon}}| \le c\epsilon^{-2}
\end{cases}
\tag{4-9}
$$

for $z \in \Sigma$. Here $\phi \ge 0$ is a smooth function in $\mathbb{R}$ with compact support in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ such that

$$\int_{\mathbb{R}} \phi(s)\, ds = 1.$$

Using arguments similar to those before using h-flow, we can conclude:

Corollary 4.8. *Let M^n be a compact smooth manifold and let g_0 be a Riemannian metric satisfying (b1)–(b3) such that the scalar curvature of g_0 on $M \setminus \Sigma$ is at least σ. Then there exists a sequence of smooth metrics g_i such that as i tends to infinity g_i converges to g_0 uniformly in M, and converges to g_0 in C^∞ norm on any compact subset of $M \setminus \Sigma$. Moreover, $\mathcal{S}_{g_i} \ge \sigma$.*

Proof. As before, choose $h = g_{0,\epsilon_0}$ for ϵ_0 small enough, one can solve the h-flow with initial data $g_{0,\epsilon}$. Let $g_\epsilon(t)$ be the solution and let $\mathcal{S}_\epsilon(t)$ be its scalar curvature. From the proof of Lemma 4.4, one can conclude that

$$
\begin{aligned}
\exp(-C_3 t^{1/2}) \int_M (\mathcal{S}_\epsilon(t) - \sigma)_-\, dv_{g_\epsilon(t)} &\le \int_M (\mathcal{S}_{g_{0,\epsilon}} - \sigma)_-\, dv_{g_{0,\epsilon}}\\
&= \int_{U(\epsilon)} (\mathcal{S}_{g_{0,\epsilon}} - \sigma)_-\, dv_{g_{0,\epsilon}}\\
&\le C_1\epsilon
\end{aligned}
$$

for some $C_1, C_3 > 0$ independent of ϵ, t. Here we used the fact that $H_- - H_+ \ge 0$. Let $\epsilon \to 0$, we conclude that the solution $g(t)$ of the h-flow with initial value g_0 has scalar curvature no less than σ. The result follows as before. $\square$

Remark 4.9. By [Miao 2002], suppose Σ is a compact orientable hypersurface, and a neighborhood of Σ is of the disjoint union of U_1, U_2 and Σ. Assume g_0 is smooth up Σ from each side U_i of Σ and such that the mean curvatures H_1, H_2 with respect to unit normals in the two sides of Σ satisfying $H_1 + H_2 \ge 0$, where unit normals are chosen to be outward pointing in each side. Then one can find a smooth structure so that (b2) and (b3) are true.

We give some applications.

Corollary 4.10. *Let (M^n, g) be a compact manifold such that M^n is the topological n-torus, g is smooth except at a point, where it has a cone singularity of the form*

$$g = dr^2 + \alpha^2 r^2 h_0$$

with $0 < \alpha \le 1$ and where h_0 is the standard metric on $\mathbb{S}^{n-1}$. Suppose the scalar curvature of g is nonnegative; then g must be flat and $\alpha = 1$.

Proof. For r small, the mean curvature of the level set $\{r\} \times \mathbb{S}^{n-1}$ with respect to the normal ∂_r is $H = (n-1)/r$. Consider the Euclidean ball $B(\alpha r)$ of radius αr with center at the origin. Then metric of the boundary is $(\alpha r)^2 h_0$. Moreover, the mean curvature is $H_0 = (n-1)/(\alpha r)$. Since $\alpha \le 1$, $H_0 \ge H$. By gluing $B(\alpha r)$ along with M along $\{r\} \times \mathbb{S}^{n-1}$, we obtain a metric with corner so that (b1)–(b3) are true by changing the smooth structure if necessary. Still denote this metric by g. By Corollary 4.8, there exist smooth metrics g_i on the new manifold with nonnegative scalar curvature such that $g_i \to g$ in C^∞ away from the singular part. By [Schoen and Yau 1979a; 1979c; Gromov and Lawson 1983], g_i is flat. Hence g must be flat away from the singular part. Let $r \to 0$, we conclude that the original metric g is flat, and we must have $\alpha = 1$. □

Similarly, one can prove the following:

Corollary 4.11. *Let (M^n, g) be a compact manifold such that M^n is the topological n-torus and g is smooth away from some compact set with codimension at least 2. Moreover, assume g is in $W^{1,p}_{\text{loc}}$ for some $p > n$. Suppose the scalar curvature of g is nonnegative; then g must be flat.*

Remark 4.12. Suppose M is asymptotically flat with nonnegative scalar curvature and with some cone singularities as in Corollary 4.10; then we still have positive mass for each end by [Miao 2002]. The proof is similar. Compare this result with the example in Proposition 2.3.

Let us consider the case that M^n is noncompact. Let g_0 be a continuous Riemannian metric on M which is smooth outside a compact set Σ. Suppose there is a family of smooth complete metrics $g_{\epsilon,0}$ on M such that $g_{\epsilon,0}$ converges uniformly to g_0 and converges smoothly on compact sets of $M \setminus \Sigma$. Assume $g_{\epsilon,0}$ has bounded curvature for all ϵ. As before, we can find $\epsilon_0 > 0$ such that if $h = g_{\epsilon_0,0}$ then there are solutions $g_\epsilon(t)$ to the h-flow with initial data $g_{\epsilon,0}$, and solution to the h-flow with initial data g_0 on some fixed interval $[0, T]$, $T > 0$. As in [Simon 2002], using [Shi 1989], we may assume that all the derivatives of the curvature of h are bounded. Moreover, $g_\epsilon(t)$ converges uniformly on compact sets of $M \times (0, T]$ and

$M \setminus \Sigma \times [0, T]$. Suppose the scalar curvature of g_0 satisfies $\mathcal{S}_{g_0} \geq \sigma$. We want to find conditions so that the scalar curvature of $g(t)$ is also bounded below by σ.

Lemma 4.13. *With the above assumptions and notation, suppose*

(i) $g_{\epsilon,0} = g_0$ *outside* $\Sigma(2\epsilon)$;

(ii) $|^h\nabla g_\epsilon(t)| \leq Ct^{-\frac{\delta}{2}}$ *and* $|^h\nabla^2 g_\epsilon(t)| \leq Ct^{-\frac{1}{2}(1+\delta)}$ *for some* C *independent of* ϵ, t;

(iii) *there is an* $R_0 > 0$ *and a* $C > 0$ *independent of* ϵ, t *such that*

$$\int_{M\setminus B(o,R_0)} |\mathcal{S}_\epsilon(t) - \sigma|\, dv_h \leq C,$$

where $B(o, R_0)$ *is the geodesic ball with respect to* h *and* $\mathcal{S}_\epsilon(t)$ *is the scalar curvature of* $g_\epsilon(t)$;

(iv) $V(\Sigma(2\epsilon), g_0) = O(\epsilon^2)$.

Then the scalar curvature $\mathcal{S}(t)$ *of* $g(t)$ *satisfies* $\mathcal{S}(t) \geq \sigma$ *for all* $t > 0$.

Proof. By [Shi 1989; Tam 2010], we can find a smooth function ρ such that

$$C_1^{-1}(r(x)+1) \leq \rho(x) \leq C_1(1 + r(x))$$

for some constant $C_1 > 0$ where $r(x)$ is the distance function to a fixed point o with respect to h. Moreover, the gradient and Hessian of ρ with respect to h are uniformly bounded. (Hence the constants in the lemma may depend also on the choice of o.)

Let $0 \leq \eta \leq 1$ be a smooth function on $\mathbb{R}$ such that $\eta = 1$ on $[0, 1]$ and $\eta = 0$ on $[2, \infty)$. We proceed as in the proofs of Lemmas 4.4 and 4.5. For $R \gg 1$, denote $\eta(\rho(x)/R)$ still by $\eta(x)$. Let $\tilde{g}_\epsilon$ be the Ricci flow corresponding to the $g_\epsilon(t)$ and let $\mathcal{S}_\epsilon(t)$ be its scalar curvature. Let $\theta > 0$ and let v be as in the proof of Lemma 4.4. We have

$$\begin{aligned}
\frac{d}{dt}\int_M \eta v\, dv_{\tilde{g}_\epsilon(t)}& \\
&\leq C_2\left(t^{-\frac{1}{2}(1+\delta)}\int_M \eta v\, dv_{\tilde{g}_\epsilon(t)} + \int_M v|\Delta\eta|\, dv_{\tilde{g}_\epsilon(t)}\right)\\
&\leq C_3\left(t^{-\frac{1}{2}(1+\delta)}\int_M \eta v\, dv_{\tilde{g}_\epsilon(t)} + t^{-\delta/2}R^{-1}\int_{M\setminus B(o,2C_1R)} (|\mathcal{S}_\epsilon(t) - \sigma| + \theta)\, dv_{\tilde{g}_\epsilon(t)}\right)
\end{aligned}$$

for some positive constants C_2, C_3 independent of t, ϵ, θ. Hence

$$\begin{aligned}
\frac{d}{dt}&\left(\exp(-C_4 t^{\frac{1}{2}(1+\delta)})\int_M \eta v\, dv_{\tilde{g}_\epsilon(t)}\right)\\
&\qquad\leq C_5 t^{-\delta/2}R^{-1}\int_{M\setminus B(o,2C_1R)} (|\mathcal{S}_\epsilon(t) - \sigma| + \theta)\, dv_{\tilde{g}_\epsilon(t)}
\end{aligned}$$

for some positive constants C_4, C_5 independent of t, ϵ, θ. Integrating from $0 < t_1 < t_2$, let $\theta \to 0$ and then let $R \to \infty$. Using condition (iii), we conclude that

$$\exp(-C_4 t^{\frac{1}{2}(1+\delta)}) \int_M (\mathcal{S}_\epsilon(t) - \sigma)_- \, dv_{\tilde{g}_\epsilon(t)}$$

is nonincreasing in t. Let $\epsilon \to 0$, we conclude that

$$\exp(-C_4 t^{\frac{1}{2}(1+\delta)}) \int_M (\mathcal{S}(t) - \sigma)_- \, dv_{g_\epsilon(t)}$$

is nonincreasing in t.

Next we proceed as in the proof of Lemma 4.5. But we need the cutoff function η. For $\epsilon > 0$ and $\theta > 0$ as in the proof of Lemma 4.5, let β, $\tilde{\beta}$ as in that proof, we have for $R \gg 1$,

$$\begin{aligned} \frac{d}{dt} F \, dv &\le C_6 \left(t^{-\frac{1}{2}(1+\delta)} + t^{-\delta} + t^{-\frac{1}{2}(1+\delta)} F + \int_M |\Delta \eta| v \tilde{\beta}^2 \, dv_{\tilde{g}_\epsilon(t)} \right) \\ &\le C_7 \biggl(t^{-\frac{1}{2}(1+\delta)} + t^{-\delta} + t^{-\frac{1}{2}(1+\delta)} F \\ &\qquad\qquad + \frac{1}{R} \int_{M \setminus B(o, 2C_1 R)} (|\mathcal{S}_\epsilon(t) - \sigma| + \theta) \, dv_{\tilde{g}_\epsilon(t)} \biggr) \end{aligned} \tag{4-10}$$

for some constants C_6, C_7 independent of ϵ, t, θ where

$$F = \int_M \eta \tilde{\beta}^2 v \, dv_{\tilde{g}_\epsilon(t)}.$$

Integrate from 0 to t and let $\theta \to 0$. We have

$$\begin{aligned} &\int_M \eta \tilde{\beta}^2 (\mathcal{S}_\epsilon(t) - \sigma)_- \, dv_{\tilde{g}_\epsilon(t)} \\ &\qquad \le C_8 \left(t^{1-\delta} + t^{\frac{1}{2}(1-\delta)} + \frac{1}{R} \int_0^t \int_{M \setminus B(o, 2C_1 R)} (|\mathcal{S}_\epsilon(s) - \sigma| \, dv_{\tilde{g}_\epsilon(s)}) \, ds \right) \end{aligned}$$

for some constant C_8 independent of ϵ, t. Here we have used the fact that $g_{\epsilon,0} = g_0$ outside $\Sigma(2\epsilon)$ and the fact that $\mathcal{S}_{g_0} \ge \sigma$. Let $R \to \infty$, using (iii), and finally let $\epsilon \to 0$, we conclude that

$$\int_M (\mathcal{S}(t) - \sigma)_- \, dv_{g(t)} \le C_8 (t^{1-\delta} + t^{\frac{1}{2}(1-\delta)}).$$

Since

$$\exp(-C_4 t^{\frac{1}{2}(1+\delta)}) \int_M (\mathcal{S}(t) - \sigma)_- \, dv_{g_\epsilon(t)}$$

is nonincreasing in t, we conclude that the lemma is true. □

5. Singular metrics realizing the nonpositive Yamabe invariant

In this section, we will apply the results in previous sections to study singular metrics on compact manifolds. Let M^n be a compact smooth manifold without boundary. Then as in the Introduction, we may define the *Yamabe invariant* $\sigma(M)$. It is well known that if $\sigma(M) \le 0$ and if g is a smooth metric which realizes $\sigma(M)$, then g is Einstein; see [Schoen 1989, pp. 126–127] for example. If $\sigma(M) > 0$, the situation is more complicated; for some recent results see [Macbeth 2017].

In this section we want to discuss the following question:

Suppose g is a continuous Riemannian metric on M which is smooth outside some compact set Σ such that the volume of g is normalized to be 1. *Suppose the scalar curvature of g satisfies $\mathcal{S}_g \ge \sigma(M)$ away from Σ. What can we say about g?*

In the case that Σ has codimension at least 2, we have the following:

Theorem 5.1. *Let M^n be a smooth compact manifold such that $\sigma(M) \le 0$. Suppose g_0 is a Riemannian metric with $V(M, g_0) = 1$ satisfying the following*:

(i) *There is a compact subset Σ such that g_0 is smooth on $M \setminus \Sigma$ with scalar curvature $\mathcal{S}_{g_0} \ge \sigma(M)$ away from Σ.*

(ii) *The metric g_0 is in $W^{1,q}_{\mathrm{loc}}$ for some $q > n$ in the sense that g_0 has weak derivative and $|g_0|_{\mathfrak{b}}, |{}^{\mathfrak{b}}\nabla g_0|_{\mathfrak{b}} \in L^q_{\mathrm{loc}}$ with respect to a smooth background metric $\mathfrak{b}$.*

(iii) *The volume $V(\Sigma(\epsilon), g_0)$ with respect to g_0 of the ϵ-neighborhood $\Sigma(\epsilon)$ of Σ is bounded by $C\epsilon^2$ for some constant C independent of ϵ. Here*

$$\Sigma(\epsilon) = \{x \in M \mid d_{g_0}(x, \Sigma) < \epsilon\}.$$

Then g_0 is Einstein on $M \setminus \Sigma$.

To prove the theorem, let (M^n, g_0) be as in the theorem. Let

$$\mathring{\mathrm{Ric}}(g_0) = \mathrm{Ric}(g_0) - \frac{\mathcal{S}_0}{n} g_0$$

be the traceless part of $\mathrm{Ric}(g_0)$ where $\mathcal{S}_0 = \mathcal{S}_{g_0}$ is the scalar curvature of g_0. Let $x_0 \in M \setminus \Sigma$. We want to prove that $\mathring{\mathrm{Ric}}(x_0) = 0$. Suppose $\mathring{\mathrm{Ric}}(g_0)(x_0) \ne 0$, then there is $r > 0$ such that $B_{x_0}(4r; g_0) \cap \Sigma = \varnothing$ and there is $c > 0$, $|\mathring{\mathrm{Ric}}(g_0)|(x_0) \ge 2c$ in $B_{x_0}(3r)$. By Corollary 4.6, we can find smooth metrics g_i such that (i) g_i converges uniformly to g_0 and converges in C^∞ norm on any compact sets in $M \setminus \Sigma$; (ii) $V(M, g_i) = 1$; (iii) the scalar curvature $\mathcal{S}_i$ of g_i satisfies $\mathcal{S}_i \ge \sigma - \delta_i$ for all i with $\delta_i \downarrow 0$. Hence we may assume that

$$|\mathring{\mathrm{Ric}}(g_i)|(x) \ge c \tag{5-1}$$

in $B_{x_0}(2r; g_i)$ for all i, and $B_{x_0}(r; g_i) \subset B_{x_0}(2r; g)$, $B_{x_0}(2r; g_i) \subset B_{x_0}(3r; g)$. We may also assume that the distance function $r_i(x)$ from x_0 with respect to g_i are smooth in $B_{x_0}(3r; g)$, provided $r > 0$ is small enough, independent of i.

Let ϕ be a smooth function on $[0, \infty)$ with $\phi \geq 0$, $\phi = 1$ on $[0, 1]$ and $\phi = 0$ on $[2, \infty)$ and such that $|\phi'|^2 \leq C\phi$, with C being an absolute constant. Let

$$h_i(x) = \phi\left(\frac{r_i(x)}{r}\right)\mathring{\mathrm{Ric}}(g_i)(x).$$

For $|\tau| > 0$, let $G_{i;\tau} = g_i + \tau h_i$. Then there is $\tau_0 > 0$ such that $G_{i;\tau}$ are smooth metrics for all i and for all $0 < |\tau| \leq \tau_0$.

In the following, $E_k = E_k(x, \tau)$ $(k = 1, 2)$ will denote a quantity such that $|E_k| \leq C|\tau|^k$ for some C independent of x, i and τ.

Lemma 5.2. *We have*

$$dv_{G_{i;t}} = dv_{g_i}(1 + E_2)$$

and

$$V(M, G_{i;t}) = 1 + E_2;$$

here dv_g denotes the volume element of metric g.

Proof. Since $g_i \to g$ uniformly on compact sets of $M \setminus \Sigma$ in C^∞ norm and since h_i is traceless, the results follow. □

We have the following general fact [Brendle and Marques 2011, Proposition 4]:

Lemma 5.3. *Let (Ω^n, g) be a smooth Riemannian manifold. Let $\bar{g} = g + h$ with $|h|_g \leq \frac{1}{2}$. Then the scalar curvatures are related as*

$$\mathcal{S}_{\bar{g}} - \mathcal{S}_g = \mathrm{div}_g(\mathrm{div}_g(h)) - \Delta_g \operatorname{tr}_g h - \langle h, \mathrm{Ric}(g)\rangle_g + F,$$

where

$$|F| \leq C\big(|\nabla h|^2 + |h|_g|\nabla^2 h|_g + |\mathrm{Ric}(g)|\,|h|_g^2\big)$$

for some constant C depending only on n. Here ∇ is the covariant derivative with respect to g.

Lemma 5.4. *Let $\mathcal{S}_i$ be the scalar curvature of g_i and $\mathcal{S}_{i;\tau}$ be the scalar curvature of $G_{i;\tau}$. Then*

$$\mathcal{S}_{i;\tau} = \mathcal{S}_i + \tau\, \mathrm{div}_{g_i}(\mathrm{div}_{g_i} h_i) - \tau\langle h_i, \mathrm{Ric}(g_i)\rangle_{g_i} + E_2(\tau).$$

Note that $\mathcal{S}_{i;\tau} = \mathcal{S}_i$ outside $B_{x_0}(2r, g_i)$ and is bounded below by a constant independent of i, τ.

Proof. This follows from Lemma 5.3, the fact that h_i is traceless, $h_i = 0$ outside $B_{x_0}(2r, g_i)$, the fact that $g_i \to g$ in C^∞ outside Σ and the fact that $\mathcal{S}_i \ge \sigma - \delta_i$. □

In the following, let

$$a = \frac{4(n-1)}{n-2}, \quad p = \frac{2n}{n-2}. \tag{5-2}$$

By the resolution of the Yamabe conjecture [Yamabe 1960; Trudinger 1968; Aubin 1976b; Schoen 1984], for each i, τ, we can find a smooth positive solution $u_{i;\tau}$ of

$$-a\Delta_{G_{i;\tau}} u_{i;\tau} + \mathcal{S}_{i;\tau} u_{i;\tau} = \lambda_{i;\tau} V_{i;\tau}^{-2/n} u_{i;\tau}^{p-1} \tag{5-3}$$

with $\lambda_{i;\tau} = Y(\mathcal{C}_{i,\tau})$ which is less than or equal to σ (in particular, it is nonpositive), where $\mathcal{C}_{i,\tau}$ is the class of smooth metrics conformal to $G_{i;\tau}$. Moreover, $u_{i;\tau}$ is normalized by

$$\int_M u_{i;\tau}^p \, dv_{G_{i;\tau}} = 1,$$

and $V_{i,\tau} = V(M, G_{i;\tau})$.

Lemma 5.5. *There is $0 < \tau_1 \le \tau_0$ independent of i such that if $0 > \tau \ge -\tau_1$, then*

$$\frac{a}{2}\int_M |{}^{(i;\tau)}\nabla u_{i;\tau}|^2_{G_{i;\tau}} \, dv_{G_{i;\tau}} - \lambda_{i;\tau} V_{i;\tau}^{-2/n} + \sigma \le -C|\tau| \int_{B_{x_0}(2r, g_i)} \phi u_{i;\tau}^2 \, dv_{g_i} + C'\delta_i + E_2(\tau)$$

for some positive constants C, C' independent of i and τ. Here ${}^{(i;\tau)}\nabla$ is the covariant derivative with respect to $G_{i;\tau}$.

Proof. For simplicity of notation, in the following we denote ${}^{(i;\tau)}\nabla$ by ∇, $G_{i;\tau}$ by G; g_i by g; $u_{i;\tau}$ by u; $\lambda_{i;\tau}$ by λ; $\mathcal{S}_{i;\tau}$ by $\mathcal{S}_G$; $\mathcal{S}_i$ by $\mathcal{S}_g$; and $V_{i;\tau}$ by V.

Multiply (5-3) by u and integrating by parts, using the fact that

$$\int_M u^p \, dv_G = 1,$$

we have

$$\begin{aligned} a\int_M |\nabla u|^2_G \, dv_G - \lambda V^{-2/n} &= -\int_M \mathcal{S}_G u^2 \, dv_G \\ &\le -\int_M \mathcal{S}_G u^2 \, dv_g + E_2(\tau)\int_M u^2 \, dv_g \end{aligned} \tag{5-4}$$

by Lemmas 5.2 and 5.4 and the fact that g_i converges in C^∞ norm in $B_{x_0}(3r, g_0) \supset B_{x_0}(g_i, 2r)$. On the other hand, by Lemma 5.4, for any $0 < \epsilon < 1$,

$$
\begin{aligned}
\text{(5-5)}\quad & -\int_M \mathcal{S}_G u^2\, dv_g \\
& \le -\int_M \mathcal{S}_g u^2\, dv_g - \tau \int_M \bigl(\operatorname{div}_g(\operatorname{div}_g h) - \langle h, \operatorname{Ric}(g)\rangle_g\bigr) u^2\, dv_g \\
& \qquad + E_2(\tau) \int_{B_{x_0}(2r;g)} u^2\, dv_g \\
& \le -\int_M \mathcal{S}_g u^2\, dv_g + C_1|\tau| \int_M u|{}^g\nabla u|_g \bigl(|\phi'||\mathring{\operatorname{Ric}}(g)|_g + \phi|{}^g\nabla \mathcal{S}_0|_g\bigr)\, dv_g \\
& \qquad - |\tau| \int_M \phi |\mathring{\operatorname{Ric}}(g)|^2 u^2\, dv_g + E_2(\tau) \int_{B_{x_0}(2r;g)} u^2\, dv_g \\
& \le (-\sigma + \delta) \int_M u^2\, dv_g + (C_2 + \epsilon^{-1})|\tau| \int_M |{}^g\nabla u|_g^2\, dv_g \\
& \qquad - C_3|\tau| \int_M \phi |\mathring{\operatorname{Ric}}(g)|^2 u^2\, dv_g + (E_2(\tau) + C_2\epsilon|\tau|) \int_{B_{x_0}(2r;g)} \phi u^2\, dv_g \\
& \le (-\sigma + \delta) \int_M u^2\, dv_g + (C_2 + \epsilon^{-1})|\tau| \int_M |{}^g\nabla u|_g^2\, dv_g \\
& \qquad + (E_1(\tau) + C_2\epsilon - C_3 c)|\tau| \int_{B_{x_0}(2r;g)} \phi u^2\, dv_g
\end{aligned}
$$

for some constants $C_1, C_2, C_3 > 0$ independent of i, τ. Here we have used the fact that $|\phi'|^2 \le C\phi$ and the fact that $\mathcal{S}_g \ge \sigma - \delta_i$ which is negative, where we denote δ_i by δ. Choose $\epsilon > 0$ so that $C_2\epsilon = \frac{1}{2}C_3 c$. Then the result follows if $\tau_1 > 0$ is small enough and independent of i, by (5-4), (5-5), the Hölder inequality, the fact that g, G are uniformly equivalent, and the fact that

$$\int_M u^p\, dv_G = 1, \quad V(M, g) = 1,$$

and

$$V(M, G) = 1 + E_2(\tau). \qquad \square$$

Let $0 > \tau_k > -\tau_1$, $\tau_k \to 0$. Since $\delta_i \to 0$, for each k we can find i_k such that $\delta_{i_k} \le \tau_k^2$, $i_k \to \infty$. Let us denote $G_{i_k;\tau_k}$ by G_k, and $u_{i_k;\tau_k}$ by u_k. We want to prove the following:

Lemma 5.6. *There is a constant $C > 0$ such that for all k,*

$$\inf_{B_{x_0}(3r, g_0)} u_k \ge C.$$

Proof of Theorem 5.1. Suppose the lemma is true then we will have a contradiction. In fact, if we denote δ_{i_k} by δ_k, since $V(M, G_k) = 1 + E_2(\tau_k)$, $\lambda \le \sigma$, by Lemma 5.5, we have

$$\begin{aligned}\frac{a}{2}\int_M |{}^{G_k}\nabla u_k|^2_{G_k}\, dv_{G_k} &\le -C_1|\tau_k| \int_{B_{x_0}(2r, g_{i_k})} \phi u_k^2\, dv_{g_{i_k}} + C_2\delta_k + C_2\tau_k^2\\ &\le -C_1|\tau_k| \int_{B_{x_0}(2r, g_{i_k})} \phi u_k^2\, dv_{g_{i_k}} + (C_2+1)\tau_k^2\end{aligned}$$

for some positive constants C_1, C_2 independent of k. By Lemma 5.6, this is impossible if k is large enough. Hence $\overset{\circ}{\mathrm{Ric}}(g_0)(x_0)$ must be zero. Theorem 5.1 then follows. □

It remains to prove Lemma 5.6. Consider the equation

$$-a\Delta u + \mathcal{S}u = \lambda u^{p-1}. \tag{5-6}$$

Lemma 5.7. *Let (M^n, g) be a smooth metric with scalar curvature $\mathcal{S} \ge -s_0$, with $s_0 \ge 0$. Let $u > 0$ be a solution of (5-6) with $\|u\|_p = 1$ and with $\lambda \le 0$. Then for any $q > p$,*

$$\|u\|_q \le C(s_0, V(M; g), n, q).$$

Proof. See [Trudinger 1968]; see also [Lee and Parker 1987, Proposition 4.4]. □

Lemma 5.8. *Using the notation of Lemma 5.6,*

(i) *for any $q > p$, there is a constant C independent of k such that*

$$\|u_k\|_{q,g_0} \le C;$$

(ii) *u_k subconverges in C^2 norm with respect to g_0 in any compact set $K \subset M \setminus \Sigma$;*

(iii) $\lim_{k\to\infty} \int_M \|{}^{g_0}\nabla u_k\|^2_{g_0}\, dv_{g_0} = 0$;

(iv) $\lim_{k\to\infty} \lambda_k = \sigma$, *where $\lambda_k = \lambda_{i_k;\tau_k}$ as in (5-3).*

Proof. Since $\mathcal{S}_{i_k;\tau_k} \ge \sigma - \delta_k$ and $\delta_k \to 0$, (i) follows from Lemma 5.7 and the fact that $C^{-1}g_0 \le G_k \le Cg_0$ for some $C > 0$ for all k.

To prove (ii), for any compact set $K \subset M\setminus\Sigma$, there is an open set $K \Subset U \subset M\setminus\Sigma$ such that G_k converges in C^∞ norm to g_0 on U. By Lemma 5.5, we conclude that $0 \le -\lambda_k \le C$ for some constant independent of k. Then by (i), and [Lee and Parker 1987, Theorem 2.4], we conclude that for any $U' \Subset U$,

$$\|u_k\|_{L_2^q(U')} \le C_1$$

for some constant C independent of k. We then use the Sobolev embedding theorem to conclude that the C^α norm of u_k are uniformly bounded in $U' \Subset U$. From this the result follows by Schauder estimates.

Parts (iii) and (iv) follow from Lemma 5.5. □

Corollary 5.9. *After passing to a subsequence, u_k converges in C^2 norm locally in $M \setminus \Sigma$ to a function $\mathfrak{u}$. Moreover, $\mathfrak{u} = 1$ in $M \setminus \Sigma$ and*

$$\mathcal{S}_{g_0} = \sigma.$$

In particular Lemma 5.6 is true.

Proof. By Lemma 5.8, after passing to a subsequence, u_k converges in C^2 norm locally in $M \setminus \Sigma$ to a function $\mathfrak{u}$. Moreover, $\mathfrak{u}$ is constant in each component of $M \setminus \Sigma$. We claim that there is $C_1 > 0$ such that $0 \le u_k \le C_1$ for all k.

Since the scalar curvature $\mathcal{S}_{G_k} \ge -s_0$ for some $s_0 > 0$ independent of k and since $\lambda_k \le 0$, we have

$$-a\Delta_{G_k} u_k - s_0 u_k \le -a\Delta_{G_k} u_k + \mathcal{S}_{G_k} u_k \le 0.$$

Moreover, $\int_M u_k^p \, dv_{G_k} = 1$ and G_k is equivalent to g_0 uniformly in k, the claim follows from mean value inequality [Gilbarg and Trudinger 1983, Theorem 8.17].

Since $u_k \to \mathfrak{u}$ almost everywhere, and G_k converges uniformly to g_0, we have

$$\int_M \mathfrak{u}^p \, dv_{g_0} = 1.$$

In particular, $\mathfrak{u} > 0$ somewhere.

Next we want to prove that $\mathfrak{u}$ is constant on M. By Lemma 5.8, there is a constant C_2 independent of k such that

$$\int_M (|^{g_0}\nabla u_k|_{g_0}^2 + u_k^2) \, dv_{g_0} \le C_2.$$

Passing to a subsequence, we may assume that u_k converges weakly in $W^{1,2}(M, g_0)$ to v say. We claim that v is constant. In fact, for any $\ell \ge 1$, the sequence $u_{\ell+k}$, $k \ge 1$, also weakly converges to v. Then we can find convex combinations of $u_{\ell+k}$ which converge to v strongly in $W^{1,2}(M, g_0)$. Namely, for any $\epsilon > 0$, there exists $\alpha_1, \ldots, \alpha_m$ with $\alpha_k \ge 0$, $\sum_{k=1}^m \alpha_k = 1$ such that if $w = \sum_{k=1}^m \alpha_k u_{\ell+k}$, then

$$\|w - v\|_{W^{1,2}(M, g_0)} \le \epsilon.$$

On the other hand, by Lemma 5.8, if ℓ is large enough, then

$$\begin{aligned}\left(\int_M |^{g_0}\nabla w|_{g_0}^2 \, dv_{g_0}\right)^{\frac12} &\le \left(\int_M \left(\sum_k \alpha_k |^{g_0}\nabla u_{\ell+k}|_{g_0}\right)^2 dv_{g_0}\right)^{\frac12} \\ &\le \sum_k \alpha_k \left(\int_M |^{g_0}\nabla u_{l+k}|_{g_0}^2 \, dv_{g_0}\right)^{\frac12} \\ &\le \epsilon.\end{aligned}$$

Hence

$$\int_M |{}^{g_0}\nabla v|^2\, dv_{g_0} \le (2\epsilon)^2.$$

This implies ${}^{g_0}\nabla v = 0$, a.e. Since $v \in W^{1,2}(M, g_0)$, we conclude that $v = c$ is a constant as claimed.

On the other hand, for any smooth function ϕ on M

$$\begin{aligned}\lim_{k\to\infty}\int_M (\langle {}^{g_0}\nabla\phi, {}^{g_0}\nabla u_k\rangle_{g_0} + \phi u_k)\, dv_{g_0} &= \int_M (\langle {}^{g_0}\nabla\phi, {}^{g_0}\nabla v\rangle_{g_0} + \phi v)\, dv_{g_0}\\ &= \int_M \phi v\, dv_{g_0}.\end{aligned}$$

Also by Lemma 5.8 again, and the fact that u_k are uniformly bounded and $u_k \to \mathfrak{u}$ a.e., we have

$$\lim_{k\to\infty}\int_M (\langle {}^{g_0}\nabla\phi, {}^{g_0}\nabla u_k\rangle_{g_0} + \phi u_k)\, dv_{g_0} = \int_M \phi\mathfrak{u}\, dv_{g_0}.$$

So

$$\int_M \phi\mathfrak{u}\, dv_{g_0} = \int_M \phi v\, dv_{g_0}.$$

Hence $\mathfrak{u} = v$ is a constant. Since $\int_M \mathfrak{u}^p\, dv_{g_0} = 1$ so $\mathfrak{u} = 1$. Since $\mathfrak{u}$ satisfies

$$-a\Delta_{g_0}\mathfrak{u} + \mathcal{S}_{g_0}\mathfrak{u} = \sigma\mathfrak{u}^p,$$

the last assertion follows. □

This completes the proof of Theorem 5.1. Next we want to discuss the case that Σ has codimension one. We have the following:

Theorem 5.10. *Let M^n be a smooth compact manifold such that $\sigma(M) \le 0$. Suppose g_0 is a Riemannian metric with $V(M, g_0) = 1$ satisfying* (b1)–(b3) *in Section 4. Then g_0 is Einstein on $M \setminus \Sigma$ and $\mathcal{S}_{g_0} = \sigma(M)$. Moreover, $H_- = H_+$.*

Proof. Let $g_i = g_{\epsilon_i,0}$ be the smooth approximation of g_0 by [Miao 2002] as given in Section 4. The fact that g_0 is Einstein outside Σ can be proved similarly as above using Corollary 4.8. It remains to prove that $H_- = H_+$. Let $\epsilon_i \to 0$ and let u_i be the positive solution of

$$-a\Delta_i u_i + \mathcal{S}_i u_i = \lambda_i u_i^{p-1}$$

normalized as

$$\int_M u_i^p\, dv_i = 1.$$

Here Δ_i is the Laplacian of g_i etc. Also $\lambda_i \le \sigma$, where $\sigma := \sigma(M)$. Suppose $H_-(z) > H_+(z)$ somewhere; then one can easily check that there is a positive constant b such that for i large enough,

$$\int_M \mathcal{S}_i \, dv_i \geq \sigma + b. \tag{5-7}$$

As before, passing to a subsequence if necessary, $u_i \to 1$ outside Σ and uniform in C^∞ norm in any compact set of $M \setminus \Sigma$. Moreover, u_i are uniformly bounded, and $\lambda_i \to \sigma$. Since $\mathcal{S}_i$ be bounded below by $-s_0$, for some $s_0 \geq 0$ and u_i is bounded from below, we have

$$\begin{aligned}
\sigma &= \lim_{i\to\infty} \lambda_i \int_M u_i^{p-1} \, dv_i \\
&= \lim_{i\to\infty} \int_M \mathcal{S}_i u_i \, dv_i \\
&\geq \lim_{i\to\infty} \int_M \mathcal{S}_i (u_i - 1) \, dv_i + \sigma + b,
\end{aligned}$$

where we have used the fact that $V(M, g_{0,\epsilon_i}) \to V(M, g_0) = 1$ and (5-7). We claim

$$\lim_{i\to\infty} \int_M \mathcal{S}_i (u_i - 1) \, dv_i = 0.$$

If the claim is true, then we have a contradiction because $b > 0$. To prove the claim, note that on $|t| \leq a$, the original metric g_0 is of the form

$$g_0(z, t) = dt^2 + g_{ij}(z, t) \, dz^i \, dz^j.$$

We assume that $g_{ij}(z, t)$ (which will be denoted by $h^t_{ij}(z)$) is uniformly equivalent to $g_{ij}(z, 0)$ (which will be denoted by $h_{ij}(z)$). For any $z \in \Sigma$ and for any $1 \geq t \geq 0$,

$$|u_i(z, a) - u_i(z, t)| \leq \int_0^a \left| \frac{\partial u_i(z, s)}{\partial s} \right| ds \leq \int_0^1 |{}^{g_0}\nabla u_i|(z, s) \, ds.$$

By the properties of $g_{0,\epsilon}$,

$$\int_{\epsilon_i^2/100 \leq |t| \leq \epsilon_i} |\mathcal{S}_i (u_i - 1)| \, dv_i = o(1) \tag{5-8}$$

because u_i are uniformly bounded. So

$$\begin{aligned}
\int_{|t| \leq \epsilon_i^2/100} & \mathcal{S}_i(z, t)(u_i(z, t) - 1) \, dv_{g_i} \\
&= \int_{|t| \leq \epsilon_i^2/100} \mathcal{S}_i(z, t)(u_i(z, 1) - 1) \, dv_{g_i} \\
&\qquad + \int_{|t| \leq \epsilon_i^2/100} \mathcal{S}_i(z, t)(u_i(z, t) - u_i(z, 1)) \, dv_{g_i} \\
&= I + II.
\end{aligned} \tag{5-9}$$

Since $u_i(z,1)\to 1$ uniformly on $z\in\Sigma$, and $\int_M |\mathcal{S}_i|\,dv_{g_i}$ is bounded, we conclude that

$$I=o(1) \tag{5-10}$$

as $i\to\infty$. On the other hand,

$$\begin{aligned}
|II| &\le \int_{|t|\le\epsilon_i^2/100} |\mathcal{S}_i(z,t)(u_i(z,t)-u_i(z,1))|\,dv_{g_i} \\
&\le c\int_{z\in\Sigma}\left(\int_{-\epsilon_i^2/100}^{\epsilon_i^2/100}\epsilon_i^{-2}\int_0^1 |\nabla u_i(z,s)|\,ds\right)dt\,dv_h \\
&\le c\int_{z\in\Sigma}\left(\int_0^a |\nabla u_i(z,s)|\,ds\right)dt\,dv_h \\
&\le c\int_M |\nabla u_i|\,dv_{g_i} \\
&= o(1)
\end{aligned} \tag{5-11}$$

by the Schwartz inequality and Lemma 5.8. The claim follows from (5-8)–(5-11). □

6. Singular Einstein metrics

In the conclusions of Theorem 5.1, one obtains metrics which are smooth and Einstein outside some singular sets. In this section, we want to prove that under certain conditions, one may introduce a smooth structure so that the Einstein metric is actually smooth. More precisely, we have the following:

Theorem 6.1. *Let M^n, $n\ge 3$, be a smooth manifold and g be a Riemannian metric on M satisfying the following conditions: There is a compact set Σ in M such that*

(i) *g is Lipschitz and g is smooth on $M\setminus\Sigma$;*

(ii) *$g=\lambda\,\mathrm{Ric}$ on $M\setminus\Sigma$ for some constant λ;*

(iii) *the codimension of Σ is larger than 1 in the sense that $V(\Sigma(\epsilon),g)=O(\epsilon^{1+\theta})$ for some $\theta>0$, where $\Sigma(\epsilon)=\{x\in M\mid d(x,\Sigma)<\epsilon\}$.*

Then for any open set U containing Σ, there is a smooth structure on M which is the same as the original smooth structure on $M\setminus U$ so that g is a smooth Einstein metric on M.

We want to construct the required smooth structure using harmonic coordinates. First recall the following.

Lemma 6.2. *Let $B(1)$ be the unit ball in $\mathbb{R}^n$ with center at the origin. Let (a_{ij}) be a symmetric matrix such that*

$$\lambda|\xi|^2\le a^{ij}\xi^i\xi^j\le\Lambda|\xi|^2,$$

for some $\Lambda > \lambda > 0$ for all $\xi \in \mathbb{R}^n$ and where a^{ij} is Lipschitz with Lipschitz constant L. Let $f \in L^\infty(B(1))$. Then the boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial x^i}\left(a^{ij}\dfrac{\partial u}{\partial x^j}\right) = f & \text{in } B(1), \\ u = 0 & \text{on } \partial B(1) \end{cases}$$

has a unique solution in $W^{2,p}(B(1))$ for any $p > 1$ with $u \in W_0^{1,p}(B(1))$. Moreover, we have

$$\|u\|_{2,p} \le C(\|u\|_p + \|f\|_p)$$

for some constant C depending only on $p, n, \lambda, \Lambda, L$. Here $\|u\|_{2,p}$ is the $W^{2,p}$ norm on $B(1)$ and $\|u\|_p$ is the L^p norm in $B(1)$.

Proof. The results follow from [Gilbarg and Trudinger 1983, Theorem 9.15, Corollary 9.13]. By taking $p > n$ and the Sobolev embedding theorem, u is continuous up to the boundary and $u = 0$ at the boundary. □

With the same assumptions and notation as in Theorem 6.1, let $q \in \Sigma$. Let $U_\delta = \{(x^1, \dots, x^n) \mid |x| < \delta\}$ be a smooth local coordinate neighborhood with q being at the origin such that g_{ij} is equivalent to the Euclidean metric and g_{ij} is Lipschitz with Lipschitz constant L

Lemma 6.3. *With the above assumptions and notation, there is $\delta > \epsilon > 0$ and functions $u^1, \dots, u^n$ on $U_\epsilon = \{(x^1, \dots, x^n) \mid |x| < \epsilon\}$ such that the mapping $(x^1, \dots, x^n) \to (u^1, \dots, u^n)$ is a local $C^{1,\alpha}$ diffeomorphism at the origin for some $0 < \alpha < 1$, $u^i \in W^{2,p}(U_\epsilon)$ for all $p > 1$ and u^i is harmonic with respect to g for $1 \le i \le n$. Moreover, u^i is smooth outside Σ.*

Proof. Let $\delta > \epsilon > 0$ to be chosen later. Fix ℓ, let $f = \Delta_g x^\ell$ which is bounded by the assumption on g_{ij}. Let $\lambda, \Lambda > 0$ be such that

$$\lambda|\xi|^2 \le g^{ij}\xi^i\xi^j \le \Lambda|\xi|^2 \tag{6-1}$$

in U_δ.

Let $y = \epsilon^{-1}x$. Consider the following boundary value problem on $B(1)$ in the y-space

$$\begin{cases} \dfrac{\partial}{\partial y^i}\left(\sqrt{g}g^{ij}\dfrac{\partial v}{\partial y^j}\right) = \epsilon^2\sqrt{g}f & \text{in } B(1), \\ v = 0 & \text{on } \partial B(1). \end{cases} \tag{6-2}$$

By Lemma 6.2, the boundary value problem has a solution v satisfying the conclusions in that lemma. Here we have used the fact that g_{ij} has Lipschitz constant bounded by ϵL and still satisfies (6-1) as functions of y. In particular, we have

$$\|v\|_{2,p;y} \le C_1(\|v\|_{p;y} + \epsilon^2).$$

Here and below, C_i will denote positive constants independent of ϵ. Let $p > n$ be fixed; then one can see that there is $1 > \alpha > 0$ such that $v \in C^{1,\alpha}(B(1))$ in the y-space and

$$\|v\|_{C^{1,\alpha}(B(1))} \le C_2(\|v\|_{p;y} + \epsilon^2) \tag{6-3}$$

for some positive constants $C_2 - C_4$ independent of ϵ.

Let $w(x) = v(\epsilon^{-1}x)$ with $x \in B(\epsilon)$ in the x-space. Then w satisfies

$$\begin{cases} \dfrac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\dfrac{\partial w}{\partial x^j}\right) = \sqrt{g} f & \text{in } B(\epsilon), \\ w = 0 & \text{on } \partial B(\epsilon) \end{cases}$$

in the x-space. Moreover, $w \in W^{2,p}(B(\epsilon))$. Let $u^\ell = w - x^\ell$. Then u^ℓ is harmonic, namely, u^ℓ satisfies

$$\begin{cases} \dfrac{1}{\sqrt{g}}\dfrac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\dfrac{\partial u^\ell}{\partial x^j}\right) = 0 & \text{in } B(\epsilon), \\ u^\ell = x^\ell & \text{on } \partial B(\epsilon). \end{cases}$$

By the maximum principle, we conclude that $|u^\ell| \le \epsilon$ and so $|w| \le 2\epsilon$, and moreover, we have

$$\sup_{B(\epsilon)} |\partial_x w| = \epsilon^{-1} \sup_{B(1)} |\partial_y v| \le C_2\epsilon^{-1}(\|v\|_{p;y} + \epsilon^2). \tag{6-4}$$

To estimate the right-hand side, multiply (6-2) by v and integrating by parts, using the Poincaré inequality, we have

$$\int_{B(1)} v^2\,dy \le C_3\epsilon^2 \int_{B(1)} |v|\,dy$$

and so

$$\begin{aligned} \|v\|_{p;y} &\le \left(\sup_{B(1)} |v|\right)^{1-2/p}\left(\int_{B(1)} v^2\,dy\right)^{1/p} \\ &\le C_4\epsilon^{1-2/p}\cdot\epsilon^{4/p} \\ &= C_4\epsilon^{1+2/p}, \end{aligned}$$

where we have used the Hölder inequality and the fact that $|v| = |w| \le 2\epsilon$. By (6-4) we conclude that

$$\sup_{B(\epsilon)} |\partial_x w| \le C_5\epsilon^{2/p}.$$

Hence

$$\frac{\partial u^\ell}{\partial x^i} = \delta_i^\ell + O(\epsilon^{2/p}).$$

From this and the fact that g is smooth outside Σ it is easy to see that the lemma is true, provided ϵ is small enough. □

Proof of Theorem 6.1. Let U be any open set containing Σ. For any $q \in \Sigma$, by Lemma 6.3, we can find smooth coordinates neighborhood $V_q \Subset U$ around q and $C^{1,\alpha}$ functions $u^1, \dots, u^n$ on V_q near q which are in $W^{2,p}(V_q)$ as functions of x. Moreover, $(x^1, \dots, x^n) \to (u^1, \dots, u^n)$ is a C^1 diffeomorphism from V_q to its image $\widetilde{V}_q$ in the u-space. Let

$$h_{ab} = g\left(\frac{\partial}{\partial u^a}, \frac{\partial}{\partial u^b}\right) = \frac{\partial x^i}{\partial u^a}\frac{\partial x^j}{\partial u^b} g_{ij}, \tag{6-5}$$

where

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

Let

$$R_{ab} = \mathrm{Ric}\left(\frac{\partial}{\partial u^a}, \frac{\partial}{\partial u^b}\right).$$

Since each u^a is harmonic, and $R_{ab} = \lambda h_{ab}$ by assumption, away from Σ for all a, b we have

$$h^{cd} h_{ab,cd} = -2\lambda h_{ab} + \partial h^{-1} * \partial h + h^{-1} * h^{-1} * \partial h * \partial h := Q(h, \partial h), \tag{6-6}$$

where $(h^{cd}) = (h_{cd})^{-1}$,

$$h_{ab,c} = \frac{\partial}{\partial u^c} h_{ab}$$

etc., and $\partial h^{-1} * \partial h$ denotes a sum of finite terms of the form

$$\left(\frac{\partial}{\partial u^c} h^{ab}\right)\left(\frac{\partial}{\partial u^f} h_{de}\right)$$

etc. By (6-5),

$$h_{ab,c} = 2\frac{\partial^2 x^i}{\partial u^a \partial u^c}\frac{\partial x^j}{\partial u^b} g_{ij} + \frac{\partial x^i}{\partial u^a}\frac{\partial x^j}{\partial u^b}\frac{\partial x^k}{\partial u^c}\frac{\partial}{\partial x^k} g_{ij}. \tag{6-7}$$

We may assume that $\widetilde{V}_q$ contains the origin which is the coordinates of q. Then by shrinking $\widetilde{V}_q$ if necessary, by Lemma 6.3, h_{ab} is bounded and $h_{ab,c}$ is in L^p for all $p > 1$ for all a, b, c as functions of u. In particular, h_{ab} is in $W^{1,p}(\widetilde{V}_q)$ for all $p > 1$. Moreover, (h^{ab}) is uniformly elliptic. Since h^{ab} is only in C^α with $0 < \alpha < 1$, we cannot apply the standard L^p estimate as in [Gilbarg and Trudinger 1983, Theorem 9.19]. Hence, we want to prove that h_{ab} is in $W^{2,p}(B(\delta))$ for all a, b for all $p > n$ and for some $\delta > 0$ in the u-space, where $B(\delta) = \{u \mid |u| < \delta\}$. Suppose this is true; then $h_{ab} \in C^{0,1}_{\mathrm{loc}}(B(\delta))$ and $\partial h \in W^{1,p}_{\mathrm{loc}}(B(\delta))$. This implies $Q(h, \partial h)$ in (6-6) is in $W^{1,p/2}_{\mathrm{loc}}(B(\delta))$. Since this is true for all $p > n$, by [Gilbarg and Trudinger 1983, Theorem 9.19], we conclude that h_{ab} is in $W^{3,p}(B(\delta))$. Continuing in this way, we conclude that $h_{ab} \in W^{k,p}_{\mathrm{loc}}(B(\delta))$ for all $k \geq 1$ and $p > n$ by a bootstrap argument. Hence h_{ab} is smooth near the origin.

It remains to prove that $h_{ab}\in W^{2,p}(B(\delta))$ for all $p>n$ for all a,b for some $\delta>0$. Fix a,b and let $w=\phi h_{ab}$ where ϕ is a smooth cutoff function in $B(2\delta)$ such that $\phi=1$ in $B(\delta)$, $\phi=0$ outside $B\big(\frac{3}{2}\delta\big)$, where $\delta>0$ is small enough so that $B(2\delta)\Subset \widetilde{V}_q$. Then away from Σ, w satisfies

$$h^{cd}w_{cd}=Q_1(h,\partial h,\phi,\partial\phi,\partial^2\phi). \tag{6-8}$$

Since Q_1 is in $L^p(B(2\delta))$ by Lemma 6.3 and (h^{cd}) is continuous and is uniformly elliptic, by [Gilbarg and Trudinger 1983, Theorem 9.15] for any $p>n$ there is $v\in W^{2,p}(B(2\delta))\cap W^{1,p}_0(B(2\delta))$ such that

$$h^{cd}v_{cd}=Q_1(h,\partial h,\phi,\partial\phi,\partial^2\phi).$$

Since $h^{cd}\in W^{1,p}(B(2\delta))$ for all p, for any smooth function η with compact support in $B(2\delta)$, we have

$$\int_{B(2\delta)}\left(h^{cd}\frac{\partial\eta}{\partial u^a}\frac{\partial v}{\partial u^b}+\eta s^d\frac{\partial v}{\partial u^d}\right)du=-\int_{B(2\delta))}\eta Q_1\,du. \tag{6-9}$$

where $s^d=\frac{\partial}{\partial u^c}h^{cd}$. We want to prove that w also satisfies this relation.

To prove the claim, note that if we consider $\Sigma\cap\widetilde{V}_q$ then the codimension of Σ in the u-space is at least $1+\theta$ for some $\theta>0$ because h_{ab} and the Euclidean metric are uniformly equivalent. As in [Lee 2013], for $\epsilon>0$ small enough, we can find a smooth function $0\le\xi_\epsilon\le 1$ in $\widetilde{V}_q$ such that $\xi_\epsilon=1$ outside $\Sigma_{2\epsilon}$ and is zero in $\Sigma_\epsilon\cap\widetilde{V}_q$ where $\Sigma_\epsilon=\{u\in\widetilde{V}_q\mid d(u,\Sigma)<\epsilon\}$ where the distance is the Euclidean distance. Moreover, $|\partial\xi_\epsilon|\le C_1\epsilon^{-1}$. Here and below C_i denotes a positive constant independent of ϵ. Now let η be a smooth function with compact support in $B(2\delta)$. Multiply (6-8) by $\eta\xi_\epsilon$ and integrating by parts, we have

$$-\int_{B(2\delta)}\eta\xi_\epsilon Q_1\,du=\int_{B(2\delta)}\left[h^{cd}\left(\xi_\epsilon\frac{\partial\eta}{\partial u^a}+\eta\frac{\partial\xi_\epsilon}{\partial u^a}\right)\frac{\partial w}{\partial u^b}+\eta\xi_\epsilon s^d\frac{\partial w}{\partial u^d}\right]du.$$

Since $w,h^{cd}\in L^{1,p}(B(2\delta))$ for all $p>1$, we have

$$\int_{B(2\delta)}|\eta(\xi_\epsilon-1)Q_1|\,du\le\left(\int_{B(2\delta)}|\eta(\xi_\epsilon-1)Q_1|^2\,du\right)^{1/2}V(\Sigma_{2\epsilon})^{1/2}\to 0$$

as $\epsilon\to 0$. Similarly, one can prove that

$$\int_{B(2\delta)}\left|h^{cd}(\xi_\epsilon-1)\frac{\partial\eta}{\partial u^a}\frac{\partial w}{\partial u^b}+\eta(\xi_\epsilon-1)s^d\frac{\partial w}{\partial u^d}\right|du\to 0$$

as $\epsilon \to 0$. On the other hand,

$$\begin{aligned}\int_{B(2\delta)} \left| h^{cd} \eta \frac{\partial \xi_\epsilon}{\partial u^a} \frac{\partial w}{\partial u^b} \right| du &\le C_2 \epsilon^{-1} \int_{\Sigma_{2\epsilon}} |\partial w| \, du \\ &\le C_3 \epsilon^{-1} \left(\int_{\Sigma_{2\epsilon}} |\partial w|^p \, du \right)^{\frac{1}{p}} (V(\Sigma(2\epsilon)))^{1-\frac{1}{p}} \\ &\le C_4 \epsilon^{-1+(1+\theta)(1-1/p)} \left(\int_{\Sigma_{2\epsilon}} |\partial w|^p \, du \right)^{\frac{1}{p}} \\ &\to 0\end{aligned}$$

as $\epsilon \to 0$ provided p is large enough. Hence we have

$$\int_{B(2\delta)} \left(h^{cd} \frac{\partial \eta}{\partial u^a} \frac{\partial w}{\partial u^b} + \eta s^d \frac{\partial w}{\partial u^d} \right) du = -\int_{B(2\delta))} \eta Q_1 \, du \tag{6-10}$$

for all smooth functions η with compact support $B(2\delta)$.

Let $\zeta = v - w$; then $v - w \in W_0^{1,p}$ for all $p > 1$ and

$$\int_{B(2\delta)} \left(h^{cd} \frac{\partial \eta}{\partial u^a} \frac{\partial \zeta}{\partial u^b} + \eta s^d \frac{\partial \zeta}{\partial u^d} \right) du = 0 \tag{6-11}$$

for all smooth functions η with compact support in $B(2\delta)$. Using the fact that $s^d \in L^p(B(2\delta))$ we can proceed as in the proof of [Gilbarg and Trudinger 1983, Theorem 8.1] to conclude that $\zeta \equiv 0$, because $s^q \in L^p(B(2\delta))$ for all $p > 1$.

To summarize we have proved that $h_{ab} \in W^{2,p}(B(2\delta))$ for all $p > n$ and h_{ab} is smooth in u for all a, b.

We can cover Σ by such harmonic coordinate neighborhoods V_q so that the components of g are smooth with respect to these coordinates. By [Taylor 2006, Theorem 2.1] one can conclude that the theorem is true. □

Corollary 6.4. *Suppose (M^n, g_0) is as in Theorem 5.1. If in addition, g_0 is Lipschitz, then there is a smooth structure on M such that g_0 is smooth and Einstein.*

7. A positive mass theorem with singular set

In this section, we will use the results in Sections 3 and 4 to study positive mass theorems on asymptotically flat manifolds with singular metrics. We want to discuss the theorem without assuming that the manifold is spin. There are different definitions for asymptotically flat manifold. For our purpose, we use the following:

Definition 7.1. An n-dimensional Riemannian manifold (M^n, g), where g is continuous, is said to be asymptotically flat (AF) if there is a compact subset K such that g is smooth on $M \setminus K$, and $M \setminus K$ has finitely many components E_k, $1 \le k \le l$, each E_k is called an end of M, such that each E_k is diffeomorphic to $\mathbb{R}^n \setminus B(R_k)$ for some Euclidean ball $B(R_k)$, and the following are true:

(i) In the standard coordinates x^i of $\mathbb{R}^n$,

$$g_{ij} = \delta_{ij} + \sigma_{ij}$$

with

$$\sup_{E_k}\left\{\sum_{s=0}^{2} |x|^{\tau+s}|\partial^s\sigma_{ij}| + [|x|^{\alpha+2+\tau}\partial\partial\sigma_{ij}]_\alpha\right\} < \infty,$$

for some $0 < \alpha \le 1$, $\tau > (n-2)/2$, where ∂f and $\partial^2 f$ are the gradient and Hessian of f with respect to the Euclidean metric, and $[f]_\alpha$ is the α-Hölder norm of f.

(ii) The scalar curvature $\mathcal{S}$ satisfies the decay condition

$$|\mathcal{S}|(x) \le C(1+d(x))^{-q}$$

for some $q > n$. Here $d(x)$ is the distance function from a fixed point in M.

The coordinate chart satisfying (i) is said to be *admissible*.

Without loss of generality, we assume that $q \le n+2$. This assumption will be used in (7-3).

In the following, for a function f defined near infinity or $\mathbb{R}^n$, and for $k \ge 0$, $f = O_k(r^{-\tau})$ refers to $\sum_{i=0}^k r^i|\partial^i f| = O(r^{-\tau})$ as $r \to \infty$, where $r = |x|$.

Definition 7.2. The Arnowitt–Deser–Misner (ADM) mass (see [Arnowitt et al. 1961]) of an end E of an AF manifold M is defined as

$$\text{(7-1)} \qquad \mathfrak{m}_{\mathrm{ADM}}(E) = \lim_{r\to\infty} \frac{1}{2(n-1)}\omega_{n-1}\int_{S_r} (g_{ij,i} - g_{ii,j})\nu^j \, d\Sigma_r^0$$

in an admissible coordinate chart where S_r is the Euclidean sphere, ω_{n-1} is the volume of the $(n-1)$-dimensional unit sphere, $d\Sigma_r^0$ is the volume element induced by the Euclidean metric, ν is the outward unit normal of S_r in $\mathbb{R}^n$ and the derivative is the ordinary partial derivative.

By [Bartnik 1986], $\mathfrak{m}_{\mathrm{ADM}}(E)$ is well-defined, i.e., it is independent of the choice of admissible charts.

For smooth metrics, without assuming the manifold is spin, we have the following positive mass theorem by Schoen and Yau [1979b; 1981; Schoen 1989]:

Theorem 7.3. *Let (M^n, g), $3 \le n \le 7$, be an AF manifold with nonnegative scalar curvature $\mathcal{S} \ge 0$. Then the ADM mass of each end is nonnegative. Moreover, if the ADM mass of one of the ends is zero, then (M^n, g) is isometric to $\mathbb{R}^n$ with the standard metric.*

We want to prove the following positive mass theorem for metrics which are smooth outside a compact set of codimension at least 2. More precisely, we want to prove the following:

Theorem 7.4. *Let (M^n, g_0) be an AF manifold with $3 \le n \le 7$, g_0 being a continuous metric on M such that*

(i) *g_0 is smooth outside a compact set Σ with codimension at least 2 as in (a4) in Section 4,*

(ii) *the scalar curvature $\mathcal{S}$ of g_0 is nonnegative outside Σ,*

(iii) *$g_0 \in W^{1,p}_{\mathrm{loc}}$ for some $p > n$ as in (a2) in Section 4,*

(iv) *on each end E, in an admissible coordinate chart,*

$$g_{ij} = \delta_{ij} + \sigma_{ij}$$

with $\sigma_{ij} = O_5(r^{-\tau})$ with $\tau > (n-2)/2$.

Then the ADM mass of each end is nonnegative. Moreover, if the mass of one of the ends is zero, then M is diffeomorphic to $\mathbb{R}^n$, and g_0 is flat outside Σ.

Remark 7.5. (a) The assumption of continuity of the metric cannot be removed. See the construction in Proposition 2.3.

(b) The case that the singular set is an embedded hypersurface has been studied in [Miao 2002; Shi and Tam 2002]; see also [McFeron and Székelyhidi 2012].

(c) In the case that the singular set has codimension larger than 1, for spin manifolds, positive mass theorems have been obtained under rather general assumptions in [Lee and LeFloch 2015]. Without the spin condition, there are also results for metrics with bounded C^2 norm and with singular set having codimension at least $n/2$ [Lee 2013].

We proceed as in [McFeron and Székelyhidi 2012]. As in Section 4, let $\epsilon > 0$, $\epsilon \to 0$. We can construct a family of metrics $g_{\epsilon,0}$ such that

(i) $g_{\epsilon,0} \to g_0$ uniformly,

(ii) $g_{\epsilon,0} = g_0$ outside $\Sigma(2\epsilon)$,

(iii) the $W^{1,p}$ norm of $g_{\epsilon,0}$ in a fixed precompact open set containing Σ is bounded by a constant independent of ϵ.

As in Section 4, we can choose $\epsilon_0 > 0$ small enough and let $h = g_{\epsilon_0,0}$. Then there is a $T > 0$ independent of ϵ such that if $0 < \epsilon \le \epsilon_0$, then there is a smooth solution $g_\epsilon(t)$ on $M \times [0, T]$ to the h-flow with initial data $g_{\epsilon,0}$. There is also a smooth solution $g(t)$ on $M \times (0, T]$ to the h-flow such that $g(t) \to g_0$ uniformly on compact sets as $t \to 0$. Moreover, Lemma 4.2 is still true with M being noncompact in this case because M is AF.

Let $\tilde{g}_\epsilon(t)$ be the corresponding solution to the Ricci flow with $\tilde{g}_\epsilon(t) = \Phi_t^*(g_\epsilon(t))$ as in the compact case in Section 4. Then we have the following:

Lemma 7.6. (i) *The metrics $g_\epsilon(t)$, $\tilde{g}_\epsilon(t)$, $g(t)$ are AF in the sense of Definition 7.1.*

(ii) *For each end E of M, $\mathfrak{m}(E)(\epsilon, t) = \mathfrak{m}(E)(\epsilon, 0) = \mathfrak{m}(E)$, where $\mathfrak{m}(E)(\epsilon, t)$ is the mass with respect to $g_\epsilon(t)$ or $\tilde{g}_\epsilon(t)$, and $\mathfrak{m}(E)(\epsilon, 0)$ is the mass with respect to $g_{\epsilon,0}$ or g_0.*

Proof. (i) First note that $C_1^{-1}h \le g_\epsilon(t) \le C_1 h$ for some constant $C_1 > 0$ independent of ϵ, t. On the other hand, by Lemma 4.2 applied to the noncompact case, we conclude that the curvature of $\tilde{g}_\epsilon(t)$ is bounded by $Ct^{-\frac{1}{2}(1+\delta)}$ for some $0 < \delta < 1$ where C, δ are independent of ϵ, t. Hence we also have $C_1^{-1}g_{\epsilon,0} \le g_\epsilon(t) \le C_1 g_{\epsilon,0}$ and $C_1^{-1}h \le \tilde{g}_\epsilon(t) \le C_1 h$, with possible larger C_1.

Using the fact that $\sigma_{ij} = O_5(r^{-\tau})$, we can proceed with some modifications as in [Dai and Ma 2007; McFeron and Székelyhidi 2012] to show that outside a fixed compact set, for $0 \le l \le 3$,

$$|{}^h\nabla^l g_\epsilon(x, t)| \le C_2 d^{-l-\tau}(x)$$

for some constant C_2 independent of ϵ, t, x, where $d(x)$ is the distance function from a fixed point with respect to h. Here we use the fact that $\sigma_{ij} = O_5(r^{-\tau})$. The proof is similar to the proof for the decay rate of scalar curvature. So we only carry out the proof for this case in more detail.

We want to prove that there is a constant $C_3 > 0$ independent of ϵ, t and a compact set K such that if $\tilde{\mathcal{S}}_\epsilon(t)$ is the scalar curvature of $\tilde{g}_\epsilon(t)$, then

$$\sup_{M\setminus K} d^q(x)|\tilde{\mathcal{S}}_\epsilon(x, t)| \le C_3. \tag{7-2}$$

We will prove this on each end. Fix ϵ. Denote the scalar curvature of $g_\epsilon(t)$ simply by $\mathcal{S}$ and curvature by Rm etc. Let E be an end which is diffeomorphic to $\mathbb{R}^n \setminus B(R)$, say. By [Simon 2002], by choosing R large enough so that $g_{\epsilon,0} = h = g_0$ outside $B(R/2)$ and g_0 is smooth there, we may assume that $|\mathrm{Rm}(g_\epsilon(t))| \le C_4$ for some constant C_4 independent of ϵ, t outside $B(R/2)$. Here we have used the fact that $g_\epsilon(t)$, $\tilde{g}_\epsilon(t)$ are uniformly equivalent.

Let g_e be the standard Euclidean metric and let $0 \le \phi \le 1$ be a fixed smooth function on $\mathbb{R}^n$ such that $\phi = 1$ in $B(R)$ and $\phi = 0$ outside $B(2R)$. Consider the metric $\phi g_e + (1 - \phi)g_\epsilon(t)$. Still denote its curvature by Rm etc.

Let ρ be a fixed function $\rho \ge 1$, $\rho = 1$ in $B(R)$, $\rho(x) = |x|$ outside $B(2R)$. Hence the gradient and the Hessian of ρ with respect to $g_\epsilon(t)$ are bounded by a constant independent of ϵ, t. We have

$$\frac{\partial}{\partial t}\mathcal{S}^2 \le \Delta\mathcal{S}^2 + C_5$$

in $B(2R)$ and

$$\frac{\partial}{\partial t}\mathcal{S}^2 = \Delta \mathcal{S}^2 + 2\mathcal{S}|\mathrm{Ric}|^2 - 2|\nabla \mathcal{S}|^2$$

outside $B(2R)$.

Let $F = \rho^{2q}\mathcal{S}^2$; then outside $B(2R)$,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)F &= \rho^{2q}(2\mathcal{S}|\mathrm{Ric}|^2 - 2|\nabla \mathcal{S}|^2) - 2\langle \nabla \rho^{2q}, \nabla \mathcal{S}^2\rangle + F\Delta \rho^{2q} \\ &\le C_6 \rho^{q-4-2\tau}\rho^q \mathcal{S} - 4q\rho^{-1}\langle \nabla\rho, \nabla F\rangle + C_6 F \\ &\le C_7 - 4q\rho^{-1}\langle \nabla\rho, \nabla F\rangle + C_7 F\end{aligned} \tag{7-3}$$

for some constants C_6, C_7 independent of ϵ, t since $q-4-2\tau < q-(n+2) \le 0$. The inequality is still true in $B(2R)$ because in $B(R)$, $\nabla\rho = 0$ and in $B(2R)\setminus B(R)$, $|\nabla\rho|$ and $|\nabla\mathcal{S}|$ are uniformly bounded. Hence if $\tilde{F} = e^{-C_7 t}F - C_7 t$, then

$$\left(\frac{\partial}{\partial t} - \Delta\right)\tilde{F} \le -4q\rho^{-1}\langle \nabla\rho, \nabla\tilde{F}\rangle. \tag{7-4}$$

Let $A > 0$ to be chosen later. Let $\eta = \exp(2At + \rho)$. Then

$$\left(\frac{\partial}{\partial t} - \Delta\right)\eta \ge 2A\eta - C\eta$$

for some constant C independent of ϵ, t. Choose $A = C$; then we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\eta \ge A\eta.$$

Let $\kappa > 0$ be any positive number; then

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\tilde{F} - \kappa\eta) \le -4q\rho^{-1}\langle \nabla\rho, \nabla\tilde{F}\rangle - \kappa A\eta.$$

Since $\tilde{F}$ has at most polynomial growth, if $\tilde{F} - \kappa\eta$ has a positive maximum, then the maximum will be attained at some point (x_0, t_0). Suppose $t_0 > 0$; then at (x_0, t_0),

$$\nabla\tilde{F} = \kappa\nabla\eta.$$

Hence at (x_0, t_0),

$$\begin{aligned}0 &\le \left(\frac{\partial}{\partial t} - \Delta\right)(\tilde{F} - \kappa\eta) \\ &\le -4q\rho^{-1}\langle \nabla\rho, \nabla\tilde{F}\rangle - \kappa A\eta \\ &= -4q\rho^{-1}\kappa\langle \nabla\rho, \nabla\eta\rangle - \kappa A\eta \\ &\le -\kappa A\eta,\end{aligned}$$

which is impossible. Hence either $\tilde{F} - \kappa\eta \le 0$, or

$$\tilde{F} - \kappa\eta \le \sup_{\mathbb{R}^n}\left(\rho^{2q}(x)\mathcal{S}^2(0)\right),$$

where $\mathcal{S}(0)$ is the scalar curvature of $\phi g_e + (1 - \phi g_0)$. Let $\kappa \to 0$, we conclude the (7-2) is true.

(ii) Since $g_{\epsilon,0} = g_0$ outside a compact set, $\mathfrak{m}(E) = \mathfrak{m}(E)(\epsilon, 0)$. On the other hand, by the fact that $\tilde{g}_\epsilon(t)$ and $\tilde{g}(t)$ are given by a diffeomorphism and by (i) and [Bartnik 1986], the mass of E is the same whether it is computed with respect to $\tilde{g}_\epsilon(t)$ or $g_\epsilon(t)$.

The fact that $\mathfrak{m}(E)(\epsilon, t) = \mathfrak{m}(E)(\epsilon, 0)$ follows from [Dai and Ma 2007]. □

Proof of Theorem 7.4. By Lemmas 4.1 and 4.13, we conclude that $g(t)$ is AF and with nonnegative scalar curvature for $t > 0$. Let E be an end. Using the notation in Lemma 7.6, by the lemma and [McFeron and Székelyhidi 2012, Theorem 14] (see also [Jauregui 2014]), the mass $\mathfrak{m}(E)(t)$ of E with respect to $g(t)$ satisfies

$$\begin{aligned}
\mathfrak{m}(E) &= \liminf_{\epsilon\to 0} \mathfrak{m}(E)(\epsilon, 0) \\
&= \liminf_{\epsilon\to 0} \mathfrak{m}(E)(\epsilon, t) \\
&\geq \mathfrak{m}(E)(t).
\end{aligned}$$

By Theorem 7.3, $\mathfrak{m}(E)(t) \geq 0$, we have $\mathfrak{m}(E) \geq 0$. If $\mathfrak{m}(E) = 0$, then $\mathfrak{m}(E)(t) = 0$ and $(M^n, g(t))$ is isometric to the Euclidean space. Since $g(t)$ converges to g_0 in C^∞ as $t \to 0$ away from Σ, g_0 is flat outside Σ. □

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ON THE DIFFERENTIABILITY ISSUE OF THE DRIFT-DIFFUSION EQUATION WITH NONLOCAL LÉVY-TYPE DIFFUSION

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We investigate the differentiability property of the drift-diffusion equation with nonlocal Lévy-type diffusion at either supercritical- or critical-type cases. Under the suitable conditions on the velocity field and the forcing term in terms of the spatial Hölder regularity, and for the initial data without regularity assumption, we show the a priori differentiability estimates for any positive time. If additionally the velocity field is divergence-free, we also prove that the vanishing viscosity weak solution is differentiable with some Hölder continuous derivatives for any positive time.

1. Introduction

We consider the following drift-diffusion equation with nonlocal diffusion:

$$\begin{cases} \partial_t \theta + (u \cdot \nabla)\theta + \mathcal{L}\theta = f & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \theta(x,0) = \theta_0(x) & \text{on } \mathbb{R}^d, \end{cases} \tag{1-1}$$

where θ is a scalar function, u is a velocity vector field of $\mathbb{R}^d$ and f is a scalar function as the forcing term. The nonlocal diffusion operator $\mathcal{L}$ is given by

$$\mathcal{L}g(x) = \text{p. v.}\int_{\mathbb{R}^d} (g(x) - g(x+y))K(y)\,\mathrm{d}y, \tag{1-2}$$

where the symmetric kernel function $K(y) = K(-y)$ defined on $\mathbb{R}^d \setminus \{0\}$ satisfies

$$\int_{\mathbb{R}^d} \min\{1, |y|^2\}|K(y)|\,\mathrm{d}y \le c_1, \tag{1-3}$$

and there exist two constants $\alpha \in (0,1]$ and $\sigma \in [0,\alpha)$ such that

$$\frac{c_2^{-1}}{|y|^{d+\alpha-\sigma}} \le K(y) \le \frac{c_2}{|y|^{d+\alpha}} \quad \text{for all } 0 < |y| \le 1, \tag{1-4}$$

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with $c_1 > 0$ and $c_2 \geq 1$ two absolute constants. In the sequel we also consider the special case that K satisfies the nonnegative condition

$$K(y) \geq 0 \quad \text{for all } y \in \mathbb{R}^d \setminus \{0\}. \tag{1-5}$$

By taking the Fourier transform on $\mathcal{L}$, we get

$$\widehat{\mathcal{L}\theta}(\xi) = A(\xi)\hat{\theta}(\xi),$$

where the symbol $A(\xi)$ is given by the Lévy–Khinchin formula

$$A(\xi) = \text{p. v.}\int_{\mathbb{R}^d} (1 - \cos(x \cdot \xi))K(x)\,\mathrm{d}x. \tag{1-6}$$

The nonlocal diffusion operator $\mathcal{L}$ defined by (1-2) with the symmetric kernel K satisfying (1-3)–(1-4) corresponds to the stable-type Lévy operator, which is the infinitesimal generator of the stable-type Lévy process (see [Chen et al. 2015; Sato 1999]). If $\sigma = 0$, the operator $\mathcal{L}$ is referred to as the stable-like Lévy operator, and in recent years many deep works have been devoted to studying various regularity problems concerning this diffusion operator (one can see [Komatsu 1995; Husseini and Kassmann 2007; Kassmann 2009; Caffarelli et al. 2011; Caffarelli and Silvestre 2011; Maekawa and Miura 2013; Dabkowski et al. 2014]). The typical example of the stable-like Lévy operator is the fractional Laplacian operator $|D|^\alpha := (-\Delta)^{\alpha/2}$ ($\alpha \in]0, 2[$), which has the following expression formula:

$$|D|^\alpha \theta(x) = c_{d,\alpha}\,\text{p. v.}\int_{\mathbb{R}^d} \frac{\theta(x) - \theta(x+y)}{|y|^{d+\alpha}}\,\mathrm{d}y, \tag{1-7}$$

with $c_{d,\alpha} > 0$ some absolute constant. The operator $\mathcal{L} = |D|^\alpha := (-\Delta)^{\alpha/2}$ ($\alpha \in (0, 2)$) is the infinitesimal generator of the symmetric stable Lévy process (see [Sato 1999]), and recently has been intensely considered in many theoretical problems. If $\sigma \neq 0$, the stable-type Lévy operator can contain more general diffusion operators. An important class is the following multiplier operators $\mathcal{L} = A(D) = A(|D|)$ defined by

$$\mathcal{L} = \frac{|D|^\alpha}{(\log(\lambda + |D|))^\mu}, \quad (\alpha \in (0, 1],\ \mu \geq 0,\ \lambda > 0), \tag{1-8}$$

and one can refer to [Dabkowski et al. 2014, Lemmas 5.1–5.2] for more details on the assumptions on $A(\xi)$ so that the kernel K satisfies (1-3)–(1-4) (the condition (1-5) can also be satisfied under some additional assumption on $A(\xi)$, see [Hmidi 2011; Miao and Xue 2015]). Recently, the logarithmic diffusion operator defined by (1-8) in many systems has attracted a lot of attention and has been variously studied (e.g., [Tao 2009; Hmidi 2011; Chae and Wu 2012; Dabkowski et al. 2014; Miao and Xue 2015]). One can also refer to [Chen et al. 2015, Example 4.2] for other important classes of stable-type Lévy operators.

Recalling that for the drift-diffusion equation (1-1) with $\mathcal{L} = |D|^\alpha$, we conventionally call the cases $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$ supercritical, critical and subcritical cases, respectively. Thus the operator $\mathcal{L}$ defined by (1-2) under the kernel conditions (1-3)–(1-4) can be viewed as the critical- and supercritical-type cases and they are the main concern in this paper.

For the drift-diffusion equation (1-1) with the fractional Laplacian operator $\mathcal{L} = |D|^\alpha$, if the velocity field is divergence-free, the $C^{1,\gamma}$-regularity improvement of weak solutions was obtained by Constantin and Wu [2008] by using the Bony's paradifferential calculus. Partially motivated by that work, without the divergence-free restriction on the velocity, Silvestre [2012b] considered the supercritical and critical cases ($\alpha \in (0, 1]$), and proved the interior $C^{1,\gamma}$-regularity of the solution provided that u and f belong to $L_t^\infty C_x^{1-\alpha+\gamma}$ ($\gamma \in (0, \alpha)$), more precisely, the author showed the following regularity estimate:

$$\|\theta\|_{L^\infty([-1/2,0];C^{1,\gamma}(B_{1/2}))} \le C(\|u\|_{L^\infty([-1,0]\times\mathbb{R}^d)} + \|f\|_{L^\infty([-1,0];C^{1-\alpha+\gamma}(B_1))}), \tag{1-9}$$

where $C > 0$ depends only on d, α and $\|u\|_{L^\infty([-1,0];C^{1-\alpha+\gamma})}$. The proof is by a locally approximate procedure where an extension derived in [Caffarelli and Silvestre 2007] plays a key role. For the drift-diffusion equation (1-1) with more general diffusion operator, so far there are not many such differentiability results. We here only mention a related work [Chen et al. 2015], where the authors considered the backward drift-diffusion equation

$$\partial_t\theta + u\cdot\nabla\theta - (\mathcal{L}+\lambda)\theta = f, \quad \theta|_{t=1}(x) = 0, \quad \lambda \ge 0, \tag{1-10}$$

with $\mathcal{L}$ defined by (1-2)–(1-4) (in fact slightly more general Lévy operator $\mathcal{L}$ considered there), and by applying a purely probabilistic method, the authors proved the $C^{1,\gamma}$-regularity of a continuous solution $\theta : [0, 1] \times \mathbb{R}^d \to \mathbb{R}$ under the conditions that u and f are C_x^δ-Hölder continuous ($\delta \in (1-\alpha+\sigma, 1)$) for each time.

If we slightly lower the regularity index in the assumption of u and f, the solution of the equations (1-1)–(1-2) may in general not have such a differentiable regularity. For the drift-diffusion equation (1-1) with $\mathcal{L} = |D|^\alpha$, Silvestre [2012a] proved that if $u \in L_t^\infty \dot{C}_x^{1-\alpha}$ for $\alpha \in (0, 1)$ and $u \in L_{t,x}^\infty$ for $\alpha = 1$, and if $f \in L_{t,x}^\infty$, then the bounded solution becomes Hölder continuous for any positive time. For the drift-diffusion equation (1-1) with stable-like Lévy operator $\mathcal{L}$, and under the divergence-free condition of u, we refer to [Chamorro and Menozzi 2016] for a similar improvement to Hölder continuous solutions (see also [Maekawa and Miura 2013] for a related result). Note that the condition $u \in L_t^\infty \dot{C}^{1-\alpha}$ is invariant under the scaling transformation $u(x, t) \mapsto \lambda^{\alpha-1}u(\lambda^\alpha t, \lambda x)$ for all $\lambda > 0$. If we further weaken the regularity condition on u in the supercritical case, the solution of (1-1)–(1-2) may not even be continuous, indeed, as proved by Silvestre, Vicol and Zlatoš in [Silvestre et al. 2013], there is a divergence-free drift $u \in L_t^\infty C_x^\delta$ with

$\delta < 1-\alpha$ so that the solution of the equation (1-1) with $\mathcal{L} = |D|^{\alpha}$ and $f = 0$ forms a discontinuity starting from smooth initial data.

In this paper, we are concerned with the differentiability property of the system (1-1)–(1-2), and if the velocity field u is divergence free, we consider the differentiability of weak solutions, which is derived by passing to a limit of the approximate system, while if u is not divergence free, we only consider the a priori differentiability estimate. We impose no regularity assumption on the nonzero initial data, and we generalize the result of Silvestre [2012b] for more general stable-type Lévy operators.

Our first result states that if the velocity field is divergence-free, then the differentiability of the vanishing viscosity weak solution can be achieved for the equations (1-1)–(1-2) under conditions (1-3)–(1-4) and suitable assumptions.

Theorem 1.1. *Let the symmetric kernel $K(y) = K(-y)$ of the diffusion operator $\mathcal{L}$ satisfy (1-3)–(1-4), and the velocity field u be divergence-free. Assume that for any given $T > 0$, the drift u, the force f and the initial data θ_0 satisfy*

$$u \in L^{\infty}([0,T], C^{\delta}(\mathbb{R}^d)) \quad \textit{for some } \delta \in (1-\alpha+\sigma, 1), \tag{1-11}$$

and

$$f \in L^{\infty}([0,T]; B^{\delta}_{p,\infty} \cap B^{\delta}_{\infty,\infty}(\mathbb{R}^d)), \quad \theta_0 \in L^p(\mathbb{R}^d) \quad \textit{for some } p \in [2,\infty). \tag{1-12}$$

Then there exists a weak solution $\theta \in L^{\infty}([0,T]; L^p(\mathbb{R}^d)) \cap L^p([0,T]; B^{\alpha-\sigma/p}_{p,p}(\mathbb{R}^d))$ which satisfies the drift-diffusion equation (1-1)–(1-2) in the distributional sense (see (3-52) below). Moreover, $\theta \in L^{\infty}((0,T], C^{1,\gamma}(\mathbb{R}^d))$ for any $\gamma \in (0, \delta+\alpha-\sigma-1)$, which precisely satisfies that for every $t' \in (0,T)$,

$$\|\theta\|_{L^{\infty}([t',T];C^{1,\gamma}(\mathbb{R}^d))} \le C t'^{-(\gamma+1+d/p)/(\alpha-\sigma)} (\|\theta_0\|_{L^p} + \|f\|_{L^{\infty}_T(B^{\delta}_{p,\infty}\cap B^{\delta}_{\infty,\infty})}), \tag{1-13}$$

with the constant C depending only on $T, \alpha, \sigma, d, \delta$ and $\|u\|_{L^{\infty}_T \dot{C}^{\delta}}$.

For our second result, we do not necessarily impose the divergence-free property of the velocity field, and we mainly focus on the a priori differentiability estimates of the drift-diffusion equation (1-1)–(1-2) under the conditions (1-3)–(1-5), or in other words, we concentrate on the uniform-in-ϵ differentiability estimates of the following ϵ-regularized drift-diffusion equation under (1-3)–(1-5)

$$\partial_t \theta + u_{\epsilon} \cdot \nabla\theta + \mathcal{L}\theta - \epsilon\Delta\theta = f_{\epsilon}, \quad \theta|_{t=0} = \theta_{0,\epsilon} = \phi_{\epsilon} * (\theta_0 1_{B_{1/\epsilon}(0)}), \tag{1-14}$$

where $u_{\epsilon} = \phi_{\epsilon} * u$, $f_{\epsilon} = \phi_{\epsilon} * f$, $\phi_{\epsilon}(x) = \epsilon^{-d}\phi(x/\epsilon)$ and ϕ is the standard mollifier. The result is as follows.

Theorem 1.2. *Let the kernel $K(y) = K(-y)$ of the diffusion operator $\mathcal{L}$ satisfy the conditions (1-3)–(1-5). Let $\theta_0 \in C_0(\mathbb{R}^d)$, with $C_0(\mathbb{R}^d)$ the space of continuous* *functions which decay to zero at infinity. Suppose that for any given $T > 0$, the drift u and the external force f satisfy*

$$u \in L^\infty([0,T]; C^\delta(\mathbb{R}^d)) \quad \textit{and} \quad f \in L^\infty([0,T]; C^\delta \cap L^2(\mathbb{R}^d)), \tag{1-15}$$

for some $\delta \in (1-\alpha+\sigma, 1)$, then the solutions $\theta^{(\epsilon)}$ of the regularized drift-diffusion equation (1-14) uniformly-in-ϵ belong to

$$L^\infty([0,T]; C_0(\mathbb{R}^d)) \cap L^\infty((0,T], C^{1,\gamma}(\mathbb{R}^d)) \quad \textit{for any } \gamma \in (0, \delta+\alpha-\sigma-1).$$

More precisely, for any $t' \in (0,T)$, we have

$$\|\theta^{(\epsilon)}\|_{L^\infty([t',T]; C^{1,\gamma}(\mathbb{R}^d))} \le C t'^{-(\gamma+1)/(\alpha-\sigma)} (\|\theta_0\|_{L^\infty} + \|f\|_{L^\infty_T C^\delta}), \tag{1-16}$$

where C is a positive constant depending only on α, σ, d, δ and $\|u\|_{L^\infty_T \dot{C}^\delta}$ and is independent of ϵ.

Theorems 1.1 and 1.2 (and Remark 1.3 below) can be applied to the regularity problem of the (weak) solution of various nonlinear drift-diffusion equations, and one can refer to the recent work [Miao and Xue 2015] for some direct applications.

The method in showing Theorems 1.1 and 1.2 is consistent with the method of paradifferential calculus used in [Constantin and Wu 2008], but is mostly in a different style; and by choosing some time function as a weight and developing the technique of weighted estimates (where Lemma 3.4 is of great use), we find that the process used here is efficient and is not sensitive to the divergence-free condition of u so that we can get rid of such an assumption in Theorem 1.2 (noticing that the method in [Constantin and Wu 2008] does not extend to the drift-diffusion equations (1-1)–(1-2) without the divergence-free property of u). We use the L^p ($p \in [2,\infty)$) framework in Theorem 1.1 and the L^∞ framework in proving Theorem 1.2, and the key diffusion effect of the Lévy-type diffusion operator (for high frequency part) is derived in Lemma 3.2 and Lemma 4.2 respectively. The iterative argument also plays an important role in the proof of both theorems, and we can see clearly how the regularity index of the solution improves step by step.

We want to point out that our approach in this paper is purely analytic, and does not use the probabilistic representations of solutions. Note also that the approach of [Silvestre 2012b] is not adopted here, and it seems rather hard (if not impossible) to extend the method of that work for the drift-diffusion equation with more general diffusion operators.

Remark 1.3 (On higher regularity). If the assumptions (1-11)–(1-12) and (1-15) hold for any $\delta > 1-\alpha+\sigma$ by removing the restriction $\delta < 1$, then by following the deduction in Subsections 3B and 4B, we infer that for the cases studied in Theorems 1.1 and 1.2, we a priori have

$$\theta \in L^\infty((0,T]; C^{[\delta+\alpha-\sigma]-1,\gamma}) \quad \text{for all } \gamma \in (0,1)$$

if $\delta+\alpha-\sigma\in\mathbb{N}^+$, and

$$\theta\in L^\infty((0,T];C^{[\delta+\alpha-\sigma],\gamma})\quad\text{for all }\ \gamma\in(0,\delta+\alpha-\sigma-[\delta+\alpha-\sigma])$$

if $\delta+\alpha-\sigma\notin\mathbb{N}^+$.

As a consequence of the above result, and if $f=0$ and $u=\mathcal{P}\theta$ in the equation (1-1) with $\mathcal{P}$ composed of zero-order pseudodifferential operators, e.g., the dissipative SQG equation which recently has been intensely considered (see [Caffarelli and Vasseur 2010; Chen et al. 2007; Constantin and Vicol 2012; Córdoba and Córdoba 2004; Dabkowski et al. 2014; Kiselev and Nazarov 2009; Kiselev et al. 2007; Wang and Zhang 2011]):

$$\partial_t\theta+u\cdot\nabla\theta+\mathcal{L}\theta=0,\quad u=(-\mathcal{R}_2\theta,\mathcal{R}_1\theta),\quad \theta(x,0)=\theta_0(x),\quad x\in\mathbb{R}^2, \tag{1-17}$$

with $\mathcal{R}_i$ $(i=1,2)$ the usual Riesz transform, we can deduce that under the assumptions of Theorems 1.1 and 1.2 with the condition on u replaced by

$$\theta\in L^\infty([0,T],C^\delta(\mathbb{R}^d))\quad\text{for some }\ \delta\in(1-\alpha+\sigma,1),$$

then the corresponding weak solution θ further belongs to $C^\infty((0,T]\times\mathbb{R}^d)$. Indeed, after obtaining the bound of $\|\theta\|_{L^\infty C^{1,\gamma}}$ (and $\|\theta\|_{L^\infty B^{1+\gamma+d/\tilde p}_{\tilde p,\infty}}$ with some $\tilde p<\infty$ in Theorem 1.1) for any $\gamma\in(0,\delta+\alpha-\sigma-1)$, from the Calderón–Zygmund theorem, we get $\nabla u\in L^\infty\dot C^\gamma$, which further leads to

$$\theta\in L^\infty C^{[1+\gamma+\alpha-\sigma]-1,\gamma'}\quad\text{for all }\ \gamma'\in(0,1)$$

if $1+\gamma+\alpha-\sigma\in\mathbb{N}$, and

$$\theta\in L^\infty C^{[1+\gamma+\alpha-\sigma],\gamma'}\quad\text{for all }\ \gamma'\in(0,\gamma+\alpha-\sigma-[\gamma+\alpha-\sigma])$$

if $1+\gamma+\alpha-\sigma\notin\mathbb{N}^+$, (in Theorem 1.1 we in fact obtain a stronger estimate on θ in terms of L^p-based Besov spaces); noting that the regularity index can be arbitrarily close to $\delta+2(\alpha-\sigma)$ by suitably choosing γ and γ', thus by the bootstrapping method, we can iteratively improve the regularity index to any large number and finally conclude the C^∞_x-smoothness of the solution. The C^∞-smoothness in $t\in(0,T]$ can be derived from equation (1-1) and Lemma 2.2.

Remark 1.4. In Theorem 1.2, if the velocity field u is divergence-free, and $\theta_0\in L^2\cap L^\infty(\mathbb{R}^d)$, and (1-15) is also assumed, then there exists a weak solution θ to the drift-diffusion equation (1-1)–(1-2) which satisfies (1-16) with θ in place of $\theta^{(\epsilon)}$. But if the velocity field is not divergence-free, and under the assumptions of Theorem 1.2, it is not so clear for the authors to pass to the limit $\epsilon\to0$ in equation (1-14) to obtain the weak solution of the drift-diffusion equations (1-1)–(1-2). Despite that, we believe that the uniform-in-ϵ differentiability estimate (1-16) is meaningful and may have its various applications.

Remark 1.5. By examining the proof of both theorems, the upper bound in (1-4) does not play an essential role in the proof of (1-13) and (1-16), which indeed can be relaxed to larger numbers. But we here include the upper bound in (1-4) is to restrict ourselves to the critical and supercritical type cases.

Remark 1.6. In Theorem 1.2, the condition on f in (1-15) can be replaced by $f \in L^\infty([0,T]; C^\delta_0(\mathbb{R}^d))$ with $C^\delta_0(\mathbb{R}^d)$ the closure of Schwartz class under the norm of Hölder space $C^\delta(\mathbb{R}^d)$, and the same uniform estimate (1-16) holds true for a suitable approximate system of the equations (1-1)–(1-2).

The outline of the paper is as follows. In Section 2, we present some preliminary knowledge on Bony's paradifferential calculus and the Besov spaces, and give a useful lemma on the stable-type Lévy operator $\mathcal{L}$. Section 3 is dedicated to the proof of Theorem 1.1, and we first show several useful auxiliary lemmas, then we prove the key a priori estimate (1-13) in the Section 3B, and then we sketch the proof of the existence part and conclude the theorem. We show Theorem 1.2 in Section 4, and the proof is also divided into three parts: the auxiliary lemmas, the a priori estimates and the uniform-in-ϵ differentiability estimates for the regularized system (1-14), which are treated in the subsections 4A – 4C respectively.

2. Preliminaries

In this preliminary section, we shall gather some notations used in this paper, collect some basic facts on Bony's paradifferential calculus and Besov spaces, and show a useful lemma on the considered Lévy operator $\mathcal{L}$.

Throughout this paper, C stands for a constant which may be different from line to line. The notation $X \lesssim Y$ means that $X \leq CY$, and $X \approx Y$ implies that $X \lesssim Y$ and $Y \lesssim X$ simultaneously. Denote $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions, $\mathcal{S}(\mathbb{R}^d)$ the Schwartz class of rapidly decreasing smooth functions, $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$ the quotient space of tempered distributions modulo polynomials. We use $\hat{g}$ of $\mathcal{F}(g)$ to denote the Fourier transform of a tempered distribution, that is, $\hat{g}(\xi) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} g(x)\,\mathrm{d}x$. For a number $a \in \mathbb{R}$, denote by $[a]$ the integer part of a.

Now we recall the so-called Littlewood–Paley operators and their elementary properties. Let (χ, φ) be a couple of smooth functions taking values on $[0,1]$ such that $\chi \in C^\infty_c(\mathbb{R}^d)$ is supported in the ball $\mathcal{B} := \{\xi \in \mathbb{R}^d, |\xi| \leq \frac{4}{3}\}$, $\varphi \in C^\infty_0(\mathbb{R}^d)$ is supported in the annulus $\mathcal{C} := \{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and satisfies that (see [Bahouri et al. 2011])

$$\chi(\xi) + \sum_{j\in\mathbb{N}} \varphi(2^{-j}\xi) = 1, \text{ for all } \xi \in \mathbb{R}^d, \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \text{ for all } \xi \in \mathbb{R}^d \setminus \{0\}.$$

For every $u \in S'(\mathbb{R}^d)$, we define the nonhomogeneous Littlewood–Paley operators as follows:

$$\Delta_{-1} f = \chi(D)u; \quad \Delta_j f = \varphi(2^{-j}D)f, \quad S_j f = \sum_{-1\le k\le j-1} \Delta_k u \quad \text{for all } j \in \mathbb{N}.$$

And the homogeneous Littlewood–Paley operators can be defined as follows:

$$\dot{\Delta}_j f := \varphi(2^{-j}D)f; \quad \dot{S}_j f := \sum_{k\in\mathbb{Z}, k\le j-1} \dot{\Delta}_k f \quad \text{for all } j \in \mathbb{Z}.$$

Also, we denote

$$\widetilde{\Delta}_j f := \Delta_{j-1} f + \Delta_j f + \Delta_{j+1} f.$$

It is clear to see that, for any f and g belonging to $S'(\mathbb{R}^d)$, from the property of the frequency supports, we have

$$\Delta_j \Delta_l f \equiv 0, \quad |j-l| \ge 2 \quad \text{and} \quad \Delta_k(S_{l-1} g \Delta_l g) \equiv 0 \quad |k-l| \ge 5.$$

Now we introduce the definition of Besov spaces. Let $s \in \mathbb{R}$, $(p, r) \in [1, +\infty]^2$. Then the inhomogeneous Besov space $B^s_{p,r}$ is defined as

$$B^s_{p,r} := \{f \in \mathcal{S}'(\mathbb{R}^d);\ \|f\|_{B^s_{p,r}} := \|\{2^{js}\|\Delta_j f\|_{L^p}\}_{j\ge -1}\|_{\ell^r} < \infty\},$$

and the homogeneous space $\dot{B}^s_{p,r}$ is given by

$$\dot{B}^s_{p,r} := \{f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d);\ \|f\|_{\dot{B}^s_{p,r}} := \|\{2^{js}\|\dot{\Delta}_j f\|_{L^p}\}_{j\in\mathbb{Z}}\|_{\ell^r(\mathbb{Z})} < \infty\}.$$

For any noninteger $s > 0$, the Hölder space $C^s = C^{[s],s-[s]}$ is equivalent to $B^s_{\infty,\infty}$ with $\|f\|_{C^s} \approx \|f\|_{B^s_{\infty,\infty}}$.

Bernstein's inequality plays an important role in the analysis involving Besov spaces.

Lemma 2.1 (see [Bahouri et al. 2011]). *Let $f \in L^a$, $1 \le a \le b \le \infty$. Then for every $(k, j) \in \mathbb{N}^2$, there exists a constant $C > 0$ independent of j such that*

$$\sup_{|\alpha|=k} \|\partial^\alpha S_j f\|_{L^b} \le C 2^{j(k+d/a-d/b)} \|S_j f\|_{L^a},$$

and

$$C^{-1} 2^{jk} \|\Delta_j f\|_{L^a} \le \sup_{|\alpha|=k} \|\partial^\alpha \Delta_j f\|_{L^a} \le C 2^{jk} \|\Delta_j f\|_{L^a}.$$

The following lemma concerning the Lévy operator $\mathcal{L}$ is useful in the proof of the existence parts.

Lemma 2.2. *Let the operator $\mathcal{L}$ be defined by (1-2) with the symmetric kernel $K(y) = K(-y)$ under the conditions (1-3)–(1-4).*

(1) *Assume that* $g \in C^{1,\gamma}(\mathbb{R}^d)$, $\gamma > 0$. *Then we have* $\mathcal{L}g \in L^\infty(\mathbb{R}^d)$ *with* $\|\mathcal{L}g\|_{L^\infty(\mathbb{R}^d)} \le C\|g\|_{C^{1,\gamma}(\mathbb{R}^d)}$.

(2) *Assume that* $h \in \mathcal{S}(\mathbb{R}^d)$ *and* $h_j(x) = 2^{jd}h(2^j x)$, $j \in \mathbb{N}$. *Then we have*

$$\|\mathcal{L}h_j\|_{L^1(\mathbb{R}^d)} \le C2^{j\alpha}. \tag{2-1}$$

(3) *Assume that* $g \in L^p(\mathbb{R}^d)$, $p \in [1,\infty]$. *Then* $\|\mathcal{L}\Delta_j g\|_{L^p} \le C2^{j\alpha}\|\widetilde{\Delta}_j g\|_{L^p}$ *for every* $j \in \mathbb{N}$ *and* $\|\mathcal{L}\Delta_{-1}g\|_{L^p} \le C\|g\|_{L^p}$.

Proof of Lemma 2.2. (1) If $\alpha \in (0,1)$, it follows from equation (1-2) that

$$\begin{aligned}\mathcal{L}g(x) &= \mathrm{p.\,v.}\int_{\mathbb{R}^d}(g(x)-g(x+y))K(y)\,\mathrm{d}y\\ &= \mathrm{p.\,v.}\int_{|y|\le 1}(g(x)-g(x+y))K(y)\,\mathrm{d}y\\ &\qquad + \int_{|y|\ge 1}(g(x)-g(x+y))K(y)\,\mathrm{d}y.\end{aligned}$$

By virtue of inequality (1-3), one has

$$\left|\mathrm{p.v.}\int_{|y|\ge 1}(g(x)-g(x+y))K(y)\mathrm{d}y\right| \le C\|g\|_{L^\infty}\int_{|y|\ge 1}|K(y)|\mathrm{d}y \le C\|g\|_{L^\infty}. \tag{2-2}$$

Thanks to the upper bound of (1-4), we have

$$\begin{aligned}\left|\mathrm{p.v.}\int_{|y|\le 1}(g(x)-g(x+y))K(y)\mathrm{d}y\right| &= \left|\int_{|y|\le 1}\int_0^1 y\cdot(\nabla g)(x+sy)K(y)\mathrm{d}s\,\mathrm{d}y\right|\\ &\le C\|\nabla g\|_{L^\infty}\int_{|y|\le 1}|y||K(y)|\mathrm{d}y\\ &\le C\|\nabla g\|_{L^\infty}\int_{|y|\le 1}|y|\frac{c_2}{|y|^{d+\alpha}}\mathrm{d}y\\ &\le C\|\nabla g\|_{L^\infty}.\end{aligned}$$

Hence for the case $\alpha \in (0,1)$, we get

$$\|\mathcal{L}g\|_{L^\infty(\mathbb{R}^d)} \le C(\|g\|_{L^\infty(\mathbb{R}^d)} + \|\nabla g\|_{L^\infty(\mathbb{R}^d)}).$$

If $\alpha = 1$, similarly as above, and by adopting the following equivalent formula of $\mathcal{L}g$ (from the symmetric condition $K(y) = K(-y)$)

$$\mathcal{L}g(x) = \int_{\mathbb{R}^d}(g(x) + y\cdot\nabla g(x)1_{\{|y|\le 1\}} - g(x+y))K(y)\,\mathrm{d}y, \tag{2-3}$$

we can prove that

$$\|\mathcal{L}g\|_{L^\infty(\mathbb{R}^d)} \le C\|g\|_{C^{1,\gamma}(\mathbb{R}^d)}.$$

Both in the cases $\alpha \in (0,1)$ and $\alpha = 1$, we conclude $\|\mathcal{L}g\|_{L^\infty(\mathbb{R}^d)} \le C\|g\|_{C^{1,\gamma}(\mathbb{R}^d)}$.

(2) If $\alpha \in (0,1)$, from (1-2), (1-4) and the Fubini theorem, we see that

$$\begin{aligned}\|\mathcal{L}h_j\|_{L^1(\mathbb{R}^d)} &\le c_2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|h_j(x)-h_j(x+y)|}{|y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad\qquad + \int_{\mathbb{R}^d}\int_{|y|\ge 1}|h_j(x)-h_j(x+y)||K(y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\le C\int_{\mathbb{R}^d}\frac{\|h_j(x)-h_j(x+y)\|_{L^1_x}}{|y|^{d+\alpha}}\,\mathrm{d}y + C\|h_j\|_{L^1_x}\int_{|y|\ge1}|K(y)|\,\mathrm{d}y \\ &\le C\|h_j\|_{\dot{B}^{\alpha}_{1,1}} + C\|h_j\|_{L^1} \le C2^{j\alpha},\end{aligned}$$

where in the last line we used the characterization of homogeneous Besov spaces (see [Bahouri et al. 2011, Theorem 2.36])

$$\|g\|_{\dot{B}^s_{p,r}} \approx \left\|\frac{\|g(x)-g(x+y)\|_{L^p}}{|y|^{\alpha}}\right\|_{L^r(\mathbb{R}^d,dy/|y|^d)} \qquad \text{for all } s\in(0,1),\ (p,r)\in[1,\infty]^2.$$

If $\alpha = 1$, we use the following equivalent formula for $\mathcal{L}g$:

$$\mathcal{L}g(x) = \int_{\mathbb{R}^d}(g(x)+y\cdot\nabla g(x)1_{\{|y|\le\epsilon\}} - g(x+y))K(y)\,\mathrm{d}y, \tag{2-4}$$

with $\epsilon > 0$. Thus by choosing $\epsilon = 2^{-j}$, we get

$$\begin{aligned}\|\mathcal{L}h_j\|_{L^1} \le c_2&\int_{\mathbb{R}^d}\int_{|y|\le 2^{-j}}\frac{|h_j(x)+y\cdot\nabla h_j(x)-h_j(x+y)|}{|y|^{d+1}}\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad + c_2\int_{\mathbb{R}^d}\int_{2^{-j}\le|y|\le1}\frac{|h_j(x)-h_j(x+y)|}{|y|^{d+1}}\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^d}\int_{|y|\ge1}|h_j(x)-h_j(x+y)||K(y)|\,\mathrm{d}y\,\mathrm{d}x \\ \le\ & C\|\nabla^2 h_j\|_{L^1}\int_{|y|\le 2^{-j}}\frac{1}{|y|^{d-1}}\,\mathrm{d}y \\ &\qquad + C\|h_j\|_{L^1}\left(\int_{|y|\ge 2^{-j}}\frac{1}{|y|^{d+1}}\,\mathrm{d}y + \int_{|y|\ge1}|K(y)|\,\mathrm{d}y\right) \\ \le\ & C2^j.\end{aligned}$$

Hence (2-1) follows for every $\alpha \in (0,1]$.

(3) Denoting $h := \mathcal{F}^{-1}(\varphi)$, $\tilde{h} := \mathcal{F}^{-1}(\chi)$, we have $\Delta_j g = h_j * g = (2^{jd}h(2^j\cdot)) * g$ ($j \in \mathbb{N}$) and $\Delta_{-1}g = \tilde{h} * g$. By virtue of the facts that $\Delta_j\widetilde{\Delta}_j = \Delta_j$ ($j\in\mathbb{N}$) and $\mathcal{L}(f*g) = (\mathcal{L}f)*g$, and thanks to the statement (2), we infer that

$$\|\mathcal{L}\Delta_j g\|_{L^p} = \|\mathcal{L}\Delta_j\widetilde{\Delta}_j g\|_{L^p} = \|(\mathcal{L}h_j)*(\widetilde{\Delta}_j g)\|_{L^p} \le C2^{j\alpha}\|\widetilde{\Delta}_j g\|_{L^p}$$

for all $j \in \mathbb{N}$, and $\|\mathcal{L}\Delta_{-1}g\|_{L^p} = \|(\mathcal{L}\tilde{h})*g\|_{L^p} \le C\|g\|_{L^p}$. □

3. Proof of Theorem 1.1

3A. ***Auxiliary lemmas.*** In this section we introduce some useful auxiliary lemmas. The first lemma is about the pointwise lower bound estimate of the Fourier symbol of the operator $\mathcal{L}$.

Lemma 3.1. *Let the operator $\mathcal{L}$ be defined by (1-2) with the kernel $K(y)=K(-y)$ satisfying (1-3)–(1-4). Then the associated symbol $A(\xi)$ given by (1-6) satisfies*

$$A(\xi)\ge C^{-1}|\xi|^{\alpha-\sigma}-C, \tag{3-1}$$

where $\alpha\in]0,1]$, $\sigma\in[0,\alpha[$ and $C=C(d,\alpha,\sigma)$ is a positive constant.

Proof of Lemma 3.1. Recalling that one has (see Equation (3.219) of [Jacob 2005])

$$|\xi|^{\alpha}=c_{d,\alpha}\,\mathrm{p.\,v.}\!\int_{\mathbb{R}^d}(1-\cos(y\cdot\xi))\frac{1}{|y|^{d+\alpha}}\,\mathrm{d}y\quad\text{for all }\alpha\in]0,2[, \tag{3-2}$$

and by virtue of (1-3)–(1-4), we get

$$\begin{aligned}A(\xi)&=\mathrm{p.\,v.}\!\int_{\mathbb{R}^d}(1-\cos(y\cdot\xi))K(y)\,\mathrm{d}y\\&\ge c_2^{-1}\int_{0<|y|\le1}(1-\cos(y\cdot\xi))\frac{1}{|y|^{d+\alpha-\sigma}}\,\mathrm{d}y-\int_{|y|\ge1}|K(y)|\,\mathrm{d}y\\&\ge c_2^{-1}\left(c_{d,\alpha}^{-1}|\xi|^{\alpha-\sigma}-\int_{|y|\ge1}\frac{1}{|y|^{d+\alpha-\sigma}}\,\mathrm{d}y\right)-c_1\\&\ge c_2^{-1}c_{d,\alpha}^{-1}|\xi|^{\alpha-\sigma}-C_{d,\alpha,\sigma}-c_1,\end{aligned}$$

which corresponds to (3-1). □

Next we derive the following lower bound estimates of some quantities involving the Lévy operator $\mathcal{L}$ given by (1-2).

Lemma 3.2. *Let $p\ge2$ and the kernel function $K(y)=K(-y)$ satisfy the conditions (1-3)–(1-4), then for every $\theta\in\mathcal{S}(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d}|\theta(x)|^{p-2}\theta(x)\mathcal{L}\theta(x)\mathrm{d}x\ge C\int_{\mathbb{R}^d}(|D|^{\frac{\alpha-\sigma}{2}}|\theta(x)|^{\frac{p}{2}})^2\mathrm{d}x-\widetilde{C}\int_{\mathbb{R}^d}|\theta(x)|^p\,\mathrm{d}x, \tag{3-3}$$

and for every $j\in\mathbb{N}$,

$$\int_{\mathbb{R}^d}\mathcal{L}(\Delta_j\theta)(|\Delta_j\theta|^{p-2}\Delta_j\theta)\,\mathrm{d}x\ge c2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^p}^p-\widetilde{C}\|\Delta_j\theta\|_{L^p}^p, \tag{3-4}$$

where the constants $c, C>0$, $\widetilde{C}\ge0$ depend only on the coefficients p,α,σ,d.

Proof of Lemma 3.2. First we claim that the following estimate holds true:

$$|\theta(x)|^{p/2-2}\theta(x)\mathcal{L}\theta(x) \geq 2/p(\mathcal{L}|\theta|^{p/2})(x) - 2\int_{|x-y|\geq 1}(|\theta(x)|^{p/2}+|\theta(y)|^{p/2})|K(x-y)|\,\mathrm{d}y. \tag{3-5}$$

Indeed, according to (1-2) and the following estimate deduced from Young's inequality

$$|\theta(x)|^{p/2-2}\theta(x)\theta(y) \leq |\theta(x)|^{p/2-1}|\theta(y)| \leq \frac{p-2}{p}|\theta(x)|^{p/2}+\frac{2}{p}|\theta(y)|^{p/2}, \tag{3-6}$$

we have

$$\begin{aligned} &|\theta(x)|^{p/2-2}\theta(x)\mathcal{L}\theta(x) \\ &\quad= \mathrm{p.\,v.}\int_{\mathbb{R}^d}(|\theta(x)|^{p/2}-|\theta(x)|^{p/2-2}\theta(x)\theta(y))K(x-y)\,\mathrm{d}y \\ &\quad= \mathrm{p.\,v.}\int_{|x-y|\leq 1}(|\theta(x)|^{p/2}-|\theta(x)|^{p/2-2}\theta(x)\theta(y))K(x-y)\,\mathrm{d}y \\ &\qquad+\int_{|x-y|\geq 1}(|\theta(x)|^{p/2}-|\theta(x)|^{p/2-2}\theta(x)\theta(y))K(x-y)\,\mathrm{d}y \\ &\quad\geq \mathrm{p.\,v.}\int_{|x-y|\leq 1}(|\theta(x)|^{p/2}-|\theta(x)|^{p/2-2}\theta(x)\theta(y))K(x-y)\,\mathrm{d}y \\ &\qquad-\frac{2p-2}{p}\int_{|x-y|\geq 1}(|\theta(x)|^{p/2}+|\theta(y)|^{p/2})|K(x-y)|\,\mathrm{d}y. \end{aligned} \tag{3-7}$$

Due to the positivity property of $K(y)$ on $0<|y|\leq 1$ and the inequality (3-6) again, we see that

$$\begin{aligned} &\mathrm{p.v.}\int_{|x-y|\leq 1}(|\theta(x)|^{p/2}-|\theta(x)|^{p/2-2}\theta(x)\theta(y))K(x-y)\mathrm{d}y \\ &\geq \mathrm{p.v.}\int_{|x-y|\leq 1}\Big(|\theta(x)|^{p/2}-\Big(\frac{p-2}{p}|\theta(x)|^{p/2}+2/p|\theta(y)|^{p/2}\Big)\Big)K(x-y)\mathrm{d}y \\ &= \frac{2}{p}\mathrm{p.v.}\int_{|x-y|\leq 1}(|\theta(x)|^{p/2}-|\theta(y)|^{p/2})K(x-y)\mathrm{d}y \\ &= \frac{2}{p}(\mathcal{L}|\theta|^{p/2})(x)-\frac{2}{p}\int_{|x-y|\geq 1}(|\theta(x)|^{p/2}-|\theta(y)|^{p/2})K(x-y)\mathrm{d}y \\ &\geq \frac{2}{p}(\mathcal{L}|\theta|^{p/2})(x)-\frac{2}{p}\int_{|x-y|\geq 1}(|\theta(x)|^{p/2}+|\theta(y)|^{p/2})|K(x-y)|\mathrm{d}y. \end{aligned} \tag{3-8}$$

Gathering the above estimates leads to (3-5).

As a consequence of (3-5), we get

$$
\begin{aligned}
(3\text{-}9)\quad & \int_{\mathbb{R}^d} |\theta(x)|^{p-2}\theta(x)\mathcal{L}\theta(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} |\theta(x)|^{p/2}|\theta(x)|^{p/2-2}\theta(x)\mathcal{L}\theta(x)\,\mathrm{d}x \\
&\geq \frac{2}{p}\int_{\mathbb{R}^d} |\theta(x)|^{p/2}(\mathcal{L}|\theta|^{p/2})(x)\,\mathrm{d}x \\
&\quad - 2\int_{\mathbb{R}^d} |\theta(x)|^{p/2}\int_{|x-y|\geq 1} (|\theta(x)|^{p/2}+|\theta(y)|^{p/2})|K(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\
&:= N_1 + N_2.
\end{aligned}
$$

In view of the Plancherel theorem and the estimate (3-1) concerning the symbol of $\mathcal{L}$, this leads to

$$
\begin{aligned}
N_1 &= \frac{2}{p}\int_{\mathbb{R}^d} \widehat{|\theta|^{p/2}}(\xi)A(\xi)\widehat{|\theta|^{p/2}}(\xi)\,\mathrm{d}\xi \\
&\geq \frac{2}{p}C_{\alpha,\sigma,d}^{-1}\int_{\mathbb{R}^d} |\xi|^{\alpha-\sigma}\widehat{|\theta|^{p/2}}(\xi)\widehat{|\theta|^{p/2}}(\xi)\,\mathrm{d}\xi - \frac{2}{p}C_{\alpha,\sigma,d}\int_{\mathbb{R}^d} \widehat{|\theta|^{p/2}}(\xi)\widehat{|\theta|^{p/2}}(\xi)\,\mathrm{d}\xi \\
&= \frac{2}{p}C_{\alpha,\sigma,d}^{-1}\int_{\mathbb{R}^d} (|\xi|^{(\alpha-\sigma)/2}\widehat{|\theta|^{p/2}}(\xi))^2\,\mathrm{d}\xi - \frac{2}{p}C_{\alpha,\sigma,d}\int_{\mathbb{R}^d} \widehat{|\theta|^{p/2}}(\xi)\widehat{|\theta|^{p/2}}(\xi)\,\mathrm{d}\xi \\
&= \frac{2}{p}C_{\alpha,\sigma,d}^{-1}\int_{\mathbb{R}^d} (|D|^{(\alpha-\sigma)/2}|\theta(x)|^{p/2})^2\,\mathrm{d}x - \frac{2}{p}C_{\alpha,\sigma,d}\int_{\mathbb{R}^d} |\theta(x)|^p\,\mathrm{d}x.
\end{aligned}
$$

The Young inequality and the condition (1-3) ensure that

$$
\begin{aligned}
-\frac{N_2}{2} &\leq \int_{\mathbb{R}^d} |\theta(x)|^{p/2}\int_{|x-y|\geq 1} |\theta(x)|^{p/2}|K(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^d} |\theta(x)|^{p/2}\int_{|x-y|\geq 1} |\theta(y)|^{p/2}|K(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^d} |\theta(x)|^p\int_{|x-y|\geq 1} |K(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad + \|\theta\|_{L^p}^{p/2}\left\|\int_{\mathbb{R}^d} |\theta(y)|^{p/2}|K(x-y)|1_{\{|x-y|\geq 1\}}\,\mathrm{d}y\right\|_{L^2_x} \\
&\leq \|\theta\|_{L^p}^p\int_{|x|\geq 1} |K(x)|\,\mathrm{d}x + \|\theta\|_{L^p}^{p/2}\||\theta(x)|^{p/2}\|_{L^2_x}\int_{|x|\geq 1} |K(x)|\,\mathrm{d}x \\
&\leq 2c_1\|\theta\|_{L^p}^p.
\end{aligned}
$$

Inserting the estimates of N_1 and N_2 into (3-9) yields the desired estimate (3-3). Recalling the following inequality (see [Chen et al. 2007, Proposition 3.1]),

$$
\||D|^\beta(|\Delta_j\theta|^{p/2})\|_{L^2}^2 \geq \tilde{c}2^{j\beta}\|\Delta_j\theta\|_{L^p}^p \quad \text{for every } \beta\in(0,2],\ p\in[2,\infty),\ j\in\mathbb{N},
$$

with a constant $\tilde{c}>0$ independent of j, then the estimate (3-4) follows by combining the above lower bound estimate with (3-3). We thus conclude Lemma 3.2. ☐

Now we can show the key a priori L^p-estimate of the drift-diffusion equations (1-1)–(1-2).

Lemma 3.3. *Let u be a smooth **divergence-free** vector field and f be a smooth forcing term. Assume that θ is a smooth solution for the drift-diffusion equations (1-1)–(1-2) under the assumptions (1-3)–(1-4) with $\theta_0 \in L^p(\mathbb{R}^d)$. Then for any $T > 0$,*

$$\max_{0\le t\le T} \|\theta(t)\|_{L^p}^p + \int_0^T \|\theta(\tau)\|^p_{\dot{B}^{(\alpha-\sigma)/p}_{p,p}}\, d\tau \le e^{C'T}\left(\|\theta_0\|^p_{L^p} + \int_0^T \|f(t)\|^p_{L^p}\,\mathrm{d}t\right), \tag{3-10}$$

with $C' \ge 0$ depending only on p, α, σ, d.

Proof of Lemma 3.3. Multiplying both sides of (1-1) by $|\theta|^{p-2}\theta(x)$ and integrating over the spatial variable, we use the divergence-free condition of u and Hölder's inequality to get

$$\frac{1}{p}\frac{d}{dt}\|\theta\|^p_{L^p} + \int_{\mathbb{R}^d} \mathcal{L}\theta(x)(|\theta|^{p-2}\theta)(x)\,\mathrm{d}x \le \|f\|_{L^p}\|\theta\|^{p-1}_{L^p}.$$

Thanks to (3-3), we have

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{L}\theta(|\theta|^{p-2}\theta)\,\mathrm{d}x &\ge C\int_{\mathbb{R}^d} (|D|^{(\alpha-\sigma)/2}|\theta(x)|^{p/2})^2\,\mathrm{d}x - \widetilde{C}\int_{\mathbb{R}^d}|\theta(x)|^p\,\mathrm{d}x \\ &\ge C\|\theta\|^p_{\dot{B}^{(\alpha-\sigma)/p}_{p,p}} - \widetilde{C}\|\theta\|^p_{L^p},\end{aligned}$$

where in the last line we have used the following inequality (see [Chamorro and Lemarié-Rieusset 2012, Theorem 2] or [Chamorro and Menozzi 2016, Theorem 5])

$$\int_{\mathbb{R}^d} (|D|^{\gamma}|\theta(x)|^{p/2})^2\,\mathrm{d}x \ge c\|\theta\|^p_{\dot{B}^{\gamma/p}_{p,p}} \quad \text{for all } \gamma \in (0,1).$$

We thus obtain

$$\frac{1}{p}\frac{d}{dt}\|\theta\|^p_{L^p} + C\|\theta\|^p_{\dot{B}^{(\alpha-\sigma)/p}_{p,p}} - \widetilde{C}\|\theta\|^p_{L^p} \le \|f\|_{L^p}\|\theta\|^{p-1}_{L^p},$$

which directly implies

$$\frac{d}{dt}\|\theta(t)\|^p_{L^p} + \|\theta\|^p_{\dot{B}^{(\alpha-\sigma)/p}_{p,p}} \le C\|\theta(t)\|^p_{L^p} + C\|f(t)\|^p_{L^p}.$$

Grönwall's inequality guarantees the desired inequality (3-10). □

The final lemma is concerned with a (time function) weighted estimate, which plays a key role in proving our main results.

Lemma 3.4. *Let $\lambda > 0$ and $0 < \mu < 1$. Then for any $t > 0$, there exists a constant C_μ depending only on μ such that*

$$\int_0^t e^{-(t-\tau)\lambda}\tau^{-\mu}\,\mathrm{d}\tau \le C_\mu \lambda^{-1} t^{-\mu}. \tag{3-11}$$

In particular, for any $t > t_0 \geq 0$, *we have*

$$\int_{t_0}^{t} e^{-(t-\tau)2^{(\alpha-\sigma)j}} (\tau - t_0)^{-\mu}\, \mathrm{d}\tau = \int_0^{t-t_0} e^{-(t-t_0-\tau)2^{(\alpha-\sigma)j}} \tau^{-\mu}\, \mathrm{d}\tau \leq C_\mu 2^{-(\alpha-\sigma)j} (t-t_0)^{-\mu}. \tag{3-12}$$

Proof of Lemma 3.4. First, by changing of the variable $(t-\tau)\lambda = s$, one deduces

$$\begin{aligned}\int_0^t e^{-(t-\tau)\lambda}\tau^{-\mu}\mathrm{d}\tau &= \lambda^{-1}\int_0^{t\lambda} e^{-s}\Big(t-\frac{s}{\lambda}\Big)^{-\mu}\mathrm{d}s \\ &= \lambda^{-1}\bigg(\int_0^{t\lambda/2} e^{-s}\Big(t-\frac{s}{\lambda}\Big)^{-\mu}\mathrm{d}s + \int_{t\lambda/2}^{t\lambda} e^{-s}\Big(t-\frac{s}{\lambda}\Big)^{-\mu}\mathrm{d}s\bigg) \\ &:= \lambda^{-1}(B_1+B_2).\end{aligned}$$

For the first term B_1, noting that $t - s/\lambda \geq \frac{1}{2}t$ for all $0 \leq s \leq t\lambda/2$, we directly get

$$B_1 \leq 2^\mu t^{-\mu} \int_0^{t\lambda/2} e^{-s}\, \mathrm{d}s \leq 2^\mu t^{-\mu} \int_0^{\infty} e^{-s}\, \mathrm{d}s \leq 2^\mu t^{-\mu}.$$

For the second term B_2, by changing of the variable $t - s/\lambda = s'$ and using the fact $t\lambda e^{-t\lambda/2} \leq C_0$, we deduce that

$$\begin{aligned}B_2 &\leq e^{-t\lambda/2}\int_{t\lambda/2}^{t\lambda}\Big(t-\frac{s}{\lambda}\Big)^{-\mu}\mathrm{d}s \\ &= t^{1-\mu}\lambda e^{-t\lambda/2}\int_0^{1/2}(s')^{-\mu}\,\mathrm{d}s' = \frac{2^{\mu-1}}{1-\mu}t^{-\mu}(t\lambda e^{-t\lambda/2}) \leq \frac{C_0 2^{\mu-1}}{1-\mu}t^{-\mu}.\end{aligned}$$

Combining the above two estimates, we obtain

$$\int_0^t e^{-(t-\tau)\lambda}\tau^{-\mu}\,\mathrm{d}\tau \leq \Big(2^\mu + \frac{C_0 2^{\mu-1}}{1-\mu}\Big)\lambda^{-1}t^{-\mu} = C_\mu \lambda^{-1} t^{-\mu},$$

which corresponds to (3-11). □

3B. ***A priori estimates.*** In this subsection, we assume θ is a smooth solution for the drift-diffusion equations (1-1)–(1-2) with smooth u and f. We shall show the estimate (1-13) and the proof consists of four steps.

Step 1: the estimation of $\|\theta\|_{L^\infty([t_0,T];B^{s_0}_{p,\infty})}$ for any $s_0 \in (0, \alpha-\sigma)$ and $t_0 \in (0, T)$.

By applying the dyadic operator Δ_j ($j \in \mathbb{N},\ j \geq 4$) to the equation of θ in (1-1), we get

$$\partial_t \Delta_j\theta + u\cdot\nabla\Delta_j\theta + \mathcal{L}\Delta_j\theta = -[\Delta_j, u\cdot\nabla]\theta + \Delta_j f, \tag{3-13}$$

where $[A, B] = AB - BA$ denotes the commutator of two operators A and B. Bony's paraproduct decomposition leads to

$$\begin{aligned}(3\text{-}14)\quad -[\Delta_j, u\cdot\nabla]\theta &= -\sum_{|k-j|\le 4}[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta \\ &\quad - \sum_{|k-j|\le 4}(\Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta) - \Delta_k u\cdot\nabla\Delta_j S_{k-1}\theta) \\ &\quad - \sum_{k\ge j-2}(\Delta_j(\Delta_k u\cdot\nabla\widetilde{\Delta}_k\theta) - \Delta_k u\cdot\nabla\Delta_j\widetilde{\Delta}_k\theta) \\ &:= I_1 + I_2 + I_3.\end{aligned}$$

Multiplying both sides of the equation (4-8) with $|\Delta_j\theta|^{p-2}\Delta_j\theta$ and integrating on the spatial variable over $\mathbb{R}^d$, we use the divergence-free property of u and the Hölder inequality to get

$$\begin{aligned}(3\text{-}15)\quad \frac{1}{p}\frac{d}{dt}\|\Delta_j\theta\|_{L^p}^p + \int_{\mathbb{R}^d}\mathcal{L}(\Delta_j\theta)(|\Delta_j\theta|^{p-2}\Delta_j\theta)\,\mathrm{d}x \\ \le (\|\Delta_j f\|_{L^p} + \|I_1\|_{L^p} + \|I_2\|_{L^p} + \|I_3\|_{L^p})\|\Delta_j\theta\|_{L^p}^{p-1}.\end{aligned}$$

According to (3-4) in Lemma 3.2, we see that

$$(3\text{-}16)\qquad \int_{\mathbb{R}^d}\mathcal{L}(\Delta_j\theta)(|\Delta_j\theta|^{p-2}\Delta_j\theta)\,\mathrm{d}x \ge c2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^p}^p - C_1\|\Delta_j\theta\|_{L^p}^p,$$

where c and C_1 are constants depending on p, α, σ, d. Inserting (3-16) into (3-15) and dividing $\|\Delta_j\theta\|_{L^p}^{p-1}$ lead to

$$\begin{aligned}(3\text{-}17)\quad \frac{d}{dt}\|\Delta_j\theta\|_{L^p} + c2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^p} \\ \le C_1\|\Delta_j\theta\|_{L^p} + \|\Delta_j f\|_{L^p} + \|I_1\|_{L^p} + \|I_2\|_{L^p} + \|I_3\|_{L^p}.\end{aligned}$$

For $\|I_1\|_{L^p}$, noting that I_1 can be expressed as

$$(3\text{-}18)\qquad I_1 = -\sum_{|k-j|\le 4}\int_{\mathbb{R}^d} h_j(x-y)(S_{k-1}u(y) - S_{k-1}u(x))\cdot\nabla\Delta_k\theta(y)\,\mathrm{d}y,$$

where $h_j(x) = 2^{jd}(\mathcal{F}^{-1}\varphi)(2^j x)$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ is the test function introduced in Section 2, thus from Hölder's inequality, Bernstein's inequality and Young's inequality, one has

$$\begin{aligned}
\|I_1\|_{L^p} &\le \sum_{|k-j|\le 4} \left\| \int_{\mathbb{R}^d} h_j(x-y)(S_{k-1}u(y)-S_{k-1}u(x))\cdot\nabla\Delta_k\theta(y)\,\mathrm{d}y \right\|_{L^p_x} \\
&\le C \sum_{|k-j|\le 4} \left\| \int_{\mathbb{R}^d} |h_j(x-y)| \|u\|_{\dot C^\delta} |x-y|^\delta |\nabla\Delta_k\theta(y)|\,\mathrm{d}y \right\|_{L^p_x} \\
&\le C\|u\|_{\dot C^\delta} \int_{\mathbb{R}^d} |h_j(x)||x|^\delta\,\mathrm{d}x \sum_{|k-j|\le 4} \|\nabla\Delta_k\theta\|_{L^p} \\
&\le C2^{-j\delta}\|u\|_{\dot C^\delta} \sum_{|k-j|\le 4} 2^k\|\Delta_k\theta\|_{L^p}.
\end{aligned} \tag{3-19}$$

By virtue of Hölder's inequality and Bernstein's inequality again, we see that

$$\begin{aligned}
\|I_2\|_{L^p} &\le \sum_{|k-j|\le 4} \|\Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta)\|_{L^p} + \sum_{|k-j|\le 4} \|\Delta_k u\cdot\nabla S_{k-1}\Delta_j\theta\|_{L^p} \\
&\le C \sum_{|k-j|\le 4} \|\Delta_k u\|_{L^\infty}\|\nabla S_{k-1}\theta\|_{L^p} + C \sum_{|k-j|\le 4} \|\Delta_k u\|_{L^\infty}\|\nabla\Delta_j\theta\|_{L^p} \\
&\le C2^{-j\delta} \sum_{|k-j|\le 4} 2^{k\delta}\|\Delta_k u\|_{L^\infty} \Bigl(\sum_{l\le j} 2^l\|\Delta_l\theta\|_{L^p}\Bigr) \\
&\le C2^{-j\delta}\|u\|_{\dot C^\delta} \Bigl(\sum_{k\le j} 2^k\|\Delta_k\theta\|_{L^p}\Bigr),
\end{aligned} \tag{3-20}$$

and by using the divergence-free property of u, we get

$$\begin{aligned}
\|I_3\|_{L^p} &\le \sum_{k\ge j-2} \|\nabla\cdot\Delta_j(\Delta_k u\widetilde{\Delta}_k\theta)\|_{L^p} + \sum_{k\ge j-2} \|\Delta_k u\cdot\nabla\widetilde{\Delta}_k\Delta_j\theta\|_{L^p} \\
&\le C \sum_{k\ge j-2} 2^j\|\Delta_k u\|_{L^\infty}\|\widetilde{\Delta}_k\theta\|_{L^p} \\
&\le C2^j \sum_{k\ge j-2} 2^{k\delta}\|\Delta_k u\|_{L^\infty} 2^{-k\delta}\|\widetilde{\Delta}_k\theta\|_{L^p} \\
&\le C\|u\|_{\dot C^\delta} 2^j \Bigl(\sum_{k\ge j-2} 2^{-k\delta}\|\Delta_k\theta\|_{L^p}\Bigr).
\end{aligned} \tag{3-21}$$

Gathering the above estimates leads to

$$\begin{aligned}
\frac{d}{dt}\|\Delta_j\theta\|_{L^p} + c2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^p} \le C_1\|\Delta_j\theta\|_{L^p} &+ \|\Delta_j f\|_{L^p} \\
&+ C\|u\|_{\dot C^\delta} 2^{-j\delta} \sum_{k\le j+4} 2^k\|\Delta_k\theta\|_{L^p} \\
&+ C\|u\|_{\dot C^\delta} 2^j \sum_{k\ge j-3} 2^{-k\delta}\|\Delta_k\theta\|_{L^p}.
\end{aligned}$$

Let $j_0 \in \mathbb{N}$ be a number chosen later (see (3-32)) which satisfies $(c/2)2^{j_0(\alpha-\sigma)} \geq C_1$, or more precisely,

$$j_0 \geq \Big[\frac{1}{\alpha-\sigma}\log_2\Big(\frac{2C_1}{c}\Big)\Big]+1. \tag{3-22}$$

We infer that for all $j \geq j_0$,

$$\begin{aligned}\frac{d}{dt}\|\Delta_j\theta\|_{L^p}+\frac{c}{2}2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^p} &\leq \|\Delta_j f\|_{L^p}+C\|u\|_{\dot{C}^\delta}2^{-j\delta}\sum_{k\leq j+4}2^k\|\Delta_k\theta\|_{L^p}\\ &\quad+C\|u\|_{\dot{C}^\delta}2^j\sum_{k\geq j-3}2^{-k\delta}\|\Delta_k\theta\|_{L^p}\\ &:=\|\Delta_j f\|_{L^p}+H_j^1+H_j^2.\end{aligned} \tag{3-23}$$

Thus Grönwall's inequality yields that for every $j \geq j_0 \geq 4$ and $t \geq 0$,

$$\begin{aligned}\|\Delta_j\theta(t)\|_{L^p} &\leq e^{-(c/2)t2^{j(\alpha-\sigma)}}\|\Delta_j\theta_0\|_{L^p}\\ &\quad+\int_0^t e^{-(c/2)(t-\tau)2^{j(\alpha-\sigma)}}(\|\Delta_j f\|_{L^p}(\tau)+H_j^1(\tau)+H_j^2(\tau))\,\mathrm{d}\tau.\end{aligned} \tag{3-24}$$

According to Lemma 3.3, we also have the L^p-estimate for equation (1-1):

$$\|\theta(t)\|_{L^p} \leq e^{Ct}\Big(\|\theta_0\|_{L^p}+\int_0^t\|f(\tau)\|_{L^p}\,\mathrm{d}t\Big). \tag{3-25}$$

Observing that for all $t>0$, $j\in\mathbb{N}$ and $s\in(0,\alpha-\sigma)$,

$$\begin{aligned}2^{js}e^{-\frac{c}{2}t2^{j(\alpha-\sigma)}}\|\Delta_j\theta_0\|_{L^p} &\leq t^{-\frac{s}{\alpha-\sigma}}\big((t2^{j(\alpha-\sigma)})^{\frac{s}{\alpha-\sigma}}e^{-\frac{c}{2}t2^{j(\alpha-\sigma)}}\big)\|\Delta_j\theta_0\|_{L^p}\\ &\leq C_{\alpha,\sigma,s}t^{-\frac{s}{\alpha-\sigma}}\|\theta_0\|_{L^p},\end{aligned} \tag{3-26}$$

thus collecting (3-24), (3-25) and (3-26) leads to

$$\begin{aligned}\|\theta(t)\|_{B^s_{p,\infty}} &\leq \sup_{j\leq j_0}2^{js}\|\Delta_j\theta(t)\|_{L^p}+\sup_{j\geq j_0}2^{js}\|\Delta_j\theta(t)\|_{L^p}\\ &\leq C2^{j_0 s}e^{Ct}(\|\theta_0\|_{L^p}+\|f\|_{L^1_tL^p})+C_{\alpha,\sigma,s}t^{-\frac{s}{\alpha-\sigma}}\|\theta_0\|_{L^p}\\ &\quad+\sup_{j\geq j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}(\|\Delta_j f\|_{L^p}(\tau)+H_j^1(\tau)+H_j^2(\tau))\,\mathrm{d}\tau.\end{aligned} \tag{3-27}$$

For the term containing $\|\Delta_j f\|_{L^p}$, we infer that for every $s \in (0, \alpha-\sigma+\delta)$,

$$
\begin{aligned}
(3\text{-}28)\quad & \sup_{j\ge j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}\|\Delta_j f\|_{L^p}(\tau)\,\mathrm{d}\tau \\
&\le C\sup_{j\ge j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s-\delta)}\|f(\tau)\|_{\dot B^{\delta}_{p,\infty}}\,\mathrm{d}\tau \\
&\le C\|f\|_{L^\infty_t\dot B^{\delta}_{p,\infty}}\sup_{j\ge j_0}2^{j(s-\delta)}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\,\mathrm{d}\tau \\
&\le C\|f\|_{L^\infty_t\dot B^{\delta}_{p,\infty}}\sup_{j\ge j_0}2^{j(s-\alpha+\sigma-\delta)} \\
&\le C\|f\|_{L^\infty_t\dot B^{\delta}_{p,\infty}}.
\end{aligned}
$$

For the term including H_j^1 in (3-27), thanks to (3-12) in Lemma 3.4, we deduce that for every $s \in (0, \alpha-\sigma)$ and $\delta \in (1-\alpha+\sigma, 1)$,

$$
\begin{aligned}
(3\text{-}29)\quad & \sup_{j\ge j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}H_j^1(\tau)\mathrm{d}\tau \\
&= C\sup_{j\ge j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|u(\tau)\|_{\dot C^{\delta}}2^{j(s-\delta)}\Big(\sum_{k\le j+4}2^k\|\Delta_k\theta(\tau)\|_{L^p}\Big)\mathrm{d}\tau \\
&\le C\|u\|_{L^\infty_t\dot C^{\delta}}\sup_{j\ge j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s-\delta)}\Big(\sum_{k\le j+4}2^{k(1-s)}\|\theta(\tau)\|_{B^s_{p,\infty}}\Big)\mathrm{d}\tau \\
&\le C\|u\|_{L^\infty_t\dot C^{\delta}}\Big(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{p,\infty}}\Big) \\
&\qquad\qquad\times\sup_{j\ge j_0}2^{j(1-\delta)}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\tau^{-\frac{s}{\alpha-\sigma}}\,\mathrm{d}\tau \\
&\le C\|u\|_{L^\infty_t\dot C^{\delta}}\Big(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{p,\infty}}\Big)t^{-\frac{s}{\alpha-\sigma}}\sup_{j\ge j_0}2^{j(1-\delta-\alpha+\sigma)} \\
&\le Ct^{-\frac{s}{\alpha-\sigma}}2^{-j_0(\delta-(1-\alpha+\sigma))}\|u\|_{L^\infty_t\dot C^{\delta}}\Big(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{p,\infty}}\Big).
\end{aligned}
$$

For the term including H_j^2 in (3-27), by using (3-12) again, we similarly get that for all $s \in (0, \alpha-\sigma)$ and $\delta \in (1-\alpha+\sigma, 1)$,

$$\begin{aligned}\text{(3-30)}\quad &\sup_{j\ge j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}H_j^2(\tau)\mathrm{d}\tau\\ &= C\sup_{j\ge j_0}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|u(\tau)\|_{\dot C^\delta}2^{j(s+1)}\Big(\sum_{k\ge j-3}2^{-k\delta}\|\Delta_k\theta(\tau)\|_{L^p}\Big)\mathrm{d}\tau\\ &\le C\|u\|_{L_t^\infty\dot C^\delta}\sup_{j\ge j_0}2^{j(s+1)}\Big(\sum_{k\ge j-3}2^{-k(\delta+s)}\Big)\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|\theta(\tau)\|_{B^s_{p,\infty}}\mathrm{d}\tau\\ &\le Ct^{-\frac{s}{\alpha-\sigma}}2^{-j_0(\delta-(1-\alpha+\sigma))}\|u\|_{L_t^\infty\dot C^\delta}\Big(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{p,\infty}}\Big).\end{aligned}$$

Plugging the estimates (3-28), (3-29), (3-30) into (3-27) yields that for any $0<s<\alpha-\sigma$ and $0<t\le T$,

$$\begin{aligned}\text{(3-31)}\quad t^{s/(\alpha-\sigma)}\|\theta(t)\|_{B^s_{p,\infty}} &\le CT^{s/(\alpha-\sigma)}e^{CT}2^{j_0}(\|\theta_0\|_{L^p}+\|f\|_{L^1_TL^p})\\ &\quad + C_{\alpha,\sigma,s}\|\theta_0\|_{L^p}+CT^{s/(\alpha-\sigma)}\|f\|_{L^\infty_T\dot B^\delta_{p,\infty}}\\ &\quad + \frac{C\|u\|_{L_T^\infty\dot C^\delta}}{2^{j_0(\delta-(1-\alpha+\sigma))}}\Big(\sup_{t\in(0,T]}t^{s/(\alpha-\sigma)}\|\theta(t)\|_{B^s_{p,\infty}}\Big).\end{aligned}$$

Now, by choosing $j_0\in\mathbb{N}$ such that $C2^{j_0(1-\alpha+\sigma-\delta)}\|u\|_{L_T^\infty\dot C^\delta}\le\frac12$ and (3-22) holds, or more precisely,

$$\text{(3-32)}\qquad j_0=\max\Big\{\Big[\frac{\log_2(2C\|u\|_{L_T^\infty\dot C^\delta})}{\delta-(1-\alpha+\sigma)}\Big],\Big[\frac{\log_2(C_1/c)}{\alpha-\sigma}\Big],4\Big\}+1,$$

we have that for all $0<s<\alpha-\sigma$,

$$\begin{aligned}\text{(3-33)}\quad &\sup_{t\in(0,T]}(t^{s/(\alpha-\sigma)}\|\theta(t)\|_{B^s_{p,\infty}})\\ &\qquad\le C(T+1)(e^{CT}2^{j_0s}(\|\theta_0\|_{L^p}+\|f\|_{L^1_TL^p})+\|f\|_{L^\infty_T\dot B^\delta_{p,\infty}}),\end{aligned}$$

which implies that for arbitrarily small $t_0\in(0,T)$ and every $s_0\in(0,\alpha-\sigma)$,

$$\begin{aligned}\text{(3-34)}\quad &\sup_{t\in[t_0,T]}\|\theta(t)\|_{B^{s_0}_{p,\infty}}\\ &\qquad\le Ct_0^{-s_0/(\alpha-\sigma)}(T+1)(e^{CT}2^{j_0s}(\|\theta_0\|_{L^p}+\|f\|_{L^1_TL^p})+\|f\|_{L^\infty_T\dot B^\delta_{p,\infty}}),\end{aligned}$$

where j_0 is given by (3-32).

Step 2: the estimation of $\|\theta\|_{L^\infty([t_1,T];B^{s_0+s_1}_{p,\infty})}$ for every $s_0,s_1\in(0,\alpha-\sigma)$ and $t_1\in(t_0,T)$.

For every $j \geq j_0$ with $j_0 \in \mathbb{N}$ satisfying (3-22) chosen later (j_0 may be different from that number in Step 1), adapting the Grönwall inequality to (3-23) over the time interval $[t_0, t]$ (for $t > t_0 > 0$) yields

$$\|\Delta_j\theta(t)\|_{L^p} \leq e^{-\frac{c}{2}(t-t_0)2^{j(\alpha-\sigma)}}\|\Delta_j\theta(t_0)\|_{L^p} + \int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}(\|\Delta_j f\|_{L^p}(\tau) + H_j^1(\tau) + H_j^2(\tau))\,\mathrm{d}\tau. \tag{3-35}$$

Noting that for $j \in \mathbb{N},\ s_0 \in (0, \alpha-\sigma)$ and every $s \in (0, \alpha-\sigma)$,

$$\begin{aligned} e^{-\frac{c}{2}(t-t_0)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}\|\Delta_j\theta(t_0)\|_{L^p} &\leq e^{-\frac{c}{2}(t-t_0)2^{j(\alpha-\sigma)}}2^{js}\|\theta(t_0)\|_{B^{s_0}_{p,\infty}} \\ &\leq C_{\alpha,\sigma,s}(t-t_0)^{-\frac{s}{\alpha-\sigma}}\|\theta(t_0)\|_{B^{s_0}_{p,\infty}}, \end{aligned} \tag{3-36}$$

thus we get that for all $t \geq t_0 > 0$,

$$\begin{aligned} &\|\theta(t)\|_{B^{s_0+s}_{p,\infty}} \\ &\leq \sup_{j\leq j_0} 2^{j(s_0+s)}\|\Delta_j\theta(t)\|_{L^p} + \sup_{j\geq j_0} 2^{j(s_0+s)}\|\Delta_j\theta(t)\|_{L^p} \\ &\leq C2^{j_0(s_0+s)}e^{Ct}(\|\theta_0\|_{L^p} + \|f\|_{L^1_tL^p}) + C_{\alpha,\sigma,s}(t-t_0)^{-\frac{s}{\alpha-\sigma}}\|\theta(t_0)\|_{B^{s_0}_{p,\infty}} \\ &\quad + \sup_{j\geq j_0}\int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}(\|\Delta_j f\|_{L^p}(\tau) + H_j^1(\tau) + H_j^2(\tau))\,\mathrm{d}\tau. \end{aligned} \tag{3-37}$$

For the term containing $\|\Delta_j f\|_{L^p}$, similarly as obtaining (3-28), we get that for every $s \in (0, \alpha-\sigma)$ and $s_0 + s < \delta + \alpha - \sigma$,

$$\begin{aligned} &\sup_{j\geq j_0}\int_{t_0}^t e^{-(c/2)(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}\|\Delta_j f\|_{L^p}\,\mathrm{d}\tau \\ &\qquad \leq C\sup_{j\geq j_0}\int_{t_0}^t e^{-(c/2)(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s-\delta)}\|f(\tau)\|_{\dot{B}^{\delta}_{p,\infty}}\,\mathrm{d}\tau \\ &\qquad \leq C\|f\|_{L^\infty_t\dot{B}^{\delta}_{p,\infty}}\sup_{j\geq j_0}2^{j(s_0+s-\delta)}\int_{t_0}^t e^{-(c/2)(t-\tau)2^{j(\alpha-\sigma)}}\,\mathrm{d}\tau \\ &\qquad \leq C\|f\|_{L^\infty_t\dot{B}^{\delta}_{p,\infty}}\sup_{j\geq j_0}2^{j(s_0+s-\alpha+\sigma-\delta)} \\ &\qquad \leq C\|f\|_{L^\infty_t\dot{B}^{\delta}_{p,\infty}}. \end{aligned} \tag{3-38}$$

For the term including H_j^1 in (3-37), by arguing as (3-29), we deduce that for every $s\in(0,\alpha-\sigma)$ and $s_0+s\le 1$,

$$
\begin{aligned}
(3\text{-}39)\quad &\sup_{j\ge j_0}\int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}H_j^1(\tau)\mathrm{d}\tau\\
&= C\sup_{j\ge j_0}\int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|u(\tau)\|_{\dot C^\delta}2^{j(s_0+s-\delta)}\Big(\sum_{-1\le k\le j+4}2^k\|\Delta_k\theta(\tau)\|_{L^p}\Big)\mathrm{d}\tau\\
&\le C\|u\|_{L_t^\infty\dot C^\delta}\\
&\qquad \sup_{j\ge j_0}\int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s-\delta)}\Big(\sum_{-1\le k\le j+4}2^{k(1-s-s_0)}\Big)\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\mathrm{d}\tau\\
&\le C\|u\|_{L_t^\infty\dot C^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\Big)\\
&\qquad\qquad \sup_{j\ge j_0}(2^{j(1-\delta)}j)\int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}(\tau-t_0)^{-\frac{s}{\alpha-\sigma}}\mathrm{d}\tau\\
&\le C\|u\|_{L_t^\infty\dot C^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s}_{p,\infty}}\Big)\\
&\qquad\qquad (t-t_0)^{-\frac{s}{\alpha-\sigma}}\sup_{j\ge j_0}(2^{-j(\delta-(1-\alpha+\sigma))}j)\\
&\le C\|u\|_{L_t^\infty\dot C^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s}_{p,\infty}}\Big)(t-t_0)^{-\frac{s}{\alpha-\sigma}}2^{-j_0\frac{\delta-(1-\alpha+\sigma)}{2}},
\end{aligned}
$$

and for $1<s_0+s<\delta+\alpha-\sigma$,

$$
\begin{aligned}
(3\text{-}40)\quad &\sup_{j\ge j_0}\int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}H_j^1(\tau)\mathrm{d}\tau\\
&\le C\|u\|_{L_t^\infty\dot C^\delta}\sup_{j\ge j_0}\int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s-\delta)}\Big(\sum_{-1\le k\le j+4}2^{k(1-s-s_0)}\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\Big)\mathrm{d}\tau\\
&\le C\|u\|_{L_t^\infty\dot C^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\Big)\\
&\qquad\qquad \sup_{j\ge j_0}2^{j(s_0+s-\delta)}\int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}(\tau-t_0)^{-\frac{s}{\alpha-\sigma}}\mathrm{d}\tau\\
&\le C\|u\|_{L_t^\infty\dot C^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\Big)\\
&\qquad\qquad (t-t_0)^{-\frac{s}{\alpha-\sigma}}\sup_{j\ge j_0}(2^{-j(\delta-(s_0+s-\alpha+\sigma))})\\
&\le C\|u\|_{L_t^\infty\dot C^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}}\Big)(t-t_0)^{-\frac{s}{\alpha-\sigma}}2^{-j_0(\delta-(s_0+s-\alpha+\sigma))}.
\end{aligned}
$$

For the term including H_j^2 in (3-37), by using (3-12) again, we estimate similarly as (3-30) to get that for all $s \in (0, \alpha - \sigma)$,

$$
\begin{aligned}
(3\text{-}41)\quad & \sup_{j\geq j_0} \int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} 2^{j(s_0+s)} H_j^2(\tau)\,\mathrm{d}\tau \\
&= C \sup_{j\geq j_0} \int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} \|u(\tau)\|_{\dot{C}^\delta} 2^{j(s_0+s+1)} \Big(\sum_{k\geq j-3} 2^{-k\delta} \|\Delta_k \theta(\tau)\|_{L^p} \Big) \mathrm{d}\tau \\
&\leq C \|u\|_{L_t^\infty \dot{C}^\delta} \sup_{j\geq j_0} 2^{j(s_0+s+1)} \\
&\qquad \Big(\sum_{k\geq j-3} 2^{-k(\delta+s_0+s)} \Big) \int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} \|\theta(\tau)\|_{B^{s_0+s}_{\infty,\infty}} \mathrm{d}\tau \\
&\leq C \|u\|_{L_t^\infty \dot{C}^\delta} \Big(\sup_{\tau\in(t_0,t]} (\tau-t_0)^{\frac{s}{\alpha-\sigma}} \|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}} \Big) \\
&\qquad \sup_{j\geq j_0} 2^{j(1-\delta)} \int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} (\tau-t_0)^{-\frac{s}{\alpha-\sigma}} \mathrm{d}\tau \\
&\leq C \|u\|_{L_t^\infty \dot{C}^\delta} \Big(\sup_{\tau\in(t_0,t]} (\tau-t_0)^{\frac{s}{\alpha-\sigma}} \|\theta(\tau)\|_{B^{s_0+s}_{p,\infty}} \Big) (t-t_0)^{-\frac{s}{\alpha-\sigma}} 2^{-j_0(\delta-(1-\alpha+\sigma))}.
\end{aligned}
$$

Inserting the estimates (3-38)–(3-41) into (3-37), we obtain that for every $t \in (t_0, T]$, $s \in (0, \alpha - \sigma)$ and $s_0 + s < \delta + \alpha - \sigma$,

$$
\begin{aligned}
(t-t_0)^{\frac{s}{\alpha-\sigma}} \|\theta(t)\|_{B^{s_0+s}_{p,\infty}} &\leq C T^{\frac{s}{\alpha-\sigma}} e^{CT} (\|\theta_0\|_{L^p} + \|f\|_{L^1_T L^p}) 2^{j_0(s_0+s)} \\
&\qquad + C_{\alpha,\sigma,s} \|\theta(t_0)\|_{B^{s_0}_{p,\infty}} + C T^{\frac{s}{\alpha-\sigma}} \|f\|_{L_t^\infty \dot{B}^\delta_{p,\infty}} \\
&+ \begin{cases} \frac{C\|u\|_{L_T^\infty \dot{C}^\delta}}{2^{j_0(\delta-(1-\alpha+\sigma))/2}} (\sup_{t\in(t_0,T]} (t-t_0)^{\frac{s}{\alpha-\sigma}} \|\theta\|_{B^{s_0+s}_{p,\infty}}), & \text{if } s_0+s\leq 1, \\ \frac{C\|u\|_{L_T^\infty \dot{C}^\delta}}{2^{j_0(\delta-(s_0+s-\alpha+\sigma))}} (\sup_{t\in(t_0,T]} (t-t_0)^{\frac{s}{\alpha-\sigma}} \|\theta\|_{B^{s_0+s}_{p,\infty}}), & \text{if } 1 < s_0+s < \delta+\alpha-\sigma. \end{cases}
\end{aligned}
$$

Hence by selecting $j_0 \in \mathbb{N}$ as

$$
(3\text{-}42)\qquad j_0 = \max\bigg\{ \bigg[\frac{2\log_2(2C\|u\|_{L_T^\infty \dot{C}^\delta})}{\delta-(1-\alpha+\sigma)} \bigg], \bigg[\frac{\log_2(2C_1/c)}{\alpha-\sigma} \bigg], 4 \bigg\} + 1
$$

if $s_0 + s \leq 1$, and

$$
(3\text{-}43)\qquad \max\bigg\{ \bigg[\frac{\log_2(2C\|u\|_{L_T^\infty \dot{C}^\delta})}{\delta-(s_0+s-\alpha+\sigma)} \bigg], \bigg[\frac{\log_2(2C_1/c)}{\alpha-\sigma} \bigg], 4 \bigg\} + 1
$$

if $1 < s_0+s < \delta+\alpha-\sigma$, we find that for all $s \in (0, \alpha-\sigma)$ and $s_0+s < \delta+\alpha-\sigma$,

$$\sup_{t\in(t_0,T]} \big((t-t_0)^{s/(\alpha-\sigma)} \|\theta(t)\|_{B^{s_0+s}_{p,\infty}}\big) \le C(T+1)(\|\theta_0\|_{L^p} + \|f\|_{L^1_T L^p}) 2^{j_0(s_0+s)} + C\|\theta(t_0)\|_{B^{s_0}_{p,\infty}} + C(T+1)\|f\|_{L^\infty_T \dot B^\delta_{p,\infty}},$$

which ensures that for any $t_1 \in (t_0, T)$ and every $s_0, s_1 \in (0, \alpha-\sigma)$ satisfying $s_0+s_1 < \delta+\alpha-\sigma$,

$$\sup_{t\in[t_1,T]} \|\theta(t)\|_{B^{s_0+s_1}_{p,\infty}} \le C(t_1-t_0)^{-\frac{s_1}{\alpha-\sigma}} \big((T+1)e^{CT}\big(\|\theta_0\|_{L^p}+\|f\|_{L^1_T L^p}\big)2^{j_0(s_0+s_1)} + \|\theta(t_0)\|_{B^{s_0}_{p,\infty}}\big) + C(t_1-t_0)^{-\frac{s_1}{\alpha-\sigma}}(T+1)\|f\|_{L^\infty_T \dot B^\delta_{p,\infty}}, \tag{3-44}$$

where j_0 is given by (3-42)–(3-43).

Step 3: the estimation of $\|\theta\|_{L^\infty([\tilde t,T];C^{1,\gamma})}$ for some $\gamma > 0$ and any $\tilde t \in (0, T)$.

If $\alpha-\sigma \in \big(\frac12, 1\big)$, we can choose appropriate indexes $s_0, s_1 \in (0, \alpha-\sigma)$ so that $1 < s_0+s_1 < \delta+\alpha-\sigma$, more precisely, denoting by

$$\nu_1 := \min\Big\{\frac{2(\alpha-\sigma)-1}{2}, \frac{\delta+\alpha-\sigma-1}{2}\Big\},$$

s_0+s_1 can be chosen so that $s_0+s_1 = 1+\nu_1$, thus in view of (3-44), we obtain that

$$\sup_{t\in[t_1,T]} \|\theta(t)\|_{B^{1+\nu_1}_{p,\infty}} \le C < \infty. \tag{3-45}$$

If $p > d/\nu_1$, then from the Besov embedding $B^{1+\nu_1}_{p,d} \hookrightarrow B^{1+\nu_1-d/p}_{\infty,\infty}$, we get the bound of $\|\theta\|_{L^\infty([\tilde t,T];C^{1,\gamma})}$ with $\tilde t = t_1$ and $\gamma = \nu_1 - d/p > 0$. If $p \le d/\nu_1$, and we have the embedding $B^{1+\nu_1}_{p,\infty} \hookrightarrow L^{p_1}$ with some $p_1 > d/\nu_1$, by repeating the above Step 1 and Step 2 with p_1 in place of p, we can obtain the estimate of $\|\theta\|_{L^\infty([t_1^1,T];B^{1+\nu_1}_{p_1,\infty})}$ with any $t_1^1 \in (t_1, T)$, which implies the bound of $\|\theta\|_{L^\infty([t_1^1,T];C^{1,\gamma})}$ with $\gamma = \nu_1 - d/p_1$. Otherwise, for $p \le d/\nu_1$ and p_1 satisfying $d/p_1 = d/p - (1+\nu_1)$ is such that $p_1 \in (p, d/\nu_1]$, as above we can obtain the bound of $\|\theta\|_{L^\infty([t_1^1,T];B^{1+\nu_1}_{p_1,\infty})}$ with any $t_1^1 \in (t_1, T)$, then if the embedding $B^{1+\nu_1}_{p_1,\infty} \hookrightarrow L^{p_2}$ with some $p_2 > d/\nu_1$, we can repeat the above Step 1 and Step 2 to conclude the proof, while if p_2 satisfying $d/p_2 = d/p_1 - (1+\nu_1) = d/p - 2(1+\nu_1)$ is still such that $p_2 \in (p_1, d/\nu_1]$, we can iterate the above steps for several times, say m times, to find some number $p_{m+1} > d/\nu_1$ and obtain the bound of $\|\theta\|_{L^\infty([t_1^{m+1},T];B^{1+\nu_1}_{p_{m+1},\infty})}$ with $t_1^{m+1} \in (t_1^m, T)$ any chosen, which further implies the bound of $\|\theta\|_{L^\infty([t_1^{m+1},T];C^{1,\gamma})}$ with $\gamma = 1+\nu_1 - d/p_{m+1}$.

For $\alpha-\sigma \in \big(0, \frac12\big]$, we need to iterate the above procedure in Step 2 more times. Assume that for some small number $t_k > 0$, $k \in \mathbb{N}$, we already have a finite bound on $\|\theta(t_k)\|_{B^{s_0+s_1+\cdots+s_k}_{p,\infty}}$ with $s_0, s_1, \dots, s_k \in (0, \alpha-\sigma)$ satisfying $s_0+s_1+\cdots+s_k \le 1$. Then by arguing as (3-44), we deduce that for any $t_{k+1} > t_k$, $s_{k+1} \in (0, \alpha-\sigma)$ satisfying $s_0+s_1+\cdots+s_{k+1} < \delta+\alpha-\sigma$,

$$
\begin{aligned}
\text{(3-46)}\quad \sup_{t\in[t_{k+1},T]} \|\theta(t)\|_{B^{s_0+s_1+\cdots+s_{k+1}}_{p,\infty}}
&\le C(t_{k+1}-t_k)^{-\frac{s_{k+1}}{\alpha-\sigma}}\big((T+1)(\|\theta_0\|_{L^p}+\|f\|_{L^1_T L^p})2^{j_0(\sum_{i=0}^{k+1} s_i)}+\|\theta(t_k)\|_{B^{\sum_{i=0}^k s_i}_{p,\infty}}\big)\\
&\quad + C(t_{k+1}-t_k)^{-\frac{s_{k+1}}{\alpha-\sigma}}(T+1)\|f\|_{L^\infty_T \dot B^\delta_{p,\infty}},
\end{aligned}
$$

where j_0 is also given by (3-42)–(3-43) with s_0+s_1 replaced by $s_0+s_1+\cdots+s_{k+1}$. Hence if $\alpha-\sigma \in (1/(k+2), 1/(k+1)]$, $k \in \mathbb{N}^+$, we can select appropriate numbers $s_0, s_1, \dots, s_{k+1} \in (0, \alpha-\sigma)$ so that $1 < s_0+s_1+\cdots+s_{k+1} < \delta+\alpha-\sigma$, or, more precisely, $s_0+s_1+\cdots+s_{k+1} = 1+\nu_{k+1}$, with

$$
\nu_{k+1} := \min\Big\{\frac{(k+2)(\alpha-\sigma)-1}{2}, \frac{\delta+\alpha-\sigma-1}{2}\Big\},
$$

and by repeating Step 2 in the above manner for $(k+1)$-times, we obtain

$$
\text{(3-47)}\qquad \sup_{t\in[t_{k+1},T]} \|\theta(t)\|_{B^{1+\nu_{k+1}}_{p,\infty}} \le C < \infty.
$$

The following deduction is similar to that stated below (3-45). If $p > d/\nu_{k+1}$, then from $B^{1+\nu_{k+1}}_{p,\infty} \hookrightarrow B^{1+\nu_{k+1}-d/p}_{\infty,\infty}$, we naturally get the estimate of $\|\theta\|_{L^\infty([t_{k+1},T];C^{1,\gamma})}$ with $\gamma = 1+\nu_{k+1}-d/p$. Otherwise, there exists a unique number $m \in \mathbb{N}$ so that

$$
\text{(3-48)}\qquad \frac{d}{p} - m(1+\nu_{k+1}) \ge \nu_{k+1}, \quad \text{and} \quad \frac{d}{p} - (m+1)(1+\nu_{k+1}) < \nu_{k+1},
$$

and by denoting $p_j \in [p, \infty)$ by

$$
\frac{d}{p_j} = \frac{d}{p} - j(1+\nu_{k+1}), \qquad j = 0, 1, 2, \dots, m,
$$

we see that $p = p_0 < p_1 < \cdots < p_m \le d/\nu_{k+1}$, thus by repeating the above process in obtaining (3-47) with p_j replaced by p_{j+1} iteratively ($j = 0, 1, \dots, m-1$), we have the bound of $\|\theta\|_{L^\infty([t^m_{k+1},T]; B^{1+\nu_{k+1}}_{p_m,\infty})}$ with any $t^m_{k+1} \in (0, T)$ (with the convention $t^0_i := t_i$ for $i = 0, 1, \dots, k+1$), which ensures that there is some $p_{m+1} > d/\nu_{k+1}$ so that $\|\theta\|_{L^\infty([t^m_{k+1},T];L^{p_{m+1}})}$ is bounded, and then iterating the above process once again leads to the estimate of $\|\theta\|_{L^\infty([t^{m+1}_{k+1},T]; B^{1+\nu_{k+1}}_{p_{m+1},\infty})}$ with any $t^{m+1}_{k+1} \in (t^m_{k+1}, T)$ and moreover implies that for $1+\gamma = 1+\nu_{k+1}-d/p_{m+1} = (m+2)(1+\nu_{k+1})-d/p$,

$$
\begin{aligned}
\|\theta\|_{L^\infty([t^{m+1}_{k+1},T];C^{1,\gamma})} &\approx \|\theta\|_{L^\infty([t^{m+1}_{k+1},T];B^{1+\gamma}_{\infty,\infty})}\\
\text{(3-49)}\qquad &\le C\bigg(\prod_{j=0}^{m+1}\prod_{i=0}^{k}(t^j_{i+1}-t^j_i)^{-\frac{s_{i+1}}{\alpha-\sigma}}(t^j_0-t^{j-1}_{k+1})^{-\frac{s_0}{\alpha-\sigma}}\bigg)\\
&\qquad (\|\theta_0\|_{L^p}+\|f\|_{L^\infty_T(B^\delta_{p,\infty}\cap B^\delta_{\infty,\infty})}),
\end{aligned}
$$

where $t_i^0 := t_i$ for $i = 0, 1, \dots, k+1$, $t_{k+1}^{-1} := 0$, $C > 0$ is a constant depending only on $p, \alpha, \sigma, \delta, d, T$ and $\|u\|_{L_T^\infty \dot{C}^\delta}$.

Therefore, for every $\alpha \in (0, 1]$, $\sigma \in [0, \alpha)$, $p \in [2, \infty)$, and for any $\tilde{t} \in (0, T)$, there is some $k \in \mathbb{N}$ so that $\alpha - \sigma \in (1/(k+2), 1/(k+1)]$, and there is some number $m \in \mathbb{N}$ so that (3-48) holds, and thus we can choose

$$t_i^j = \frac{j(k+2)+i+1}{(k+2)(m+2)}\tilde{t} \quad \text{for } i = 0, 1, \dots, k+1,\ j = 0, 1, 2, \dots, m+1,$$

and appropriate $s_0, s_1, \dots, s_{k+1} \in (0, \alpha - \sigma)$ such that $s_0 + s_1 + \cdots + s_{k+1} = 1 + \nu_{k+1}$. Then we use (3-49) to get that for some $\gamma > 0$,

$$\|\theta\|_{L^\infty([\tilde{t}, T]; C^{1,\gamma}(\mathbb{R}^d))} \le C\tilde{t}^{-\frac{\gamma+1+d/p}{\alpha-\sigma}}\big(\|\theta_0\|_{L^p} + \|f\|_{L_T^\infty(B^\delta_{p,\infty} \cap B^\delta_{\infty,\infty})}\big), \tag{3-50}$$

with the constant C depending only on p, α, σ, δ, T, d and $\|u\|_{L_T^\infty \dot{C}^\delta}$.

Step 4: the estimation of $\|\theta\|_{L^\infty([t', T]; C^{1,\gamma})}$ for any $\gamma \in (0, \delta + \alpha - \sigma - 1)$ and any $t' \in (0, T)$.

This is achieved by pursuing the above iteration process more times. In fact, for any $\gamma \in (0, \delta + \alpha - \sigma - 1)$, there exists some $\tilde{p} < \infty$ so that $\gamma + d/\tilde{p} < \delta + \alpha - \sigma - 1$, and according to the above Step 3, we may suppose that there is already a bound of $\|\theta\|_{L^\infty([t'/2, T]; B^{s'}_{\tilde{p},\infty})}$ with some $1 < s' < 1 + \gamma + d/\tilde{p}$, but by repeating the deduction in Steps 1–2 for several times and due to the increment of regularity index s in each time belonging to $(0, \alpha - \sigma)$, we can derive an upper bound of $\|\theta\|_{L^\infty([t', T]; B^{1+\gamma+d/\tilde{p}}_{\tilde{p},\infty})}$, which also satisfies (3-50) with t' in place of $\tilde{t}$.

3C. ***The existence part.*** We consider the approximate system

$$\begin{cases} \partial_t\theta + (u_\epsilon \cdot \nabla)\theta + \mathcal{L}\theta - \epsilon\Delta\theta = f_\epsilon, \\ u_\epsilon := \phi_\epsilon * u, \quad f_\epsilon = \phi_\epsilon * f, \quad \theta|_{t=0} = \theta_{0,\epsilon} := \phi_\epsilon * \theta_0, \end{cases} \tag{3-51}$$

where $\phi_\epsilon(x) = \epsilon^{-d}\phi(\epsilon^{-1}x)$ for all $x \in \mathbb{R}^d$, and $\phi \in C_c^\infty(\mathbb{R}^d)$ is a test function supported on the ball $B_1(0)$ satisfying $0 \le \phi \le 1$, $\phi \equiv 1$ on $B_{1/2}(0)$ and $\int_{\mathbb{R}^d} \phi \, \mathrm{d}x = 1$.

Due to that for all $s \ge 0$, $\|\theta_{0,\epsilon}\|_{B^s_{p,2}(\mathbb{R}^d)} \lesssim_\epsilon \|\theta_0\|_{L^p(\mathbb{R}^d)}$ and $\|u_\epsilon\|_{L_T^\infty C^s(\mathbb{R}^d)} \lesssim_\epsilon \|u\|_{L_T^\infty C^\delta}$ and $\|f_\epsilon\|_{L_T^\infty B^s_{p,2}} \lesssim_\epsilon \|f\|_{L_T^\infty B^\delta_{p,\infty}}$, by using a classical procedure (the operator $\mathcal{L}$ can be treated as Lemma 3.2), we obtain a smooth approximate solution $\theta^{(\epsilon)} \in C([0, T]; B^s_{p,2}(\mathbb{R}^d)) \cap C^1([0, T]; C_b^\infty(\mathbb{R}^d))$, $s > d/p + 1$ for the system (3-51).

Notice that we have the following uniform-in-ϵ estimates that $\|\theta_{0,\epsilon}\|_{L^p} \le \|\theta_0\|_{L^p}$, $\|u_\epsilon\|_{L_T^\infty C^\delta} \le \|u\|_{L_T^\infty C^\delta}$ and $\|f_\epsilon\|_{L_T^\infty(B^\delta_{p,\infty} \cap B^\delta_{\infty,\infty})} \le \|f\|_{L_T^\infty(B^\delta_{p,\infty} \cap B^\delta_{\infty,\infty})}$. According to Lemma 3.3, we infer that the solutions $\theta^{(\epsilon)}$ uniformly-in-ϵ belong to the space $L^\infty([0, T]; L^p(\mathbb{R}^d)) \cap L^p([0, T]; B^{(\alpha-\sigma)/p}_{p,p}(\mathbb{R}^d))$. From the system (3-51), we also claim that $\partial_t\theta^{(\epsilon)} \in L^p([0, T]; B^{-2}_{p,p}(\mathbb{R}^d))$ uniformly in $\epsilon > 0$. Indeed, it is derived from the following uniform-in-ϵ estimates:

$$\|f_\epsilon\|_{L^p_T B^{-2}_{p,p}} \le C\|f_\epsilon\|_{L^p_T L^p} \le C\|f\|_{L^p_T L^p} \le C T^{1/p}\|f\|_{L^\infty_T L^p},$$

and (thanks to Lemma 2.2(3))

$$\begin{aligned}\|\mathcal{L}\theta^{(\epsilon)}\|_{L^p_T B^{-2}_{p,p}} &\le C\|\mathcal{L}\Delta_{-1}\theta^{(\epsilon)}\|_{L^p_T L^p} + \sum_{j\in\mathbb{N}} 2^{-2j}\|\mathcal{L}\Delta_j\theta^{(\epsilon)}\|_{L^p_T L^p} \\ &\le C\|\theta^{(\epsilon)}\|_{L^p_T L^p} + C\sum_{j\in\mathbb{N}} 2^{-2j}2^{j\alpha}\|\theta^{(\epsilon)}\|_{L^p_T L^p} \\ &\le C T^{1/p}\|\theta^{(\epsilon)}\|_{L^\infty_T L^p},\end{aligned}$$

and $\|\Delta\theta^{(\epsilon)}\|_{L^p_T B^{-2}_{p,p}} \le C\|\theta^{(\epsilon)}\|_{L^p_T B^0_{p,p}} \le C T^{1/p}\|\theta^{(\epsilon)}\|_{L^\infty_T L^p}$, and

$$\begin{aligned}\|(u_\epsilon\cdot\nabla)\theta^{(\epsilon)}\|_{L^p_T B^{-2}_{p,p}} &\le C\|u_\epsilon\theta^{(\epsilon)}\|_{L^p_T B^0_{p,\infty}} \le C T^{1/p}\|u_\epsilon\|_{L^\infty_T L^\infty}\|\theta^{(\epsilon)}\|_{L^\infty_T L^p} \\ &\le C T^{1/p}\|u\|_{L^\infty_T L^\infty}\|\theta^{(\epsilon)}\|_{L^\infty_T L^p}.\end{aligned}$$

Since the embedding $B^{\alpha-\sigma/p}_{p,p} \hookrightarrow L^p$ is locally compact, the classical Aubin–Lions lemma (see, e.g., [Constantin and Foias 1988, Lemma 8.4]) ensures the strong convergence of $\theta^{(\epsilon)}$ (up to the subsequence, still denoting by $\theta^{(\epsilon)}$) to θ in $L^p_T L^p_{\mathrm{loc}}$. From Fatou's lemma, we get $\theta \in L^\infty([0,T]; L^p(\mathbb{R}^d)) \cap L^p([0,T]; B^{\alpha-\sigma/p}_{p,p}(\mathbb{R}^d))$. Noticing also that from $u \in L^\infty_T C^\delta$ we have $u_\epsilon \to u$ in $L^\infty_T L^\infty$ as $\epsilon \to 0$, by using Hölder's inequality, it is not hard to check that for any test function $\varphi \in C^\infty_c(\mathbb{R}^d \times [0,T])$ (assuming $\operatorname{supp}\varphi \subseteq \mathcal{O}\times[0,T]$ with a compact set $\mathcal{O} \subseteq \mathbb{R}^d$),

$$\begin{aligned}&\left|\int_0^T\int_{\mathbb{R}^d}\theta^{(\epsilon)}u_\epsilon\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\mathbb{R}^d}\theta u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\qquad\le \left|\int_0^T\int_{\mathbb{R}^d}(\theta^{(\epsilon)}-\theta)u_\epsilon\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\right| + \left|\int_0^T\int_{\mathbb{R}^d}\theta(u_\epsilon-u)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\qquad\le C\|\theta^{(\epsilon)}-\theta\|_{L^p_T L^p(\mathcal{O})}\|u_\epsilon\|_{L^\infty_T L^\infty}\|\nabla\varphi\|_{L^{p/(p-1)}_{t,x}} \\ &\qquad\qquad + \|u_\epsilon-u\|_{L^\infty_T L^\infty}\|\theta\|_{L^\infty_T L^p}\|\nabla\varphi\|_{L^1_T L^{p/(p-1)}} \\ &\qquad\le C\|\theta^{(\epsilon)}-\theta\|_{L^p_T L^p(\mathcal{O})}\|u\|_{L^\infty_T L^\infty} + C\|u_\epsilon-u\|_{L^\infty_T L^\infty}\|\theta\|_{L^\infty_T L^p} \\ &\qquad\to 0 \quad \text{as } \epsilon \to 0.\end{aligned}$$

By passing ϵ to 0 in (3-51), from $f_\epsilon \to f$ in $L^\infty_T L^p$ and $\theta_{0,\epsilon} \to \theta_0$ in L^p, we deduce that θ is a distributional solution of (1-1) such that for every $\varphi \in C_c^\infty(\mathbb{R}^d \times [0, T])$

$$\begin{aligned} \int_{\mathbb{R}^d} \theta(x,t)\varphi(x,t)\,\mathrm{d}x - \int_{\mathbb{R}^d} \theta_0(x)\varphi(x,0)\,\mathrm{d}x - \int_0^t\int_{\mathbb{R}^d} \theta(x,\tau)\partial_\tau\varphi(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ = \int_0^t\int_{\mathbb{R}^d} (u\theta)(x,\tau)\nabla\varphi(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ - \int_0^t\int_{\mathbb{R}^d} \theta(x,\tau)\mathcal{L}^*\varphi(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\ + \int_0^t\int_{\mathbb{R}^d} f(x,\tau)\varphi(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau, \end{aligned} \tag{3-52}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$.

Moreover, from Lemma 3.3, the weak solution θ also satisfies

$$\max_{0\le t\le T} \|\theta(t)\|_{L^p} \le e^{C'T}(\|\theta_0\|_{L^p} + \|f\|_{L^p_T L^p}), \tag{3-53}$$

with some constant $C' = C'(p, \alpha, \sigma, d)$. Moreover, by repeating the process in Section 3B for the approximate system (3-51) and using the Fatou lemma, we get

$$\theta \in C((0, T]; C^{1,\gamma}(\mathbb{R}^d))$$

for any $\gamma \in (0, \delta+\alpha-1-\sigma)$. Therefore, we conclude Theorem 1.1.

4. Proof of Theorem 1.2

4A. *Auxiliary lemmas.* Before proceeding with the main proof, we introduce several auxiliary lemmas. First is the maximum principle for the drift-diffusion equations (1-1)–(1-2).

Lemma 4.1. *Let the vector field u and the forcing term f be smooth. Assume that*

$$\theta \in L^\infty([0, T]; H^s(\mathbb{R}^d))$$

($s > d/2+1$) *is a smooth solution for the drift-diffusion equations* (1-1)–(1-2) *under the assumptions of K* (1-3)–(1-5). *Then we have*

$$\max_{0\le t\le T} \|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty} + \int_0^T \|f(t)\|_{L^\infty}\,\mathrm{d}t. \tag{4-1}$$

Proof of Lemma 4.1. Thanks to the nonnegative condition (1-5), the proof is similar to [Córdoba and Córdoba 2004, Theorem 4.1]. We here sketch the proof for the sake of completeness. Since $\theta(\cdot\,, t) \in H^s$ with $s > d/2+1$ for any $0 \le t \le T$, there exists a point $x_t \in \mathbb{R}^d$ where $|\theta|$ attains its maximum value;with no loss of generality we set

$$\theta(x_t, t) = \|\theta(t)\|_{L^\infty}.$$

It should be noted that $\nabla_x \theta(x_t, t) = 0$ and due to $K(y) \geq 0$ we find

$$(\mathcal{L}\theta)(x_t, t) = \mathrm{p.\,v.}\int_{\mathbb{R}^n} (\theta(x_t, t) - \theta(x_t + y, t))K(y)dy \geq 0.$$

We thus get

$$\frac{d}{dt}\|\theta(t)\|_{L^\infty} \leq \partial_t\theta(x_t, t) \leq \|f(t)\|_{L^\infty} \quad \text{for all } 0 \leq t \leq T.$$

Integrating in time yields the desired estimate (4-1). □

The second is the maximum principle with diffusion effect for the following frequency localized drift-diffusion equation

$$\partial_t \Delta_j\theta + u \cdot \nabla\Delta_j\theta + \mathcal{L}\Delta_j\theta = g, \quad j \in \mathbb{N}, \tag{4-2}$$

where the operator $\mathcal{L}$ defined by (1-2) with the symmetric kernel K satisfying (1-3)–(1-5).

Lemma 4.2. *Assume that u and f are suitably smooth functions, and θ is a smooth solution to the equation (4-2) satisfying $\Delta_j\theta \in C_0(\mathbb{R}^d)$ for all $t > 0$ and $j \in \mathbb{N}$. Then there exist absolute positive constants c and C depending only on α, σ, d such that*

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^\infty} + c2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^\infty} \leq C\|\Delta_j\theta\|_{L^\infty} + \|g\|_{L^\infty}. \tag{4-3}$$

Proof of Lemma 4.2. Denote by $\theta_j := \Delta_j\theta$, and from $\theta_j(t) \in C_0(\mathbb{R}^d)$ for $j \in \mathbb{N}$, there exists a point $x_{t,j} \in \mathbb{R}^d$ such that $|\theta_j(t, x_{t,j})| = \|\theta_j\|_{L^\infty} > 0$. Without loss of generality, we assume $\theta_j(t, x_{t,j}) = \|\theta_j\|_{L^\infty} > 0$ (otherwise, we consider the equation of $-\theta_j$ and replace θ_j by $-\theta_j$ in the following deduction). Now by using (1-2), (1-5), (1-7) and the fact $\theta(t, x_{t,j}) - \theta(t, x_{t,j} + y) \geq 0$, we get

$$\begin{aligned}
\mathcal{L}\theta_j(x_{t,j}) &= \mathrm{p.\,v.}\int_{\mathbb{R}^d} (\theta_j(x_{t,j}) - \theta_j(x_{t,j} + y))K(y)\,\mathrm{d}y \\
&= \mathrm{p.\,v.}\int_{|y|\leq 1} (\theta_j(x_{t,j}) - \theta_j(x_{t,j} + y))K(y)\,\mathrm{d}y \\
&\qquad\qquad + \int_{|y|>1} (\theta_j(x_{t,j}) - \theta_j(x_{t,j} + y))K(y)\,\mathrm{d}y
\end{aligned}$$

which then gives

$$
\begin{aligned}
\text{(4-4)}\quad \mathcal{L}\theta_j(x_{t,j}) = &\geq c_2^{-1}\,\text{p. v.}\!\int_{|y|\le 1} \frac{\theta_j(x_{t,j})-\theta_j(x_{t,j}+y)}{|y|^{d+\alpha-\sigma}}\,\mathrm{d}y \\
&\qquad + \int_{|y|>1} (\theta_j(x_{t,j})-\theta_j(x_{t,j}+y))K(y)\,\mathrm{d}y \\
&\geq c_2^{-1}\,\text{p. v.}\!\int_{\mathbb{R}^d} \frac{\theta_j(x_{t,j})-\theta_j(x_{t,j}+y)}{|y|^{d+\alpha-\sigma}}\,\mathrm{d}y \\
&\qquad - c_2^{-1}\int_{|y|>1} \frac{\theta_j(x_{t,j})-\theta_j(x_{t,j}+y)}{|y|^{d+\alpha-\sigma}}\,\mathrm{d}y \\
&\geq c_2^{-1}c_{d,\alpha}^{-1}|D|^{\alpha-\sigma}\theta_j(x_{t,j}) - 2c_2^{-1}\|\theta_j\|_{L^\infty}\int_{|y|>1}\frac{1}{|y|^{d+\alpha-\sigma}}\mathrm{d}y \\
&\geq c_2^{-1}c_{d,\alpha}^{-1}|D|^{\alpha-\sigma}\theta_j(x_{t,j}) - C\|\theta_j\|_{L^\infty}.
\end{aligned}
$$

According to [Wang and Zhang 2011, Lemma 3.4], we have

$$
\text{(4-5)}\qquad |D|^{\alpha-\sigma}\theta_j(x_{t,j}) \geq \tilde{c}2^{j(\alpha-\sigma)}\|\theta_j\|_{L^\infty},
$$

with some generic constant $\tilde{c}>0$. Inserting (4-5) into (4-4) yields

$$
\text{(4-6)}\qquad \mathcal{L}\theta_j(x_{t,j}) \geq c2^{j(\alpha-\sigma)}\|\theta_j\|_{L^\infty} - C\|\theta_j\|_{L^\infty}.
$$

Hence, by arguing as Lemma 3.2 of the same work and using the fact $\nabla\theta_j(t,x_{t,j})=0$, we get

$$
\begin{aligned}
\text{(4-7)}\qquad \frac{d}{dt}\|\theta_j\|_{L^\infty} &\leq \partial_t\theta_j(t,x_{t,j}) \\
&= -u(t,x_{t,j})\cdot\nabla\theta_j(t,x_{t,j}) - \mathcal{L}\theta_j(t,x_{t,j}) + g(t,x_{t,j}) \\
&\leq -c2^{j(\alpha-\sigma)}\|\theta_j\|_{L^\infty} + C\|\theta_j\|_{L^\infty} + \|g\|_{L^\infty},
\end{aligned}
$$

which finishes the proof of (4-3). □

4B. ***A priori estimates.*** In this subsection, we assume θ is a smooth solution with suitable spatial decay for the drift-diffusion equations (1-1)–(1-2) with sufficiently smooth u and f. We intend to show the key a priori differentiability estimate. The proof is divided into four steps.

Step 1: the estimation of $\|\theta\|_{L^\infty([t_0,T];C^s(\mathbb{R}^d))}$ for any $s\in(1-\delta,\alpha-\sigma)$ and $t_0\in(0,T)$.

For every $j\in\mathbb{N}$ and $j\geq 4$, applying the inhomogeneous dyadic operator Δ_j to the equation (1-1), we get

$$
\text{(4-8)}\quad \partial_t\Delta_j\theta + u\cdot\nabla\Delta_j\theta + \mathcal{L}\Delta_j\theta = \Delta_j f - [\Delta_j, u\cdot\nabla]\theta = \Delta_j f + I_1 + I_2 + I_3,
$$

where I_1–I_3 defined by (3-14) are the Bony's decomposition of the commutator term $-[\Delta_j, u\cdot\nabla]\theta$. Taking advantage of Lemma 4.2 in the frequency localized equation (3-13), we get

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^\infty}+c2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^\infty} \le C_1\|\Delta_j\theta\|_{L^\infty}+\|I_1\|_{L^\infty}+\|I_2\|_{L^\infty}+\|I_3\|_{L^\infty}+\|\Delta_j f\|_{L^\infty}. \tag{4-9}$$

Similarly to the derivation of (3-19) and (3-20), we see that

$$\|I_1\|_{L^\infty}\le C2^{-j\delta}\|u\|_{\dot C^\delta}\sum_{|k-j|\le 4}2^k\|\Delta_k\theta\|_{L^\infty}, \tag{4-10}$$

and

$$\|I_2\|_{L^\infty}\le C2^{-j\delta}\|u\|_{\dot C^\delta}\Bigl(\sum_{k\le j}2^k\|\Delta_k\theta\|_{L^\infty}\Bigr), \tag{4-11}$$

and for $\|I_3\|_{L^\infty}$, by virtue of Hölder's inequality and Bernstein's inequality, we find

$$\begin{aligned}\|I_3\|_{L^\infty}&\le\sum_{k\ge j-2}\|\Delta_j(\Delta_k u\cdot\nabla\widetilde\Delta_k\theta)\|_{L^\infty}+\sum_{k\ge j-2}\|\Delta_k u\cdot\nabla\widetilde\Delta_k\Delta_j\theta\|_{L^\infty}\\&\le C\sum_{k\ge j-2}\|\Delta_k u\|_{L^\infty}2^k\|\widetilde\Delta_k\theta\|_{L^\infty}\\&\le C\sum_{k\ge j-2}2^{k(1-\delta)}2^{k\delta}\|\Delta_k u\|_{L^\infty}\|\widetilde\Delta_k\theta\|_{L^\infty}\\&\le C\|u\|_{\dot C^\delta}\Bigl(\sum_{k\ge j-3}2^{k(1-\delta)}\|\Delta_k\theta\|_{L^\infty}\Bigr).\end{aligned} \tag{4-12}$$

Inserting the upper estimates (4-10)–(4-12) into (4-9), we have

$$\begin{aligned}\frac{d}{dt}\|\Delta_j\theta\|_{L^\infty}+c2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^\infty}&\le C_2\|\Delta_j\theta\|_{L^\infty}+\|\Delta_j f\|_{L^\infty}+C\|u\|_{\dot C^\delta}2^{-j\delta}\sum_{k\le j+4}2^k\|\Delta_k\theta\|_{L^\infty}\\&\quad+C\|u\|_{\dot C^\delta}\sum_{k\ge j-3}2^{k(1-\delta)}\|\Delta_k\theta\|_{L^\infty}.\end{aligned} \tag{4-13}$$

In particular, by some $j_1\in\mathbb{N}$ chosen later (see (4-24)) so that $c2^{j_1(\alpha-\sigma)}\ge 2C_2$, or more precisely

$$j_1\ge\Bigl[\frac{1}{\alpha-\sigma}\log_2\Bigl(\frac{2C_2}{c}\Bigr)\Bigr]+1, \tag{4-14}$$

we see that for $j \ge j_1$,

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^\infty}+\frac{c}{2}2^{j(\alpha-\sigma)}\|\Delta_j\theta\|_{L^\infty} \le \|\Delta_j f\|_{L^\infty}+C\|u\|_{\dot{C}^\delta}2^{-j\delta}\sum_{k\le j+4}2^k\|\Delta_k\theta\|_{L^\infty} + C\|u\|_{\dot{C}^\delta}\sum_{k\ge j-3}2^{k(1-\delta)}\|\Delta_k\theta\|_{L^\infty} := \|\Delta_j f\|_{L^\infty}+F_j^1+F_j^2. \tag{4-15}$$

Consequently, Grönwall's inequality guarantees that for every $j \ge j_1$ and $t \ge 0$,

$$\|\Delta_j\theta(t)\|_{L^\infty} \le e^{-\frac{c}{2}t2^{j(\alpha-\sigma)}}\|\Delta_j\theta_0\|_{L^\infty} + \int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}(\|\Delta_j f\|_{L^\infty}(\tau)+F_j^1(\tau)+F_j^2(\tau))\,\mathrm{d}\tau. \tag{4-16}$$

On the other hand, we have the classical maximum principle (4-1) for (1-1):

$$\|\theta(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty}+\int_0^t \|f(\tau)\|_{L^\infty}\,\mathrm{d}t. \tag{4-17}$$

By arguing as (3-26), we get that for all $t > 0$, $j \in \mathbb{N}$ and $s \in (0, \alpha-\sigma)$,

$$2^{js}e^{-\frac{c}{2}t2^{j(\alpha-\sigma)}}\|\Delta_j\theta_0\|_{L^\infty} \le C_{\alpha,\sigma,s}t^{-\frac{s}{\alpha-\sigma}}\|\theta_0\|_{L^\infty}, \tag{4-18}$$

we gather (4-16) and (4-17) to obtain

$$\begin{aligned}\|\theta(t)\|_{C^s} &\approx \|\theta(t)\|_{B^s_{\infty,\infty}} \\ &\le \sup_{j\le j_1}2^{js}\|\Delta_j\theta(t)\|_{L^\infty}+\sup_{j\ge j_1}2^{js}\|\Delta_j\theta(t)\|_{L^\infty} \\ &\le C2^{j_1 s}(\|\theta_0\|_{L^\infty}+\|f\|_{L^1_tL^\infty})+C_{\alpha,\sigma,s}t^{-\frac{s}{\alpha-\sigma}}\|\theta_0\|_{L^\infty} \\ &\quad +\sup_{j\ge j_1}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}(\|\Delta_j f\|_{L^\infty}(\tau)+F_j^1(\tau)+F_j^2(\tau))\mathrm{d}\tau.\end{aligned} \tag{4-19}$$

For the term containing $\|\Delta_j f\|_{L^\infty}$ and F_j^1, in a similar way as obtaining (3-28) and (3-29), we obtain that for every $s \in (0, \alpha-\sigma+\delta)$ and $\delta \in (1-\alpha+\sigma, 1)$,

$$\sup_{j\ge j_1}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}\|\Delta_j f\|_{L^\infty}(\tau)\,\mathrm{d}\tau \le C\|f\|_{L^\infty_t\dot{C}^\delta}\sup_{j\ge j_1}2^{j(s-\alpha+\sigma-\delta)} \le C\|f\|_{L^\infty_t\dot{C}^\delta}, \tag{4-20}$$

and

$$\sup_{j\ge j_1}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}F_j^1(\tau)\,\mathrm{d}\tau \le Ct^{-\frac{s}{\alpha-\sigma}}2^{j_1(1-\alpha+\sigma-\delta)}\|u\|_{L_t^\infty\dot{C}^\delta}\Big(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{\infty,\infty}}\Big). \tag{4-21}$$

For the term including F_j^2 in (4-19), by using (3-12) again, we similarly get that for all $s\in(1-\delta,\alpha-\sigma)$ and $\delta\in(1-\alpha+\sigma,1)$,

$$\begin{aligned}
\sup_{j\ge j_1}&\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{js}F_j^2(\tau)\,\mathrm{d}\tau \\
&= C\sup_{j\ge j_1}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|u(\tau)\|_{\dot{C}^\delta}2^{js}\Big(\sum_{k\ge j-3}2^{k(1-\delta)}\|\Delta_k\theta(\tau)\|_{L^\infty}\Big)\,\mathrm{d}\tau \\
&\le C\|u\|_{L_t^\infty\dot{C}^\delta}\sup_{j\ge j_1}2^{js}\Big(\sum_{k\ge j-3}2^{k(1-\delta-s)}\Big)\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|\theta(\tau)\|_{B^s_{\infty,\infty}}\,\mathrm{d}\tau \\
&\le C\|u\|_{L^\infty\dot{C}^\delta}\Big(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{\infty,\infty}}\Big) \\
&\qquad\qquad \sup_{j\ge j_1}2^{j(1-\delta)}\int_0^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\tau^{-\frac{s}{\alpha-\sigma}}\,\mathrm{d}\tau \\
&\le Ct^{-\frac{s}{\alpha-\sigma}}2^{-j_1(\delta-(1-\alpha+\sigma))}\|u\|_{L_t^\infty\dot{C}^\delta}\Big(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{\infty,\infty}}\Big).
\end{aligned}\tag{4-22}$$

Inserting the estimates (4-20), (4-21), (4-22) into (4-19) yields that for any $1-\delta<s<\alpha-\sigma$ and $0<t\le T$,

$$\begin{aligned}
t^{\frac{s}{\alpha-\sigma}}&\|\theta(t)\|_{B^s_{\infty,\infty}} \\
&\le Ct^{\frac{s}{\alpha-\sigma}}(\|\theta_0\|_{L^\infty}+\|f\|_{L_t^1L^\infty})2^{j_1s}+C_{\alpha,\sigma,s}\|\theta_0\|_{L^\infty}Ct^{\frac{s}{\alpha-\sigma}}\|f\|_{L_t^\infty\dot{C}^\delta} \\
&\qquad +C2^{j_1(1-\alpha+\sigma-\delta)}\|u\|_{L_t^\infty\dot{C}^\delta}\Big(\sup_{\tau\in(0,t]}\tau^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^s_{\infty,\infty}}\Big) \\
&\le CT^{\frac{s}{\alpha-\sigma}}(\|\theta_0\|_{L^\infty}+\|f\|_{L_T^1L^\infty})2^{j_1s}+C_{\alpha,\sigma,s}\|\theta_0\|_{L^\infty}+CT^{\frac{s}{\alpha-\sigma}}\|f\|_{L_T^\infty\dot{C}^\delta} \\
&\qquad +C2^{-j_1(\delta-(1-\alpha+\sigma))}\|u\|_{L_T^\infty\dot{C}^\delta}\Big(\sup_{t\in(0,T]}t^{\frac{s}{\alpha-\sigma}}\|\theta(t)\|_{B^s_{\infty,\infty}}\Big).
\end{aligned}\tag{4-23}$$

Since $1-\alpha+\sigma-\delta>0$, by further choosing j_1 such that $C2^{j_1(1-\alpha+\sigma-\delta)}\|u\|_{L_T^\infty\dot{C}^\delta}\le\frac{1}{2}$ and (4-14) holds, or more precisely,

$$j_1=\max\bigg\{\bigg[\frac{\log_2 2C\|u\|_{L_T^\infty\dot{C}^\delta}}{\delta-(1-\alpha+\sigma)}\bigg],\bigg[\frac{\log_2(2C_2/c)}{\alpha-\sigma}\bigg],4\bigg\}+1, \tag{4-24}$$

we have that for all $1-\delta < s < \alpha-\sigma$,

$$\sup_{t\in(0,T]} (t^{\frac{s}{\alpha-\sigma}} \|\theta(t)\|_{B^s_{\infty,\infty}}) \le C(T+1)(2^{j_1 s}(\|\theta_0\|_{L^\infty} + \|f\|_{L^1_T L^\infty}) + \|f\|_{L^\infty_T \dot{C}^\delta}), \tag{4-25}$$

which implies that for arbitrarily small $t_0 \in (0, T)$ and every $s_0 \in (1-\delta, \alpha-\sigma)$,

$$\sup_{t\in[t_0,T]} \|\theta(t)\|_{B^{s_0}_{\infty,\infty}} \le C t_0^{-\frac{s_0}{\alpha-\sigma}} (T+1)(2^{j_1 s}(\|\theta_0\|_{L^\infty} + \|f\|_{L^1_T L^\infty}) + \|f\|_{L^\infty_T \dot{C}^\delta}), \tag{4-26}$$

with j_1 given by (4-24).

Step 2: the estimation of $\|\theta\|_{L^\infty([t_1,T];B^{s_0+s_1}_{\infty,\infty})}$ for $s_0 \in (1-\delta, \alpha-\sigma)$, $s_1 \in (0, \alpha-\sigma)$ and any $t_1 \in (t_0, T)$.

For every $j \ge j_1$ with $j_1 \in \mathbb{N}$ satisfying (4-14) chosen later (j_1 is slightly different from that number in Step 1), applying the Grönwall inequality to (4-15) over the time interval $[t_0, t]$ (for $t > t_0 > 0$) gives

$$\|\Delta_j\theta(t)\|_{L^\infty} \le e^{-(c/2)(t-t_0)2^{j(\alpha-\sigma)}} \|\Delta_j\theta(t_0)\|_{L^\infty} + \int_{t_0}^t e^{-(c/2)(t-\tau)2^{j(\alpha-\sigma)}} (\|\Delta_j f\|_{L^\infty} + F_j^1 + F_j^2)(\tau)\, \mathrm{d}\tau. \tag{4-27}$$

Noticing that for $j \in \mathbb{N}$, $s_0 \in (1-\delta, \alpha-\sigma)$ and all $s \in (0, \alpha-\sigma)$,

$$\begin{aligned} e^{-\frac{c}{2}(t-t_0)2^{j(\alpha-\sigma)}} 2^{j(s_0+s)} \|\Delta_j\theta(t_0)\|_{L^\infty} &\le e^{-\frac{c}{2}(t-t_0)2^{j(\alpha-\sigma)}} 2^{js} \|\theta(t_0)\|_{B^{s_0}_{\infty,\infty}} \\ &\le C_{\alpha,\sigma,s}(t-t_0)^{-\frac{s}{\alpha-\sigma}} \|\theta(t_0)\|_{B^{s_0}_{\infty,\infty}}, \end{aligned} \tag{4-28}$$

by arguing as (4-19) we obtain that for all $t \ge t_0 > 0$,

$$\begin{aligned} \|\theta(t)\|_{B^{s_0+s}_{\infty,\infty}} &\le \sup_{j\le j_1} 2^{j(s_0+s)} \|\Delta_j\theta(t)\|_{L^\infty} + \sup_{j\ge j_1} 2^{j(s_0+s)} \|\Delta_j\theta(t)\|_{L^\infty} \\ &\le C 2^{j_1(s_0+s)} (\|\theta_0\|_{L^\infty} + \|f\|_{L^1_t L^\infty}) + C_{\alpha,\sigma,s}(t-t_0)^{-\frac{s}{\alpha-\sigma}} \|\theta(t_0)\|_{B^{s_0}_{\infty,\infty}} \\ &\quad + \sup_{j\ge j_1} \int_{t_0}^t e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}} 2^{j(s_0+s)} (\|\Delta_j f\|_{L^\infty}(\tau) + F_j^1(\tau) + F_j^2(\tau))\, \mathrm{d}\tau. \end{aligned} \tag{4-29}$$

In a similar fashion as the estimation of (3-38), (3-39)–(3-40), we find that for every $s \in (0, \alpha-\sigma)$ and $s_0 + s < \delta + \alpha - \sigma$,

$$\sup_{j\ge j_1} \int_{t_0}^t e^{-(c/2)(t-\tau)2^{j(\alpha-\sigma)}} 2^{j(s_0+s)} \|\Delta_j f\|_{L^\infty}(\tau)\, \mathrm{d}\tau \le C \|f\|_{L^\infty_t \dot{B}^\delta_{\infty,\infty}}, \tag{4-30}$$

and

$$\sup_{j\geq j_1}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}F_j^1(\tau)\,\mathrm{d}\tau \leq \frac{C\|u\|_{L_t^\infty \dot{C}^\delta}}{2^{j_1(\delta-(1-\alpha+\sigma))/2}}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta\|_{B^{s_0+s}_{\infty,\infty}}\Big)(t-t_0)^{-\frac{s}{\alpha-\sigma}} \tag{4-31}$$

if $0<s_0+s\leq 1$, and

$$\sup_{j\geq j_1}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}F_j^1(\tau)\,\mathrm{d}\tau \leq \frac{C\|u\|_{L_t^\infty \dot{C}^\delta}}{2^{j_1(\delta-(s_0+s-\alpha+\sigma))}}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta\|_{B^{s_0+s}_{\infty,\infty}}\Big)(t-t_0)^{-\frac{s}{\alpha-\sigma}} \tag{4-32}$$

if $1<s_0+s<\delta+\alpha-\sigma$. For the term including F_j^2 in (4-29), by using (3-12) again and the fact that $s_0\in(1-\delta,\alpha-\sigma)$, we get that for all $s\in(0,\alpha-\sigma)$,

$$\begin{aligned}
&\sup_{j\geq j_1}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}2^{j(s_0+s)}F_j^2(\tau)\mathrm{d}\tau \\
&= C\sup_{j\geq j_1}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|u(\tau)\|_{\dot{C}^\delta}2^{j(s_0+s)}\Big(\sum_{k\geq j-3}2^{k(1-\delta)}\|\Delta_k\theta(\tau)\|_{L^\infty}\Big)\mathrm{d}\tau \\
&\leq C\|u\|_{L_t^\infty\dot{C}^\delta}\sup_{j\geq j_1}2^{j(s_0+s)}\Big(\sum_{k\geq j-3}2^{k(1-\delta-s_0-s)}\Big)\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}\|\theta(\tau)\|_{B^{s_0+s}_{\infty,\infty}}\mathrm{d}\tau \\
&\leq C\|u\|_{L_t^\infty\dot{C}^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{\infty,\infty}}\Big) \\
&\qquad\qquad \sup_{j\geq j_1}2^{j(1-\delta)}\int_{t_0}^{t} e^{-\frac{c}{2}(t-\tau)2^{j(\alpha-\sigma)}}(\tau-t_0)^{-\frac{s}{\alpha-\sigma}}\mathrm{d}\tau \\
&\leq C\|u\|_{L_t^\infty\dot{C}^\delta}\Big(\sup_{\tau\in(t_0,t]}(\tau-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(\tau)\|_{B^{s_0+s}_{\infty,\infty}}\Big)(t-t_0)^{-\frac{s}{\alpha-\sigma}}2^{-j_1(\delta-(1-\alpha+\sigma))}.
\end{aligned} \tag{4-33}$$

Plugging the estimates (4-30)–(4-33) into (4-29), and in a similar way as obtaining (4-23), we have that for every $t\in(t_0,T]$, $s\in(0,\alpha-\sigma)$ and $s_0+s<\delta+\alpha-\sigma$,

$$\begin{aligned}
(t-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(t)\|_{B^{s_0+s}_{\infty,\infty}} &\leq CT^{\frac{s}{\alpha-\sigma}}(\|\theta_0\|_{L^\infty}+\|f\|_{L^1_TL^\infty})2^{j_1(s_0+s)} \\
&\quad + C_{\alpha,\sigma,s}\|\theta(t_0)\|_{B^{s_0}_{\infty,\infty}} + CT^{\frac{s}{\alpha-\sigma}}\|f\|_{L_t^\infty\dot{C}^\delta} \\
&\quad + \text{additional term}
\end{aligned} \tag{4-34}$$

where the additional term is given by

$$\frac{C\|u\|_{L_T^\infty\dot{C}^\delta}}{2^{j_1(\delta-(1-\alpha+\sigma))/2}}\Big(\sup_{t\in(t_0,T]}(t-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(t)\|_{B^{s_0+s}_{\infty,\infty}}\Big)$$

if $s_0+s\le 1$, and

$$\frac{C\|u\|_{L^\infty_T\dot C^\delta}}{2^{j_1(\delta+\alpha-\sigma-(s_0+s))}}\Big(\sup_{t\in(t_0,T]}(t-t_0)^{\frac{s}{\alpha-\sigma}}\|\theta(t)\|_{B^{s_0+s}_{\infty,\infty}}\Big)$$

if $1<s_0+s<\delta+\alpha-\sigma$. Hence we choose $j_1\in\mathbb{N}$ as

$$j_1=\max\Big\{\Big[\frac{2\log_2 2C\|u\|_{L^\infty_T\dot C^\delta}}{\delta-(1-\alpha+\sigma)}\Big],\Big[\frac{\log_2(2C_2/c)}{\alpha-\sigma}\Big],4\Big\}+1 \tag{4-35}$$

if $s_0+s\le 1$, and

$$j_1=\max\Big\{\Big[\frac{\log_2 2C\|u\|_{L^\infty_T\dot C^\delta}}{\delta+\alpha-\sigma-(s_0+s)}\Big],\Big[\frac{\log_2(2C_2/c)}{\alpha-\sigma}\Big],4\Big\}+1 \tag{4-36}$$

if $1<s_0+s<\delta+\alpha-\sigma$. We thus find that for all $s\in(0,\alpha-\sigma)$ and $s_0+s<\delta+\alpha-\sigma$,

$$\begin{aligned}\sup_{t\in(t_0,T]}((t-t_0)^{s/(\alpha-\sigma)}\|\theta(t)\|_{B^{s_0+s}_{\infty,\infty}})\le C(T+1)(\|\theta_0\|_{L^\infty}+\|f\|_{L^1_TL^\infty})2^{j_1(s_0+s)}\\ +C\|\theta(t_0)\|_{B^{s_0}_{\infty,\infty}}+C(T+1)\|f\|_{L^\infty_T\dot C^\delta},\end{aligned} \tag{4-37}$$

which specially guarantees that for any $t_1>t_0>0$ (which may be arbitrarily close to t_0) and every $s_0\in(1-\delta,\alpha-\sigma)$, $s_1\in(0,\alpha-\sigma)$ satisfying $s_0+s_1<\delta+\alpha-\sigma$,

$$\begin{aligned}&\sup_{t\in[t_1,T]}\|\theta(t)\|_{B^{s_0+s_1}_{\infty,\infty}}\\ &\le C(t_1-t_0)^{-\frac{s_1}{\alpha-\sigma}}((T+1)(\|\theta_0\|_{L^\infty}+\|f\|_{L^1_TL^\infty})2^{j_1(s_0+s_1)}+\|\theta(t_0)\|_{B^{s_0}_{\infty,\infty}})\\ &\qquad+C(t_1-t_0)^{-\frac{s_1}{\alpha-\sigma}}(T+1)\|f\|_{L^\infty_T\dot C^\delta},\end{aligned} \tag{4-38}$$

with j_1 given by (4-35)–(4-36).

Step 3:the estimation of $\|\theta\|_{L^\infty([\tilde t,T];C^{1,\gamma})}$ for some $\gamma>0$ and any $\tilde t\in(0,T)$.

If $\alpha-\sigma\in\big(\frac12,1\big)$, we can select appropriate $s_0\in(1-\delta,\alpha-\sigma)$, $s_1\in(0,\alpha-\sigma)$ so that $1<s_0+s_1<\delta+\alpha-\sigma$, thus from (4-38) we obtain that for $\gamma=s_0+s_1-1>0$,

$$\sup_{t\in[t_1,T]}\|\theta(t)\|_{C^{1,\gamma}}\approx\sup_{t\in[t_1,T]}\|\theta(t)\|_{B^{s_0+s_1}_{\infty,\infty}}\le C,$$

with C the bound on the right-hand side of (4-38).

For the remained scope $\alpha-\sigma\in(0,\frac12]$, we have to iterate the above procedure in Step 2 for more times. Assume that for some small number $t_k>0$, $k\in\mathbb{N}$, we have a finite bound on $\|\theta(t_k)\|_{B^{s_0+s_1+\cdots+s_k}_{\infty,\infty}}$ with $s_0\in(1-\delta,\alpha-\sigma)$, $s_1,\dots,s_k\in(0,\alpha-\sigma)$ satisfying $s_0+s_1+\cdots+s_k\le 1$, then by arguing as (4-38), we infer that for any

$t_{k+1} > t_k,\ s_{k+1} \in (1-\delta, \alpha-\sigma)$ satisfying $s_0 + s_1 + \cdots + s_{k+1} < \delta + \alpha - \sigma$,

$$\sup_{t\in[t_{k+1},T]} \|\theta(t)\|_{B^{s_0+s_1+\cdots+s_{k+1}}_{\infty,\infty}} \le C(t_{k+1}-t_k)^{-\frac{s_{k+1}}{\alpha-\sigma}} \Big((T+1)(\|\theta_0\|_{L^\infty} + \|f\|_{L^1_T L^\infty})2^{j_1(\sum_{i=0}^{k+1} s_i)} + \|\theta(t_k)\|_{B^{\sum_{i=0}^{k} s_i}_{\infty,\infty}}\Big) + C(t_{k+1}-t_k)^{-\frac{s_{k+1}}{\alpha-\sigma}}(T+1)\|f\|_{L^\infty_T \dot{C}^\delta}, \tag{4-39}$$

where j_1 is also given by (4-35)–(4-36) with s_0+s_1 replaced by $s_0+s_1+\cdots+s_{k+1}$. Hence if $\alpha-\sigma \in (1/(k+2), 1/(k+1)]$, $k \in \mathbb{N}^+$, we can choose appropriate numbers $s_0, s_1, \ldots, s_{k+1} \in (1-\delta, \alpha-\sigma)$ so that $1 < s_0 + s_1 + \cdots + s_{k+1} < \delta+\alpha-\sigma$, and by repeating the above process for $(k+1)$-times, we deduce that for $\gamma = s_0 + s_1 + \cdots + s_{k+1} - 1 > 0$,

$$\sup_{t\in[t_{k+1},T]} \|\theta(t)\|_{C^{1,\gamma}} \approx \sup_{t\in[t_{k+1},T]} \|\theta(t)\|_{B^{s_0+s_1+\cdots+s_{k+1}}_{\infty,\infty}} \le C\bigg(\prod_{i=0}^{k}(t_{i+1}-t_i)^{-\frac{s_{i+1}}{\alpha-\sigma}} t_0^{-\frac{s_0}{\alpha-\sigma}}\bigg)(\|\theta_0\|_{L^\infty} + \|f\|_{L^\infty_T C^\delta}), \tag{4-40}$$

with C a finite constant depending on α, σ, δ, T, d and $\|u\|_{L^\infty_T \dot{C}^\delta}$.

Hence for every $\alpha \in (0,1]$, $\sigma \in [0,\alpha)$, and for any $\tilde{t} \in (0,T)$, there is some $k \in \mathbb{N}$ so that $\alpha-\sigma \in (1/(k+2), 1/(k+1)]$, and we can choose $t_i = (i+1)/(k+2)\tilde{t}$ for $i = 0, 1, \ldots, k+1$ and appropriate numbers $s_0 \in (1-\delta, \alpha-\sigma)$, $s_1, \ldots, s_{k+1} \in (0, \alpha-\sigma)$ such that $1 < s_0 + s_1 + \cdots + s_{k+1} < \delta + \alpha - \sigma$, thus from (4-40) we deduce that for some $\gamma > 0$,

$$\|\theta\|_{L^\infty([\tilde{t},T];C^{1,\gamma}(\mathbb{R}^d))} \le C\tilde{t}^{-(\gamma+1)/(\alpha-\sigma)}(\|\theta_0\|_{L^\infty} + \|f\|_{L^\infty_T C^\delta}), \tag{4-41}$$

with the constant C depending only on α, σ, δ, T, d and $\|u\|_{L^\infty_T \dot{C}^\delta}$.

Step 4:the estimation of $\|\theta\|_{L^\infty([\tilde{t},T];C^{1,\gamma})}$ for any $\gamma \in (0, \delta+\alpha-\sigma-1)$ and any $t' \in (0,T)$.

After obtaining the estimate of $\|\theta\|_{L^\infty([t'/2,T];B^{\tilde{s}}_{\infty,\infty})}$ with some $1 < \tilde{s} < 1+\gamma$ for any $\gamma \in (0, \delta+\alpha-\sigma-1)$, we can repeat the deduction in Steps 1–2 for several times and due to the increment of regularity index s at each time belonging to $(0, \alpha-\sigma)$, we can derive an upper bound of $\|\theta\|_{L^\infty([t',T];B^{1+\gamma}_{\infty,\infty})}$ by establishing (4-41) with t' in place of $\tilde{t}$.

4C. ***Uniform-in-ϵ differentiability estimates of the regularized system.*** We consider the approximate system

$$\begin{cases} \partial_t\theta + (u_\epsilon\cdot\nabla)\theta + \mathcal{L}\theta - \epsilon\Delta\theta = f_\epsilon, \\ u_\epsilon := \phi_\epsilon * u, \quad f_\epsilon := \phi_\epsilon * f, \\ \theta|_{t=0} = \theta_{0,\epsilon} := \phi_\epsilon * (\theta_0 1_{B_{1/\epsilon}(0)}). \end{cases} \tag{4-42}$$

Here $1_\Omega(x)$ is the standard indicator function on the set Ω and $\phi_\epsilon = \epsilon^{-d}\phi(\epsilon^{-1}x) \in C_c^\infty(\mathbb{R}^d)$ is the function introduced in Section 3C.

Due to $\theta_0 \in C_0(\mathbb{R}^d)$, we see that $\theta_{0,\epsilon} = \phi_\epsilon * (\theta_0 1_{B_{1/\epsilon}(0)})$ is smooth for every $\epsilon > 0$, and $\|\theta_{0,\epsilon}\|_{H^s(\mathbb{R}^d)} \lesssim_\epsilon \|\theta_0\|_{L^\infty(\mathbb{R}^d)}$ for all $s \geq 0$. Similarly from $u \in L^\infty([0,T]; C^\delta(\mathbb{R}^d))$ and $f(t) \in C^\delta \cap L^2(\mathbb{R}^d)$ for every $t \in [0,T]$, we get $u_\epsilon \in L^\infty([0,T]; C^s(\mathbb{R}^d))$ for all $s \geq \delta$ and $f_\epsilon \in L^\infty([0,T]; H^s(\mathbb{R}^d))$ for all $s \geq 0$. Hence, for every $\epsilon > 0$, by the classical method (e.g., [Miao and Xue 2015, Proposition 7.1]), we obtain an approximate solution $\theta^{(\epsilon)} \in C([0,T]; H^s(\mathbb{R}^d)) \cap C^1((0,T]; C_b^\infty(\mathbb{R}^d))$, $s > d/2+1$ for the system (4-42).

Since we have the uniform-in-ϵ estimates that $\|\theta_{0,\epsilon}\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}$, $\|u_\epsilon\|_{L_T^\infty C^\delta} \leq \|u\|_{L_T^\infty C^\delta}$ and $\|f_\epsilon\|_{L_T^\infty C^\delta} \leq \|f\|_{L_T^\infty C^\delta}$, we consider the equation of $\theta^{(\epsilon)}$ and by arguing as (4-41) and Step 4 in the above subsection, we can derive the uniform-in-ϵ estimate of $\|\theta^{(\epsilon)}\|_{L^\infty([t',T]; C^{1,\gamma}(\mathbb{R}^d))}$ with any $\gamma \in (0, \delta+\alpha-\sigma-1)$ and $t' \in (0,T)$.

Therefore, we have finished the proof of Theorem 1.2.

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