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
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## THREE-DIMENSIONAL SOL MANIFOLDS AND COMPLEX KLEINIAN GROUPS

WALDEMAR BARRERA, RENE GARCIA-LARA AND JUAN NAVARRETE

**We give a topological description of the quotient space  $\Omega(G)/G$ , in the case when  $G \subset \mathrm{PSL}(3, \mathbb{C})$  is a discrete subgroup acting on  $\mathbb{P}_{\mathbb{C}}^2$  and the maximum number of complex projective lines in general position contained in the Kulkarni limit set  $\Lambda(G) = \mathbb{P}_{\mathbb{C}}^2 \setminus \Omega(G)$  is equal to 4. Moreover, we give a topological description of the quotient space  $\Omega(G)/G$  in the case when  $G$  is a lattice of the Heisenberg group.**

### 1. Introduction

Complex Kleinian groups were introduced by José Seade and Alberto Verjovsky [2001]. A complex Kleinian group  $G$  is a subgroup of  $\mathrm{PSL}(n+1, \mathbb{C})$  acting properly and discontinuously on a nonempty  $G$ -invariant open subset of  $\mathbb{P}_{\mathbb{C}}^n$ . We remark that complex Kleinian groups are discrete subgroups of  $\mathrm{PSL}(n+1, \mathbb{C})$  but the converse is not necessarily true; for example, the group  $\mathrm{PSL}(3, \mathbb{Z})$  is a discrete subgroup of  $\mathrm{PSL}(3, \mathbb{C})$  which is not a complex Kleinian group. See [Barrera et al. 2014].

There is no standard definition of limit set for a complex Kleinian group, we use the following three notions: Kulkarni limit set, Myrberg limit set, or the complement of a maximal region of discontinuity which are discussed in detail in [Barrera et al. 2016]. However by some additional hypotheses on the action of  $G$  on the projective plane, all these concepts of limit set are equivalent; see [Barrera et al. 2011a]. In the classical theory of Kleinian groups, there is a theorem which states that discrete infinite subgroups contain one, two, or infinity points in its limit set. On the other hand, Angel Cano and José Seade show that every infinite discrete subgroup of  $\mathrm{PSL}(3, \mathbb{C})$  has a complex projective line contained in its limit set (see [Cano and Seade 2014]), in consequence, the limit set of infinite subgroups of  $\mathrm{PSL}(3, \mathbb{C})$  is an uncountable subset of  $\mathbb{P}_{\mathbb{C}}^2$ .

Thus, it is natural to say that  $G \subset \mathrm{PSL}(3, \mathbb{C})$  is an *elementary complex Kleinian group* whenever its limit set contains a finite number of complex projective lines; see [Cano et al. 2013]. There is another kind of group whose limit set contains

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infinitely many complex projective lines but only finitely many in general position. We call these groups *elementary complex Kleinian groups of type II*.

In [Barrera et al. 2011b], the authors give an algebraic characterization of those complex Kleinian groups such that the maximum number of complex projective lines in general position contained in its Kulkarni limit set is equal to 4. In this article we describe the topology of the quotient space of these groups. In fact, we prove the following theorem:

**Theorem 1.1.** *Let  $G \subset \mathrm{PSL}(3, \mathbb{C})$  be a torsion-free complex Kleinian group, such that the maximum number of complex projective lines in general position contained in its Kulkarni limit set is equal to 4, then:*

- (i) *The group  $G$  is isomorphic to a lattice of the group  $\mathrm{Sol}$  (see Section 2F).*
- (ii) *If  $\Omega_0$  is a  $G$ -invariant connected component of the Kulkarni discontinuity region of  $G$ , then  $\Omega_0/G$  is diffeomorphic to  $(\mathrm{Sol}/G) \times \mathbb{R}$ .*

**Corollary 1.2.** *There is a countable number of nonisomorphic complex Kleinian groups such that the maximum number of complex projective lines in general position contained in its Kulkarni limit set is equal to 4.*

**Corollary 1.3.** *Under the hypotheses of Theorem 1.1,  $\Omega_0/G$  is a fiber bundle with base  $\mathbb{S}^1$  and fiber  $\mathbb{T}^2 \times \mathbb{R}$ .*

**Theorem 1.4.** *If  $G$  is a lattice on the three-dimensional real Heisenberg group  $\mathcal{H}$ , then there exists a  $G$ -invariant open set  $\Omega \subset \mathbb{P}_{\mathbb{C}}^2$  such that  $\Omega/G$  is diffeomorphic to  $(\mathcal{H}/G) \times \mathbb{R}$ .*

This article is organized in the following way: In Section 2 we include some basic preliminaries about complex Kleinian groups, and a brief survey on complex Kleinian groups, such that the maximum number of complex lines contained in their Kulkarni limit set is equal to 4 [Barrera et al. 2011b]. In Section 3 we give an explicit smooth foliation of the bidisc  $\mathbb{H} \times \mathbb{H}$  where the leaves are diffeomorphic copies of  $\mathrm{Sol}$ . In Section 4, we study the geometry of the leaves and we show that the bidisc  $\mathbb{H} \times \mathbb{H}$  is diffeomorphic to  $\mathrm{Sol} \times \mathbb{R}$ . Moreover this diffeomorphism is  $G$ -equivariant, where  $G$  denotes an hyperbolic toral group. In Section 5 we do some explicit computations to determine the Riemannian metrics of the leaves. For each leaf we obtain an isometric embedding of the group  $\mathrm{Sol}$  to  $\mathbb{H} \times \mathbb{H}$ . Finally, we give a proof of Theorem 1.1.

Corollaries 1.2 and 1.3 are a consequence of Theorem 1.1 and [de la Harpe 2000, Proposition 30]. In Section 6 we give a proof of Theorem 1.4, the procedure is similar to the proof of Theorem 1.1, except that we have not a  $G$ -equivariant diffeomorphism between  $\mathbb{C} \times \mathbb{H}$  and  $\mathcal{H} \times \mathbb{R}$ . However the proof can be done because the natural action of  $G$  on  $\mathbb{C} \times \mathbb{H}$  translated to  $\mathcal{H} \times \mathbb{R}$  is the classical action on the first factor of  $G$  on  $\mathcal{H}$ , and it is the trivial action on the second factor.

## 2. Preliminaries

The purpose of this section is to provide some definitions and results about complex Kleinian groups that will be helpful to the reader. For more details see [Cano et al. 2013; Barrera et al. 2011b; 2011a].

**2A. Projective geometry.** We recall that the complex projective plane  $\mathbb{P}_{\mathbb{C}}^2$  is defined as the orbit space of the usual scalar multiplication action of the Lie group  $\mathbb{C}^*$  in  $\mathbb{C}^3 \setminus \{\mathbf{0}\}$  and it is denoted by

$$\mathbb{P}_{\mathbb{C}}^2 := (\mathbb{C}^3 \setminus \{\mathbf{0}\})/\mathbb{C}^*.$$

This is a compact connected complex 2-dimensional manifold. Let  $[\cdot]: \mathbb{C}^3 \setminus \{\mathbf{0}\} \rightarrow \mathbb{P}_{\mathbb{C}}^2$  be the quotient map. If  $\beta = \{e_1, e_2, e_3\}$  is the standard basis of  $\mathbb{C}^3$ , we write  $[e_j] = e_j$ , for  $j = 1, 2, 3$ , and if  $\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$  then we write  $[\mathbf{z}] = [z_1 : z_2 : z_3]$ . Also,  $\ell \subset \mathbb{P}_{\mathbb{C}}^2$  is said to be a complex line if  $[\ell]^{-1} \cup \{\mathbf{0}\}$  is a complex linear subspace of dimension 2. Given two distinct points  $[\mathbf{z}], [\mathbf{w}] \in \mathbb{P}_{\mathbb{C}}^2$ , there is a unique complex projective line passing through  $[\mathbf{z}]$  and  $[\mathbf{w}]$ . This kind of complex projective line is called a *line*, for short, and it is denoted by  $\overleftrightarrow{[\mathbf{z}], [\mathbf{w}]}$ . Consider the action of  $\mathbb{C}^*$  on  $\text{GL}(3, \mathbb{C})$  given by the usual scalar multiplication, then

$$\text{PGL}(3, \mathbb{C}) = \text{GL}(3, \mathbb{C})/\mathbb{C}^*$$

is a Lie group whose elements are called projective transformations. Letting  $[[\cdot]]: \text{GL}(3, \mathbb{C}) \rightarrow \text{PGL}(3, \mathbb{C})$  be the quotient map,  $g \in \text{PGL}(3, \mathbb{C})$  and  $\mathbf{g} \in \text{GL}(3, \mathbb{C})$ , we say that  $\mathbf{g}$  is a lift of  $g$  if  $[[\mathbf{g}]] = g$ . One can show that  $\text{PGL}(3, \mathbb{C})$  is a Lie group which acts transitively, effectively and by biholomorphisms on  $\mathbb{P}_{\mathbb{C}}^2$  by  $[[\mathbf{g}]]([\mathbf{w}]) = [\mathbf{g}(\mathbf{w})]$ , where  $\mathbf{w} \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$  and  $\mathbf{g} \in \text{GL}(3, \mathbb{C})$ .

We could have considered the action of the cube roots of unity  $\{1, \omega, \omega^2\} \subset \mathbb{C}^*$  on  $\text{SL}(3, \mathbb{C})$  given by the usual scalar multiplication, then

$$\text{PSL}(3, \mathbb{C}) = \text{SL}(3, \mathbb{C})/\{1, \omega, \omega^2\} \cong \text{PGL}(3, \mathbb{C}).$$

We denote by  $\text{M}_{3 \times 3}(\mathbb{C})$  the space of all  $3 \times 3$  matrices with entries in  $\mathbb{C}$  equipped with the standard topology. The quotient space

$$\text{End}(3, \mathbb{C}) := (\text{M}_{3 \times 3}(\mathbb{C}) \setminus \{\mathbf{0}\})/\mathbb{C}^*$$

is called the space of *pseudoprojective maps* of  $\mathbb{P}_{\mathbb{C}}^2$  and it is naturally identified with the projective space  $\mathbb{P}_{\mathbb{C}}^8$ . Since  $\text{GL}(3, \mathbb{C})$  is an open, dense,  $\mathbb{C}^*$ -invariant set of  $\text{M}_{3 \times 3}(\mathbb{C}) \setminus \{\mathbf{0}\}$ , we get that the space of pseudoprojective maps of  $\mathbb{P}_{\mathbb{C}}^2$  is a compactification of  $\text{PGL}(3, \mathbb{C})$  (or  $\text{PSL}(3, \mathbb{C})$ ). As in the case of projective maps, if  $\mathbf{s}$  is an element in  $\text{M}_{3 \times 3}(\mathbb{C}) \setminus \{\mathbf{0}\}$ , then  $[\mathbf{s}]$  denotes the equivalence class of the matrix  $\mathbf{s}$  in the space of pseudoprojective maps of  $\mathbb{P}_{\mathbb{C}}^2$ . Also, we say that  $\mathbf{s} \in \text{M}_{3 \times 3}(\mathbb{C}) \setminus \{\mathbf{0}\}$  is a lift of the pseudoprojective map  $S$  whenever  $[\mathbf{s}] = S$ .

Let  $S$  be an element in  $(M_{3 \times 3}(\mathbb{C}) \setminus \{\mathbf{0}\})/\mathbb{C}^*$  and  $s$  a lift to  $M_{3 \times 3}(\mathbb{C}) \setminus \{\mathbf{0}\}$  of  $S$ . The matrix  $s$  induces a nonzero linear transformation  $s : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ , which is not necessarily invertible. Let  $\text{Ker}(s) \subsetneq \mathbb{C}^3$  be its kernel and let  $\text{Ker}(S)$  denote its projectivization to  $\mathbb{P}_{\mathbb{C}}^2$ , taking into account that  $\text{Ker}(S) := \emptyset$  whenever  $\text{Ker}(s) = \{(0, 0, 0)\}$ .

## 2B. Discontinuous actions on $\mathbb{P}_{\mathbb{C}}^2$ .

**Definition 2.1.** Let  $G \subset \text{PSL}(3, \mathbb{C})$  be a group. We say that  $G$  is a *complex Kleinian group* if it acts properly and discontinuously on an open nonempty  $G$ -invariant set  $U \subset \mathbb{P}_{\mathbb{C}}^2$ , meaning that for each pair of compact subsets  $C, D \subset U$ , the set

$$\{g \in G : g(C) \cap D \neq \emptyset\}$$

is *finite*.

One of the main difficulties in deciding whether a group  $G$  is Kleinian complex is to find an open set verifying the definition above. In order to give an answer to this problem we study two mathematical concepts: the equicontinuity set of  $G$  and the Kulkarni discontinuity region of  $G$ . Now, we discuss each of these concepts.

**2C. The equicontinuity set.** The concept of equicontinuity has long been studied in mathematics. For convenience to the reader, we include the definition and notation that we use in this work.

**Definition 2.2.** The *equicontinuity set* for a family  $\mathcal{F}$  of endomorphisms of  $\mathbb{P}_{\mathbb{C}}^2$ , denoted  $\text{Eq}(\mathcal{F})$ , is defined as the set of points  $z \in \mathbb{P}_{\mathbb{C}}^2$  for which there is an open neighborhood  $U$  of  $z$  such that  $\{f|_U : f \in \mathcal{F}\}$  is a normal family.

This modern approach and ideas on this concept were studied by Angel Cano in his Ph.D thesis. However, thanks to a reference by Ravi Kulkarni to works of Myrberg, we found that some of these results had already been discovered, in an arcane mathematical language. However, it is fair to acknowledge Angel Cano for rediscovering these results and applying them successfully to the theory of complex Kleinian groups.

**Definition 2.3.** Let  $G \subset \text{PSL}(3, \mathbb{C})$  be a discrete group. If

$$G' = \{S \text{ is a pseudoprojective map of } \mathbb{P}_{\mathbb{C}}^2 : S \text{ is a cluster point of } G\},$$

then the *Myrberg limit set* (see [Myrberg 1925]) is defined as the set

$$\Lambda_{\text{Myr}}(G) = \bigcup_{S \in G'} \text{Ker}(S).$$

Myrberg [1925] shows that  $G$  acts properly and discontinuously on  $\mathbb{P}_{\mathbb{C}}^2 \setminus \Lambda_{\text{Myr}}(G)$ .

**Theorem 2.4** [Barrera et al. 2011a]. *If  $G \subset \text{PSL}(3, \mathbb{C})$  is a discrete group, then:*

- (i) *The group  $G$  acts properly and discontinuously on  $\text{Eq}(G)$ .*

(ii) *The equicontinuity set of  $G$  satisfies:*

$$\text{Eq}(G) = \mathbb{P}_{\mathbb{C}}^2 \setminus \Lambda_{\text{Myr}}(G)$$

(iii) *If  $U$  is an open  $G$ -invariant subset such that  $\mathbb{P}_{\mathbb{C}}^2 \setminus U$  contains at least three complex lines in general position, then  $U \subset \text{Eq}(G)$ .*

**2D. Kulkarni discontinuity region.** Kulkarni [1978] defined a limit set for groups of homeomorphisms acting on a locally compact Hausdorff space. For the reader's convenience, we explain this construction in the context of projective spaces.

**Definition 2.5.** Let  $G \subset \text{PSL}(3, \mathbb{C})$  be a subgroup.

- The set  $L_0(G)$  is defined as the closure of the set of points in  $\mathbb{P}_{\mathbb{C}}^2$  with infinite isotropy group.
- The set  $L_1(G)$  is defined as the closure of the set of cluster points of the orbit  $Gz$ , where  $z$  runs over  $\mathbb{P}_{\mathbb{C}}^2 \setminus L_0(G)$ .
- The set  $L_2(G)$  is defined as the closure of the set of cluster points of the family of compact sets  $\{g(K) : g \in G\}$ , where  $K$  runs over all the compact subsets of

$$\mathbb{P}_{\mathbb{C}}^2 \setminus (L_0(G) \cup L_1(G)).$$

The *Kulkarni limit set* of  $G$  is defined as

$$\Lambda_{\text{Kul}}(G) = L_0(G) \cup L_1(G) \cup L_2(G).$$

The *Kulkarni discontinuity region* of  $G$  is defined as:

$$\Omega(G) = \mathbb{P}_{\mathbb{C}}^2 \setminus \Lambda_{\text{Kul}}(G).$$

Kulkarni [1978] proves that  $G$  acts properly and discontinuously on the set  $\Omega(G)$ . However,  $\Omega(G)$  is not necessarily the maximal open subset of  $\mathbb{P}_{\mathbb{C}}^2$  where  $G$  acts properly and discontinuously.

We notice that the Kulkarni limit set is a generalization of the classical limit set for a discrete subgroup of hyperbolic isometries acting on the sphere at infinity of hyperbolic space. In general, it is very hard to give an explicit computation of the Kulkarni limit set. In [Navarrete 2006; 2008], we can find these computations for the cyclic subgroups of  $\text{PSL}(3, \mathbb{C})$  and for discrete subgroups of  $\text{PU}(2, 1)$  acting on the complex projective plane  $\mathbb{P}_{\mathbb{C}}^2$ .

We could define the limit set as the complement of a maximal open set where the group acts properly and discontinuously, but in general there is no canonical way to build this  $G$ -invariant open set. On the other hand, when we ensure the existence of this maximal open set, this notion of limit set has good properties. See [Barrera et al. 2014].

**2E. Four lines complex Kleinian groups.** This section is devoted to complex Kleinian groups of  $\mathrm{PSL}(3, \mathbb{C})$  such that the maximum number of complex projective lines in general position contained in its Kulkarni limit set is equal to 4. For simplicity we call groups of this kind *four lines complex Kleinian groups*. In [Barrera et al. 2011b], the authors give an algebraic characterization of four lines complex Kleinian groups. For the reader's convenience, we reproduce briefly the main ideas and the notation used there.

Letting  $A \in \mathrm{SL}(2, \mathbb{Z})$ , with  $|\mathrm{tr}(A)| > 2$ , we define the following discrete subgroup of  $\mathrm{PSL}(3, \mathbb{C})$ , called *hyperbolic toral group*.

$$G_A = \left\{ \begin{pmatrix} A^k & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} : \mathbf{b} \in M(2 \times 1, \mathbb{Z}), k \in \mathbb{Z} \right\},$$

The group  $G_A$  is a four lines complex Kleinian group and moreover if  $G$  is a four lines complex Kleinian group, then there exists a hyperbolic toral group  $G_A$  such that  $[G : G_A] \leq 8$ .

It is possible to conjugate the group  $G_A$  to a group, still denoted by  $G_A$ , where each element is of the form

$$\begin{pmatrix} \lambda^k & 0 & ny_0 + mx_0 \\ 0 & \lambda^{-k} & nx_0 + mz_0 \\ 0 & 0 & 1 \end{pmatrix}$$

where  $k, m$  and  $n$  run over  $\mathbb{Z}$  and  $\lambda$  is one of the eigenvalues of  $A$ . At this point it is not hard to see that the Kulkarni discontinuity region consists of four disjoint copies of  $\mathbb{H}^\pm \times \mathbb{H}^\pm$ , where  $\mathbb{H}^+$  is the upper half plane and  $\mathbb{H}^-$  is the lower half plane.

**2F. The Sol geometries.** Sol is one of the eight geometries defined by William Thurston in his famous program of geometrization of compact three manifolds. The group Sol is defined as the space  $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$  equipped with the group operation

$$\left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, t_1 \right) \cdot \left( \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, t_2 \right) = \left( \begin{pmatrix} x_1 + e^{t_1} x_2 \\ y_1 + e^{-t_1} y_2 \end{pmatrix}, t_1 + t_2 \right).$$

In fact, it is a Lie group and it is equipped with the left-invariant Riemannian metric:  $ds^2 = e^{2t} dx^2 + e^{-2t} dy^2 + dt^2$ . An interesting fact about the Sol geometries is given by the following theorem of [de la Harpe 2000], which we state for convenience:

**Proposition 2.6.** *Let  $A, B$  in  $\mathrm{GL}(2, \mathbb{Z})$  be two matrices with traces of absolute value strictly larger than 2. The semidirect products  $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ , and  $\mathbb{Z}^2 \rtimes_B \mathbb{Z}$  considered as the matrix groups*

$$\left\{ \begin{pmatrix} A^k & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} : \mathbf{b} \in M(2 \times 1, \mathbb{Z}), k \in \mathbb{Z} \right\},$$



$$\left\{ \begin{pmatrix} B^k & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix} : \mathbf{b} \in M(2 \times 1, \mathbb{Z}), k \in \mathbb{Z} \right\}$$

are isomorphic if and only if  $A$  is conjugate in  $\mathrm{GL}(2, \mathbb{Z})$  to  $B$  or  $B^{-1}$ , and they are quasi-isometric in all cases.

The quotient spaces  $\mathrm{Sol}/(\mathbb{Z}^2 \rtimes_A \mathbb{Z})$  are examples of compact three manifolds where the topological type is determined by the fundamental group. For more details about this subject see [Scott 1983; Thurston 1997; de la Harpe 2000].

### 3. Foliation of $\mathbb{H} \times \mathbb{H}$ by $\mathrm{Sol}$

In [Barrera et al. 2011b], the authors introduce the concept of hyperbolic toral groups. These groups are matrix groups where the elements are given by

$$\begin{pmatrix} \lambda^k & 0 & ny_0 + mx_0 \\ 0 & \lambda^{-k} & nx_0 + mz_0 \\ 0 & 0 & 1 \end{pmatrix},$$

where  $\lambda$  is a fixed real number,  $|\lambda| \neq 1$  and  $k, n, m$  run over  $\mathbb{Z}$ . It is not hard to check this group is isomorphic to  $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ . Moreover, a continuous version of this group is given in the following way. Let  $A \in \mathrm{SL}(2, \mathbb{Z})$  be a hyperbolic automorphism with Jordan form

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

and consider the set of matrices of the form

$$\begin{pmatrix} \lambda^t & 0 & x \\ 0 & \lambda^{-t} & y \\ 0 & 0 & 1 \end{pmatrix},$$

where  $t, x, y$  run over  $\mathbb{R}$ . It is not hard to check that this group is isomorphic to  $\mathrm{Sol} = \mathbb{R}^2 \rtimes \mathbb{R}$ . For convenience for our computations we use this representation of the Lie group  $\mathrm{Sol}$ . In the sequel, we will use the product metric in  $\mathbb{H} \times \mathbb{H}$ , where we endow each copy of  $\mathbb{H}$  with a metric homothetic to the hyperbolic metric by a factor of  $\frac{1}{2}$ :

$$\frac{dx_1^2 + dy_1^2}{2y_1^2} + \frac{dx_2^2 + dy_2^2}{2y_2^2}.$$

**Proposition 3.1.** *Let  $z_1, z_2 \in \mathbb{H}$ . We define a natural action of  $\mathrm{Sol}$  in  $\mathbb{H} \times \mathbb{H}$  by*

$$\begin{pmatrix} \lambda^t & 0 & x \\ 0 & \lambda^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda^t z_1 + x \\ \lambda^{-t} z_2 + y \\ 1 \end{pmatrix}.$$

The natural action of the group  $\text{Sol}$  on  $\mathbb{H} \times \mathbb{H}$  satisfies the following:

- (i) The action is free.
- (ii) For each  $z = (z_1, z_2) \in \mathbb{H} \times \mathbb{H}$  the function  $f_z : \text{Sol} \rightarrow \mathbb{H} \times \mathbb{H}$  defined by  $f_z(g) = gz$  is a smooth embedding.
- (iii) If  $\{e_1, e_2, e_3, e_4\}$  is the canonical basis of  $\mathbb{R}^4$ , then

$$X = y_1 e_2 + y_2 e_4$$

is one of the two normal unitary vector fields to the embedding  $f_z(\text{Sol})$  in  $\mathbb{H} \times \mathbb{H}$ , which therefore is a smooth vector field globally defined.

*Proof.* (i) Let  $z = (z_1, z_2) \in \mathbb{H} \times \mathbb{H}$ , and assume that  $\gamma \in \text{Sol}$  is such that  $\gamma \cdot z = z$ . Let  $z_k = x_k + iy_k$ , where  $k = 1, 2$ . Taking imaginary parts in the action, we get

$$\lambda^t y_1 = y_1, \quad \lambda^{-t} y_2 = y_2,$$

so  $\lambda^t = \lambda^{-t} = 1$ , because imaginary parts can not be null. Taking real parts, we obtain  $x_1 + x = x_1$  and  $x_2 + y = x_2$ , then  $x = y = 0$ .

(ii) From the definition of the action, it is clear that  $f_z$  is smooth in  $t, x, y$ , which parametrizes  $\text{Sol}$ . By straightforward computations, we have that  $df_z$  has the Jacobian matrix given by

$$[df_z] = \begin{pmatrix} \ln(\lambda) \lambda^t x_1 & 1 & 0 \\ \ln(\lambda) \lambda^t y_1 & 0 & 0 \\ -\ln(\lambda) \lambda^{-t} x_2 & 0 & 1 \\ -\ln(\lambda) \lambda^{-t} y_2 & 0 & 0 \end{pmatrix}.$$

Since  $y_1 > 0$ , the Jacobian matrix has rank 3. Therefore,  $f_z$  is an immersion.

If  $z_k = x_y + iy_k$ , and  $z' = (x'_1, y'_1, x'_2, y'_2) \in \mathbb{H} \times \mathbb{H}$ , define  $t$  such that  $\lambda^t = y'_1/y_1$ , and  $(x', y')$  such that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x'_1 - \frac{y'_1 x_1}{y_1} \\ x'_2 - \frac{y'_2 y_2}{y'_1} \end{pmatrix}.$$

These values for  $(t, x', y')$  define a mapping  $F$ , from  $\mathbb{H} \times \mathbb{H}$  to  $\mathbb{R}^3 \cong \text{Sol}$ , such that  $F \circ f_z = \text{Id}$ . Note that  $F$  is a left continuous inverse for  $f_z$ , and hence,  $f_z$  is an homeomorphism.

(iii) The given formula for the product metric implies that  $X$  is unitary. By the form of the Jacobian matrix, the tangent space to the leaf passing through  $z = (z_1, z_2) \in \mathbb{H} \times \mathbb{H}$  is spanned by the vectors,  $e_1, e_3, \lambda^t x_1 e_1 + \lambda^t y_1 e_2 - \lambda^{-t} x_2 e_3 - \lambda^{-t} y_2 e_4$ .

A straightforward computation shows that  $X = \lambda^t y_1 e_2 + \lambda^{-t} y_2 e_4$  is orthogonal to the spanning tangent vectors. Finally, the result is obtained by taking  $t = 0$ .  $\square$

By this theorem, if we vary  $z$ , we obtain a foliation  $f_z(\text{Sol})$  of  $\mathbb{H} \times \mathbb{H}$  by copies of Sol. We proceed to show that this foliation is globally rectifiable, in the sense that it induces a diffeomorphism to  $\mathbb{R}^3 \times \mathbb{R}$ , such that the hyperplanes  $\mathbb{R}^3 \times \{t\}$  correspond to the leaves, and are diffeomorphic to Sol. For details on the theory of foliations, the reader can consult [Candel and Conlon 2000].

#### 4. Geometry of the leaves

In the previous section, we described how Sol induces a foliation in the space  $\mathbb{H} \times \mathbb{H}$  and gave an explicit formula for a smooth vector field  $X$ , normal to any leaf of the foliation in a product metric, which is homothetic to the canonical metric in each hyperbolic factor. In this section, we study the dynamics of the integral curves for this normal field. Let

$$\psi(t) = (z_1(t), z_2(t))$$

be an integral curve of the field, where  $z_k = x_k + iy_k$  is as before. From the definition of  $X$ , it follows that the integral curves satisfy the set of equations

$$\dot{x}_k = 0, \quad \dot{y}_k = y_k.$$

These equations can be readily solved to get constant solutions in the real part of each copy of the hyperbolic, and exponentials in the imaginary parts. The flow of the normal field defines a one-parameter family of diffeomorphisms in  $\mathbb{H} \times \mathbb{H}$ , denoted by  $\psi_t(z_1, z_2)$ , where

$$\psi_t(z_1, z_2) = (x_1, e^t y_1, x_2, e^t y_2), \quad t \in \mathbb{R},$$

and it satisfies the following:

**Proposition 4.1.** (i) *The flow  $\psi_t$  rules  $\mathbb{H} \times \mathbb{H}$  by geodesics.*

(ii) *The action of  $f_z$  is **equivariant** with the action induced by the flow, that is,*

$$\psi_s \circ f_z = f_{\psi_s(z)}.$$

*Proof.* (i) Both curves

$$(x_1, e^t y_1) \quad \text{and} \quad (x_2, e^t y_2)$$

correspond to a parametrization of a vertical geodesic in  $\mathbb{H}$  with respect to the hyperbolic metric. Since the metric we consider is homothetic to the standard hyperbolic metric, with a constant factor, these parametrizations correspond to geodesics with respect to this metric as well. Since the metric in  $\mathbb{H} \times \mathbb{H}$  is a product, the result follows (see [Gallot et al. 2004]).

(ii)  $\psi_s \circ f_z(t, x, y) = (\lambda^t x_1 + x, \lambda^t e^s y_1, \lambda^{-t} x_2 + y, \lambda^{-t} e^s y_2),$

which is the same expression obtained calculating  $f_{\psi_s(z)}$ . □

Observe that if we reparametrize the presentation of  $\text{Sol}$  we have used so far by the change of coordinates

$$(t', x', y') \mapsto \left( \frac{t'}{\ln(\lambda)}, x', y' \right),$$

we recover the original description of the group as given in [Scott 1983], and that under this reparametrization, we can analyze the geometry of the foliation in simpler terms, i.e., we can and will assume that

$$\psi_s \circ f_z(t, x, y) = (e^t x_1 + x, e^{t+s} y_1, e^{-t} x_2 + y, e^{-t+s} y_2)$$

in order to analyze the metric properties of the foliation.

**Proposition 4.2.** *Let  $z = (iy_1, iy_2)$ . Consider the leaf  $f_z : \text{Sol} \rightarrow \mathbb{H} \times \mathbb{H}$ , then the pullback metric is*

$$dt^2 + \frac{e^{-2t}}{2y_1^2} dx^2 + \frac{e^{2t}}{2y_2^2} dy^2.$$

*In particular, if  $y_1 = y_2 = 1/\sqrt{2}$ ,  $f_z$  is an isometric embedding of  $\text{Sol}$  into  $\mathbb{H} \times \mathbb{H}$ .*

*Proof.* We have

$$f_z(t, x, y) = (x, e^t y_1, y, e^{-t} y_2).$$

Therefore, the Jacobian matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ e^t y_1 & 0 & 0 \\ 0 & 0 & 1 \\ -e^{-t} y_2 & 0 & 0 \end{pmatrix}.$$

Applying the product metric to the basis vectors  $e^t y_1 e_2 - e^{-t} y_2 e_4$ ,  $e_1$ ,  $e_3$  we get the result.  $\square$

In the sequel, unless otherwise established,  $z_0$  will denote the special point  $1/\sqrt{2}(i, i)$ .

**Corollary 4.3.** *The leaves  $f_{\psi_s(z_0)} : \text{Sol} \rightarrow \mathbb{H} \times \mathbb{H}$  can be identified with  $\text{Sol}$ , up to a homothety in the direction spanned by the  $x, y$  coordinates.*

*Proof.* We have

$$\psi_s \circ f_{z_0}(t, x, y) = \left( x, \frac{1}{\sqrt{2}} e^{t+s}, y, \frac{1}{\sqrt{2}} e^{-t+s} \right).$$

If we pullback the induced metric to  $\text{Sol}$ , we get

$$dt^2 + e^{-2(t+s)} dx^2 + e^{2(t-s)} dy^2.$$

Define  $F_s : \text{Sol} \rightarrow \text{Sol}$  by  $F_s(t, x, y) = (t, e^s x, e^s y)$ . Another pullback with  $F_s$  turns the induced metric into the standard metric in  $\text{Sol}$ .  $\square$

**Proposition 4.4.** *The foliation is globally rectifiable: there is a diffeomorphism  $\Psi : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{H} \times \mathbb{H}$ , such that each hyperplane  $\mathbb{R}^3 \times \{c\}$  is diffeomorphic to a leaf.*

*Proof.* Sol is diffeomorphic to  $\mathbb{R}^3$  in a natural way. Any  $\gamma \in \text{Sol}$  is uniquely determined by a triplet  $(t, x, y)$ . Define  $\Psi : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{H} \times \mathbb{H}$  by  $\Psi(t, x, y, s) = \psi_s \circ f_{z_0}(\gamma)$ . The function  $\Psi$  is injective because the action is free. Given  $z' \in \mathbb{H} \times \mathbb{H}$ , there is a leaf going through it, and since  $\psi_s(z_0)$  traverses all the leaves, there exists a number  $s$ , such that  $\psi_{-s}(z')$  is in the leaf passing through  $z_0$ . Let  $\gamma \in \text{Sol}$  be such that  $\psi_{-s}(z') = f_{z_0}(\gamma)$ . Therefore,

$$z' = \psi_s \circ f_{z_0}(\gamma),$$

which implies that  $\Psi$  is also surjective. Finally, the Jacobian of  $\Psi$  is

$$[d\Psi] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ e^{t+s}/\sqrt{2} & 0 & 0 & e^{t+s}/\sqrt{2} \\ 0 & 0 & 1 & 0 \\ -e^{-t+s}/\sqrt{2} & 0 & 0 & e^{-t+s}/\sqrt{2} \end{pmatrix},$$

which is nondegenerated. By the inverse function theorem,  $\Psi$  is a diffeomorphism. The last claim follows from the fact that  $\psi_s$  maps leaves onto leaves.  $\square$

**Corollary 4.5.** *The previous diffeomorphism can be modified, such that it not only maps the foliation to a Cartesian product globally, but also maps each leaf in the foliation isometrically to Sol.*

*Proof.* The pullback of the metric in  $\mathbb{H} \times \mathbb{H}$  with the previous diffeomorphism is

$$dt^2 + e^{-2(t+s)}dx^2 + e^{2(t-s)}dy^2 + ds^2,$$

which is analogous to the expression in Corollary 4.3. Let

$$\tilde{\Psi}(t, x, y, s) = \Psi(t, e^s x, e^s y, s).$$

$\tilde{\Psi}$  is a leaf-preserving diffeomorphism such that, for fixed  $s$ , it isometrically maps Sol into the leaf  $\mathbb{R}^3 \times \{s\}$ .  $\square$

## 5. Extrinsic geometry

**Proposition 5.1.** *Integral curves of the normal field  $X$  are geodesics.*

*Proof.* We previously found that the integral curves of the field are given by  $\gamma(t) = (x_1, e^t y_1, x_2, e^t y_2)$ . Let  $\phi(t)$  be a smooth curve in  $\mathbb{H}$  with the homothetic metric. Then,

$$\|\dot{\phi}(t)\|^2 = \frac{\dot{x}^2 + \dot{y}^2}{2y^2},$$

which is half the standard hyperbolic square length [Gallot et al. 2004]. Therefore, a curve minimizes hyperbolic arc length if and only if it minimizes the homothetic metric arc length, i.e., geodesics in both cases are the same. It is a well known fact that the vertical curves  $(x_k, e^t y_k)$  are geodesics in hyperbolic space. Finally, since  $\gamma$  can be projected in two geodesics and the metric is a product,  $\gamma$  is a geodesic in  $\mathbb{H} \times \mathbb{H}$  (see 3.15 in [Gallot et al. 2004]).  $\square$

**Proposition 5.2.** *There are isometries in  $\mathbb{H} \times \mathbb{H}$  acting transitively and sending leaves onto leaves.*

*Proof.* We work in the  $\mathbb{R}^3 \times \mathbb{R}$  picture with the  $\tilde{\Psi}$  isometry. By straightforward calculations we have that the mappings

$$(t, x, y, s) \mapsto (t + t', e^{t'+s'}x + x', e^{-t'+s'}y + y', s + s')$$

are isometries. The first claim comes from the fact that given a pair of points  $(t_k, x_k, y_k, s_k)$ , there exists exactly one such isometry sending one onto another. That this isometry sends leaves onto leaves is obvious, since under this diffeomorphism, they correspond to hypersurfaces where  $s$  is constant.  $\square$

We aim to calculate the distance between any pair of leaves. Recall that in any metric space, the distance from a point  $p$  to a set  $S \neq \emptyset$  is given by the expression

$$d(p, S) = \inf\{d(p, x) : x \in S\}.$$

See [Munkres 2000] for details.

**Proposition 5.3.** *The separation between two leaves in  $\mathbb{H} \times \mathbb{H}$  is constant. Moreover, if leaves are parametrized with the normal field affine parameter, then leaves' separation is given by the difference  $|s - s'|$  between the parameters corresponding to any leaf.*

*Proof.* A point in a leaf can be parametrized as

$$\left(x, \frac{e^{s+t}}{\sqrt{2}}, y, \frac{e^{s-t}}{\sqrt{2}}\right),$$

where  $x, y, t$  are arbitrary, and  $s$  is the parameter corresponding to the leaf. Given a second point in another leaf, say,

$$(x', e^{s'+t'}/\sqrt{2}, y', e^{s'-t'}/\sqrt{2}),$$

and since the metric is a product, we can find a geodesic minimizing the arc length in  $\mathbb{H} \times \mathbb{H}$ , such that, in each factor  $\mathbb{H}$ , the distance is also minimized [Gallot et al. 2004]. On the other hand, the metric we use in each factor of  $\mathbb{H} \times \mathbb{H}$  is half the hyperbolic

distance, for which a well-known formula gives us the distance [Anderson 2005]. Let  $\rho_k$  denote the distance in each factor with our metric, then,

$$\cosh(\sqrt{2}\rho_1) = 1 + \frac{2(x - x')^2 + (e^{s+t} - e^{s'+t'})^2}{2e^{s+t}e^{s'+t'}},$$

$$\cosh(\sqrt{2}\rho_2) = 1 + \frac{2(y - y')^2 + (e^{s-t} - e^{s'-t'})^2}{2e^{s-t}e^{s'-t'}},$$

where the  $\sqrt{2}$  factor within the hyperbolic cosine is due to the factor relating the standard hyperbolic metric with ours. The previous expression shows that, in order to get the minimum distance,  $x'$  must be equal to  $x$  and  $y'$  to  $y$ . Simplifying the previous expressions for such values of  $x'$  and  $y'$ , we find

$$\cosh(\sqrt{2}\rho_1) = \cosh(s - s' + t - t'),$$

$$\cosh(\sqrt{2}\rho_2) = \cosh(s - s' + t' - t).$$

Therefore,

$$\rho_1 = \frac{|s - s' + t - t'|}{\sqrt{2}}, \quad \rho_2 = \frac{|s - s' + t' - t|}{\sqrt{2}},$$

and the distance in the product metric is given by  $\sqrt{\rho_1^2 + \rho_2^2}$ . In order for this distance to be a minimum, a short analysis shows that one must take  $t' = t$ , and the statement follows.  $\square$

**Proposition 5.4.** *The principal curvatures of each leaf are  $-1$  with multiplicity two, and  $0$ . The principal directions are determined by the integral curves of the vectors  $\partial_x$ ,  $\partial_y$ ,  $\partial_t$  respectively.*

*Proof.* Recall the principal curvatures and directions for an orientable submanifold  $M$  are determined by the *shape operator*,  $S$ , which, in codimension 1, can be regarded as the mapping  $TM \rightarrow TM$  given by  $v_x \mapsto \nabla_{v_x} X$ , where  $X$  is the normal field to the manifold, compatible with orientation (see [Spivak 1979, Chapter 1]). Here, the principal directions and curvatures are the shape operator eigenvectors, and eigenvalues. Consider a leaf embedded in  $\mathbb{H} \times \mathbb{H}$ ,

$$\left(x, \frac{e^{-t-s}}{\sqrt{2}}, y, \frac{e^{t-s}}{\sqrt{2}}\right),$$

with normal field  $X = x_2\partial_2 + x_4\partial_4$ , where  $x_2 = e^{-t-s}/\sqrt{2}$  and  $x_4 = e^{t-s}/\sqrt{2}$ . A calculation shows that

$$\nabla X = -dx_1 \otimes \partial_1 - dx_3 \otimes \partial_3,$$

i.e., the shape operator is diagonal, once expressed in the base for the tangent space to the leaf, spanned by the coordinate vectors  $\partial_1$ ,  $\partial_3$ , and the vector  $-x_2\partial_2 + x_4\partial_4$ , with eigenvalues  $\{-1, -1, 0\}$  counted with multiplicity.  $\square$

**5A. Proof of Theorem 1.1.** Let  $G$  be a complex Kleinian group with a maximum of four lines in general position contained in its limit set, then  $G$  acts properly and discontinuously in four disjoint copies of  $\mathbb{H} \times \mathbb{H}$ . Without loss of generality we can assume that  $\mathbb{H} \times \mathbb{H}$  is  $G$ -invariant. By Proposition 3.1, if

$$\psi : \text{Sol} \times \mathbb{R} \rightarrow \mathbb{H} \times \mathbb{H}$$

is a  $G$ -equivariant diffeomorphism, then  $(\mathbb{H} \times \mathbb{H})/G$  is diffeomorphic to  $(\text{Sol}/G) \times \mathbb{R}$ . We notice the topological type is perfectly determined by the group  $G$ . In fact, the group  $G$  is the fundamental group of the manifold  $(\mathbb{H} \times \mathbb{H})/G$ . We remember the Kulkarni discontinuity region is equal to four disjoint copies of  $\mathbb{H} \times \mathbb{H}$ , hence  $\Omega/G$  is equal to four disjoint copies of  $(\mathbb{H} \times \mathbb{H})/G$ . We remark that if  $G$  represents a lattice of the Lie group  $\text{Sol}$ , then  $\text{Sol}/G$  is a compact 3 manifold. This last statement implies in some sense that  $\text{Sol}/G$  is the compact heart of  $(\mathbb{H} \times \mathbb{H})/G$ .

## 6. The Heisenberg group

Given a symplectic vector space,  $V$ , with symplectic form  $\omega$ , recall the *Heisenberg* group,  $\mathcal{H}$ , is the space  $V \times \mathbb{R}$ , with the product operation given by

$$(v, t) * (w, s) = (v + w, t + s + \omega(v, w)).$$

An account of this group in the context of complex hyperbolic geometry can be found in [Cano et al. 2013]. If  $V$  is of dimension 2, and  $\{\partial_p, \partial_q\}$  is a symplectic base for  $V$ , that is,  $\omega(\partial_p, \partial_q) = 1$ , a well-known fact from Lie group theory is that there is a faithful representation  $\mathcal{H} \rightarrow \text{SL}(3, \mathbb{R})$  [Binz and Pods 2008], given by

$$(p\partial_p + q\partial_q)t \rightarrow \begin{pmatrix} 1 & p & t + \frac{1}{2}pq \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}.$$

We will use this representation and identify  $\mathcal{H}$  with a subgroup of  $\text{SL}(3, \mathbb{R})$ . Therefore, we will identify  $\mathcal{H}$  with  $\mathbb{R}^3$ , with group structure,

$$(a, b, c) * (a', b', c') = (a + a', b + b', c + c' + ab'),$$

which corresponds to the matrix product

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix}.$$

With these identifications, there is a natural left action  $\mathcal{H} \curvearrowright \mathbb{C} \times \mathbb{H}$ :

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \\ 1 \end{pmatrix} = \begin{pmatrix} z + aw + c \\ w + b \\ 1 \end{pmatrix},$$

which we will denote by  $(a, b, c) * (z, w)$ .



**Proposition 6.1.** *The action of  $\mathcal{H}$  in  $\mathbb{C} \times \mathbb{H}$  is free.*

*Proof.* If  $(a, b, c) * (z, w) = (z, w)$ , then

$$\begin{aligned} z + aw + c &= z, \\ w + b &= w. \end{aligned}$$

From this linear system, one deduces that  $a = b = c = 0$ . □

**Proposition 6.2.** *For fixed  $(z, w) \in \mathbb{C} \times \mathbb{H}$ , the orbit  $h \in \mathcal{H} \mapsto h * (z, w)$  defines a differentiable embedding  $\mathcal{H} \hookrightarrow \mathbb{C} \times \mathbb{H}$ .*

*Proof.* The map is injective, since the action is free. Let  $w = p + qi$ , the Jacobian matrix of the mapping in  $(a, b, c) \in \mathcal{H}$  is given by

$$\begin{pmatrix} p & 0 & 1 \\ q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since the Jacobian has rank 3, the action defines a local diffeomorphism, and hence an embedding. □

Therefore, the action of  $\mathcal{H}$  defines a foliation of  $\mathbb{C} \times \mathbb{H}$ , in analogy with the foliation of  $\mathbb{H} \times \mathbb{H}$  generated by Sol.

**Proposition 6.3.** *Consider  $\mathbb{C} \times \mathbb{H}$  as a subset of  $\mathbb{R}^4$ , but with the product metric of the euclidean metric in  $\mathbb{C}$  and the hyperbolic metric in  $\mathbb{H}$ . If  $e_1, \dots, e_4$  denote the canonical coordinates in  $\mathbb{R}^4$ , and  $(p, q)$  denotes the coordinates in  $\mathbb{H}$ , then the vector field  $X = qe_4$  is unitary and orthogonal to any leaf of the foliation generated by  $\mathcal{H}$ .*

*Proof.* From Theorem 2.4, the vector fields  $pe_1 + qe_2$ ,  $e_3$ ,  $e_1$  generate the tangent space to the orbit of  $(z, w) \in \mathbb{C} \times \mathbb{H}$ , where  $w = p + qi$ . Since the metric is a product,  $X$  is orthogonal to  $pe_1 + qe_2$  and  $e_1$ . Moreover, the metric in  $\mathbb{H}$  is conformal to the euclidean, and therefore  $X$  is orthogonal to  $e_3$ . Finally,  $qe_4$  is unitary in the hyperbolic metric. □

**Corollary 6.4.** *Let  $(z, w) \in \mathbb{C} \times \mathbb{H}$ ,  $z = x + yi$  and  $w = p + qi$ . The integral curves of  $X$  are geodesics.*

*Proof.* The integral curves of  $X$  are constant in the first factor, and vertical straight lines in the hyperbolic factor. □

Although in this case, the action induced by the normal field  $X$  is not equivariant, we can describe in a precise way the quotients  $(\mathbb{C} \times \mathbb{H})/\Gamma$ , where  $\Gamma$  is a discrete subgroup of  $\mathcal{H}$ . Moreover, if  $\Gamma$  acts properly discontinuously in  $\mathbb{C} \times \mathbb{H}$ , it has to act in the same way in Heisenberg, because the slices  $\mathcal{H} \times \{qi\}$  are preserved. This is a general property of Lie groups that we prove in the following lemma.

**Lemma 6.5.** *Let  $X$  and  $Y$  be two locally compact spaces. If  $\Gamma \curvearrowright X \times Y$ , and the action of  $g \in \Gamma$  can be decomposed as  $g \cdot (x, y) = (g \cdot x, y)$  then  $\Gamma$  acts properly discontinuously in  $X$  if and only if it acts properly discontinuously in  $X \times Y$ .*

*Proof.* Let  $K \subset X$  be a compact set. Fix  $y \in Y$ . With the product topology,  $K \times \{y\}$  is a compact set in  $X \times Y$ . One can easily verify the equality

$$\{g \in \Gamma : g \cdot K \cap K \neq \emptyset\} = \{g \in \Gamma : g \times 1 \cdot K \times \{y\} \cap K \times \{y\} \neq \emptyset\}.$$

If  $\Gamma$  acts properly discontinuously in  $X \times Y$ , the previous equality implies that it acts properly discontinuously in  $X$ . On the other hand, if  $K \subset X \times Y$  is compact, the product topology together with the local compactity implies that we can find an open set  $U \times V$ , with  $U \subset X$  and  $V \subset Y$ , such that  $\bar{U}$  is compact in  $X$ ,  $\bar{V}$  is compact in  $Y$ , and  $K \subset U \times V$ . We have the contention

$$\{g \in \Gamma : gK \cap K \neq \emptyset\} \subset \{g \in \Gamma : g \cdot l(\bar{U} \times \bar{V}) \cap \bar{U} \times \bar{V} \neq \emptyset\}$$

Take  $g \in \Gamma$  and  $(x, y) \in \bar{U} \times \bar{V}$ , such that  $g \cdot (x, y) \in \bar{U} \times \bar{V}$ . Since  $g \cdot (x, y) = (g \cdot x, y)$ , it follows that  $g \cdot x \in \bar{U}$ . Therefore, the second set in the previous contention is at the same time contained in

$$\{g \in \Gamma : g \cdot \bar{U} \cap \bar{U} \neq \emptyset\}.$$

If  $\Gamma$  acts properly discontinuously in  $X$ , this set has to be finite, and the same must be true for the set of intersections in  $X \times Y$ , that is,  $\Gamma$  acts properly discontinuously in  $X$ .  $\square$

**Proposition 6.6.**  $\mathbb{C} \times \mathbb{H}$  is diffeomorphic to  $\mathcal{H} \times \mathbb{R}$ , where, up to diffeomorphism,  $\mathcal{H}$  acts on the first factor only.

*Proof.* Let  $\gamma = (a, b, c) \in \mathcal{H}$  and take  $(0, qi) \in \mathbb{C} \times \mathbb{H}$ . We can describe the orbits  $\gamma \cdot (0, qi)$  explicitly:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ qi \\ 1 \end{pmatrix} = \begin{pmatrix} aqi + c \\ qi + b \\ 1 \end{pmatrix}.$$

Therefore, there is exactly one  $(0, qi)$  in each orbit of the group action. Define  $\Psi : \mathcal{H} \times \mathbb{R} \rightarrow \mathbb{C} \times \mathbb{H}$  as

$$\Psi(\gamma, q) = \gamma \cdot (0, qi).$$

It can be shown that  $\Psi$  is bijective. It is a diffeomorphism, since an explicit computation shows that  $d\Psi$  maps the canonical vectors  $T_{(\gamma, q)}\mathcal{H} \times \mathbb{R} \cong \mathbb{R}^4 \rightarrow T_{\gamma \cdot (0, qi)}\mathbb{C} \times \mathbb{H} \cong \mathbb{R}^4$ :

$$\{\partial_1, \dots, \partial_4\} \mapsto \{q\partial_2, \partial_3, \partial_1, a\partial_2 + \partial_4\}.$$

The last assertion follows since the action is associative, i.e.,  $\gamma' \cdot (\gamma \cdot (0, qi)) = (\gamma' \cdot \gamma) \cdot (0, qi)$ , and therefore, preserves the imaginary part on the second factor.  $\square$

**6A. Proof of Theorem 1.4.** The proof is analogous to that of Theorem 1.1, the only difference is that we need the technical Lemma 6.5. A consequence of Theorem 1.4 is the following corollary:

**Corollary 6.7.** *If  $\Gamma < \mathcal{H}$  is a discrete subgroup acting properly and discontinuously in  $\mathbb{C} \times \mathbb{H}$ , up to diffeomorphism,  $(\mathbb{C} \times \mathbb{H})/\Gamma \cong (\mathcal{H}/\Gamma) \times \mathbb{R}$ , and the quotient  $\mathcal{H}/\Gamma$  is a manifold whose fundamental group is  $\pi_1(\mathcal{H}/\Gamma) \cong \Gamma$ .*

**Example 6.8.** Let  $\mathcal{H}_{\mathbb{Z}} < \mathcal{H}$  be the discrete subgroup of Heisenberg matrices with integer coefficients. It can be shown that the unit cube  $K_{\mathbb{C}} = [0, 1]^3 \subset \mathcal{H}$  is a fundamental region for the action of  $\mathcal{H}_{\mathbb{Z}}$  [Lukyanenko 2014]. The quotient  $\mathcal{H}_{\mathbb{Z}} \backslash \mathcal{H}$  is an example of a nilmanifold, whose fundamental group is

$$\mathcal{H}_{\mathbb{Z}} \cong \langle m, n, k : [m, n] = k^4 \rangle;$$

see [Lukyanenko 2014]. In view of the previous results,  $\mathcal{H}_{\mathbb{Z}}$  acts properly and discontinuously in  $\mathbb{C} \times \mathbb{H}$ , and the quotient  $(\mathbb{C} \times \mathbb{H})/\mathcal{H}_{\mathbb{Z}}$  is a product of a nilmanifold times  $\mathbb{R}$ , whose fundamental group has the previous presentation.

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# ON PERIODIC POINTS OF SYMPLECTOMORPHISMS ON SURFACES

MARTA BATORÉO

**We construct a symplectic flow on a surface of genus  $g \geq 2$ ,  $\Sigma_{g \geq 2}$ , with exactly  $2g - 2$  hyperbolic fixed points and no other periodic orbits. Moreover, we prove that a (strongly nondegenerate) symplectomorphism of  $\Sigma_{g \geq 2}$  isotopic to the identity has infinitely many periodic points if there exists a fixed point with nonzero mean index. From this result, we obtain two corollaries, namely that such a symplectomorphism of  $\Sigma_{g \geq 2}$  with an elliptic fixed point or with strictly more than  $2g - 2$  fixed points has infinitely many periodic points provided that the flux of the isotopy is “irrational”.**

## 1. Introduction and main results

In this paper, we construct a symplectic flow  $\psi^t$  on a closed surface with genus  $g \geq 2$ ,  $\Sigma_{g \geq 2}$ , having exactly  $2g - 2$  hyperbolic fixed points and no other periodic points. This is a genuine flow and it satisfies an “irrationality” condition on its flux; see property (1-1). This construction yields the computation of the Floer–Novikov homology when (1-1) holds. With this information and assuming (1-1), we prove that a (strongly nondegenerate) symplectomorphism  $\phi$  on  $\Sigma_{g \geq 2}$  (connected to the identity by an isotopy  $\phi_t$ ) possessing a fixed point with nonzero mean index has infinitely many periodic points. As a consequence of this result, we see that the presence of an elliptic fixed point or of strictly more than  $2g - 2$  fixed points implies the existence of infinitely many periodic points.

We are interested in symplectomorphisms which are not Hamiltonian. However our results fit in the context of a conjecture of B. Z. Gürel [2013; 2014] which suggests that the presence of an *unnecessary* fixed point of a Hamiltonian diffeomorphism guarantees the existence of infinitely many periodic points. There, *unnecessary* is viewed from a homological or geometrical perspective. The results in [Gürel 2013; 2014] support the conjecture when the fixed point is unnecessary from a homological viewpoint. From the geometrical perspective, the conjecture is supported, e.g., by the result in [Ginzburg and Gürel 2014] where V. L. Ginzburg and

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B. Z. Gürel prove that, for a vast class of symplectic manifolds (which includes the complex projective spaces  $\mathbb{CP}^n$ ), a Hamiltonian diffeomorphism with a hyperbolic fixed point has infinitely many periodic points.

Furthermore, the conjecture by Gürel is a variant of a conjecture by H. Hofer and E. Zehnder [1994, page 263] claiming that “every Hamiltonian map on a compact symplectic manifold  $(M, \omega)$  possessing more fixed points than necessarily required by the V. Arnold conjecture possesses always infinitely many periodic points”. For instance, the conjecture in [Hofer and Zehnder 1994] on  $\mathbb{CP}^n$  claims that a nondegenerate Hamiltonian diffeomorphism has infinitely many periodic points if it fixes more than  $n + 1$  points. This was motivated by the result of J. Franks [1988] stating that an area-preserving diffeomorphism on  $S^2$  with more than two fixed points has infinitely many periodic points (see also [Franks 1992; 1996; Le Calvez 1999; Bramham and Hofer 2012; Collier et al. 2012; Kerman 2012] for symplectic topological proofs).

Recall that a Hamiltonian diffeomorphism on a closed surface with genus  $g \geq 1$  always has infinitely many periodic points. This statement was conjectured to hold on the torus by C. Conley in a lecture given on April 6th 1984, in the University of Wisconsin. This was later proved in [Hingston 2009] and it has been generalized to a vast class of symplectic manifolds; see Ginzburg’s proof [2010] and, e.g., [Ginzburg and Gürel 2012; Hein 2012; Ginzburg et al. 2015] for more contributions.

The background discussed so far concerns Hamiltonian diffeomorphisms. For symplectomorphisms which need not be Hamiltonian, H. V. Lê and K. Ono [1995] proved a version of Arnold’s conjecture for nondegenerate symplectomorphisms. A lower bound for the number of fixed points of a symplectomorphism is given by the sum of the Betti numbers of the Novikov homology of a closed 1-form representing the cohomology class given by the flux of an isotopy connecting the identity to the symplectomorphism. Observe that this lower bound may be zero as in the case of the 2-torus. Moreover, when the flux of the isotopy is zero, the Novikov homology associated to the flux is the ordinary homology of  $M$  and, in this case, i.e., when the symplectomorphism is Hamiltonian, this is the statement of Arnold’s conjecture.

There is also an analogue of the result by Ginzburg and Gürel [2014] which claims that if a symplectomorphism (satisfying some conditions on its flux) has a hyperbolic fixed point, then there are infinitely many periodic points. If the hyperbolic fixed point corresponds to a contractible periodic orbit, the result is proved in [Batoréo 2015] for some class of manifolds which includes, for instance, the product of  $\mathbb{CP}^n$  with a  $2m$ -dimensional torus,  $\mathbb{CP}^n \times \mathbb{T}^{2m}$ , with  $m \leq n$  (or  $\mathbb{CP}^n \times P^{2m}$ , with  $P^{2m}$  a symplectically aspherical  $2m$ -manifold). The case when the hyperbolic periodic orbit is noncontractible was proved in [Batoréo 2017] and it holds, for example, on the product spaces  $\mathbb{CP}^n \times \Sigma_{g \geq 2}$ . We point out that the existence of infinitely many periodic points is guaranteed by the presence of a

hyperbolic fixed point on  $\mathbb{CP}^n$  and on  $\mathbb{CP}^n \times \Sigma_{g \geq 2}$ . However, such a result does not hold on  $\Sigma_{g \geq 2}$ .

In fact, Theorem 1.1 and the construction of Section 3A give a symplectomorphism with finitely many hyperbolic fixed points and no other periodic points; see [Katok and Hasselblatt 1995, Exercise 14.6.1]. The number of fixed points of this symplectomorphism is exactly  $2g - 2$ , which is the lower bound for the number of fixed points of a diffeomorphism given by the Lefschetz fixed point theorem. We prove the presence of infinitely many periodic points of a (strongly nondegenerate) symplectomorphism on  $\Sigma_{g \geq 2}$  (with an “irrationality” assumption on its flux) provided the existence of a fixed point with nonzero mean index (see Theorem 1.4). Such a condition is satisfied if the fixed point is elliptic (see Theorem 1.3) or if the number of fixed points is strictly greater than  $2g - 2$  (see Theorem 1.5).

In Sections 1A and 1B, we state the main theorems of this paper. The theorems in Section 1A refer to the existence of the symplectomorphism with exactly  $2g - 2$  fixed points and no other periodic points (Theorem 1.1) and to the computation of the Floer–Novikov homology of symplectomorphisms satisfying condition (1-1) (Theorem 1.2). In Section 1B, we state the theorems which give sufficient conditions for the existence of infinitely many periodic points of symplectomorphisms with flux as in (1-1) (Theorems 1.3–1.5). The remaining sections are organized as follows: in Section 2, we present the definitions and known results used in the statements and proofs of our theorems, in Section 3, we prove the results stated in Section 1A and, in Section 4, we prove the theorems stated in Section 1B.

**1A. Existence of a symplectomorphism with exactly  $2g - 2$  hyperbolic fixed points and no other periodic points.** Consider a closed surface  $\Sigma$  with genus  $g$  greater than or equal to 2 and a symplectic form  $\omega$  on  $\Sigma$ . The first cohomology group  $H^1(\Sigma; \mathbb{R})$  of the surface  $\Sigma$  can be identified with  $\mathbb{R}^{2g}$  and hence the image of  $[\phi_t] \in \widetilde{\text{Symp}}_0(\Sigma, \omega)$  under the flux homomorphism (see Section 2B) can be viewed as a  $2g$ -tuple,

$$(u_1, v_1, \dots, u_g, v_g) \in \mathbb{R}^{2g},$$

where  $\widetilde{\text{Symp}}_0(\Sigma, \omega)$  is the universal covering of the identity component of the group of symplectomorphisms on  $\Sigma$ . Moreover, the kernel of the flux homomorphism is given by the universal covering of the group of Hamiltonian diffeomorphisms,  $\widetilde{\text{Ham}}(\Sigma, \omega)$ . We recall that the flux homomorphism

$$\text{Flux} : \widetilde{\text{Symp}}_0(\Sigma, \omega) \rightarrow H^1(\Sigma, \mathbb{R}); \quad [\phi_t] \mapsto \left[ \int_0^1 \omega(X_t, \cdot) dt \right]$$

descends to a homomorphism

$$\text{Flux} : \text{Symp}_0(\Sigma, \omega) \rightarrow H^1(\Sigma, \mathbb{R}); \quad \phi \mapsto \left[ \int_0^1 \omega(X_t, \cdot) dt \right]$$

since  $\Sigma$  is atoroidal (see Section 2B). If a symplectomorphism  $\phi$  satisfies

$$(1-1) \quad \text{Flux}(\phi) = (u_1, v_1, \dots, u_g, v_g) \\ \text{with } u_i \neq 0 \text{ and } \frac{v_i}{u_i} \in \mathbb{R} \setminus \mathbb{Q}, \quad \text{for all } i = 1, \dots, g,$$

we say that it satisfies the *flux condition*.

**Remark 1.** If  $\phi$  satisfies the flux condition (1-1), then  $\phi^k$  also satisfies the flux condition (for all  $k \in \mathbb{N}$ ).

Our first main result is the following:

**Theorem 1.1.** *Given  $(u_1, v_1, \dots, u_g, v_g) \in \mathbb{R}^{2g}$  such that*

$$u_i \neq 0 \quad \text{and} \quad v_i/u_i \in \mathbb{R} \setminus \mathbb{Q}, \quad i = 1, \dots, g,$$

*there exists a symplectic flow*

$$\psi_{(u_1, v_1, \dots, u_g, v_g)}^t : \Sigma \rightarrow \Sigma$$

*with no periodic orbits other than (exactly)  $2g - 2$  hyperbolic fixed points and*

$$\text{Flux}(\psi_{(u_1, v_1, \dots, u_g, v_g)}^t|_{t \in [0, 1]}) = (u_1, v_1, \dots, u_g, v_g).$$

Denote by  $\text{HFN}_*(\phi)$  the Floer–Novikov homology of a symplectomorphism  $\phi$  of  $\Sigma$  isotopic to the identity (see Section 2D for the definition). Using the construction of the (genuine) flow  $\psi_{(u_1, v_1, \dots, u_g, v_g)}^t$  given by the previous theorem, we compute  $\text{HFN}_*(\phi)$  for nondegenerate symplectomorphisms  $\phi$  satisfying (1-1) (see Theorem 1.2). In the following theorem, one can take any ring (e.g.,  $\mathbb{Z}$  or  $\mathbb{Q}$ ) as the ground ring  $\mathbb{F}$ . In this paper, for the sake of simplicity, all complexes and homology groups are defined over the ground field  $\mathbb{F} = \mathbb{Z}_2$ .

**Theorem 1.2.** *Let  $\phi \in \text{Symp}_0(\Sigma, \omega)$  be nondegenerate and satisfying the flux condition (1-1). Then the Floer–Novikov homology of  $\phi$  is given by*

$$(1-2) \quad \text{HFN}_r(\phi) = \begin{cases} \mathbb{F}^{2g-2} \otimes \Lambda_\theta & \text{if } r = 0, \\ 0 & \text{if } r \neq 0. \end{cases}$$

We point out that Lê and Ono [1995, Theorem 8.1] proved that, for a certain class of symplectic manifolds, if the flux of the isotopy is sufficiently small, then the Floer–Novikov homology of the isotopy may be computed by the Novikov homology of a closed 1-form representing the flux of the isotopy. Namely, on  $\Sigma$ , [Lê and Ono 1995, Theorem 8.1] states that there exists  $\varepsilon > 0$  such that if  $\|\theta\|_{C^1} < \varepsilon$ , then

$$\text{HFN}_*(\phi) \simeq \text{HN}_{*+1}(\theta),$$

where  $[\theta] = \text{Flux}(\phi)$ . In Theorem 1.2, in contrast, the flux of  $\phi$  is not assumed to be small.



We will now compute the Novikov homology of  $\theta$  when  $[\theta] = \text{Flux}(\phi) = (u_1, v_1, \dots, u_g, v_g)$  with  $u_1, v_1, \dots, u_g, v_g \in \mathbb{R}$  rationally independent. Consider the homomorphism  $\pi_1(\Sigma) \rightarrow \mathbb{R}$  defined by

$$(1-3) \quad \gamma \mapsto \int_{\gamma} \theta$$

which we also denote by  $[\theta]$ . Since  $u_1, v_1, \dots, u_g, v_g$  are rationally independent, the kernel  $\ker([\theta])$  is the commutator  $[\pi_1(\Sigma), \pi_1(\Sigma)]$  of the fundamental group  $\pi_1(\Sigma)$ . Let  $\pi : \tilde{\Sigma} \rightarrow \Sigma$  be the covering space associated to the homomorphism  $[\theta]$ , i.e.,  $\tilde{\Sigma}$  is the maximal free abelian covering of  $\Sigma$ . Then there exists a function  $\bar{f} : \tilde{\Sigma} \rightarrow \mathbb{R}$  such that  $\pi^*\theta = d\bar{f}$ . We recall that the Novikov complex of  $\theta$  is defined in the same way as the Morse complex of  $\bar{f}$ ; see, e.g., [Lê and Ono 1995], namely Section 6 and Appendix C, and [Ono 2006].

As mentioned in the example of [Lê and Ono 1995, Section 7], the Betti numbers of  $\tilde{\Sigma}$  are 0,  $2g - 2$  and 0. Hence, by [Lê and Ono 1995, Theorem 8.1], for  $\|\theta\|_{C^1}$  sufficiently small,

$$\text{HFN}_*(\phi) \simeq \begin{cases} \mathbb{F}^{2g-2} \otimes \Lambda_{\theta} & \text{if } * = 0, \\ 0 & \text{if } * \neq 0, \end{cases}$$

which coincides with the computations in Theorem 1.2.

Notice that, when  $u_1, v_1, \dots, u_g, v_g \in \mathbb{R}$  are rationally independent, the sum of the Betti numbers of the Novikov homology of  $\theta$ , with  $[\theta] = (u_1, v_1, \dots, u_g, v_g)$ , is  $2g - 2$  (regardless of whether  $\|\theta\|$  is sufficiently small or not) and hence the lower bound given by the main theorem in [Lê and Ono 1995, page 156] is attained by the symplectic flow given by Theorem 1.1.

**Remark 2.** We observe that:

- (1) Due to conventions on the indices, the Floer–Novikov homology in this paper is the Floer–Novikov homology considered in [Lê and Ono 1995] with the degree shifted by  $n = 1$ .
- (2) On  $\Sigma$ , the Novikov rings  $\Lambda_{\theta, \omega}$  and  $\Lambda_{\theta}$  in [Lê and Ono 1995] are isomorphic and hence

$$\text{Nov}_*(\theta) \otimes_{\Lambda_{\theta}} \Lambda_{\theta, \omega} \simeq \text{Nov}_*(\theta).$$

- (3)  $\varepsilon > 0$  is taken small enough so that the conditions in [Ono 2006, Definition 3.9] are satisfied. See also [Ono 2006, Theorem 3.12].

**Remark 3** (noncontractible orbits). In this paper, the Floer–Novikov homology is defined for contractible periodic orbits (as in [Ono 2006]), unless explicitly stated otherwise. If the fixed points of the symplectomorphisms correspond to noncontractible periodic orbits, take the Floer–Novikov homology for noncontractible periodic orbits defined in [Burghlelea and Haller 2001]. In that case, the

Floer–Novikov homology of a nondegenerate  $\phi \in \text{Symp}_0(\Sigma, \omega)$  satisfying (1-1) is  $\text{HFN}_*(\phi, \zeta) = 0$ , where  $\zeta$  is a nontrivial free homotopy class of loops in  $\Sigma$ . See Remark 17.

**1B. Existence of infinitely many periodic points.** Consider a strongly nondegenerate symplectomorphism  $\phi$  (see page 27 for the definition) on a closed surface  $\Sigma$  (with genus  $g \geq 2$ ) satisfying the flux condition (1-1). The following theorem gives a condition under which  $\phi$  has infinitely many periodic points.

**Theorem 1.3.** *Let  $\phi \in \text{Symp}_0(\Sigma, \omega)$  be strongly nondegenerate. Suppose  $\phi$  satisfies the flux condition (1-1) and that  $\phi$  has an elliptic fixed point. Then  $\phi$  has infinitely many periodic points.*

**Remark 4.** If  $x_0$  corresponds to a noncontractible periodic orbit, Theorem 1.3 remains valid. See Remark 18.

Theorem 1.3 follows from a more general result:

**Theorem 1.4.** *Let  $\phi \in \text{Symp}_0(\Sigma, \omega)$  be strongly nondegenerate. Suppose  $\phi$  satisfies the flux condition (1-1) and that  $\phi$  has a fixed point  $x_0$  such that its mean index  $\Delta(x_0)$  is not zero. Then  $\phi$  has infinitely many periodic points.*

In Section 4, we prove, more precisely, that if  $\phi$  has finitely many fixed points, then every large prime is a simple period, i.e., a period of a simple (noniterated) orbit. (In particular, the number of simple periods less than or equal to  $k$  is of order at least  $k/\log(k)$ .) One of the main tools used in the proof of this theorem is Floer–Novikov homology and the proof relies on Theorem 1.2. Another consequence of Theorem 1.4 is Theorem 1.5, which gives a sufficient condition on the number of fixed points of  $\phi$  for the existence of infinitely many periodic points of  $\phi$ .

**Theorem 1.5.** *Let  $\phi \in \text{Symp}_0(\Sigma, \omega)$  be strongly nondegenerate and suppose it satisfies the flux condition (1-1). If the number of fixed points of  $\phi$  is (strictly) greater than  $2g - 2$ , then  $\phi$  has infinitely many periodic points.*

## 2. Preliminaries

Consider a closed surface  $\Sigma$  with genus  $g \geq 2$  and a symplectic structure  $\omega$  on  $\Sigma$ . In this section, we follow [Burghlea and Haller 2001; Ginzburg and Gürel 2015; Lê and Ono 1995; Ono 2006; Salamon and Zehnder 1992].

**2A. A covering space of the space of contractible loops.** Let  $\mathcal{L}\Sigma$  be the space of contractible loops in  $\Sigma$  and  $\Omega\Sigma$  be the space of based contractible loops in  $\Sigma$ . The map  $ev : \mathcal{L}\Sigma \rightarrow \Sigma$  defined by  $x \mapsto x(0)$  is a fibration with fiber  $\Omega\Sigma$  (see, e.g., [Hu 1959, page 83] for the details). It induces a long exact sequence on the homotopy groups and part of it is given by

$$\pi_1(\Omega\Sigma) \rightarrow \pi_1(\mathcal{L}\Sigma) \rightarrow \pi_1(\Sigma).$$

Since this fibration admits a section consisting of constant loops,

$$\pi_1(\mathcal{L}\Sigma) \cong \pi_1(\Omega\Sigma) \oplus \pi_1(\Sigma).$$

With the identification  $\pi_1(\Omega\Sigma) \equiv \pi_2(\Sigma)$  (see, e.g., [Adams 1978, pages 5–7] for the details) and since  $\pi_2(\Sigma) = 0$ , we have

$$\pi_1(\mathcal{L}\Sigma) \cong \pi_1(\Sigma).$$

Let  $\theta$  be a closed 1-form on  $\Sigma$  and consider the homomorphism

$$(2-1) \quad [\bar{\theta}] : \pi_1(\mathcal{L}\Sigma) \rightarrow \mathbb{R}$$

induced by the homomorphism  $[\theta] : \pi_1(\Sigma) \rightarrow \mathbb{R}$  defined by (1-3). Moreover, take the covering  $\pi : \tilde{\Sigma} \rightarrow \Sigma$  associated with  $\ker([\theta]) \leq \pi_1(\Sigma)$ . When  $\ker([\theta]) = 0$ ,  $\tilde{\Sigma}$  is the universal covering of  $\Sigma$ . Choose a function  $\bar{f} : \tilde{\Sigma} \rightarrow \mathbb{R}$  such that  $d\bar{f} = \pi^*\theta$ .

Denote by  $\tilde{\mathcal{L}}\Sigma$  the covering space of  $\mathcal{L}\Sigma$  associated with  $\ker([\bar{\theta}]) \leq \pi_1(\mathcal{L}\Sigma)$ . The deck transformation group of  $p : \tilde{\mathcal{L}}\Sigma \rightarrow \mathcal{L}\Sigma$  is

$$(2-2) \quad \Gamma := \frac{\pi_1(\mathcal{L}\Sigma)}{\ker([\bar{\theta}])} \cong \frac{\pi_1(\Sigma)}{\ker([\theta])}.$$

Following [Ono 2006], an element of the covering space  $\tilde{\mathcal{L}}\Sigma$  can be described as an equivalence class (for a relation  $\sim$ ) of a loop  $\tilde{x}$  in  $\tilde{\Sigma}$  where the relation  $\sim$  is defined by  $\tilde{x} \sim \tilde{y}$  if

$$(2-3) \quad \pi \circ \tilde{x} = \pi \circ \tilde{y}$$

and

$$(2-4) \quad \bar{f}(\tilde{x}(o)) = \bar{f}(\tilde{y}(o))$$

where  $o$  is the base point of  $S^1$ , i.e.,  $1 \in \partial D^2 \subset \mathbb{C}$ . We observe that conditions (2-3) and (2-4) are equivalent to  $\tilde{x} = \tilde{y}$  and, hence,  $\tilde{\mathcal{L}}\Sigma$  is in fact the space  $\mathcal{L}\tilde{\Sigma}$  of contractible loops in  $\tilde{\Sigma}$ .

**Remark 5.** The homomorphisms  $\mathcal{I}_\omega$  and  $\mathcal{I}_{c_1}$  defined by [Ono 2006] are identically zero when  $M = \Sigma$ , since  $\pi_2(\Sigma) = 0$ . Moreover, the homomorphism  $\mathcal{I}_\eta$  from that paper is the map  $[\bar{\theta}]$  in (2-1).

**2B. Symplectomorphisms and periodic orbits.** We denote by  $\text{Symp}(\Sigma, \omega)$  the group of symplectomorphisms of  $(\Sigma, \omega)$  and by  $\text{Symp}_0(\Sigma, \omega)$  the component of the identity in  $\text{Symp}(\Sigma, \omega)$ .

Let  $\phi \in \text{Symp}_0(\Sigma, \omega)$  and consider  $\phi_t$  a symplectic isotopy connecting the identity  $\phi_0 = \text{id}$  to  $\phi_1 = \phi$  and define a vector field  $X_t$  by

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t.$$

The flux homomorphism is defined on the universal covering  $\widetilde{\text{Symp}}_0(\Sigma, \omega)$  of  $\text{Symp}_0(\Sigma, \omega)$  by

$$\text{Flux} : \widetilde{\text{Symp}}_0(\Sigma, \omega) \rightarrow H^1(\Sigma; \mathbb{R}); \quad [\phi_t] \mapsto \left[ \int_0^1 \omega(X_t, \cdot) dt \right].$$

This homomorphism is surjective, its kernel is given by  $\widetilde{\text{Ham}}(\Sigma, \omega)$ , i.e., the universal covering of the group of Hamiltonian diffeomorphisms (see [McDuff and Salamon 1995]) and, when  $g \geq 2$ , (see [Kędra 2000]) it descends to a homomorphism

$$\text{Flux} : \text{Symp}_0(\Sigma, \omega) \rightarrow H^1(\Sigma, \mathbb{R}); \quad \phi \mapsto \left[ \int_0^1 \omega(X_t, \cdot) dt \right].$$

**Remark 6** [McDuff and Salamon 1995, pages 316–317]. Under the usual identification of  $H^1(\Sigma; \mathbb{R})$  with  $\text{Hom}(\pi_1(\Sigma), \mathbb{R})$ , the cohomology class  $\text{Flux}([\phi_t])$  corresponds to the homomorphism

$$\pi_1(\Sigma) \rightarrow \mathbb{R}; \quad \gamma \mapsto \int_0^1 \int_0^1 \omega(X_t(\gamma(s)), \dot{\gamma}(s)) ds dt,$$

for  $\gamma : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow \Sigma$ . Geometrically, the value of  $\text{Flux}([\phi_t])$  on the loop  $\gamma$  is the symplectic area swept by the path  $\gamma$  under the isotopy  $\phi_t$ .

Denote by  $\theta$  a closed 1-form such that  $\text{Flux}([\phi]) = [\theta] \in H^1(\Sigma; \mathbb{R})$ .

Lê and Ono [1995, Lemma 2.1] proved that  $\{\phi_t\}$  can be deformed through symplectic isotopies (keeping the end points fixed) so that the cohomology classes  $[\omega(X'_t, \cdot)]$ , for all  $t \in [0, 1]$ , and  $\text{Flux}([\phi'_t]) = [\theta]$  are the same (where  $X'_t$  is the vector field associated with the deformed symplectic isotopies  $\phi'_t$ ). Namely, each element in  $\widetilde{\text{Symp}}_0(\Sigma, \omega)$  admits a representative symplectic isotopy generated by a smooth path of closed 1-forms  $\theta_t$  on  $\Sigma$  whose cohomology class is identically equal to the flux, i.e.,

$$-\omega(X'_t, \cdot) = \theta + dh_t =: \theta_t$$

for some Hamiltonian  $h_t : \Sigma \rightarrow \mathbb{R}$ ,  $t \in S^1$ , that is 1-periodic in time.

The fixed points of  $\phi = \phi_1$  are in one-to-one correspondence with 1-periodic solutions of the differential equation

$$(2-5) \quad \dot{x}(t) = X_{\theta_t}(t, x(t)),$$

where  $X_{\theta_t}$  is defined by  $\omega(X_{\theta_t}, \cdot) = -\theta_t$ . From now on we denote the vector field  $X_{\theta_t}$  also by  $X_t$ .

A 1-periodic solution  $x$  of (2-5) is called *nondegenerate* if 1 is not an eigenvalue of the linearized return map  $d\phi_{x(0)} : T_{x(0)}\Sigma \rightarrow T_{x(0)}\Sigma$ . If all 1-periodic orbits of  $X_t$  are nondegenerate, then the associated symplectomorphism  $\phi$  is called *nondegenerate* and if all periodic orbits of  $X_t$  are nondegenerate then  $\phi$  is called *strongly*

*nondegenerate*. Moreover, if all periodic orbits of  $X_t$  are nondegenerate, then the set  $\mathcal{P}(\theta_t)$  of 1-periodic solutions of (2-5) is finite.

The set  $\mathcal{P}(\theta_t)$  coincides with the zero set of the closed 1-form defined on the space of contractible loops on  $\Sigma$ ,  $\mathcal{L}\Sigma$ , by

$$\begin{aligned}\alpha_{\{\phi_t\}}(x, \xi) &= \int_0^1 \omega(\dot{x} - X_t, \xi) dt \\ &= \int_0^1 \omega(\dot{x}, \xi) + \theta_t(x(t))(\xi) dt \\ &= \int_0^1 \omega(\dot{x}, \xi) dt + \int_0^1 (\theta + dh_t)(\xi) dt\end{aligned}$$

where  $x \in \mathcal{L}\Sigma$  and  $\xi \in T_x \mathcal{L}\Sigma$  (i.e.,  $\xi$  is a tangent vector field along the loop  $x$  or, equivalently,  $\xi(t) \in T_{x(t)} \Sigma$ ).

A primitive function of the pull-back of the 1-form  $\alpha_{\{\phi_t\}}$  to the covering space  $\tilde{\mathcal{L}}\Sigma$  (defined in Section 2A) is given by

$$\mathcal{A}_{\{\phi_t\}}(\tilde{x}) := - \int_{D^2} v^* \omega + \int_0^1 (\bar{f} + h_t \circ \pi)(\tilde{x}(t)) dt$$

where  $v : D^2 \rightarrow \Sigma$  is some disc in  $\Sigma$  with  $\pi \circ \tilde{x} = v|_{\partial D^2}$ . Notice that the right-hand side is independent of the choice of the disc  $v$ .

**2C. The mean index and the Conley–Zehnder index.** For every continuous path  $\Phi : [0, 1] \rightarrow \text{Sp}(2)$  of  $2 \times 2$  symplectic matrices such that  $\Phi(0) = \text{Id}$ , the mean index  $\Delta(\Phi)$  measures, roughly speaking, the total rotation angle swept by certain eigenvalues on the unit circle. We describe this index (and the Conley–Zehnder index) explicitly.

Let  $A$  be a symplectic matrix in  $\text{Sp}(2)$ . Then it has two eigenvalues  $\lambda_1$  and  $\lambda_2$  such that either  $\lambda_i \in S^1 \subset \mathbb{C}$  or  $\lambda_i \in \mathbb{R} \setminus \{-1, 1\}$ , where  $i = 1, 2$ , and  $\lambda_1 \lambda_2 = 1$ . We denote the spectrum of  $A$ , i.e., the set of eigenvalues of  $A$ , by  $\sigma(A)$ .

If  $1 \notin \sigma(A)$ , we say  $A$  is *nondegenerate*. We distinguish two cases of nondegenerate matrices:

- The eigenvalues are real ( $\sigma(A) \subset \mathbb{R} \setminus \{-1, +1\}$ ). Then  $0 < \lambda_1 < 1 < \lambda_2 = \lambda_1^{-1}$  or  $\lambda_1 < -1 < \lambda_2 = \lambda_1^{-1} < 0$ . In this case,  $A$  is called *hyperbolic*.
- The eigenvalues are on the unit circle ( $\sigma(A) \subset S^1 \setminus \{1\}$ ) in which case  $A$  is called *elliptic*.

Set

$$\rho(A) = \begin{cases} e^{i\nu} & \text{if } A \text{ is conjugate to a rotation by an angle } \nu \in (-\pi, \pi), \\ 1 & \text{if } \sigma(A) \subset \mathbb{R}_{>0}, \\ -1 & \text{if } \sigma(A) \subset \mathbb{R}_{<0}. \end{cases}$$

This function  $\rho : \mathrm{Sp}(2) \rightarrow S^1$  is continuous, invariant by conjugation and equal to  $\det_{\mathbb{C}} : U(1) \rightarrow S^1$  on  $U(1)$ . When

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

we have  $\rho(A) = -1$ . Then, given a path  $\Phi : [0, 1] \rightarrow \mathrm{Sp}(2)$ , there is a continuous function  $\eta(\cdot)$  such that  $\rho(\Phi(t)) = e^{i\eta(t)}$  measuring the rotation of certain unit eigenvalues and the *mean index* of  $\Phi$  is defined by

$$\Delta(\Phi) := \frac{\eta(1) - \eta(0)}{\pi}.$$

Denote the set of nondegenerate matrices in  $\mathrm{Sp}(2)$  by  $\mathrm{Sp}(2)^*$ . This set has two connected components

$$\mathrm{Sp}(2)^+ := \{A \in \mathrm{Sp}(2)^* : \det(A - I) > 0\}$$

and

$$\mathrm{Sp}(2)^- := \{A \in \mathrm{Sp}(2)^* : \det(A - I) < 0\}.$$

**Remark 7.** The set  $\mathrm{Sp}(2)^+$  consists of matrices in  $\mathrm{Sp}(2)^*$  which are elliptic or hyperbolic with negative eigenvalues and  $\mathrm{Sp}(2)^-$  is the set of matrices in  $\mathrm{Sp}(2)^*$  which are hyperbolic with positive eigenvalues.

Define the matrices

$$W^+ := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathrm{Sp}(2)^+ \quad \text{and} \quad W^- := \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \in \mathrm{Sp}(2)^-.$$

For  $A \in \mathrm{Sp}(2)^*$ , consider a path  $\Psi_A : [0, 1] \rightarrow \mathrm{Sp}(2)^*$  connecting  $A \in \mathrm{Sp}(2)^\pm$  to  $W^\pm$ . Then, the *Conley–Zehnder index* of  $\Phi$  with  $\Phi(1) \in \mathrm{Sp}(2)^*$  is, by definition,

$$\mu_{\mathrm{CZ}}(\Phi) := \Delta(\Phi \# \Psi_{\Phi(1)}) \in \mathbb{Z},$$

where  $\Phi \# \Psi_{\Phi(1)}$  is the concatenation of the paths  $\Phi$  and  $\Psi_{\Phi(1)}$  in  $\mathrm{Sp}(2)$ .

The mean index and the Conley–Zehnder index of  $\Phi$  satisfy the relation

$$0 \neq |\Delta_{\{\phi_i\}}(\Phi) - \mu_{\mathrm{CZ}}(\Phi)| < 1$$

when  $\Phi(1)$  is nondegenerate. We recall some properties of the indices where we assume  $\Phi(1) \in \mathrm{Sp}(2)^*$  and  $-1 \notin \sigma(\Phi(1))$ ; see Remark 8.

**Result 1.** • If  $\Phi(1)$  is elliptic, then  $\Delta(\Phi) \neq 0$ .

- If  $\Phi(1)$  is hyperbolic then  $\Delta(\Phi) \in \mathbb{Z}$ . Equivalently, if  $\Delta(\Phi) \in \mathbb{R} \setminus \mathbb{Z}$ , then  $\Phi(1)$  is elliptic.

**Result 2.** If  $\Phi(1)$  is elliptic, then  $\mu_{\mathrm{CZ}}(\Phi)$  is an odd integer. Equivalently, if  $\mu_{\mathrm{CZ}}(\Phi)$  is an even integer, then  $\Phi(1)$  is hyperbolic.

**Result 3.** If  $\Phi(1)$  is hyperbolic, then  $\Delta(\Phi) = \mu_{\text{CZ}}(\Phi)$ . Moreover, the eigenvalues of  $\Phi(1)$  are positive if and only if  $\mu_{\text{CZ}}(\Phi)$  is even.

**Remark 8.** In the main theorems of this paper, we assume that  $\Phi(1)$  is strongly nondegenerate and, hence,  $-1 \notin \sigma(\Phi(1))$ .

For every  $x \in \mathcal{P}(\theta_t)$ , there is a well-defined mean index and, when  $x$  is nondegenerate, the Conley–Zehnder index of  $x$  is also well-defined. In fact, for  $\tilde{x} \in \tilde{\mathcal{L}}\Sigma$ , there is a well-defined, up to homotopy,  $\mathbb{C}$ -vector bundle trivialization of  $x^*T\Sigma$ , and the linearized flow along  $x \in \mathcal{P}(\theta_t)$ ,

$$d\phi_t : T_{x(0)}\Sigma \rightarrow T_{x(t)}\Sigma,$$

can be viewed as a symplectic path,

$$(2-6) \quad \Phi : [0, 1] \rightarrow \text{Sp}(2).$$

Then the mean index  $\Delta_{\phi_t}$  is defined by

$$\Delta_{\{\phi_t\}}(\tilde{x}) := \Delta(\Phi)$$

and the Conley–Zehnder index  $\mu_{\text{CZ}}$  is defined, for nondegenerate orbits  $x$ , by

$$\mu_{\text{CZ}}(\tilde{x}) := \mu_{\text{CZ}}(\Phi).$$

Since  $\Sigma$  is aspherical, the indices are independent of the lift  $\tilde{x}$  of  $x$  and we write

$$\Delta_{\{\phi_t\}}(x) \quad \text{and} \quad \mu_{\text{CZ}}(x)$$

for the mean index and the Conley–Zehnder index of  $x$ , respectively.

These indices satisfy the properties

$$(2-7) \quad \Delta_{\{\phi_t^k\}}(x^k) = k \Delta_{\{\phi_t\}}(x)$$

and

$$(2-8) \quad |\Delta_{\{\phi_t\}}(x) - \mu_{\text{CZ}}(x)| < 1 \quad (\text{when } x \text{ is nondegenerate}).$$

Furthermore, we say that a nondegenerate periodic orbit  $x \in \mathcal{P}(\theta_t)$  is elliptic, or hyperbolic, if the endpoint of the associated symplectic path as in (2-6) is elliptic, or hyperbolic, respectively. Moreover, the stated results hold for a periodic orbit  $x$  if they are satisfied by the corresponding symplectic path  $\Phi$ , as in (2-6). For instance, the claim for orbits corresponding to the first part of Result 1 enunciates that if  $x$  is an elliptic orbit for  $\phi$ , then its mean index is not zero.

**Remark 9** (noncontractible orbits). Let  $\zeta$  be a free homotopy class of maps  $S^1 \rightarrow \Sigma$ . Fix a reference loop  $z$  in  $\zeta$  and a trivialization of  $TM$  along  $z$ . They give rise to a well defined, up to homotopy,  $\mathbb{C}$ -vector bundle trivialization of  $x^*TM$  for every  $x \in \mathcal{L}_\zeta M$  and, for a 1-periodic orbit of  $\phi$ , the linearized flow along  $x$ ,

$$d\phi_t : T_{x(0)}M \rightarrow T_{x(t)}M,$$

can be viewed as a symplectic path  $\Phi : [0, 1] \rightarrow \text{Sp}(2n)$ . Consider the abelian principal covering  $\tilde{\mathcal{L}}_\zeta \Sigma$  with structure group

$$\Gamma_\zeta := \frac{\pi_1(\mathcal{L}_\zeta \Sigma)}{\ker([\bar{\theta}]} ,$$

where  $[\bar{\theta}] : \pi_1(\mathcal{L}_\zeta \Sigma) \rightarrow \mathbb{R}$ . The mean index and the Conley–Zehnder index are defined as above and, since  $\Sigma$  is atoroidal, in this case the indices are also independent of the lifts.

**2D. The Floer–Novikov homology.** In this section, we revisit the definition of the Floer–Novikov homology for contractible nondegenerate periodic orbits.

Consider a smooth almost complex structure  $J$  on  $\Sigma$  compatible with  $\omega$ , i.e., such that

$$g(X, Y) := \omega(X, JY)$$

defines a Riemannian metric on  $\Sigma$ . We will denote by  $\mathcal{J}$  the set of almost complex structures compatible with  $\omega$ . Choose  $J \in \mathcal{J}$  and let  $\tilde{g}$  denote the induced weak Riemannian metric on  $\mathcal{L}\Sigma$  given by

$$\tilde{g}(X_x, Y_x) = \int_{S^1} g(X_x(t), Y_x(t)) dt,$$

where  $X_x$  and  $Y_x$  are vector fields along  $x$ . A gradient flow line is a mapping  $u : \mathbb{R} \times S^1 \rightarrow \Sigma$  satisfying

$$(2-9) \quad \partial_s u(s, t) + J(\partial_t u(s, t) - X_t(u(s, t))) = 0.$$

The maps  $u : \mathbb{R} \rightarrow \mathcal{L}\Sigma$  which satisfy (2-9) with boundary conditions

$$(2-10) \quad \lim_{s \rightarrow \pm\infty} \tilde{u}(s, t) = \tilde{x}_\pm(t),$$

for some lift  $\tilde{u} : \mathbb{R} \rightarrow \tilde{\mathcal{L}}\Sigma$  of  $u$ , can be seen as connecting orbits between  $\tilde{x}_-$  and  $\tilde{x}_+$ . We denote by  $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$  the space of finite energy solutions of (2-9) and (2-10). The energy of a connecting orbit in this space is given by

$$E(u) := \int_{\mathbb{R} \times S^1} |\partial_s u|_g^2 ds dt = \mathcal{A}_{\{\phi_t\}}(\tilde{x}_+) - \mathcal{A}_{\{\phi_t\}}(\tilde{x}_-)$$

when  $x_-$  and  $x_+$  are nondegenerate. The space  $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$  is a smooth manifold of dimension  $\mu_{\text{CZ}}(x_+) - \mu_{\text{CZ}}(x_-)$ . It admits a natural  $\mathbb{R}$ -action given by reparametrization. For nondegenerate  $x, y \in \mathcal{P}(\theta_t)$  such that

$$\mu_{\text{CZ}}(x) - \mu_{\text{CZ}}(y) = 1,$$

we have that  $\mathcal{M}(\tilde{x}, \tilde{y})/\mathbb{R}$  is finite and set

$$n_2(\tilde{x}, \tilde{y}) := \#\mathcal{M}(\tilde{x}, \tilde{y})/\mathbb{R} \quad \text{modulo } 2.$$



Denote by  $\mathcal{P}_k$  the set of elements  $\tilde{x} \in \tilde{\mathcal{L}}\Sigma$  such that  $x \in \mathcal{P}(\theta_t)$  and  $\mu_{\text{CZ}}(x) = k$ . Consider the chain complex where the  $k$ -th chain group  $C_k$  consists of all formal sums

$$\sum \xi_{\tilde{x}} \cdot \tilde{x}$$

with  $\tilde{x} \in \mathcal{P}_k$ ,  $\xi_{\tilde{x}} \in \mathbb{Z}_2$  and such that, for all  $c \in \mathbb{R}$ , the set

$$\{\tilde{x} \mid \xi_{\tilde{x}} \neq 0, \mathcal{A}_{\{\phi_t\}}(\tilde{x}) > c\}$$

is finite. Denote by

$$\Lambda_\theta = \Lambda(\Gamma, [\bar{\theta}], \mathbb{F})$$

the Novikov ring associated with the group  $\Gamma$  (defined in (2-2)) and the weighting homomorphism  $[\bar{\theta}]$  (defined in (2-1)) with values in the field  $\mathbb{F} = \mathbb{Z}_2$ ; see [Hofer and Salamon 1995, Section 4]. The chain group  $C_k$  is a torsion-free module over the algebra  $\Lambda_\theta$ . The rank of this module is the number of elements of  $\mathcal{P}_k$ ; see [Lê and Ono 1995, Lemma 4.2]. For a generator  $\tilde{x}$  in  $C_k$ , the boundary operator  $\partial_k$  is defined as

$$\partial_k(\tilde{x}) = \sum_{\mu_{\text{CZ}}(\tilde{y})=k-1} n_2(\tilde{x}, \tilde{y}) \tilde{y}.$$

Since  $\partial_k$  is invariant under the action of  $\Gamma$ , we extend  $\partial_k$  as a  $\Lambda_\theta$ -linear map from  $C_k$  to  $C_{k-1}$ . The boundary operator  $\partial$  satisfies  $\partial^2 = 0$ . The homology groups

$$\text{HFN}_k(\{\phi_t\}, J) = \frac{\ker \partial_k}{\text{im } \partial_{k+1}}$$

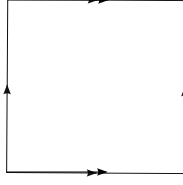
are called the Floer–Novikov homology groups and they are graded  $\Lambda_\theta$ -modules.

Moreover, this homology is invariant under exact deformations of the closed form  $\theta_t$  (see [Lê and Ono 1995, Theorem 4.3]) and hence two paths with the same flux have isomorphic associated Floer–Novikov homology groups.

**Remark 10** (Floer–Novikov homology for noncontractible orbits). As mentioned in the introduction, the Floer–Novikov homology is defined for orbits which lie in some free homotopy class  $\zeta$ . Here, we refer the reader to [Burghilea and Haller 2001] for the details and point out that the Conley–Zehnder index defined in that paper when  $\zeta = 0$  may result in a shift of the standard grading of the Floer–Novikov homology by an even integer; see [Burghilea and Haller 2001, Remark 3.4].

### 3. Proofs of Theorems 1.1 and 1.2

In this section, we construct a flow  $\psi^t$  with exactly  $2g-2$  hyperbolic fixed points and no other periodic orbits on a surface  $\Sigma$  with genus  $g \geq 2$ . This proves Theorem 1.1, and yields the Floer–Novikov homology of a symplectomorphism satisfying property (1-1) and hence also establishes Theorem 1.2.



**Figure 1.** Torus:  $[0, 1] \times [0, 1]$ .

**3A. Construction of a symplectic flow with exactly  $2g - 2$  hyperbolic fixed points and no other periodic orbits.** We start with the case when  $\Sigma$  is a surface of genus  $g = 2$ . The construction has three steps.

In the first step, take two 2-tori  $\mathbb{T}_1$  and  $\mathbb{T}_2$  and the linear flow  $\phi_i^t$  on each torus  $\mathbb{T}_i$  ( $i = 1, 2$ ):

$$\phi_i^t(x_i, y_i) = (tu_i x_i, tv_i y_i) \quad \text{with} \quad u_i \neq 0 \quad \text{and} \quad \frac{v_i}{u_i} \in \mathbb{R} \setminus \mathbb{Q}, \quad i = 1, 2.$$

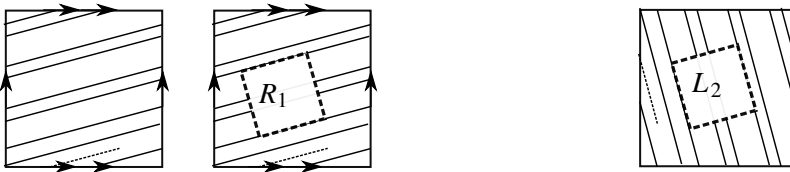
Here  $x_i, y_i$  are the coordinates on  $\mathbb{T}_i = \mathbb{R}^2/\mathbb{Z}^2$ ,  $i = 1, 2$ .

Representing each torus by a square  $[0, 1] \times [0, 1]$ , where the sides  $\{0\} \times [0, 1]$  and  $[0, 1] \times \{0\}$  are identified with  $\{1\} \times [0, 1]$  and  $[0, 1] \times \{1\}$ , respectively (see Figure 1), consider a square  $R_1$  in  $\mathbb{T}_1$  such that two parallel sides are segments of a linear flow line (of  $\phi_1^t$ ) with length  $\varepsilon > 0$  and a square  $L_2$  in  $\mathbb{T}_2$  where two parallel sides are segments of a linear flow line (of  $\phi_2^t$ ) with length  $\varepsilon > 0$  (see Remark 11). In Figure 2, there are three pictures. The two on the left refer to torus  $\mathbb{T}_1$ . The first one represents a flow line of  $\phi_1^t$  (with slope  $v_1/u_1$ ) and the second one shows the square  $R_1$  where two of its sides are segments of the represented flow line. The picture on the right refers to the torus  $\mathbb{T}_2$  where a flow line of  $\phi_2^t$  (with slope  $v_2/u_2$ ) is represented together with the square  $L_2$ .

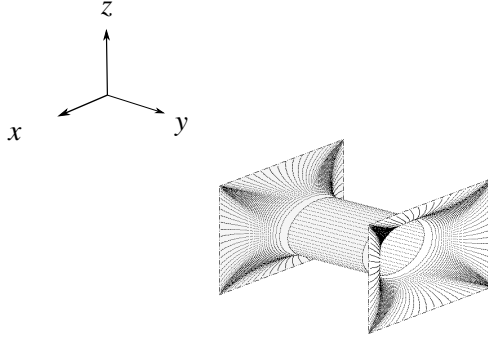
**Remark 11.** In the current case, where  $g = 2$ ,  $\varepsilon$  is small enough so that the squares  $R_1$  and  $L_2$  are inside the square  $[0, 1] \times [0, 1]$ . See Remark 16 for the general case.

**Remark 12.** In order to distinguish the boundaries of the squares from the interiors of the squares, we denote by  $R_1$  and  $L_2$  their boundaries and by  $\mathring{R}_1$  and  $\mathring{L}_2$  their interiors.

In the second step, consider a surface  $P$  obtained by a homotopy between a circle (of radius  $\varepsilon/4$ ) and a square (with side length equal to  $\varepsilon$ ) and a surface  $U$



**Figure 2.** Tori  $\mathbb{T}_1$  (left) and  $\mathbb{T}_2$  (right) and linear flow lines.

**Figure 3.** Surface  $U$ .

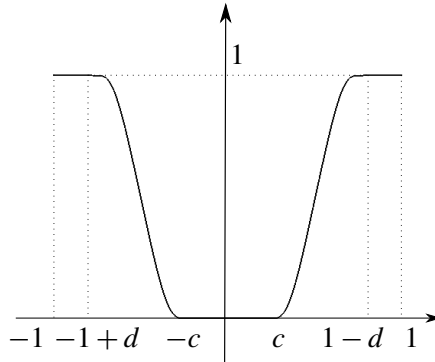
defined piecewise, in the middle, by a (horizontal) cylinder with radius  $\varepsilon/4$  together with a surface  $P$  at each end (with circles identified) as shown in Figure 3. For  $(x, y, z) \in U$ , we have  $-\varepsilon/2 \leq x, z \leq \varepsilon/2$  and  $-1 \leq y \leq 1$ . The boundary of  $U$  is the disjoint union of two squares  $S^L$  and  $S^R$  which lie in the planes  $\{y = -1\}$  and  $\{y = 1\}$ , respectively. Let  $H : U \rightarrow [-\varepsilon/2, \varepsilon/2] \subset \mathbb{R}$  be a smooth function defined by

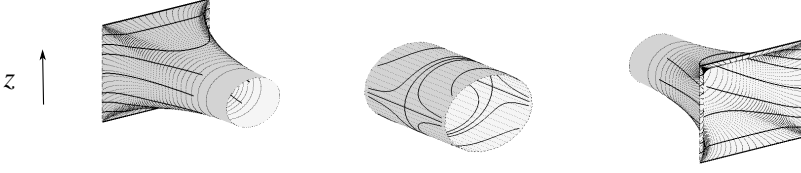
$$(3-1) \quad H(x, y, z) = (1 - \beta(y))yz + \beta(y)z, \quad \text{for } (x, y, z) \in U,$$

where  $\beta : [-1, 1] \rightarrow [0, 1]$  is a smooth function which is 0 when  $y$  is in  $(-c, c)$ , 1 when  $y$  is in  $[-1, -1 + d) \cup (1 - d, 1]$  and strictly monotone in  $(-1 + d, -c) \cup (c, 1 - d)$  with  $0 < c < 1 - d$ ,  $d < 0$ . (See Figure 4 and Remark 14 for the choice of the real numbers  $c$  and  $d$ .)

The Hamiltonian flow lines of  $H$  are depicted in Figure 5. The picture on the left shows the Hamiltonian flow lines in  $U$  when  $y$  is *near*  $-1$ , in the middle are the Hamiltonian flow lines in  $U$  when  $y$  is *near* 0 and on the right are the Hamiltonian flow lines in  $U$  when  $y$  is *near* 1.

**Remark 13.** Here, “ $y$  is near  $-1$ ” means that  $y \in [-1, -1 + d)$ . Similarly, “ $y$  is near 0” means  $y \in (-c, c)$  and “ $y$  is near 1” means  $y \in (1 - d, 1]$ .

**Figure 4.** Function  $\beta$ .



**Figure 5.** Flow lines of the Hamiltonian  $H$  on the surface  $U$ .

In the last step,

- cut off  $\mathring{R}_1$  from  $\mathbb{T}_1$  and  $\mathring{L}_2$  from  $\mathbb{T}_2$ ,
- identify  $R_1$  with  $S^L$  so that the sides of  $R_1$  given by segments of a flow line correspond to the sides of  $S^L$  determined by  $z = \pm\varepsilon/2$  (see Figure 6), and
- identify  $L_2$  with  $S^R$  so that the sides of  $L_2$  given by segments of a flow line correspond to the sides of  $S^R$  determined by  $z = \pm\varepsilon/2$ .

This construction yields a closed surface  $\Sigma$  of genus 2 with a symplectic flow  $\psi^t : \Sigma \rightarrow \Sigma$  which coincides with

- the linear flow  $\phi_1^t$  on  $\mathbb{T}_1 \setminus \mathring{R}_1$ ,
- the linear flow  $\phi_2^t$  on  $\mathbb{T}_2 \setminus \mathring{L}_2$ ,
- the Hamiltonian flow of  $H$  on  $U$ .

Each flow line of  $\psi^t$  lies entirely either

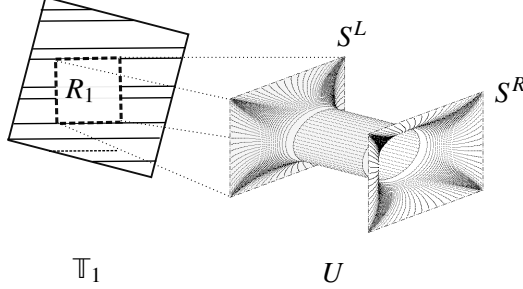
- (1) on the circle  $U \cap \{y = 0\}$ ,
- (2) on  $\mathbb{T}_1 \setminus \mathring{R}_1 \cup (U \cap \{y < 0\}) =: V^-$ , or
- (3) on  $\mathbb{T}_2 \setminus \mathring{L}_2 \cup (U \cap \{y > 0\}) =: V^+$ .

We observe that a flow line of  $\psi^t$  does not intersect both  $V^-$  and  $V^+$ . In (1),  $\psi^t$  has two hyperbolic fixed points and no other periodic orbits. In (2),  $\psi^t$  has no periodic orbits. In fact, by construction, when a flow line of  $\psi^t$  given by  $\phi_1^t$  reaches  $R_1$ , it will either

- stay on  $U$  and converge to one of the hyperbolic fixed points, or
- cross  $R_1$  again after some time and continue in the same flow line of  $\phi_1^t$  when exiting  $\mathbb{T}_1 \setminus \mathring{R}_1$  (since at  $S^L$  the Hamiltonian is given by the height function).

This property together with the fact that  $\phi_1^t$  is an irrational linear flow imply the nonexistence of (long) periodic orbits of  $\psi^t$  on  $V^-$ . Case (3) is similar to (2) and there are no periodic orbits of  $\psi^t$  on  $V^+$ .

**Remark 14.** In the function  $\beta$ , the real numbers  $c$  and  $d$  are selected so that  $c < 0.5 < 1 - d$  and  $c$  and  $1 - d$  are close enough so that the flow  $\psi^t$  has the above properties. For instance, we may choose  $c = d = 0.4$ .



**Figure 6.** Identification of  $R_1$  with  $S^L$ .

Therefore, we have obtained a symplectic flow on  $\Sigma$  with exactly two hyperbolic fixed points, no other periodic orbits. Let us see that the flux of this symplectic flow is given by  $(u_1, v_1, u_2, v_2)$ .

Recall that the fundamental group  $\pi_1(\Sigma)$  of a surface of genus 2 is given by the group

$$\langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1 \rangle$$

with generators  $a_1, b_1, a_2, b_2$  and relation  $[a_1, b_1][a_2, b_2] = 1$ , where  $[a, b] = aba^{-1}b^{-1}$  is the commutator of  $a$  and  $b$ . Consider the following loops in  $\Sigma$ :

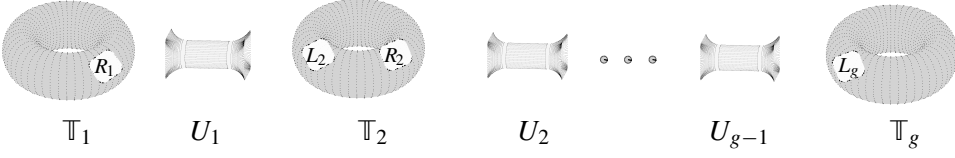
- $\gamma_1$ , such that  $[\gamma_1] = [a_1]$  in  $\pi_1(\Sigma)$  and it corresponds to a vertical line in  $\mathbb{T}_1$  such that  $\psi^t \circ \gamma_1$  does not intersect  $\mathring{R}_1 \cup R_1$  for all  $t \in [0, 1]$ ,
- $\gamma_2$ , such that  $[\gamma_2] = [b_1]$  in  $\pi_1(\Sigma)$  and it corresponds to a horizontal line in  $\mathbb{T}_1$  such that  $\psi^t \circ \gamma_2$  does not intersect  $\mathring{R}_1 \cup R_1$  for all  $t \in [0, 1]$ ,
- $\gamma_3$ , such that  $[\gamma_3] = [a_2]$  in  $\pi_1(\Sigma)$  and it corresponds to a vertical line in  $\mathbb{T}_2$  such that  $\psi^t \circ \gamma_3$  does not intersect  $\mathring{L}_2 \cup L_2$  for all  $t \in [0, 1]$ , and
- $\gamma_4$ , such that  $[\gamma_4] = [b_2]$  in  $\pi_1(\Sigma)$  and it corresponds to a horizontal line in  $\mathbb{T}_2$  such that  $\psi^t \circ \gamma_4$  does not intersect  $\mathring{L}_2 \cup L_2$  for all  $t \in [0, 1]$ .

**Remark 15.** We may have to take  $\varepsilon > 0$  (in the definitions of the squares  $R_1$  and  $L_2$ ) sufficiently small so that the above conditions on the loops  $\gamma_i$  are satisfied.

The area swept by  $\gamma_i$  ( $i = 1, 2$ ) under  $\psi_{(u_1, v_1, u_2, v_2)}^1$  is the area swept by  $\gamma_i$  under  $\phi_1^t$  and hence it is  $u_1$  when  $i = 1$  and  $v_1$  when  $i = 2$ . The area swept by  $\gamma_i$  ( $i = 3, 4$ ) under  $\psi_{(u_1, v_1, u_2, v_2)}^1$  is the area swept by  $\gamma_i$  under  $\phi_2^t$  and hence it is  $u_2$  when  $i = 3$  and  $v_2$  when  $i = 4$ . Therefore, the flux of the symplectic flow  $\psi_{(u_1, v_1, u_2, v_2)}^t$  ( $t \in [0, 1]$ ) is  $(u_1, v_1, u_2, v_2)$ ; recall Remark 6.

The general case, where  $\Sigma$  is a surface of genus  $g \geq 2$ , is similar to the case where  $g = 2$ . Take  $g$  copies of 2-tori,  $\mathbb{T}_1, \dots, \mathbb{T}_g$ , and the linear flow on each  $\mathbb{T}_i$ :

$$\phi_i^t(x_i, y_i) = (tu_i x_i, tv_i y_i) \quad \text{with} \quad u_i \neq 0 \quad \text{and} \quad \frac{v_i}{u_i} \in \mathbb{R} \setminus \mathbb{Q}, \quad i = 1, \dots, g.$$



**Figure 7.** Construction of the surface with genus  $g$ .

On each torus  $\mathbb{T}_i$  (viewed as a square, as above), consider two squares  $R_i$  and  $L_i$  such that

- $\mathring{R}_g \cup R_g = \emptyset$  and  $\mathring{L}_1 \cup L_1 = \emptyset$ ,
- $\mathring{R}_i \cup R_i$  and  $\mathring{L}_i \cup L_i$  are disjoint,
- two parallel sides of  $R_i$  are segments of a flow line of  $\phi_i^t$  in  $\mathbb{T}_i$  ( $i \neq g$ ),
- two parallel sides of  $L_i$  are segments of a flow line of  $\phi_i^t$  in  $\mathbb{T}_i$  ( $i \neq 1$ ), and
- the length of the sides of each square is  $\varepsilon$ .

**Remark 16.** In the general case, where  $g \geq 2$ ,  $\varepsilon$  is small enough so that

$$(\mathring{R}_i \cup R_i) \dot{\cup} (\mathring{L}_i \cup L_i)$$

is inside the square  $[0, 1] \times [0, 1]$ .

Let  $U_i$ , with  $i = 1, \dots, g-1$  be  $g-1$  copies of the surface  $U$  and the corresponding functions  $H_i : U_i \rightarrow [-\varepsilon/2, \varepsilon/2] \subset \mathbb{R}$  defined as in (3-1). Much as in the case where  $g = 2$ , we denote the boundary components of  $U_i$  by  $S_i^L$  and  $S_i^R$ . For each  $i = 1, \dots, g$  (see Figure 7),

- cut off  $\mathring{R}_i$  and  $\mathring{L}_i$  from  $\mathbb{T}_i$ ,
- identify  $R_i$  with  $S_i^L$  so that the sides of  $R_i$  given by segments of a flow line correspond to the sides of  $S_i^L$  determined by  $z = \pm\varepsilon/2$ , and
- identify  $L_i$  with  $S_i^R$  so that the sides of  $L_i$  given by segments of a flow line correspond to the sides of  $S_i^R$  determined by  $z = \pm\varepsilon/2$ .

We have thus obtained a closed surface  $\Sigma$  with genus  $g \geq 2$  and a symplectic flow on  $\Sigma$

$$\psi_{(u_1, v_1, \dots, u_g, v_g)}^t : \Sigma \rightarrow \Sigma$$

which coincides with

- the linear flow  $\phi_i^t$  on  $\mathbb{T}_i \setminus (\mathring{R}_i \cup \mathring{L}_i)$ ,  $i = 1, \dots, g$ ,
- the Hamiltonian flow of  $H_i$  on  $U_i$ ,  $i = 1, \dots, g-1$ .

Arguing as in the case  $g = 2$ , we obtain Theorem 1.1.

**3B. The Floer–Novikov homology of symplectomorphisms satisfying the flux condition (1-1).** Consider  $\phi \in \text{Symp}_0(\Sigma, \omega)$  such that

$$\text{Flux}(\phi) = (u_1, v_1, \dots, u_g, v_g) \text{ with } u_i \neq 0 \text{ and } \frac{v_i}{u_i} \notin \mathbb{Q} \text{ for all } i = 1, \dots, g.$$

Then  $\text{Flux}(\phi) = \text{Flux}(\psi_{(u_1, v_1, \dots, u_g, v_g)}^t|_{t \in [0, 1]})$ , where  $\psi_{(u_1, v_1, \dots, u_g, v_g)}^t$  is the symplectic flow constructed in Section 3A with flux equal to  $(u_1, v_1, \dots, u_g, v_g)$ .

The symplectic flow  $\psi_{(u_1, v_1, \dots, u_g, v_g)}^t$  has  $2g - 2$  hyperbolic fixed points. Then the mean index and the Conley–Zehnder index of the fixed points are 0. Since there are no other periodic orbits, we have that  $C_0$  is the only nontrivial group of the (Floer–Novikov) chain complex and it is generated by  $2g - 2$  fixed points. Hence, the Floer–Novikov homology of

$$\psi = \psi_{(u_1, v_1, \dots, u_g, v_g)}^1$$

is given by

$$\text{HFN}_r(\psi) = \begin{cases} \mathbb{F}^{2g-2} \otimes \Lambda_\theta & \text{if } r = 0, \\ 0 & \text{if } r \neq 0. \end{cases}$$

Since  $\text{Flux}(\phi) = \text{Flux}(\psi_{(u_1, v_1, \dots, u_g, v_g)}^t|_{t \in [0, 1]})$ , Theorem 1.2 follows by the comment on page 31 after the definition of the Floer–Novikov homology.

**Remark 17** (the noncontractible case of Theorem 1.2). Since  $\psi_{(u_1, v_1, \dots, u_g, v_g)}^t$  has no noncontractible periodic orbits, the Floer–Novikov homology for noncontractible orbits of a strongly nondegenerate  $\phi$  is  $\text{HFN}_*(\phi, \zeta) = 0$  for any nontrivial free homotopy class of loops  $\zeta$ .

#### 4. Proofs of Theorems 1.3–1.5

**4A. Proofs of Theorems 1.3 and 1.4.** Theorem 1.3 follows from Theorem 1.4 and the first case of Result 1. Let us then prove Theorem 1.4.

Assume  $\phi$  has finitely many fixed points. Let  $S$  be the finite set of fixed points  $y$  of  $\phi$  such that  $\Delta_{\{\phi_t\}}(y) \neq \Delta_{\{\phi_t\}}(x_0)$ . If  $S \neq \emptyset$ , then define

$$\tau_0 := \min\{k > 1 : k|\Delta_{\{\phi_t\}}(x_0) - \Delta_{\{\phi_t\}}(y)| > 3 \text{ for all } y \in S\},$$

otherwise take  $\tau_0 := 2$ .

The proof goes by contradiction. Let  $\tau$  be a prime integer greater than  $\tau_0$  such that all  $\tau$ -periodic points are iterations of fixed points. We show that, with these assumptions,  $x_0^\tau$ , the  $\tau$ -th iteration of  $x_0$ , contributes nontrivially to the Floer–Novikov homology in degree  $\mu := \mu_{\text{CZ}}(x_0^\tau) \neq 0$  which contradicts Theorem 1.2.

If  $x_0^\tau$  connects to  $y^\tau$ , some  $\tau$ -th iteration of a fixed point  $y$  of  $\phi$ , by a solution of the Floer–Novikov Equation (2-9), then

$$(4-1) \quad |\mu_{\text{CZ}}(x_0^\tau) - \mu_{\text{CZ}}(y^\tau)| = 1.$$

If  $y \in S$ , then

$$(4-2) \quad \tau |\Delta_{\{\phi_t\}}(x_0) - \Delta_{\{\phi_t\}}(y)| > 3$$

and we obtain the following contradiction:

$$1 = |\mu_{\text{CZ}}(x_0^\tau) - \mu_{\text{CZ}}(y^\tau)| \geq \tau |\Delta_{\{\phi_t\}}(x_0) - \Delta_{\{\phi_t\}}(y)| - 2 > 1,$$

where the first inequality follows from (2-7) and (2-8) and the last inequality follows from (4-2). If  $y \notin S$ , then  $\Delta_{\{\phi_t\}}(x_0^\tau) = \Delta_{\{\phi_t\}}(y^\tau)$  by (2-7) and  $\mu_{\text{CZ}}(x_0^\tau) = \mu_{\text{CZ}}(y^\tau)$  by (2-8) which contradicts (4-1). Hence,  $x_0^\tau$  is not connected to any  $y^\tau$  which implies that  $\text{HFN}_\mu(\phi^\tau) \neq 0$  where  $\mu := \mu_{\text{CZ}}(x_0^\tau)$ .

If  $\mu$  were 0, then  $x_0^\tau$  would be hyperbolic (by Result 2). Then we would have that  $\Delta_{\{\phi_t\}}(x_0^\tau) = \mu_{\text{CZ}}(x_0^\tau) = 0$  (by Result 3) which implies, by (2-7), that  $\Delta_{\{\phi_t\}}(x_0) = 0$ . This contradicts our assumption on  $x_0$ . Therefore,  $\mu \neq 0$  and we obtained the wanted contradiction.  $\square$

**Remark 18** (the noncontractible cases of Theorems 1.3 and 1.4).

- In Theorem 1.3, the result still holds true if the elliptic periodic orbit corresponding to  $x_0$  is noncontractible. In this case, we choose  $\tau$  as above, fix the free homotopy class  $\tau\zeta$ , where  $\zeta$  is the free homotopy class of the loop corresponding to  $x_0$ , consider  $x_0^\tau$  as the reference loop in  $\tau\zeta$  and work with the (noncontractible) Floer–Novikov homology  $\text{HFN}(\phi^\tau, \tau\zeta)$ . (Recall Remarks 9 and 10.) By Theorem 1.2 and Remark 10, the Floer–Novikov homology  $\text{HFN}_*(\phi^\tau, \tau\zeta)$  is 0 when  $*$  is an odd integer. Since  $x_0$  is elliptic, its Conley–Zehnder index  $\mu_{\text{CZ}}(x_0)$  is odd (by Result 2). Moreover, using the above argument,  $x_0^\tau$  is not connected to any  $y^\tau$  which implies that  $x_0^\tau$  contributes nontrivially to the Floer–Novikov homology in some odd degree. If the Conley–Zehnder index  $\mu_{\text{CZ}}(x_0^\tau)$  were even, then  $x_0^\tau$  would be hyperbolic and  $\mu_{\text{CZ}}(x_0^\tau) = \Delta(x_0^\tau) = \tau \Delta(x_0)$  would be even. Since  $\tau$  is odd, the mean index  $\Delta(x_0)$  would also be even and, by (2-8), the Conley–Zehnder index  $\mu_{\text{CZ}}(x_0) = \Delta(x_0)$  would be even. Hence  $x_0$  would be hyperbolic contradicting the hypothesis on  $x_0$ . The result then follows.

- In Theorem 1.4, if the fixed point  $x_0$  with nonzero mean index corresponds to a noncontractible periodic orbit with nontrivial homotopy class  $\zeta$  and its  $\tau$ -th iterations, with  $\tau$  a prime integer, lie in nontrivial homotopy classes  $\tau\zeta$ , then  $\phi$  has infinitely many periodic points. These points correspond to periodic orbits which lie in the free homotopy classes formed by iterations of the orbit corresponding to  $x_0$ . In this case, the proof is essentially the same as in the contractible case. (The last paragraph is not needed.) Recall Remark 17.

**4B. Proof of Theorem 1.5.** Suppose the number of fixed points of  $\phi$  is greater than  $2g - 2$ . By (1-2), there exist  $2g - 2$  fixed points  $x_1, \dots, x_{2g-2}$  of  $\phi$  which contribute nontrivially to the Floer–Novikov homology of  $\phi$ . If there exists  $j \in \{1, \dots, 2g - 2\}$



such that  $\Delta_{\{\phi_i\}}(x_j) \neq 0$ , then, by Theorem 1.4, the result follows. If not, then  $\Delta_{\{\phi_i\}}(x_i) = 0$  for all  $i = 1, \dots, 2g - 2$ . Take a fixed point  $x$  such that  $x \neq x_i$  ( $i = 1, \dots, 2g - 2$ ). Either  $\mu_{\text{CZ}}(x) = 0$ ,  $\mu_{\text{CZ}}(x) = 1$  or  $\mu_{\text{CZ}}(x) = 2$ .

Let us first consider the case  $\mu_{\text{CZ}}(x) = 0$ . By (1-2), there exists  $y \in C_1$  such that  $y$  is connected to  $x$  by a solution of the Floer–Novikov equation (2-9). Then, either  $y$  is elliptic or  $y$  is hyperbolic. If  $y$  is elliptic, the result follows by Theorem 1.3. If  $y$  is hyperbolic, then  $\Delta_{\{\phi_i\}}(y) = \mu_{\text{CZ}}(y) = 1 \neq 0$  and the result follows by Theorem 1.4.

Assume now  $\mu_{\text{CZ}}(x) = 1$ . Then, the result follows by the same argument used for  $y$  in the previous step.

Finally, assume  $\mu_{\text{CZ}}(x) = 2$ . Then, by (2-8), we have that  $\Delta_{\{\phi_i\}}(x) \neq 0$  and the result follows by Theorem 1.4.  $\square$

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# MIXING PROPERTIES FOR HOM-SHIFTS AND THE DISTANCE BETWEEN WALKS ON ASSOCIATED GRAPHS

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Let  $\mathcal{H}$  be a finite connected undirected graph and  $\mathcal{H}_{\text{walk}}^2$  be the graph of bi-infinite walks on  $\mathcal{H}$ ; two such walks  $\{x_i\}_{i \in \mathbb{Z}}$  and  $\{y_i\}_{i \in \mathbb{Z}}$  are said to be adjacent if  $x_i$  is adjacent to  $y_i$  for all  $i \in \mathbb{Z}$ . We consider the question: Given a graph  $\mathcal{H}$ , when is the diameter (with respect to the graph metric) of  $\mathcal{H}_{\text{walk}}^2$  finite? Such questions arise while studying mixing properties of hom-shifts (shift spaces which arise as the space of graph homomorphisms from the Cayley graph of  $\mathbb{Z}^d$  with respect to the standard generators to  $\mathcal{H}$ ) and are the subject of this paper.

## 1. Introduction

Let  $\mathcal{A}$  be a finite set called the alphabet. A shape is a finite subset of  $\mathbb{Z}^d$  and a pattern is a function from a shape to the alphabet  $\mathcal{A}$ . Given a finite set of patterns  $\mathcal{F}$  called a forbidden list, a shift of finite type (SFT)  $X_{\mathcal{F}} \subset \mathcal{A}^{\mathbb{Z}^d}$  is the set of configurations in which patterns from  $\mathcal{F}$  and their translates do not appear. There is a natural topology on  $X_{\mathcal{F}}$  coming from the product of the discrete topology on  $\mathcal{A}$  making it a compact metrisable space;  $\mathbb{Z}^d$  acts on it by translation of configurations making it a dynamical system. The study of SFTs for  $d \geq 2$  is rife with numerous undecidability issues. It is not even decidable if an SFT is nonempty [Berger 1966]. It follows immediately that most nontrivial properties of SFTs are undecidable (Proposition 3.2). In this paper we study an important class of SFTs called hom-shifts, for which, a priori, many such issues do not arise.

By  $\mathbb{Z}^d$  we will mean both the group and its Cayley graph with respect to standard generators. Given any SFT,  $X_{\mathcal{F}}$ , we can assume by a standard recoding argument that  $X_{\mathcal{F}}$  is in fact a nearest neighbour SFT (possibly for a different alphabet  $\mathcal{A}$ ), meaning  $\mathcal{F}$  consists of patterns on edges and vertices of  $\mathbb{Z}^d$ . Let  $\text{Hom}(\mathcal{G}, \mathcal{H})$  denote the set of all graph homomorphisms from  $\mathcal{G}$  to  $\mathcal{H}$ . An SFT  $X$  is called a hom-shift if  $X = \text{Hom}(\mathbb{Z}^d, \mathcal{H})$  for some graph  $\mathcal{H}$ ; it is denoted by  $X_{\mathcal{H}}^d$ . Alternatively, a hom-shift can be described as a nearest neighbour SFT which is “symmetric” and “isotropic”,

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that is, if  $v, w \in \mathcal{A}$  are forbidden to sit next to each other in some coordinate direction, then they are forbidden to sit next to each other in all coordinate directions. It follows that a hom-shift  $X_{\mathcal{H}}^d$  is nonempty if and only if  $\mathcal{H}$  has at least one edge. An introduction to SFTs and hom-shifts can be found in Section 2.

Many important SFTs arise as hom-shifts like the hard square shift and the  $n$ -coloured chessboard. In this paper we study certain mixing properties of hom-shifts: topological mixing, block-gluing and strong irreducibility and relate them to some natural questions in graph theory. The mixing conditions studied in this paper are introduced in Section 3. For further background consider [Boyle et al. 2010].

An SFT  $X$  is said to be topologically mixing (or just mixing) if any two patterns appearing in  $X$  can coappear in a configuration in  $X$  provided the corresponding shapes are far enough apart (the distance depending on the patterns). Clearly, a hom-shift  $X_{\mathcal{H}}^d$  is not mixing if  $\mathcal{H}$  is bipartite; the pattern on any partite class of  $\mathbb{Z}^d$  is mapped into a partite class of  $\mathcal{H}$ . It turns out that this is essentially the only obstruction. We prove in Proposition 3.1 that a hom-shift  $X_{\mathcal{H}}^d$  is mixing if and only if  $\mathcal{H}$  is a connected undirected graph which is not bipartite; further if  $\mathcal{H}$  is bipartite then it still satisfies a similar mixing condition but we may need to translate one of the two patterns by a unit coordinate vector. In the heart of the analysis is the following simple idea: we say that two finite walks,  $\{v_i\}_{i=1}^n$  and  $\{w_i\}_{i=1}^n$ , are adjacent if  $v_i$  is adjacent to  $w_i$  for all  $i$ . We show that for all  $n$  and finite connected graphs  $\mathcal{H}$ , the graph of finite walks of length  $n$  is connected.

However we find that the diameter of the graph of finite walks on a graph  $\mathcal{H}$  of length  $n$  might increase with  $n$ . Whether the diameter remains bounded or not relates to another important mixing property called the phased block-gluing property: we say that an SFT  $X$  is block-gluing if there is an  $n \in \mathbb{N}$  such that any two patterns on rectangular shapes in  $X$  can coexist in a configuration in  $X$  provided that they are separated by distance  $n$ . Strong irreducibility (SI) is a similar (though a much stronger) mixing property where there is no restriction on the shape of the patterns.

Again we observe that if the graph  $\mathcal{H}$  is bipartite then  $X_{\mathcal{H}}^d$  is neither block-gluing nor SI. To remedy the situation we introduce the phased block-gluing and the phased SI properties in Section 4 which are similar to the usual block-gluing and SI properties but there is a fixed finite set  $S \subset \mathbb{Z}^d$  by elements of which we are allowed to translate one of the two patterns. We prove in Propositions 4.1 and 4.2 that if  $\mathcal{H}$  is not bipartite then if  $X_{\mathcal{H}}^d$  is phased block-gluing it is block-gluing, and if  $X_{\mathcal{H}}^d$  is phased SI then it is SI. Further if  $\mathcal{H}$  is bipartite and  $X_{\mathcal{H}}^d$  is phased block-gluing or phased SI then the set  $S$  can be chosen to be the origin and any of the coordinate unit vectors. This is done by relating the mixing conditions with some natural graph theoretic questions.

The study of the phased block-gluing property for the  $d$ -dimensional shift space  $X_{\mathcal{H}}^d$  relates to a natural graph structure on  $X_{\mathcal{H}}^{d-1}$ , namely,  $x, y \in X_{\mathcal{H}}^{d-1}$  are said to

be adjacent if  $x_{\vec{i}}$  is adjacent to  $y_{\vec{i}}$  for all  $\vec{i}$  in  $\mathbb{Z}^{d-1}$ . Denote the graph thus obtained by  $\mathcal{H}_{\text{walk}}^d$ . In Proposition 4.1 we prove that  $X_{\mathcal{H}}^d$  is phased block-gluing if and only if the diameter of  $\mathcal{H}_{\text{walk}}^d$  is finite.

It can be proved using the ideas of graph folding in [Nowakowski and Winkler 1983; Brightwell and Winkler 2000] that if  $\mathcal{H}$  is a tree then the space  $X_{\mathcal{H}}^d$  is phased SI. This turns out to be a characterisation for the phased SI property for a large class of graphs: a graph is called four-cycle free if it is connected, it has no self-loops and the four-cycle,  $C_4$ , is not a subgraph. In Section 5 we prove for four-cycle free graphs  $\mathcal{H}$  that  $X_{\mathcal{H}}^d$  is phased block-gluing/phased SI if and only if  $\mathcal{H}$  is a tree. Surprisingly the proof goes via lifts to the universal cover of the graph; in fact following [Wrochna 2015] we prove the results for a more general class of graphs called the four-cycle hom-free graphs (defined in Section 5). In Section 5A we discuss why this characterisation fails when the four-cycle hom-free restriction is removed. The paper concludes with a long list of open questions (Section 6).

Let us summarise. Results regarding decidability among hom-shifts and shifts of finite type are Proposition 2.2, Corollary 2.3 and Proposition 3.2; in Sections 6A and 6G we mention some related open questions. In the proof of Proposition 3.1 and in Proposition 4.1 we reformulate transitivity, mixing and block-gluing in terms of walks on graphs. Proposition 3.1 gives necessary and sufficient conditions for transitivity and mixing. Section 5 discusses the mixing properties for hom-shifts where the corresponding graph is four-cycle hom-free.

We end the introduction with the question which is the cornerstone for this line of research; this we are unable to address. For a more detailed discussion, look at Section 6A.

**Question.** Is it decidable whether a hom-shift is SI/block-gluing?

## 2. SFTs and hom-shifts

Let  $\mathcal{A}$  be a finite set which we refer to as the *alphabet* with the discrete topology; we give the set  $\mathcal{A}^{\mathbb{Z}^d}$  the product topology making it a compact metrizable space. By  $\mathbb{Z}^d$  we will mean both the Cayley graph of  $\mathbb{Z}^d$  with respect to standard generators and the group. The elements of  $\mathcal{A}^{\mathbb{Z}^d}$  are called *configurations* while elements of  $\mathcal{A}^B$  for some finite set  $B$  are called *patterns*. Given a configuration  $x$ , let  $x_{\vec{i}} := x(\vec{i})$  and given a pattern  $a \in \mathcal{A}^B$  and  $\vec{i} \in B$ , let  $a_{\vec{i}} := a(\vec{i})$ .

There is a natural action of  $\mathbb{Z}^d$  on  $\mathcal{A}^{\mathbb{Z}^d}$ : for all  $\vec{i} \in \mathbb{Z}^d$  let

$$\sigma^{\vec{i}} : \mathcal{A}^{\mathbb{Z}^d} \rightarrow \mathcal{A}^{\mathbb{Z}^d} \text{ given by } (\sigma^{\vec{i}}(x))_{\vec{j}} := x_{\vec{i}+\vec{j}}$$

denote the *shift-action*. A *shift space* is a closed set of configurations  $X \subset \mathcal{A}^{\mathbb{Z}^d}$  which is invariant under the shift-action, meaning  $\sigma^{\vec{i}}(X) = X$  for all  $\vec{i} \in \mathbb{Z}^d$ . Alternatively, it can also be defined using forbidden patterns: a set of configurations  $X$  is a shift

space if and only if there is a set of patterns  $\mathcal{F}$  such that

$$X = X_{\mathcal{F}} := \{x \in \mathcal{A}^{\mathbb{Z}^d} : \text{patterns from } \mathcal{F} \text{ do not appear in any shift of } x\}.$$

Look at [Lind and Marcus 1995, Chapter 6] for the proof of the equivalence when  $d = 1$ ; the proof is similar in higher dimensions. In a similar fashion the shift map extends to patterns:

$$\sigma^{\vec{i}} : \mathcal{A}^F \rightarrow \mathcal{A}^{F-\vec{i}} \quad \text{given by} \quad (\sigma^{\vec{i}}(a))_{\vec{j}} := x_{\vec{i}+\vec{j}} \quad \text{for } F \subset \mathbb{Z}^d \text{ and } \vec{j} \in F - \vec{i}.$$

Let  $\vec{0}$  be the origin and  $\{\vec{e}_1^d, \vec{e}_2^d, \dots, \vec{e}_d^d\}$  denote the standard generators of  $\mathbb{Z}^d$ . We drop the superscript when it is obvious from the context. Given  $a, b \in \mathcal{A}$  we denote by  $\langle a, b \rangle^i \in \mathcal{A}^{\{\vec{0}, \vec{e}_i\}}$  the pattern

$$\langle a, b \rangle_{\vec{0}}^i := a, \quad \langle a, b \rangle_{\vec{e}_i}^i = b.$$

Let us look at a few examples:

- (1) Let  $\mathcal{A} = \{0, 1\}$  and  $\mathcal{F} = \{\langle 1, 1 \rangle^i : 1 \leq i \leq d\}$ . Then

$$X_{\mathcal{F}} = \{x \in \{0, 1\}^{\mathbb{Z}^d} : \text{no two appearances of 1 in } x \text{ are adjacent}\}.$$

This is called the *hard square shift*.

- (2) Let  $\mathcal{A} = \{1, 2, \dots, n\}$  and  $\mathcal{F} = \{\langle j, j \rangle^i : 1 \leq i \leq d, 1 \leq j \leq n\}$ . Then

$$X_{\mathcal{F}} = \{x \in \{1, 2, \dots, n\}^{\mathbb{Z}^d} : \text{adjacent symbols in } x \text{ are distinct}\}.$$

This is called the *n-coloured chessboard*.

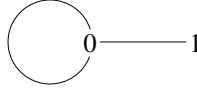
- (3) Let  $d = 1$ ,  $\mathcal{A} = \{0, 1\}$  and  $\mathcal{F} = \{10^{2i-1}1 : i \in \mathbb{Z}\}$ . Then

$$X_{\mathcal{F}} = \{x \in \{0, 1\}^{\mathbb{Z}} : \text{the separation between successive 1s is even}\}.$$

This is called the *even shift*.

Note that in the hard square shift the forbidden list  $\mathcal{F}$  consists of  $d$  elements while in the even shift the forbidden list  $\mathcal{F}$  consists of infinitely many elements. It can in fact be proven that  $\mathcal{F}$  cannot be chosen finite for the even shift.

A shift space  $X$  is called a *shift of finite type* (SFT) if there exists a finite set of forbidden patterns  $\mathcal{F}$  such that  $X = X_{\mathcal{F}}$ . Thus the hard square shift is an SFT while the even shift is not an SFT. Further if  $\mathcal{F}$  can be chosen to be a set of patterns on edges and vertices of  $\mathbb{Z}^d$  then  $X$  is called a *nearest neighbour shift of finite type*. Any SFT can be “recoded” into a nearest neighbour SFT: Given shift spaces  $X$  and  $Y$ , a continuous map  $f : X \rightarrow Y$  which commutes with the shift-action, that is,  $f \circ \sigma^{\vec{i}} = \sigma^{\vec{i}} \circ f$  is called a *sliding block code*. A *factor map* is a sliding block code which is surjective while a *conjugacy* is a sliding block code which is bijective. The inverse of a conjugacy is also a conjugacy; thus conjugacies determine an equivalence relation. Any shift space conjugate to an SFT is also an SFT. Further



**Figure 1.** Graph for the hard square shift.

given an SFT,  $X$ , a simple construction gives us a nearest neighbour SFT,  $Y$ , which is conjugate to  $X$  [Schmidt 1998].

A *periodic configuration* is a configuration  $x \in \mathcal{A}^{\mathbb{Z}^d}$  such that there exists some  $n \in \mathbb{N}$  such that  $\sigma^{n\vec{e}_i}(x) = x$  for all  $1 \leq i \leq d$ . Some fundamental properties of nearest neighbour SFTs are undecidable for  $d \geq 2$ ; for instance there is no algorithm to decide, given a finite set  $\mathcal{F}$ , whether  $X_{\mathcal{F}}$  is nonempty [Berger 1966; Robinson 1971]. Let us review a few salient features of the proof: Fix  $d \geq 2$ . Given a Turing machine  $T$  there is a finite alphabet  $\mathcal{A}_T$  and a finite forbidden list  $\mathcal{F}_T$  such that  $X_{\mathcal{F}_T}^d$  is nonempty if and only if  $T$  does not halt starting on the empty input. Since the halting problem for Turing machines is undecidable, the nonemptiness problem for SFTs (and hence nearest neighbour SFTs) is also undecidable. Further  $X_{\mathcal{F}_T}^d$  has no periodic configurations; this shall be useful later.

All the graphs  $\mathcal{H}$  in this paper are undirected, without multiple edges and have no isolated vertices.

$X \subset \mathcal{A}^{\mathbb{Z}^d}$  is called a *hom-shift* if there exists a finite undirected graph  $\mathcal{H}$  such that  $X = \text{Hom}(\mathbb{Z}^d, \mathcal{H})$ . Alternatively, these are exactly the nearest neighbour SFTs which are symmetric and isotropic, meaning nearest neighbour SFTs which are invariant under the automorphism group of  $\mathbb{Z}^d$  (as a graph). These correspond to vertex shifts in  $d = 1$  defined by an undirected graph [Lind and Marcus 1995, Chapter 2].

For an undirected graph  $\mathcal{H}$  (finite or not) we denote

$$X_{\mathcal{H}}^d := \text{Hom}(\mathbb{Z}^d, \mathcal{H}).$$

Clearly  $X_{\mathcal{H}}^d$  is nonempty if and only if  $\mathcal{H}$  is nonempty. Let  $K_n$  denote the complete graph on  $n$  vertices  $\{1, 2, 3, \dots, n\}$ . Then  $X_{K_n}^d$  is the  $n$ -coloured chessboard. If  $\mathcal{H}$  is the graph given by Figure 1 then  $X_{\mathcal{H}}^d$  is the hard square shift.

We shall frequently use the cartesian product on graphs: Given graphs  $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$  and  $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ ,  $\mathcal{H}_1 \square \mathcal{H}_2$  is the graph with vertex set  $\mathcal{V}_1 \times \mathcal{V}_2$  where  $(v_1, v_2) \sim_{\mathcal{H}_1 \square \mathcal{H}_2} (w_1, w_2)$  if and only if  $v_1 = w_1$  and  $v_2 \sim_{\mathcal{H}_2} w_2$  or  $v_1 \sim_{\mathcal{H}_1} w_1$  and  $v_2 = w_2$ . By  $\square_{j=1}^r \mathcal{H}_j$  we mean the graph  $\mathcal{H}_1 \square \mathcal{H}_2 \square \dots \square \mathcal{H}_r$ .

For a shift space  $X \subset \mathcal{A}^{\mathbb{Z}^d}$ , the *language* for  $X$  is given by

$$\mathcal{L}(X) := \{a \in \mathcal{A}^B : B \subset \mathbb{Z}^d \text{ is finite and there exists } x \in X \text{ such that } x|_B = a\}.$$

These are called the set of *globally allowed patterns* in  $X$ . On the other hand, if the shift space  $X$  is given by a forbidden list  $\mathcal{F}$ , then a pattern  $a$  is called *locally allowed* if no element of  $\mathcal{F}$  appears in the shifts of  $a$ . For shifts of finite type, it

is not decidable whether a locally allowed pattern is globally allowed [Robinson 1971]. For hom-shifts, it is in fact decidable; this follows from Proposition 2.1.

A *shape* is a finite subset of  $\mathbb{Z}^d$ . For a shape  $A \subset \mathbb{Z}^d$  we write  $\mathcal{L}_A(X) := \mathcal{L}(X) \cap \mathcal{A}^A$ . We will often denote an element  $a \in \mathcal{A}^A$  by  $\langle a \rangle_A$  instead to emphasise the domain of the pattern. By a *rectangular shape*  $A \subset \mathbb{Z}^d$  we mean that  $A = \square_{j=1}^d I_j$  for some finite intervals  $I_j \subset \mathbb{Z}$ . A *rectangular pattern* in  $X$  is a pattern in  $\mathcal{L}_A(X)$  for some rectangular shape  $A$ . The following proposition implies that periodic configurations are dense in hom-shifts.

**Proposition 2.1** (extension of (possibly infinite) rectangular patterns). *Let  $\mathcal{H}$  be an undirected graph and  $A = \square_{i=1}^d I_i$  where the  $I_i$  are intervals in  $\mathbb{Z}$ . Then for all homomorphisms  $a \in \text{Hom}(A, \mathcal{H})$  there exists a configuration  $x \in X_{\mathcal{H}}^d$  such that  $x|_A = a$ . If  $A$  is a finite set then  $x$  can be chosen to be periodic.*

Here is the idea: Let us first observe this for a finite  $A$ . If any of the side-lengths of  $A$  is 1 then we extend it to a pattern  $\tilde{a}$  on a bigger rectangular shape by “stacking shifts” of the pattern  $a$ . Then we reflect the pattern obtained about its faces to obtain a pattern  $b$  on a still bigger rectangular shape and finally tile  $\mathbb{Z}^d$  by this new pattern to obtain a periodic configuration. Some of the details are provided in part (2) of the proof of [Chandgotia 2017, Lemma 8.2]. Although the proof there is for the case when  $\mathcal{H}$  is a tree, it carries forward without any change to our context.

Now if  $A$  is an (infinite) rectangular shape then by compactness of shift spaces and a standard limiting argument (taking a sequence of rectangular patterns which approximate the given pattern and considering the corresponding sequence of configurations extending them), the result for finite rectangular patterns implies the proposition.

In the following, by a given nearest neighbour SFT  $X$ , we mean a given finite list of patterns  $\mathcal{F}$  on edges and vertices of  $\mathbb{Z}^d$  such that  $X = X_{\mathcal{F}}$ .

**Proposition 2.2.** *Fix  $d \geq 2$ . Let  $\mathcal{C}$  be a set of SFTs for which periodic points are dense for all  $X \in \mathcal{C}$ . It is undecidable whether an SFT is conjugate to some  $X \in \mathcal{C}$ .*

*Proof.* Let  $X \in \mathcal{C}$ . Recall the properties of the SFT  $X_{\mathcal{F}_T}$ , which was constructed given a Turing machine  $T$ . We can assume (possibly after a change in alphabet for  $X$ ) that the underlying alphabets for  $X$  and  $X_{\mathcal{F}_T}$  are disjoint for all Turing machines  $T$ . Then  $X \cup X_{\mathcal{F}_T}$  is a nearest neighbour SFT for every Turing machine  $T$ ; since the  $X_{\mathcal{F}_T}$  do not have periodic points, periodic points are dense in  $X \cup X_{\mathcal{F}_T}$  if and only if  $X_{\mathcal{F}_T}$  is empty.

We claim that this implies  $X \cup X_{\mathcal{F}_T}$  is conjugate to a member of  $\mathcal{C}$  if and only if  $X_{\mathcal{F}_T}$  is empty. Clearly, if  $X_{\mathcal{F}_T}$  is empty then  $X \cup X_{\mathcal{F}_T} \in \mathcal{C}$ . Now suppose  $X_{\mathcal{F}_T}$  is not empty. Since it does not have periodic points, periodic points are not dense in  $X \cup X_{\mathcal{F}_T}$  and hence it cannot be conjugate to a member of  $\mathcal{C}$ .



Thus it is undecidable whether  $X \cup X_{\mathcal{F}_T}$  is conjugate to an element of  $\mathcal{C}$  proving, more generally, that it is undecidable whether a nearest neighbour SFT is conjugate to an element of  $\mathcal{C}$ .  $\square$

**Corollary 2.3.** *It is undecidable whether a shift space  $X$  is conjugate to a hom-shift for  $d \geq 2$ .*

This follows immediately from Propositions 2.1 and 2.2.

### 3. Some mixing conditions for hom-shifts

In this section we introduce some topological mixing conditions for shift spaces in  $d \geq 2$ . This introduction will be far from comprehensive; for more background consider [Boyle et al. 2010].

Given  $A, B \subset \mathbb{Z}^d$  let

$$d_\infty(A, B) := \min_{\vec{i} \in A, \vec{j} \in B} |\vec{i} - \vec{j}|_\infty \text{ where } |\cdot|_\infty \text{ is the } l_\infty \text{ norm on } \mathbb{R}^d.$$

A shift space  $X$  is *topologically mixing* or just *mixing* if for all  $\langle a \rangle_A, \langle b \rangle_B \in \mathcal{L}(X)$  there exists  $n \in \mathbb{N}$  such that for all  $\vec{i} \in \mathbb{Z}^d$ ,  $|\vec{i}|_\infty \geq n$  there is  $x \in X$  satisfying  $x|_A = a$  and  $\sigma^{\vec{i}}(x)|_B = b$ . A shift space  $X$  is *transitive* if for all  $\langle a \rangle_A, \langle b \rangle_B \in \mathcal{L}(X)$  there exists  $x \in X$  and  $\vec{i} \in \mathbb{Z}^d$  such that  $x|_A = a$  and  $\sigma^{\vec{i}}(x)|_B = b$ .

In this section we shall prove the following result:

**Proposition 3.1.** *Let  $d \geq 2$  and  $\mathcal{H}$  be a finite undirected graph. Then  $X_{\mathcal{H}}^d$  is transitive if and only if  $\mathcal{H}$  is connected. Further it is mixing if and only if  $\mathcal{H}$  is connected and not bipartite.*

Before we proceed with the proof, we shall consider a few more standard mixing conditions. A stronger mixing property which is also the main theme of this paper is the *block-gluing property*: a shift space  $X$  is said to be *block-gluing* if there exists an  $n \in \mathbb{N}$  such that for all rectangular patterns  $\langle a \rangle_A, \langle b \rangle_B \in \mathcal{L}(X)$  satisfying  $d_\infty(A, B) \geq n$  there exists  $x \in X$  such that  $x|_A = a$  and  $x|_B = b$ . A still stronger mixing condition is the following: a shift space  $X$  is called *strongly irreducible* (SI) if there exists  $n \in \mathbb{N}$  such that for all  $\langle a \rangle_A, \langle b \rangle_B \in \mathcal{L}(X)$  satisfying  $d_\infty(A, B) \geq n$  there exists  $x \in X$  such that  $x|_A = a$  and  $x|_B = b$ .

The hard square shift  $X$  is SI for  $n=2$ : given shapes  $A, B$  such that  $d_\infty(A, B) \geq 2$  and  $a \in \mathcal{L}_A(X), b \in \mathcal{L}_B(X)$ , then  $x \in X$  given by

$$x_{\vec{i}} := \begin{cases} a_{\vec{i}} & \text{if } \vec{i} \in A, \\ b_{\vec{i}} & \text{if } \vec{i} \in B, \\ 0 & \text{otherwise} \end{cases}$$

satisfies  $x|_A = a$  and  $x|_B = b$ . We will give a large class of examples in this paper of hom-shifts which are block-gluing and of hom-shifts which are mixing but not

block-gluing (Theorem 5.3). We will also give an example of a hom-shift which is (phased) block-gluing but not (phased) SI in Section 5A; the phased properties are introduced in Section 4.

**Proposition 3.2.** *Let  $d \geq 2$ . It is undecidable whether an SFT is transitive/mixing/block-gluing/SI.*

The proof is very similar to the proof of Proposition 2.2. Let  $X$  be the hard square shift and consider for every Turing machine  $T$  the SFT,  $X_{\mathcal{F}_T}$  (with alphabet disjoint from  $\{0, 1\}$ ); it is undecidable whether  $X_{\mathcal{F}_T}$  is empty. Further  $X \cup X_{\mathcal{F}_T}$  is transitive/mixing/block-gluing/SI if and only if  $X_{\mathcal{F}_T}$  is empty; thus the proposition follows.

Now let us return to Proposition 3.1. Suppose  $\mathcal{H}$  is not connected. Let  $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$  where  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are disjoint. Then  $X_{\mathcal{H}}^d = X_{\mathcal{H}_1}^d \cup X_{\mathcal{H}_2}^d$  where  $X_{\mathcal{H}_1}^d$  and  $X_{\mathcal{H}_2}^d$  are nonempty shift spaces over disjoint alphabets proving that  $X_{\mathcal{H}}^d$  is not transitive. Also if  $\mathcal{H}$  is bipartite then  $X_{\mathcal{H}}^d$  is not mixing since for a given  $x \in X_{\mathcal{H}}^d$  and all even vertices  $\vec{i} \in \mathbb{Z}^d$ , the  $x_{\vec{i}}$  belong to the same partite class.

To prove the other direction we will use some auxiliary constructions; the idea used for the proof of this proposition will be useful later as well.

A walk  $p$  in a graph  $\mathcal{H}$  is a (finite, infinite or bi-infinite) sequence of vertices  $\{p_i\}$  in  $\mathcal{H}$  satisfying  $p_i \sim_{\mathcal{H}} p_{i+1}$  for all  $i$ . A walk of length  $k$  is a finite walk  $p = (p_0, p_1, \dots, p_k)$ ; let  $|p|$  denote the length of  $p$ . Denote by  $[i, j]$  the induced subgraph of  $\mathbb{Z}$  on  $\{i, i+1, \dots, j\}$ . For every  $n \in \mathbb{Z}^+$  and  $d \geq 2$  let

$$B_n^{d-1} := \square_{j=1}^{d-1} [-n, n],$$

that is, the  $l^\infty$  ball of radius  $n$  in  $\mathbb{Z}^{d-1}$ . Consider the graph

$$\mathcal{H}_{n, \text{walk}}^d := (\text{Hom}(B_n^{d-1}, \mathcal{H}), \mathcal{E}_{n, \text{walk}}^d)$$

where

$$\mathcal{E}_{n, \text{walk}}^d := \{(x, y) : x_{\vec{i}} \sim_{\mathcal{H}} y_{\vec{i}} \text{ for all } \vec{i} \in B_n^{d-1}\}.$$

As with homotopies in algebraic topology, there is a walk from  $p$  to  $q$  in  $\mathcal{H}_{n, \text{walk}}^d$  of length  $k$  if and only if there is a graph homomorphism  $a : B_n^{d-1} \square [0, k] \rightarrow \mathcal{H}$  such that  $a_{\vec{i}, 0} = p_{\vec{i}}$  and  $a_{\vec{i}, k} = q_{\vec{i}}$  for all  $\vec{i} \in B_n^{d-1}$ . We will use this correspondence frequently throughout the paper. Connectivity of the graph  $\mathcal{H}_{n, \text{walk}}^d$  is related to the transitivity/mixing property via the following lemma:

**Lemma 3.3.** *Let  $d \geq 2$  and  $\mathcal{H}$  be a finite undirected graph. If  $\mathcal{H}_{n, \text{walk}}^d$  is connected for all  $n \in \mathbb{Z}^+$  then  $X_{\mathcal{H}}^d$  is transitive. Further if  $\mathcal{H}_{n, \text{walk}}^d$  is connected and not bipartite for all  $n \in \mathbb{Z}^+$  then  $X_{\mathcal{H}}^d$  is mixing.*

*Proof.* Let  $A, B \subset \mathbb{Z}^d$  be finite sets and  $\langle a \rangle_A, \langle b \rangle_B \in \mathcal{L}(X_{\mathcal{H}}^d)$  be given and suppose  $\mathcal{H}_{n, \text{walk}}^d$  is connected. We need to prove that there exists some  $\vec{i} \in \mathbb{Z}^d$  such that

$x|_A = a$  and  $(\sigma^{\vec{i}}(x))|_B = b$ . By shifting the patterns if necessary and extending them to  $B_n^d$  for some large enough  $n > 1$  we can assume  $A = B = B_n^d$ . By the hypothesis we know that  $\mathcal{H}_{n,\text{walk}}^d$  is connected so there is a walk of length  $k$  for some  $k \in \mathbb{N}$  from  $a|_{B_n^{d-1} \square \{n\}}$  to  $b|_{B_n^{d-1} \square \{-n\}}$ ; here the graphs  $B_n^{d-1} \square \{-n\}$  and  $B_n^{d-1} \square \{n\}$  are identified with  $B_n^{d-1}$ . As observed earlier, this gives us a homomorphism  $c : B_n^{d-1} \square [n, n+k] \rightarrow \mathcal{H}$  such that  $c_{\vec{i},n} = a_{\vec{i},n}$  and  $c_{\vec{i},n+k} = b_{\vec{i},-n}$ . “Pasting together” the configurations  $a$  and  $b$  to  $c$  we get a homomorphism

$$l : B_n^{d-1} \square [-n, 3n+k] \rightarrow \mathcal{H}$$

with  $l|_{B_n^d} = a$  and

$$l|_{B_n^d + (2n+k)\vec{e}_d} = (\sigma^{-(2n+k)\vec{e}_d}(b)).$$

By Proposition 2.1 we see that  $X_{\mathcal{H}}^d$  is transitive.

For mixing, assume that  $\mathcal{H}_{n,\text{walk}}^d$  is connected and not bipartite. As before, let  $\langle a \rangle_{B_n^d}, \langle b \rangle_{B_n^d} \in \mathcal{L}(X_{\mathcal{H}}^d)$ . Choose an integer  $k$  such that for all  $a', b' \in \mathcal{H}_{n,\text{walk}}^d$  there is a walk from  $a'$  to  $b'$  of length  $r$  for all  $r \geq k$ . Let  $\vec{i} = (i_1, i_2, \dots, i_d)$  such that  $|\vec{i}|_{\infty} \geq k + 2n$ ; without the loss of generality assume  $i_d \geq k + 2n$ . Extend  $a$  and  $b$  periodically to get extensions  $\tilde{a}, \tilde{b}$  on  $\mathbb{Z}^{d-1} \square [-n, n]$ . There is a walk in  $\mathcal{H}_{n,\text{walk}}^d$  from  $\tilde{a}|_{B_n^{d-1} \square \{n\}}$  to  $(\sigma^{-\vec{i}}(\tilde{b}))|_{B_n^{d-1} \square \{-n+i_d\}}$  of length  $i_d - 2n$ ; thus we obtain a homomorphism  $l' : B_n^{d-1} \square [-n, n+i_d] \rightarrow \mathcal{H}$  such that

$$l'|_{B_n^d} = \tilde{a}|_{B_n^d} \quad \text{and} \quad l'|_{B_n^{d-1} \square [-n+i_d, n+i_d]} = (\sigma^{-\vec{i}}(\tilde{b}))|_{B_n^{d-1} \square [-n+i_d, n+i_d]}.$$

By periodically extending  $l'$  we get a homomorphism  $\tilde{l} : \mathbb{Z}^{d-1} \square [-n, n+i_d] \rightarrow \mathcal{H}$  such that

$$\tilde{l}|_{\mathbb{Z}^{d-1} \square [-n, n]} = \tilde{a} \quad \text{and} \quad (\sigma^{\vec{i}}(\tilde{l}))|_{\mathbb{Z}^{d-1} \square [-n, n]} = \tilde{b}.$$

By Proposition 2.1 the proof is complete.  $\square$

*Proof of Proposition 3.1.* Fix  $d \geq 2$ . We have already shown that if  $\mathcal{H}$  is not connected then  $X_{\mathcal{H}}^d$  is not transitive. Let  $\mathcal{H}$  be a connected graph. By Lemma 3.3 we need to prove that the graph  $\mathcal{H}_{n,\text{walk}}^d$  is connected for all  $n \in \mathbb{Z}^+$ . When  $n=0$ , then  $B_n^d$  consists of a single vertex; the connectivity of  $\mathcal{H}_{0,\text{walk}}^d$  is exactly the connectivity of the graph  $\mathcal{H}$ . Now fix  $n \geq 1$ . The argument will follow by induction on  $d$ .

**Base case:** Let  $p, q \in \mathcal{H}_{n,\text{walk}}^2$ . Consider a walk  $r$  (say of length  $k$ ) in  $\mathcal{H}$  from  $p_n$  to  $q_{-n}$ . Let  $s : [-n, 3n+k] \rightarrow \mathcal{H}$  be the walk “joining”  $p, r$  and  $q$ ; formally, let

$$s_i := \begin{cases} p_i & \text{if } i \in [-n, n], \\ r_{i-n} & \text{if } i \in [n, n+k], \\ q_{i-2n-k} & \text{if } i \in [n+k, 3n+k]. \end{cases}$$

By “stacking together the shifts” of the pattern  $s$  we get a walk in  $\mathcal{H}_{n,\text{walk}}^2$  from  $p$  to  $q$ ; formally, let  $p^i \in \mathcal{H}_{n,\text{walk}}^2$  be given by  $p_t^i := s_{i+t}$  for  $t \in [-n, n]$  and

$i \in [0, 2n + k]$ . Then  $p^0 = p$ ,  $p^{2n+k} = q$  and

$$p_t^i = s_{i+t} \sim_{\mathcal{H}} s_{i+t+1} = p_t^{i+1},$$

proving that  $p^i \sim_{\mathcal{H}_{n,\text{walk}}^2} p^{i+1}$ .

**The induction step:** Let's assume the conclusion for some  $d \geq 2$ . Let  $p, q \in \mathcal{H}_{n,\text{walk}}^{d+1}$ . By the induction hypothesis there exists a walk  $r^0, r^1, \dots, r^k$  in  $\mathcal{H}_{n,\text{walk}}^d$  from  $p|_{[-n,n]^{d-1} \square \{n\}}$  to  $q|_{[-n,n]^{d-1} \square \{-n\}}$  for some  $k$ . Let

$$s : [-n, n]^{d-1} \square [-n, 3n + k] \rightarrow \mathcal{H}$$

be a graph homomorphism obtained by “joining”  $p, r^0, r^1, \dots, r^k$  and  $q$ ; formally, let

$$s_{\vec{j},i} := \begin{cases} p_{\vec{j},i} & \text{if } i \in [-n, n], \\ r_{\vec{j}}^{i-n} & \text{if } i \in [n, n+k], \\ q_{\vec{j},i-2n-k} & \text{if } i \in [n+k, 3n+k], \end{cases}$$

for all  $\vec{j} \in [-n, n]^{d-1}$ . As in the base case, by “stacking together the shifts” of the pattern  $s$  we get a walk from  $p$  to  $q$  in  $\mathcal{H}_{n,\text{walk}}^{d+1}$ . This proves that  $X_{\mathcal{H}}^d$  is transitive.

If  $\mathcal{H}$  is bipartite with partite classes  $V_1, V_2$  and  $x \in X_{\mathcal{H}}^d$  then  $x_{\vec{0}} \in V_1$  if and only if  $x_{\vec{i}} \in V_1$  for all even vertices  $\vec{i} \in \mathbb{Z}^d$ ; thus  $X_{\mathcal{H}}^d$  isn't mixing. For the other direction assume that  $\mathcal{H}$  is connected and not bipartite. By the first part of the proof the graph  $\mathcal{H}_{n,\text{walk}}^d$  is connected. Further since  $\mathcal{H}$  is not bipartite it has an odd cycle. Thus one obtains an odd cycle in  $\mathcal{H}_{n,\text{walk}}^d$  for all  $n$ ; hence it is also not bipartite. By Lemma 3.3, the proof is complete.  $\square$

Observe that the proof of Proposition 3.1 gives us a bound on the diameter in the graph metric of  $\mathcal{H}_{n,\text{walk}}^{d+1}$  given the diameter of  $\mathcal{H}_{n,\text{walk}}^d$ . Specifically

$$(3-1) \quad \text{diam}(\mathcal{H}_{n,\text{walk}}^{d+1}) \leq 2n + \text{diam}(\mathcal{H}_{n,\text{walk}}^d)$$

for all  $d \geq 0$ ; here  $\mathcal{H}_{n,\text{walk}}^0$  is interpreted as the graph  $\mathcal{H}$ . We are interested in cases where  $\text{diam}(\mathcal{H}_{n,\text{walk}}^{d+1})$  is uniformly bounded for all  $n$ .

The following corollary follows from arguments in the proofs of Lemma 3.3 and Proposition 3.1.

**Corollary 3.4.** *Let  $\mathcal{H}$  be a finite undirected graph. The following are equivalent:*

- (1)  $\mathcal{H}$  is connected.
- (2)  $X_{\mathcal{H}}^d$  is transitive for some  $d \in \mathbb{N}$ .
- (3)  $X_{\mathcal{H}}^d$  is transitive for all  $d \in \mathbb{N}$ .
- (4)  $\mathcal{H}_{n,\text{walk}}^d$  is connected for all  $n$  and  $d$ .
- (5)  $\mathcal{H}_{n,\text{walk}}^d$  is connected for some  $n$  and  $d$ .

Let  $\mathcal{H}$  be a bipartite connected graph with partite classes  $V_1, V_2$ . Then  $X_{\mathcal{H}}^d = X_1 \cup X_2$  where

$$X_i := \{x \in X_{\mathcal{H}}^d : x_{\vec{0}} \in V_i\}.$$

To prove that if  $\mathcal{H}$  is connected and not bipartite then  $X_{\mathcal{H}}^d$  is mixing, note that the only place we used the fact that the graph  $\mathcal{H}$  is not bipartite is to conclude that  $\mathcal{H}_{n,\text{walk}}^d$  is also not bipartite. If  $\mathcal{H}$  is connected and bipartite then  $\mathcal{H}_{n,\text{walk}}^d$  is also connected and bipartite; there exists  $K \in \mathbb{N}$  such that for any  $k > K$  and  $p, q \in \mathcal{H}_{n,\text{walk}}^d$  there is a walk from  $p$  to  $q$  of length either  $k$  or  $k+1$ . It follows that  $X_1$  and  $X_2$  are mixing SFTs for the  $(2\mathbb{Z})^d$  action. So we have the following proposition:

**Corollary 3.5.** *If  $\mathcal{H}$  is a bipartite connected graph then  $X_{\mathcal{H}}^d$  is a disjoint union of two conjugate mixing SFTs with respect to the  $(2\mathbb{Z})^d$  action.*

This is reminiscent of the case for  $d = 1$ , where if  $X$  is an irreducible SFT of period  $p$  then it can be written as a disjoint union of  $p$  conjugate mixing SFTs with respect to the  $p\mathbb{Z}$  action; see [Lind and Marcus 1995, Exercise 4.5.6]. We shall state similar conclusions in Corollary 4.3 for some stronger mixing properties. We remark that the group  $(2\mathbb{Z})^d$  (which is of index  $2^d$  in  $\mathbb{Z}^d$ ) can be replaced by any subgroup contained in the same partite class as  $\vec{0}$  in these results. However for the ease of notation and understanding, we will work with the group  $(2\mathbb{Z})^d$  instead.

#### 4. The phased block-gluing and SI property for hom-shifts

From here on the graph  $\mathcal{H}$  is connected unless stated otherwise. The graph metric on  $\mathcal{H}$  is denoted by  $d_{\mathcal{H}}$ . The block-gluing property is too restrictive: if  $\mathcal{H}$  is bipartite then  $X_{\mathcal{H}}^d$  is not even mixing. With this in view, we define the following:

A shift space  $X$  is said to be *phased block-gluing* if there exists an  $n \in \mathbb{N}$  and a finite set  $S \subset \mathbb{Z}^d$  such that for all rectangular patterns  $\langle a \rangle_A, \langle b \rangle_B \in \mathcal{L}(X)$  satisfying  $d_{\infty}(A, B) \geq n$  there exists  $x \in X$  such that  $x|_A = a$  and  $\sigma^{\vec{i}}(x)|_B = b$  for some  $\vec{i} \in S$ . The set  $S$  will be called a *gluing set* of  $X$  and  $n$  will be called a *gluing distance*. Observe that although the phased block-gluing property is defined for finite rectangular patterns  $\langle a \rangle_A, \langle b \rangle_B$ , it immediately applies (by using the compactness of shift spaces) to infinite rectangular patterns as well.

From here on fix  $d \geq 2$  unless mentioned otherwise. We will now construct some auxiliary graphs which will be useful in the study of the phased block-gluing property. Let  $\mathcal{H}_{\text{walk}}^d = (X_{\mathcal{H}}^{d-1}, \mathcal{E}_{\text{walk}}^d)$  be the graph where

$$\mathcal{E}_{\text{walk}}^d = \{(x, y) : x_{\vec{i}} \sim_{\mathcal{H}} y_{\vec{i}} \text{ for all } \vec{i} \in \mathbb{Z}^{d-1}\}.$$

Given symbols  $v, w$  we denote by  $(v, w)^{\infty, d-1} \in \{v, w\}^{\mathbb{Z}^{d-1}}$  the checkerboard configuration given by

$$(v, w)_{\vec{i}}^{\infty, d-1} := \begin{cases} v & \text{if } \vec{i} \text{ is in the same partite class as } \vec{0}, \\ w & \text{otherwise.} \end{cases}$$

Similarly  $v^{\infty, d-1}$  is the constant configuration given by

$$v_{\vec{i}}^{\infty, d-1} := v \text{ for all } \vec{i} \in \mathbb{Z}^{d-1}.$$

Let us look at a few examples.

- (1) If  $\mathcal{H}$  is a graph with a single edge and vertices  $v, w$  then  $X_{\mathcal{H}}^{d-1}$  consists only of the two checkerboard patterns  $(v, w)^{\infty, d-1}$  and  $(w, v)^{\infty, d-1}$  which are connected to each other in  $\mathcal{H}_{\text{walk}}^d$ .
- (2) Let  $\mathcal{H}$  be the graph in Figure 1 (the graph for the hard square shift). Since  $0, 1 \sim_{\mathcal{H}} 0$ , for all  $x \in X_{\mathcal{H}}^{d-1}$ ,

$$x \sim_{\mathcal{H}_{\text{walk}}^d} 0^{\infty, d-1}.$$

In general, if  $\mathcal{H}$  is a graph with a vertex  $\star$  such that  $\star \sim_{\mathcal{H}} v$  for all  $v \in \mathcal{H}$  (in other words, if the hom-shift  $X_{\mathcal{H}}^{d-1}$  has a so-called safe symbol) then  $x \sim_{\mathcal{H}_{\text{walk}}^d} \star^{\infty, d-1}$  for all  $x \in X_{\mathcal{H}}^{d-1}$ .

The usual graph metric on  $\mathcal{H}_{\text{walk}}^d$  is denoted by  $d_{\mathcal{H}}^w$ . Further we say  $d_{\mathcal{H}}^w(x, y) := \infty$  if there is no finite walk from  $x$  to  $y$ . The *diameter* of  $\mathcal{H}_{\text{walk}}^d$  is denoted by

$$\text{diam}(\mathcal{H}_{\text{walk}}^d) := \sup_{x, y \in \mathcal{H}_{\text{walk}}^d} d_{\mathcal{H}}^w(x, y).$$

The diameter of the graph  $\mathcal{H}_{\text{walk}}^d$  measures the maximum distance required to transition between two configurations in  $X_{\mathcal{H}}^{d-1}$ . Recall the graphs  $\mathcal{H}_{n, \text{walk}}^d$ . They may be thought to “approximate” the graph  $\mathcal{H}_{\text{walk}}^d$ ; in fact it follows easily that

$$\text{diam}(\mathcal{H}_{\text{walk}}^d) = \infty \text{ if and only if } \lim_{n \rightarrow \infty} \text{diam}(\mathcal{H}_{n, \text{walk}}^d) = \infty.$$

The proof is left to the reader. Look also at Section 6C.

As mentioned previously with respect to the graphs  $\mathcal{H}_{n, \text{walk}}^d$ , there is a correspondence between walks  $x = p^0, p^1, \dots, p^k = y$  in  $\mathcal{H}_{\text{walk}}^d$  from  $x$  to  $y$  of length  $k$  and  $\tilde{x} \in \text{Hom}(\mathbb{Z}^{d-1} \square [0, k], \mathcal{H})$  satisfying  $\tilde{x}_{\vec{i}, 0} = x_{\vec{i}}$  and  $\tilde{x}_{\vec{i}, k} = y_{\vec{i}}$ . We will use this and similar correspondences throughout the paper.

While the graphs  $\mathcal{H}_{n, \text{walk}}^d$  were useful in analysing the mixing and transitivity of the hom-shifts  $X_{\mathcal{H}}^d$  (as in Proposition 3.1), the graph  $\mathcal{H}_{\text{walk}}^d$  relates to the phased block-gluing property by the following proposition:

**Proposition 4.1.** *Let  $\mathcal{H}$  be a finite, undirected graph. Then:*

- (1)  $X_{\mathcal{H}}^d$  is block-gluing if and only if there exists an  $n \in \mathbb{N}$  such that for all  $x, y \in X_{\mathcal{H}}^{d-1}$  there exists a walk of length  $n$  in  $\mathcal{H}_{\text{walk}}^d$  starting at  $x$  and ending at  $y$ .
- (2)  $X_{\mathcal{H}}^d$  is phased block-gluing if and only if  $\text{diam}(\mathcal{H}_{\text{walk}}^d) < \infty$ .

- (3) If  $\mathcal{H}$  is bipartite and  $X_{\mathcal{H}}^d$  is phased block-gluing then the gluing set can be chosen to be  $\{0, \vec{e}_i\}$  for all  $1 \leq i \leq d$ .
- (4) If  $\mathcal{H}$  is not bipartite and  $X_{\mathcal{H}}^d$  is phased block-gluing then  $X_{\mathcal{H}}^d$  is block-gluing.

*Proof of Proposition 4.1(1).* Suppose that  $X_{\mathcal{H}}^d$  is block-gluing with gluing distance  $n$ . Let  $x, y \in X_{\mathcal{H}}^{d-1}$ . We can identify them as elements of  $\text{Hom}(\mathbb{Z}^{d-1} \square \{0\}, \mathcal{H})$  and  $\text{Hom}(\mathbb{Z}^{d-1} \square \{n\}, \mathcal{H})$  respectively. By the block-gluing property there exists  $z \in X_{\mathcal{H}}^d$  for which  $z|_{\mathbb{Z}^{d-1} \square \{0\}} = x$  and  $z|_{\mathbb{Z}^{d-1} \square \{n\}} = y$ . Equivalently we have found a walk of length  $n$  in  $\mathcal{H}_{\text{walk}}^d$  from  $x$  to  $y$ .

Conversely suppose that for all  $x, y \in X_{\mathcal{H}}^{d-1}$  there exists a walk of length  $n$  starting at  $x$  and ending at  $y$ . Since we can always lengthen such a walk by revisiting a configuration adjacent to  $y$ , it follows that for all  $x, y \in X_{\mathcal{H}}^{d-1}$ ,  $m \geq n$ , there is a walk of length  $m$  from  $x$  to  $y$ .

We would like to prove that  $X_{\mathcal{H}}^d$  is block-gluing with block-gluing distance  $n$ . Let  $\langle a \rangle_A, \langle b \rangle_B$  be two rectangular patterns in  $X_{\mathcal{H}}^d$  such that  $d_{\infty}(A, B) = m$ . Using the symmetry and isotropy in hom-shifts and translating the patterns (if necessary), by Proposition 2.1 we can assume  $A \subset \mathbb{Z}^{d-1} \square [-r, r]$  and  $B \subset \mathbb{Z}^{d-1} \square [m+r, m+r+k]$  for some  $r, k \in \mathbb{N}$ . Consider

$$\tilde{y} \in \text{Hom}(\mathbb{Z}^{d-1} \square [-r, r], \mathcal{H}) \quad \text{and} \quad \tilde{z} \in \text{Hom}(\mathbb{Z}^{d-1} \square [m+r, m+r+k], \mathcal{H})$$

such that  $\tilde{y}|_A = a$  and  $\tilde{z}|_B = b$ . Then there exists a walk  $p^0, p^1, \dots, p^m$  from  $\tilde{y}|_{\mathbb{Z}^{d-1} \square \{r\}}$  to  $\tilde{z}|_{\mathbb{Z}^{d-1} \square \{m+r\}}$  in  $\mathcal{H}_{\text{walk}}^d$ . Hence we get a homomorphism

$$\tilde{x} \in \text{Hom}(\mathbb{Z}^{d-1} \square [-r, m+r+k], \mathcal{H})$$

such that  $\tilde{x}|_{\mathbb{Z}^{d-1} \square [-r, r]} = \tilde{y}$  and  $\tilde{x}|_{\mathbb{Z}^{d-1} \square [m+r, m+r+k]} = \tilde{z}$ . By Proposition 2.1 there exists  $x \in X_{\mathcal{H}}^d$  such that  $x|_A = a$  and  $x|_B = b$ .  $\square$

In the following proof, by  $|\cdot|_1$  we mean the  $l_1$  metric on  $\mathbb{R}^d$ .

*Proof of Proposition 4.1(2).* Suppose that  $X_{\mathcal{H}}^d$  is phased block-gluing with gluing distance  $n$  and gluing set  $S$ . Choose  $m \geq n$  large enough such that  $m > |\vec{i}|_1$  for all  $\vec{i} \in S$ . Let  $x, y \in X_{\mathcal{H}}^{d-1}$  be given. As before we identify  $x$  and  $y$  as configurations in  $\text{Hom}(\mathbb{Z}^{d-1} \square \{0\}, \mathcal{H})$  and  $\text{Hom}(\mathbb{Z}^{d-1} \square \{m\}, \mathcal{H})$  respectively. By the phased block-gluing property there exists  $z \in X_{\mathcal{H}}^d$  such that  $z|_{\mathbb{Z}^{d-1} \square \{0\}} = x$  and  $\sigma^{\vec{i}}(z)|_{\mathbb{Z}^{d-1} \square \{m\}} = y$  for some  $\vec{i} \in S$ . Write  $\vec{i} = (\vec{i}^f, i_d)$  where  $\vec{i}^f \in \mathbb{Z}^{d-1}$ . Then

$$z_{\vec{j}, m+i_d} = y_{\vec{j}-\vec{i}^f} \quad \text{for all } \vec{j} \in \mathbb{Z}^{d-1}.$$

Thus we have obtained a walk from  $x$  to  $\sigma^{-\vec{i}^f}(y)$  in  $\mathcal{H}_{\text{walk}}^d$  of length  $m + i_d$ . By using the fact that  $z' \sim_{\mathcal{H}_{\text{walk}}^d} \sigma^{\vec{e}_j^{d-1}}(z')$  for all  $1 \leq j \leq d-1$  and  $z' \in X_{\mathcal{H}}^{d-1}$  we get a walk from  $\sigma^{-\vec{i}^f}(y)$  to  $y$  of length  $|\vec{i}^f|_1$ . Thus

$$\text{diam}(\mathcal{H}_{\text{walk}}^d) \leq \max_{\vec{i} \in S} (m + |\vec{i}|_1).$$

Now let us prove the converse. Suppose  $\text{diam}(\mathcal{H}_{\text{walk}}^d) < n < \infty$ . Let  $1 \leq j \leq d$ ,  $S = \{\vec{0}, \vec{e}_j^d\}$  and let  $\langle a \rangle_A, \langle b \rangle_B \in \mathcal{L}(X_{\mathcal{H}}^d)$  be rectangular patterns such that  $d_{\infty}(A, B) = m \geq n + 1$ . We can assume  $A \subset \mathbb{Z}^{d-1} \square [-r, r]$  and  $B \subset \mathbb{Z}^{d-1} \square [m+r, m+r+k]$  for some  $r, k \in \mathbb{N}$ . Consider

$$\tilde{y} \in \text{Hom}(\mathbb{Z}^{d-1} \square [-r, r], \mathcal{H}) \quad \text{and} \quad \tilde{z} \in \text{Hom}(\mathbb{Z}^{d-1} \square [m+r, m+r+k], \mathcal{H})$$

such that  $\tilde{y}|_A = a$  and  $\tilde{z}|_B = b$ . There is a walk of length either  $m-1$  or  $m$  from  $\tilde{y}|_{\mathbb{Z}^{d-1} \square \{r\}}$  to  $\tilde{z}|_{\mathbb{Z}^{d-1} \square \{m+r\}}$  since there is always a walk of length 2 from any vertex in  $\mathcal{H}_{\text{walk}}^d$  to itself.

**Case 1:** *A walk of length  $m$  is found.* We get  $\tilde{x} \in \text{Hom}(\mathbb{Z}^{d-1} \square [-r, m+r+k], \mathcal{H})$  such that  $\tilde{x}|_{\mathbb{Z}^{d-1} \square [-r, r]} = \tilde{y}$  and  $\tilde{x}|_{\mathbb{Z}^{d-1} \square [m+r, m+r+k]} = \tilde{z}$ . By Proposition 2.1 there exists  $x \in X_{\mathcal{H}}^d$  such that  $x|_A = a$  and  $x|_B = b$ .

**Case 2:** *A walk of length  $m-1$  is found.* This is similar to the previous case; just replace the pattern  $\tilde{z}$  by  $\sigma^{-\vec{e}_j^d}(\tilde{z})$ .  $\square$

*Proof of Proposition 4.1(3).* Note that we have proved that the phased block-gluing property for  $X_{\mathcal{H}}^d$  implies that  $\text{diam}(\mathcal{H}_{\text{walk}}^d) < \infty$  and that  $\text{diam}(\mathcal{H}_{\text{walk}}^d) < \infty$  implies that  $X_{\mathcal{H}}^d$  has the phased block-gluing property where the gluing set  $S$  can be chosen to be  $\{\vec{0}, \vec{e}_i\}$  for  $1 \leq i \leq d$ . Thus, if  $X_{\mathcal{H}}^d$  is phased block-gluing then the gluing set  $S$  can be chosen to be  $\{\vec{0}, \vec{e}_i\}$  for  $1 \leq i \leq d$ .  $\square$

*Proof of Proposition 4.1(4).* Suppose  $\mathcal{H}$  is a finite, undirected graph which is not bipartite and  $X_{\mathcal{H}}^d$  is phased block-gluing. If  $\mathcal{H}$  is a single vertex with a self-loop then  $\mathcal{H}_{\text{walk}}^d$  is a single configuration with a self-loop as well; there is nothing to prove. If  $\mathcal{H}$  is not a single vertex with a self-loop then since  $\mathcal{H}$  is not bipartite there exist cycles of even and odd length in  $\mathcal{H}$  and (hence) in  $\mathcal{H}_{\text{walk}}^d$ . Thus the graph  $\mathcal{H}_{\text{walk}}^d$  is aperiodic.

Moreover since  $X_{\mathcal{H}}^d$  is phased block-gluing, from Proposition 4.1(2) we know that  $\mathcal{H}_{\text{walk}}^d$  has finite diameter. Since  $\mathcal{H}_{\text{walk}}^d$  is aperiodic and has finite diameter, from standard arguments (see [Durrett 2010, Lemma 6.6.3]) one can prove that the adjacency matrix of the graph  $\mathcal{H}_{\text{walk}}^d$  is primitive, meaning, there exists  $m \in \mathbb{N}$  such that for every  $x, y \in X_{\mathcal{H}}^{d-1}$  there exists a walk of length  $m$  from  $x$  to  $y$  in  $\mathcal{H}_{\text{walk}}^d$ . By Proposition 4.1(1), the proof is complete.  $\square$

In exactly the same way, the phased SI property can also be defined: a shift space  $X$  is said to be *phased SI* if there exists an  $n \in \mathbb{N}$  and a finite set  $S \subset \mathbb{Z}^d$  such that for all patterns  $\langle a \rangle_A, \langle b \rangle_B \in \mathcal{L}(X)$  satisfying  $d_{\infty}(A, B) \geq n$  there exists  $x \in X$  such that  $x|_A = a$  and  $\sigma^{\vec{i}}(x)|_B = b$  for some  $\vec{i} \in S$ .  $S$  will be called an *SI gluing set* of  $X$  and  $n$  will be called an *SI gluing distance*.

**Proposition 4.2.** *Let  $\mathcal{H}$  be a finite, undirected graph. Then:*



- (1) If  $\mathcal{H}$  is bipartite and  $X_{\mathcal{H}}^d$  is phased SI, the SI gluing set can be chosen to be  $\{\vec{0}, \vec{e}_i\}$  for all  $1 \leq i \leq d$ .
- (2) If  $\mathcal{H}$  is not bipartite and  $X_{\mathcal{H}}^d$  is phased SI then it is SI.

Since the arguments for the proof of this proposition are similar to those in the proof of Proposition 4.1, we will not repeat them here. Roughly speaking, in Proposition 4.1 we obtained the result by translating the question into one about walks on the auxiliary graphs  $\mathcal{H}_{\text{walk}}^d$ . For SI we can use the following simple equivalence instead: Given a set  $A \subset \mathbb{Z}^d$  let

$$\partial_r A = \{\vec{i} \in \mathbb{Z}^d \setminus A : |\vec{i} - \vec{j}|_1 \leq r \text{ for some } \vec{j} \in A\}.$$

A nearest neighbour SFT  $X$  is SI if and only if there is an  $N \in \mathbb{N}$  such that for all  $n \geq N$ , finite  $A \subset \mathbb{Z}^d$  and  $\langle a \rangle_A, \langle b \rangle_{\partial_n A \setminus \partial_{n-1} A} \in \mathcal{L}(X)$ , there exists  $x \in X$  such that  $x|_A = a$  and  $x|_{\partial_n A \setminus \partial_{n-1} A} = b$ .

As in Corollary 3.5 we can also conclude:

**Corollary 4.3.** *Let  $\mathcal{H}$  be a bipartite finite undirected graph. If  $X_{\mathcal{H}}^d$  is phased block-gluing/phased SI then  $X_{\mathcal{H}}^d$  is a union of two disjoint conjugate SFTs with respect to the  $(2\mathbb{Z})^d$  action which are block-gluing/SI respectively.*

This follows from the fact that for a phased block-gluing/phased SI hom-shift, the gluing set/SI gluing set can be chosen to be  $\{0, \vec{e}_i\}$  for all  $1 \leq i \leq d$ . The proof is left to the reader.

We will need the following “monotonicity” result:

**Proposition 4.4.** *Let  $\mathcal{H}$  be a finite undirected graph and  $d_1 < d_2$ . If  $X_{\mathcal{H}}^{d_1}$  is not phased block-gluing/phased SI then  $X_{\mathcal{H}}^{d_2}$  is not phased block-gluing/phased SI.*

Let us see this for the phased block-gluing property; the proof for the phased SI property uses similar ideas. Suppose that  $X_{\mathcal{H}}^{d_1}$  is not phased block-gluing. Fix  $n \in \mathbb{N}$ . By Proposition 4.1 we know that  $\text{diam}(\mathcal{H}_{\text{walk}}^{d_1}) = \infty$ . Thus there exist  $x, y \in X_{\mathcal{H}}^{d_1-1}$  such that  $d_{\mathcal{H}}^w(x, y) \geq n$ . By Proposition 2.1 there exist  $x^1, y^1 \in X_{\mathcal{H}}^{d_2-1}$  such that  $x_{(\vec{i}, \vec{0})}^1 = x_{\vec{i}}^1$  and  $y_{(\vec{i}, \vec{0})}^1 = y_{\vec{i}}^1$  for all  $\vec{i} \in \mathbb{Z}^{d_1-1}$ . Now given a walk (if it exists),

$$x^1, x^2, \dots, x^k = y^1,$$

from  $x^1$  to  $y^1$  in  $\mathcal{H}_{\text{walk}}^{d_2}$ ,

$$x^1|_{\mathbb{Z}^{d_1-1} \square \{\vec{0}\}}, x^2|_{\mathbb{Z}^{d_1-1} \square \{\vec{0}\}}, \dots, x^k|_{\mathbb{Z}^{d_1-1} \square \{\vec{0}\}}$$

is a walk in  $\mathcal{H}_{\text{walk}}^{d_1}$  (up to identification of  $\mathbb{Z}^{d_1-1} \square \{\vec{0}\}$  with  $\mathbb{Z}^{d_1-1}$ ). Hence

$$d_{\mathcal{H}}^w(x^1, y^1) \geq n.$$

Since  $n$  was arbitrary we have proven that  $\text{diam}(\mathcal{H}_{\text{walk}}^{d_2}) = \infty$  proving that  $X_{\mathcal{H}}^{d_2}$  is not phased block-gluing.

We end this section with a few minor structural remarks. Let  $C_n$  denote the  $n$ -cycle with vertices  $\{0, 1, 2, \dots, n-1\}$ . The phased SI/phased block-gluing property for transitive hom-shifts is not stable under containment: For instance we will prove that  $X_{C_3}^2$  is not phased block-gluing in Theorem 5.3. However  $X_{\text{Edge}}^2$  and  $X_{K_4}^2$  are both phased SI [Briceño 2014] where Edge is the induced subgraph on a pair of vertices in  $C_3$  and  $C_3$  is isomorphic to an induced subgraph of  $K_4$ . The mixing properties are however preserved under certain products:

The tensor product of graphs  $\mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$  and  $\mathcal{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ , denoted by  $\mathcal{H}_1 \times \mathcal{H}_2$ , is the graph with vertex set  $\mathcal{V}_1 \times \mathcal{V}_2$  and  $(v_1, v_2) \sim_{\mathcal{H}_1 \times \mathcal{H}_2} (w_1, w_2)$  if  $v_1 \sim_{\mathcal{H}_1} w_1$  and  $v_2 \sim_{\mathcal{H}_2} w_2$ .

**Proposition 4.5.** *Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be graphs such that  $X_{\mathcal{H}_1}^d$  and  $X_{\mathcal{H}_2}^d$  are phased SI/phased block-gluing. Let  $\mathcal{H}$  be a connected component of  $\mathcal{H}_1 \times \mathcal{H}_2$ . Then  $X_{\mathcal{H}}^d$  is also phased SI/phased block-gluing.*

We understand the case of the cartesian product to a much lesser extent and it might be of interest for future work.

*Proof.* There are three separate cases to consider: neither  $\mathcal{H}_1$  nor  $\mathcal{H}_2$  is bipartite, exactly one of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  is bipartite and finally both  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are bipartite. The proofs for the three cases are similar given the following well known observations: If  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are connected graphs which are not bipartite then  $\mathcal{H}_1 \times \mathcal{H}_2$  is connected and bipartite. If exactly one of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  is bipartite and both are connected then  $\mathcal{H}_1 \times \mathcal{H}_2$  is also bipartite and connected. If both  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are bipartite and connected then  $\mathcal{H}_1 \times \mathcal{H}_2$  has two graph components, both are connected bipartite graphs.

Since these three cases are very similar we shall only prove the theorem for the case where both  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are not bipartite. Let  $X_{\mathcal{H}_1}^d$  and  $X_{\mathcal{H}_2}^d$  be phased SI (and hence SI given Proposition 4.2). Let  $(x^1, y^1), (x^2, y^2) \in X_{\mathcal{H}_1 \times \mathcal{H}_2}^d$ . Let  $n$  be the maximum of the SI gluing distances for  $X_{\mathcal{H}_1}^d$  and  $X_{\mathcal{H}_2}^d$ . Let  $A, B \subset \mathbb{Z}^d$  such that they are separated by distance  $n$ . Then there exists  $(x, y) \in X_{\mathcal{H}_1 \times \mathcal{H}_2}^d$  such that  $x|_A = x^1|_A$ ,  $x|_B = x^2|_B$ ,  $y|_A = y^1|_A$  and  $y|_B = y^2|_B$ . The proof for the block-gluing property follows the same idea; we need to restrict to rectangular shapes  $A$  and  $B$ .  $\square$

Finally we observe that the lack of the block-gluing property is equivalent to the graph  $\mathcal{H}_{\text{walk}}^d$  being disconnected:

**Proposition 4.6.** *Let  $\mathcal{H}$  be a finite undirected graph. Then  $\text{diam}(\mathcal{H}_{\text{walk}}^d) = \infty$  if and only if  $\mathcal{H}_{\text{walk}}^d$  is disconnected.*

*Proof.* We will prove the proposition in the case when  $\mathcal{H}$  is not bipartite; the proof for the bipartite case is similar and left to the reader. Let  $\text{diam}(\mathcal{H}_{\text{walk}}^d) = \infty$ . Then either  $\mathcal{H}_{\text{walk}}^d$  is disconnected or for all  $n \in \mathbb{N}$  there exist configurations  $x^n, y^n \in X_{\mathcal{H}}^{d-1}$

such that  $d_{\mathcal{H}}^w(x^n, y^n) \geq n$ . By choosing a large enough subpattern from these configurations it follows that there exists  $k_n \in \mathbb{N}$  and  $a^n, b^n \in \mathcal{H}_{k_n, \text{walk}}^d$  such that the shortest walk from  $a^n$  to  $b^n$  is of length greater than or equal to  $n$ . Since  $\mathcal{H}$  is not bipartite, by Proposition 3.1, the hom-shift  $X_{\mathcal{H}}^d$  is mixing. Thus there exist  $x, y \in X_{\mathcal{H}}^{d-1}$  such that there exists  $\vec{i}_n \in \mathbb{Z}^{d-1}$  satisfying

$$\sigma^{\vec{i}_n}(x)|_{B_{k_n}^{d-1}} = a^n, \quad \sigma^{\vec{i}_n}(y)|_{B_{k_n}^{d-1}} = b^n \quad \text{for all } n \in \mathbb{N}.$$

It follows that  $d_{\mathcal{H}}^w(x, y) = \infty$  implying that  $\mathcal{H}_{\text{walk}}^d$  is disconnected.

For the other direction, if  $\mathcal{H}_{\text{walk}}^d$  is disconnected then its diameter is infinite; this follows from the definition of the diameter.  $\square$

### 5. Phased mixing properties for four-cycle hom-free graphs

We say that an undirected graph  $\mathcal{H}$  is a *four-cycle hom-free graph* if for all graph homomorphisms  $f : C_4 \rightarrow \mathcal{H}$  either  $f(0) = f(2)$  or  $f(1) = f(3)$ . Let us begin by unravelling the definition.

**Proposition 5.1.** *An undirected graph  $\mathcal{H}$  is four-cycle hom-free if and only if  $C_4$  is not a subgraph of  $\mathcal{H}$  and if  $v \in \mathcal{H}$  has a self-loop then  $w_1, w_2 \sim_{\mathcal{H}} v$  and  $w_1, w_2 \neq v$  implies  $w_1 \not\sim_{\mathcal{H}} w_2$ .*

*Proof.* Let us see the forward direction; the arguments for the backward direction are similar in nature and left to the reader. Suppose  $\mathcal{H}$  is four-cycle hom-free. Since there exists no graph homomorphism  $f \in \text{Hom}(C_4, \mathcal{H})$  which is an embedding, the graph  $C_4$  is not a subgraph of  $\mathcal{H}$ . Now suppose the vertex  $v \in \mathcal{H}$  has a self-loop,  $w_1, w_2 \sim_{\mathcal{H}} v$  and  $w_1, w_2 \neq v$ . Consider the map  $f' : C_4 \rightarrow \mathcal{H}$  given by  $f'(0) = f'(1) := v$ ,  $f'(2) := w_1$ ,  $f'(3) := w_2$ ; it is a graph homomorphism if and only if  $w_1 \sim_{\mathcal{H}} w_2$ . But for the map  $f'$ ,  $f'(0) \neq f'(2)$  and  $f'(1) \neq f'(3)$ . Thus by the four-cycle hom-free property of  $\mathcal{H}$  it follows that  $f'$  is not a graph homomorphism from where it follows that  $w_1 \not\sim_{\mathcal{H}} w_2$ .  $\square$

It follows from Proposition 5.1 that a graph  $\mathcal{H}$  without self-loops is four-cycle hom-free if and only if it is a *four-cycle free graph* in the sense of [Chandgotia 2017], that is,  $C_4$  is not a subgraph of  $\mathcal{H}$ . It was observed in [Chandgotia 2017] that a homomorphism from  $\mathbb{Z}^d$  to  $\mathcal{H}$  can be lifted to the universal cover  $\mathcal{H}_{\text{uni}}$  (defined below). This includes graphs  $\mathcal{H}$  which are trees and cycles  $C_n$  for  $n \neq 4$ . A particular case is that of  $n = 3$ ;  $X_{C_3}^d$  is the space of proper 3-colourings of  $\mathbb{Z}^d$ .

This condition was studied in [Wrochna 2015] in the context of reconfiguration problems; we remark that the so-called fundamental groupoid in that paper is intimately related to the universal cover of  $\mathcal{H}$ . If  $\mathcal{H} = C_3$  then the lifts correspond to the so-called height functions [Lieb 1967].

In addition it follows from Proposition 5.1 that the graph for the hard square shift (Figure 1) satisfies the hypothesis. For trees with loops, we refer to [Briceño and Pavlov 2017, Proposition 8.1 and its corollaries] for related results.

In this section we describe a procedure for deciding the mixing conditions of  $X_{\mathcal{H}}^d$  for a four-cycle-hom-free graph. For this we require a notion of folding in graphs: We say that a vertex  $v$  *folds* into  $w$  if  $N_{\mathcal{H}}(v) \subset N_{\mathcal{H}}(w)$ . In this case  $\mathcal{H} \setminus \{v\}$  is called a *fold* of the graph  $\mathcal{H}$ . A graph is called *stiff* if it does not have any nontrivial folds. Starting with a finite graph  $\mathcal{H}$  we can obtain a stiff graph by a sequence of folds; stiff graphs thus obtained are the same up to graph isomorphism [Brightwell and Winkler 2000, Theorem 4.4]. A graph  $\mathcal{H}$  is called *dismantlable* if there exists a sequence of graphs  $\mathcal{H} = \mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$  such that  $\mathcal{H}_{i+1}$  is a fold of the graph  $\mathcal{H}_i$  for every  $i$  and  $\mathcal{H}_n$  is a vertex with or without self-loop. If  $\mathcal{H}$  is a connected dismantlable graph which is not an isolated vertex then it follows that the stiff graph obtained by successive folds of  $\mathcal{H}$  is a vertex with a self-loop. A graph  $\mathcal{H}$  is called *bipartite-dismantlable* if there exists a sequence of graphs  $\mathcal{H} = \mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$  such that  $\mathcal{H}_{i+1}$  is a fold of the graph  $\mathcal{H}_i$  for every  $i$  and  $\mathcal{H}_n$  is either a single edge or a single vertex with a self-loop. Graph folding was introduced in [Nowakowski and Winkler 1983] to study cop-win graphs; later in [Brightwell and Winkler 2000] it was observed that folding preserves a lot of properties of the graphs. Since a fold of a graph  $\mathcal{H}$  is bipartite if and only if  $\mathcal{H}$  is bipartite it follows that if a graph  $\mathcal{H}$  is bipartite-dismantlable, then it is dismantlable if and only if  $\mathcal{H}$  is not bipartite.

The following proposition essentially follows from arguments similar to those in the proof of [Brightwell and Winkler 2000, Theorem 4.1] and we omit them here:

**Proposition 5.2.** *Let  $\mathcal{H}$  be a bipartite-dismantlable graph. Then  $X_{\mathcal{H}}^d$  is phased SI. If  $\mathcal{H}$  is bipartite-dismantlable and  $X_{\mathcal{H}}^d$  is SI then  $\mathcal{H}$  is dismantlable.*

We can now state the main result of this section.

**Theorem 5.3.** *Let  $\mathcal{H}$  be a four-cycle hom-free graph. The following are equivalent:*

- (a)  $X_{\mathcal{H}}^d$  is phased SI.
- (b)  $X_{\mathcal{H}}^d$  is phased block-gluing.
- (c)  $\mathcal{H}$  is bipartite-dismantlable.

The four-cycle hom-free condition is necessary for these equivalences; we will discuss this further after the proof of Theorem 5.3.

Since phased SI is stronger than phased block-gluing, clearly (a) implies (b) and by Proposition 5.2, (c) implies (a). To complete the proof of the theorem we need to prove (b) implies (c). For this we need to introduce the universal cover. For more details, look at [Chandgotia 2017; Angluin 1980; Stallings 1983].

A graph homomorphism  $\phi : \mathcal{H}' \rightarrow \mathcal{H}$  is called a *graph covering* if it is surjective and for all  $v \in \mathcal{H}$ , the restricted map  $\phi|_{N_{\mathcal{H}'}(v)}$  is bijective onto  $N_{\mathcal{H}}(\phi(v))$ ; the

induced map from  $X_{\mathcal{H}'}^d$  to  $X_{\mathcal{H}}^d$  is denoted by  $\tilde{\phi}$ . There is some subtlety here. Undirected graphs  $\mathcal{H}$  can be viewed as 1-CW-complexes where the vertices form 0-cells and the edges form the 1-cells of the complex. If  $\mathcal{H}$  has no self-loops, then clearly the condition for a map  $\phi : \mathcal{H}' \rightarrow \mathcal{H}$  to be a graph covering implies that it is a topological covering as well. However a topological covering space of a graph  $\mathcal{H}$  viewed as a 1-CW-complex may be different from the covering graph of  $\mathcal{H}$  when  $\mathcal{H}$  has a self-loop. For instance, let  $\mathcal{H}$  be a graph with a single vertex and a self-loop and  $\mathcal{H}'$  be a graph with exactly one edge connecting two vertices;  $\mathcal{H}'$  is a covering graph of  $\mathcal{H}$  however  $\mathcal{H}$  is homeomorphic to  $S^1$  as a CW-complex and its only covering spaces are itself and  $\mathbb{R}$ ; neither of these are homeomorphic to  $\mathcal{H}'$ .

To avoid confusion, by a covering space of  $\mathcal{H}$  we mean the usual topological covering space of  $\mathcal{H}$  and by a covering graph of  $\mathcal{H}$  we mean it in the sense as defined above; these two notions coincide if  $\mathcal{H}$  has no self-loops.

A *universal covering graph* of  $\mathcal{H}$ , denoted by  $\mathcal{H}_{\text{uni}}$  is a covering graph of  $\mathcal{H}$  which is a tree; this is unique up to graph isomorphism. Alternatively it can be defined as the connected covering graph  $(\mathcal{H}_{\text{uni}}, \phi_{\text{uni}})$  satisfying the following (universal) property: given a covering graph map  $\phi : \mathcal{H}' \rightarrow \mathcal{H}$  there exists a covering graph map  $\phi' : \mathcal{H}_{\text{uni}} \rightarrow \mathcal{H}'$  such that  $\phi \circ \phi' = \phi_{\text{uni}}$ . There is an explicit construction of these graphs: A *nonbacktracking walk* in a graph  $\mathcal{H}$  is a finite walk in which subsequent steps do not use the same edge, that is, walks  $p_1, p_2, \dots, p_n$  such that  $(p_i, p_{i+1}) \neq (p_{i+2}, p_{i+1})$ . Fix a vertex  $v \in \mathcal{H}$ .  $\mathcal{H}_{\text{uni}}$  is the graph where the vertex set is the set of nonbacktracking walks in  $\mathcal{H}$  starting at the vertex  $v$  and two nonbacktracking walks  $p$  and  $q$  are adjacent in the graph if one extends the other by a single step. Choosing a different starting vertex  $v$  gives us a graph isomorphic to  $\mathcal{H}_{\text{uni}}$ . It is a tree and the covering graph map  $\phi_{\text{uni}} : \mathcal{H}_{\text{uni}} \rightarrow \mathcal{H}$  is given by

$$\phi_{\text{uni}}(p) := \text{terminal vertex of } p.$$

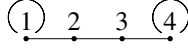
Let us look at a few examples. Nonbacktracking walks in a tree cannot visit the same vertex twice and there is a unique nonbacktracking walk joining two distinct vertices. Hence the universal cover of a tree  $\mathcal{H}$  is isomorphic to  $\mathcal{H}$ . The nonbacktracking walks in the graph  $C_n$  starting at 0 are the finite prefixes of the periodic walks

$$0, 1, 2, 3, \dots, n-1, 0, 1, \dots, \quad \text{and} \quad 0, n-1, n-2, \dots, 1, 0, n-1, \dots$$

Thus the universal covering graph of  $C_n$  is  $\mathbb{Z}$  and the covering graph map is  $(\text{mod } n) : \mathbb{Z} \rightarrow C_n$ .

Another important class of examples are the *barbell graphs*  $\text{Bar}_n$  for  $n > 2$  with vertices  $\{1, 2, 3, \dots, n\}$  and, as seen in (Figure 2), edges

$$\{(1, 1), (1, 2), (2, 3), \dots, (n-1, n), (n, n)\}.$$



**Figure 2.** Barbell graph for  $n = 4$ .

The nonbacktracking walks on  $\text{Bar}_n$  starting at 1 are the finite prefixes of the periodic walks

$$(1, 1, 2, 3, \dots, n-1, n, n, n-1, n-2, \dots, 2, 1, 1, \dots) \text{ and}$$

$$(1, 2, 3, \dots, n-1, n, n, n-1, n-2, \dots, 2, 1, 1, \dots),$$

proving that  $(\text{Bar}_n)_{\text{uni}} = \mathbb{Z}$ . Thus though the cycles  $C_n$  and the barbells  $\text{Bar}_n$  seem unrelated a priori, their universal covers are the same. By Proposition 5.5 it will follow that the corresponding hom-shifts are related to each other. The fact that  $\text{Bar}_n$  does not satisfy the block-gluing property has been essentially observed in [Briceño and Pavlov 2017].

Let  $\mathcal{H}$  be the graph for the hard square shift (given by Figure 1). The nonbacktracking walks starting at the vertex 1 are  $(1)$ ,  $(1, 0)$ ,  $(1, 0, 0)$  and  $(1, 0, 0, 1)$ . Thus  $\mathcal{H}_{\text{uni}}$  is isomorphic to the graph in Figure 3.

The universal covers of a graph are so-called normal covers [Hatcher 2002, Chapter 1]:

**Proposition 5.4.** *Let  $\mathcal{H}$  be a finite undirected graph. For all  $v', v'' \in \mathcal{H}_{\text{uni}}$  satisfying  $\phi_{\text{uni}}(v') = \phi_{\text{uni}}(v'')$  there is an automorphism  $\psi$  of  $\mathcal{H}_{\text{uni}}$  such that  $\phi_{\text{uni}} \circ \psi = \phi_{\text{uni}}$  and  $\psi(v') = v''$ .*

A lift of a configuration  $x \in X_{\mathcal{H}}^d$  is a configuration  $x' \in X_{\mathcal{H}_{\text{uni}}}^d$  such that  $\tilde{\phi}_{\text{uni}}(x') = x$ .

**Proposition 5.5.** *Let  $\mathcal{H}$  be a four-cycle hom-free graph. For all homomorphisms  $x \in X_{\mathcal{H}}^{d-1}$ , there exists a unique lift  $x' \in X_{\mathcal{H}_{\text{uni}}}^{d-1}$  up to a choice of  $x'_0$ . Further the induced map  $\tilde{\phi}_{\text{uni}}$  is a graph covering map from  $(\mathcal{H}_{\text{uni}})_{\text{walk}}^d$  to  $\mathcal{H}_{\text{walk}}^d$ .*

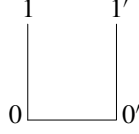
The proof of the first part of the proposition can be found in [Chandgotia 2017, Proposition 6.2]; the proof there is for four-cycle free graphs but it carries over for four-cycle hom-free graphs. For the second part, the same approach works with the added observation that  $x \sim_{\mathcal{H}_{\text{walk}}^d} y$  if and only if the configuration  $z : \mathbb{Z}^{d-1} \square [0, 1] \rightarrow \mathcal{H}$  given by

$$z_{\vec{i}, t} := \begin{cases} x_{\vec{i}} & \text{if } t = 0, \\ y_{\vec{i}} & \text{if } t = 1 \end{cases}$$

is a graph homomorphism.

The proposition has immediate consequences for the phased block-gluing property:

**Corollary 5.6.** *Let  $\mathcal{H}$  be a four-cycle hom-free graph. Then  $\text{diam}(\mathcal{H}_{\text{walk}}^d) < \infty$  if and only if  $\mathcal{H}_{\text{uni}}$  is finite.*



**Figure 3.** Graph for the lift of the hard square shift.

The proof shows  $\text{diam}(\mathcal{H}_{\text{walk}}^d) < \infty$  for some  $d \geq 2$  if and only if  $\text{diam}(\mathcal{H}_{\text{walk}}^d) < \infty$  for all  $d \geq 2$ ; look also at Section 6D.

*Proof.* Suppose  $\mathcal{H}_{\text{uni}}$  is a finite graph (and hence a finite tree). By Proposition 5.2 and Proposition 4.1(2) we get that  $\text{diam}((\mathcal{H}_{\text{uni}})^d_{\text{walk}}) < \infty$ . Let  $x, y \in X_{\mathcal{H}}^{d-1}$  and  $x', y'$  be lifts of  $x, y$  in  $\mathcal{H}_{\text{uni}}$ . There is a finite walk from  $x'$  to  $y'$  in  $(\mathcal{H}_{\text{uni}})^d_{\text{walk}}$ . By applying the induced map  $\tilde{\phi}_{\text{uni}}$  to each step of the walk we get a walk of the same length from  $x$  to  $y$  in  $\mathcal{H}_{\text{walk}}^d$ . Thus  $\text{diam}(\mathcal{H}_{\text{walk}}^d) \leq \text{diam}((\mathcal{H}_{\text{uni}})^d_{\text{walk}}) < \infty$ .

Now suppose that  $\mathcal{H}_{\text{uni}}$  is an infinite graph (and hence an infinite tree). By Proposition 4.4 it is sufficient to prove that  $\text{diam}(\mathcal{H}_{\text{walk}}^2) = \infty$ . Consider  $x' \in X_{\mathcal{H}_{\text{uni}}}^1$  such that  $x'|_{\mathbb{N}}$  does not visit the same vertex twice; since  $\mathcal{H}_{\text{uni}}$  is a bounded degree infinite graph such an  $x'$  exists. Let  $x := \tilde{\phi}_{\text{uni}}(x')$  and consider  $y := (v, w)^{\infty, 1}$  for some edge  $v \sim_{\mathcal{H}} w$ . Suppose that there is a walk from  $x$  to  $(v, w)^{\infty, 1}$  in  $\mathcal{H}_{\text{walk}}^2$ . By Proposition 5.5 it lifts to a unique walk from  $x'$  to  $y' = (v', w')^{\infty, 1}$  in  $(\mathcal{H}_{\text{uni}})^2_{\text{walk}}$  for some  $v', w' \in \mathcal{H}_{\text{uni}}$ .

Let  $i_0 \in \mathbb{N}$  be such that  $d_{\mathcal{H}_{\text{uni}}}(x'_{i_0}, v') := \min_{i \in \mathbb{N}} d_{\mathcal{H}_{\text{uni}}}(x'_i, v') =: t$ . Since  $\mathcal{H}_{\text{uni}}$  is a tree it follows that  $d_{\mathcal{H}_{\text{uni}}}(x'_{i_0}, x'_i) = i - i_0$  for all  $i \geq i_0$  and in fact

$$d_{\mathcal{H}_{\text{uni}}}(x'_i, v') = i - i_0 + t$$

for all  $i \geq i_0$ . Therefore,

$$d_{\mathcal{H}_{\text{uni}}}^w(x', (v', w')^{\infty, 1}) = \infty$$

which leads to a contradiction and completes the proof.  $\square$

*Proof of Theorem 5.3.* Let  $\mathcal{H}$  be a four-cycle hom-free graph. We are left to prove that (b) implies (c). By Corollary 5.6 it is sufficient to prove that if  $\mathcal{H}_{\text{uni}}$  is finite then  $\mathcal{H}$  is bipartite-dismantlable.

Now suppose that  $\mathcal{H}_{\text{uni}}$  is a finite tree and hence is bipartite-dismantlable. We want to prove that  $\mathcal{H}$  is bipartite-dismantlable. Suppose  $v'$  folds into  $w'$  in  $\mathcal{H}_{\text{uni}}$ , that is,  $N_{\mathcal{H}_{\text{uni}}}(v') \subset N_{\mathcal{H}_{\text{uni}}}(w')$ . Let  $v := \phi_{\text{uni}}(v')$  and  $w := \phi_{\text{uni}}(w')$ . By Proposition 5.4 it follows that for all  $v'' \in \mathcal{H}_{\text{uni}}$  satisfying  $\phi_{\text{uni}}(v'') = v$  there is an automorphism  $\psi$  of  $\mathcal{H}_{\text{uni}}$  for which  $\phi_{\text{uni}} \circ \psi = \phi_{\text{uni}}$  and  $\psi(v') = v''$ . Thus for  $w'' := \psi(w')$  we have that  $\phi_{\text{uni}}(w'') = w$  and  $v''$  folds into  $w''$ . Since  $v'$  and  $w'$  have common neighbours and  $\phi_{\text{uni}}$  is a covering map it follows that  $v \neq w$ ; in fact that  $v$  folds into  $w$ . By folding all  $v''$  which satisfy  $\phi_{\text{uni}}(v'') = v$  we get  $(\mathcal{H} \setminus \{v\})_{\text{uni}}$ . The proof can be completed by induction on  $|\mathcal{H}|$ .  $\square$

**5A. Why is the four-cycle hom-free condition necessary?** Some of the implications of Theorem 5.3 fail without the four-cycle hom-free assumption. We know that (a) implies (b) for all shift spaces and by Proposition 5.2, (c) implies (a). Let us see why the other implications do not hold:

(1) Neither (a) nor (b) implies (c): Here we see why the phased SI property in hom-shifts does not imply that the corresponding graph is bipartite-dismantlable. Let  $K_n$  denote the complete graph with  $n$  vertices,  $1, 2, \dots, n$ . It is mentioned in [Briceño 2014] that  $X_{K_n}^d$  is SI for  $n \geq 2d + 1$ ; note that there is no folding possible in  $K_n$  and hence it is not bipartite-dismantlable (except for  $n = 2$ ). Yet  $X_{K_n}^d$  is block-gluing for  $n \geq 4$  and  $d \in \mathbb{N}$ ; this is proved in Proposition 5.7. The argument given here is by Ronnie Pavlov; similar arguments appear in [Schmidt 1995, Section 4.4].

A vertex in  $\mathbb{Z}^{d-1}$  is called *even* if it is in the same partite class as  $\vec{0}$  and *odd* otherwise.

**Proposition 5.7.** *For  $n \geq 4$ ,  $\text{diam}((K_n)_{\text{walk}}^d) \leq 4$ .*

By Proposition 4.1 this implies that  $X_{K_n}^d$  is block-gluing for  $n \geq 4$ .

*Proof.* Let  $x \in X_{K_n}^{d-1}$ . Let  $y \in X_{K_n}^{d-1}$  be a homomorphism given by

$$y_{\vec{i}} = \begin{cases} 1 & \text{if } \vec{i} \text{ is even and } x_{\vec{i}} \neq 1, \\ 2 & \text{if } \vec{i} \text{ is even and } x_{\vec{i}} = 1, \\ 3 & \text{if } \vec{i} \text{ is odd and } x_{\vec{i}} \neq 3, \\ 4 & \text{if } \vec{i} \text{ is odd and } x_{\vec{i}} = 3. \end{cases}$$

Clearly  $x \sim_{\mathcal{H}_{\text{walk}}^d} y$  and  $y \sim_{\mathcal{H}_{\text{walk}}^d} (3, 1)^{\infty, d-1}$  (which is the checkerboard pattern of 3s and 1s which has value 3 at entry  $\vec{0}$ ). Therefore  $d_{K_n}^w(x, (3, 1)^{\infty, d-1}) \leq 2$ . Hence  $\text{diam}((K_n)_{\text{walk}}^d) \leq 4$ .  $\square$

(2) (b) does not imply (a): Here we show the existence of a hom-shift which is phased block-gluing but not phased SI. It was mentioned to the authors by Raimundo Briceño (2014) that  $X_{K_4}^3$  is not phased SI (while by Proposition 5.7 it is phased block-gluing). Here we shall give another example; this will be an instance of a large class of hom-shifts with the phased block-gluing property (Section 6B). Let  $\mathcal{H}$  be the graph given by Figure 4. We will prove that  $X_{\mathcal{H}}^d$  is phased block-gluing for all  $d \geq 2$  but not phased SI even for  $d = 2$ . Let us first observe why is  $X_{\mathcal{H}}^2$  not phased SI. Fix  $n \in \mathbb{N}$  and let  $L$  be the shape given by

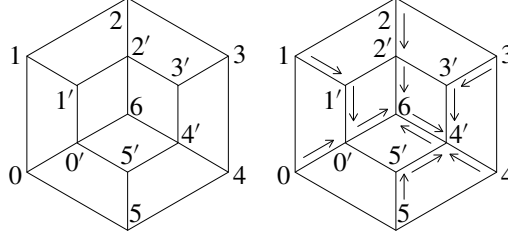
$$L := \{(i, 0), (n, i) : 0 \leq i \leq n\}.$$

Let  $x \in X_{\mathcal{H}}^2$  be given by

$$x_{(j,k)} := j + k(\text{mod } 6).$$

Observe that for all  $i \in \mathbb{Z}$ ,  $i + 1(\text{mod } 6)$  is the unique vertex in  $\mathcal{H}$  adjacent to both  $i(\text{mod } 6)$  and  $i + 2(\text{mod } 6)$ . It follows that  $x_{(j+1,k)}$  is the unique vertex





**Figure 4.** On the left: Graph  $\mathcal{H}$  for a hom-shift which is phased block-gluing but not phased SI. On the right: A graph homomorphism  $f : \mathcal{H} \rightarrow \mathcal{H}$  such that  $f(v) \sim_{\mathcal{H}} v$  and  $f^3(\mathcal{H})$  is a single edge.

adjacent to  $x_{(j,k)}$  and  $x_{(j+1,k+1)}$  for all  $(j,k) \in \mathbb{Z}^2$  which implies that if  $y \in X_{\mathcal{H}}^2$  is a configuration such that  $x|_L = y|_L$  then  $x|_{[0,n] \square [0,n]} = y|_{[0,n] \square [0,n]}$ . Thus  $X_{\mathcal{H}}^2$  is not phased SI.

Now we will prove that  $X_{\mathcal{H}}^d$  is phased block-gluing for all  $d \geq 2$ . Consider the map  $f : \mathcal{H} \rightarrow \mathcal{H}$  given by Figure 4 and  $d \geq 2$ : For all  $v \in \mathcal{H}$ ,  $f(v)$  is defined to be the head of the arrow starting at  $v$ . Observe that  $f$  is a graph homomorphism such that  $f(v) \sim_{\mathcal{H}} v$  for all  $v \in \mathcal{H}$  and  $f^3(\mathcal{H})$  is the edge joining vertices  $4'$  and  $6$ . Thus for all  $x \in X_{\mathcal{H}}^{d-1}$ ,  $f \circ x \sim_{\mathcal{H}_{\text{walk}}^d} x$  and  $f^3 \circ x$  is either  $(4', 6)^{\infty, d-1}$  or  $(6, 4')^{\infty, d-1}$  proving

$$d_{\mathcal{H}}^w(x, (4', 6)^{\infty, d-1}) \leq 4$$

and hence  $\text{diam}(\mathcal{H}_{\text{walk}}^d) \leq 8$ .

## 6. Further directions

### 6A. Decidability of the fixed block-gluing distance.

**Question.** Fix  $n \in \mathbb{N}$  and  $d \geq 2$ . Is there an algorithm to decide, for undirected graphs  $\mathcal{H}$ , whether  $\text{diam}(\mathcal{H}_{\text{walk}}^d) = n$ ?

Let us see how such an algorithm may be constructed for certain dimensions. Fix  $n \in \mathbb{N}$  and a graph  $\mathcal{H}$ . Recall, as in Section 3 the graph  $\mathcal{H}_{n, \text{walk}}^2$  for which the vertices are homomorphisms from  $[-n, n]$  to  $\mathcal{H}$ ; two such homomorphisms  $x, y$  are adjacent if  $x_i \sim_{\mathcal{H}} y_i$  for all  $i$ . Consider the  $(d-1)$ -dimensional hom-shift constructed using this graph:  $X_{\mathcal{H}_{n, \text{walk}}^2}^{d-1}$ . This makes notation onerous so we denote these shift spaces by  $X_{\mathcal{H}, n}^{d-1}$ . Let

$$X_{\mathcal{H}, \text{TB}}^{d-1} := \{(x, y) \in X_{\mathcal{H}}^{d-1} \times X_{\mathcal{H}}^{d-1} : \text{there is a walk of even length from } x_0 \text{ to } y_0\}.$$

Observe that if  $\mathcal{H}$  is not bipartite then  $X_{\mathcal{H}, \text{TB}}^{d-1} = X_{\mathcal{H}}^{d-1} \times X_{\mathcal{H}}^{d-1}$ ; if it is bipartite then we further require that  $x_0$  and  $y_0$  are in the same partite class. There is a natural

map  $\pi_{\mathcal{H},n}^{d-1} : X_{\mathcal{H},n}^{d-1} \rightarrow X_{\mathcal{H},\text{TB}}^{d-1}$  given by  $\pi_{\mathcal{H},n}^{d-1}(z) := (x, y)$  where

$$x_{\vec{i}} := z_{\vec{i}}(n), \quad y_{\vec{i}} := z_{\vec{i}}(-n).$$

This construction is related with the phased block-gluing property via the following proposition:

**Proposition 6.1.** *Let  $\mathcal{H}$  be an undirected graph. Then  $X_{\mathcal{H}}^d$  is phased block-gluing for some block-gluing distance  $2n$  if and only if the map  $\pi_{\mathcal{H},n}^{d-1}$  is surjective.*

*Proof.* By the proof of Proposition 4.1,  $X_{\mathcal{H}}^d$  is phased block-gluing for distance  $2n$  if and only if for all  $x, y \in X_{\mathcal{H}}^{d-1}$  there exists a walk either from  $x$  to  $y$  or from  $x$  to  $\sigma^{\vec{e}_1}(y)$  of length  $2n$ ; equivalently, for all  $x, y \in X_{\mathcal{H}}^{d-1}$  there exists  $z \in X_{\mathcal{H},n}^{d-1}$  such that either  $\pi_{\mathcal{H},n}^{d-1}(z) = (x, y)$  or  $\pi_{\mathcal{H},n}^{d-1}(z) = (x, \sigma^{\vec{e}_1}(y))$ . Consider a pair

$$(x', y') \in X_{\mathcal{H},\text{TB}}^{d-1}.$$

The distance between  $x'$  and  $y'$  is even. Thus,

$$\pi_{\mathcal{H},n}^{d-1}(z') \neq (x, \sigma^{\vec{e}_1}(y))$$

for  $z' \in X_{\mathcal{H},n}^{d-1}$ , and there exists  $z'' \in X_{\mathcal{H},n}^{d-1}$  such that  $\pi_{\mathcal{H},n}^{d-1}(z'') = (x, y)$ , completing the proof.  $\square$

**Theorem 6.2.** *It is decidable whether a hom-shift in two dimensions is block-gluing for distance  $n$ .*

Recall, a shift space is called a *sofic shift* if it is the image of an SFT under a sliding block-code.

*Proof.* We will verify this only in the case when  $n$  is even; for odd  $n$ , the proof is similar. By Proposition 6.1 it is equivalent to verify that

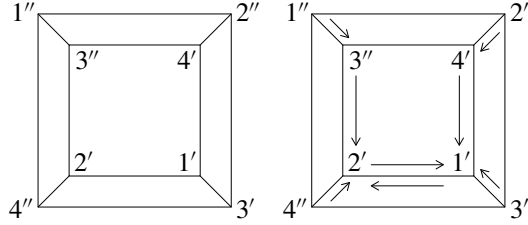
$$\text{Image}(\pi_{\mathcal{H},n/2}^1) = X_{\mathcal{H},\text{TB}}^1.$$

Now  $X_{\mathcal{H},\text{TB}}^1$  is an SFT (and hence sofic) and  $\text{Image}(\pi_{\mathcal{H},n/2}^1)$  is sofic; there are well-known algorithms to decide whether two sofic shifts are the same; [Lind and Marcus 1995, Theorem 3.4.13]. This proves that it is decidable whether a hom-shift in two dimensions is block-gluing for block-gluing distance  $n$ .  $\square$

Since it is undecidable whether a higher dimensional SFT is nonempty it automatically follows that it is undecidable whether two  $(d-1)$ -dimensional sofic shifts are equal for  $d \geq 3$ . However even for  $d = 2$  we do not know the answer to the following questions:

**Question.** Fix  $n \in \mathbb{N}$ . Is it decidable whether the SI gluing distance for a hom-shift is less than or equal to  $n$ ?

**Question.** Is the phased block-gluing/phased SI property decidable for hom-shifts?



**Figure 5.** A cover of  $K_4$  on the left and its collapsing map on the right.

**6B. The gluing property for general boards  $\mathcal{G}$ .** Our construction of the graph  $\mathcal{H}_{\text{walk}}^d$  was motivated by the study of the block-gluing property. The question of whether  $\text{diam}(\mathcal{H}_{\text{walk}}^d) < \infty$  can be viewed as a certain “reconfiguration” problem. A natural extension of the question is the following: Let  $\mathcal{G}$  be a connected undirected graph without self-loops. Consider the graph

$$\mathcal{H}_{\text{walk}}^{\mathcal{G}} := (\text{Hom}(\mathcal{G}, \mathcal{H}), \mathcal{E}_{\text{walk}}^{\mathcal{G}}) \text{ where } \mathcal{E}_{\text{walk}}^{\mathcal{G}} := \{(x, y) : x_i \sim_{\mathcal{H}} y_i \text{ for all } i \in \mathcal{G}\}.$$

**Question.** For which graphs  $\mathcal{H}$  is  $\text{diam}(\mathcal{H}_{\text{walk}}^{\mathcal{G}}) < \infty$  for all undirected graphs  $\mathcal{G}$ ?

For a reconfiguration problem of a similar nature, a characterisation was given in [Brightwell and Winkler 2000]: We say that  $\text{Hom}(\mathcal{G}, \mathcal{H})$  satisfies the pivot property if for all  $x, y \in \text{Hom}(\mathcal{G}, \mathcal{H})$  which differ only at finitely many sites there exists a sequence  $x = x^1, x^2, \dots, x^n = y \in \text{Hom}(\mathcal{G}, \mathcal{H})$  such that  $x^i, x^{i+1}$  differ at most at one site. Brightwell and Winkler proved that the pivot property is satisfied by  $\text{Hom}(\mathcal{G}, \mathcal{H})$  for all graphs  $\mathcal{G}$  if and only if  $\mathcal{H}$  is dismantlable. We wonder if a characterisation of similar nature exists in our case as well. In the following we provide a large class of graphs  $\mathcal{H}$  for which  $\text{diam}(\mathcal{H}_{\text{walk}}^{\mathcal{G}}) < \infty$  for all connected undirected graphs  $\mathcal{G}$ .

We say that  $\mathcal{H}$  is *collapsible* if there exists a graph homomorphism  $f : \mathcal{H} \rightarrow \mathcal{H}$  such that  $f(v) \sim_{\mathcal{H}} v$  for all  $v \in \mathcal{H}$  and there exists  $n \in \mathbb{N}$  such that  $f^n(\mathcal{H})$  is either an edge or a vertex with a self-loop;  $f$  is called a *collapsing map*. If  $\mathcal{H}$  is a collapsible graph,  $\text{diam}(\mathcal{H}_{\text{walk}}^{\mathcal{G}}) < \infty$  for all graphs  $\mathcal{G}$  (see Section 5A(2)).

While one may feel that the proof that  $\text{diam}((K_n)_{\text{walk}}^d) < \infty$  for all  $n \geq 4$  in Proposition 5.7 is of a very different nature from that for the collapsible graphs, it can be shown that they are intimately related. Consider the covering graph map  $\phi : \mathcal{H} \rightarrow K_4$  given by  $\phi(v') = \phi(v'') = v$  for all  $v \in [1, 4]$  where  $\mathcal{H}$  is as shown in Figure 5. As in Proposition 5.5, it is easy to see that for all homomorphisms  $x \in X_{K_4}^{d-1}$ , there exists a unique lift  $x' \in X_{\mathcal{H}}^{d-1}$  up to a choice of  $x'_0$ . Further the induced map  $\tilde{\phi}$  is a graph covering map from  $(\mathcal{H})_{\text{walk}}^d$  to  $(K_4)_{\text{walk}}^d$ . One can thereby conclude  $\text{diam}(\mathcal{H}_{\text{walk}}^d) < \infty$  if and only if  $\text{diam}((K_4)_{\text{walk}}^d) < \infty$ . But the map  $f : \mathcal{H} \rightarrow \mathcal{H}$  given by Figure 5 is a collapsing map proving  $\text{diam}((K_4)_{\text{walk}}^d) < \infty$ .

**6C. The growth rate of the diameter of  $\mathcal{H}_{n,\text{walk}}^d$ .** We write that a sequence  $a_n$  is equal to  $\Theta(n)$  if there exists  $c, C > 0$  such that  $cn \leq a_n \leq Cn$ .

**Conjecture.** If  $\mathcal{H}$  is a finite undirected graph,  $\text{diam}(\mathcal{H}_{\text{walk}}^d) = \infty$  if and only if  $\text{diam}(\mathcal{H}_{n,\text{walk}}^d) = \Theta(n)$ .

This was also conjectured by Ronnie Pavlov and Michael Schraudner, who showed that this is true in several examples (personal communication from Raimundo Briceño, 2015). From (3-1) we get a natural upper bound on the diameter:

$$\text{diam}(\mathcal{H}_{n,\text{walk}}^d) \leq \text{diam}(\mathcal{H}) + 2n(d-1).$$

If  $\mathcal{H}$  is a four-cycle hom-free graph and  $d \geq 2$  then it can be proved  $\text{diam}(\mathcal{H}_{\text{walk}}^d) = \infty$  if and only if  $\text{diam}(\mathcal{H}_{n,\text{walk}}^d) = \Theta(n)$ . We will prove the conjecture in the case when  $\mathcal{H}$  is a four-cycle hom-free graph.

Suppose that  $\text{diam}(\mathcal{H}_{n,\text{walk}}^d) = \Theta(n)$ . Since  $\text{diam}(\mathcal{H}_{n,\text{walk}}^d)$  is increasing in  $n$  and converges to  $\text{diam}(\mathcal{H}_{\text{walk}}^d)$ , it follows that  $\text{diam}(\mathcal{H}_{\text{walk}}^d) = \infty$ .

For the other direction assume that  $\text{diam}(\mathcal{H}_{\text{walk}}^d) = \infty$ . Since  $\text{diam}(\mathcal{H}_{n,\text{walk}}^d)$  is increasing in  $d$ , it is sufficient to prove that  $\text{diam}(\mathcal{H}_{n,\text{walk}}^d) = \Theta(n)$  for  $d = 2$ . By Corollary 5.6,  $\mathcal{H}_{\text{uni}}$  is infinite. As in the proof of the corollary, let  $x' \in X_{\mathcal{H}_{\text{uni}}}^1$  be such that  $x'|_{\mathbb{N}}$  does not visit the same vertex twice and let  $x := \tilde{\phi}_{\text{uni}}(x')$ . Then  $d_{\mathcal{H}_{\text{uni}}}(x'_i, x'_j) = |i - j|$  for all  $i, j \in \mathbb{N}$  implying that for all vertices  $v' \in \mathcal{H}_{\text{uni}}$ , there exists  $i \in [0, 2n]$  such that  $d_{\mathcal{H}_{\text{uni}}}(x_i, v') \geq n$ . This implies that the shortest walk in  $\mathcal{H}_{n,\text{walk}}^d$  from  $x|_{[0,2n]}$  to  $(v, w)^{\infty,1}|_{[0,2n]}$  for all edges  $v \sim_{\mathcal{H}} w$  is of length at least  $n$ . This proves that  $\text{diam}(\mathcal{H}_{n,\text{walk}}^2) = \Theta(n)$ .

## 6D. Dependence on dimension.

**Problem.** Construct a graph  $\mathcal{H}$  for which  $\text{diam}(\mathcal{H}_{\text{walk}}^2) < \infty$  but  $\text{diam}(\mathcal{H}_{\text{walk}}^3) = \infty$ .

In this paper we mention two large collections of graphs where  $\text{diam}(\mathcal{H}_{\text{walk}}^d) < \infty$  for all  $d$ : bipartite-dismantlable graphs (as in Section 5) and collapsible graphs (as in Section 6B). However in all such examples, we find that  $\text{diam}(\mathcal{H}_{\text{walk}}^d) < \infty$  for all  $d$ . To find examples for the problem above, we would have to find a way to prove that  $\text{diam}(\mathcal{H}_{\text{walk}}^2) < \infty$  in a fundamentally different way.

By Proposition 4.1, the problem stated above is equivalent to the problem of finding a graph  $\mathcal{H}$  for which  $X_{\mathcal{H}}^2$  is block-gluing but  $X_{\mathcal{H}}^3$  is not block-gluing. We note that the answer to the analogue of this problem for SI is known:  $X_{K_4}^2$  is SI [Briceño 2014] but  $X_{K_4}^3$  is not SI (personal communication, 2014).

## 6E. Block-gluing for periodic points.

**Problem.** Construct a graph  $\mathcal{H}$  such that  $d_{\mathcal{H}}^w(x, y) < \infty$  for all periodic points  $x, y \in X_{\mathcal{H}}^{d-1}$  but  $\text{diam}(\mathcal{H}_{\text{walk}}^d) = \infty$ .

If  $\text{diam}(\mathcal{H}_{\text{walk}}^d) = \infty$ , by Proposition 4.6 there exists some  $x, y \in X_{\mathcal{H}}^{d-1}$  such that  $d_{\mathcal{H}}^w(x, y) = \infty$ , however it is not clear if  $x, y$  can be chosen periodic. Such periodic points can be chosen if  $\mathcal{H}$  is four-cycle free: By Corollary 5.6,  $\mathcal{H}_{\text{uni}}$  is infinite and  $\mathcal{H}$  is not a tree. Let

$$u_0, u_1, \dots, u_{k-1}, u_k = u_0$$

be a simple cycle in  $\mathcal{H}$  for some  $k > 2$ . Consider  $x \in X_{\mathcal{H}}^2$  given by  $x_i := u_{i \pmod k}$  for all  $i \in \mathbb{Z}$ ;  $x$  is periodic. Let  $x' \in X_{\mathcal{H}_{\text{uni}}}^1$  be any lift of  $x$ . Since  $x_i \neq x_{i+2}$  for all  $i \in \mathbb{Z}$  it follows that  $x'_i \neq x'_{i+2}$  for all  $i \in \mathbb{Z}$ ; because  $\mathcal{H}_{\text{uni}}$  is a tree, this implies that  $x'$  does not visit the same vertex twice. As in the proof of Corollary 5.6 it follows that  $d_{\mathcal{H}}^w(x, (v, w)^{\infty, 1}) = \infty$  for all  $v \sim_{\mathcal{H}} w$ .

**6F. Measures of maximal entropy and Markov chains on  $\mathcal{H}_{\text{walk}}^2$ .** Given a shift space  $X$  and  $b \in \mathcal{L}_B(X)$  for some  $B \subset \mathbb{Z}^2$ , denote by

$$[b]_B := \{x \in X : x|_B = b\}$$

the corresponding cylinder set. One of the motivations for studying the graph  $\mathcal{H}_{\text{walk}}^d$  is also to understand the measures of maximal entropy on the space  $X_{\mathcal{H}}^d$ . Let us talk about the case  $d = 2$ . There is a natural correspondence between stochastic processes  $\nu$  on  $\mathcal{H}_{\text{walk}}^2$  and probability measures  $\mu$  on  $X_{\mathcal{H}}^2$  given by

$$\nu(X_j^i = a_{i,j} \text{ for } (i, j) \in B) := \mu([a]_B) \quad \text{for } B \subset \mathbb{Z}^2 \text{ finite and } a \in \mathcal{L}_B(X_{\mathcal{H}}^2).$$

For this subsection the necessary background for measures of maximal entropy can be gathered from [Ruelle 2004; Burton and Steif 1994] and for Markov chains from [Durrett 2010, Chapter 6]. Let  $\mathcal{H}$  be a finite undirected graph and  $\mu$  be an ergodic measure of maximal entropy for  $X_{\mathcal{H}}^2$ . Consider the Markov chain  $\nu$  on  $\mathcal{H}_{\text{walk}}^2$  obtained by the “Markovisation” of  $\mu$  (look also at [Bowen 2008, Chapter 1]): Let  $\pi$  be the probability measure on  $X_{\mathcal{H}}^1$  given by marginalising  $\mu$  to the vertical line  $\{0\} \square \mathbb{Z}$ . Consider the probability (also called Markov) kernel on  $(\mathcal{H}_{\text{walk}}^2, \mathcal{B})$ ,  $\kappa : X_{\text{walk}}^1 \times \mathcal{B} \rightarrow [0, 1]$  given by

$$\kappa(x, [y]_{-n,n}) := \mu(X_{(1,i)} = y_i \text{ for } i \in [-n, n] | X_{(0,i)} = x_i \text{ for } i \in \mathbb{Z});$$

it is well defined for  $\pi$ -almost every  $x$ .

Since  $\mu$  is a shift-invariant probability measure it follows that  $\pi$  is a stationary measure for the kernel  $\kappa$ . It can be proved that the measure  $\tilde{\mu}$  on  $X_{\mathcal{H}}^2$  corresponding to the Markov chain  $\nu$  is also a measure of maximal entropy.

**Conjecture.** Let  $\mathcal{H}$  be a finite undirected graph and  $\mu$  be an ergodic measure of maximal entropy on  $X_{\mathcal{H}}^2$ . Then the stochastic process on  $\mathcal{H}_{\text{walk}}^2$  corresponding to  $\mu$  is a Markov chain.

A study of random walks on the graph  $(C_3)_{n, \text{walk}}^2$  can be found in [Boissard et al. 2015].

## 6G. When is an SFT conjugate to a hom-shift?

**Question.** Let  $d = 1$ . Is it decidable whether an SFT is conjugate to a hom-shift?

For  $d \geq 2$  we have already observed in Corollary 2.3 that it is undecidable whether an SFT is conjugate to a hom-shift.

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# SIMULTANEOUS CONSTRUCTION OF HYPERBOLIC ISOMETRIES

MATT CLAY AND CAGLAR UYANIK

**Given isometric actions by a group  $G$  on finitely many  $\delta$ -hyperbolic metric spaces, we provide a sufficient condition that guarantees the existence of a single element in  $G$  that is hyperbolic for each action. As an application we prove a conjecture of Handel and Mosher regarding relatively fully irreducible subgroups and elements in the outer automorphism group of a free group.**

## 1. Introduction

A  $\delta$ -hyperbolic space is a geodesic metric space where geodesic triangles are  $\delta$ -slim: the  $\delta$ -neighborhood of any two sides of a geodesic triangle contains the third side. Such spaces were introduced by Gromov [1987] as a coarse notion of negative curvature for geodesic metric spaces and since then have evolved into an indispensable tool in geometric group theory.

There is a classification of isometries of  $\delta$ -hyperbolic metric spaces analogous to the classification of isometries of hyperbolic space  $\mathbb{H}^n$  into elliptic, hyperbolic and parabolic. Of these, hyperbolic isometries have the best dynamical properties and are often the most desired. For example, typically they can be used to produce free subgroups in a group acting on a  $\delta$ -hyperbolic space [Gromov 1987, 5.3B]; see also [Bridson and Haefliger 1999, III.Γ.3.20]. Another application is to show that a certain element does not have fixed points in its action on some set. Indeed, if the set naturally sits inside a  $\delta$ -hyperbolic metric space and the given element acts as a hyperbolic isometry then it has no fixed points (in a strong sense). This strategy has been successfully employed for the curve complex of a surface and for the free factor complex of a free group by several authors [Clay et al. 2012; Clay and Pettet 2012; Dowdall and Taylor 2018; Fujiwara 2015; Gültepe 2017; Horbez 2016; Mangahas 2013; Taylor 2014].

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We consider the situation of a group acting on finitely many  $\delta$ -hyperbolic spaces and produce a sufficient condition that guarantees the existence of a single element in the group that is a hyperbolic isometry for each of the spaces. Of course, a necessary condition is that for each of the spaces there is some element of the group that is a hyperbolic isometry. Thus we are concerned with when we may reverse the quantifiers:  $\forall \exists \rightsquigarrow \exists \forall$ . Our main result is the following theorem.

**Theorem 5.1.** *Suppose that  $\{X_i\}_{i=1,\dots,n}$  is a collection of  $\delta$ -hyperbolic spaces,  $G$  is a group and for each  $i = 1, \dots, n$  there is a homomorphism  $\rho_i : G \rightarrow \text{Isom}(X_i)$  such that*

- (1) *there is an element  $f_i \in G$  such that  $\rho_i(f_i)$  is hyperbolic; and*
- (2) *for each  $g \in G$ , either  $\rho_i(g)$  has a periodic orbit or is hyperbolic.*

*Then there is an  $f \in G$  such that  $\rho_i(f)$  is hyperbolic for all  $i = 1, \dots, n$ .*

**Remark 1.1.** Since the completion of this paper we have been alerted to the fact that Theorem 5.1 should follow from random walk techniques developed in [Björklund and Hartnick 2011; Maher and Tiozzo 2016]. Here we provide an elementary and constructive proof.

Essentially, we assume that there are no parabolic isometries and that elliptic isometries are relatively tame.

As an application of our main theorem we prove a conjecture of Handel and Mosher which involves exactly the same type of quantifier reversing:  $\forall \exists \rightsquigarrow \exists \forall$ . Consider a finitely generated subgroup  $\mathcal{H} < \text{IA}_N(\mathbb{Z}/3) < \text{Out}(F_N)$  and a maximal  $\mathcal{H}$ -invariant filtration of  $F_N$ , the free group of rank  $N$ , by free factor systems,

$$\emptyset = \mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \dots \sqsubset \mathcal{F}_m = \{[F_N]\}$$

(see Section 6). Handel and Mosher [2013a, Theorem D] prove that for each multi-edge extension  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$  there exists some  $\varphi_i \in \mathcal{H}$  that is irreducible with respect to  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ . They conjecture that there exists a single  $\varphi \in \mathcal{H}$  that is irreducible with respect to each multi-edge extension  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ . We show that this is indeed the case.

**Theorem 6.6.** *For each finitely generated subgroup  $\mathcal{H} < \text{IA}_N(\mathbb{Z}/3) < \text{Out}(F_N)$  and each maximal  $\mathcal{H}$ -invariant filtration by free factor systems,*

$$\emptyset = \mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \dots \sqsubset \mathcal{F}_m = \{[F_N]\},$$

*there is an element  $\varphi \in \mathcal{H}$  such that for each  $i = 1, \dots, m$  such that  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$  is a multi-edge extension,  $\varphi$  is irreducible with respect to  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ .*

Our paper is organized as follows. Section 2 contains background on  $\delta$ -hyperbolic spaces and their isometries. In Section 3 we generalize a construction from [Clay

and Pettet 2012] that is useful in constructing hyperbolic isometries. This result is Theorem 3.1. We examine certain cases that will arise in the proof of the main theorem to see how to apply Theorem 3.1 in Section 4. The proof of Theorem 5.1 constitutes Section 5. The application to  $\text{Out}(F_N)$  appears in Section 6.

## 2. Background on $\delta$ -hyperbolic spaces

In this section we recall basic notions and facts about  $\delta$ -hyperbolic spaces, their isometries and their boundaries. The reader familiar with these topics can safely skip this section, with the exception of Definition 2.8. References for this section are [Alonso et al. 1991; Bridson and Haefliger 1999; Kapovich and Benakli 2002].

**2A.  $\delta$ -hyperbolic spaces.** We recall the definition of a  $\delta$ -hyperbolic space given in the Introduction.

**Definition 2.1.** Let  $(X, d)$  be a geodesic metric space. A geodesic triangle with sides  $\alpha$ ,  $\beta$  and  $\gamma$  is  $\delta$ -*slim* if for each  $x \in \alpha$ , there is some  $y \in \beta \cup \gamma$  such that  $d(x, y) \leq \delta$ . The space  $X$  is said to be  $\delta$ -*hyperbolic* if every geodesic triangle is  $\delta$ -slim.

There are several equivalent definitions that we will use in the sequel. The first of these is *insize*. Let  $\Delta$  be the geodesic triangle with vertices  $x$ ,  $y$  and  $z$  and sides  $\alpha$  from  $y$  to  $z$ ,  $\beta$  from  $z$  to  $x$  and  $\gamma$  from  $x$  to  $y$ . There exist unique points  $\hat{\alpha} \in \alpha$ ,  $\hat{\beta} \in \beta$  and  $\hat{\gamma} \in \gamma$ , called the *internal points* of  $\Delta$ , such that

$$d(x, \hat{\beta}) = d(x, \hat{\gamma}), \quad d(y, \hat{\gamma}) = d(y, \hat{\alpha}) \quad \text{and} \quad d(z, \hat{\alpha}) = d(z, \hat{\beta}).$$

The *insize* of  $\Delta$  is the diameter of the set  $\{\hat{\alpha}, \hat{\beta}, \hat{\gamma}\}$ .

Another notion makes use of the so-called *Gromov product*:

$$(2-1) \quad (x \cdot y)_w = \frac{1}{2}(d(x, w) + d(w, y) - d(x, y)).$$

The Gromov product is said to be  $\delta$ -*hyperbolic* (with respect to  $w \in X$ ) if for all  $x, y, z \in X$ ,

$$(x \cdot z)_w \geq \min\{(x \cdot y)_w, (y \cdot z)_w\} - \delta.$$

**Proposition 2.2** [Alonso et al. 1991, Proposition 2.1; Bridson and Haefliger 1999, III.H.1.17 and III.H.1.22]. *The following are equivalent for a geodesic metric space  $X$ :*

- (1) *There is a  $\delta_1 \geq 0$  such that every geodesic triangle in  $X$  is  $\delta_1$ -slim, i.e.,  $X$  is  $\delta_1$ -hyperbolic.*
- (2) *There is a  $\delta_2 \geq 0$  such that every geodesic triangle in  $X$  has insize at most  $\delta_2$ .*
- (3) *There is a  $\delta_3 \geq 0$  such that for some (equivalently any)  $w \in X$ , the Gromov product is  $\delta_3$ -hyperbolic.*

Henceforth, when we say  $X$  is a  $\delta$ -hyperbolic space we assume that  $\delta$  is large enough to satisfy each of the above conditions.

**2B. Boundaries.** There is a useful notion of a boundary for a  $\delta$ -hyperbolic space that plays the role of the “sphere at infinity” for  $\mathbb{H}^n$ . This space is defined using equivalence classes of certain sequences of points in  $X$  and the Gromov product. Fix a basepoint  $w \in X$ .

**Definition 2.3.** We say a sequence  $(x_n) \subseteq X$  *converges to infinity* if  $(x_i \cdot x_j)_w \rightarrow \infty$  as  $i, j \rightarrow \infty$ . Two such sequences  $(x_n), (y_n)$  are equivalent if  $(x_i \cdot y_j)_w \rightarrow \infty$  as  $i, j \rightarrow \infty$ . The *boundary of  $X$* , denoted  $\partial X$ , is the set of equivalence classes of sequences  $(x_n) \subseteq X$  that converge to infinity.

One can show that the notion of “converges to infinity” and the subsequent equivalence relation do not depend on the choice of basepoint  $w \in X$  [Kapovich and Benakli 2002]. The definition of the Gromov product in (2-1) extends to boundary points  $\hat{x}, \hat{y} \in \partial X$  by

$$(\hat{x} \cdot \hat{y})_w = \inf \{ \liminf_n (x_n \cdot y_n)_w \},$$

where the infimum is over sequences  $(x_n) \in \hat{x}, (y_n) \in \hat{y}$ . If  $y \in X$  then we set

$$(\hat{x} \cdot y)_w = \inf \{ \liminf_n (x_n \cdot y)_w \},$$

where the infimum is over sequences  $(x_n) \in \hat{x}$ . For  $x \in X$ , the Gromov product  $(x \cdot \hat{y})_w$  is defined analogously. Let  $\bar{X} = X \cup \partial X$ .

We will make use of the following properties of the Gromov product on  $\bar{X}$ .

**Proposition 2.4** [Alonso et al. 1991, Lemma 4.6; Bridson and Haefliger 1999, III.H.3.17]. *Let  $X$  be a  $\delta$ -hyperbolic space.*

- (1) *If  $x, y \in \bar{X}$  then  $(x \cdot y)_w = \infty \iff x = y \in \partial X$ .*
- (2) *If  $\hat{x} \in \partial X$  and  $(x_n) \subseteq X$  then  $(\hat{x} \cdot x_n)_w \rightarrow \infty$  as  $n \rightarrow \infty \iff (x_n) \in \hat{x}$ .*
- (3) *If  $\hat{x}, \hat{y} \in \partial X$  and  $(x_n) \in \hat{x}, (y_n) \in \hat{y}$  then*

$$(\hat{x} \cdot \hat{y})_w \leq \liminf_n (x_n \cdot y_n)_w \leq (\hat{x} \cdot \hat{y})_w - 2\delta.$$

- (4) *If  $x, y, z \in \bar{X}$  then*

$$(x \cdot z)_w \geq \min\{(x \cdot y)_w, (y \cdot z)_w\} - \delta.$$

**Proposition 2.5** [Alonso et al. 1991, Proposition 4.8]. *The following collection of subsets of  $\bar{X}$  forms a basis for a topology:*

- (1)  $B(x, r) = \{y \in X \mid d(x, y) < r\}$  for each  $x \in X$  and  $r > 0$ .
- (2)  $N(\hat{x}, k) = \{y \in \bar{X} \mid (\hat{x} \cdot y)_w > k\}$  for each  $\hat{x} \in \partial X$  and  $k > 0$ .

**2C. Isometries.** As mentioned in the Introduction, there is a classification of isometries of a  $\delta$ -hyperbolic space  $X$  into elliptic, parabolic and hyperbolic; see

[Gromov 1987, 8.1.B]. We will not make use of parabolic isometries and so do not give the definition here.

**Definition 2.6.** An isometry  $f \in \text{Isom}(X)$  is *elliptic* if for any  $x \in X$ , the set  $\{f^n x \mid n \in \mathbb{Z}\}$  has bounded diameter.

An isometry  $f \in \text{Isom}(X)$  is *hyperbolic* if for any  $x \in X$  there is a  $t > 0$  such that

$$t|m - n| \leq d(f^m x, f^n x)$$

for all  $m, n \in \mathbb{Z}$ . In this case, one can show the sequence  $(f^n x) \subseteq X$  converges to infinity and the equivalence class it defines in  $\partial X$  is independent of  $x \in X$ . This point in  $\partial X$  is called the *attracting fixed point* of  $f$ . The *repelling fixed point* of  $f$  is the attracting fixed point of  $f^{-1}$  and is represented by the sequence  $(f^{-n} x) \subseteq X$ .

The action of a hyperbolic isometry  $f \in \text{Isom}(X)$  on  $\bar{X}$  has “North-South dynamics.”

**Proposition 2.7** [Gromov 1987, 8.1.G]. *Suppose that  $f \in \text{Isom}(X)$  is a hyperbolic isometry and that  $U_+, U_- \subset \bar{X}$  are disjoint neighborhoods of the attracting and repelling fixed points of  $f$  respectively. There exists an  $N \geq 1$  such that for  $n \geq N$ :*

$$f^n(\bar{X} - U_-) \subseteq U_+ \quad \text{and} \quad f^{-n}(\bar{X} - U_+) \subseteq U_-.$$

We will make use of the following definition.

**Definition 2.8.** Suppose  $X$  is a  $\delta$ -hyperbolic space and  $f, g \in \text{Isom}(X)$  are hyperbolic isometries. Let  $A_+, A_-$  be the attracting and repelling fixed points of  $f$  in  $\partial X$  and let  $B_+$  and  $B_-$  be the attracting and repelling fixed points of  $g$  in  $\partial X$ . We say  $f$  and  $g$  are *independent* if

$$\{A_+, A_-\} \cap \{B_+, B_-\} = \emptyset.$$

Hyperbolic isometries that are not independent are said to be *dependent*.

### 3. A recipe for hyperbolic isometries

In this section we prove the principal tool used in the proof of the main result of this article, producing a single element in the given group that is hyperbolic for each action. The idea is to start with elements  $f$  and  $g$  that are hyperbolic for different actions and then combine them into a single element  $f^a g^b$  that is hyperbolic for both actions. A theorem of Clay and Pettet shows that if  $g$  does not send the attracting fixed point of  $f$  to the repelling fixed point, then  $f^a g$  is hyperbolic in the first action for large enough  $a$ . We can reverse the roles to get that  $f g^b$  is hyperbolic in the second action for large enough  $b$ . In order to simultaneously work with powers for both  $f$  and  $g$ , we need a uniform version of this result. That is the content of the next theorem, which generalizes [Clay and Pettet 2012, Theorem 4.1].

**Theorem 3.1.** *Suppose  $X$  is a  $\delta$ -hyperbolic space and  $f \in \text{Isom}(X)$  is a hyperbolic isometry with attracting and repelling fixed points  $A_+$  and  $A_-$  respectively. Fix disjoint neighborhoods  $U_+$  and  $U_-$  in  $\bar{X}$  for  $A_+$  and  $A_-$  respectively. Then there is an  $M \geq 1$  such that if  $m \geq M$  and  $g \in \text{Isom}(X)$  then  $f^m g$  is a hyperbolic isometry whenever  $gU_+ \cap U_- = \emptyset$ .*

The proof follows along the lines of [Clay and Pettet 2012, Theorem 4.1]. In the following two lemmas we assume the hypotheses of Theorem 3.1. The first lemma is obvious in the hypothesis of [Clay and Pettet 2012, Theorem 4.1] but requires a proof in this setting.

**Lemma 3.2.** *Given a point  $x \in U_+ \cap X$ , there are constants  $t > 0$  and  $C \geq 0$  such that if  $g \in \text{Isom}(X)$  is such that  $gU_+ \cap U_- = \emptyset$  then  $d(x, f^m g x) \geq mt - C$  for all  $m \geq 0$ .*

*Proof.* Let  $A = \{f^n x \mid n \in \mathbb{Z}\}$  and for  $z \in X$  let

$$d_z = \inf\{d(x', z) \mid x' \in A\}.$$

As  $f$  is a hyperbolic isometry, there is a constant  $\tau \geq 1$  such that

$$\frac{1}{\tau}|m - n| \leq d(f^m x, f^n x) \leq \tau|m - n|.$$

This shows that for any  $z \in X$  the set  $\pi_z = \{x' \in A \mid d(x', z) = d_z\}$  is nonempty and finite.

**Claim 1.** *There is a constant  $D \geq 0$  such that for any  $z \in X$  and  $x_z \in \pi_z$ ,*

$$d(x, z) \geq d(x, x_z) + d(x_z, z) - D.$$

*Proof of Claim 1.* Fix a point  $x_z \in \pi_z$  and geodesics  $\alpha$  from  $x_z$  to  $x$ ,  $\beta$  from  $z$  to  $x_z$  and  $\gamma$  from  $z$  to  $x$ . Let  $\Delta$  be the geodesic triangle formed with these segments and  $\hat{\alpha} \in \alpha$ ,  $\hat{\beta} \in \beta$  and  $\hat{\gamma} \in \gamma$  be the internal points of  $\Delta$ . These points satisfy the equalities

$$d(z, \hat{\beta}) = d(z, \hat{\gamma}) = a,$$

$$d(x, \hat{\gamma}) = d(x, \hat{\alpha}) = b,$$

$$d(x_z, \hat{\alpha}) = d(x_z, \hat{\beta}) = c.$$

As the insize of geodesic triangles is bounded by  $\delta$  in a  $\delta$ -hyperbolic space, we have that  $d(\hat{\alpha}, \hat{\beta}), d(\hat{\beta}, \hat{\gamma}), d(\hat{\gamma}, \hat{\alpha}) \leq \delta$ . By the Morse lemma [Bridson and Haefliger 1999, III.H.1.7], there is a constant  $R$ , only depending on  $\tau$  and  $\delta$ , and a point  $y \in A$  such that  $d(\hat{\alpha}, y) \leq R$ . Thus we have

$$d(z, y) \leq d(z, \hat{\beta}) + d(\hat{\beta}, \hat{\alpha}) + d(\hat{\alpha}, y) \leq a + \delta + R.$$

As  $x_z \in \pi_z$ , we have

$$a + c = d(x_z, z) \leq d(z, y) \leq a + \delta + R,$$

and so  $c \leq \delta + R$ . Letting  $D = 2\delta + 2R$  we compute

$$\begin{aligned} d(x, z) &= a + b = (b + c) + (a + c) - 2c \\ &\geq d(x, x_z) + d(x_z, z) - D. \end{aligned}$$

□

**Claim 2.** *There is a constant  $M_0 \in \mathbb{Z}$  such that if  $z \notin U_-$  and  $f^m x \in \pi_z$  then  $m \geq M_0$ .*

*Proof of Claim 2.* Let  $x_z = f^m x \in \pi_z$  and without loss of generality assume that  $m \leq 0$ . Using the constant  $D$  from Claim 1 we have:

$$\begin{aligned} (x_z \cdot z)_x &= \frac{1}{2}(d(x, x_z) + d(x, z) - d(x_z, z)) \\ &\geq d(x, x_z) - D/2. \end{aligned}$$

Suppose that  $i \leq m$  and let  $\alpha$  be a geodesic from  $f^i x$  to  $x$ . The Morse lemma implies that there is a  $y \in \alpha$  such that  $d(x_z, y) \leq R$ . Therefore,

$$\begin{aligned} d(x, x_z) + d(x_z, f^i x) &\leq d(x, y) + d(y, f^i x) + 2R \\ &= d(x, f^i x) + 2R. \end{aligned}$$

Hence for such  $i$  we have:

$$\begin{aligned} (x_z \cdot f^i x)_x &= \frac{1}{2}(d(x, x_z) + d(x, f^i x) - d(x_z, f^i x)) \\ &\geq d(x, x_z) - R. \end{aligned}$$

This shows that  $(x_z \cdot A_-)_x \geq d(x, x_z) - R - 2\delta$  and so for  $K = \max\{D/2, R + 2\delta\}$  we have

$$(z \cdot A_-)_x \geq \min\{(x_z \cdot z)_x, (x_z \cdot A_-)_x\} - \delta \geq d(x, x_z) - K - \delta.$$

As  $z \notin U_-$ , the Gromov product  $(z \cdot A_-)_x$  is bounded independently of  $z$  and hence  $d(x, x_z)$  is also bounded. □

Now we will finish the proof of the lemma. Fix a point  $x_g \in \pi_{gx}$ . Clearly we have  $f^m x_g \in \pi_{f^m gx}$  for  $m \geq 0$ . As  $gx \notin U_-$ , by Claim 2 we have  $x_g = f^{M_0+n} x$  for some  $n \geq 0$  and therefore,

$$\begin{aligned} d(x, f^m x_g) &= d(x, f^{M_0+n+m} x) \geq d(x, f^{m+n} x) - d(x, f^{M_0} x) \\ &\geq \frac{1}{\tau} m - \tau |M_0|. \end{aligned}$$

As  $f^m x_g \in \pi_{f^m gx}$ , Claim 1 implies

$$\begin{aligned} d(x, f^m gx) &\geq d(x, f^m x_g) + d(f^m x_g, f^m gx) - D \\ &\geq \frac{1}{\tau} m - (\tau |M_0| + D). \end{aligned}$$

Since the constants  $\tau$ ,  $D$  and  $M_0$  only depend on  $f$ ,  $x$  and the open neighborhoods  $U_+$  and  $U_-$ , the lemma is proven. □

The next lemma replaces Lemma 4.3 in [Clay and Pettet 2012] and its proof is a small modification of the proof there.

**Lemma 3.3.** *Fix  $x \in X \cap U_+$  and for  $m \geq 0$  let  $\alpha_m$  be a geodesic connecting  $x$  to  $f^m gx$ . Then there is an  $\epsilon \geq 0$  and  $M_1 \geq 0$  such that for  $m \geq M_1$  the concatenation of the geodesics  $\alpha_m \cdot f^m g\alpha_m$  is a  $(1, \epsilon)$ -quasigeodesic.*

*Proof.* Let  $d_m = d(x, f^m gx)$ .

As  $gU_+ \cap U_- = \emptyset$  we have  $U_+ \cap g^{-1}U_- = \emptyset$  and so the Gromov product  $(g^{-1}f^{-m}x \cdot f^m x)_x$  is bounded independent of  $g$  and  $m \geq M_1$  for some constant  $M_1$ . Indeed, by Proposition 2.5 there is a  $k \geq 0$  such that  $N(A_+, k) \subseteq U_+$  and  $M_1 \geq 0$  such that  $f^{-m}x \in U_-$  and  $f^m x \in N(A_+, k+2\delta)$  for  $m \geq M_1$ . Thus,  $(A_+ \cdot g^{-1}f^{-m}x)_x \leq k$  and so  $(g^{-1}f^{-m}x \cdot f^m x)_x \leq k + \delta$  as

$$\min\{(A_+ \cdot f^m x)_x, (g^{-1}f^{-m}x \cdot f^m x)_x\} - \delta \leq (A_+ \cdot g^{-1}f^{-m}x)_x \leq k,$$

for  $m \geq M_1$ .

By making  $M_1$  larger, we can assume that for  $m \geq M_1$  we have

$$f^m(\bar{X} - U_-) \subseteq N(A_+, k+4\delta),$$

by Proposition 2.7. Since  $gx, x \notin U_-$ , we have  $f^m gx, f^m x \in N(A_+, k+4\delta)$  and so  $(f^m xg \cdot f^m x)_x \geq k+3\delta$ . Hence  $(g^{-1}f^{-m}x \cdot f^m gx)_x \leq k+2\delta$  as

$$\min\{(g^{-1}f^{-m}x \cdot f^m gx)_x, (f^m gx \cdot f^m x)_x\} - \delta \leq (g^{-1}f^{-m}x \cdot f^m x)_x \leq k + \delta.$$

Therefore for  $C = k+2\delta$  and  $m \geq M_1$  we have:

$$\begin{aligned} d(x, f^m g f^m gx) &= d(g^{-1}f^{-m}x, g f^m x) \\ &\geq d(g^{-1}f^{-m}x, x) + d(x, f^m gx) - 2C \\ &= 2d_m - 2C. \end{aligned}$$

The proof now proceeds exactly as that of Lemma 4.3 in [Clay and Pettet 2012].  $\square$

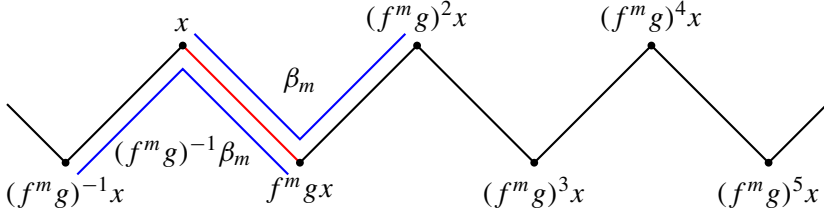
*Proof of Theorem 3.1.* Using Lemmas 3.2 and 3.3, the proof of Theorem 3.1 proceeds exactly like that of Theorem 4.1 in [Clay and Pettet 2012]. We repeat the argument here.

Fix  $x \in U_+ \cap X$ , and let  $t > 0$  and  $C \geq 0$  be the constants from Lemma 3.2, and  $\epsilon > 0$  and  $M_1 \geq 0$  be the constants from Lemma 3.3. For  $m \geq M_1$  we set  $L_m = d(x, f^m gx) \geq mt - C$ . As in Lemma 3.3, let  $\alpha_m : [0, L_m] \rightarrow X$  be a geodesic connecting  $x$  to  $f^m gx$ , and let  $\beta_m = \alpha_m \cdot f^m g\alpha_m$ . Then define a path  $\gamma : \mathbb{R} \rightarrow X$  by:

$$\gamma = \cdots (f^m g)^{-1} \beta_m \bigcup_{\alpha_m} \beta_m \bigcup_{f^m g\alpha_m} f^m g\beta_m \bigcup_{(f^m g)^2\alpha_m} (f^m g)^2\beta_m \cdots$$

See Figure 1.





**Figure 1.** The path  $\gamma$  in the proof of Theorem 3.1.

By Lemma 3.3,  $\gamma$  is an  $L_m$ -local  $(1, \epsilon)$ -quasigeodesic and hence for  $m$  large enough,  $\gamma$  is a  $(\lambda', \epsilon')$ -quasigeodesic from some  $\lambda' \geq 1$  and  $\epsilon' \geq 0$ ; see [Bridson and Haefliger 1999, III.H.1.7 and III.H.1.13] or [Clay and Pettet 2012, Theorem 4.4].

Let  $N$  be such that  $t = \frac{1}{\lambda'} L_m N - \epsilon' > 0$ . Then for any  $k \neq \ell \in \mathbb{Z}$  we have

$$d((f^m g)^{Nk} x, (f^m g)^{N\ell} x) \geq \frac{1}{\lambda'} L_m N |k - \ell| - \epsilon' \geq t |k - \ell|.$$

Thus  $(f^m g)^N$  is hyperbolic and therefore so is  $f^m g$ .  $\square$

We conclude this section with an application of Theorem 3.1 to dependent hyperbolic isometries; [Clay and Pettet 2012, Theorem 4.1] would suffice as well.

**Proposition 3.4.** *Suppose  $X$  is a  $\delta$ -hyperbolic space and  $f, g \in \text{Isom}(X)$  are dependent hyperbolic isometries. There is an  $N \geq 0$  such that if  $n \geq N$  then  $fg^n$  is hyperbolic.*

*Proof.* Let  $A_+, A_-, B_+, B_- \in \partial X$  be the attracting and repelling fixed points for  $f$  and  $g$ , respectively. Then  $fB_+ \neq B_-$  as one of these points is fixed by  $f$ . Thus there are neighborhoods  $V_+$  and  $V_-$  for  $B_+$  and  $B_-$ , respectively, in  $\bar{X}$  such that  $fV_+ \cap V_- = \emptyset$ . Let  $N$  be the constant from Theorem 3.1 applied to this setup after interchanging the roles of  $f$  and  $g$ . Hence  $g^n f$ , and therefore also its conjugate  $fg^n$ , are hyperbolic when  $n \geq N$ .  $\square$

#### 4. Finding neighborhoods

We now need to understand when we can find neighborhoods satisfying the hypotheses of Theorem 3.1 for all powers (or, at least, many powers) of a given  $g$ . There are two cases that we examine: first when  $g$  has a fixed point and second when  $g$  is hyperbolic.

**Proposition 4.1.** *Suppose  $X$  is a  $\delta$ -hyperbolic space and  $f \in \text{Isom}(X)$  is a hyperbolic isometry with attracting and repelling fixed points  $A_+$  and  $A_-$  in  $\partial X$ . Suppose  $g \in \text{Isom}(X)$  has a fixed point and consider a sequence of elements  $(g_k)_{k \in \mathbb{N}} \subseteq \langle g \rangle$ . Then either*

- (1) *there are disjoint neighborhoods  $U_+$  and  $U_-$  of  $A_+$  and  $A_-$ , respectively, and a constant  $M \geq 1$  such that if  $k \geq M$  then  $g_k U_+ \cap U_- = \emptyset$ ; or*

(2) there is a subsequence  $(g_{k_n})$  so that  $g_{k_n}A_+ \rightarrow A_-$ .

Further, if  $gA_- = A_-$  then (1) holds.

*Proof.* Let  $p \in X$  be such that  $gp = p$ . Thus  $g_k p = p$  for all  $k \in \mathbb{N}$ .

Fix a system of decreasing disjoint neighborhoods  $U_-^k$  of  $A_-$  and  $U_+^k$  of  $A_+$  indexed by the natural numbers so that:

$$\begin{aligned} (x \cdot A_+)_p &\geq k + \delta, & \text{for } x \in U_+^k, & \text{ and} \\ (x \cdot A_-)_p &\geq k + \delta, & \text{for } x \in U_-^k. \end{aligned}$$

This implies that for any two points  $x, x' \in U_+^k$  we have that

$$(x \cdot x')_p \geq \min\{(x \cdot A_+)_p, (x' \cdot A_+)_p\} - \delta \geq k.$$

Likewise for any two points  $y, y' \in U_-^k$  we have  $(y \cdot y')_p \geq k$ .

For each  $n \in \mathbb{N}$ , define

$$I_n = \{k \in \mathbb{N} \mid g_k U_+^n \cap U_-^n \neq \emptyset\}.$$

If  $I_n$  is a finite set for some  $n$ , then (1) holds for the neighborhoods  $U_- = U_-^n$  and  $U_+ = U_+^n$  where  $M = \max I_n + 1$ .

Otherwise, there is a strictly increasing sequence  $(k_n)_{n \in \mathbb{N}}$  such that  $k_n \in I_n$ . Hence, for each  $n \in \mathbb{N}$ , there is an element  $x_n \in U_+^{k_n}$  such that  $g_{k_n} x_n \in U_-^{k_n}$ . In particular,

$$(4-1) \quad (g_{k_n} x_n \cdot A_-)_p \geq n + \delta.$$

On the other hand, since  $x_n \in U_+^{k_n}$  and  $g_{k_n}$  fixes the point  $p$ , we have

$$\begin{aligned} (4-2) \quad (g_{k_n} x_n \cdot g_{k_n} A_+)_p &= (g_{k_n} x_n \cdot g_{k_n} A_+)_{g_{k_n} p} \\ &= (x_n \cdot A_+)_p \geq n + \delta. \end{aligned}$$

Combining (4-1) and (4-2), we get  $(g_{k_n} A_+ \cdot A_-)_p \geq n$  for any  $n \in \mathbb{N}$ . Hence (2) holds.

Now suppose that  $gA_- = A_-$ . As  $A_+ \neq A_-$ , there is a constant  $D \geq 0$  such that  $(f^{-k} p \cdot f^k p)_p \leq D$  for all  $k \in \mathbb{N}$ . For any  $n \in \mathbb{Z}$ , we have  $(f^{-k} p \cdot g^n f^{-k} p)_p \rightarrow \infty$  as  $k \rightarrow \infty$ . In particular, for each  $n \in \mathbb{Z}$ , there is a constant  $K_n \geq 0$  such that  $(f^{-k} p \cdot g^n f^{-k} p)_p \geq D + \delta$  for  $k \geq K_n$ . Therefore  $(g^n f^{-k} p \cdot f^k p)_p \leq D + \delta$  for  $k \geq K_n$  as:

$$(f^{-k} p \cdot f^k p)_p \geq \min\{(f^{-k} p \cdot g^n f^{-k} p)_p, (g^n f^{-k} p \cdot f^k p)_p\} - \delta.$$

As  $gp = p$ , we have  $(f^{-k} p \cdot g^n f^k p)_p = (g^{-n} f^{-k} p \cdot f^k p)_p$  and so we see that  $(f^{-k} p \cdot g^n f^k p)_p \leq D + \delta$  for  $k \geq K_{-n}$ . This shows that (2) cannot hold if  $gA_- = A_-$ .  $\square$

**Proposition 4.2.** *Suppose  $X$  is a  $\delta$ -hyperbolic space and  $f, g \in \text{Isom}(X)$  are independent hyperbolic isometries. There are disjoint neighborhoods  $U_+$  and  $U_-$  of  $A_+$  and  $A_-$  and an  $N \geq 1$  such that if  $k \geq N$  then  $g^k U_+ \cap U_- = \emptyset$ .*

*Proof.* Let  $A_+, A_-, B_+, B_- \in \partial X$  be the attracting and repelling fixed points for  $f$  and  $g$ , respectively. As  $f$  and  $g$  are independent, the set  $\{A_-, A_+, B_-, B_+\}$  consists of four distinct points. Take mutually disjoint open neighborhoods  $U_-, U_+, V_-, V_+$  of  $A_-, A_+, B_-, B_+$ , respectively. The North-South dynamics of the action of  $g$  on  $\bar{X}$  implies that there exists an  $N \geq 1$  such that  $g^k(\bar{X} - V_-) \subset V_+$  for all  $k \geq N$ . In particular,  $g^k U_+ \subseteq V_+$  and since  $V_+ \cap U_- = \emptyset$  we see that  $g^k U_+ \cap U_- = \emptyset$  for  $k \geq N$ .  $\square$

## 5. Simultaneously producing hyperbolic isometries

We can now apply the above propositions via a careful induction to prove the main result.

**Theorem 5.1.** *Suppose that  $\{X_i\}_{i=1,\dots,n}$  is a collection of  $\delta$ -hyperbolic spaces,  $G$  is a group and for each  $i = 1, \dots, n$  there is a homomorphism  $\rho_i : G \rightarrow \text{Isom}(X_i)$  such that*

- (1) *there is an element  $f_i \in G$  such that  $\rho_i(f_i)$  is hyperbolic; and*
- (2) *for each  $g \in G$ , either  $\rho_i(g)$  has a periodic orbit or is hyperbolic.*

*Then there is an  $f \in G$  such that  $\rho_i(f)$  is hyperbolic for all  $i = 1, \dots, n$ .*

*Proof.* We will prove this by induction. The case  $n = 1$  obviously holds by hypothesis.

For  $n \geq 2$ , by induction there is an  $f \in G$  such that for  $i = 1, \dots, n-1$  the isometry  $\rho_i(f) \in \text{Isom}(X_i)$  is hyperbolic. For  $i = 1, \dots, n-1$ , let  $A_+^i, A_-^i \in \partial X_i$  be the attracting and repelling fixed points of the hyperbolic isometry  $\rho_i(f)$ . By hypothesis, there is a  $g \in G$  so that  $\rho_n(g) \in \text{Isom}(X_n)$  is hyperbolic. Let  $B_+, B_- \in \partial X_n$  be the attracting and repelling fixed points of the hyperbolic isometry  $\rho_n(g)$ . Our goal is to find  $a, b \in \mathbb{N}$  so that  $\rho_i(f^a g^b)$  is hyperbolic for each  $i = 1, \dots, n$ .

We begin with some simplifications. If  $\rho_n(f) \in \text{Isom}(X_n)$  is hyperbolic then there is nothing to prove, so assume that  $\rho_n(f)$  has a periodic orbit, and so after replacing  $f$  by a power we have that  $f$  has a fixed point. By replacing  $g$  with a power if necessary, we can assume that for  $i = 1, \dots, n-1$  the isometry  $\rho_i(g)$  is either the identity or has infinite order. In fact, we can assume that  $\rho_i(g)$  has infinite order. Indeed, if  $\rho_i(g)$  is the identity, then for all  $a, b \in \mathbb{N}$  we have  $\rho_i(f^a g^b) = \rho_i(f^a)$ , which is hyperbolic by the inductive hypothesis. Hence any powers for  $f$  and  $g$  that work for all other indices between 1 and  $n-1$  necessarily work for this index  $i$  as well. Again, by replacing  $g$  with a power if necessary, we can assume that for each  $i = 1, \dots, n-1$  either  $\rho_i(g)A_-^i = A_-^i$  or  $\rho_i(g^b)A_-^i \neq A_-^i$  for each  $b \in \mathbb{Z} - \{0\}$ .

Finally, replacing  $g$  with a further power necessary, we can assume that for each  $i = 1, \dots, n-1$  if  $\rho_i(g)$  is not hyperbolic, then it has a fixed point. Analogously, by replacing  $f$  with a power if necessary, we can assume that the isometry  $\rho_n(f)$  has infinite order and that either  $\rho_n(f)B_- = B_-$  or  $\rho_n(f^a)B_- \neq B_-$  for  $a \in \mathbb{Z} - \{0\}$ .

There are various scenarios depending on the dynamics of the isometries  $\rho_i(g)$  and  $\rho_n(f)$ .

Let  $E \subseteq \{1, \dots, n-1\}$  be the subset where the isometry  $\rho_i(g)$  has a fixed point. Let  $H = \{1, \dots, n-1\} - E$ ; this is of course the subset where  $\rho_i(g)$  is hyperbolic. For  $i \in H$ , let  $B_+^i, B_-^i \in \partial X_i$  be the attracting and repelling fixed points of the hyperbolic isometry  $\rho_i(g)$ . We further identify the subset  $H' \subseteq H$  where  $\rho_i(f)$  and  $\rho_i(g)$  are independent.

We first deal with the spaces where  $\rho_i(g)$  is hyperbolic. To this end, fix  $i \in H$ .

If  $i \in H'$ , then by Proposition 4.2 there are disjoint neighborhoods  $U_+^i, U_-^i \subset \bar{X}_i$  of  $A_+^i$  and  $A_-^i$ , respectively, and an  $N_i$  so that for  $k \geq N_i$  we have  $\rho_i(g^k)U_+^i \cap U_-^i = \emptyset$ . Applying Theorem 3.1 with the neighborhoods  $U_+$  and  $U_-$ , there is an  $M_i$  so that for  $a \geq M_i$  and  $b \geq N_i$  the element  $\rho_i(f^a g^b)$  is hyperbolic.

If  $i \in H - H'$  then, by Proposition 3.4, for each  $a \in \mathbb{N}$  there is a constant  $C_i(a) \geq 0$  such that the isometry  $\rho_i(f^a g^b)$  is hyperbolic if  $b \geq C_i(a)$ .

To create a uniform statement in the sequel, for  $i \notin H'$  (including  $i \in E$ ), set  $C_i(a) = 0$  for all  $a \in \mathbb{N}$ . Also, set  $M_i = N_i = 0$  for  $i \in H - H'$ .

Summarizing the situation so far, we let  $M_0 = \max\{M_i \mid i \in H\}$  and  $N_0 = \max\{N_i \mid i \in H\}$ . Then, at this point, we know that if  $i \in H$ ,  $a \geq M_0$  and  $b \geq N_0$  then the element  $\rho_i(f^a g^b)$  is hyperbolic so long as  $b \geq C_i(a)$ .

Next we deal with the spaces where  $\rho_i(g)$  has a fixed point. To this end, fix  $i \in E$ .

Let  $E' \subseteq E$  be the subset where condition (1) of Proposition 4.1 holds using  $\rho_i(g_k) = \rho_i(g^{N_0+k})$ . The analysis here is similar to the case when  $i \in H'$ . By assumption, for  $i \in E'$ , there are disjoint neighborhoods  $U_+^i, U_-^i \subset \bar{X}_i$  of  $A_+^i$  and  $A_-^i$ , respectively, and an  $N_i$  so that for  $k \geq N_i$  we have  $\rho_i(g_k)U_+^i \cap U_-^i = \emptyset$ . Applying Theorem 3.1 with the neighborhoods  $U_+^i$  and  $U_-^i$ , there is an  $M_i$  so that for  $a \geq M_i$  the element  $\rho_i(f^a g^b)$  is hyperbolic if  $b \geq N_i$ .

To summarize again, let  $M_1 = \max\{M_i \mid i \in H \cup E'\}$  and  $N_1 = \max\{N_i \mid i \in H \cup E'\}$ . Then at this point, if  $i \in H \cup E'$ ,  $a \geq M_1$  and  $b \geq N_1$  then the element  $\rho_i(f^a g^b)$  is hyperbolic so long as  $b \geq C_i(a)$ .

It remains to deal with  $E - E'$ ; enumerate this set by  $\{i_1, \dots, i_\ell\}$ . As condition (1) of Proposition 4.1 does not hold for  $\rho_{i_1}(g_k) = \rho_{i_1}(g^{N_0+k})$  acting on  $X_{i_1}$ , there is a subsequence  $(g^{k_n}) \subseteq (g^{N_0+k})$  such that  $\rho_{i_1}(g^{k_n})A_+^{i_1} \rightarrow A_-^{i_1}$ . By iteratively passing to subsequences of  $(g^{k_n})$ , we can assume that for all  $i \in E - E'$ , either the sequence of points  $(\rho_i(g^{k_n})A_+^i) \subseteq \partial X_i$  converges or is discrete.

Notice that for  $i \in E - E'$ , the final statement of Proposition 4.1 implies that  $\rho_i(g)A_-^i \neq A_-^i$ . Coupling this with one of our earlier simplifications, we have

that  $\rho_i(g^b)A_-^i \neq A_-^i$  for all  $b \in \mathbb{Z} - \{0\}$ . Hence, there is a  $K \in \mathbb{N}$  such that for any  $i \in E - E'$  the sequence  $(g^{K+k_n})$  either satisfies  $\rho_i(g^{K+k_n})A_+^i \rightarrow p_i \neq A_-^i$ , or  $(\rho_i(g^{K+k_n})A_+^i) \subset \partial X_i$  is discrete. Indeed, suppose  $\rho_i(g^{k_n})A_+^i \rightarrow p_i$  (nothing new is being claimed in the discrete case). If  $p_i$  is not in  $\{\rho_i(g^k)A_-^i\}_{k \in \mathbb{Z}}$ , then neither is  $\rho_i(g^K)p_i$  for any  $K \in \mathbb{N}$  so  $\rho_i(g^{K+k_n})A_+^i \rightarrow \rho_i(g^K)p_i \neq A_-^i$ . Else, if  $p_i = \rho_i(g^{K_i})A_-^i$ , then for  $K \neq -K_i$  we have

$$\rho_i(g^{K+k_n})A_+^i \rightarrow \rho_i(g^{K+K_i})A_-^i \neq A_-^i.$$

So by taking  $K \in \mathbb{N}$  to avoid the finitely many such  $-K_i$  we see that the claim holds. Without loss of generality, we can assume that  $K \geq N_1$ .

Hence for each  $i \in E - E'$ , by Proposition 4.1, there are disjoint neighborhoods  $U_+^i, U_-^i \subset \bar{X}$  of  $A_+^i$  and  $A_-^i$ , respectively, and an  $N_i$  so that for  $n \geq N_i$  we have  $\rho_i(g^{K+k_n})U_+^i \cap U_-^i = \emptyset$ . Applying Theorem 3.1 with the neighborhoods  $U_+^i$  and  $U_-^i$ , there is an  $M_i$  so that for  $a \geq M_i$  the element  $\rho_i(f^a g^{K+k_n})$  is hyperbolic if  $n \geq N_i$ .

Putting all of this together, let  $M_2 = \max\{M_i \mid 1 \leq i \leq n-1\}$  and let  $N_2 = \max\{N_i \mid i \in E - E'\}$ . Thus for all  $i = 1, \dots, n-1$ , if  $a \geq M_2$ , and  $n \geq N_2$  then  $\rho_i(f^a g^{K+k_n})$  is hyperbolic so long as  $K + k_n \geq C_i(a)$ . (Notice  $K + k_n \geq K \geq N_1$  by assumption.)

We now work with the action on the space  $X_n$ . Interchanging the roles of  $f$  and  $g$  and arguing as above using Proposition 4.1 to the sequence of isometries  $(\rho_n(f^\ell))$  we obtain a subsequence  $(f^{\ell_m}) \subseteq (f^\ell)$  and constants  $M_3$  and  $N_3$  so that  $\rho_n(f^{\ell_m} g^b)$  is hyperbolic if  $m \geq M_3$  and  $b \geq N_3$ .

Fix some  $m \geq M_3$  large enough so that  $a = \ell_m \geq M_2$  and let

$$C = \max\{C_i(a) \mid 1 \leq i \leq n-1\}.$$

Now for  $n \geq N_2$  large enough so that  $b = K + k_n \geq \max\{C, N_3\}$  we have that  $\rho_i(f^a g^b)$  is hyperbolic for  $i = 1, \dots, n$  as desired.  $\square$

## 6. Application to $\text{Out}(F_N)$

Let  $F_N$  be a free group of rank  $N \geq 2$ . A *free factor system* of  $F_N$  is a finite collection  $\mathcal{A} = \{[A_1], [A_2], \dots, [A_K]\}$  of conjugacy classes of subgroups of  $F_N$ , such that there exists a free factorization

$$F_N = A_1 * \dots * A_K * B,$$

where  $B$  is a (possibly trivial) subgroup, called a *cofactor*. There is a natural partial ordering among the free factor systems:  $\mathcal{A} \subseteq \mathcal{B}$  if for each  $[A] \in \mathcal{A}$  there is a  $[B] \in \mathcal{B}$  such that  $gAg^{-1} < B$  for some  $g \in F_N$ . In this case, we say that  $\mathcal{A}$  is *contained* in  $\mathcal{B}$  or  $\mathcal{B}$  is an *extension* of  $\mathcal{A}$ .

Recall, the *reduced rank* of a subgroup  $A < F_N$  is defined as

$$\underline{\text{rk}}(A) = \min\{0, \text{rk}(A) - 1\}.$$

We extend this to a free factor system by addition:

$$\underline{\text{rk}}(\mathcal{A}) = \sum_{k=1}^K \underline{\text{rk}}(A_k),$$

where  $\mathcal{A} = \{[A_1], [A_2], \dots, [A_K]\}$ . An extension  $\mathcal{A} \subseteq \mathcal{B}$  is called a *multi-edge extension* if  $\underline{\text{rk}}(\mathcal{B}) \geq \underline{\text{rk}}(\mathcal{A}) + 2$ .

The group  $\text{Out}(F_N)$  naturally acts on the set of free factor systems as follows. Given  $\mathcal{A} = \{[A_1], [A_2], \dots, [A_K]\}$ , and  $\varphi \in \text{Out}(F_N)$  choose a representative  $\Phi \in \text{Aut}(F_N)$  of  $\varphi$ , a realization  $F_N = A_1 * \dots * A_K * B$  of  $\mathcal{A}$  and define  $\varphi(\mathcal{A})$  to be the free factor system  $\{[\Phi(A_1)], \dots, [\Phi(A_K)]\}$ . Given a free factor system  $\mathcal{A}$  consider the subgroup  $\text{Out}(F_N; \mathcal{A})$  of  $\text{Out}(F_N)$  that stabilizes the free factor system  $\mathcal{A}$ . The group  $\text{Out}(F_N; \mathcal{A})$  is called the *outer automorphism group of  $F_N$  relative to  $\mathcal{A}$* , or the *relative outer automorphism group* if the free factor system  $\mathcal{A}$  is clear from context. If  $\mathcal{A} = \{[A]\}$ , there is a well-defined restriction homomorphism  $\text{Out}(F_N; \mathcal{A}) \rightarrow \text{Out}(A)$ , which we denote by  $\varphi \mapsto \varphi|_A$  [Handel and Mosher 2013b, Fact 1.4].

For a subgroup  $\mathcal{H} < \text{Out}(F_N)$  and  $\mathcal{H}$ -invariant free factor systems  $\mathcal{F}_1 \subseteq \mathcal{F}_2$ , we say that  $\mathcal{H}$  is *irreducible with respect to the extension  $\mathcal{F}_1 \subseteq \mathcal{F}_2$*  if for any  $\mathcal{H}$ -invariant free factor system  $\mathcal{F}$  such that  $\mathcal{F}_1 \subseteq \mathcal{F} \subseteq \mathcal{F}_2$ , it follows that either  $\mathcal{F} = \mathcal{F}_1$  or  $\mathcal{F} = \mathcal{F}_2$ . We sometimes say that  $\mathcal{H}$  is *relatively irreducible* if the extension is clear from the context. The subgroup  $\mathcal{H}$  is *relatively fully irreducible* if each finite index subgroup  $\mathcal{H}' < \mathcal{H}$  is relatively irreducible. For an individual element  $\varphi \in \text{Out}(F_N)$ , we say that  $\varphi$  is relatively (fully) irreducible if the cyclic subgroup  $\langle \varphi \rangle$  is relatively (fully) irreducible.

In close analogy with Ivanov's classification [1992] of subgroups of mapping class groups, in a series of papers Handel and Mosher gave a classification of finitely generated subgroups of  $\text{Out}(F_N)$  [2013a; 2013b; 2013c; 2013d; 2013e].

**Theorem 6.1** [Handel and Mosher 2013a, Theorem D]. *For each finitely generated subgroup  $\mathcal{H} < \text{IA}_N(\mathbb{Z}/3) < \text{Out}(F_N)$ , each maximal  $\mathcal{H}$ -invariant filtration by free factor systems*

$$\emptyset = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_m = \{[F_N]\},$$

*and each  $i = 1, \dots, m$  such that  $\mathcal{F}_{i-1} \subset \mathcal{F}_i$  is a multi-edge extension, there exists  $\varphi \in \mathcal{H}$  which is irreducible with respect to  $\mathcal{F}_{i-1} \subset \mathcal{F}_i$ .*

Here,  $\text{IA}_N(\mathbb{Z}/3)$  is the finite index subgroup of  $\text{Out}(F_N)$  which is the kernel of the natural surjection

$$p : \text{Out}(F_N) \rightarrow H^1(F_N, \mathbb{Z}/3) \cong GL(N, \mathbb{Z}/3).$$

For elements in  $\text{IA}_N(\mathbb{Z}/3)$ , irreducibility is equivalent to full irreducibility hence in the above statement we can also conclude that  $\varphi$  is fully irreducible [Handel and Mosher 2013a, Theorem B].

Handel and Mosher conjecture that there is a single  $\varphi \in \mathcal{H}$  which is (fully) irreducible for each multi-edge extension  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$  [Handel and Mosher 2013a, Remark following Theorem D]. The goal of this section is to prove this conjecture. Invoking theorems of Handel–Mosher and Horbez–Guirardel, this is (essentially) an immediate application of Theorem 5.1. We state the setup and their theorems now.

**Definition 6.2.** Let  $\mathcal{A}$  be a free factor system of  $F_N$ . The *complex of free factor systems of  $F_N$  relative to  $\mathcal{A}$* , denoted  $\mathcal{FF}(F_N; \mathcal{A})$ , is the geometric realization of the partial ordering  $\sqsubseteq$  restricted to proper free factor systems that properly contain  $\mathcal{A}$ .

If  $\mathcal{A} = \{[A_1], [A_2], \dots, [A_K]\}$  is a free factor system for  $F_N$ , its *depth* is defined as:

$$D_{\mathcal{FF}}(\mathcal{A}) = (2N - 1) - \sum_{k=1}^K (2 \operatorname{rk}(A_k) - 1).$$

The free factor system  $\mathcal{A}$  is *nonexceptional* if  $D_{\mathcal{FF}}(\mathcal{A}) \geq 3$ .

**Theorem 6.3** [Handel and Mosher 2014, Theorem 1.2]. *For any nonexceptional free factor system  $\mathcal{A}$  of  $F_N$ , the complex  $\mathcal{FF}(F_N; \mathcal{A})$  is positive-dimensional, connected and  $\delta$ -hyperbolic.*

Although the group  $\operatorname{Out}(F_N)$  does not act on  $\mathcal{FF}(F_N; \mathcal{A})$ , the natural subgroup  $\operatorname{Out}(F_N; \mathcal{A})$  associated to the free factor system  $\mathcal{A}$  acts on  $\mathcal{FF}(F_N; \mathcal{A})$  by simplicial isometries. In a companion paper Handel and Mosher characterize the elements of  $\operatorname{Out}(F_N; \mathcal{A})$  that act as a hyperbolic isometry of  $\mathcal{FF}(F_N; \mathcal{A})$ :

**Theorem 6.4** [Handel and Mosher  $\geq 2018$ ]. *For any nonexceptional free factor system  $\mathcal{A}$  of  $F_N$ ,  $\varphi \in \operatorname{Out}(F_N; \mathcal{A})$  acts as a hyperbolic isometry on  $\mathcal{FF}(F_N; \mathcal{A})$  if and only if  $\varphi$  is fully irreducible with respect to  $\mathcal{A} \sqsubset \{[F_N]\}$ .*

**Remark 6.5.** An alternative proof of Theorem 6.4 is given by Guirardel and Horbez [2017] using the description of the boundary of the relative free factor complex. Further, with a slight modification of the definition of the relative free factor complex, both Handel and Mosher and Guirardel and Horbez can additionally prove that the theorem holds for the only remaining multi-edge configuration which is when  $\mathcal{A} = \{[A_1], [A_2], [A_3]\}$  and  $F_N = A_1 * A_2 * A_3$ . Yet another proof of Theorem 6.4 is given by Radhika Gupta [2016] using dynamics on relative outer space and relative currents.

We are now ready to prove our application:

**Theorem 6.6.** *For each finitely generated subgroup  $\mathcal{H} < \operatorname{IA}_N(\mathbb{Z}/3) < \operatorname{Out}(F_N)$  and each maximal  $\mathcal{H}$ -invariant filtration by free factor systems*

$$\emptyset = \mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \dots \sqsubset \mathcal{F}_m = \{[F_N]\},$$

*there is an element  $\varphi \in \mathcal{H}$  such that for each  $i = 1, \dots, m$  such that  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$  is a multi-edge extension,  $\varphi$  is irreducible with respect to  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ .*

*Proof.* Let  $I$  be the subset of indices  $i$  such that  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$  is a multi-edge extension.

Given  $i \in I$ , since  $\mathcal{H} < \text{IA}_N(\mathbb{Z}/3)$ , each component of  $\mathcal{F}_{i-1}$  and  $\mathcal{F}_i$  is  $\mathcal{H}$ -invariant [Handel and Mosher 2013c, Lemma 4.2]. Moreover, by the argument at the beginning of Section 2.1 in [Handel and Mosher 2013e], since  $\mathcal{H}$  is irreducible with respect to  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$  (this follows from maximality of the filtration) there is precisely one component  $[B_i] \in \mathcal{F}_i$  that is not a component of  $\mathcal{F}_{i-1}$ . Let  $\widehat{\mathcal{A}}_i$  be the maximal subset of  $\mathcal{F}_{i-1}$  such that  $\widehat{\mathcal{A}}_i \sqsubset \{[B_i]\}$ . Notice that this extension is again multi-edge, indeed  $\text{rk}(B_i) - \text{rk}(\widehat{\mathcal{A}}_i) = \text{rk}(\mathcal{F}_i) - \text{rk}(\mathcal{F}_{i-1})$ . The system  $\widehat{\mathcal{A}}_i$  can be represented by  $\{[A_{i,1}], \dots, [A_{i,K_i}]\}$  where  $A_{i,k} < B_i$  for each  $k$ . Let  $\mathcal{A}_i$  be the free factor system in the subgroup  $B_i$  consisting of the conjugacy classes in  $B_i$  of the subgroups  $A_{i,k}$ . Then a given  $\varphi \in \mathcal{H}$  is irreducible with respect to  $\widehat{\mathcal{A}}_i \sqsubset \{[B_i]\}$ , equivalently  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$  as the remaining components are the same, if and only if the restriction  $\varphi|_{B_i} \in \text{Out}(B_i; \mathcal{A}_i)$  is irreducible relative to  $\mathcal{A}_i$ .

For  $i \in I$ , let  $X_i = \mathcal{FF}(B_i; \mathcal{A}_i)$  and consider the action homomorphism

$$\rho_i : \mathcal{H} \rightarrow \text{Isom}(X_i)$$

defined by  $\rho_i(\varphi) = \varphi|_{B_i}$ . These spaces are  $\delta$ -hyperbolic for some  $\delta$  by Theorem 6.3, and by the above discussion and Theorem 6.4,  $\rho_i(\varphi)$  is a hyperbolic isometry if  $\varphi \in \mathcal{H}$  is irreducible with respect to  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ . If  $\rho_i(\varphi)$  is not irreducible with respect to  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ , then  $\rho_i(\varphi)$  fixes a point in  $X_i$ . By Theorem 6.1, for each  $i \in I$ , there exists some  $\varphi_i \in \mathcal{H}$  that is irreducible with respect to  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$  and hence  $\rho_i(\varphi_i)$  is a hyperbolic isometry.

We are now in the model situation of Theorem 5.1. We conclude that there is a  $\varphi \in \mathcal{H}$  such that  $\rho_i(\varphi)$  is a hyperbolic isometry for all  $i \in I$ . By the above discussion, this means that  $\varphi$  is (fully) irreducible with respect to  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$  for each  $i \in I$  as desired.  $\square$

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## A LOCAL WEIGHTED AXLER–ZHENG THEOREM IN $\mathbb{C}^n$

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The well-known Axler–Zheng theorem characterizes compactness of finite sums of finite products of Toeplitz operators on the unit disk in terms of the Berezin transform of these operators. Subsequently this theorem was generalized to other domains and appeared in different forms, including domains in  $\mathbb{C}^n$  on which the  $\bar{\partial}$ -Neumann operator  $N$  is compact. In this work we remove the assumption on  $N$ , and we study weighted Bergman spaces on smooth bounded pseudoconvex domains. We prove a local version of the Axler–Zheng theorem characterizing compactness of Toeplitz operators in the algebra generated by symbols continuous up to the boundary in terms of the behavior of the Berezin transform at strongly pseudoconvex points. We employ a Forelli–Rudin type inflation method to handle the weights.

### 1. Introduction

**1.1. History.** In the theory of Bergman space operators on the open unit disk  $\mathbb{D}$ , the Axler–Zheng theorem [1998] provides an important characterization of compactness of a large class of operators in terms of their Berezin transforms. Specifically this theorem states that if  $S$  is a finite sum of finite products of Toeplitz operators on the Bergman space  $A^2(\mathbb{D})$  whose symbols are in  $L^\infty(\mathbb{D})$ , then  $S$  is compact if and only if  $BS(z)$ , the Berezin transform of  $S$ , tends to 0 as  $|z| \rightarrow 1$ . This theorem has been extended by Suárez [2007] to include all operators in the Toeplitz algebra in the unit ball in  $\mathbb{C}^n$ . Engliš [1999] extended the Axler–Zheng theorem to irreducible bounded symmetric domains and the unit polydisk. Mitkovski, Suárez and Wick [Mitkovski et al. 2013] proved a weighted version of Suárez’s result on the unit ball in  $\mathbb{C}^n$ . Using the techniques of several complex variables, Čučković and Şahutoğlu [2013] proved a version of the Axler–Zheng theorem on smooth bounded pseudoconvex domains on which the  $\bar{\partial}$ -Neumann operator is compact. The use of the  $\bar{\partial}$  techniques required that the operators in their theorem belong to the algebra  $\mathcal{T}(\bar{\Omega})$  which is the norm closed algebra generated by  $\{T_\phi : \phi \in C(\bar{\Omega})\}$ . Recently, Kreutzer [2014] generalized Čučković and Şahutoğlu’s result in a more abstract setting.

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Our aim is to extend the previous result of Čučković and Şahutoğlu in two ways: Firstly, we want to remove the hypothesis of the compactness of the  $\bar{\partial}$ -Neumann operator on  $\Omega$ . We also want to consider weighted Bergman spaces. Our main theorem gives a local version of the Axler–Zheng theorem for a wide class of domains in  $\mathbb{C}^n$ . The novelty of our approach is to use the inflation of the domain argument pioneered by Forelli and Rudin [1974] and Ligočka [1989]. The second important ingredient is the B-regularity of the inflated domain which will give us the compactness of  $\bar{\partial}$ , thus replacing the assumption on the compactness of the  $\bar{\partial}$ -Neumann operator. As a corollary we obtain a weighted version of the Axler–Zheng theorem for strongly pseudoconvex domains, which itself is a new result.

**1.2. Preliminaries.** Let  $\Omega$  be a  $C^1$ -smooth bounded pseudoconvex domain in  $\mathbb{C}^n$  with a defining function  $\rho$ . We denote the boundary of  $\Omega$  by  $b\Omega$ . Let  $L^2(\Omega, (-\rho)^r)$  denote the square integrable functions on  $\Omega$  with respect to the measure  $(-\rho)^r dV$  where  $dV$  denotes the Lebesgue measure,  $r \geq 0$ , and

$$A^2(\Omega, (-\rho)^r) = \{f \in L^2(\Omega, (-\rho)^r) : f \text{ is holomorphic}\}.$$

Since  $A^2(\Omega, (-\rho)^r)$  is a closed subspace of  $L^2(\Omega, (-\rho)^r)$ , a bounded orthogonal projection,

$$P_r : L^2(\Omega, (-\rho)^r) \rightarrow A^2(\Omega, (-\rho)^r),$$

(called Bergman projection) exists.  $P_r$  is an integral operator of the form

$$P_r(f)(z) = \int_{\Omega} K^r(z, \zeta) f(\zeta) (-\rho)^r dV$$

for  $f \in L^2(\Omega, (-\rho)^r)$ . The integral kernel  $K^r(z, \xi)$  is called the Bergman kernel and the normalized Bergman kernel  $k_z^r(\xi)$  is defined as

$$k_z^r(\xi) = \frac{K^r(\xi, z)}{\sqrt{K^r(z, z)}}.$$

When  $r = 0$  we drop the superscript  $r$ ; that is,  $K = K_{\Omega}$  denotes the unweighted Bergman kernel and  $k_z$  denotes the unweighted normalized Bergman kernel. For a bounded operator  $T$  on  $A^2(\Omega, (-\rho)^r)$ , the Berezin transform  $B_r T$  of  $T$  is defined as

$$B_r T(z) = \langle T k_z^r, k_z^r \rangle_r,$$

where  $\langle \cdot, \cdot \rangle_r$  is the inner product on  $A^2(\Omega, (-\rho)^r)$ .

For  $\phi \in L^{\infty}(\Omega)$ , the weighted Toeplitz operator  $T_{\phi}^r$  and the weighted Hankel operator  $H_{\phi}^r$  are defined as

$$\begin{aligned} T_{\phi}^r &= P^r M_{\phi}, \\ H_{\phi}^r &= (I - P^r) M_{\phi}, \end{aligned}$$

where  $M_{\phi} : A^2(\Omega, (-\rho)^r) \rightarrow L^2(\Omega, (-\rho)^r)$  denotes the multiplication by  $\phi$ .

We use  $\mathcal{T}(\bar{\Omega}, (-\rho)^r)$  to denote the norm closed subalgebra of bounded linear operators on  $A^2(\Omega, (-\rho)^r)$  generated by the set of Toeplitz operators

$$\{T_\phi^r : \phi \in C(\bar{\Omega})\}.$$

For  $\phi \in L^\infty$  we define  $B_r\phi = B_r T_\phi$ .

In this paper we look at weighted Hankel and Toeplitz operators on various domains and various weighted spaces. Whenever we need to clarify where these operators are defined, we will use appropriate subscripts and superscripts. In particular, when we need to emphasize the underlying domain we will write  $P^\Omega$ ,  $K_\Omega(z, \xi)$ ,  $H_\phi^\Omega$ , and  $T_\phi^\Omega$ , where the Bergman spaces are unweighted. When we have weighted spaces and we need to indicate the domain and the weight we will write  $P^{\Omega, r}$ ,  $K_\Omega^r(z, \xi)$ ,  $H_\phi^{\Omega, r}$ , and  $T_\phi^{\Omega, r}$ .

**1.3. Main result.** We start with the following two definitions that capture the local structure of the main theorem. To motivate the following definition, if  $f_j \rightarrow f$  weakly in  $A^2(\Omega)$  then for any point  $p \in b\Omega$  and  $r > 0$  one can show that  $f_j \rightarrow f$  weakly in  $A^2(\Omega \cap B(p, r))$  where  $B(p, r)$  is the open ball centered at  $p$  with radius  $r$ .

**Definition 1.** Let  $r \geq 0$  and  $\Omega$  be a  $C^2$ -smooth bounded pseudoconvex domain in  $\mathbb{C}^n$  with a defining function  $\rho$ . Furthermore, let  $\{f_j\} \subset A^2(\Omega, (-\rho)^r)$  be a sequence and  $f \in A^2(\Omega, (-\rho)^r)$ . We say that  $\{f_j\}$  converges to  $f$  *weakly about strongly pseudoconvex points* if:

- (i)  $f_j \rightarrow f$  weakly in  $A^2(\Omega, (-\rho)^r)$  as  $j \rightarrow \infty$ .
- (ii) When  $\Gamma_\Omega$ , the set of the weakly pseudoconvex points in  $b\Omega$ , is nonempty, there exists an open neighborhood  $U$  of  $\Gamma_\Omega$  such that  $\|f_j - f\|_{L^2(U \cap \Omega, (-\rho)^r)} \rightarrow 0$  as  $j \rightarrow \infty$ .

We note that on strongly pseudoconvex domains, sequences converging weakly about strongly pseudoconvex points and weakly convergent sequences coincide.

**Definition 2.** Let  $r$ ,  $\Omega$ , and  $\rho$  be as above. Furthermore, let  $T : A^2(\Omega, (-\rho)^r) \rightarrow A^2(\Omega, (-\rho)^r)$  be a bounded linear operator. We say that  $T$  is *compact about strongly pseudoconvex points* if  $Tf_j \rightarrow Tf$  in  $A^2(\Omega, (-\rho)^r)$  whenever  $f_j \rightarrow f$  weakly about strongly pseudoconvex points.

**Remark 3.** As shown in Proposition 13 below, it is interesting that any Hankel operator with a symbol continuous on the closure of the domain is compact about strongly pseudoconvex points.

With the help of these two definitions, we state our main result as follows.

**Theorem 4.** Let  $r$  be a nonnegative real number,  $\Omega$  be a  $C^2$ -smooth bounded pseudoconvex domain in  $\mathbb{C}^n$  with a defining function  $\rho$ , and  $T \in \mathcal{T}(\bar{\Omega}, (-\rho)^r)$ .

Then  $T$  is compact about strongly pseudoconvex points on  $A^2(\Omega, (-\rho)^r)$  if and only if  $\lim_{z \rightarrow p} B_r T(z) = 0$  for any strongly pseudoconvex point  $p \in b\Omega$ .

If  $\Omega$  is a strongly pseudoconvex domain then we have the following corollary.

**Corollary 5.** *Let  $r$  be a nonnegative real number,  $\Omega$  be a  $C^2$ -smooth bounded strongly pseudoconvex domain in  $\mathbb{C}^n$  with a defining function  $\rho$ , and  $T$  be an element of  $\mathcal{T}(\bar{\Omega}, (-\rho)^r)$ . Then  $T$  is compact on  $A^2(\Omega, (-\rho)^r)$  if and only if  $\lim_{z \rightarrow p} B_r T(z) = 0$  for any  $p \in b\Omega$ .*

**Remark 6.** In the case of the unit ball  $\mathbb{B}^n$  in  $\mathbb{C}^n$  and  $\rho(z) = |z|^2 - 1$ , we partially recover [Mitkovski et al. 2013, Theorem 1.1]. Unlike the arguments on the unit ball, the proof of Corollary 5 does not require any explicit form for the weight or the weighted Bergman kernel.

## 2. Proof of Theorem 4

In this section, before we prove Theorem 4, we present some propositions and lemmas that encapsulate the technical details of the proof.

**Proposition 7.** *Let  $\Omega$  be a  $C^2$ -smooth bounded pseudoconvex domain in  $\mathbb{C}^n$  and  $\{T_j\}$  be a sequence of operators compact about strongly pseudoconvex points that converges to  $T$  in the operator norm. Then  $T$  is compact about strongly pseudoconvex points.*

*Proof.* Let  $\{f_j\}$  be a sequence in  $A^2(\Omega, (-\rho)^r)$  that converges to 0 weakly about strongly pseudoconvex points. Since  $f_j \rightarrow 0$  weakly there exists  $C > 0$  such that

$$\sup\{\|f_j\| : j = 1, 2, 3, \dots\} \leq C.$$

Then for any  $k$  we have

$$\|Tf_j\| \leq \|(T - T_k)f_j\| + \|T_k f_j\| \leq C\|T - T_k\| + \|T_k f_j\|.$$

Let  $\varepsilon > 0$  be given. Since  $T_j \rightarrow T$  in the operator norm, we choose  $k_\varepsilon$  such that  $\|T - T_{k_\varepsilon}\| \leq \varepsilon$ . Then

$$\limsup_{j \rightarrow \infty} \|Tf_j\| \leq C\varepsilon + \limsup_{j \rightarrow \infty} \|T_{k_\varepsilon} f_j\| \leq C\varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary, we conclude that  $Tf_j \rightarrow 0$ . That is,  $T$  is compact about strongly pseudoconvex points.  $\square$

One of the key ideas in the proof is to use an inflated domain over  $\Omega$  to understand the weighted Bergman spaces. For this purpose, unless stated otherwise, for the rest of the paper,  $\Omega$  will be a bounded pseudoconvex domain in  $\mathbb{C}^n$  with  $C^2$ -smooth boundary,  $\rho$  will be a defining function for  $\Omega$ , and

$$(1) \quad \Omega_r^p = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^p : z \in \Omega \text{ and } \rho(z) + |w_1|^{2p/r} + \dots + |w_p|^{2p/r} < 0\},$$

where  $p$  is a positive integer and  $r$  is a real number such that  $0 < r \leq p$ . For a function  $f \in A^2(\Omega, (-\rho)^r)$ , we let  $F(z, w) = f(z)$  be the trivial extension of  $f$  to  $\Omega_r^p$ . It easily follows from an iterated integral argument that  $F \in A^2(\Omega_r^p)$ .

The following proposition is interesting in its own right as it gives a relationship between the Bergman kernels of the inflated domain and base.

**Proposition 8.** *Using the notation above,*

$$K_\Omega^r(z, \xi) = c_{p,r} K_{\Omega_r^p}(z, 0; \xi, 0),$$

where  $c_{p,r} = \int_{|\tilde{w}_1|^{2p/r} + \dots + |\tilde{w}_p|^{2p/r} < 1} dV(\tilde{w})$  and  $K_\Omega^r(z, \xi)$  is the weighted Bergman kernel of  $\Omega$  with weight  $(-\rho)^r$ .

*Proof.* We will follow a standard inflation argument (see, for instance, [Forelli and Rudin 1974; Ligocka 1989]). Since  $\Omega_r^p$  is a Hartogs domain with base  $\Omega$ , the Bergman kernel of  $\Omega_r^p$  can be written as

$$K_{\Omega_r^p}(z, w; \xi, \eta) = K_{\Omega_r^p}(z, 0; \xi, 0) + \sum_{|J| \geq 1} K_J(z, \xi) w^J \bar{\eta}^J,$$

where  $J$  is a multiindex with nonnegative entries. Then for  $f \in A^2(\Omega, (-\rho)^r)$  and  $z \in \Omega$  we have ( $F$  below is the trivial extension of  $f$ )

$$(2) \quad f(z) = \int_{\Omega_r^p} K_{\Omega_r^p}(z, 0; \xi, 0) F(\xi, \eta) dV(\xi, \eta) + \sum_{|J| \geq 1} \int_{\Omega_r^p} K_J(z, \xi) w^J \bar{\eta}^J F(\xi, \eta) dV(\xi, \eta).$$

However, the integrals under the sum on the right-hand side above all vanish.

Using the change of variables  $\tilde{w}_j = w_j / (-\rho(z))^{r/2p}$ , one can compute that

$$(3) \quad \int_{|w_1|^{2p/r} + \dots + |w_p|^{2p/r} < -\rho(z)} dV(w) = (-\rho(z))^r \int_{|\tilde{w}_1|^{2p/r} + \dots + |\tilde{w}_p|^{2p/r} < 1} dV(\tilde{w}).$$

We denote

$$(4) \quad c_{p,r} = \int_{|\tilde{w}_1|^{2p/r} + \dots + |\tilde{w}_p|^{2p/r} < 1} dV(\tilde{w}).$$

Then using (2), (3), and (4) we get

$$\begin{aligned} f(z) &= \int_{\Omega_r^p} K_{\Omega_r^p}(z, 0; \xi, 0) F(\xi, \eta) dV(\xi, \eta) \\ &= c_{p,r} \int_{\Omega} K_{\Omega_r^p}(z, 0; \xi, 0) f(\xi) (-\rho(\xi))^r dV(\xi). \end{aligned}$$

Therefore,  $c_{p,r} K_{\Omega_r^p}(z, 0; \xi, 0) = K_\Omega^r(z, \xi)$ . □

For a  $C^2$ -smooth function  $\rho$  around a point  $P \in \mathbb{C}^n$ ,  $X = (x_1, \dots, x_n) \in \mathbb{C}^n$ , and  $Y = (y_1, \dots, y_n) \in \mathbb{C}^n$ , we define the complex Hessian of  $\rho$  at  $P$  as

$$H_\rho(P; X, Y) = \sum_{j,k=1}^n \frac{\partial^2 \rho(P)}{\partial z_j \partial \bar{z}_k} x_j \bar{y}_k.$$

Furthermore, we use the notation  $H_\rho(P; X) = H_\rho(P; X, X)$ .

**Lemma 9.** *Let  $\Omega$  be a  $C^2$ -smooth bounded pseudoconvex domain in  $\mathbb{C}^n$ ,  $z_0 \in b\Omega$  be a strongly pseudoconvex point, and  $\Omega_r^p$  be defined as in (1). Then there exists  $s > 0$  such that  $(z, w) \in b\Omega_r^p$  is strongly pseudoconvex for  $|z - z_0| < s$  and  $w_k \neq 0$  for all  $1 \leq k \leq p$ .*

*Proof.* Let  $\tilde{\rho}(z, w) = \rho(z) + \lambda(w)$  where  $\lambda(w) = |w_1|^{2p/r} + \dots + |w_p|^{2p/r}$  and  $p \geq r$  an integer. Then  $\tilde{\rho}$  is a  $C^2$ -smooth function. Assume that  $Q = (z, w) \in b\Omega_r^p$  is near  $z_0$  and  $X$  is a complex tangential vector to  $b\Omega_r^p$  at  $Q$ . Then  $X$  can be written as  $X = X_n + X_p$  where  $X_n$  and  $X_p$  are the components of  $X$  in the  $z$  and  $w$  variables, respectively. Then

$$H_{\tilde{\rho}}(Q; X) = H_\rho(z; X_n) + H_{\tilde{\rho}}(Q; X_n, X_p) + H_{\tilde{\rho}}(Q; X_p, X_n) + H_\lambda(w; X_p).$$

However,  $H_{\tilde{\rho}}(Q; X_n, X_p) = H_{\tilde{\rho}}(Q; X_p, X_n) = 0$  as  $z$  and  $w$  are decoupled in  $\tilde{\rho}$ . Then

$$H_{\tilde{\rho}}(Q; X) = H_\rho(z; X_n) + H_\lambda(w; X_p).$$

Let  $\pi$  denote the projection from a neighborhood of  $b\Omega$  in  $\mathbb{C}^n$  onto  $b\Omega$ . Then  $X_n = X_t + X_v$  where  $X_t$  is a tangential vector to  $b\Omega$  at  $\pi z$  and  $X_v$  is a vector complex normal to  $b\Omega$  at  $\pi z$ . Then

$$H_\rho(z; X_n) = H_\rho(z; X_t) + H_\rho(z; X_t, X_v) + H_\rho(z; X_v, X_t) + H_\rho(z; X_v).$$

We note that the complex Hessian  $H_\rho$  changes continuously and  $w \rightarrow 0$  as  $z \rightarrow z_0$  (here we assume  $(z, w) \in b\Omega_r^p$ ). Furthermore,  $X_v \rightarrow 0$  as  $z \rightarrow z_0$  (as the complex normal to  $b\Omega$  at  $z_0$  is parallel to the complex normal to  $b\Omega_r^p$  at  $(z_0, 0)$ ). Then, using the fact that  $z_0$  is a strongly pseudoconvex point, we conclude that there exists  $s > 0$  so that

$$H_\rho(z; X_n) \geq \frac{H_\rho(\pi z; X_t)}{2} > 0$$

for  $|z - z_0| < s$  and  $X_t \neq 0$ . Also  $H_\lambda(w; X_p) > 0$  whenever  $X_p \neq 0$  and  $w_k \neq 0$  for all  $k$  as  $\lambda$  is strictly plurisubharmonic whenever  $w_k \neq 0$  for all  $k$ . Therefore,  $H_{\tilde{\rho}}(Q; X) > 0$  for  $Q = (z, w) \in b\Omega_r^p$  such that  $|z - z_0| < s$  and  $w_k \neq 0$  for all  $k$ .  $\square$

The following corollary follows from the previous lemma together with the fact that  $\Omega_r^p$  has  $C^2$ -smooth boundary for  $0 < r \leq p$ .



**Corollary 10.** *Let  $\Omega$  be a  $C^2$ -smooth bounded pseudoconvex domain in  $\mathbb{C}^n$ ,  $z_0 \in b\Omega$  be a strongly pseudoconvex point, and  $\Omega_r^p$  be defined as in (1). Then there exists  $\varepsilon > 0$  such that  $B((z_0, 0), \varepsilon) \cap \Omega_r^p$  is pseudoconvex.*

Next we will prove some statements about compactness of single Toeplitz and Hankel operators.

**Lemma 11.** *Let  $\phi \in L^\infty(\Omega)$ ,  $\{f_j\}$  be a bounded sequence in  $A^2(\Omega, (-\rho)^r)$  and  $F_j$  be the trivial extension of  $f_j$  to  $\Omega_r^p$  for each  $j$  where  $\Omega_r^p$  is defined as in (1). Assume that  $\{H_\phi^{\Omega_r^p} F_j\}$  is convergent in  $L^2(\Omega_r^p)$ . Then  $\{H_\phi^{\Omega, r} f_j\}$  is convergent in  $L^2(\Omega, (-\rho)^r)$ .*

*Proof.* We will abuse the notation and denote the trivial extension of  $\phi$  to  $\Omega_r^p$  by  $\phi$ . We assume that  $\{H_\phi^{\Omega_r^p} F_j\}$  is convergent (and hence Cauchy). Let

$$G_j(z, w) = (H_\phi^{\Omega_r^p} F_j)(z, w)$$

and  $g_j(z) = G_j(z, 0)$ . Then  $G_j$  is holomorphic in  $w$  because

$$\frac{\partial G_j}{\partial \bar{w}_k} = \frac{\partial}{\partial \bar{w}_k} (I - P^{\Omega_r^p})(F_j \phi) = \frac{\partial (F_j \phi)}{\partial \bar{w}_k} = 0$$

for all  $j$  and  $1 \leq k \leq p$ . We note that  $\partial(F_j \phi)/\partial \bar{w}_k = 0$  as  $F_j \phi$  is independent of  $w_k$ . Then  $|G_j(z, w) - G_k(z, w)|^2$  is subharmonic in  $w$  and using the mean value property for subharmonic functions together with (3) and (4) one can show that

$$\begin{aligned} & |g_j(z) - g_k(z)|^2 \\ & \leq \frac{1}{c_{p,r}(-\rho(z))^r} \int_{|w_1|^{2p/r} + \dots + |w_p|^{2p/r} < -\rho(z)} |G_j(z, w) - G_k(z, w)|^2 dV(w) \end{aligned}$$

for  $j = 1, 2, \dots$  and  $z \in \Omega$ . By integrating over  $\Omega$  we get

$$c_{p,r} \|g_j - g_k\|_{L^2_{(0,1)}(\Omega, (-\rho)^r)}^2 \leq \|G_j - G_k\|_{L^2_{(0,1)}(\Omega_r^p)}^2$$

for  $j, k = 1, 2, \dots$ . Then  $\{g_j\}$  is a Cauchy sequence in  $L^2_{(0,1)}(\Omega, (-\rho)^r)$  (and hence convergent) because  $\|G_j - G_k\|_{L^2_{(0,1)}(\Omega_r^p)} \rightarrow 0$  as  $j, k \rightarrow \infty$ .

Let  $h_j(z) = P^{\Omega_r^p}(\phi F_j)(z, 0)$ . Then

$$c_{r,p} \|h_j\|_{L^2(\Omega, (-\rho)^r)}^2 \leq \|P^{\Omega_r^p}(\phi F_j)\|_{L^2(\Omega_r^p)}^2 \leq \|\phi F_j\|_{L^2(\Omega_r^p)}^2 = c_{r,p} \|\phi f_j\|_{L^2(\Omega, (-\rho)^r)}^2 < \infty$$

for each  $j$ . Hence,  $h_j \in A^2(\Omega, (-\rho)^r)$  and  $(I - P^{\Omega, r})h_j = 0$  for all  $j$ . We get equality between the last terms above because  $F_j$  and  $\phi$  are independent of  $w$ . Now

$$\begin{aligned} (I - P^{\Omega, r})g_j &= (I - P^{\Omega, r})(\phi f_j - P^{\Omega_r^p}(\phi F_j)(\cdot, 0)) \\ &= (I - P^{\Omega, r})(\phi f_j) - (I - P^{\Omega, r})(h_j) = H_\phi^{\Omega, r} f_j. \end{aligned}$$

Therefore, the sequence  $\{H_\phi^{\Omega, r} f_j\}$  is convergent in  $L^2(\Omega, (-\rho)^r)$ .  $\square$

**Lemma 12.** *Let  $r$  be a nonnegative real number and  $\Omega$  be a  $C^2$ -smooth bounded pseudoconvex domain in  $\mathbb{C}^n$  with a defining function  $\rho$ . Assume that  $\phi \in C(\bar{\Omega})$  such that  $\phi(z) = 0$  if  $z$  is a strongly pseudoconvex point in  $b\Omega$ . Then  $T_\phi^r$  is compact about strongly pseudoconvex points on  $A^2(\Omega, (-\rho)^r)$ .*

*Proof.* Let  $\{f_j\}$  be a sequence in  $A^2(\Omega, (-\rho)^r)$  that (without loss of generality) converges to 0 weakly about strongly pseudoconvex points. Then  $f_j \rightarrow 0$  weakly as  $j \rightarrow \infty$  and there is a neighborhood  $U$  of weakly pseudoconvex points in  $b\Omega$  such that

$$\|f_j\|_{L^2(U \cap \Omega, (-\rho)^r)} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Using the uniform boundedness principle and the fact that  $f_j \rightarrow 0$  weakly we conclude that the sequence  $\{f_j\}$  is bounded in  $A^2(\Omega, (-\rho)^r)$ . Furthermore, Cauchy estimates together with Montel's theorem (and the fact that  $f_j \rightarrow 0$  weakly) imply that  $\{f_j\}$  converges to zero uniformly on compact subsets of  $\Omega$ . Using the fact that  $\phi = 0$  on strongly pseudoconvex points, one can show that  $\phi f_j \rightarrow 0$  in  $A^2(\Omega, (-\rho)^r)$ . Therefore,  $T_\phi^r f_j \rightarrow 0$  in  $A^2(\Omega, (-\rho)^r)$ . That is,  $T_\phi^r$  is compact about strongly pseudoconvex points on  $A^2(\Omega, (-\rho)^r)$ .  $\square$

Let  $\Omega$  be a domain in  $\mathbb{C}^n$ . Then  $z \in b\Omega$  is said to have a holomorphic (plurisubharmonic) peak function if there exists a holomorphic (plurisubharmonic)  $f$  that is continuous on  $\bar{\Omega}$  such that  $f(z) = 1$  and  $|f(w)| < 1$  (or  $f(w) < 1$  if  $f$  is plurisubharmonic) for  $w \in \bar{\Omega} \setminus \{z\}$ .

Next we show that any Hankel operator with a symbol continuous on the closure of the domain is compact about strongly pseudoconvex points.

**Proposition 13.** *Let  $r$  be a nonnegative real number,  $\Omega$  be a  $C^2$ -smooth bounded pseudoconvex domain in  $\mathbb{C}^n$  with a defining function  $\rho$ , and  $\phi \in C(\bar{\Omega})$ . Then*

$$H_\phi^r : A^2(\Omega, (-\rho)^r) \rightarrow L^2(\Omega, (-\rho)^r)$$

*is compact about strongly pseudoconvex points.*

*Proof.* We will prove more (see Corollary 14 below). First of all, for any  $\phi \in C(\bar{\Omega})$  there exists  $\{\phi_j\} \subset C^1(\bar{\Omega})$  such that  $\phi_j \rightarrow \phi$  uniformly on  $\bar{\Omega}$  as  $j \rightarrow \infty$ . Furthermore,  $\{H_{\phi_j}^r\}$  converges to  $H_\phi^r$  in the operator norm and, by Proposition 7, if  $H_{\phi_j}^r$  is compact about strongly pseudoconvex points for every  $j$  then so is  $H_\phi^r$ . Therefore, for the rest of the proof we will assume that  $\phi \in C^1(\bar{\Omega})$ . Secondly, the proof for  $r = 0$  does not require the inflation argument in the next paragraph and hence it is easier than the case  $r > 0$ . Since both proofs are similar, except for the inflation argument, in the rest of the proof, we will assume that  $r > 0$ .

Let  $z_0 \in b\Omega$  be a strongly pseudoconvex point. Then, by Corollary 10, the domain  $B((z_0, 0), \varepsilon) \cap \Omega_r^p$  is pseudoconvex for small  $\varepsilon$ . Let  $\varepsilon > 0$  be such that

$X_0 = b\Omega \cap \overline{B((z_0, 0), \varepsilon)} \subset \mathbb{C}^n$  consists of strongly pseudoconvex points. Let us define

$$Y = b\Omega_r^p \cap \overline{B((z_0, 0), \varepsilon)} \cap \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^p : w_k = 0 \text{ for some } 1 \leq k \leq p\},$$

$$X_j = b\Omega_r^p \cap \overline{B((z_0, 0), \varepsilon)} \cap \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^p : |w_k| \geq 1/j \text{ for all } 1 \leq k \leq p\},$$

for  $j = 1, 2, 3, \dots$ . Then  $X_0$  is B-regular as any point in  $X_0$  has a holomorphic (hence plurisubharmonic) peak function on  $\Omega \subset \mathbb{C}^n$ . The same function (by extending it trivially) is also a plurisubharmonic peak function on  $\Omega_r^p \subset \mathbb{C}^{n+p}$ . Hence,  $X_0 \times \{0\}$  is B-regular as a compact set in  $\mathbb{C}^{n+p}$ . Furthermore, Lemma 9 implies that we can shrink  $\varepsilon$ , if necessary, so that  $X_j$ 's are composed of strongly pseudoconvex points for  $j \geq 1$ . Hence,  $X_j$  is B-regular for every  $j = 0, 1, 2, \dots$ .

Next we will apply a similar idea to  $Y$  in lower dimensions. Let us define  $Y_1 = \bigcup_{m=1}^p Y_1^m$  where

$$Y_1^m = b\Omega_r^p \cap \overline{B((z_0, 0), \varepsilon)} \cap \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^p : w_k = 0 \text{ for } k \neq m\}.$$

We can write  $Y_1^m$  as the union of  $X_0 \times \{0\}$  together with the compact sets

$$b\Omega_r^p \cap \overline{B((z_0, 0), \varepsilon)} \cap \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^p : |w_m| \geq 1/j, w_k = 0 \text{ for } k \neq m\}$$

for  $j = 1, 2, 3, \dots$ . However, we can think of the sets above as subsets in  $\mathbb{C}^n \times \mathbb{C}$  and (by Lemma 9) they are composed of strongly pseudoconvex points. Hence, they are B-regular. Then [Sibony 1987, Proposition 1.9] implies that each  $Y_1^m$  is B-regular as it is a countable union of B-regular sets. Hence, applying Sibony's proposition again, we conclude that  $Y_1$  is B-regular. Similarly, we can define  $Y_2 \subset Y$  as a countable union of compact sets where all but at most two  $w_k$  are equal to 0. Using the same reasoning above adopted for  $Y_2$  we can conclude that  $Y_2$  is B-regular. In a similar fashion, we can define  $Y_l$  for  $1 \leq l \leq p-1$  and prove that all of them are B-regular. Hence  $Y = \left(\bigcup_{l=1}^p Y_l\right) \cup (X_0 \times \{0\})$  is B-regular. Then

$$b(\Omega_r^p \cap B((z_0, 0), \varepsilon)) \subset Y \cup \left(\bigcup_{j=1}^{\infty} X_j\right) \cup (X_0 \times \{0\}) \cup bB((z_0, 0), \varepsilon)$$

is B-regular (satisfies property (P) in Catlin's terminology) and, therefore, the  $\bar{\partial}$ -Neumann operator on  $\Omega_r^p \cap B((z_0, 0), \varepsilon)$  is compact [Straube 2010, Theorem 4.8; Catlin 1984]. Then  $H_{\phi}^{\Omega_r^p \cap B((z_0, 0), \varepsilon)}$  is compact (see [Straube 2010, Proposition 4.1]) and Lemma 11 implies that  $H_{\phi}^{\Omega \cap B((z_0, 0), \varepsilon), r}$  is compact.

Next we will use local compact solution operators to show that  $H_{\phi}^r$  is compact about strongly pseudoconvex points. Let  $\{f_j\} \subset A^2(\Omega, (-\rho)^r)$  be a sequence weakly convergent about strongly pseudoconvex points. Then there exists an open neighborhood  $U$  of the set of weakly pseudoconvex points in  $b\Omega$  such that

(i)  $\{f_j\}$  is weakly convergent,

(ii)  $\|f_j - f_k\|_{L^2(U \cap \Omega, (-\rho)^r)} \rightarrow 0$  as  $j, k \rightarrow \infty$ .

Let us choose  $\{p_k : k = 1, \dots, m\} \subset b\Omega \setminus U$  and  $\varepsilon_k > 0$  (for  $k = 1, \dots, m$ ) such that

- (i)  $b\Omega \setminus U \subset \bigcup_{k=1}^m B(p_k, \varepsilon_k)$ ,
- (ii)  $H_\phi^{k,r} = H_\phi^{B(p_k, \varepsilon_k) \cap \Omega, r}$  is compact on  $A^2(B(p_k, \varepsilon_k) \cap \Omega, (-\rho)^r)$  for  $k = 1, \dots, m$ .

Let us choose a strongly pseudoconvex domain  $\Omega_{-1} \Subset \Omega$  and smooth cut-off functions  $\chi_{-1} \in C_0^\infty(\Omega_{-1})$ ,  $\chi_0 \in C_0^\infty(U)$ , and  $\chi_k \in C_0^\infty(B(p_k, \varepsilon))$  for  $k = 1, \dots, m$  such that  $\sum_{k=-1}^m \chi_k \equiv 1$  on  $\bar{\Omega}$ .

Let  $H_\phi^{-1,r} = H_\phi^{\Omega_{-1}, r}$ ,  $H_\phi^{0,r} = H_\phi^{U \cap \Omega, r}$ , and  $g_j = \sum_{k=-1}^m \chi_k H_\phi^{k,r} f_j$ . We note that  $H_\phi^{-1,r}$  is compact as  $\Omega_{-1} \Subset \Omega$  is strongly pseudoconvex (and  $\rho < 0$  on the closure of  $\Omega_{-1}$ );  $\{H_\phi^{0,r} f_j\}$  is convergent as  $\{f_j\}$  is convergent in  $L^2(U \cap \Omega, (-\rho)^r)$ ; and by the previous part of this proof,  $H_\phi^{k,r}$  is compact for each  $k = 1, \dots, m$ . Therefore,  $\{g_j\}$  is convergent in  $L^2(\Omega, (-\rho)^r)$ . Furthermore,

$$\bar{\partial} g_j = f_j \bar{\partial} \phi + \sum_{k=-1}^m (\bar{\partial} \chi_k) H_\phi^{k,r} f_j.$$

Then  $\{\sum_{k=-1}^m (\bar{\partial} \chi_k) H_\phi^{k,r} f_j\}$  is a convergent sequence of  $\bar{\partial}$ -closed  $(0, 1)$ -forms as both  $\bar{\partial} g_j$  and  $f_j \bar{\partial} \phi$  are  $\bar{\partial}$ -closed. Let  $Z^r : L^2_{(0,1)}(\Omega, (-\rho)^r) \rightarrow L^2(\Omega, (-\rho)^r)$  be a bounded linear solution operator to  $\bar{\partial}$  (see [Hörmander 1965]). Let

$$h_j = g_j - Z^r \sum_{k=-1}^m (\bar{\partial} \chi_k) H_\phi^{k,r} f_j.$$

Then  $\{h_j\}$  is convergent and  $\bar{\partial} h_j = f_j \bar{\partial} \phi$ . So by taking projection on the orthogonal complement of  $A^2(\Omega, (-\rho)^r)$  we get  $(I - P^r)h_j = H_\phi^r f_j$ . Therefore,  $\{H_\phi^r f_j\}$  is convergent.  $\square$

Using the proof of the proposition above we get the following corollary.

**Corollary 14.** *Let  $r$  be a nonnegative real number and  $\Omega$  be a  $C^2$ -smooth bounded pseudoconvex domain in  $\mathbb{C}^n$  with a defining function  $\rho$ . Assume that  $\Omega$  satisfies property (P) of Catlin (or B-regularity of Sibony). Then*

- (i)  $\bar{\partial}$  has a compact solution operator on  $K^2_{(0,1)}(\Omega, (-\rho)^r)$ , the weighted  $\bar{\partial}$ -closed  $(0, 1)$ -forms,
- (ii)  $H_\phi^r : A^2(\Omega, (-\rho)^r) \rightarrow L^2(\Omega, (-\rho)^r)$  is compact for all  $\phi \in C(\bar{\Omega})$ .

*Proof.* Since (ii) follows from (i), we will only prove (i). By a theorem of Diederich and Fornæss [1977], there exists a  $C^2$ -smooth defining function  $\rho_1$  and  $0 < \eta \leq 1$  such that  $-(-\rho_1)^\eta$  is a strictly plurisubharmonic exhaustion function for  $\Omega$ . Since  $\rho_1$  and  $\rho$  are comparable on  $\bar{\Omega}$  it is enough to prove that  $\bar{\partial}$  has a compact solution operator on  $K^2_{(0,1)}(\Omega, (-\rho_1)^r)$ .

Let  $s = r/\eta \geq 0$  and  $q$  be an integer such that  $s \leq q$ . We define

$$\Omega_s^q = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^q : -(-\rho_1(z))^\eta + \lambda(w) < 0\},$$

where  $\lambda(w) = |w_1|^{2q/s} + \cdots + |w_q|^{2q/s}$ . Then  $-(-\rho_1)^n + \lambda$  is a bounded  $C^2$ -smooth plurisubharmonic function and  $\Omega_s^q$  is pseudoconvex. Furthermore, the first part of the proof of Proposition 13 shows that  $\Omega_s^q$  satisfies property (P).

Let  $\{f_j\}$  be a bounded sequence in  $K_{(0,1)}^2(\Omega, (-\rho_1)^r)$ . Then  $\{F_j\}$  is a bounded sequence in  $K_{(0,1)}^2(\Omega_s^q)$ . As shown in the first part of this proof,  $\Omega_s^q$  is a bounded (not necessarily  $C^2$ -smooth) pseudoconvex domain with property (P). Then  $\{\bar{\partial}^* N^{\Omega_s^q} F_j\}$  has a convergent subsequence in  $L^2(\Omega_s^q)$  where  $N^{\Omega_s^q}$  is the  $\bar{\partial}$ -Neumann operator on  $L_{(0,1)}^2(\Omega_s^q)$ . By the proof of Proposition 8 and the fact that  $\bar{\partial}^* N^{\Omega_s^q} F_j$  is holomorphic in  $w$ , we conclude that  $\bar{\partial}^* N^{\Omega_s^q} F_j(\cdot, 0) \in L^2(\Omega, (-\rho_1)^r)$ . Further,  $\bar{\partial} \bar{\partial}^* N^{\Omega_s^q} F_j(\cdot, 0) = f_j$  for all  $j$  and  $\{\bar{\partial}^* N^{\Omega_s^q} F_j(\cdot, 0)\}$  has a convergent subsequence in  $L^2(\Omega, (-\rho_1)^r)$ . Therefore,  $\bar{\partial}$  has a compact solution operator  $R \bar{\partial}^* N^{\Omega_s^q} E$  on  $K_{(0,1)}^2(\Omega, (-\rho_1)^r)$  where  $E$  is the trivial extension operator and  $R$  is the restriction from  $\Omega_s^q$  onto  $\Omega$ .  $\square$

The following lemma is essentially contained in the proof of [Arazy and Engliš 2001, Proposition 1.3]. We present it here for the convenience of the reader.

**Lemma 15.** *Let  $r$  be a nonnegative real number,  $\Omega$  be a bounded domain in  $\mathbb{C}^n$ , and  $\phi \in C(\bar{\Omega})$ . Assume that  $z_0 \in b\Omega$  has a holomorphic peak function. Then*

$$\lim_{z \rightarrow z_0} B_r T_\phi^r(z) = \phi(z_0).$$

*Proof.* First, we prove that for any neighborhood  $U$  of  $z_0$ ,

$$(5) \quad \int_{\Omega \setminus U} |k_z^r(w)|^2 (-\rho(w))^r dV(w) \rightarrow 0 \quad \text{as } z \rightarrow z_0.$$

Indeed, for given  $U$  and  $\varepsilon > 0$  first we choose a holomorphic peak function  $g$  such that  $|g(w)| \leq \varepsilon$  for all  $w \in \Omega \setminus U$ . This can be simply done by taking a high enough power of the holomorphic peak function  $g$ . Then we choose  $\delta > 0$  such that if  $|z - z_0| < \delta$  and  $z \in \Omega$  then  $|g(z)| > 1 - \varepsilon$ . In this case,

$$\begin{aligned} \int_U |k_z^r(w)|^2 (-\rho(w))^r dV(w) &\geq \int_U |g(w)| |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\geq \left| \int_\Omega g(w) |k_z^r(w)|^2 (-\rho(w))^r dV(w) \right| \\ &\quad - \left| \int_{\Omega \setminus U} g(w) |k_z^r(w)|^2 (-\rho(w))^r dV(w) \right| \\ &\geq |g(z)| - \int_{\Omega \setminus U} |g(w)| |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\geq 1 - \varepsilon - \varepsilon \int_{\Omega \setminus U} |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\geq 1 - 2\varepsilon, \end{aligned}$$

whenever  $|z - z_0| < \delta$ . This implies that for a given neighborhood  $U$  and  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $|z - z_0| < \delta$  then

$$\int_{\Omega \setminus U} |k_z^r(w)|^2 (-\rho(w))^r dV(w) \leq \varepsilon.$$

This gives (5).

Now for  $\varepsilon > 0$ , we choose a neighborhood  $U$  of  $z$  such that  $|\phi(w) - \phi(z_0)| \leq \varepsilon$  for all  $w \in U$ . Then for this neighborhood  $U$  and the same  $\varepsilon$  we choose  $\delta > 0$  such that if  $|z - z_0| < \delta$  then  $\int_{\Omega \setminus U} |k_z^r(w)|^2 (-\rho(w))^r dV(w) \leq \varepsilon / (1 + 2\|\phi\|_{L^\infty})$ . In this case,

$$\begin{aligned} |B_r T_\phi^r(z) - \phi(z_0)| &\leq \int_{\Omega} |\phi(w) - \phi(z_0)| |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &= \int_U |\phi(w) - \phi(z_0)| |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\quad + \int_{\Omega \setminus U} |\phi(w) - \phi(z_0)| |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\leq \varepsilon \int_U |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\quad + 2\|\phi\|_{L^\infty} \int_{\Omega \setminus U} |k_z^r(w)|^2 (-\rho(w))^r dV(w) \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

This indeed concludes  $\lim_{z \rightarrow z_0} B_r T_\phi^r(z) = \phi(z_0)$ .  $\square$

We note that on any bounded domain, we have (see [Čučković and Şahutoğlu 2014, Lemma 1])

$$T_{\phi_2}^r T_{\phi_1}^r = T_{\phi_2 \phi_1}^r - H_{\phi_2}^{r*} H_{\phi_1}^r.$$

Using the fact above inductively one can prove the following lemma.

**Lemma 16.** *Let  $r$  be a nonnegative real number and  $\Omega$  be a  $C^1$ -smooth bounded domain in  $\mathbb{C}^n$  with a defining function  $\rho$ . Suppose  $\phi_1, \dots, \phi_m \in L^\infty(\Omega)$ . Then*

$$\begin{aligned} T_{\phi_m}^r T_{\phi_{m-1}}^r \cdots T_{\phi_2}^r T_{\phi_1}^r &= T_{\phi_m \phi_{m-1} \cdots \phi_2 \phi_1}^r - T_{\phi_m}^r T_{\phi_{m-1}}^r \cdots T_{\phi_3}^r H_{\phi_2}^{r*} H_{\phi_1}^r \\ &\quad - T_{\phi_m}^r T_{\phi_{m-1}}^r \cdots T_{\phi_4}^r H_{\phi_3}^{r*} H_{\phi_2 \phi_1}^r - \cdots - H_{\phi_m}^{r*} H_{\phi_{m-1} \cdots \phi_2 \phi_1}^r \\ &= T_{\phi_m \phi_{m-1} \cdots \phi_2 \phi_1}^r + S^r, \end{aligned}$$

where  $S^r$  is a finite sum of finite products of operators and each product starts with a Hankel operator.

Therefore, if the symbols  $\phi_1, \dots, \phi_m$  are continuous on  $\bar{\Omega}$  we can write

$$(6) \quad T_\phi^r \cdots T_{\phi_m}^r = T_{\phi_1 \cdots \phi_m}^r + S^r,$$

where  $S^r$  is a finite sum of finite products of operators such that each product starts with a Hankel operator with symbol continuous on  $\bar{\Omega}$ .

We state the lemma below for general weights  $\mu(z)$  (not only the ones of the form  $(-\rho)^k$ ) that are nonnegative (can vanish on the boundary) and continuous on  $\Omega$ . The weights of this form are called admissible weights (see [Pasternak-Winiarski 1990]) and the corresponding weighted Bergman projections and kernels are well defined. We say two weights  $\mu_1$  and  $\mu_2$  are comparable if there exists  $c > 0$  such that  $c^{-1}\mu_1 < \mu_2 < c\mu_1$  on  $\Omega$ .

**Lemma 17.** *Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $\mu_1$  and  $\mu_2$  be comparable admissible weights. Let  $k_z^{\mu_j}$  be the normalized Bergman kernel corresponding to  $\mu_j$  for  $j = 1, 2$ , and  $z_0 \in b\Omega$ . Then  $k_z^{\mu_1} \rightarrow 0$  weakly as  $z \rightarrow z_0$  if and only if  $k_z^{\mu_2} \rightarrow 0$  weakly as  $z \rightarrow z_0$ .*

*Proof.* It is enough to show one direction. So we will show that if  $k_z^{\mu_1} \rightarrow 0$  weakly as  $z \rightarrow z_0$  then  $k_z^{\mu_2} \rightarrow 0$  weakly as  $z \rightarrow z_0$ . Since  $\mu_1$  and  $\mu_2$  are equivalent measures we have  $A^2(\Omega, d\mu_1) = A^2(\Omega, d\mu_2)$  and there exists  $C > 1$  such that

$$\frac{\|f\|_{\mu_1}}{C} \leq \|f\|_{\mu_2} \leq C\|f\|_{\mu_1}$$

for all  $f \in A^2(\Omega, d\mu_1)$ . We remind the reader that for  $z \in \Omega$  we have

$$K_{\mu_j}(z, z) = \sup\{|f(z)|^2 : \|f\|_{\mu_j} \leq 1\},$$

where  $K_{\mu_j}$  is the Bergman kernel corresponding to  $\mu_j$ . Then  $K_{\mu_1}$  and  $K_{\mu_2}$  are equivalent on the diagonal in the sense that there exists  $D = C^2 > 1$  such that

$$\frac{K_{\mu_1}(z, z)}{D} \leq K_{\mu_2}(z, z) \leq DK_{\mu_1}(z, z).$$

Now we assume that  $k_z^{\mu_1} \rightarrow 0$  weakly as  $z \rightarrow z_0$ . Let us fix  $f \in A^2(\Omega, d\mu_1)$ . Then we have

$$\frac{f(z)}{\sqrt{K_{\mu_1}(z, z)}} = \langle f, k_z^{\mu_1} \rangle_{\mu_1} \rightarrow 0 \quad \text{as } z \rightarrow z_0.$$

Then

$$\langle f, k_z^{\mu_2} \rangle_{\mu_2} = \frac{f(z)}{\sqrt{K_{\mu_2}(z, z)}} \rightarrow 0 \quad \text{as } z \rightarrow z_0.$$

Therefore, we showed that if  $k_z^{\mu_1} \rightarrow 0$  weakly as  $z \rightarrow z_0$  then  $k_z^{\mu_2} \rightarrow 0$  weakly as  $z \rightarrow z_0$ .  $\square$

Let  $\Omega$  be a pseudoconvex domain in  $\mathbb{C}^n$  and  $z_0 \in b\Omega$ . Then we call  $z_0$  a *bumping point* if for any  $\delta > 0$  there exists a pseudoconvex domain  $\Omega_1$  such that  $\{z_0\} \cup \Omega \subset \Omega_1 \subset \Omega \cup B(z_0, \delta)$ .

**Lemma 18.** *Let  $r$  be a nonnegative real number,  $\Omega$  be a bounded pseudoconvex domain in  $\mathbb{C}^n$  with Lipschitz boundary, and  $z_0 \in b\Omega$  be a bumping point. Then  $k_z^r \rightarrow 0$  weakly as  $z \rightarrow z_0$ .*

*Proof.* By Lemma 17, without loss of generality, we assume that  $\rho$  denotes the negative distance to the boundary of  $\Omega$ .

Let us fix  $f \in A^2(\Omega, (-\rho)^r)$  and choose  $r_1, r_2 > 0$  so that  $0 < r_1 < r_2$  and the outward unit vector  $\nu$  is transversal to  $B(z_0, 2r_2) \cap b\Omega$ . Since  $z_0$  is a bumping point we choose a bounded pseudoconvex domain  $\Omega_1$  such that

$$\{z_0\} \cup \Omega \subset \Omega_1 \subset \Omega \cup B(z_0, r_1).$$

So even though  $\Omega_1$  contains a small neighborhood of  $z_0$ , we have

$$\Omega \setminus B(z_0, r_1) = \Omega_1 \setminus B(z_0, r_1).$$

Let us choose  $\chi \in C_0^\infty(B(z_0, r_2))$  such that  $\chi \equiv 1$  on a neighborhood of  $\overline{B(z_0, r_1)}$ . For  $\varepsilon > 0$  small we define  $f_\varepsilon(z) = f(z - \varepsilon\nu)$  and  $g_\varepsilon = (1 - \chi)f + \chi f_\varepsilon$ . Then

- (i)  $f_\varepsilon \in A^2(\Omega \cap B(z_0, r_2), (-\rho)^r)$  and  $f_\varepsilon \rightarrow f$  in  $L^2(\Omega \cap B(z_0, r_2), (-\rho)^r)$ ,
- (ii)  $g_\varepsilon|_{\Omega \cap B(z_0, r_2)}$  is  $C^\infty$ -smooth and  $g_\varepsilon \rightarrow f$  in  $L^2(\Omega, (-\rho)^r)$  as  $\varepsilon \rightarrow 0$ .

Let  $\rho_1$  and  $\text{Supp}(\bar{\partial}\chi)$  denote the negative distance to the boundary of  $\Omega_1$  and the support of  $\bar{\partial}\chi$ , respectively. Then  $\text{Supp}(\bar{\partial}\chi) \cap \Omega = \text{Supp}(\bar{\partial}\chi) \cap \Omega_1$  and  $-\rho$  and  $-\rho_1$  are equivalent on the support of  $\bar{\partial}\chi$ . We note that  $g_\varepsilon$  is well defined on  $\Omega$  and not on  $\Omega_1$ . However,  $\bar{\partial}g_\varepsilon = 0$  on  $\Omega \cap B(z_0, r_1)$  as  $\bar{\partial}\chi = 0$  on  $B(z_0, r_1)$  for all small  $\varepsilon > 0$ . Hence  $\bar{\partial}g_\varepsilon$  can be extended trivially to be defined on  $\Omega_1$  as a  $\bar{\partial}$ -closed  $(0, 1)$ -form on  $\Omega_1$ . Then there exists  $C > 0$  such that

$$\|\bar{\partial}g_\varepsilon\|_{L^2(\Omega_1, (-\rho_1)^r)} \leq C \|f - f_\varepsilon\|_{L^2(\Omega \cap B(z_0, r_2), (-\rho)^r)} \|\bar{\partial}\chi\|_{L^\infty(B(z_0, r_2))} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Next we will use Hörmander's theorem [1965] with the plurisubharmonic exponential weight  $-r \log(-\rho_1)$ . We note that  $-\log(-\rho_1)$  is plurisubharmonic because  $\Omega_1$  is pseudoconvex. Then using Hörmander's theorem we get a constant  $c_{\Omega_1} > 0$  (depending on  $\Omega_1$ ) and  $h_\varepsilon \in L^2(\Omega_1)$  such that  $\bar{\partial}h_\varepsilon = \bar{\partial}g_\varepsilon$  and

$$\|h_\varepsilon\|_{L^2(\Omega_1, (-\rho_1)^r)} \leq c_{\Omega_1} \|\bar{\partial}g_\varepsilon\|_{L^2(\Omega_1, (-\rho_1)^r)}.$$

Furthermore, since  $\bar{\partial}$  is elliptic on the interior and  $\bar{\partial}g_\varepsilon$  is  $C^\infty$ -smooth on  $\Omega_1$ , we have  $h_\varepsilon \in C^\infty(\Omega_1)$ .

We define  $\tilde{f}_n = g_{1/n} - h_{1/n}$  on  $\Omega$ . We note that while  $\bar{\partial}g_\varepsilon$  is even defined on all of  $\Omega_1$ ,  $g_\varepsilon$  and, hence,  $\tilde{f}_{1/n}$  in general, are not. Then we have

- (i)  $\tilde{f}_n \in A^2(\Omega, (-\rho)^r)$  and  $\tilde{f}_n \rightarrow f$  in  $A^2(\Omega, (-\rho)^r)$ ,
- (ii)  $\tilde{f}_n|_{\Omega \cap B(z_0, r_1)} \in C^\infty(\overline{\Omega \cap B(z_0, r_1)})$ .



So  $\{\tilde{f}_n\}$  is a sequence converging to  $f$  and each member of the sequence is smooth up to the boundary of  $\Omega$  on a neighborhood of  $z_0$ .

Finally, we will show weak convergence of  $k_z^r$  to 0 as  $z \rightarrow z_0$ .

$$\begin{aligned} |\langle f, k_z^r \rangle_{A^2(\Omega, (-\rho)^r)}| &\leq |\langle f - \tilde{f}_n, k_z^r \rangle_{A^2(\Omega, (-\rho)^r)}| + |\langle \tilde{f}_n, k_z^r \rangle_{A^2(\Omega, (-\rho)^r)}| \\ &\leq \|f - \tilde{f}_n\|_{L^2(\Omega, (-\rho)^r)} + \frac{|\tilde{f}_n(z)|}{\sqrt{K_r(z, z)}}. \end{aligned}$$

The first term on the right-hand side can be made arbitrarily small for large enough  $n$ , because  $\|f - \tilde{f}_n\|_{L^2(\Omega, (-\rho)^r)} \rightarrow 0$  as  $n \rightarrow \infty$ . So for  $\delta > 0$  given we choose  $n_\delta$  so that  $\|f - \tilde{f}_{n_\delta}\|_{L^2(\Omega, (-\rho)^r)} \leq \delta$ . Then since  $\tilde{f}_{n_\delta}$  is  $C^\infty$ -smooth on  $\overline{\Omega} \cap B(z_0, r_1)$  (and  $K_r(z, z) \rightarrow \infty$  as  $z \rightarrow z_0$ ), we conclude that  $|\tilde{f}_{n_\delta}(z)|/\sqrt{K_r(z, z)} \rightarrow 0$  as  $z \rightarrow z_0$ . Hence,  $\limsup_{z \rightarrow z_0} |\langle f, k_z^r \rangle| \leq \delta$  for arbitrary  $\delta > 0$ . Therefore,  $k_z^r \rightarrow 0$  weakly as  $z \rightarrow z_0$ .  $\square$

Now we are ready to prove Theorem 4.

*Proof of Theorem 4.* In the case  $r = 0$ , the proof of the theorem simplifies greatly as inflation and the related techniques are unnecessary. So we will prove the more difficult case,  $r > 0$ .

First we assume that  $T$  is compact about strongly pseudoconvex points. Let  $\Omega_r^p$  be defined as in (1) and  $z_0 \in b\Omega$  be a strongly pseudoconvex point. Since small  $C^2$ -perturbations of strongly pseudoconvex points stay pseudoconvex,  $z_0$  is a bumping point for  $\Omega$ . Then Lemma 18 implies that  $k_z^r \rightarrow 0$  weakly as  $z \rightarrow z_0$ . Furthermore, there exists an open neighborhood  $U$  of  $z_0$  such that weakly pseudoconvex points are contained in  $b\Omega \setminus \overline{U}$ ; and, as in the proof of (5), one can show that

$$\|k_z^r\|_{L^2(\Omega \setminus \overline{U}, (-\rho)^r)} \rightarrow 0 \text{ as } z \rightarrow z_0.$$

Therefore,  $\{k_z^r\}$  converges to 0 weakly about strongly pseudoconvex points as  $z \rightarrow z_0$ . Moreover, since  $T$  is compact about strongly pseudoconvex points (such operators map sequences of holomorphic functions weakly convergent about strongly pseudoconvex points to convergent sequences) we conclude that

$$B_r T(z) = \langle T k_z^r, k_z^r \rangle_{A^2(\Omega, (-\rho)^r)} \rightarrow 0$$

as  $z \rightarrow z_0$ .

Next we prove the other direction. As a first step we assume that  $T$  is a finite sum of finite products of Toeplitz operators on  $A^2(\Omega, (-\rho)^r)$  with symbols continuous on  $\overline{\Omega}$ . Furthermore, we assume that

$$\lim_{z \rightarrow z_0} B_r T(z) = 0$$

for any strongly pseudoconvex point  $z_0 \in b\Omega$ .

Lemma 16 implies that

$$(7) \quad T = T_\phi^r + S^r,$$

where  $\phi \in C(\overline{\Omega})$  and  $S^r$  is a sum of operators that start with a Hankel operator with symbol continuous on  $\overline{\Omega}$ .

Lemma 15 implies that

$$(8) \quad \lim_{z \rightarrow z_0} B_r T_\phi^r(z) = \phi(z_0)$$

as strongly pseudoconvex points have holomorphic peak functions (see [Range 1986, Theorem 1.13 in Ch VI]).

By Proposition 13, the operator  $H_\psi^r$  is compact about strongly pseudoconvex points for any  $\psi \in C(\overline{\Omega})$ . Then  $H_\psi^r k_z^r \rightarrow 0$  as  $z \rightarrow z_0$  for any  $\psi \in C(\overline{\Omega})$  because, as proven in the first part of this proof,  $k_z^r \rightarrow 0$  weakly about strongly pseudoconvex points as  $z \rightarrow z_0$ . Hence,  $B_r S^r(z) \rightarrow 0$  as  $z \rightarrow z_0$ . Combining this with (7) and (8) we can conclude that

$$\phi(z_0) = \lim_{z \rightarrow z_0} B_r T(z) = 0.$$

Since  $z_0$  was an arbitrary strongly pseudoconvex point, we have  $\phi = 0$  on all the strongly pseudoconvex boundary points. Then Lemma 12 and the fact that  $S^r$  is compact about strongly pseudoconvex points imply that  $T$  is compact about strongly pseudoconvex points.

Finally, we assume  $T \in \mathcal{T}(\overline{\Omega}, (-\rho)^r)$ . Then, using Lemma 16, for every  $\varepsilon > 0$  there exists  $\phi_\varepsilon \in C(\overline{\Omega})$  and an operator  $S_\varepsilon^r$ , compact about strongly pseudoconvex points, such that

$$\|T + T_{\phi_\varepsilon}^r + S_\varepsilon^r\| \leq \varepsilon.$$

Then for  $z \in \Omega$  we have

$$\begin{aligned} |B_r T(z) + B_r T_{\phi_\varepsilon}^r(z) + B_r S_\varepsilon^r(z)| &= |\langle T k_z^r + T_{\phi_\varepsilon}^r k_z^r + S_\varepsilon^r k_z^r, k_z^r \rangle_r| \\ &\leq \|T + T_{\phi_\varepsilon}^r + S_\varepsilon^r\| \\ &\leq \varepsilon. \end{aligned}$$

Since  $B_r S_\varepsilon^r(z) \rightarrow 0$  and  $B_r T_{\phi_\varepsilon}^r(z) \rightarrow \phi_\varepsilon(z_0)$  as  $z \rightarrow z_0$  (and we assume that  $B_r T(z) \rightarrow 0$  as  $z \rightarrow z_0$ ), we have  $|\phi_\varepsilon(z_0)| \leq \varepsilon$ . That is,  $|\phi_\varepsilon|$  is less than or equal to  $\varepsilon$  on strongly pseudoconvex points of  $\Omega$ . We choose  $\psi_\varepsilon \in C(\overline{\Omega})$  such that  $\psi_\varepsilon = 0$  on strongly pseudoconvex boundary points of  $\Omega$  and

$$\sup\{|\psi_\varepsilon(z) - \phi_\varepsilon(z)| : z \in \overline{\Omega}\} \leq 2\varepsilon.$$

Then Lemma 12 implies that  $T_{\psi_\varepsilon}^r$  is compact about strongly pseudoconvex points and

$$\|T_{\phi_\varepsilon}^r - T_{\psi_\varepsilon}^r\| \leq 2\varepsilon.$$

Hence

$$\|T + T_{\psi_\varepsilon}^r + S_\varepsilon^r\| \leq \|T + T_{\phi_\varepsilon}^r + S_\varepsilon^r\| + \|T_{\psi_\varepsilon}^r - T_{\phi_\varepsilon}^r\| \leq 3\varepsilon.$$

Therefore,  $T$  is in the norm closure of the compact about strongly pseudoconvex points operators. Finally, Proposition 7 implies that  $T$  is compact about strongly pseudoconvex points.  $\square$

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# MONOTONICITY AND RADIAL SYMMETRY RESULTS FOR SCHRÖDINGER SYSTEMS WITH FRACTIONAL DIFFUSION

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We consider a nonlinear Schrödinger system with fractional diffusion

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) + A(x)u(x) = v^p(x) & \text{in } \Omega, \\ (-\Delta)^{\beta/2}v(x) + B(x)v(x) = u^q(x) & \text{in } \Omega, \\ u(x) \geq 0, v(x) \geq 0 & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \Omega^C, \end{cases}$$

where  $\Omega$  is an unbounded parabolic domain. We first establish a narrow region principle. Using this principle and a direct method of moving planes, we obtain the monotonicity of nonnegative solutions and the Liouville-type result for the nonlinear Schrödinger system with fractional diffusion. We also obtain the radially symmetric result of positive solutions for the system in the unit ball when  $A(x)$  and  $B(x)$  are constants.

## 1. Introduction

We are interested in the following nonlinear Schrödinger system with fractional diffusion:

$$(1-1) \quad \begin{cases} (-\Delta)^{\alpha/2}u(x) + A(x)u(x) = v^p(x) & \text{in } \Omega, \\ (-\Delta)^{\beta/2}v(x) + B(x)v(x) = u^q(x) & \text{in } \Omega, \\ u(x) \geq 0, v(x) \geq 0 & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \Omega^C, \end{cases}$$

where  $\alpha, \beta \in (0, 2)$ ,  $p, q > 1$ ,  $A(x)$  and  $B(x)$  are bounded from below and  $\Omega$  is an unbounded parabolic domain in  $\mathbb{R}^n$  defined by

$$\Omega = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > |x'|^2, x' = (x_1, x_2, \dots, x_{n-1})\}.$$

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*Keywords:* fractional Schrödinger system, narrow region principle, direct method of moving planes, monotonicity, radially symmetric.

Here,  $(-\Delta)^{\alpha/2}$  and  $(-\Delta)^{\beta/2}$  are nonlocal pseudodifferential operators defined by

$$(1-2) \quad (-\Delta)^{\alpha/2} u(x) = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} dy,$$

$$(1-3) \quad (-\Delta)^{\beta/2} v(x) = C_{n,\beta} P.V. \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x - y|^{n+\beta}} dy,$$

where  $P.V.$  stands for the Cauchy principal value, and  $C_{n,\alpha}$ ,  $C_{n,\beta}$  are normalization positive constants. Let

$$F = L_\alpha \cap C_{loc}^{1,1}(\Omega), \quad G = L_\beta \cap C_{loc}^{1,1}(\Omega),$$

where

$$L_\alpha = \left\{ u \mid u \in L_{loc}^1, \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty \right\},$$

and

$$L_\beta = \left\{ v \mid v \in L_{loc}^1, \int_{\mathbb{R}^n} \frac{|v(x)|}{1 + |x|^{n+\beta}} dx < \infty \right\}.$$

For  $u \in F$ ,  $v \in G$ , the integral on the left-hand side of the equations in (1-1) is well defined (see [Chen et al. 2017b]).

Linear and nonlinear equations and systems involving the fractional Laplacian have received growing attention in recent years. It can be used to model diverse physical phenomena, such as turbulence and water waves, molecular dynamics, and pseudorelativistic boson stars (see [Bouchaud and Georges 1990], [Caffarelli and Vasseur 2010], [Constantin 2006], [Tarasov and Zaslavsky 2006]). The operator  $(-\Delta)^{\alpha/2}$  can also be used in mathematical finance (see [Applebaum 2009], [Bertoin 1996]). But they are still much less understood than nonfractional counterparts.

When  $\alpha = \beta = 2$ ,  $A(x) = B(x) = 0$ , (1-1) becomes the classical Lane–Emden system:

$$(1-4) \quad \begin{cases} -\Delta u = v^p, \\ -\Delta v = u^q. \end{cases}$$

When  $1/(p+1) + 1/(q+1) > (n-2)/n$ , the system (1-4) has no positive radial solutions in all dimension (see [Mitidieri 1996]). D. G. de Figueiredo and P. L. Felmer [1994] studied a Liouville type theorem for (1-4) by introducing superharmonic functions when  $n \geq 3$ . The main tool they used is the method of moving planes. For  $n = 3$ , J. Serrin and H. Zou [1996] proved that the system (1-4) has no positive solutions when  $1/(p+1) + 1/(q+1) > (n-2)/n$  under assumption that  $(u, v)$  has at most polynomial growth at infinity. After Serrin's work, there are some interesting works about Lane–Emden systems and related Schrödinger systems on whole space and half space; see [Montaru and Souplet 2014; Poláčik et al. 2007; Souplet 2009].

For classical semilinear elliptic system, the symmetry and monotonicity of positive solutions have been widely studied (see [Busca and Sirakov 2000; Chen and Li 2010; Liu and Ma 2012; 2013; Ma and Liu 2010]). A powerful tool to obtain these properties of such equations and systems is the method of moving planes, which was introduced by Alexandrov [1962]. Serrin [1971] and Gidas, Ni, Nirenberg [Gidas et al. 1979; 1981] adapted this method in partial differential equations and made great contributions to improving this method.

As we know, the fractional Laplacian is nonlocal; that is, it is not differentiable pointwise, but is globally integrable with respect to a singular kernel. The nonlocality causes the main difficulty in studying corresponding problems. To circumvent this difficulty, Caffarelli and Silvestre [2007] introduced the extension method that reduced this nonlocal problem in  $\mathbb{R}^n$  into a local one in  $\mathbb{R}_+^{n+1}$  through constructing a Dirichlet to Neumann operator of a degenerate elliptic equation. This extension method has been applied successfully to study equations involving the fractional Laplacian, and a series of fruitful results have been obtained; see [Brändle et al. 2013; Chen and Zhu 2016].

Due to technical restrictions, they have to assume  $\alpha \geq 1$ . It seems that this condition cannot be weakened if one wants to carry out the method of moving planes on the extended equation. Actually, the case  $0 < \alpha < 1$  can be treated by considering the corresponding integral equation. Using the method of moving planes (or spheres) in integral forms [Chen and Li 2009; Chen et al. 2005a; 2005b; 2006; 2015; Fall and Weth 2016; Fang and Zhang 2013; Li and Ma 2008; Ma and Chen 2008; Ma and Zhao 2008, one can obtain the radial symmetry properties of the fractional Laplacian equation. For the fractional Laplacian system, here we mention the work by Zhuo, Chen, Cui and Yuan [Zhuo et al. 2016]. They considered the system

$$(1-5) \quad (-\Delta)^{\alpha/2} u_i(x) = f_i(u_1(x), u_2(x), \dots, u_m(x)), \quad x \in \mathbb{R}^n, \quad i = 1, 2, \dots, m.$$

By establishing the equivalence between (1-5) and its corresponding integral system, the authors obtained the symmetry result and the nonexistence of positive solutions.

Either by extension or by integral equations, one needs to impose extra conditions on the solutions. Can one carry out the method of moving planes directly on fractional equation? The answer was provided in [Jarohs and Weth 2016] by Jarohs and Weth. They introduced antisymmetric maximum principles and applied them to carry out the method of moving planes directly on nonlocal problems to show the symmetry of solutions. However, their maximum principles only apply to bounded regions.

Recently, Chen, Li and Li [Chen et al. 2017b] developed a direct method of moving planes to study the fractional Laplacian, which worked directly on the nonlocal operator. The key ingredients of this method are the antisymmetric properties. They used this property to develop some techniques needed in the direct method of moving planes in the whole space  $\mathbb{R}^n$  and the upper half space  $\mathbb{R}_+^n$ .

such as the narrow region principle, decay at infinity. The direct method of moving planes is very useful. This method has been applied to fully nonlinear fractional order-equations and systems in [Chen et al. 2017a]. In [Cheng et al. 2017], the authors considered the symmetry and monotonicity properties for positive solutions of fractional Laplacian equations by the direct method. Using the spirit of direct method of moving planes in [Chen et al. 2017b], Cai and Mei [2017] studied the fractional Lane–Emden system in  $\mathbb{R}^n$  and obtain the symmetry properties and Liouville-type result of positive solutions. Liu and Ma [2016] studied symmetry properties of the general fractional Laplacian system:

$$(1-6) \quad \begin{cases} (-\Delta)^{\alpha/2} u(x) = f(u, v) & \text{in } \mathbb{R}^n, \\ (-\Delta)^{\alpha/2} v(x) = g(u, v) & \text{in } \mathbb{R}^n, \\ u(x) \geq 0, v(x) \geq 0 & \text{in } \mathbb{R}^n \end{cases}$$

under a strong decay condition on the solutions at infinity.

The goal of this paper is to generalize the direct method of moving planes to the Schrödinger system. We first establish the narrow region principle for Schrödinger systems with fractional diffusion. We write  $x = (x', x_n) \in \mathbb{R}^n$  with  $x' = (x_1, x_2, \dots, x_{n-1})$ . Assume  $A(x)$  and  $B(x)$  are independent of  $x_n$ , that is,

$$A(x) = A(x'), \quad B(x) = B(x').$$

Let

$$T_\lambda = \{x \in \mathbb{R}^n \mid x_n = \lambda, \lambda \in \mathbb{R}, \lambda > 0\}$$

be the moving plane and denote

$$H_\lambda = \{x \in \mathbb{R}^n \mid x_n < \lambda\}, \quad \Sigma_\lambda = \{x \in \Omega \mid 0 < x_n < \lambda\}.$$

For each point  $x = (x', x_n) \in \Sigma_\lambda$ , let  $x^\lambda = (x', 2\lambda - x_n)$  be the reflection point about the plane  $T_\lambda$ . Denote

$$U_\lambda(x) = u(x^\lambda) - u(x) = u_\lambda(x) - u(x), \quad V_\lambda(x) = v(x^\lambda) - v(x) = v_\lambda(x) - v(x).$$

It follows that for  $x \in \Sigma_\lambda$ ,

$$(1-7) \quad \begin{aligned} (-\Delta)^{\alpha/2} U_\lambda(x) &= (-\Delta)^{\alpha/2} u_\lambda(x) - (-\Delta)^{\alpha/2} u(x) \\ &= p\xi^{p-1}(x)V_\lambda(x) - A(x')U_\lambda(x), \end{aligned}$$

and

$$(1-8) \quad \begin{aligned} (-\Delta)^{\beta/2} V_\lambda(x) &= (-\Delta)^{\beta/2} v_\lambda(x) - (-\Delta)^{\beta/2} v(x) \\ &= q\eta^{q-1}(x)U_\lambda(x) - B(x')V_\lambda(x), \end{aligned}$$

where  $\xi(x)$  is between  $v_\lambda(x)$  and  $v(x)$  and  $\eta(x)$  is between  $u_\lambda(x)$  and  $u(x)$ . It is obvious that  $U_\lambda(x)$  and  $V_\lambda(x)$  satisfy the antisymmetry property:

$$(1-9) \quad U_\lambda(x^\lambda) = -U_\lambda(x), \quad V_\lambda(x^\lambda) = -V_\lambda(x), \quad x \in H_\lambda.$$



**Lemma 1.1** (narrow region principle). *Let  $(u, v) \in F \times G$  be a nonnegative solution of system (1-1) with  $0 < \alpha, \beta < 2$ . Assume that  $1 < p, q < \infty$ ,  $A(x) = A(x')$  and  $B(x) = B(x')$  are bounded from below in  $\Omega$ , where  $x = (x', x_n) \in \Omega$ ,  $x' = (x_1, x_2, \dots, x_{n-1})$ . Then for all systems (1-7) and (1-8) and for sufficiently small  $\delta$ ,*

- (i) *if there exists  $x_1^* \in \Sigma_{\lambda, \delta} = \{x \in \Sigma_\lambda \mid \lambda - \delta < x_n < \lambda\}$  satisfying  $U_\lambda(x_1^*) = \min_{x \in \bar{\Sigma}_\lambda} U_\lambda(x) < 0$ , then*

$$V_\lambda(x_1^*) < U_\lambda(x_1^*) < 0;$$

- (ii) *if there exists  $x_2^* \in \Sigma_{\lambda, \delta} = \{x \in \Sigma_\lambda \mid \lambda - \delta < x_n < \lambda\}$  satisfying  $V_\lambda(x_2^*) = \min_{x \in \bar{\Sigma}_\lambda} V_\lambda(x) < 0$ , then*

$$U_\lambda(x_2^*) < V_\lambda(x_2^*) < 0.$$

Based on Lemma 1.1, we can obtain the following result.

**Theorem 1.2.** *Let  $(u, v) \in F \times G$  be a nonnegative solution of system (1-1) with  $0 < \alpha, \beta < 2$ . If  $1 < p, q < \infty$ ,  $A(x) = A(x')$  and  $B(x) = B(x')$  are bounded from below in  $\Omega$ , where  $x = (x', x_n) \in \Omega$ ,  $x' = (x_1, x_2, \dots, x_{n-1})$ . Then  $(u, v)$  is monotonically increasing in  $x_n$ .*

When  $\alpha = \beta = 2$ , this result is contained in the series papers of Berestycki, Caffarelli and Nirenberg [Berestycki and Nirenberg 1992; Berestycki et al. 1993; 1996; 1997] and they have used the classical method of moving planes. Hence our result by using the direct method of moving planes due to [Chen et al. 2017b] and [Jarohs and Weth 2016] can be considered as an extension of theirs to the nonlocal system.

As an immediate application, we obtain the following Liouville type result.

**Corollary 1.3.** *Let  $(u, v) \in F \times G$  be a nonnegative solution of system (1-1) with  $0 < \alpha, \beta < 2$ . Assume that  $1 < p, q < \infty$ ,  $A(x) = A(x')$  and  $B(x) = B(x')$  are bounded from below in  $\Omega$ , where  $x = (x', x_n) \in \Omega$ ,  $x' = (x_1, x_2, \dots, x_{n-1})$ . If*

$$(1-10) \quad \lim_{x \rightarrow \infty} u(x) = 0, \quad \lim_{x \rightarrow \infty} v(x) = 0,$$

*Then  $u \equiv 0, v \equiv 0$ .*

We consider the system (1-1) when  $A(x) = A$  and  $B(x) = B$ , where  $A, B$  are two constants in the unit ball  $\mathbb{B}_1(0)$  and obtain the radial symmetry and monotonicity of positive solutions.

**Theorem 1.4.** *Assume  $(u, v) \in L_\alpha \cap C_{loc}^{1,1}(\mathbb{B}_1(0)) \times L_\beta \cap C_{loc}^{1,1}(\mathbb{B}_1(0))$  is a positive solution of the following system*

$$(1-11) \quad \begin{cases} (-\Delta)^{\alpha/2} u(x) + Au(x) = v^p(x) & \text{in } \mathbb{B}_1(0), \\ (-\Delta)^{\beta/2} v(x) + Bv(x) = u^q(x) & \text{in } \mathbb{B}_1(0), \\ u(x) \geq 0, v(x) \geq 0 & \text{in } \mathbb{B}_1(0), \\ u(x) = v(x) = 0 & \text{on } \mathbb{B}_1^C(0), \end{cases}$$

with  $0 < \alpha, \beta < 2$  and  $1 < p, q < \infty$ . Then each positive solution  $(u(x), v(x))$  must be radially symmetric and monotone decreasing about the origin.

The paper is organized as follows. Section 2 is devoted to proving Lemma 1.1, the narrow region principle for (1-1). In Section 3, we study the monotonicity of positive solutions of (1-1) in  $\Omega$  and prove Theorem 1.2. Finally, the proof of Theorem 1.4 will be presented in Section 4. Note that in the following,  $C$  will be a positive constant which can be different from line to line.

## 2. Preliminaries

In this section, we will prove Lemma 1.1, which plays an important role in the proof of Theorem 1.2 and Theorem 1.4.

*Proof.* (i) Without loss of generality, let

$$x_1^* \in \Sigma_{\lambda, \delta} \quad \text{and} \quad U_\lambda(x_1^*) = \min_{x \in \Sigma_\lambda} U_\lambda(x) < 0.$$

It follows that

$$\begin{aligned} (-\Delta)^{\alpha/2} U_\lambda(x_1^*) &= C_{n, \alpha} P.V. \int_{\mathbb{R}^n} \frac{U_\lambda(x_1^*) - U_\lambda(y)}{|x_1^* - y|^{n+\alpha}} dy \\ &= C_{n, \alpha} P.V. \left( \int_{H_\lambda} \frac{U_\lambda(x_1^*) - U_\lambda(y)}{|x_1^* - y|^{n+\alpha}} dy + \int_{\mathbb{R}^n \setminus H_\lambda} \frac{U_\lambda(x_1^*) - U_\lambda(y)}{|x_1^* - y|^{n+\alpha}} dy \right) \\ &= C_{n, \alpha} P.V. \left( \int_{H_\lambda} \frac{U_\lambda(x_1^*) - U_\lambda(y)}{|x_1^* - y|^{n+\alpha}} dy + \int_{H_\lambda} \frac{U_\lambda(x_1^*) - U_\lambda(y^\lambda)}{|x_1^* - y^\lambda|^{n+\alpha}} dy \right). \end{aligned}$$

Note that  $|x_1^* - y| \leq |x_1^* - y^\lambda|$  when  $x_1^*, y \in \Sigma_\lambda$ , and using the antisymmetry of  $U_\lambda(x)$ , we have

$$\begin{aligned} (-\Delta)^{\alpha/2} U_\lambda(x_1^*) &\leq C_{n, \alpha} P.V. \left( \int_{H_\lambda} \frac{U_\lambda(x_1^*) - U_\lambda(y)}{|x_1^* - y^\lambda|^{n+\alpha}} dy + \int_{H_\lambda} \frac{U_\lambda(x_1^*) + U_\lambda(y)}{|x_1^* - y^\lambda|^{n+\alpha}} dy \right) \\ &= C_{n, \alpha} P.V. \int_{H_\lambda} \frac{2U_\lambda(x_1^*)}{|x_1^* - y^\lambda|^{n+\alpha}} dy. \end{aligned}$$

Let  $D = B_{2\delta}(x_1^*) \cap H_\lambda$ . Then we obtain

$$(2-1) \quad \int_{H_\lambda} \frac{1}{|x_1^* - y^\lambda|^{n+\alpha}} dy \geq \int_D \frac{1}{|x_1^* - y^\lambda|^{n+\alpha}} dy \geq C \int_{B_{2\delta}(x_1^*)} \frac{1}{|x_1^* - y^\lambda|^{n+\alpha}} dy \geq \frac{C}{\delta^\alpha}.$$

Thus,

$$(-\Delta)^{\alpha/2} U_\lambda(x_1^*) \leq \frac{C U_\lambda(x_1^*)}{\delta^\alpha} < 0.$$

According to (1-7), we get

$$p\xi^{p-1}(x_1^*)V_\lambda(x_1^*) - A(x_1^*)U_\lambda(x_1^*) = (-\Delta)^{\alpha/2} U_\lambda(x_1^*) \leq \frac{C}{\delta^\alpha} U_\lambda(x_1^*) < 0.$$

Note that  $A(x) = A(x')$  is bounded from below and  $\xi(x)$  is between  $v_\lambda(x)$  and  $v(x)$ ,  $v(x) \in G = L_\beta \cap C_{loc}^{1,1}(\Omega)$ , hence, for  $\delta$  sufficiently small, we have

$$(2-2) \quad V_\lambda(x_1^*) < \frac{A(x_1^*) + \frac{C}{\delta^\alpha}}{pv^{p-1}(x_1^*)} U_\lambda(x_1^*) < U_\lambda(x_1^*) < 0.$$

(ii) If there exists  $x_2^* \in \Sigma_{\lambda,\delta} = \{x \in \Sigma_\lambda \mid \lambda - \delta < x_n < \lambda\}$  such that

$$V_\lambda(x_2^*) = \min_{x \in \Sigma_\lambda} V_\lambda(x) < 0,$$

similarly to the proof of (i), we can obtain

$$(2-3) \quad (-\Delta)^{\beta/2} V_\lambda(x_2^*) \leq \frac{C V_\lambda(x_2^*)}{\delta^\beta} < 0.$$

Note that  $B(x)$  is bounded from below and  $u(x) \in F = L_\alpha \cap C_{loc}^{1,1}(\Omega)$ , hence, for  $\delta$  sufficiently small, according to (1-8), we have

$$(2-4) \quad U_\lambda(x_2^*) < \frac{B(x_2^*) + \frac{C}{\delta^\beta}}{qu^{q-1}(x_2^*)} V_\lambda(x_2^*) < V_\lambda(x_2^*) < 0.$$

Thus we have completed the proof of Lemma 1.1.  $\square$

### 3. The proof of Theorem 1.2

In this section, we will carry out the direct method of moving planes on the solution  $(u(x), v(x))$  along  $x_n$  direction to prove Theorem 1.2.

*Proof.* The proof of Theorem 1.2 is divided into two steps.

**Step 1:** We show that for  $\lambda > 0$  sufficiently close to zero,

$$(3-1) \quad U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda.$$

If (3-1) does not hold, then there exists a point  $x_1^* \in \Sigma_\lambda$  such that  $U_\lambda(x_1^*) < 0$ . Without loss of generality, we assume

$$U_\lambda(x_1^*) = \min_{x \in \Sigma_\lambda} U_\lambda(x) < 0.$$

By Lemma 1.1, for  $\lambda > 0$  sufficiently close to zero,

$$(3-2) \quad V_\lambda(x_1^*) < \frac{A(x_1^*) + \frac{C}{\lambda^\alpha}}{pv^{p-1}(x_1^*)} U_\lambda(x_1^*) < 0.$$

Note that  $V_\lambda(x) = 0$  for  $x \in T_\lambda$  and  $V_\lambda(x) \geq 0$  for  $x \in \partial \Sigma_\lambda$ , hence, there exists  $x_2^* \in \Sigma_\lambda$  such that

$$V_\lambda(x_2^*) = \min_{x \in \Sigma_\lambda} V_\lambda(x) < 0.$$

Similarly to the proof of Lemma 1.1, for  $\lambda > 0$  sufficiently close to zero, we have

$$(3-3) \quad U_\lambda(x_2^*) < \frac{B(x_2^*) + \frac{C}{\lambda^\beta}}{qu^{q-1}(x_2^*)} V_\lambda(x_2^*).$$

Therefore, together with (3-2) and (3-3), we get

$$(3-4) \quad \begin{aligned} V_\lambda(x_1^*) &< \frac{A(x_1^*) + \frac{C}{\lambda^\alpha}}{pv^{p-1}(x_1^*)} U_\lambda(x_1^*) \leq \frac{A(x_1^*) + \frac{C}{\lambda^\alpha}}{pv^{p-1}(x_1^*)} U_\lambda(x_2^*) \\ &< \frac{(A(x_1^*) + \frac{C}{\lambda^\alpha}) (B(x_2^*) + \frac{C}{\lambda^\beta})}{pv^{p-1}(x_1^*) qu^{q-1}(x_2^*)} V_\lambda(x_2^*) \\ &\leq \frac{(A(x_1^*) + \frac{C}{\lambda^\alpha}) (B(x_2^*) + \frac{C}{\lambda^\beta})}{pv^{p-1}(x_1^*) qu^{q-1}(x_2^*)} V_\lambda(x_1^*). \end{aligned}$$

Because  $A(x) = A(x')$  and  $B(x) = B(x')$  are bounded from below and  $V_\lambda(x_1^*) < 0$ , therefore, (3-4) is a contradiction for  $\lambda > 0$  sufficiently close to zero and the proof of Step 1 is completed.

Step 1 provides a starting point. We start from such a small  $\lambda$  and move the plane  $T_\lambda$  up continuously in the direction of  $x_n$ -axis to its limiting position as long as (3-1) holds. Define

$$(3-5) \quad \lambda_0 = \sup\{\lambda > 0 \mid U_\mu(x) \geq 0, V_\mu(x) \geq 0, x \in \Sigma_\mu; \mu \leq \lambda\}.$$

**Step 2:** We prove

$$(3-6) \quad \lambda_0 = +\infty.$$

Before proceeding further, we investigate some properties of  $U_{\lambda_0}(x)$  and  $V_{\lambda_0}(x)$  for  $x \in \Sigma_{\lambda_0}$ .

**Proposition 3.1.** *If  $U_{\lambda_0}(x) \equiv 0$ , then  $V_{\lambda_0}(x) \equiv 0$ . If  $V_{\lambda_0}(x) \equiv 0$ , then  $U_{\lambda_0}(x) \equiv 0$ .*

*Proof.* If  $U_{\lambda_0}(x) \equiv 0$ , then (1-7) becomes

$$0 = (-\Delta)^{\alpha/2} U_{\lambda_0}(x) = p\xi^{p-1}(x) V_{\lambda_0}(x),$$

Obviously, we get  $V_{\lambda_0}(x) \equiv 0$ . □

**Proposition 3.2.** *If  $U_{\lambda_0}(x) \not\equiv 0$  or  $V_{\lambda_0}(x) \not\equiv 0$ , then  $U_{\lambda_0}(x) > 0$  and  $V_{\lambda_0}(x) > 0$  for all  $x \in \Sigma_{\lambda_0}$ .*

*Proof.* Since we know that  $U_{\lambda_0}(x) \geq 0$ ,  $x \in \Sigma_{\lambda_0}$ . If  $U_{\lambda_0}(x) > 0$  does not hold, we assume that there exists some point  $\tilde{x} \in \Sigma_{\lambda_0}$  such that  $U_{\lambda_0}(\tilde{x}) = 0$ .

$$\begin{aligned}
 (-\Delta)^{\alpha/2} U_{\lambda_0}(\tilde{x}) &= C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{U_{\lambda_0}(\tilde{x}) - U_{\lambda_0}(y)}{|\tilde{x} - y|^{n+\alpha}} dy \\
 &= C_{n,\alpha} P.V. \left( \int_{H_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|\tilde{x} - y|^{n+\alpha}} dy + \int_{\mathbb{R}^n \setminus H_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|\tilde{x} - y|^{n+\alpha}} dy \right) \\
 &= C_{n,\alpha} P.V. \left( \int_{H_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|\tilde{x} - y|^{n+\alpha}} dy + \int_{H_{\lambda_0}} \frac{-U_{\lambda_0}(y^\lambda)}{|\tilde{x} - y^\lambda|^{n+\alpha}} dy \right) \\
 (3-7) \quad &= C_{n,\alpha} P.V. \int_{H_{\lambda_0}} \left( \frac{1}{|\tilde{x} - y^\lambda|^{n+\alpha}} - \frac{1}{|\tilde{x} - y|^{n+\alpha}} \right) U_{\lambda_0}(y) dy.
 \end{aligned}$$

Note that  $|\tilde{x} - y^\lambda| > |\tilde{x} - y|$ , hence,

$$(3-8) \quad (-\Delta)^{\alpha/2} U_{\lambda_0}(\tilde{x}) < 0.$$

On the other hand, according to (1-7),

$$(3-9) \quad (-\Delta)^{\alpha/2} U_{\lambda_0}(\tilde{x}) = p\xi^{p-1}(\tilde{x}) V_{\lambda_0}(\tilde{x}) - A(\tilde{x}) U_{\lambda_0}(\tilde{x}) = p\xi^{p-1}(\tilde{x}) V_{\lambda_0}(\tilde{x}) \geq 0.$$

Evidently, this is contradictory to (3-8). Consequently, we obtain  $U_{\lambda_0}(x) > 0$  for all  $x \in \Sigma_{\lambda_0}$ . Then by Proposition 3.1, we get  $V_{\lambda_0}(x) \not\equiv 0$ . Similarly to above, we obtain  $V_{\lambda_0}(x) > 0$  for all  $x \in \Sigma_{\lambda_0}$ , completing the proof.  $\square$

Now, we start to prove (3-6). If  $\lambda_0 < +\infty$ , we will show

$$(3-10) \quad U_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0}.$$

Then by Proposition 3.1,  $V_{\lambda_0}(x) \equiv 0$ ,  $x \in \Sigma_{\lambda_0}$ . Thus, we obtain

$$(3-11) \quad u(x', 2\lambda_0) = u(x', 0),$$

$$(3-12) \quad v(x', 2\lambda_0) = v(x', 0).$$

But the left-hand side of (3-11) is positive, and the right-hand side of (3-11) is equal to zero. This is contradictory. The same holds for  $v(x)$ . Hence, (3-6) holds.

In the following, we will prove (3-10). If  $U_{\lambda_0}(x) \not\equiv 0$ ,  $x \in \Sigma_{\lambda_0}$ , by Proposition 3.2, we have  $U_{\lambda_0}(x) > 0$ ,  $V_{\lambda_0}(x) > 0$ ,  $x \in \Sigma_{\lambda_0}$ . Hence, for small  $\delta > 0$ , there exists a positive constant  $c_0$  such that

$$U_{\lambda_0}(x) \geq c_0 > 0, \quad V_{\lambda_0}(x) \geq c_0 > 0, \quad x \in \bar{\Sigma}_{\lambda_0-\delta}.$$

Since  $U_\lambda(x)$  and  $V_\lambda(x)$  depends on  $\lambda$  continuously, there exists  $\varepsilon > 0$  and  $\varepsilon < \delta$  such that for all  $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ ,

$$(3-13) \quad U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \bar{\Sigma}_{\lambda_0-\delta}.$$

When  $x \in \Sigma_\lambda \setminus \bar{\Sigma}_{\lambda_0-\delta}$ , we also have  $U_\lambda(x) \geq 0$ ,  $V_\lambda(x) \geq 0$ . If not, without loss of generality, we assume that there exists a point  $\bar{x}_1 \in \Sigma_\lambda \setminus \bar{\Sigma}_{\lambda_0-\delta}$  such that

$$U_\lambda(\bar{x}_1) = \min_{x \in \Sigma_\lambda \setminus \bar{\Sigma}_{\lambda_0-\delta}} U_\lambda(x) < 0.$$

According to Lemma 1.1,

$$V_\lambda(\bar{x}_1) < \frac{A(\bar{x}_1) + \frac{C}{(\delta+\varepsilon)^\alpha}}{pv^{p-1}(\bar{x}_1)} U_\lambda(\bar{x}_1) < 0.$$

Hence, there exists  $\bar{x}_2 \in \Sigma_\lambda \setminus \bar{\Sigma}_{\lambda_0-\delta}$  such that

$$V_\lambda(\bar{x}_2) = \min_{x \in \Sigma_\lambda \setminus \bar{\Sigma}_{\lambda_0-\delta}} V_\lambda(x) < 0.$$

By Lemma 1.1 again,

$$U_\lambda(\bar{x}_2) < \frac{B(\bar{x}_2) + \frac{C}{(\delta+\varepsilon)^\beta}}{qu^{q-1}(\bar{x}_2)} V_\lambda(\bar{x}_2) < 0.$$

Similarly to (3-4),

$$U_\lambda(\bar{x}_2) < \frac{\left(A(\bar{x}_1) + \frac{C}{(\delta+\varepsilon)^\alpha}\right)}{pv^{p-1}(\bar{x}_1)} \frac{\left(B(\bar{x}_2) + \frac{C}{(\delta+\varepsilon)^\beta}\right)}{qu^{q-1}(\bar{x}_2)} U_\lambda(\bar{x}_2).$$

Since  $A(x) = A(x')$  and  $B(x) = B(x')$  are bounded from below and  $U_\lambda(\bar{x}_2) < 0$ , this is contradictory for  $\delta$  and  $\varepsilon$  sufficiently small. Therefore, we obtain

$$(3-14) \quad U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \setminus \bar{\Sigma}_{\lambda_0-\delta}.$$

Combining (3-13) and (3-14), we get that for  $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ ,

$$(3-15) \quad U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda.$$

(3-15) indicates that the plane  $T_{\lambda_0}$  can be moved up further. We have reached a contradiction with the definition of  $\lambda_0$ . Hence, we must have  $U_{\lambda_0}(x) \equiv 0$ .

We have shown that  $\lambda_0 = +\infty$  and  $U_{\lambda_0}(x) \geq 0$ ,  $V_{\lambda_0}(x) \geq 0$ . It indicates that  $u(x)$  and  $v(x)$  are monotonically increasing in  $x_n$ , which completes the proof of Theorem 1.2.  $\square$

*Proof of Corollary 1.3.* We have shown that  $u(x)$  and  $v(x)$  are monotonically increasing in  $x_n$ . In terms of  $u(0) = v(0) = 0$  and the condition (1-10), we derive

$$u(x) \equiv 0, \quad v(x) \equiv 0 \quad \text{in } \Omega.$$

Thus we have completed the proof.  $\square$

#### 4. The proof of Theorem 1.4

In this section, we will apply Lemma 1.1 to prove Theorem 1.4. We consider the case  $A(x) = A$  and  $B(x) = B$  in system (1-1), where  $A, B$  are constants. In this case, (1-1) becomes (1-11).

Choose any direction to be the  $x_1$  direction. We write  $x = (x_1, x') \in \mathbb{R}^n$  with  $x' = (x_2, x_3, \dots, x_n)$ . Let

$$\hat{T}_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda, \lambda \in \mathbb{R}, \lambda > -1\}$$

be the moving planes and denote

$$\hat{H}_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}, \quad \hat{\Sigma}_\lambda = \{x \in \mathbb{B}_1(0) \mid -1 < x_1 < \lambda\}.$$

For each point  $x = (x_1, x') \in \hat{\Sigma}_\lambda$ , let  $x^\lambda = (2\lambda - x_1, x')$  be the reflection point about the plane  $\hat{T}_\lambda$  and  $U_\lambda(x)$ ,  $V_\lambda(x)$  defined as before. Then it follows that for  $x \in \hat{\Sigma}_\lambda$ ,

$$(4-1) \quad (-\Delta)^{\alpha/2} U_\lambda(x) = p\xi^{p-1}(x) V_\lambda(x) - A U_\lambda(x),$$

and

$$(4-2) \quad (-\Delta)^{\beta/2} V_\lambda(x) = q\eta^{q-1}(x) U_\lambda(x) - B V_\lambda(x),$$

where  $\xi(x)$  is between  $v_\lambda(x)$  and  $v(x)$  and  $\eta(x)$  is between  $u_\lambda(x)$  and  $u(x)$ .

*Proof of Theorem 1.4. Step 1:* We show that for  $\lambda > -1$  sufficiently close to  $-1$ ,

$$(4-3) \quad U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \hat{\Sigma}_\lambda.$$

The proof is almost the same as Step 1 in the proof of Theorem 1.2.

Step 1 provides a starting point. We start from such a small  $\lambda$  and move the plane  $\hat{T}_\lambda$  continuously in the direction of  $x_1$ -axis to its limiting position as long as (4-3) holds.

**Step 2:** Define

$$(4-4) \quad \lambda_0 = \sup\{\lambda > -1 \mid U_\mu(x) \geq 0, V_\mu(x) \geq 0, x \in \hat{\Sigma}_\mu; \mu \leq \lambda\}.$$

We will prove

$$(4-5) \quad \lambda_0 = 0.$$

If  $\lambda_0 < 0$ , we will show that the plane  $\hat{T}_\lambda$  can be moved further right. That is, there exists  $\varepsilon > 0$  such that for all  $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ ,

$$(4-6) \quad U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \hat{\Sigma}_\lambda.$$

This is a contradiction with the definition of  $\lambda_0$ . Hence, we must have  $\lambda_0 = 0$ .

The proof of (4-6) is composed of two parts.

**(a):** We show that for  $\varepsilon > 0$ ,  $\delta > 0$  and  $\varepsilon < \delta$ , when  $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ ,

$$(4-7) \quad U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \bar{\Sigma}_{\lambda_0 - \delta}.$$

We know that

$$U_{\lambda_0}(x) \geq 0, \quad V_{\lambda_0}(x) \geq 0, \quad x \in \hat{\Sigma}_{\lambda_0}.$$

In fact, if  $\lambda_0 < 0$ , we must have

$$U_{\lambda_0}(x) > 0, \quad V_{\lambda_0}(x) > 0, \quad x \in \hat{\Sigma}_{\lambda_0}.$$

If  $U_{\lambda_0}(x) > 0$  does not hold, we assume that there exists some point  $\tilde{x} \in \hat{\Sigma}_{\lambda_0}$  such that  $U_{\lambda_0}(\tilde{x}) = 0$ .

$$\begin{aligned} (-\Delta)^{\alpha/2} U_{\lambda_0}(\tilde{x}) &= C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{U_{\lambda_0}(\tilde{x}) - U_{\lambda_0}(y)}{|\tilde{x} - y|^{n+\alpha}} dy \\ &= C_{n,\alpha} P.V. \int_{\hat{H}_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|\tilde{x} - y|^{n+\alpha}} dy + \int_{\mathbb{R}^n \setminus \hat{H}_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|\tilde{x} - y|^{n+\alpha}} dy \\ &= C_{n,\alpha} P.V. \int_{\hat{H}_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|\tilde{x} - y|^{n+\alpha}} dy + \int_{\hat{H}_{\lambda_0}} \frac{-U_{\lambda_0}(y^\lambda)}{|\tilde{x} - y^\lambda|^{n+\alpha}} dy \\ &= C_{n,\alpha} P.V. \int_{\hat{H}_{\lambda_0}} \left( \frac{1}{|\tilde{x} - y^\lambda|^{n+\alpha}} - \frac{1}{|\tilde{x} - y|^{n+\alpha}} \right) U_{\lambda_0}(y) dy. \end{aligned}$$

Note that  $|\tilde{x} - y^\lambda| > |\tilde{x} - y|$ , hence,

$$(4-8) \quad (-\Delta)^{\alpha/2} U_{\lambda_0}(\tilde{x}) < 0.$$

On the other hand, according to (4-1),

$$(-\Delta)^{\alpha/2} U_{\lambda_0}(\tilde{x}) = p\xi^{p-1}(\tilde{x})V_\lambda(\tilde{x}) - AU_\lambda(\tilde{x}) = p\xi^{p-1}(\tilde{x})V_\lambda(\tilde{x}) \geq 0.$$

This is contradictory to (4-8). Consequently, we obtain  $U_{\lambda_0}(x) > 0$  for all  $x \in \hat{\Sigma}_{\lambda_0}$ . Similarly, we can show  $V_{\lambda_0}(x) > 0$  for all  $x \in \hat{\Sigma}_{\lambda_0}$ . Hence, for small  $\delta > 0$ , there exists a positive constant  $c_0$  such that

$$U_{\lambda_0}(x) \geq c_0 > 0, \quad V_{\lambda_0}(x) \geq c_0 > 0, \quad x \in \bar{\Sigma}_{\lambda_0 - \delta}.$$

Since  $U_\lambda(x)$  and  $V_\lambda(x)$  depend on  $\lambda$  continuously, there exists  $\varepsilon > 0$  with  $\varepsilon < \delta$  such that for all  $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ ,

$$U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \bar{\Sigma}_{\lambda_0 - \delta}.$$

**(b):** Using Lemma 1.1 and similarly to the proof of (3-14), we get

$$(4-9) \quad U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0 \quad x \in \hat{\Sigma}_\lambda \setminus \bar{\Sigma}_{\lambda_0 - \delta}.$$



Together with **(a)** and **(b)**, we prove (4-6) is true for all  $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ . Thus, we obtain  $\lambda_0 = 0$  and  $U_{\lambda_0}(x) \geq 0$ ,  $V_{\lambda_0}(x) \geq 0$ ,  $x \in \hat{\Sigma}_{\lambda_0}$ .

Similarly, we move the plane  $T_\lambda$  from 1 to the left and show that

$$U_{\lambda_0}(x) \leq 0, V_{\lambda_0}(x) \leq 0, \quad x \in \hat{\Sigma}_{\lambda_0}.$$

Then we obtain that

$$\lambda_0 = 0 \quad \text{and} \quad U_{\lambda_0}(x) \equiv 0, V_{\lambda_0}(x) \equiv 0, \quad x \in \hat{\Sigma}_{\lambda_0}.$$

This indicates that  $u(x)$  and  $v(x)$  are symmetric about  $T_0$ . Since the  $x_1$  direction can be chosen arbitrarily, we have actually shown that  $u(x)$  and  $v(x)$  are radially symmetric about the origin. Thus, we have completed the proof of Theorem 1.4.  $\square$

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# MODULI SPACES OF STABLE PAIRS

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**We construct a moduli space of stable pairs over a smooth projective variety, parametrizing morphisms from a fixed coherent sheaf to a varying sheaf of fixed topological type, subject to a stability condition. This generalizes the notion used by Pandharipande and Thomas, following Le Potier, where the fixed sheaf is the structure sheaf of the variety. We then describe the relevant deformation and obstruction theories. We also show the existence of the virtual fundamental class in special cases.**

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## 1. Introduction

The past couple of decades of research have highlighted the importance of moduli spaces of *decorated* sheaves, which are sheaves with additional structure, such as one or more sections. Moduli spaces of rank two vector bundles with a section on a Riemann surface  $X$ ,

$$E \rightarrow X \quad \text{and} \quad \alpha : \mathcal{O}_X \rightarrow E,$$

were used in [Thaddeus 1994] to deduce an important invariant of the moduli space of sheaves, the Verlinde number. More recently, Pandharipande and Thomas [2009; 2010] studied stable pairs  $(E, \alpha)$ , where  $E$  is a sheaf with dimension 1 support, on a Calabi–Yau threefold. They showed that invariants of this moduli space are closely related to the Gromov–Witten invariants of the Calabi–Yau threefold.

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We would like to broaden our perspective and replace the structure sheaf by a general coherent sheaf. Subject to a stability condition, we would like to parametrize morphisms of coherent sheaves,

$$\alpha : E_0 \rightarrow E,$$

where  $E_0$  is a fixed coherent sheaf. We will denote such a morphism as a pair

$$(E, \alpha).$$

Let us set up the problem. We will work over an algebraically closed field  $k$  of characteristic 0. We denote by  $X$  a smooth projective variety of dimension  $n$ , with a fixed polarization  $\mathcal{O}_X(1)$ . We fix a coherent sheaf  $E_0$  on  $X$ . Let  $P$  be a fixed polynomial of degree  $d \leq n$ . Let  $\delta \in \mathbb{Q}[m]$  be 0 or a polynomial with a positive leading coefficient; this will play the role of parameter for stability conditions.

When  $\delta$  is large, i.e.,  $\deg \delta \geq \deg P$ , a pair  $(E, \alpha)$ , such that the Hilbert polynomial of  $E$  equals  $P$ , is *stable* if  $E$  is pure and the support of  $\text{coker } \alpha$  has dimension strictly smaller than  $d$ . This is the most significant case geometrically. In this case, the moduli space of stable pairs is closely related to Grothendieck's Quot scheme. But intersection theory on the moduli space of stable pairs is expected to be more tractable than that on the Quot scheme. This is because we impose the purity condition on the sheaves underlying stable pairs, which allows us to avoid some large dimensional components.

The moduli space of stable pairs in the large  $\delta$  case is expected to have interesting applications to the enumerative geometry of higher rank sheaves on a surface  $X$ . In particular, a potential application is towards the strange duality conjecture. The conjecture over curves was proved [Belkale 2008; Marian and Oprea 2007] by studying intersection theory on related Grassmannians and Quot schemes. It is reasonable to expect that a similar method using the moduli space of stable pairs will work for the surface case.

The study of stable pairs by Pandharipande and Thomas was built on Le Potier's work [1993] on coherent systems. The moduli space of coherent systems was also used to study the Donaldson numbers of the moduli space of sheaves [He 1998]. A *coherent system* on  $X$  is a pair  $(\Gamma, E)$ , where  $E$  is a coherent sheaf and  $\Gamma \subset H^0(X, E)$  is a subspace of global sections. A pair  $(E, \alpha : \mathcal{O}_X \rightarrow E)$  can be viewed as a coherent system  $(k\langle \alpha \rangle, E)$ . However, when  $\mathcal{O}_X$  is replaced by, for example,  $\mathcal{O}_X^{\oplus 2}$ , the pair can no longer be viewed as a coherent system, because the map

$$H^0(\alpha) : k^{\oplus 2} \rightarrow H^0(E)$$

may not be injective. Aside from this issue, there is yet another difference between pairs and coherent systems: while the morphism  $\alpha$  is part of the data of the pair, the coherent system only remembers the image of  $H^0(\alpha)$ . Consequently, when one

tries to parametrize  $\alpha : E_0 \rightarrow E$  for general  $E_0$ , Le Potier's construction does not automatically apply. But the main ingredients of constructing the moduli space remain the same: Grothendieck's Quot scheme [1961b] and Mumford's geometric invariant theory [Mumford et al. 1994].

**Theorem 1.1** (existence of moduli spaces). *For the moduli functor  $S_{E_0}(P, \delta)$  of  $S$ -equivalence classes of  $\delta$ -semistable pairs, there exists a projective coarse moduli space  $S_{E_0}(P, \delta)$ . The moduli functor  $S_{E_0}^s(P, \delta)$  of equivalence classes of  $\delta$ -stable pairs is represented by an open subscheme  $S_{E_0}^s(P, \delta)$  of  $S_{E_0}(P, \delta)$ .*

Deformation-obstruction theory of stable pairs is very similar to that of the Quot scheme. For a quotient  $q : E_0 \twoheadrightarrow F$ , let  $G = \ker q$ , then we have a short exact sequence,

$$0 \rightarrow G \rightarrow E_0 \rightarrow F \rightarrow 0.$$

The deformation space and the obstruction space are  $\mathrm{Hom}(G, F)$  and  $\mathrm{Ext}^1(G, F)$ . Notice that  $G$  is quasi-isomorphic to the cochain complex  $J^\bullet = \{E_0 \rightarrow F\}$ , and the deformation space and the obstruction space of this quotient are isomorphic to  $\mathrm{Hom}(J^\bullet, F)$  and  $\mathrm{Ext}^1(J^\bullet, F)$ , respectively.

The deformation-obstruction problem of stable pairs has a similar answer. Let  $\mathrm{Art}_k$  be the category of local Artinian  $k$ -algebras with residue field  $k$ . Let  $A, B \in \mathrm{Ob} \mathrm{Art}_k$  and

$$0 \rightarrow K \rightarrow B \xrightarrow{\sigma} A \rightarrow 0$$

be a *small* extension, i.e.,  $\mathfrak{m}_B K = 0$ . Suppose  $(E, \alpha)$  is a stable pair. Let  $I^\bullet$  denote the following cochain complex concentrating at degree 0 and 1:

$$I^\bullet = \{E_0 \xrightarrow{\alpha} E\}.$$

**Theorem 1.2** (deformation-obstruction). *Suppose  $\alpha_A : E_0 \otimes_k A \rightarrow E_A$  is a morphism over  $X_A = X \times_{\mathrm{Spec} k} \mathrm{Spec} A$  extending  $\alpha$ , where  $E_A$  is a coherent sheaf flat over  $A$ . There is a class,*

$$\mathrm{ob}(\alpha_A, \sigma) \in \mathrm{Ext}^1(I^\bullet, E \otimes K),$$

*such that there exists a flat extension of  $\alpha_A$  over  $X_B$  if and only if  $\mathrm{ob}(\alpha_A, \sigma) = 0$ . If extensions exist, the space of extensions is a torsor under*

$$\mathrm{Hom}(I^\bullet, E \otimes K).$$

In some special cases,  $\mathrm{Ext}^i(I^\bullet, E) \neq 0$  only when  $i = 0, 1$ . In these cases, we will demonstrate the existence of the virtual fundamental class, which is important for the study of intersection theory on the moduli spaces.

**Theorem 1.3** (virtual fundamental class). *Suppose  $X$  is a surface,  $E_0$  is torsion-free,  $\deg P = 1$ , and  $\deg \delta \geq 1$ . Then the moduli space  $S_{E_0}(P, \delta)$  of stable pairs admits a virtual fundamental class.*

The virtual fundamental class can be used to define invariants of the surface. Kool and Thomas [2014a; 2014b] studied stable pairs invariants with  $E_0 \cong \mathcal{O}_X$  on surfaces, using the reduced obstruction theory, which is necessary. We will address the intersection theory of the moduli space of stable pairs on a surface in future work.

After this project was completed, we learned about the article [Wandel 2015] where the stability condition for pairs had been defined. When  $\deg \delta < \deg P$ , Theorem 1.1 of this paper had been stated as the main theorem, Theorem 3.8, in [Wandel 2015]. In the large  $\delta$  case,

$$\deg \delta \geq \deg P,$$

the linearized ample line bundle needs to be chosen differently, as in (4-4), for the GIT construction. In this paper, the construction of the moduli space focuses on the large  $\delta$  case, which is geometrically important but has not been treated in [Wandel 2015]. The construction is carried out from a basic level. For example, Lemma 3.5 is shown for characterizing stability in terms of global sections instead of Hilbert polynomials. As preparation, Section 2 introduces the notion of stable pair and states basic properties of pairs. Section 3 studies the boundedness of the family of stable pairs. Proofs of statements that have been proved in [Wandel 2015] are omitted. This paper also contains, in Section 5, the deformation-obstruction theory, captured by Theorem 1.2, which holds for all  $\delta$ 's, small or large. Section 6 shows the existence of the virtual fundamental class in special geometries (see Theorem 1.3). Section 7 gives examples of smooth moduli spaces and calculates their topological Euler characteristics.

We recently learned that the stable pair moduli space for  $\deg \delta \geq \deg P$  was also previously studied in [Kollár 2008], where it appears as the moduli space of quotient husks. The author constructed it as a bounded proper separated algebraic space, and used it to study an analogue of the flattening decomposition theorem for reflexive hulls. The current paper settles affirmatively the question raised in [Kollár 2008] regarding the projectivity of the space.

We finally note that once it is constructed for  $\deg \delta < \deg P$ , the moduli space is available in an indirect way for  $\deg \delta \geq \deg P$  as well. This follows from two facts: the set of critical values<sup>1</sup> is finite and the largest critical polynomial  $\delta_{\max}$  has degree  $< \deg P$ . Let  $\delta'$  be of degree  $\deg P - 1$  and larger than  $\delta_{\max}$ . Then, for any  $\delta$  with  $\deg \delta \geq \deg P$ , we have  $S_{E_0}(P, \delta) \cong S_{E_0}(P, \delta')$ . Although this observation is not made in [Wandel 2015], the author proves the set of critical  $\delta$ 's is finite.

This indirect argument does not, however, yield the linearized ample line bundle for  $S_{E_0}(P, \delta)$  with  $\deg \delta \geq \deg P$ . For stability polynomials  $\delta'$  with  $\deg \delta' < \deg P$ , the linearization depends directly on  $\delta'$ ; the highest critical polynomial  $\delta_{\max}$  cannot

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<sup>1</sup>A critical value is a value such that as  $\delta$  crosses over it, the moduli space  $S_{E_0}(P, \delta)$  changes.



be determined explicitly, however, since the boundedness which underlies the finiteness of the set of critical stability values is itself not explicit.

For some applications, it is nevertheless important to know the line bundle explicitly. A natural problem to study next is that of wall-crossing formulas, using Thaddeus' master space [Thaddeus 1996; Mochizuki 2009]. The construction of the master space requires the linearized ample line bundle. So, it is important to construct the moduli space directly via GIT and obtain the ample line bundle. We will address the problem of wall-crossing formulas in future work.

## 2. Basic properties of stable pairs

**2A. Preliminaries on coherent sheaves.** For a coherent sheaf  $E$  on  $(X, \mathcal{O}_X(1))$ , we denote by  $P_E$  its *Hilbert polynomial*. Recall that we can write the Hilbert polynomial in the form

$$P_E(m) = \sum_{i=0}^d a_i(E) \frac{m^i}{i!},$$

where  $d$  is the dimension of the support of  $E$ , which we simply write as  $\dim E$ , and  $a_i(E) \in \mathbb{Q}$ . We denote by

$$r(E) = a_d(E)$$

the *multiplicity* of  $E$ . The *reduced* Hilbert polynomial is

$$p_E = \frac{P_E}{r(E)}.$$

The *slope* of  $E$  is

$$\mu(E) = \frac{a_{d-1}(E)}{a_d(E)}.$$

A coherent sheaf  $E$  is *pure* if there is no subsheaf with lower dimensional support. It is *semistable* (respectively, *slope-semistable*) if it is pure and there is no subsheaf with larger reduced Hilbert polynomial (respectively, slope). For a pure sheaf, there is a *Harder–Narasimhan filtration* with respect to the slope

$$0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_l = E,$$

where  $E_{t+1}/E_t$  is slope semistable and  $\mu(E_t/E_{t-1}) > \mu(E_{t+1}/E_t)$ , for  $t \in [1, l-1]$ . We shall denote  $\mu_{\max}(E) = \mu(E_1)$  and  $\mu_{\min}(E) = \mu(E_l/E_{l-1})$ .

To construct the moduli space via GIT, the first step is to prove a boundedness result. For our convenience, we group a sequence of boundedness results here.

**Theorem 2.1** (Grothendieck). *Suppose  $F$  is a pure coherent  $\mathcal{O}_X$ -module of dimension  $d$ . Then:*

- (i) *The slopes of nonzero coherent subsheaves are bounded above.*

- (ii) *The family of subsheaves  $F' \subset F$  with slopes bounded below, such that the quotient  $F/F'$  is pure and of dimension  $d$ , is bounded.*

We can also make a statement similar to the second assertion about the boundedness of quotients. For the proof of this basic theorem, see [Grothendieck 1961b, Lemma 2.5].

Let  $Y$  be the scheme-theoretic support of a pure sheaf  $E$  of dimension  $d$  and multiplicity  $r$ . We include the following results discussed in [Le Potier 1993].

**Lemma 2.2.** *The degree of  $Y$  is no larger than  $r^2$ .*

*Proof.* This is clear from an equivalent definition of multiplicity [Le Potier 1993, Definition 2.1].  $\square$

**Lemma 2.3.** *The minimum slope  $\mu_{\min}(\mathcal{O}_Y)$  is bounded below by a constant determined by  $n$ ,  $r$ , and  $d$ .*

*Proof.* See [Le Potier 1993, Lemma 2.12].  $\square$

The following statement is crucial to our proof of boundedness.

**Theorem 2.4** [Simpson 1994, Theorem 1.1]. *Let  $C$  be a rational constant. The family of pure coherent sheaves  $E$  with Hilbert polynomial  $P_E = P$ , such that  $\mu_{\max}(E) \leq C$ , is bounded.*

Bounding  $\mu_{\max}$  from above is equivalent to bounding  $\mu_{\min}$  from below, because the Hilbert polynomial is additive in a short exact sequence.

We will also need the following statement.

**Lemma 2.5** [Simpson 1994, Corollary 1.7]. *Suppose  $F$  is a slope semistable sheaf of dimension  $d$ , multiplicity  $r$  and slope  $\mu$ . There is a constant  $C$  depending on  $r$  and  $d$  such that<sup>2</sup>*

$$\frac{h^0(F)}{r} \leq \frac{1}{d!}([\mu + C]_+)^d.$$

**2B. Stable pairs.** Let  $E_0$  be a coherent sheaf on  $X$ . Let  $P$  be a polynomial of degree  $d$ , and  $\delta$  be 0 or a polynomial with a positive leading coefficient.

**Definition 2.6.** A pair  $(E, \alpha)$  (of type  $P$ ) on  $X$  consists of a coherent sheaf  $E$  with Hilbert polynomial  $P$  and a morphism  $\alpha : E_0 \rightarrow E$ . A subpair  $(E', \alpha')$  consists of a coherent subsheaf  $E' \subset E$  and a morphism  $\alpha' : E_0 \rightarrow E'$ , such that

$$\begin{cases} \iota \circ \alpha' = \alpha & \text{if } E' \supset \text{im } \alpha, \\ \alpha' = 0 & \text{otherwise.} \end{cases}$$

Here,  $\iota$  denotes the inclusion  $E' \hookrightarrow E$ . A quotient pair  $(E'', \alpha'')$  consists of a coherent quotient sheaf  $q : E \rightarrow E''$  and a morphism  $\alpha'' = q \circ \alpha : E_0 \rightarrow E''$ .

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<sup>2</sup> $[x]_+ = \max\{0, x\}$ .

We say a pair  $(E, \alpha)$  has dimension  $d$  if  $\dim E = d$ .

A morphism  $\phi : (E, \alpha) \rightarrow (F, \beta)$  of pairs is a morphism of sheaves  $\phi : E \rightarrow F$  such that there is a constant  $b \in k$ , where  $\phi \circ \alpha = b\beta$ . By this definition, subpairs and quotient pairs can be viewed as morphisms. For simplicity, we shall use the notation  $\phi$  for both the morphism of pairs and that of their underlying sheaves.

A short exact sequence of pairs,

$$0 \rightarrow (E', \alpha') \xrightarrow{\iota} (E, \alpha) \xrightarrow{q} (E'', \alpha'') \rightarrow 0,$$

consists of a short exact sequence of sheaves,  $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ , such that  $(E', \alpha')$  is a subpair and  $(E'', \alpha'')$  the corresponding quotient pair. More precisely,  $\alpha'' = q \circ \alpha$  if  $\alpha' = 0$ , and  $\alpha'' = 0$  if  $\iota \circ \alpha' = \alpha$ .

The *Hilbert polynomial* of a pair  $(E, \alpha)$  is

$$P_{(E, \alpha)} = P_E + \epsilon(\alpha)\delta$$

and the *reduced Hilbert polynomial* of the pair is

$$p_{(E, \alpha)} = p_E + \frac{\epsilon(\alpha)\delta}{r(E)}.$$

Here,

$$\epsilon(\alpha) = \begin{cases} 1 & \text{if } \alpha \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the Hilbert polynomial is additive in a short exact sequence of pairs.

**Definition 2.7.** A pair  $(E, \alpha)$  is  $\delta$ -stable if

- (i)  $E$  is pure;
- (ii)  $p_{(E', \alpha')} < p_{(E, \alpha)}$  for every proper subpair  $(E', \alpha')$ .

Semistability is defined similarly, replacing the strong inequality by the corresponding weak inequality.

Assuming purity, the second condition is equivalent to that for every proper quotient pair  $(E'', \alpha'')$  of dimension  $d$ ,  $p_{(E'', \alpha'')} > p_{(E, \alpha)}$ .

**Convention.** In the rest of this paper, if stability is characterized by a strong inequality, semistability can be characterized by the corresponding weak inequality. So, in such a case, we will only make the statement for stability.

When the context is clear, we will omit  $\delta$  and only say a pair is stable or semistable.

Clearly, a pair  $(E, 0)$  is (semi)stable if and only if  $E$  is (semi)stable as a coherent sheaf. We will call a pair  $(E, \alpha)$  *nondegenerate* if  $\alpha \neq 0$ . We are primarily interested in nondegenerate semistable pairs, which we are going to parametrize.

A family of pairs parametrized by a scheme  $T$  is a morphism of sheaves

$$\alpha_T : \pi_X^* E_0 \rightarrow \mathcal{E}$$

over  $T \times X$ , such that  $\mathcal{E}$  is flat over  $T$ . Here,  $\pi_X$  is the projection  $T \times X \rightarrow X$ . Two families  $\alpha_T : \pi_X^* E_0 \rightarrow \mathcal{E}$  and  $\beta_T : \pi_X^* E_0 \rightarrow \mathcal{F}$  are equivalent if there is an isomorphism

$$\psi : \mathcal{E} \rightarrow \mathcal{F} \quad \text{such that } \psi \circ \alpha_T = \beta_T.$$

In the large  $\delta$  regime, semistable pairs have some special features.

**Lemma 2.8.** *When  $\deg \delta \geq \deg P$ , there is no nondegenerate strictly semistable pair, i.e., every nondegenerate semistable pair is stable.*

*Proof.* Suppose  $(G, \alpha')$  is a subpair of a semistable  $(E, \alpha)$ , such that  $p_{(G, \alpha')} = p_{(E, \alpha)}$ , that is,

$$p_G + \frac{\epsilon(\alpha')\delta}{r(G)} = p_E + \frac{\delta}{r(E)}.$$

Consider the leading coefficients. Because  $\deg \delta \geq d$ , we have  $\epsilon(\alpha') = 1$  and  $r(G) = r(E)$ . Thus,  $p_E = p_G$ . Therefore,  $P_E = P_G$ , which implies that  $G = E$ . Hence,  $(G, \alpha') = (E, \alpha)$ . We have shown that  $(E, \alpha)$  is not strictly semistable.  $\square$

We also have a reinterpretation of the stability condition.

**Lemma 2.9.** *Suppose  $E$  is a pure coherent sheaf with Hilbert polynomial  $P_E = P$  and multiplicity  $r(E) = r$ . If  $\deg \delta \geq d = \deg P$ , then a pair  $(E, \alpha)$  is stable if and only if for every proper subpair  $(G, \alpha')$ ,*

$$\frac{P_G}{2r(G) - \epsilon(\alpha')} < \frac{P}{2r - \epsilon(\alpha)}.$$

*Proof.* When  $\deg \delta \geq d$ , for any proper subpair  $(G, \alpha')$ , the inequality

$$p_G + \epsilon(\alpha') \frac{\delta}{r(G)} < p_E + \epsilon(\alpha) \frac{\delta}{r}$$

is equivalent to

$$(2-1) \quad \frac{\epsilon(\alpha')}{r(G)} \leq \frac{\epsilon(\alpha)}{r}, \quad \text{and in case of equality, } p_G < p_E.$$

The latter can be easily seen to be equivalent to

$$\frac{r(G)}{2r(G) - \epsilon(\alpha')} \leq \frac{r}{2r - \epsilon(\alpha)}, \quad \text{and in case of equality, } p_G < p_E.$$

This last condition is equivalent to the inequality in the statement.  $\square$

Moreover, there is a geometric characterization of stability.

**Lemma 2.10.** *If  $\deg \delta \geq \deg P$ , then  $(E, \alpha)$  is stable if and only if  $E$  is pure and  $\dim \operatorname{coker} \alpha < \deg P$ .*

This is essentially [Wandel 2015, Proposition 1.12]. The author stated the result for the case where  $\deg \delta \geq \dim X$  while his argument actually showed the same result under a weaker assumption  $\deg \delta \geq \deg P$ .

Pairs share some similar properties of sheaves.

**Lemma 2.11.** *Suppose  $\phi : (E, \alpha) \rightarrow (F, \beta)$  is a nonzero morphism of pairs.*

- (i) *Suppose  $(E, \alpha)$  and  $(F, \beta)$  are  $\delta$ -semistable pairs of dimension  $d$ . Then  $P(E, \alpha) \leq P(F, \beta)$ .*
- (ii) *If  $(E, \alpha)$  and  $(F, \beta)$  are  $\delta$ -stable with the same reduced Hilbert polynomial,  $\phi$  induces an isomorphism between  $E$  and  $F$ . In particular,  $\operatorname{End}((E, \alpha)) \cong k$  for a stable pair  $(E, \alpha)$ .*

*Proof.* (i) Let  $\alpha''$  be  $\phi \circ \alpha : E_0 \rightarrow \operatorname{im} \phi$ . Then  $(\operatorname{im} \phi, \alpha'')$  is a quotient pair of  $(E, \alpha)$  and a subpair of  $(F, \beta)$ . Thus,

$$(2-2) \quad P(E, \alpha) \leq P(\operatorname{im} \phi, \alpha'') \leq P(F, \beta).$$

(ii) Suppose not, then  $\ker \phi \neq 0$  or  $\operatorname{im} \phi \neq F$ . We also have the inequalities (2-2). But two equalities do not hold simultaneously, which contradicts the fact that the two stable pairs have the same reduced Hilbert polynomial. Therefore,  $\ker \phi = 0$  and  $\operatorname{im} \phi = F$ . Thus,  $\phi$  is an isomorphism of coherent sheaves. Clearly, the inverse also provides an inverse of pairs. In particular,  $\operatorname{End}((E, \alpha))$  is a finite-dimensional associative division algebra over the algebraically closed field  $k$ , and hence is  $k$ .  $\square$

The second part of the lemma is essentially [Wandel 2015, Lemma 1.6].

**Proposition 2.12** (Harder–Narasimhan filtration). *Let  $(E, \alpha)$  be a pair where  $E$  is pure of dimension  $d$ . Then there is a unique filtration by subpairs*

$$0 \subsetneq (G_1, \alpha_1) \subsetneq (G_2, \alpha_2) \subsetneq \cdots \subsetneq (G_l, \alpha_l) = (E, \alpha)$$

*with  $\operatorname{gr}_i = (G_i, \alpha_i)/(G_{i-1}, \alpha_{i-1})$  such that*

- (i)  *$\operatorname{gr}_i$  is  $\delta$ -semistable of dimension  $d$  for all  $i$ ;*
- (ii)  *$p_{\operatorname{gr}_i} > p_{\operatorname{gr}_{i+1}}$  for all  $i$ .*

*We call this filtration the **Harder–Narasimhan filtration** of the pair.*

The proof is similar to the proof of the existence and uniqueness of the Harder–Narasimhan filtration of a pure sheaf [Shatz 1977, Theorem 1].

Evidently, in the filtration, there is only one nonzero  $\alpha_i$ . In the case where  $\deg \delta \geq d$ , only  $\alpha_1$  is nonzero.

**Proposition 2.13** (Jordan–Hölder filtration). *Let  $(E, \alpha)$  be a semistable pair. There is a filtration*

$$0 \subsetneq (F_1, \alpha_1) \subsetneq (F_2, \alpha_2) \subsetneq \cdots \subsetneq (F_l, \alpha_l) = (E, \alpha),$$

*such that each factor  $\text{gr}_i = (F_i, \alpha_i)/(F_{i-1}, \alpha_{i-1})$  is stable with reduced Hilbert polynomial  $p_{(E, \alpha)}$ . Moreover,  $\text{gr}(E, \alpha) = \bigoplus_i \text{gr}_i$  does not depend on the filtration.*

*Proof.* Since we have Lemma 2.11, the proof proceeds the same way as the argument for Jordan–Hölder filtrations of a semistable sheaf, see, e.g., [Huybrechts and Lehn 1997, Proposition 1.5.2].  $\square$

Two semistable pairs are *S-equivalent*, if they have isomorphic Jordan–Hölder factors.

Let

$$\mathcal{S}_{E_0}(P, \delta) : \text{Sch}/_k \rightarrow \text{Set}$$

denote the moduli functor of S-equivalent nondegenerate semistable pairs of type  $P$ .

Let

$$\mathcal{S}_{E_0}^s(P, \delta)$$

denote the moduli functor of equivalence classes of nondegenerate stable pairs.

### 3. Boundedness

In order to construct the moduli space via GIT, we first need to prove that the family of semistable pairs is bounded. As mentioned in the introduction, the case where  $\deg \delta < \deg P$  has been treated in [Wandel 2015]. So, in this section and the next, we will focus on the case

$$\deg \delta \geq \deg P.$$

We will show boundedness using Theorem 2.4, by studying the  $\mu_{\min}$ 's of sheaves underlying semistable pairs.

**Lemma 3.1.** *Fix the Hilbert polynomial  $P$ . Assume  $\deg \delta \geq \deg P$ . Suppose  $(E, \alpha)$  is a pair, which is semistable for some  $\delta$ , with  $P_E = P$ . Then,  $\mu_{\min}(E)$  is bounded below by a constant depending on  $P$  and  $X$ .*

*Proof.* Let  $(E, \alpha)$  be a semistable pair. By Lemma 2.10,

$$(3-1) \quad \dim \text{coker } \alpha < d.$$

Choose an  $m$  large enough such that  $E_0(-m)$  is generated by global sections. Let  $Y$  be the scheme-theoretic support of  $E$ . The morphism  $\alpha$  factors through  $E_0|_Y$ . We have the sequence of morphisms

$$H^0(E_0(m)) \otimes \mathcal{O}_Y(-m) \twoheadrightarrow E_0|_Y \rightarrow E \twoheadrightarrow \text{gr}_s E,$$

where the last morphism is the surjection from  $E$  onto the last factor of the Harder–Narasimhan filtration with respect to the slope. By (3-1), the composition is nonzero. Therefore,

$$\begin{aligned}\mu_{\min}(E) &= \mu(\operatorname{gr}_s E) \geq \mu_{\min}(H^0(E_0(m)) \otimes \mathcal{O}_Y(-m)) \\ &= \mu_{\min}(\mathcal{O}_Y(-m)) = \mu_{\min}(\mathcal{O}_Y) - m,\end{aligned}$$

where the last term is bounded below, by Lemma 2.3. Thus,  $\mu_{\min}(E)$  is bounded below by a constant, which depends on  $X$  and  $P$ .  $\square$

**Remark 3.2.** The lemma also holds for  $\deg \delta < \deg P$ . Moreover, the constant can be chosen to be independent of  $\delta$ .

Combining Lemma 3.1 and Theorem 2.4, we obtain the following boundedness result.

**Proposition 3.3.** *Fix the Hilbert polynomial  $P$ . The family*

$$\{E \mid (E, \alpha) \text{ is a semistable pair of type } P \text{ with respect to some } \delta\}$$

*of coherent sheaves on  $X$  is bounded.*

For a bounded family of pure pairs, the family of factors of their Harder–Narasimhan filtrations is bounded:

**Lemma 3.4.** *Suppose  $\Phi : \pi_X^* E_0 \rightarrow \mathcal{E}$  over  $T \times X$  is a flat family of pure pairs over  $X$  parametrized by a finite type scheme  $T$ . For a closed point  $t \in T$ , let  $\mathcal{E}(t) = \mathcal{E}|_{\operatorname{Spec} k(t) \times X}$  and  $\Phi(t)$  be the corresponding morphism. Then, the family of the Harder–Narasimhan factors of  $(\mathcal{E}(t), \Phi(t))$ , for all  $t \in T$ , is bounded.*

The following proof is very similar to the proof of the corresponding statement about the boundedness of Harder–Narasimhan factors of pure sheaves [Huybrechts and Lehn 1997, Theorem 2.3.2]. We do not assume  $\deg \delta \geq \deg P$  in this proof.

*Proof.* We can assume  $T$  to be integral. Define  $A$  as the set of 2-tuples  $(P'', \epsilon'')$ , such that there is a point  $t \in T$  and a pure quotient  $q : \mathcal{E}(t) \twoheadrightarrow E''$  with Hilbert polynomial  $P_{E''} = P''$  and  $\epsilon'' = \epsilon(q \circ \Phi(t))$ , which destabilizes  $(\mathcal{E}(t), \Phi(t))$ :

$$p'' + \frac{\epsilon'' \delta}{r''} < p + \frac{\epsilon(\Phi(s)) \delta}{r}.$$

Here,  $p$  and  $p''$  denote the corresponding reduced Hilbert polynomials, and  $r$  and  $r''$  denote the multiplicities. From this inequality, we know that  $\mu(E'')$  is bounded above by a constant determined by  $P$  and  $\delta$ . Therefore,  $A$  is a finite set by Theorem 2.1.

If this set is empty, then all pairs are semistable. Then, we are done. Otherwise, we define a total order  $\preceq$  on  $A$  as:

$$(P_1, \epsilon_1) \preceq (P_2, \epsilon_2)$$

if  $p_1 + \epsilon_1 \delta / r_1 \leq p_2 + \epsilon_2 \delta / r_2$ , and in the case of equality,  $P_1 \geq P_2$ . Let us consider whether there is a  $(P_-, \epsilon_-)$ , which is minimal with respect to the total order  $\leq$  and satisfies the condition that for a generic point  $t \in T$ , there is a pure quotient  $q : \mathcal{E}(t) \rightarrow F$  with

$$(3-2) \quad P_F = P_- \quad \text{and} \quad \epsilon(q \circ \Phi(t)) = \epsilon_-.$$

If there is no such  $(P_-, \epsilon_-)$ , then generically, say over the open subscheme  $U \subset T$ , pairs are already semistable.

If there is such a  $(P_-, \epsilon_-)$ , let  $U \subset T$  be the open family having quotients satisfying the condition (3-2). The minimal Harder–Narasimhan factors of pairs in  $U$  are parametrized by a subscheme of  $\text{Quot}^{P_-}(\mathcal{E})$ . To parametrize all the Harder–Narasimhan factors of pairs parametrized by  $U$ , we can iterate the above process for the kernel, which is flat, of the universal quotient over  $\text{Quot}^{P_-}(\mathcal{E})$ . This process will terminate due to multiplicity.

Then, we can run the same algorithm for pairs parametrized by irreducible components of the complement  $T \setminus U$ . Because  $T$  is noetherian, the process will terminate.

We have thus parametrized the Harder–Narasimhan factors by a finite sequence of Quot schemes.  $\square$

The following statement enables us to handle the semistability condition via spaces of global sections, instead of Hilbert polynomials.

**Lemma 3.5.** *Fix  $P$  and  $\delta$  with  $\deg \delta \geq \deg P$ . Then there is an  $m_0 \in \mathbb{Z}_{>0}$ , such that for any integer  $m \geq m_0$  and any pair  $(E, \alpha)$ , where  $E$  is pure with  $P_E = P$  and multiplicity  $r(E) = r$ , the following assertions are equivalent.*

- i) *The pair  $(E, \alpha)$  is stable.*
- ii)  *$P_E(m) \leq h^0(E(m))$ , and for any proper subpair  $(G, \alpha')$  where  $G$  is of multiplicity  $r(G)$ ,*

$$\frac{h^0(G(m))}{2r(G) - \epsilon(\alpha')} < \frac{h^0(E(m))}{2r - \epsilon(\alpha)}.$$

- iii) *For any proper quotient pair  $(F, \alpha'')$  where  $F$  is of dimension  $d$  and multiplicity  $r(F)$ ,*

$$\frac{h^0(F(m))}{2r(F) - \epsilon(\alpha'')} > \frac{P(m)}{2r - \epsilon(\alpha)}.$$

The proof is modified from that of a similar statement in [Le Potier 1993].

*Proof.* The proof will proceed as follows: i)  $\Rightarrow$  ii)  $\Rightarrow$  iii)  $\Rightarrow$  i). The integer  $m_0$  will be determined in the course of the proof, nonexplicitly.



i)  $\Rightarrow$  ii) The family of sheaves underlying semistable pairs with a fixed Hilbert polynomial is bounded. Thus, there is  $m_0 \in \mathbb{N}$  such that for any integer  $m \geq m_0$ , we have  $H^i(E(m)) = 0$  for all  $i > 0$ . In particular,  $P(m) = h^0(E(m))$ .

In the course of proving the boundedness, we also proved that  $\mu_{\max}(E)$  is bounded above, say  $\mu_{\max}(E) \leq \mu$ . For a proper subpair  $(G, \alpha')$  of multiplicity  $r(G)$ , consider the Harder–Narasimhan filtration of  $G$  with respect to the slope. Let us denote the multiplicity and the slope of the  $i$ -th grading by  $r'_i$  and  $\mu'_i$ . Then, we have  $\mu'_i \leq \mu$ . Notice that  $r'_i$  is positive and bounded above by  $r$ , which implies that there are only finitely many possible  $r'_i$ 's and  $\mu'_i$ 's. Let  $\nu = \mu_{\min}(G)$ . By Lemma 2.5 and an easy calculation, we can find a constant  $B$  depending on  $r$  and  $d$ , such that<sup>3</sup>

$$(3-3) \quad \frac{h^0(G(m))}{r(G)} \leq \frac{1}{d!} \left( \left(1 - \frac{1}{r}\right) ([\mu + m + B]_+)^d + \frac{1}{r} ([\nu + m + B]_+)^d \right).$$

Choose a constant  $A > 0$ , which is larger than all roots of  $P$ . Replace  $m_0$  by  $\max\{m_0, A\}$ . Then

$$h^0(E(m)) = P(m) \geq \frac{r}{d!} (m - A)^d, \quad \text{for all } m \geq m_0.$$

Suppose  $\nu_0$  is an integer such that

$$B + \mu \left(1 - \frac{1}{r}\right) + \frac{\nu_0}{r} < -A.$$

Enlarging  $m_0$  if necessary, we have

$$(3-4) \quad \frac{1}{d!} \left( \left(1 - \frac{1}{r}\right) ([\mu + m + B]_+)^d + \frac{1}{r} ([\nu_0 + m + B]_+)^d \right) < \frac{P(m)}{r}, \quad \text{for all } m \geq m_0,$$

by considering the first and the second leading coefficients.

Thus, when  $m \geq m_0$  and  $\nu \leq \nu_0$ , combining (3-3) and (3-4), we get

$$(3-5) \quad h^0(G(m)) < \frac{r(G)}{r} h^0(E(m)) \leq \frac{2r(G) - \epsilon(\alpha')}{2r - \epsilon(\alpha)} h^0(E(m)).$$

The last weak inequality is a consequence of (2-1).

We are left to consider the case where  $\nu > \nu_0$ . First, notice that we can assume  $E/G$  to be pure. If not, consider the saturation of  $G$  in  $E$ , namely, the smallest  $\bar{G} \supset G$ , such that  $E/\bar{G}$  is pure. If we can prove the inequality in ii) for  $\bar{G}$ , then it's also true for  $G$ , since  $r(G) = r(\bar{G})$  and  $h^0(G(m)) \leq h^0(\bar{G}(m))$ . Since  $\mu(G) \geq \nu > \nu_0$ , the family of such  $G$  is bounded, by Theorem 2.1. So, there are only finitely many Hilbert polynomials of the form  $P_G$  for such  $G$ . Moreover, we can enlarge  $m_0$  again, if necessary, such that for  $m \geq m_0$ ,  $P_G(m) = h^0(G(m))$  and

$$\frac{P_G}{2r(G) - \epsilon(\alpha')} < \frac{P}{2r - \epsilon(\alpha)} \iff \frac{P_G(m)}{2r(G) - \epsilon(\alpha')} < \frac{P(m)}{2r - \epsilon(\alpha)}.$$

<sup>3</sup>To obtain this inequality, one can also see [Huybrechts and Lehn 1997, Corollary 3.3.8].

Therefore, by Lemma 2.9 and (3-5),

$$\frac{h^0(G(m))}{2r(G) - \epsilon(\alpha')} < \frac{h^0(E(m))}{2r - \epsilon(\alpha)}.$$

ii)  $\Rightarrow$  iii) From a proper quotient pair  $(F, \alpha'')$ , we can get a short exact sequence

$$0 \rightarrow (G, \alpha') \rightarrow (E, \alpha) \rightarrow (F, \alpha'') \rightarrow 0.$$

We thus obtain an exact sequence

$$(3-6) \quad 0 \rightarrow H^0(G(m)) \rightarrow H^0(E(m)) \rightarrow H^0(F(m)).$$

Therefore,  $h^0(F(m)) \geq h^0(E(m)) - h^0(G(m))$ . Notice that  $r(E) = r(G) + r(F)$  and  $\epsilon(\alpha) = \epsilon(\alpha') + \epsilon(\alpha'')$ . Thus,

$$\frac{h^0(F(m))}{2r(F) - \epsilon(\alpha'')} \geq \frac{h^0(E(m)) - h^0(G(m))}{(2r - \epsilon(\alpha)) - (2r(G) - \epsilon(\alpha'))} > \frac{h^0(E(m))}{2r - \epsilon(\alpha)} \geq \frac{P(m)}{2r - \epsilon(\alpha)}.$$

iii)  $\Rightarrow$  i) Take the Harder–Narasimhan filtration of  $E$  with respect to the slope. Suppose  $F$  is the last factor, then  $\mu(F) = \mu_{\min}(E)$ , denoted as  $\mu''$ . By Lemma 2.5,

$$(3-7) \quad \frac{h^0(F(m))}{r(F)} \leq \frac{1}{d!} ([\mu'' + m + C]_+)^d.$$

Let  $(F, \alpha'')$  be the induced quotient pair. If  $\epsilon(\alpha'') \neq 0$ , then  $(E, \alpha)$  is stable, since in the Harder–Narasimhan filtration, only the first morphism is nonzero. So, assume  $\epsilon(\alpha'') = 0$ . Then

$$\frac{P(m)}{r} < \frac{2P(m)}{2r - \epsilon(\alpha)} < \frac{h^0(F(m))}{2r(F)} \leq \frac{1}{d!} ([\mu'' + m + C]_+)^d.$$

If  $m \geq m_0$ , the preceding inequality with  $P(m)/r \geq (m - A)^d/d!$  implies that  $m - A \leq \mu'' + m + C$ . Therefore,  $\mu_{\min}(E) = \mu'' \geq -A - C$ . Thus, the family of coherent sheaves satisfying the third condition for some  $m \geq m_0$  is bounded.

Let  $\text{gr}_s = (\text{gr}_s E, \text{gr}_s \alpha)$  denote the last Harder–Narasimhan factor of the pair  $(E, \alpha)$ . Then

$$\frac{h^0(\text{gr}_s E(m))}{2r(\text{gr}_s E) - \epsilon(\text{gr}_s \alpha)} > \frac{P(m)}{2r - 1}.$$

By Lemma 3.4, enlarging  $m_0$  if necessary, we can assume that, for all  $m \geq m_0$ ,

$$(i) \quad h^0(\text{gr}_s E(m)) = P_{\text{gr}_s E}(m);$$

$$(ii) \quad \frac{P_{\text{gr}_s E}(m)}{2r(\text{gr}_s E) - \epsilon(\text{gr}_s \alpha)} > \frac{P(m)}{2r - 1} \iff \frac{P_{\text{gr}_s}}{2r(\text{gr}_s E) - \epsilon(\text{gr}_s \alpha)} > \frac{P}{2r - 1}.$$

Therefore,  $\epsilon(\text{gr}_i \alpha)/r(\text{gr}_s E) \geq 1/r$ , which implies  $\epsilon(\text{gr}_s \alpha) = 1$ . Thus,  $s = 1$ , which means  $(E, \alpha)$  is semistable, and thus stable.  $\square$

Replacing the strong inequalities by weak inequalities, we conclude that the lemma is also true.

#### 4. Construction of the moduli space

Fix the smooth projective variety  $(X, \mathcal{O}_X(1))$ , the coherent sheaf  $E_0$ , the Hilbert polynomial  $P$ , and the stability condition  $\delta$ .

By the boundedness results proven in the last section, there is an  $N \in \mathbb{Z}$  such that for any integer  $m > N$ , the following conditions are satisfied:

- (i)  $E_0(m)$  is globally generated.
- (ii)  $E(m)$  is globally generated and has no higher cohomology for every  $E$  appearing in a  $\delta$ -semistable pair (Proposition 3.3). Similar results hold for their Harder–Narasimhan factors (Lemma 3.4).
- (iii) The three assertions in Lemma 3.5 are equivalent.

Fix such an  $m$  and let  $V$  be a vector space such that

$$\dim V = P(m).$$

Suppose  $(E, \alpha)$  is a semistable pair, then  $E$  can be viewed as a quotient

$$q : V \otimes \mathcal{O}_X(-m) \twoheadrightarrow E.$$

Another datum of the pair is the morphism  $\alpha$ . It gives rise to a linear map

$$\sigma : H^0(E_0(m)) \rightarrow H^0(E(m)) \cong V.$$

Thus, a semistable pair gives rise to the following diagram:

$$\begin{array}{ccccc} K_0 & \xhookrightarrow{\iota} & H^0(E_0(m)) \otimes \mathcal{O}_X(-m) & \xrightarrow{\text{ev}} & E_0 \\ & & \downarrow \sigma & & \downarrow \alpha \\ & & V \otimes \mathcal{O}_X(-m) & \xrightarrow{q} & E \end{array}$$

Here,  $\iota$  is the kernel of the evaluation map  $\text{ev}$ . Conversely, we can obtain a pair from a quotient  $q$  and a linear map  $\sigma$  as long as  $q \circ \sigma \circ \iota = 0$ . Also notice that  $\sigma = 0$  if and only if  $\alpha = 0$ .

We will study the following spaces:

$$\mathbb{P} = \mathbb{P}(\text{Hom}(H^0(E_0(m)), V)) = \text{Proj}(H^0(E_0(m)) \otimes V^\vee),$$

$$\mathcal{Q} = \text{Quot}_X^P(V \otimes \mathcal{O}_X(-m)).$$

The second space is Grothendieck's Quot scheme, parametrizing quotients of  $V \otimes \mathcal{O}_X(-m)$  with Hilbert polynomial  $P$ . This is motivated by a similar construction

in [Huybrechts and Lehn 1995a; 1995b]. Spaces  $\mathbb{P}$  and  $Q$  are fine moduli spaces, with the following universal families:

$$(4-1) \quad H^0(E_0(m)) \otimes \mathcal{O}_{\mathbb{P}} \rightarrow V \otimes \mathcal{O}_{\mathbb{P}}(1),$$

$$(4-2) \quad V \otimes \mathcal{O}_X(-m) \rightarrow \mathcal{E}.$$

Let

$$Z \subset \mathbb{P} \times Q$$

be the locally closed subscheme of points  $\xi = ([\sigma], [q])$  such that

- (i)  $q \circ \sigma \circ \iota = 0$ ;
- (ii)  $E$  is pure;
- (iii) the quotient  $q$  induces an isomorphism of vector spaces  $V \xrightarrow{\sim} H^0(E(m))$ .

There is a natural  $\mathrm{SL}(V)$ -action on  $\mathbb{P} \times Q$ :

$$([\sigma], [q]).g = ([g^{-1} \circ \sigma], [q \circ g]),$$

for  $g \in \mathrm{SL}(V)$  and  $([\sigma], [q]) \in \mathbb{P} \times Q$ . It can be easily checked that this indeed defines a right action. It is clear that  $Z$  is invariant under this action. The closure  $\bar{Z}$  of  $Z \subset \mathbb{P} \times Q$  is invariant as well.

For a very large  $l$ , there is an  $\mathrm{SL}(V)$ -equivariant embedding,

$$Q = \mathrm{Quot}_X^P(V \otimes \mathcal{O}_X(-m)) \hookrightarrow \mathrm{Grass}(V \otimes H^0(\mathcal{O}_X(l-m)), P(l)),$$

$$[q : V \otimes \mathcal{O}_X(-m) \twoheadrightarrow E] \mapsto [H^0(q(l)) : V \otimes H^0(\mathcal{O}_X(l-m)) \twoheadrightarrow H^0(E(l))].$$

The standard very ample line bundle on the Grassmannian is  $\mathrm{SL}(V)$ -linearized. Let  $\mathcal{O}_Q(1)$  be its pullback to  $Q$ . The line bundle  $\mathcal{O}_{\mathbb{P}}(1)$  is also  $\mathrm{SL}(V)$ -linearized. Thus, for positive integers  $n_1$  and  $n_2$ , the following line bundle is  $\mathrm{SL}(V)$ -linearized:

$$L = \mathcal{O}_{\mathbb{P}}(n_1) \boxtimes \mathcal{O}_Q(n_2).$$

We are going to construct the moduli space by taking the GIT quotient of  $\bar{Z}$ , eliminating the extra information coming from identifying  $V$  and  $H^0(E(m))$ . A key step is to relate the  $\delta$ -stability condition to the GIT-stability condition with respect to  $L$ , which will occupy a large part of this section.

An application of the Hilbert–Mumford criterion shows the following lemma. It is very similar to [Wandel 2015, Proposition 4.3]. For the proof of the lemma, see [Lin 2016, Lemma 12].

**Lemma 4.1.** *For  $l$  very large, let  $\xi = ([\sigma], [q]) \in \bar{Z}$  be a point with associated morphism  $\alpha : E_0 \rightarrow E$ . Then the following two conditions are equivalent:*

- (i)  $\xi$  is GIT-stable with respect to  $L$ .

(ii) For any nontrivial proper subspace  $W \subsetneq V$ , let  $G = q(W \otimes \mathcal{O}_X(-m))$ . Then

$$(4-3) \quad P_G(l) > \frac{n_1}{n_2} \left( \epsilon_W(\sigma) - \frac{\dim W}{\dim V} \right) + P(l) \frac{\dim W}{\dim V}.$$

Here,  $\epsilon_W(\sigma)$  is either 1 or 0 depending on whether  $W$  contains  $\text{im } \sigma$  or not.

GIT-semistability can also be characterized by the corresponding weak inequality.

Now, let

$$(4-4) \quad \frac{n_1}{n_2} = \frac{P(l)}{2r}.$$

We fix an  $l$  such that

- (i) Lemma 4.1 holds;
- (ii) (4-3) holds if and only if it holds as an inequality of polynomials in  $l$ :

$$(4-5) \quad P_G > \frac{n_1}{n_2} \left( \epsilon_W(\sigma) - \frac{\dim W}{\dim V} \right) + P \frac{\dim W}{\dim V}.$$

We can ask for the second condition because the family of such  $G$ 's is bounded.

In defining  $Z$ , we required the quotient to be pure. When we take the closure, we may include quotients which are not pure. But the following statement imposes restrictions.

**Corollary 4.2.** *If  $([\sigma], [q]) \in \bar{Z}$  is GIT-semistable, then  $H^0(q(m)) : V \rightarrow H^0(E(m))$  is injective and for any coherent subsheaf  $G \subset E$  such that  $\dim G \leq d - 1$ ,  $H^0(G(m)) = 0$ .*

*Proof.* Let  $W$  be the kernel of  $H^0(q(m)) : V \rightarrow H^0(E(m))$ , then for the image  $G$  we have

$$G = q(W \otimes \mathcal{O}_X(-m)) = 0.$$

The inequality (4-5) forces  $\dim W$  to be zero, otherwise the right-hand side of the inequality is a positive polynomial while the left-hand side is 0.

Suppose  $G \subset E$  such that  $\dim G \leq d - 1$ . If we let  $W = H^0(G(m))$ , then  $q(W \otimes \mathcal{O}_X(-m)) \subset G$ . By the inequality (4-5), we have  $\dim W = 0$ , otherwise the right-hand side will be a positive polynomial of degree no less than  $d$ , while the left hand side is of degree  $\leq d - 1$ .  $\square$

We are ready to relate the  $\delta$ -stability condition to the GIT-stability condition.

**Proposition 4.3.** *Let  $([\sigma], [q])$  be in  $\bar{Z}$  and  $(E, \alpha)$  be the corresponding pair. The following two assertions are equivalent:*

- (i)  $([\sigma], [q])$  is GIT-(semi)stable with respect to  $L$ .
- (ii)  $(E, \alpha)$  is (semi)stable and  $q$  induces an isomorphism  $V \xrightarrow{\sim} H^0(E(m))$ .

Recall that when  $\deg \delta \geq \deg P$ , there are no strictly semistable pairs.

*Proof.* First, assume that a point  $([\sigma], [q]) \in \bar{Z}$  is GIT-semistable. Denote the quotient by

$$q : V \otimes \mathcal{O}(-m) \rightarrow E.$$

Then by Corollary 4.2, we know that the induced linear map  $V \rightarrow H^0(E(m))$  is injective. The sheaf  $E$  can be deformed to a pure sheaf since  $([\sigma], [q])$  is in the closure of  $Z$ . By [Huybrechts and Lehn 1997, Proposition 4.4.2], there is an exact sequence,

$$0 \rightarrow T_{d-1}(E) \rightarrow E \xrightarrow{\phi} F,$$

where  $T_{d-1}(E)$  is the maximal dimension  $d-1$  subsheaf of  $E$  and such that  $P_F = P_E = P$ . According to Corollary 4.2, the exact sequence provides an injective linear map,

$$H^0(E(m)) \hookrightarrow H^0(F(m)).$$

For any dimension  $d$  quotient  $\pi : F \rightarrow F''$ , let  $G$  be the kernel of  $\pi \circ \phi$ ,

$$0 \rightarrow G \rightarrow E \xrightarrow{\pi \circ \phi} F'' \rightarrow 0.$$

Let  $W = V \cap H^0(G(m))$ . Then we have

$$(4-6) \quad h^0(F''(m)) \geq h^0(E(m)) - h^0(G(m)) \geq \dim V - \dim W.$$

Let  $r'' = r(F'')$ . Let's consider the leading coefficients of the two sides of (4-3), viewed as polynomials in  $l$ . (This is where the argument diverges, depending on the degree of  $\delta$ . Here, we focus on the case where  $\deg \delta \geq d$ .) Then

$$(4-7) \quad (2r(G) - \epsilon_W(\sigma)) \dim V \geq (2r - 1) \dim W.$$

Combining ((4-6), (4-7)), we have

$$\frac{h^0(F''(m))}{2r'' - \epsilon(\pi \circ \phi \circ \alpha)} \geq \frac{\dim V}{2r - 1} \cdot \frac{2r'' - (1 - \epsilon_W(\sigma))}{2r'' - \epsilon(\pi \circ \phi \circ \alpha)} \geq \frac{P(m)}{2r - 1}.$$

To prove the second inequality, notice that, when  $\epsilon(\pi \circ \phi \circ \alpha) = 0$ ,  $\text{im } \alpha \subset G$ . Therefore  $\text{im } \sigma \subset H^0(G(m))$ . Thus,  $\text{im } \sigma \subset W$ .

According to Lemma 3.5, the pair  $(F, \phi \circ \alpha)$  is semistable. Therefore, by our choice of  $m$ ,  $h^0(F(m)) = P(m)$ . We have the following commutative diagram:

$$\begin{array}{ccccc} V \otimes \mathcal{O}_X(-m) & \xrightarrow{\sim} & H^0(E(m)) \otimes \mathcal{O}_X(-m) \text{ ev} & \xrightarrow{\sim} & H^0(F(m)) \otimes \mathcal{O}_X(-m) \text{ ev} \\ & \searrow q & \downarrow & & \downarrow \\ & & E & \xrightarrow{\phi} & F \end{array}$$

So,  $\phi$  is surjective. Since they have the same Hilbert polynomial, it is an isomorphism. Therefore,  $(E, \alpha)$  is a semistable pair.

Next, we assume that  $(E, \alpha)$  is semistable, thus stable, and  $q(m)$  induces an isomorphism between global sections. For any nontrivial proper subspace  $W \subsetneq V$ , let

$$G = q(W \otimes \mathcal{O}(-m))$$

and  $(G, \alpha')$  the corresponding subpair. If  $(G, \alpha') = (E, \alpha)$ , the inequality in Lemma 4.1 holds. Assume that  $(G, \alpha')$  is a proper subpair. According to Lemma 3.5, we have

$$\frac{h^0(G(m))}{2r(G) - \epsilon(\alpha')} < \frac{h^0(E(m))}{2r - 1}.$$

From the commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & H^0(G(m)) \\ \downarrow & & \downarrow \\ V & \xrightarrow{\cong} & H^0(E(m)) \end{array}$$

we know that  $\dim W \leq h^0(G(m))$ . Thus,

$$\frac{\dim W}{2r(G) - \epsilon(\alpha')} < \frac{h^0(E(m))}{2r - 1}.$$

Therefore,

$$r(G) > \frac{1}{2}\epsilon(\alpha') - \frac{1}{2} \cdot \frac{\dim W}{\dim V} + r \frac{\dim W}{\dim V},$$

which implies the inequality in Lemma 4.1, since  $\epsilon(\alpha') \geq \epsilon_W(\sigma)$ . Hence,  $([\sigma], [q])$  is GIT-stable.  $\square$

We still need the following lemma, which will help us identify closed orbits. A pair is *polystable* if it is isomorphic to a direct sum of stable pairs, degenerate or not, with the same reduced Hilbert polynomial.

**Lemma 4.4.** *The closures of orbits of two points,  $([\sigma_1], [R_1])$  and  $([\sigma_2], [R_2])$ , in  $\bar{Z}^{ss}$  intersect if and only if their associated semistable pairs  $(E_1, \alpha_1)$  and  $(E_2, \alpha_2)$  have the same Jordan–Hölder factors. The orbit of a point  $([\sigma], [q])$  is closed if and only if the associated pair  $(E, \alpha)$  is polystable.*

The proof is similar to that of [Huybrechts and Lehn 1997, Theorem 4.3.3], using the following lemma on semicontinuity.

**Lemma 4.5** (semicontinuity). *Suppose  $(\mathcal{F}, \alpha)$  and  $(\mathcal{G}, \beta)$  over  $X_T = T \times X$  are two flat families of pairs, with Hilbert polynomials  $P_{\mathcal{F}}$  and  $P_{\mathcal{G}}$ , parametrized by a scheme  $T$  of finite type over  $k$ . Then, the following function is semicontinuous:*

$$t \mapsto \dim_k \operatorname{Hom}_{\{t\} \times X}((\mathcal{F}_t, \alpha_t), (\mathcal{G}_t, \beta_t)).$$

The proof is modified from that of [Huybrechts and Lehn 1995a, Lemma 3.4].

*Proof.* The space  $\text{Hom}((\mathcal{F}_t, \alpha_t), (\mathcal{G}_t, \beta_t))$  is related to the pullback in the diagram

$$\begin{array}{ccc} C_t & \dashrightarrow & k \\ \downarrow & & \downarrow \cdot \beta_t \\ \text{Hom}(\mathcal{F}_t, \mathcal{G}_t) & \xrightarrow{\circ \alpha_t} & \text{Hom}(E_0, \mathcal{G}_t) \end{array}$$

in the sense that it satisfies the equality

$$\dim \text{Hom}((\mathcal{F}_t, \alpha_t), (\mathcal{G}_t, \beta_t)) = \dim C_t - 1 + \epsilon(\beta_t).$$

By our flatness assumption,  $\beta_t$  is either always zero or never zero. Thus, it is enough to show that  $C_t$  is a fiber of a common coherent  $\mathcal{O}_T$ -module, as  $t$  varies. Since the question is local on  $T$ , assume  $T = \text{Spec } A$ , where  $A$  is a  $k$ -algebra.

It is shown in the proof of [Huybrechts and Lehn 1995a, Lemma 3.4] that there is a bounded-above complex  $M_{E_0}^\bullet$  of finite type free  $A$ -modules, such that for any  $A$ -module  $M$ ,

$$(4-8) \quad h^i(M_{E_0}^\bullet \otimes_A M) \cong \text{Ext}_{X_T}^i(\pi_X^* E_0, \mathcal{G} \otimes_A M).$$

Similarly, there is such an  $M_{\mathcal{F}}^\bullet$  that

$$(4-9) \quad h^i(M_{\mathcal{F}}^\bullet \otimes_A M) \cong \text{Ext}_{X_T}^i(\mathcal{F}, \mathcal{G} \otimes_A M).$$

The morphism  $\alpha$  induces a morphism of complexes, which is still denoted as  $\alpha : M_{\mathcal{F}}^\bullet \rightarrow M_{E_0}^\bullet$ . The morphism  $\beta$  induces a morphism  $\beta : A \rightarrow M_{E_0}^\bullet$ . Thus, there is a morphism,

$$\psi = (\alpha, -\beta) : M_{\mathcal{F}}^\bullet \oplus A \rightarrow M_{E_0}^\bullet.$$

Then the mapping cone  $C(\psi)$  fits in the distinguished triangle

$$C(\psi)[-1] \rightarrow M_{\mathcal{F}}^\bullet \oplus A \rightarrow M_{E_0}^\bullet \rightarrow C(\psi).$$

Taking the long exact sequence, we have

$$0 \rightarrow h^{-1}(C(\psi)) \rightarrow \text{Hom}_{X_T}(\mathcal{F}, \mathcal{G}) \oplus A \rightarrow \text{Hom}_{X_T}(\pi_X^* E_0, \mathcal{G}) \rightarrow \cdots$$

Thus, we have the following fiber diagram:

$$\begin{array}{ccc} h^{-1}(C(\psi)) & \longrightarrow & A \\ \downarrow & & \downarrow \beta \\ \text{Hom}_{X_T}(\mathcal{F}, \mathcal{G}) & \xrightarrow{\alpha} & \text{Hom}_{X_T}(\pi_X^* E_0, \mathcal{G}) \end{array}$$

Therefore, together with (4-8) and (4-9) and the isomorphism  $\text{Ext}_{X_T}^i(\mathcal{F}, \mathcal{G} \otimes k(t)) \cong \text{Ext}_{X_t}^i(\mathcal{F}_t, \mathcal{G}_t)$ , we know  $C_t \cong h^{-1}(C(\psi)) \otimes k(t)$ .  $\square$

We can now prove the existence of the moduli space.



*Proof of Theorem 1.1.* Let

$$S = S_{E_0}(P, \delta) = \bar{Z}^{ss} // SL(V)$$

be the GIT quotient. This is a projective scheme. We will show that this is the coarse moduli space of S-equivalence classes of semistable pairs.

Suppose we are given a family of semistable pairs parametrized by  $T$ :

$$\beta : \pi_X^* E_0 \rightarrow \mathcal{F}.$$

Let  $\pi$  be the projection from  $T \times X$  onto  $T$ . Let  $m$  be chosen as before, then  $\pi_*(\mathcal{F}(m))$  is locally free of rank  $P(m) = \dim V$  and we obtain a morphism over  $T$ :

$$\pi_*(\beta(m)) : \pi_*(\pi_X^* E_0(m)) \rightarrow \pi_*(\mathcal{F}(m)).$$

Therefore, there is an open affine cover  $T = \bigcup T_i$ , such that  $\pi_*(\mathcal{F}(m))|_{T_i}$  is free of rank  $P(m)$  over each  $T_i$ . Choose an isomorphism over  $T_i$ :

$$\omega_i : V \otimes \mathcal{O}_{T_i} \rightarrow \pi_*(\mathcal{F}(m))|_{T_i}.$$

Then  $\omega_i^{-1} \circ \pi_*(\beta(m))$  induces a morphism  $T_i \rightarrow \mathbb{P}$ . Also, the quotient

$$\text{ev} \circ \pi^*(\omega_i) : V \otimes \mathcal{O}_X(-m) \xrightarrow{\cong} \pi^* \pi_*(\mathcal{F}(m)) \otimes \mathcal{O}_X(-m) \rightarrow \mathcal{F}$$

over  $T_i \times X$  induces a morphism  $T_i \rightarrow Q$ . Thus, they induce a morphism  $f_i : T_i \rightarrow \mathbb{P} \times Q$ . By the definition of  $Z$  and Proposition 4.3,  $f_i$  factors through  $\bar{Z}^{ss}$ . Therefore, we obtain unambiguously a morphism,

$$f_\beta : T \rightarrow S.$$

Thus, we have a natural transformation,

$$S = S_{E_0}(P, \delta) \rightarrow \text{Mor}(-, S).$$

Suppose there is a natural transformation,

$$(4-10) \quad S \rightarrow \text{Mor}(-, N).$$

Let  $T = \bar{Z}^{ss}$ . Universal families (4-1) and (4-2) induce

$$H^0(E_0(m)) \otimes \mathcal{O}_X(-m) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}}(1) \otimes \mathcal{O}_X(-m) \rightarrow \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(1).$$

Over  $T$ , the composition induces a family,

$$(4-11) \quad \pi_X^* E_0 \rightarrow \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(1),$$

and thus an element in  $S(T)$ . This in turn produces a map  $T = \bar{Z}^{ss} \rightarrow N$ . Because (4-10) is a natural transformation, this map is  $SL(V)$ -equivariant, with the action

on  $N$  being trivial. According to properties of a quotient, the map factors uniquely through  $S$ . Therefore, we have the following commutative diagram of functors:

$$\begin{array}{ccc} S & \longrightarrow & \text{Mor}(-, S) \\ & \searrow & \downarrow \\ & & \text{Mor}(-, N) \end{array}$$

Moreover, closed points in  $S$  are in bijection with  $S$ -equivalence classes of semistable pairs, according to Lemma 4.4. Thus,  $S$  is the coarse moduli space.

Let us consider the open set  $\bar{Z}^s \subset \bar{Z}^{ss}$  of stable points. The geometric quotient

$$\bar{Z}^s \rightarrow \bar{Z}^s / \text{SL}(V) = S_{E_0}^s(P, \delta) = S^s$$

provides a quasiprojective scheme parametrizing equivalence classes of stable pairs. We shall prove this quotient to be a principal  $\text{PGL}(V)$ -bundle. It is enough to show that the stabilizers are products of the identity matrix and roots of unity.

Suppose a point  $([\sigma], [q]) \in \bar{Z}^s$  gives rise to a stable pair  $\alpha : E_0 \rightarrow E$  and  $([\sigma], [q])$  is fixed by  $g \in \text{SL}(V)$ , that is,  $[\sigma] = [g^{-1} \circ \sigma]$   $[q] = [q \circ g]$ . Then there is a scalar  $a \in k^\times$ , such that  $g^{-1} \circ \sigma = a\sigma$ , and there is an isomorphism  $\phi : E \rightarrow E$ , such that  $\phi \circ q = q \circ g$ . Therefore,

$$\phi \circ \alpha \circ \text{ev} = a\alpha \circ \text{ev} : H^0(E_0(m)) \otimes \mathcal{O}_X(-m) \rightarrow E.$$

So,  $\phi \circ \alpha = a\alpha$ . Thus,  $\phi$  is a multiplication by a nonzero scalar, by Lemma 2.11. In the diagram

$$\begin{array}{ccc} V & \xrightarrow{H^0(q(m))} & H^0(E(m)) \\ g \downarrow & & \downarrow H^0(\phi(m)) \\ V & \xrightarrow{H^0(q(m))} & H^0(E(m)) \end{array}$$

the horizontal arrows are isomorphisms and the right vertical arrow is a multiplication by a nonzero scalar. Therefore,  $g$  is also a multiplication by a nonzero scalar. Because  $g$  lies in  $\text{SL}(V)$ , it is the product of a root of unity and the identity matrix.

In the family (4-11),  $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(1)$  is  $\text{SL}(V)$ -equivariant. Although the actions of the center of  $\text{SL}(V)$  on  $\mathcal{O}_{\mathbb{P}}(1)$  and  $\mathcal{E}$  are not trivial, its action on  $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(1)$  is. Thus,  $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(1)$  is  $\text{PGL}(V)$ -equivariant. Therefore, the restriction of (4-11) to  $\bar{Z}^s \times X$  descends to  $S^s \times X$  to give a universal family of pairs. Hence,  $S^s$  represents the functor  $S_{E_0}^s(P, \delta)$ .  $\square$

**Remark 4.6.** The construction above can be carried out in the relative case. By [Grothendieck 1961b, Lemma 2.5], the boundedness result still holds. According to [Seshadri 1977], the GIT construction works in the relative setting. More concretely, let  $T$  be a  $k$ -scheme of finite type,  $X \rightarrow T$  a flat projective morphism, and  $\mathcal{E}_0$  a

coherent  $\mathcal{O}_X$ -module flat over  $T$ . Then, there is a relative moduli space of  $\delta$ -semistable pairs  $\mathcal{S}_{\mathcal{E}_0}(P, \delta)$  which is projective over  $T$ . There is an open subscheme  $\mathcal{S}_{\mathcal{E}_0}^s(P, \delta) \subset \mathcal{S}_{\mathcal{E}_0}(P, \delta)$  parametrizing stable pairs. Moreover, fibers  $\mathcal{S}_{\mathcal{E}_0}(P, \delta)_t$  over closed points are moduli spaces of semistable pairs on  $X_t$ .

## 5. Deformation and obstruction theories

This section is devoted to the proof of Theorem 1.2, following [Huybrechts and Lehn 1997; Inaba 2002]. In Section 5A, we will outline the construction of the obstruction class and identify the deformation space. In Section 5B, we will fill in the proofs.

**5A. Constructions.** Suppose  $(E, \alpha)$  is a stable pair and

$$0 \rightarrow K \rightarrow B \xrightarrow{\sigma} A \rightarrow 0$$

is a short exact sequence, where  $A, B \in \mathcal{A}rt_k$  are local Artinian  $k$ -algebras with residue field  $k$ , such that  $\mathfrak{m}_B K = 0$ . Suppose

$$\alpha_A : E_0 \otimes A \rightarrow E_A$$

over

$$X_A = X \times \operatorname{Spec} A$$

is a (flat) extension of  $(E, \alpha)$ . Let

$$I_A^\bullet = \{E_0 \otimes A \rightarrow E_A\}$$

denote the complex positioned at 0 and 1. We would like to extend  $(E_A, \alpha_A)$  to a pair  $(E_B, \alpha_B)$  over  $X_B$ . This is similar to deforming a sheaf or a perfect complex. But we need to fix  $E_0$ .

We take two locally free resolutions  $P^\bullet \xrightarrow{\sim} E_0$  and  $Q_A^\bullet \xrightarrow{\sim} E_A$  and lift  $\alpha_A$  to a morphism of complexes  $\alpha_A^\bullet : P^\bullet \otimes A \rightarrow Q_A^\bullet$ . Then, we have the following commutative diagram:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d_P^{-2} \otimes A} & P^{-1} \otimes A & \xrightarrow{d_P^{-1} \otimes A} & P^0 \otimes A & \longrightarrow & E_0 \otimes A \longrightarrow 0 \\ & & \downarrow \alpha_A^{-1} & & \downarrow \alpha_A^0 & & \downarrow \alpha_A \\ \cdots & \xrightarrow{d_{Q_A}^{-2}} & Q_A^{-1} & \xrightarrow{d_{Q_A}^{-1}} & Q_A^0 & \longrightarrow & E_A \longrightarrow 0 \end{array}$$

where

$$(5-1) \quad P^i = V^i \otimes \mathcal{O}_X(-m_i) \quad \text{and} \quad Q_A^i = W^i \otimes \mathcal{O}_{X_A}(-n_i).$$

Here,  $V^i$  and  $W^i$  are vector spaces and  $m_i, n_i \in \mathbb{N}$ . Then,  $Q^\bullet = Q_A^\bullet \otimes_A k$  is a resolution of  $E$ , because  $E_A$  is flat over  $A$ .

We can view the morphism  $\alpha_A$  as a morphism between complexes concentrated at degree 0, then  $I_A^\bullet$  can be viewed as a mapping cone  $I_A^\bullet \cong C(\alpha_A)[-1] \cong C(\alpha_A^\bullet)[-1]$ . For the sake of notation, we write down the mapping cone explicitly:

$$\cdots \rightarrow P^{-1} \otimes A \oplus Q_A^{-2} \xrightarrow{d_A^{-2}} P^0 \otimes A \oplus Q_A^{-1} \xrightarrow{d_A^{-1}} Q_A^0 \rightarrow 0,$$

where

$$(5-2) \quad d_A^i = \begin{pmatrix} -d_P^{i+1} \otimes A & 0 \\ \alpha_A^{i+1} & d_{Q_A}^i \end{pmatrix}.$$

We lift  $d_{Q_A}^i$  to  $d_{Q_B}^i$ , getting a sequence  $(Q_B^i, d_{Q_B}^i)_{i \leq 0}$ , where

$$Q_B^i = W^i \otimes \mathcal{O}_{X_B}(-n_i).$$

We also lift  $\alpha_A^i : P^i \otimes A \rightarrow Q_A^i$  to  $\alpha_B^i : P^i \otimes B \rightarrow Q_B^i$ . We then obtain a sequence

$$(5-3) \quad (P^{i+1} \otimes B \oplus Q_B^i, d_B^i)_{i \leq 0},$$

where  $d_B^i$  is similar to  $d_A^i$  in (5-2). This is not necessarily a complex:

$$(5-4) \quad d_B^i \circ d_B^{i-1} = \begin{pmatrix} 0 & 0 \\ -\alpha_B^{i+1} \circ (d_P^i \otimes B) + d_{Q_B}^i \circ \alpha_B^i & d_{Q_B}^i \circ d_{Q_B}^{i-1} \end{pmatrix}$$

may not vanish. But when it is a complex,  $(Q_B^\bullet, d_{Q_B}^\bullet)$  forms a complex and  $\alpha_B^\bullet : P^\bullet \otimes B \rightarrow Q_B^\bullet$  is a morphism of complexes. Thus,

$$H^0(\alpha_B^\bullet) : E_0 \otimes B \rightarrow H^0(Q_B^\bullet, d_{Q_B}^\bullet)$$

provides a flat extension of  $\alpha_A$ , according to Lemma 5.1, which will be stated and proved in the next subsection.

The lower row of (5-4) constitutes a map

$$(5-5) \quad P^\bullet[1] \otimes B \oplus Q_B^\bullet \rightarrow Q_B^\bullet[2].$$

When restricted to  $X_A$ , it becomes zero. Moreover,  $m_B K = 0$ . The map above induces a map<sup>4</sup>

$$(5-6) \quad (\omega_P^\bullet, \omega_Q^\bullet) : C(\alpha^\bullet) \rightarrow Q_B^\bullet[2] \otimes_B K \cong Q^\bullet[2] \otimes_k K.$$

We claim that  $(\omega_P^\bullet, \omega_Q^\bullet)$  is a morphism of complexes, which will be proven, see Lemma 5.3. This induces a class, which will be shown to be the obstruction class

$$(5-7) \quad \text{ob}(\alpha_A, \sigma) = [(\omega_P^\bullet, \omega_Q^\bullet)] \in \text{Hom}_{K(X)}(C(\alpha^\bullet), Q^\bullet[2] \otimes_k K).$$

<sup>4</sup>The argument used to deduce (5-6) from (5-5) will be applied repeatedly.

To identify  $\text{Hom}_{K(X)}(C(\alpha^\bullet), Q^\bullet[2] \otimes K)$  with  $\text{Ext}^1(I^\bullet, E \otimes K)$  in the theorem, we only need to take (5-1) to be very negative such that  $H^i(X, E(m_j)) = 0$  and  $H^i(X, E(n_j)) = 0$ , for all  $i > 0$  and  $j \leq 0$ . Then

$$\text{Ext}^1(I^\bullet, E \otimes K) \cong \text{Hom}_{K(X)}(C(\alpha^\bullet), E[2] \otimes K) \cong \text{Hom}_{K(X)}(C(\alpha^\bullet), Q^\bullet[2] \otimes K).$$

Suppose we have two extensions  $\alpha_B : E_0 \otimes B \rightarrow E_B$  and  $\beta_B : E_0 \otimes B \rightarrow F_B$ , which arise from the following liftings:

$$\begin{aligned} \{d_{E_B}^i : Q_B^i &\rightarrow Q_B^{i+1}, \alpha_B^i : P^i \otimes B \rightarrow Q_B^i\}, \\ \{d_{F_B}^i : Q_B^i &\rightarrow Q_B^{i+1}, \beta_B^i : P^i \otimes B \rightarrow Q_B^i\}. \end{aligned}$$

The differences  $d_{E_B}^i - d_{F_B}^i$  and  $\alpha_B^i - \beta_B^i$  induce a morphism of complexes

$$(5-8) \quad (f_P^\bullet, f_Q^\bullet) : C(\alpha^\bullet) \rightarrow Q^\bullet[1] \otimes K.$$

This induces a class

$$v = [(f_P^\bullet, f_Q^\bullet)] \in \text{Hom}_{K(X)}(C(\alpha^\bullet), Q^\bullet[1] \otimes K) \cong \text{Ext}^1(I^\bullet, E \otimes K).$$

Conversely, given  $\alpha_B$  and  $(f_P^\bullet, f_Q^\bullet)$ , we can produce another extension  $\beta_B$ .

Moreover,  $\alpha_B$  and  $\beta_B$  are equivalent if and only if  $v = 0$ .

**5B. Proofs.** In this subsection, we fill in the proofs of several claims we made in Section 5A. We will assume the independence of choices in 5B1 and provide proofs of independence in 5B2. To simplify the notation, we will sometimes omit the superscripts in maps between complexes, such as  $\alpha^\bullet$  and  $\alpha^i$ .

**5B1. Obstruction classes.** We first show that  $\text{ob}(\alpha_A, \sigma)$  defined in (5-7) is an obstruction class.

Suppose an extension  $(E_B, \alpha_B)$  exists. The definition of  $\text{ob}(\alpha_A, \sigma)$  does not depend on the choice of the resolution  $Q_A^\bullet$ . We can assume  $(E_B, \alpha_B)$  arises by lifting  $d_{Q_A}^i$  and  $\alpha_A^i$ , making  $Q_B^\bullet$  into a complex and  $\alpha_B^\bullet$  a morphism of complexes. Then,  $(\omega_P^\bullet, \omega_Q^\bullet) = 0$ . Thus,  $\text{ob}(\alpha_A, \sigma) = 0$ .

Conversely, suppose  $\text{ob}(\alpha_A, \sigma) = 0$ . It is enough to show that  $(\omega_P^\bullet, \omega_Q^\bullet) = 0$ , after possible modifications of the liftings. The vanishing of  $\text{ob}(\alpha_A, \sigma)$  is equivalent to  $(\omega_P^\bullet, \omega_Q^\bullet)$  being homotopic to 0. Let  $(g_P^\bullet, g_Q^\bullet)$  be a homotopy. By abuse of notation, let  $\iota$  denote inclusions

$$\iota : Q_B^i \otimes K \hookrightarrow Q_B^i.$$

Similarly,  $\pi$  denotes the corresponding quotients,

$$\pi : P^i \otimes B \twoheadrightarrow P^i \quad \text{and} \quad \pi : Q_B^i \twoheadrightarrow Q^i.$$

We can replace  $\alpha_B$  and  $d_{Q_B}$  by

$$\alpha_B - \iota \circ g_P \circ \pi \quad \text{and} \quad d_{Q_B} - \iota \circ g_Q \circ \pi,$$

then the new  $(\omega_P^\bullet, \omega_Q^\bullet)$  is zero.

The following well-known lemma is central to our argument. For completeness, we give a proof here.

**Lemma 5.1.** *Let  $(Q_A^\bullet, d_{Q_A}^\bullet)$  be a sequence of the form  $Q_A^i \cong W^i \otimes \mathcal{O}_{X_A}(-n_i)$ ,  $i \leq 0$ , such that*

$$(Q_A^\bullet, d_{Q_A}^\bullet) \otimes_A k \cong (Q^\bullet, d^\bullet)$$

*is a resolution of  $E$ . If  $(Q_A^\bullet, d_{Q_A}^\bullet)$  is a complex, then it is exact except at the 0-th place and the cohomology  $H^0(Q_A^\bullet, d_{Q_A}^\bullet)$  is an extension of  $E$  flat over  $A$ .*

*Proof.* There is a short exact sequence of complexes

$$0 \rightarrow Q_A^\bullet \otimes_A \mathfrak{m}_A \rightarrow Q_A^\bullet \rightarrow Q^\bullet \rightarrow 0.$$

First, let  $n$  be the least integer such that  $\mathfrak{m}_A^n = 0$ . We shall show that for  $0 \leq i \leq n$ ,  $Q_A^\bullet \otimes A/\mathfrak{m}_A^i$  is exact except at the 0-th place, by induction on  $i$  decreasingly. Tensor  $Q_A^\bullet$  over  $A$  with the short exact sequence

$$0 \rightarrow \mathfrak{m}_A^{n-1} \rightarrow \mathfrak{m}_A^{n-2} \rightarrow \mathfrak{m}_A^{n-2}/\mathfrak{m}_A^{n-1} \rightarrow 0,$$

whose last term is a direct sum of copies of  $k$ . On the other hand,  $Q_A^\bullet \otimes \mathfrak{m}_A^{n-1} \cong Q^\bullet \otimes_k \mathfrak{m}_A^{n-1}$ . We deduce that the complexes  $Q_A^\bullet \otimes \mathfrak{m}_A^{n-1}$  and  $Q_A^\bullet \otimes \mathfrak{m}_A^{n-2}/\mathfrak{m}_A^{n-1}$  are exact except at the 0-th places. So, from the associated long exact sequence,  $Q_A^\bullet \otimes \mathfrak{m}_A^{n-2}$  is also exact except at the 0-th place. Inductively, we can prove this for  $Q_A^\bullet$ .

Next, let  $E_A = H^0(Q_A^\bullet, d_{Q_A}^\bullet)$ . We shall show that  $E_A \otimes A/\mathfrak{m}_A^i$  is flat for  $1 \leq i \leq n$ , by induction on  $i$ .

Of course  $E_A \otimes_A A/\mathfrak{m}_A \cong E$  is flat over  $A/\mathfrak{m}_A \cong k$ . Tensor the short exact sequence

$$(5-9) \quad 0 \rightarrow \mathfrak{m}_A/\mathfrak{m}_A^2 \rightarrow A/\mathfrak{m}_A^2 \rightarrow A/\mathfrak{m}_A \rightarrow 0$$

by  $Q_A^\bullet$  over  $A$ . Since the ideal  $\mathfrak{m}_A/\mathfrak{m}_A^2$  is square-zero, we have the short exact sequence of complexes

$$0 \rightarrow Q^\bullet \otimes_k \mathfrak{m}_A/\mathfrak{m}_A^2 \rightarrow Q_A^\bullet \otimes_A A/\mathfrak{m}_A^2 \rightarrow Q^\bullet \rightarrow 0.$$

The associated long exact sequence degenerates to

$$(5-10) \quad 0 \rightarrow E \otimes \mathfrak{m}_A/\mathfrak{m}_A^2 \rightarrow E_A \otimes A/\mathfrak{m}_A^2 \rightarrow E \rightarrow 0.$$

Therefore,  $E_A \otimes_A A/\mathfrak{m}_A^2$  is flat over  $A/\mathfrak{m}_A^2$ , according to Lemma 5.2. Replacing (5-9) by

$$0 \rightarrow \mathfrak{m}_A^2/\mathfrak{m}_A^3 \rightarrow A/\mathfrak{m}_A^3 \rightarrow A/\mathfrak{m}_A^2 \rightarrow 0,$$

we can repeat this argument. Inductively, we can prove  $E_A$  is flat over  $A$ .

Similar to obtaining (5-10), we also have the short exact sequence

$$0 \rightarrow E_A \otimes \mathfrak{m}_A \rightarrow E_A \rightarrow E \rightarrow 0.$$

So,  $E_A$  is an extension of  $E$  flat over  $A$ .  $\square$

For the reader's convenience, we include the following basic lemma about flatness. For a proof, see [Hartshorne 2010, Proposition 2.2].

**Lemma 5.2.** *Let  $B \rightarrow A$  be a surjective homomorphism of Noetherian rings whose kernel  $K$  is square zero. Then a  $B$ -module  $M'$  is flat over  $B$  if and only if  $M = M' \otimes_B A$  is flat over  $A$  and the natural map  $M \otimes_A K \rightarrow M'$  is injective.*

**Lemma 5.3.** *The map (5-6) is a morphism of complexes.*

*Proof.* We have two equalities

$$(5-11) \quad -\alpha_B \circ d_P \otimes B + d_{Q_B} \circ \alpha_B = \iota \circ \omega_P \circ \pi \quad \text{and} \quad d_{Q_B} \circ d_{Q_B} = \iota \circ \omega_Q \circ \pi.$$

The map (5-6) is indeed a morphism: one can show that

$$\iota \circ \left( d_Q \otimes K \circ (\omega_P, \omega_Q) - (\omega_P, \omega_Q) \begin{pmatrix} -d_P & 0 \\ \alpha & d_Q \end{pmatrix} \right) \circ \pi = 0.$$

Because  $\iota$  is injective and  $\pi$  is surjective,  $(\omega_P, \omega_Q)$  commutes with differentials.<sup>5</sup>  $\square$

**5B2. Obstructions: independence of choices.** We now show that  $\text{ob}(\alpha_A, \sigma)$  is independent of various choices we have made:  $\alpha_A^\bullet$ ,  $\alpha_B^\bullet$ ,  $d_{Q_B}^\bullet$ , and  $Q_A^\bullet$ .

To start, if we choose a different lifting  $\alpha_A^\bullet$  of  $\alpha_A$ , then  $(\omega_P^\bullet, \omega_Q^\bullet)$  only differs by a homotopy.

We next show that the morphism  $(\omega_P^\bullet, \omega_Q^\bullet)$  is independent of liftings  $\alpha_B$  and  $d_{Q_B}$ , modulo homotopy.

Let  $\alpha_B'$  and  $d_{Q_B}'$  be different liftings, giving rise to  $(\omega_P', \omega_Q')$ . The differences  $\alpha_B - \alpha_B'$  and  $d_{Q_B} - d_{Q_B}'$  induce a map, which will be shown to be a homotopy,

$$(h_P^\bullet, h_Q^\bullet) : P^\bullet[1] \oplus Q^\bullet \rightarrow Q^\bullet[1] \otimes_k K.$$

We have the following equalities:

$$(5-12) \quad \iota \circ h_P \circ \pi = \alpha_B - \alpha_B' \quad \text{and} \quad \iota \circ h_Q \circ \pi = d_{Q_B} - d_{Q_B}'.$$

Combining (5-11) and (5-12), we obtain

$$\begin{aligned} \omega_P - \omega_P' &= -h_P \circ d_P + d_Q \otimes K \circ h_P + h_Q \circ \alpha, \\ \omega_Q - \omega_Q' &= d_Q \otimes K \circ h_Q + h_Q \circ d_Q. \end{aligned}$$

<sup>5</sup>The trick using  $\iota$  and  $\pi$  will be applied repeatedly.

Therefore,

$$(\omega_P, \omega_Q) - (\omega'_P, \omega'_Q) = d_Q \otimes K \circ (h_P, h_Q) + (h_P, h_Q) \begin{pmatrix} -d_P & 0 \\ \alpha & d_Q \end{pmatrix},$$

which means  $(\omega_P^\bullet, \omega_Q^\bullet)$  and  $(\omega'_P^\bullet, \omega'_Q^\bullet)$  are homotopic.

Finally, we show the independence of  $Q_A^\bullet$ .

Let  $(R_A^\bullet, d_{R_A}^\bullet)$  be another very negative resolution of the form:

$$R_A^i = W^{i'} \otimes \mathcal{O}_{X_A}(-n'_i).$$

Then, there is a lifting of the identity map  $q_A^\bullet : Q_A^\bullet \rightarrow R_A^\bullet$ , unique up to homotopy.

Let

$$\beta_A^\bullet = q_A^\bullet \circ \alpha_A^\bullet : P^\bullet \otimes A \rightarrow R_A^\bullet.$$

Moreover, there is a morphism

$$\text{diag}(\text{id}, q_A^\bullet) : C(\alpha_A^\bullet) \rightarrow C(\beta_A^\bullet).$$

Lift  $q_A^\bullet$  and  $\beta_A^\bullet$  to  $q_B^\bullet : Q_B^\bullet \rightarrow R_B^\bullet$  and  $\beta_B^\bullet : P^\bullet \otimes B \rightarrow R_B^\bullet$ . Then, we have a map of sequences

$$\text{diag}(\text{id}, q_B^\bullet) : P^\bullet[1] \otimes B \oplus Q_B^\bullet \rightarrow P^\bullet[1] \otimes B \oplus R_B^\bullet.$$

This fits in the following square, which is not necessarily commutative,

$$(5-13) \quad \begin{array}{ccc} P^\bullet[1] \otimes B \oplus Q_B^\bullet & \longrightarrow & Q_B^\bullet[2] \\ \text{diag}(\text{id}, q_B^\bullet) \downarrow & & \downarrow q_B^\bullet \\ P^\bullet[1] \otimes B \oplus R_B^\bullet & \longrightarrow & R_B^\bullet[2] \end{array}$$

Here, the two horizontal maps are defined as in (5-5). The square above induces

$$\begin{array}{ccc} P^\bullet[1] \oplus Q^\bullet & \xrightarrow{(\omega_P^\bullet, \omega_Q^\bullet)} & Q^\bullet[2] \otimes K \\ \text{diag}(\text{id}, q^\bullet) \downarrow & & \downarrow q^\bullet \\ P^\bullet[1] \oplus R^\bullet & \xrightarrow{(\bar{\omega}_P^\bullet, \bar{\omega}_R^\bullet)} & R^\bullet[2] \otimes K \end{array}$$

To show that  $\text{ob}(\alpha_A, \sigma)$  is independent of the resolution, it is enough to show that the two compositions in the square above differ by a homotopy. This is because, if they differ by a homotopy, two classes  $[(\omega_P^\bullet, \omega_Q^\bullet)]$  and  $[(\bar{\omega}_P^\bullet, \bar{\omega}_R^\bullet)]$  are identified via the isomorphism

$$\text{Hom}_{K(X)}(C(\alpha^\bullet), Q^\bullet[2] \otimes K) \cong \text{Hom}_{K(X)}(C(\beta^\bullet), R^\bullet[2] \otimes K).$$

Indeed, the difference  $d_{R_B} \circ q_B - q_B \circ d_{Q_B}$  and  $\beta_B - q_B \circ \alpha_B$  induce maps

$$\tau^\bullet : Q^\bullet \rightarrow R^\bullet[1] \otimes K \quad \text{and} \quad \nu^\bullet : P^\bullet \rightarrow Q^\bullet \otimes K.$$



There are the following equalities:

$$(5-14) \quad d_{R_B} \circ q_B - q_B \circ d_{Q_B} = \iota \circ \tau \circ \pi \quad \text{and} \quad \beta_B - q_B \circ \alpha_B = \iota \circ \nu \circ \pi.$$

Combining (5-11) and (5-14), we know that the difference of two compositions in (5-13) is

$$\begin{aligned} \iota \circ ((\bar{\omega}_P, \bar{\omega}_R) \circ \text{diag}(\text{id}, q) - q \circ (\omega_P, \omega_Q)) \circ \pi \\ = \iota \circ \left( (\nu, \tau) \circ \begin{pmatrix} -d_P & 0 \\ \alpha & d_Q \end{pmatrix} + d_R \otimes K \circ (\nu, \tau) \right) \circ \pi. \end{aligned}$$

Thus,  $(\nu^\bullet, \tau^\bullet)$  is a homotopy.

**5B3. Deformations.** Assume that the obstruction class  $\text{ob}(\alpha_A, \sigma)$  vanishes.

Suppose there are two extensions:

$$\alpha_B : E_0 \otimes B \rightarrow E_B \quad \text{and} \quad \beta_B : E_0 \otimes B \rightarrow F_B.$$

Resolve  $E_B$  and  $F_B$  by two very negative complex with identical terms but different differentials:  $(Q_B^\bullet, d_{E_B}^\bullet)$  and  $(Q_B^\bullet, d_{F_B}^\bullet)$ . Then, lift  $\alpha_B$  and  $\beta_B$ :

$$\begin{array}{ccc} P^\bullet \otimes B & \xrightarrow{\sim} & E_0 \otimes B \\ \downarrow \alpha_B^\bullet & & \downarrow \alpha_B \\ (Q_B^\bullet, d_{E_B}^\bullet) & \xrightarrow{\sim} & E_B \end{array} \quad \text{and} \quad \begin{array}{ccc} P^\bullet \otimes B & \xrightarrow{\sim} & E_0 \otimes B \\ \downarrow \beta_B^\bullet & & \downarrow \beta_B \\ (Q_B^\bullet, d_{F_B}^\bullet) & \xrightarrow{\sim} & F_B \end{array}$$

The differences  $d_{E_B}^i - d_{F_B}^i$  and  $\alpha_B^i - \beta_B^i$  induce maps

$$f_Q^i : Q^i \rightarrow Q^{i+1} \otimes K \quad \text{and} \quad f_P^i : P^i \rightarrow Q^i \otimes K.$$

One can show that these provide a morphism of complexes

$$(5-15) \quad (f_P^\bullet, f_Q^\bullet) : C(\alpha^\bullet) \rightarrow Q^\bullet[1] \otimes K.$$

Thus, they induce a class  $v$  defined by

$$v = [(f_P^\bullet, f_Q^\bullet)] \in \text{Ext}^1(I^\bullet, E \otimes K).$$

Conversely, if we are given an extension  $(E_B, \alpha_B)$  and a class  $v$  represented by  $(f_P, f_Q)$ , then

$$\beta_B = \alpha_B - \iota \circ f_P \circ \pi \quad \text{and} \quad d_{F_B} = d_{E_B} - \iota \circ f_Q \circ \pi$$

produce a morphism of complexes  $P^\bullet \otimes B \rightarrow (Q_B^\bullet, d_{F_B}^\bullet)$ . This induces an extension of  $(E_A, \alpha_A)$ :

$$(F_B, \beta_B) = (H^0(Q_B^\bullet, d_{F_B}^\bullet), H^0(\beta_B^\bullet)).$$

If we choose a different resolution  $R_B^\bullet$  and define  $(\bar{f}_P^\bullet, \bar{f}_R^\bullet)$  similarly as in (5-8), then  $[(f_P^\bullet, f_Q^\bullet)]$  and  $[(\bar{f}_P^\bullet, \bar{f}_R^\bullet)]$  are identified under the isomorphism

$$\mathrm{Hom}_{K(X)}(P^\bullet[1] \oplus Q^\bullet, Q^\bullet[1] \otimes K) \cong \mathrm{Hom}_{K(X)}(P^\bullet[1] \oplus R^\bullet, R^\bullet[1] \otimes K).$$

So,  $v$  is independent of the resolution  $Q_B^\bullet$ .

We next show that the difference of two equivalent extensions gives a zero class  $v$ . Indeed, suppose  $\alpha_B$  and  $\beta_B$  are equivalent, then by Lemma 2.11, there is a constant  $z \in B$  such that  $\beta_B = z\alpha_B$ . Denote the image of  $z$  in  $k$  as  $\bar{z}$ . We have proven that  $v$  is independent of resolutions. So, for our convenience, we take the same resolution  $Q_B^\bullet$  for  $E_B$  and  $F_B$ , and take  $\beta^\bullet = z\alpha^\bullet$ . Then  $f_Q^\bullet = 0$ . Furthermore,  $f_P^\bullet$  in (5-8) is homotopic to zero via homotopy

$$(0, 1 - \bar{z}) : P^{i+1} \oplus Q^i \rightarrow Q^i \otimes K.$$

Thus, the associated  $v = 0$ .

It remains to prove that if  $(h_P^\bullet, h_Q^\bullet)$  is a homotopy between  $(f_P^\bullet, f_Q^\bullet)$  and zero, then  $\alpha_B$  and  $\beta_B$  are equivalent. One can actually check:

- (i)  $\mathrm{id} - \iota \circ h_Q \circ \pi : (Q_B^\bullet, d_{E_B}^\bullet) \rightarrow (Q_B^\bullet, d_{F_B}^\bullet)$  is a morphism of complexes.
- (ii)  $(\mathrm{id} - \iota \circ h_Q \circ \pi) \circ \alpha_B = \beta_B - d_{F_B} \circ \iota \circ h_P \circ \pi - \iota \circ h_P \circ \pi \circ d_P \otimes B$ .

Hence, there is a morphism  $\phi$  commuting two families of stable pairs  $\alpha_B$  and  $\beta_B$ . Therefore, by Lemma 2.11, this is an isomorphism.

## 6. Stable pairs on surfaces

In this section, we assume that  $(X, \mathcal{O}_X(1))$  is a smooth projective surface,  $E_0$  is torsion-free,  $P$  and  $\delta$  are of degree 1. We shall demonstrate that in these cases, the moduli space of stable pairs admits a virtual fundamental class, proving Theorem 1.3.

To show the existence of the virtual fundamental class, it suffices to show that the obstruction theory is *perfect* [Behrend and Fantechi 1997; Li and Tian 1998]. That is, there is a two-term complex of locally free sheaves resolving the deformation and obstruction sheaves. In order to do this, we essentially need to show that there are no higher obstructions, which is guaranteed by the following lemma.

**Lemma 6.1.** *Fix a stable pair  $(E, \alpha)$ . Let  $I^\bullet$  denote the complex  $\{E_0 \xrightarrow{\alpha} E\}$  positioned at 0 and 1. Then*

$$\mathrm{Ext}^i(I^\bullet, E) = 0, \quad \text{unless } i = 0, 1.$$

*Proof.* The stable pair fits into an exact sequence

$$0 \rightarrow K \rightarrow E_0 \rightarrow E \rightarrow Q \rightarrow 0,$$

which can be written as a distinguished triangle

$$K \rightarrow I^\bullet \rightarrow Q[-1] \rightarrow K[1].$$

Notice that  $K$  is torsion-free and  $Q$  is 0-dimensional. Apply the functor  $\text{Hom}(-, E)$  to this triangle. The associated long exact sequence is

$$0 \rightarrow \text{Hom}(Q, E) \rightarrow \text{Ext}^{-1}(I^\bullet, E) \rightarrow 0 \rightarrow \dots \\ \rightarrow 0 \rightarrow \text{Ext}^2(I^\bullet, E) \rightarrow \text{Ext}^2(K, E) \rightarrow 0.$$

Since  $Q$  is 0-dimensional and  $E$  is pure,  $\text{Hom}(Q, E) = 0$ . Thus,  $\text{Ext}^{-1}(I^\bullet, E) = 0$ . The kernel  $K$  is torsion-free, so  $\text{Ext}^2(K, E) \cong \text{Hom}(E, K \otimes \omega_X)^\vee = 0$ . Therefore,  $\text{Ext}^2(I^\bullet, E) = 0$ .  $\square$

Using this lemma, the expected dimension of the moduli space can be easily calculated via Hirzebruch–Riemann–Roch, knowing invariants of  $E_0$ .

Now, let

$$\mathbb{I}^\bullet = \{\pi_X^* E_0 \xrightarrow{\tilde{\alpha}} \mathbb{E}\}$$

be the universal pair, according to Theorem 1.1. By Theorem 1.2, the deformation sheaf and the obstruction sheaf are calculated by

$$R\pi_* R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{E}).$$

Take a finite complex  $P^\bullet$  of locally free sheaves resolving  $\mathbb{E}$  and a finite complex  $Q^\bullet$  of very negative locally free sheaves resolving  $\mathbb{I}^\bullet$ . Take a finite, very negative locally free resolution  $A^\bullet$  of  $(Q^\bullet)^\vee \otimes P^\bullet$ . Then

$$(6-1) \quad R\pi_* R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{E}) \cong R\pi_* R\mathcal{H}om(Q^\bullet, P^\bullet) \cong R\pi_* A^\bullet.$$

Denote this complex on the moduli space as  $B^\bullet$ . By Grothendieck–Verdier duality [Hartshorne 1966; Conrad 2000],

$$B^\bullet = R\pi_* A^\bullet \cong R\pi_* R\mathcal{H}om(A^{\bullet\vee} \otimes \omega_X, \omega_X) \\ \cong R\mathcal{H}om(R\pi_*(A^{\bullet\vee} \otimes \omega_X)[-2], \mathcal{O}).$$

Moreover, notice that

$$R\pi_*(A^{\bullet\vee} \otimes \omega_X) = \pi_*(A^{\bullet\vee} \otimes \omega_X)$$

is a complex of locally free sheaves, due to the negativity of  $A^j$ 's. Thus,  $B^\bullet$  is a complex of locally free sheaves as well. Denote the differentials as  $d^i$ 's.

Next, we show that  $B^\bullet$  can be truncated to degree 0 and 1. The cohomologies of  $B^\bullet$  concentrate at degree 0 and 1, by Lemma 6.1. Suppose  $B^i$  for an  $i \geq 2$  is the last term that is nonzero. Both  $B^i$  and  $B^{i-1}$  are locally free, then  $\ker d^{i-1}$  is also locally free. Replace  $B^i$  by zero and  $B^{i-1}$  by  $\ker d^{i-1}$ . We get a new complex of locally free sheaves, which is quasi-isomorphic to  $B^\bullet$ . Inductively, we can trim  $B^\bullet$  down to degree 1. On the other side, suppose  $B^j$  for a  $j < 0$  is the first term that is nonzero. Then,  $d^j$  is injective fiberwise. Therefore,  $\text{coker } d^j$  is flat [Grothendieck 1961a, (10.2.4), Chapter 0], thus locally free. Hence, we can replace  $B^{j-1}$  by zero and  $B^j$

by  $\text{coker } d^j$  to get a new complex of locally free sheaves. Inductively,  $B^\bullet$  becomes a complex concentrated in degree 0 and 1, with cohomologies the deformation sheaf and the obstruction sheaf. Namely, we have the following exact sequence on  $S_{E_0}(P, \delta)$ :

$$0 \rightarrow \mathcal{D}ef \rightarrow B^0 \rightarrow B^1 \rightarrow \mathcal{O}bs \rightarrow 0,$$

where  $B^0$  and  $B^1$  are locally free.

Therefore, the moduli space admits a virtual fundamental class.

## 7. Examples

In this section, we study examples of moduli spaces of dimension 1 stable pairs over K3 surfaces. Let  $(X, \mathcal{O}_X(1))$  be a polarized K3 surface,  $P$  be a Hilbert polynomial of degree 1, and  $\delta$  be a positive polynomial of degree larger than 1. Let  $E_0$  be a fixed coherent sheaf over  $X$ . Then a pair  $(E, \alpha)$ , such that  $P_E = P$ , is stable if  $E$  is pure and  $\text{coker } \alpha$  has dimension 0, by Lemma 2.10.

Let  $H = c_1(\mathcal{O}(1)) \in H_2(X, \mathbb{Z})$ . Suppose the schematic support of  $E$ , which is a curve, has arithmetic genus  $h$ . There are two discrete invariants of  $E$ <sup>6</sup>:

$$(7-1) \quad \beta_h = c_1(E) \in H_2(X, \mathbb{Z}) \quad \text{and} \quad \chi(E) = 1 - h + d.$$

They are related to the Hilbert polynomial by

$$P_E(m) = (\beta_h.H)m + 1 - h + d.$$

So, with the Hilbert polynomial fixed, there are only finitely many possible  $\beta_h$ 's. The moduli space decomposes as a disjoint union:

$$S_{E_0}(P, \delta) = \coprod_{\beta_h} S_{E_0}(\beta_h, 1 - h + d),$$

where  $S_{E_0}(\beta_h, 1 - h + d)$  denote the moduli space of stable pairs satisfying (7-1).

Let  $C_h$  be a representative in the class  $\beta_h$ ; then the linear system  $|C_h|$  is isomorphic to  $\mathbb{P}^h$ . Let

$$\mathcal{C}_h \subset |C_h| \times X$$

be the universal curve.

When  $E_0 \cong \mathcal{O}_X$ , by [Pandharipande and Thomas 2010, Proposition B.8],

$$S_{\mathcal{O}_X}(\beta_h, 1 - h + d) \cong \mathcal{C}_h^{[d]},$$

where  $\mathcal{C}_h^{[d]}$  is the relative Hilbert scheme of points. If there is an ample line bundle  $H$  such that

$$(7-2) \quad C_h.H = \min\{L.H \mid L \in \text{Pic}(X), L.H > 0\},$$

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<sup>6</sup>There is a slight abuse of notation concerning  $\beta$  and  $d$ , but this is unlikely to cause confusion.

then  $S_{\mathcal{O}_X}(\beta_h, 1 - h + d)$  is a smooth scheme of dimension  $h + d$ , see [Kawai and Yoshioka 2000, Lemmas 5.117 and 5.175] or [Pandharipande and Thomas 2010, Proposition C.2].

The moduli space is not smooth in general for a higher rank  $E_0$ . For example, assume  $E_0 \cong \mathcal{O}_X^{\oplus 2}$  and the stable pair  $(E, \alpha : \mathcal{O}_X^{\oplus 2} \rightarrow E)$  maps a summand  $\mathcal{O}_X$  to 0. Then, the deformation space of this stable pair is isomorphic to

$$\mathrm{Hom}(\mathcal{O}_X \rightarrow E, E) \oplus H^0(E).$$

The dimension of  $\mathrm{Hom}(\mathcal{O}_X \rightarrow E, E)$  is  $h + d$ , while  $h^0(E)$  may vary as  $E$  varies. But when  $d$  is large, we do expect the moduli space to be smooth for higher rank  $E_0$ .

**Proposition 7.1.** *Suppose  $\beta_h$  is irreducible, i.e.,  $\beta_h$  is not a sum of two curve classes, and  $d > 2h - 2$ . Then the moduli space  $S_{\mathcal{O}_X^{\oplus r}}(\beta_h, 1 - h + d)$  is smooth of dimension  $rd + (r - 2)(1 - h) + 1$ .*

*Proof.* Apply the functor  $\mathrm{Hom}(-, E)$  to

$$I^\bullet \rightarrow \mathcal{O}_X^{\oplus r} \rightarrow E \rightarrow I^\bullet[1].$$

According to Lemma 6.1, the associated long exact sequence is

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}(E, E) \rightarrow H^0(X, E)^{\oplus r} \rightarrow \mathrm{Hom}(I^\bullet, E) \rightarrow \\ \mathrm{Ext}^1(E, E) \rightarrow H^1(X, E)^{\oplus r} \rightarrow \mathrm{Ext}^1(I^\bullet, E) \rightarrow \mathrm{Ext}^2(E, E) \rightarrow 0. \end{aligned}$$

Since  $\beta_h$  is irreducible,  $E$  is stable. Therefore,  $\mathrm{ext}^2(E, E) = \mathrm{hom}(E, E) = 1$ . When  $d > 2h - 2$ , by Serre duality,  $h^1(X, E) = h^1(C, E) = 0$  where  $C$  is the support of  $E$ . Thus, the tangent space  $\mathrm{Hom}(I^\bullet, E)$  has constant dimension  $\chi(I^\bullet, E) + 1 = rd + (r - 2)(1 - h) + 1$ .  $\square$

For every  $h \geq 0$ , there exists a K3 surface  $X_h$  and a curve class  $\beta_h \in H_2(X_h, \mathbb{Z})$ , such that  $\beta_h \cdot \beta_h = 2h - 2$  and (7-2) is satisfied, see [Kawai and Yoshioka 2000, Remark 5.110]. For each  $h \geq 0$ , we fix such  $X_h$  and  $\beta_h$ .

Kawai and Yoshioka [2000, Corollary 5.85] calculated the generating series of topological Euler characteristics of the moduli spaces.

**Theorem 7.2** (Kawai–Yoshioka). *For  $0 < |q| < |y| < 1$ , the generating series of topological Euler characteristics is*

$$\begin{aligned} \sum_{h=0}^{\infty} \sum_{d=0}^{\infty} \chi_{\mathrm{top}}(S_{\mathcal{O}_{X_h}}(\beta_h, 1 - h + d)) q^{h-1} y^{1-h+d} \\ = \left( (y^{-1/2} - y^{1/2})^2 q \prod_{n=1}^{\infty} (1 - q^n)^{20} (1 - q^n y)^2 (1 - q^n y^{-1})^2 \right)^{-1}. \end{aligned}$$

Next, we consider stable pairs over  $X_h$  of the form

$$\alpha : L_h \rightarrow E,$$

where  $L_h$  is a line bundle with the first Chern class  $c_1(L_h) = l\beta_h$ . Such a stable pair is equivalent to  $\mathcal{O}_X \rightarrow E \otimes L_h^{-1}$ . Notice that  $c_1(E \otimes L_h^{-1}) = \beta_h$  and  $\chi(E \otimes L_h^{-1}) = 1 - h + d - 2l(h - 1)$ . Therefore,

$$S_{L_h}(\beta_h, 1 - h + d) \cong S_{\mathcal{O}_X}(\beta_h, 1 - h + d - 2l(h - 1)).$$

If  $\alpha \neq 0$ , then  $d \geq 2l(h - 1)$ . The generating series is

$$\begin{aligned} & \sum_{h=0}^{\infty} \sum_{d=2l(h-1)}^{\infty} \chi_{\text{top}}(S_{L_h}(\beta_h, 1 - h + d)) q^{h-1} y^{d+1-h} \\ &= \sum_{h=0}^{\infty} \sum_{d=0}^{\infty} \chi_{\text{top}}(S_{\mathcal{O}_X}(\beta_h, 1 - h + d)) (qy^{2l})^{h-1} y^{d+1-h} \\ &= \left( (y^{-\frac{1}{2}} - y^{\frac{1}{2}})^2 qy^{2l} \prod_{n=1}^{\infty} (1 - q^n y^{2nl})^{20} (1 - q^n y^{2nl+1})^2 (1 - q^n y^{2nl-1})^2 \right)^{-1}. \end{aligned}$$

Now, we consider stable pairs over  $X_h$  of the form

$$\alpha : \bigoplus_i L_{i,h} \rightarrow E,$$

where  $L_{i,h}$  is a line bundle with  $c_1(L_{i,h}) = l_i \beta_h$ . The proof of Proposition 7.1 can also show that the moduli space is smooth when  $d$  is large compared to  $l_i$  and  $h$ . Let  $\mathbb{G}_m$  act on direct summands with distinct weights; then there is a natural  $\mathbb{G}_m$ -action on the moduli space  $S_{X_h}^{\oplus L_{i,h}}(\beta_h, 1 - h + d)$ . A morphism  $\bigoplus L_{i,h} \rightarrow E$  is fixed under the action if and only if exactly one summand  $L_{i,h}$  is mapped to  $E$  nontrivially. Thus, we have the following the fixed loci:

$$S_{\bigoplus L_{i,h}}(\beta_h, 1 - h + d)^{\mathbb{G}_m} \cong \coprod_i S_{L_{i,h}}(\beta_h, 1 - h + d).$$

When  $\alpha \neq 0$ ,  $d \geq \min\{2l_i(h - 1)\}$ . To calculate the Euler characteristics, we can use the localization formula, even when the moduli space is not smooth [Lawson and Yau 1987]. Then,

$$\begin{aligned} & \sum_h \sum_d \chi_{\text{top}}(S_{\bigoplus L_{i,h}}(\beta_h, 1 - h + d)) q^{h-1} y^{d+1-h} \\ &= \sum_i \left( (y^{-\frac{1}{2}} - y^{\frac{1}{2}})^2 qy^{2l_i} \prod_{n=1}^{\infty} (1 - q^n y^{2nl_i})^{20} (1 - q^n y^{2nl_i+1})^2 (1 - q^n y^{2nl_i-1})^2 \right)^{-1}. \end{aligned}$$

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# SPARK DEFICIENT GABOR FRAMES

ROMANOS-DIOGENES MALIKIOSIS

The theory of Gabor frames of functions defined on finite abelian groups was initially developed in order to better understand the properties of Gabor frames of functions defined over the reals. However, during the last twenty years the topic has acquired an interest of its own. One of the fundamental questions asked in this finite setting is on the existence of full spark Gabor frames. In a previous paper, we proved the existence of such frames when the underlying group is finite cyclic, and constructed some examples. In this paper, we resolve the noncyclic case; in particular, we show that there can be no full spark Gabor frames of windows defined on finite abelian noncyclic groups. We also prove that all eigenvectors of certain unitary matrices in the Clifford group in odd dimensions generate spark deficient Gabor frames. Finally, similarities between the uncertainty principles concerning the finite-dimensional Fourier transform and the short-time Fourier transform are discussed.

## 1. Introduction

The Gabor frame of a function  $f \in L^2(\mathbb{R})$  is the set of all time-frequency translates of  $f$ , that is, the set of all functions of the form  $e^{2\pi ixy} f(x-t)$ , for  $y, t \in \mathbb{R}$ , and it is a fundamental concept in time-frequency analysis and frame theory [Pfander 2013]. The function  $f$  usually represents a signal,  $t$  the time delay, and the pointwise multiplication by  $e^{2\pi ixy}$  is the frequency “shift”. Through sampling and periodization [Christensen 2003], one passes to the finite version of a Gabor frame, namely the shift-frequency translates of a complex function defined on a finite cyclic group. Even though finite-dimensional Gabor frames were studied in order to analyze the properties of continuous signals, they later developed an interest of their own.

Up to multiplication by roots of unity, a finite-dimensional Gabor frame is the same as a *Weyl–Heisenberg* orbit, and this terminology is much more prevalent in mathematical physics and quantum information theory. A conjecture by Zauner [1999] states that for every dimension  $N$  there are vectors (called “fiducials”) whose

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WH orbit is equiangular. This means that the expression  $|\langle u, v \rangle|$  is constant for every pair of distinct vectors  $u, v$  within this orbit. This is also known as the SIC-POVM problem which has attracted a lot of attention lately due to the vast connections to scientific areas such as quantum cryptography [Renes 2005], quantum tomography [Scott 2006], and algebraic number theory, especially Hilbert's 12th problem for real quadratic fields [Appleby et al. 2013; 2015; 2016; 2017]. Such a WH orbit would then produce the maximal possible number of vectors in  $\mathbb{C}^N$  that are pairwise equiangular, namely  $N^2$  [Strohmer and Heath 2003]. Yet another terminology that appears for this phenomenon is *maximal equiangular tight frame* (or maximal ETF for short) [Fickus 2009], which is a special case of the packing problem in the setting of projective spaces. The interest in algebraic construction of families of ETFs has also increased due to applications to signal processing [Fickus et al. 2012; Iverson et al. 2016; Jasper et al. 2014].

A conjecture by Heil, Ramanathan, and Topiwala [Heil et al. 1996] states that any finite set of a Gabor frame of a nonzero  $f \in L^2(\mathbb{R})$  is linearly independent, and it is still open. Similar questions can be raised when the function  $f$  is defined on a finite abelian group  $G$ . In this case, the Gabor frame consists of  $|G|^2$  elements in a  $|G|$ -dimensional space, so it is not possible that they are linearly independent. Instead, we require that any selection of  $|G|$  vectors is linearly independent, which is the definition of the full spark property.

**Definition 1.1.** Let

$$U = \{u_1, \dots, u_M\} \subseteq \mathbb{C}^N$$

with  $M \geq N$ . The set  $U$  is called *full spark* when every selection of  $N$  vectors from  $U$  is linearly independent; otherwise,  $U$  is called *spark deficient*.

The discrete analogue of the HRT conjecture claims that the Gabor frame of  $f \in \mathbb{C}^G$  is full spark for almost all  $f$ , when  $G$  is cyclic [Lawrence et al. 2005]. This problem has been completely solved by the author [Malikiosis 2015]. While the techniques utilized to attack the HRT conjecture are analytic in nature, various algebraic techniques are needed for the discrete counterpart, such as Chebotarev's theorem on Fourier minors. The idea of the proof is as follows: consider  $f$  as a column vector in  $\mathbb{C}^N$ , where  $|G| = N$ , and consider the  $N \times N^2$  matrix whose columns are precisely the elements of the Gabor frame of  $f$ , denoted by  $V_f$ . The Gabor frame generated by  $f$  is then full spark if and only if every  $N \times N$  minor of  $V_f$  is nonzero. Every such minor is a homogeneous polynomial on the coordinates of  $f$ ; the basic ingredient of the proof is to show that there is a monomial appearing with nonzero coefficient in every such minor. For the case where  $N$  is a prime, this was accomplished in [Lawrence et al. 2005] through Chebotarev's theorem, which asserts that every minor of the  $N \times N$  discrete Fourier matrix is nonzero. For the case where  $N$  is composite, a probabilistic argument by the author [Malikiosis 2015]

was used in order to show the existence of monomials with nonzero coefficient in every minor. Furthermore, the author proved that almost every  $f \in \mathbb{C}^G$  generates a full spark Gabor frame, and explicitly constructed such frames, while previous proofs were only existential.

For noncyclic groups, it was previously only known that full spark Gabor frames *do not exist* for functions defined on the Klein group,  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  [Pfander 2013]. We shall extend this argument to any finite abelian noncyclic group, in the following way: first, we show that the full spark property is hereditary with respect to the group. Therefore, in order to show that no full spark Gabor frame exists, it suffices to restrict our attention to groups of the form  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ , for  $p$  odd prime. Thus is proved the first main result of this paper:

**Theorem 1.2.** *Let  $G$  be a finite abelian, noncyclic group. Then, for any  $f \in \mathbb{C}^G$ , the Gabor frame generated by  $f$  is spark deficient.*

In relation to the SIC-POVM problem we will revisit the cyclic case and prove that all eigenvectors of Clifford unitaries whose (projective) order is not coprime to the dimension  $N$ , for  $N$  odd, generate spark deficient Gabor frames, extending some results in [Dang et al. 2013]. This shows that there is not in principle any relation between these two basic properties of a Gabor frame, namely equiangularity and the full spark property.

Lastly, we investigate a possible connection between uncertainty principles with respect to the discrete and short-time Fourier transforms. Uncertainty principles provide a measure of localization of signals whose various transforms (e.g., Fourier) are well-localized. When these signals are defined over a finite abelian group, localization is usually measured by the size of the support, leading to classical and new versions of uncertainty principles with respect to the Fourier transform [Meshulam 2006; Tao 2005]. This sort of principle appears in applications to sparse signal recovery, and sparse matrix identification [Candès et al. 2006; Krahmer et al. 2008; Pfander 2007], among others.

The paper is organized as follows: in Section 2, we will give the definitions and the necessary background related to the results of this paper. In Section 3, we will prove that full spark Gabor frames do not exist over finite abelian noncyclic groups. Section 4 revisits the cyclic case, where we find some special vectors that generate spark deficient Gabor frames, and Section 5 deals with uncertainty principles.

## 2. Background

**2A. Notation.** Throughout this note,  $G$  will denote a finite abelian group written additively, and  $\mathbb{C}^G$  will denote the set of all complex valued functions defined on  $G$ . An element  $f \in \mathbb{C}^G$  will interchangeably be viewed as a vector in  $\mathbb{C}^N$ , where  $N = |G|$ , and as a function  $f : G \rightarrow \mathbb{C}$ .  $\mathbb{C}^N$  is equipped with an inner product  $\langle \cdot, \cdot \rangle$ ,

defined as follows:

$$\langle x, y \rangle = \sum_{i=1}^N x_i \bar{y}_i,$$

for  $x = (x_1, \dots, x_N)$ ,  $y = (y_1, \dots, y_N)$ . Only in Section 4B will we use the bra-ket notation,  $\langle x|y \rangle$ , with the caution that complex conjugation is taken on the coordinates of  $x$ . We remind that  $|x \rangle$  denotes a column vector in  $\mathbb{C}^N$ , and  $\langle x|$  is its conjugate transpose; hence  $|x \rangle \langle x|$  is the 1-dimensional projector onto  $|x \rangle$ .

Furthermore, we decided to use  $\mathbb{Z}/N\mathbb{Z}$  for the ring of residues mod  $N$ , and reserve  $\mathbb{Z}_p$  for the ring of  $p$ -adic integers. Similarly,  $\mathbb{Q}_p$  denotes the field of  $p$ -adic rational numbers.

For any  $f \in \mathbb{C}^N$  denote by  $\hat{f}$  the (unnormalized) Fourier transform of  $f$ ; that is,  $\hat{f} = W_N f$ , where  $W_N = (\omega^{ij})_{i,j=0}^{N-1}$ , the character table of  $\mathbb{Z}/N\mathbb{Z}$ , with  $\omega = e^{2\pi i/N}$ , and finally, let  $\|f\|_0$  denote the cardinality of the support of  $f$ .

Two operators  $U$  and  $V$  on  $\mathbb{C}^N$  will be equal up to a phase if  $U = e^{i\theta} V$ ; this will also be denoted as

$$U \doteq V.$$

The *projective* order of an operator  $U$  is then defined to be the smallest nonnegative integer  $m$  for which  $U^m \doteq I$ . Finally, the conjugate transpose of  $U$  is denoted by  $U^*$ .

**2B. Definitions.** For any  $x \in G$  and  $\xi \in \widehat{G}$ , define the operators  $T_x, M_\xi : \mathbb{C}^G \rightarrow \mathbb{C}^G$ , with  $T_x f(g) = f(g - x)$  and  $M_\xi f(g) = \xi(g) f(g)$ , for any  $f \in \mathbb{C}^G$ ,  $g \in G$ . The  $T_x$  are called *translation operators*, and the  $M_\xi$  *modulation operators*. For any  $\lambda = (x, \xi) \in G \times \widehat{G}$  the operators  $\pi(\lambda) = M_\xi T_x$  are called *time-frequency shift operators*. We have

$$M_\xi T_x = \xi(x) T_x M_\xi,$$

or, in other words,  $M_\xi$  and  $T_x$  commute up to a phase. From this fact we get a faithful projective representation

$$\rho : G \times \widehat{G} \rightarrow \text{PGL}(\mathbb{C}^G),$$

which is also irreducible [Feichtinger et al. 2009; Pfander 2013].

For a subset  $\Lambda \subseteq G \times \widehat{G}$  and  $f \in \mathbb{C}^G \setminus \{0\}$ , the set

$$(f, \Lambda) = \{\pi(\lambda) f \mid \lambda \in \Lambda\}$$

is called a *Gabor system*; if it spans  $\mathbb{C}^G$ , it is called a *Gabor frame*. This certainly happens when  $\Lambda = G \times \widehat{G}$  due to the irreducibility of  $\rho$ ; in this case, it is also called a *Weyl–Heisenberg orbit*.

**Definition 2.1.** A set  $\Phi$  of  $M$  vectors in  $\mathbb{C}^N$  is called a *frame* if it spans  $\mathbb{C}^N$ . In this case, we must have  $M \geq N$ . The *spark* of  $\Phi$ , denoted by  $\text{sp}(\Phi)$ , is the size of the smallest linearly dependent subset of  $\Phi$ .

A frame  $\Phi$  is full spark if and only if every set of  $N$  elements of  $\Phi$  is a basis, or equivalently  $\text{sp}(\Phi) = N + 1$ , otherwise it is spark deficient. Other definitions are also found in literature; for example, in this case we also say that the vectors of  $\Phi$  are in *general linear position*, or also that  $\Phi$  possesses the *Haar property* [Pfander 2013].

**Definition 2.2.** For a window  $\varphi \in \mathbb{C}^G$ ,  $|G| = N$ , let  $V_\varphi$  denote the  $N \times N^2$  matrix whose columns are the shift-frequency translates of  $\varphi$ , also called the *synthesis operator*. The operator  $V_\varphi^* : \mathbb{C}^G \rightarrow \mathbb{C}^{G \times \widehat{G}}$  is called the *analysis operator*, or the *short-time Fourier transform* with window  $\varphi$ , defined by

$$V_\varphi^* f = (\langle f, M_\xi T_x \varphi \rangle)_{(x, \xi) \in G \times \widehat{G}}.$$

The term “window” makes much more sense in the continuous setting, whence it originated. In signal processing, one analyzes a signal  $f \in L^2(\mathbb{R})$  by integrating against elements of a frame (e.g., Gabor frames, wavelets, etc.) generated by a well-localized function  $\varphi$ . Typical examples of well-localized functions include functions supported on an interval (thus examining the given function on a small *window of time*), or with very fast decay, such as Gaussian functions; it should be emphasized that Gabor himself first applied Gabor frames on Gaussian window functions [Gabor 1946; Pfander 2013].

This term carries on to the discrete setting as well, however, we should note that the terms “window”, “vector”, and “function” (defined over a finite abelian group) are interchangeable in what follows.

**2C. Gabor systems of  $|G| = N$  vectors.** The Gabor system  $(f, \Lambda)$  with  $|\Lambda| = N$  is linearly independent if and only if the determinant of the matrix whose columns consist of the coordinates of the vectors  $\pi(\lambda)f$ ,  $\lambda \in \Lambda$ , is nonzero. This matrix is denoted by  $D_\Lambda$ , and is well-defined up to permutation of its columns. The determinant is denoted by  $P_\Lambda = \det(D_\Lambda)$ , and is well-defined up to a sign, so it makes sense to ask whether  $P_\Lambda$  is nonzero or not.

The most important property of  $P_\Lambda$ , however, is the fact that it is a homogeneous polynomial of degree  $N$  in  $N$  variables, when the coordinates of  $f$  are viewed as independent variables. So, the existence of an element  $f$  such that  $(f, \Lambda)$  is linearly independent happens precisely when  $P_\Lambda$  is a nonzero polynomial. Investigating the properties of these polynomials  $P_\Lambda$  sheds light on the existence of Gabor frames in general linear position.

A first crucial observation regarding linear independence, comes from the following:

**Proposition 2.3.** *There is a full spark Gabor frame defined over  $G$ , if and only if, for every  $\Lambda \subseteq G \times \widehat{G}$  with  $|\Lambda| = N$  there is an  $f \in \mathbb{C}^G$  such that  $(f, \Lambda)$  is linearly independent. Moreover, either all windows  $\varphi \in \mathbb{C}^G$  generate spark deficient Gabor frames, or almost all windows generate full spark Gabor frames.*

*Proof.* One direction follows from definition: if  $(f, G \times \widehat{G})$  is full spark, then obviously every Gabor system  $(f, \Lambda)$  is linearly independent, for  $|\Lambda| = N$ . On the other hand, if for every  $\Lambda \subseteq G \times \widehat{G}$  with  $|\Lambda| = N$  there is some  $f \in \mathbb{C}^G$  such that  $(f, \Lambda)$  is linearly independent, this means that all such polynomials  $P_\Lambda$  are nonzero. The zero set of every such polynomial is of Lebesgue measure zero, and since they are finitely many, this yields that almost any  $f \in \mathbb{C}^G$  avoids the zero set of these polynomials, hence  $(f, G \times \widehat{G})$  is full spark.

For the second part, we observe that if at least one of the polynomials  $P_\Lambda$  is zero, then all Gabor frames defined over  $G$  are spark deficient. Otherwise, as we have already shown, almost all Gabor frames are full spark.  $\square$

**2D. The Weyl–Heisenberg and Clifford groups.** We restrict our attention to cyclic groups  $G = \mathbb{Z}/N\mathbb{Z}$  of odd order, for convenience, as the results of this subsection will only be used towards the construction of spark deficient Gabor frames over cyclic groups. The group generated by the translation and modulation operators is

$$\{\omega^k M^b T^a \mid a, b, k \in \mathbb{Z}/N\mathbb{Z}\},$$

where  $\omega = e^{2\pi i/N}$ ,  $T = T_1$  (see Section 2B) and  $M$  is the operator with the property  $Mf(g) = \omega^g f(g)$  for all  $g \in \mathbb{Z}/N\mathbb{Z}$  and  $f \in \mathbb{C}^N$ , and is called the *Weyl–Heisenberg* group of  $G$ . Sometimes these representatives over the center are considered [Appleby 2005; Dang et al. 2013; Zauner 1999]:

$$D_\lambda = \tau^{\lambda_1 \lambda_2} T^{\lambda_1} M^{\lambda_2},$$

where  $\lambda = (\lambda_1, \lambda_2) \in (\mathbb{Z}/N\mathbb{Z})^2$ ,  $\tau = \omega^{(N+1)/2}$ .

It is known that all irreducible projective representations of  $(\mathbb{Z}/N\mathbb{Z})^2$  of dimension  $N$  are unitarily equivalent to  $\rho$  [Weyl 1931] (see also [Feichtinger et al. 2009, Proposition 3.2]). The normalizer of the Weyl–Heisenberg group in the group of unitary matrices in  $N$  dimensions is called the *Clifford group*, denoted by  $C(N)$ . The quotient of  $C(N)$  by the Weyl–Heisenberg group is isomorphic to  $\text{SL}(2, \mathbb{Z}/N\mathbb{Z})$ , hence  $\rho$  can be extended to a faithful irreducible projective representation of  $(\mathbb{Z}/N\mathbb{Z})^2 \rtimes \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$ , which we shall also denote by  $\rho$ , abusing notation. Restricting this representation to the right factor,  $\text{SL}(2, \mathbb{Z}/N\mathbb{Z})$ , we get a projective representation  $F \mapsto U_F$ , for  $F \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$ . The unitary matrices  $U_F$  act on the Weyl–Heisenberg group by conjugation:

$$U_F D_\lambda U_F^* = D_{F\lambda}.$$

More precisely, the following is true:

**Theorem 2.4** [Appleby 2005, Theorem 1,  $N$  odd]. *There is a unique isomorphism*

$$f : (\mathbb{Z}/N\mathbb{Z})^2 \rtimes \text{SL}(2, \mathbb{Z}/N\mathbb{Z}) \rightarrow C(N)/I(N)$$

with the property  $UD_\lambda U^* = \omega^{[\varphi, F\lambda]} D_{F\lambda}$  for any  $U \in f(\varphi, F)$ , where  $I(N)$  is the center of  $C(N)$ , and  $[\varphi, \chi] = \varphi_2\chi_1 - \varphi_1\chi_2$ .

This yields the following theorem:

**Theorem 2.5.** *For  $N$  odd, there is a unique faithful irreducible projective representation of  $(\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$  of dimension  $N$ , up to unitary equivalence.*

*Proof.* Let  $\rho$  be the standard representation of  $(\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$  defined in the beginning of this subsection, and let  $\tilde{\rho}$  denote another representation of dimension  $N$  with the same properties. By Weyl's theorem [Feichtinger et al. 2009; Weyl 1931], we may assume without loss of generality that

$$\rho|_{(\mathbb{Z}/N\mathbb{Z})^2} = \tilde{\rho}|_{(\mathbb{Z}/N\mathbb{Z})^2}.$$

Since the image of  $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$  acts by conjugation on the image of  $(\mathbb{Z}/N\mathbb{Z})^2$ , the image of  $\tilde{\rho}$  will also be contained in  $C(N)$ . According to Theorem 2.4, for any  $F \in \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ ,  $\rho(F)$  and  $\tilde{\rho}(F)$  should differ by an element of  $\rho((\mathbb{Z}/N\mathbb{Z})^2)$ , that is

$$\tilde{\rho}(F) \doteq D_\varphi U_F.$$

We will investigate the possibilities of  $\varphi$  when  $F = S$  or  $T$ , the generators of  $\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ ,

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which satisfy  $S^2 = (ST)^3 = -I$ , as well as when  $F = -I$ . Assume therefore, that

$$\tilde{\rho}(T) \doteq D_\chi U_T, \quad \tilde{\rho}(S) \doteq D_\psi U_S, \quad \tilde{\rho}(-I) \doteq D_\mu U_{-I}.$$

Since  $\tilde{\rho}(S)^2 \doteq \tilde{\rho}(-I)$ , we must have

$$\tilde{\rho}(-I) \doteq D_\mu U_{-I} \doteq (D_\psi U_S)^2 = D_{\psi+S\psi} U_{-I},$$

hence

$$\mu = (I + S)\psi.$$

On the other hand,  $\tilde{\rho}(-T) \doteq \tilde{\rho}(-I)\tilde{\rho}(T) \doteq \tilde{\rho}(T)\tilde{\rho}(-I)$ , whence

$$D_{\chi+T\mu} U_{-T} \doteq \tilde{\rho}(T)\tilde{\rho}(-I) \doteq \tilde{\rho}(-I)\tilde{\rho}(T) \doteq D_{\mu-\chi} U_{-T},$$

therefore

$$2\chi = (I - T)\mu,$$

thus

$$2\chi = (I - T)(I + S)\psi.$$

Now, let

$$\lambda = -(I - S)^{-1}\psi = -2^{-1}(I + S)\psi,$$

so that

$$\chi = -(I - T)\lambda, \quad \psi = -(I - S)\lambda.$$

Then,

$$D_\lambda(D_\chi U_T)D_\lambda^* \doteq D_{\lambda+\chi-T\lambda}U_T = U_T$$

and

$$D_\lambda(D_\psi U_S)D_\lambda^* \doteq D_{\lambda+\psi-S\lambda}U_S = U_S,$$

while obviously  $D_\lambda D_\varphi D_\lambda^* \doteq D_\varphi$ , thus proving that

$$D_\lambda \tilde{\rho}(\varphi, F) D_\lambda^* \doteq \rho(\varphi, F),$$

for all  $(\varphi, F) \in \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z}) \rtimes (\mathbb{Z}/N\mathbb{Z})^2$ , or in other words,  $\rho$  and  $\tilde{\rho}$  are unitarily equivalent, completing the proof.  $\square$

Another way to obtain such a representation is the following: let

$$N = p_1^{r_1} \cdots p_s^{r_s}$$

be the prime factorization of  $N$ . By the Chinese remainder theorem we obtain

$$(\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z}) \cong \prod_{i=1}^s (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{Z}/p_i^{r_i}\mathbb{Z}),$$

and let

$$\mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z}) \ni F \mapsto (F_i)_{1 \leq i \leq s} \in \prod_{i=1}^s \mathrm{SL}(2, \mathbb{Z}/p_i^{r_i}\mathbb{Z})$$

be the natural map according to the isomorphism above; that is,  $F_i$  is the matrix obtained by reducing the entries of  $F$  mod  $p_i^{r_i}$ . Assuming that  $V_i \cong \mathbb{C}^{p_i^{r_i}}$  is the faithful irreducible projective representation of  $(\mathbb{Z}/p_i^{r_i}\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{Z}/p_i^{r_i}\mathbb{Z})$  constructed as above, then we also see that  $V_1 \otimes V_2 \otimes \cdots \otimes V_s$  is also a faithful irreducible representation of  $(\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$  (see [Serre 1977, Theorem 10]), and hence unitarily equivalent to the standard one. This shows that  $U_F$  is, up to unitary equivalence, equal to the Kronecker product of the  $U_{F_i}$ , thus

$$(2-1) \quad \mathrm{Tr} U_F = \prod_{i=1}^s \mathrm{Tr} U_{F_i},$$

a fact also pointed out in [Dang et al. 2013].

### 3. Gabor frames over noncyclic groups

First we show that the full spark property is hereditary.

**Lemma 3.1.** *Let  $G$  be a finite abelian group and  $H$  a subgroup, such that no windows defined on  $H$  generate full spark Gabor frames. Then, there exist no windows defined on  $G$  that generate full spark Gabor frames.*



*Proof.* By hypothesis, there exists a set of pairs  $(h_i, \xi_i) \in H \times \widehat{H}$ ,  $1 \leq i \leq |H|$ , such that the vectors  $M_{\xi_i} T_{h_i} \varphi$  are linearly dependent for any choice of  $\varphi \in \mathbb{C}^H$ . Now, extend the characters  $\xi_i$  to  $G$  in all possible ways. In this way, we obtain pairs in  $G \times \widehat{G}$  of the form  $(h, \xi)$ , where  $h = h_i$  and  $\xi|_H = \xi_i$ , for some  $i$ ; the number of these pairs is exactly  $|G|$ , as there are  $|G/H|$  ways to extend a character of  $H$  to a character of  $G$ .

Next, consider an arbitrary window  $\psi \in \mathbb{C}^G$ . Since the vectors  $M_{\xi_i} T_{h_i} \psi|_H$  are linearly dependent on  $\mathbb{C}^H$ , there is a nonzero vector  $f \in \mathbb{C}^H$  such that all inner products  $\langle M_{\xi_i} T_{h_i} \psi_H, \bar{f} \rangle$  equal 0. Denote by  $F$  the unique window of  $\mathbb{C}^G$  for which we have  $F|_H = f$  and  $\text{supp}(F) \subseteq H$  (also a nonzero window). Then, for all  $i$  and  $\xi \in \widehat{G}$  with  $\xi|_H = \xi_i$  we have

$$\begin{aligned} \langle M_{\xi} T_{h_i} \psi, \bar{F} \rangle &= \sum_{g \in G} \xi(g) \psi(g - h_i) F(g) \\ &= \sum_{h \in H} \xi_i(h) \psi(h - h_i) f(h) = \langle M_{\xi_i} T_{h_i} \psi_H, \bar{f} \rangle = 0, \end{aligned}$$

which shows that these  $|G|$  pairs  $(h, \xi) \in G \times \widehat{G}$  always give linearly dependent vectors, as desired.  $\square$

Since we wish to prove that there exist no windows over any finite abelian noncyclic group that generate full spark Gabor frames, it suffices to do so for groups of the form  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ , for  $p$  prime, due to the fundamental theorem of finite abelian groups; if such a group is noncyclic, then it must have a subgroup of this form. Thus, Theorem 1.2 follows directly from Theorem 3.3, which establishes the result for groups of the form  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ .

When  $p = 2$ , this has already been proven, therefore by Lemma 3.1 we know that any window defined on a group containing a copy of the Klein group as a subgroup cannot generate a full spark Gabor frame. We provide an alternative proof of this statement, more in line with the proof of Lemma 3.1, which also gives us an estimate on the minimum value of  $\|V_{\varphi}^* f\|_0$ .

**Theorem 3.2.** *Let  $G$  be a finite abelian group that has a subgroup isomorphic to the Klein-four group. Then, there are no Gabor frames  $(f, G \times \widehat{G})$  in general linear position; furthermore, we have*

$$\min \|V_{\varphi}^* f\|_0 \leq N^2 - 3N/2.$$

*Proof.* Let  $K$  be the subgroup of  $G$  isomorphic to the Klein-four group. For  $f \in \mathbb{C}^G$  define  $\bar{f}$  satisfying  $\bar{f}(g) = \overline{f(g)}$  for all  $g \in G$ , and define on  $\mathbb{C}^G$  an inner product given by

$$\langle f, h \rangle = \sum_{g \in G} f(g) \bar{h}(g).$$

By standard character theory, there are three nontrivial characters on  $K$ , and each one of them extends to  $N/4$  characters on  $G$ , where  $N = |G|$ . In total, there are  $3N/4$  characters on  $G$  whose restriction on  $K$  is nontrivial.

Let  $\xi$  be such a character, and let  $f \in \mathbb{C}^G \setminus \{0\}$  be arbitrary. Let  $a \in K$  be such that  $\xi(a) = -1$ ; there are two such elements of  $K$ , and so we consider the Gabor system consisting of time-frequency translates of the form

$$M_\xi T_a f, \quad \xi \text{ nontrivial on } K, a \in K \text{ with } \xi(a) = -1.$$

This system has  $3N/2 > N$  elements; we will show that each one of them is orthogonal to  $\bar{f}$ , and therefore the full Gabor frame  $(f, G \times \widehat{G})$  cannot be in general linear position. Indeed,

$$\begin{aligned} \langle M_\xi T_a f, \bar{f} \rangle &= \sum_{g \in G} \xi(g) f(g-a) f(g) \\ &= \sum_{g \in G} \xi(g+a) f(g) f(g+a) \\ &= \sum_{g \in G} \xi(g) \xi(a) f(g-a) f(g) \\ &= - \sum_{g \in G} \xi(g) f(g-a) f(g) \\ &= - \langle M_\xi T_a f, \bar{f} \rangle, \end{aligned}$$

so  $\langle M_\xi T_a f, \bar{f} \rangle = 0$ . This also shows that  $\|V_f \bar{f}\|_0 \leq N^2 - 3N/2$ , proving the second part of the theorem.  $\square$

**Theorem 3.3.** *There are no full spark Gabor frames over  $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ , for  $p$  prime.*

*Proof.* The case  $p = 2$  has already been proven, so we may assume that  $p$  is odd. As in the previous two proofs, we consider an arbitrary window  $z \in \mathbb{C}^G$ , and then try to find a nonzero vector that is orthogonal to at least  $|G| = p^2$  shift-frequency translates of  $z$ . In order to find this desirable set of translates, we arrange the coordinates of  $z$  in an array; here, we identify  $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$  with the finite field  $\mathbb{F}_q$ ,  $q = p^2$ , and  $\theta \in \mathbb{F}_q \setminus \mathbb{F}_p$ :

$$(3-1) \quad \begin{pmatrix} z_0 & z_\theta & \cdots & z_{-\theta} \\ z_1 & z_{\theta+1} & \cdots & z_{-\theta+1} \\ \vdots & \vdots & \ddots & \vdots \\ z_{-1} & z_{-\theta-1} & \cdots & z_{-\theta-1} \end{pmatrix}.$$

We denote this  $p \times p$  matrix by  $Z$ . The column vectors in  $\mathbb{C}^{\mathbb{F}_p}$  from left to right are denoted by  $Z_0, Z_\theta, \dots, Z_{-\theta}$ , respectively, and similarly, the row vectors by

$Z'_0, Z'_1, \dots, Z'_{p-1}$ . Next, consider the vector  $x \in \mathbb{C}^{\mathbb{F}_q}$  whose matrix representation is precisely  $X = (\text{adj } Z)^*$ , where  $\text{adj } Z$  denotes the *adjugate* matrix of  $Z$ ; we denote its columns by  $X_0, X_\theta, \dots, X_{-\theta}$  and its rows by  $X'_0, X'_1, \dots, X'_{p-1}$ . The vector  $x$  could be zero, however this happens for a set of Lebesgue measure zero. In particular,  $x$  is zero precisely when all the  $(p-1) \times (p-1)$  minors of  $Z$  are zero, but all of them are nonzero polynomials on the coordinates of  $z$ , which shows that for almost all choices of  $z$ ,  $x$  is nonzero. If we prove that the Gabor frames with windows  $z$  possessing that property are spark deficient, then by Proposition 2.3 we get that all Gabor frames over  $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$  are spark deficient.

We have

$$\det Z \cdot I = Z^T \bar{X} = Z \bar{X}^T.$$

The  $(a, b)$  entry of  $Z^T \bar{X}$  is  $\langle Z_{a\theta}, X_{b\theta} \rangle$ , and similarly for  $Z \bar{X}^T$  is  $\langle Z'_a, X'_b \rangle$ . We thus obtain

$$(3-2) \quad \langle Z_{a\theta}, X_{b\theta} \rangle = \langle Z'_a, X'_b \rangle = \delta_{ab} \det Z,$$

for every  $a, b \in \mathbb{F}_p$ , where  $\delta_{ab}$  is the usual Kronecker delta. Then, for every  $a \in \theta\mathbb{F}_p^*$  and  $\xi \in \widehat{\mathbb{F}_q}$  with  $\xi|_{\mathbb{F}_p} = \mathbf{1}_{\mathbb{F}_p}$ , we get, due to (3-2),

$$\langle M_\xi T_a z, x \rangle = \sum_{b \in \mathbb{F}_p} \xi(b\theta) \langle Z_{b\theta-a}, X_{b\theta} \rangle = 0.$$

This number of shift-frequency translates is  $p(p-1)$ , and we have just established that  $x$  is orthogonal to all of them. Furthermore, if  $a \in \mathbb{F}_p^*$  and  $\xi \in \widehat{\mathbb{F}_q}$  with  $\xi|_{\mathbb{F}_p} = \mathbf{1}_{\theta\mathbb{F}_p}$ , we also get, due to (3-2),

$$\langle M_\xi T_a z, x \rangle = \sum_{b \in \mathbb{F}_p} \xi(b) \langle Z'_{b-a}, X'_b \rangle = 0.$$

So far we have  $2p(p-1) > p^2$  translates of  $z$  orthogonal to  $x$ , so this already takes care of the spark deficiency of any Gabor frame over  $G$ . We will find more translates orthogonal to  $x$ ; let's put  $a = 0$  and  $\xi \in \widehat{\mathbb{F}_q}$  with  $\xi|_{\mathbb{F}_p} = \mathbf{1}_{\mathbb{F}_p}$ , but  $\xi \neq \mathbf{1}_{\mathbb{F}_q}$ . Then, again by (3-2), we have

$$\langle M_\xi z, x \rangle = \sum_{b \in \mathbb{F}_p} \xi(b\theta) \langle Z_{b\theta}, X_{b\theta} \rangle = \det Z \sum_{b \in \mathbb{F}_p} \xi(b\theta) = 0,$$

since  $\xi|_{\theta\mathbb{F}_p} \neq \mathbf{1}_{\theta\mathbb{F}_p}$ . This number of pairs is exactly  $p-1$ .

Next, we still consider  $a = 0$ , but  $\xi \in \widehat{\mathbb{F}_q}$  satisfies with  $\xi|_{\mathbb{F}_p} \neq \mathbf{1}_{\mathbb{F}_p}$  and  $\xi|_{\theta\mathbb{F}_p} = \mathbf{1}_{\theta\mathbb{F}_p}$ . Then,

$$\langle M_\xi z, x \rangle = \sum_{b \in \mathbb{F}_p} \xi(b) \langle Z'_b, X'_b \rangle = \det Z \sum_{b \in \mathbb{F}_p} \xi(b) = 0,$$

by (3-2), thus giving us another  $p-1$  orthogonal shift-frequency translates of  $z$  orthogonal to  $x$ . In total, there are  $2(p+1)(p-1) = 2p^2 - 2$  such translates, thus concluding the proof.  $\square$

#### 4. Spark deficient Gabor frames over cyclic groups

Here we revisit the cyclic case. As has already been proven by the author [Malikiosis 2015], almost all windows generate full spark Gabor frames, so the spark deficient Gabor frames are generated by exceptional vectors. When the order of the group is an odd, square-free integer, then all eigenvectors of certain unitaries belonging to the Clifford group generate spark deficient Gabor frames [Dang et al. 2013]. The motivation behind this result in [Dang et al. 2013] was to establish a connection between equiangularity of a Gabor frame (SIC-POVM existence) and full spark, if any. In three dimensions, the family of SIC-POVMs generated by vectors of the form  $(0, 1, -e^{i\theta})$  is always spark deficient, and Lane Hughston [2007] first established a connection between the linear dependencies that arise from this SIC-POVM for  $\theta = 0$  or  $2\pi/9$  and the inflection points of an elliptic curve. In general, it was proven in [Dang et al. 2013] that when  $N$  is an odd, square-free integer divisible by 3, all eigenvectors of the *Zauner unitary matrix* generate spark deficient Gabor frames. Zauner's conjecture [1999] states that an eigenvector of this matrix generates a SIC-POVM, i.e., a maximal equiangular tight frame. If it is true, then for all odd, square-free dimensions, this equiangular tight frame is not full spark. This is another example that showcases the difference between a nice algebraic property of a Gabor frame (full spark) and a nice geometric one (equiangularity). For unit norm tight frames in general, this is further explained in [King 2015]; see also [Fickus et al. 2012; Jasper et al. 2014] where an infinite family of spark deficient equiangular tight frames is constructed, of arbitrarily high dimension.

When  $N$  is not divisible by 3, it is not known whether this SIC-POVM is also full spark or not. For example, it is full spark when  $N = 8$ . The first construction of a full spark Gabor frame in eight dimensions was given in [Dang et al. 2013].<sup>1</sup>

Concerning the eigenvectors of other Clifford unitaries, they also generate spark deficient Gabor frames as long as the (projective) order of the matrix divides  $N$ . We will extend the results of [Dang et al. 2013, Section 7], “Generalization to other symplectic unitaries”, to all odd dimensions  $N$  and unitaries whose order is not coprime to  $N$ .

**Theorem 4.1.** *Let  $N$  be an odd integer. Then, any eigenvector of the unitary  $U_F$  generates a spark deficient Gabor frame, where  $F \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$  and*

$$\gcd(\text{ord}(F), N) > 1.$$

This is a direct consequence of the following theorem from [Dang et al. 2013], slightly rephrased in order to accommodate the terminology of this paper, with the simple observation that if  $\text{ord}(F) = n$  and  $\gcd(n, N) = d > 1$ , then the eigenvectors

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<sup>1</sup>Explicit construction of a full spark Gabor frame in every dimension was later shown by the author [Malikiosis 2015].

of  $U_F$  are also eigenvectors of  $e^{i\theta} U_F^{n/d} = e^{i\theta} U_{F^{n/d}}$  (the phase  $e^{i\theta}$  is arbitrary), while  $\text{ord}(F^{n/d}) = d > 1$ , thus  $\text{ord}(F^{n/d})$  divides  $N$ .

We call  $\mathbf{x} \in (\mathbb{Z}/N\mathbb{Z})^2$   $F$ -full, if the vectors  $\mathbf{x}, F\mathbf{x}, \dots, F^{n-1}\mathbf{x}$  are all distinct, where  $F \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$  and  $n = \text{ord}(F)$ .

**Theorem 4.2** [Dang et al. 2013, Theorem 5, odd version]. *Let  $N$  be an odd positive integer and  $F \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$ , and let  $n = \text{ord}(F)$ . Suppose*

- (1)  $n > 1$ ,
- (2)  $n$  divides  $N$ ,
- (3)  $\text{Tr } U_F \neq 0$ ,
- (4) *there exist  $N$  distinct points in  $(\mathbb{Z}/N\mathbb{Z})^2$  that are  $F$ -full.*

*Then all eigenvectors of  $U_F$  generate spark deficient Gabor frames.*

Conditions (3) and (4) always hold when  $N$  is odd, as the following two lemmata show; this was proven in [Dang et al. 2013] for  $N$  odd square-free.<sup>2</sup>

**Lemma 4.3.** *Let  $N$  be an odd positive integer. Let  $F \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$  be arbitrary. Then the number of  $F$ -full points in  $(\mathbb{Z}/N\mathbb{Z})^2$  is  $\geq N\varphi(N)$ , where  $\varphi$  is Euler's function.*

**Lemma 4.4.** *Let  $N$  be an odd positive integer. Then  $|\text{Tr}(U_F)| \geq 1$  for all  $F \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$ .*

We will dedicate the rest of this section to the proofs of these two lemmata. For basic facts about the field of  $p$ -adic numbers,  $\mathbb{Q}_p$  and its algebraic extensions, we refer the reader to [Cassels 1986; Neukirch 1999].

**4A. Proof of Lemma 4.3.** Let  $F \in \text{SL}(2, \mathbb{Z}/N\mathbb{Z})$  and  $F_i \in \text{SL}(2, \mathbb{Z}/p_i^{r_i}\mathbb{Z})$  be the reduction of  $F \bmod p_i^{r_i}$ ,  $1 \leq i \leq s$ , and similarly, with  $\mathbf{x} \in (\mathbb{Z}/N\mathbb{Z})^2$  and  $\mathbf{x}_i \in \text{SL}(2, \mathbb{Z}/p_i^{r_i}\mathbb{Z})$ . It is not hard to show that if each  $\mathbf{x}_i$  is  $F_i$ -full, then  $\mathbf{x}$  is  $F$ -full, a fact also shown in [Dang et al. 2013]. By multiplicativity of the Euler function, it suffices to consider  $N = p^r$ , a power of an odd prime.

The case  $r = 1$  was treated in [Dang et al. 2013]. The technique was to find the Jordan canonical form of  $F$ , considering a quadratic extension of the field  $\mathbb{Z}/p\mathbb{Z}$  if necessary (i.e.,  $\mathbb{F}_{p^2}$ ); then we can control the powers of  $F$  and can count the points in  $(\mathbb{Z}/p\mathbb{Z})^2$  that are  $F$ -full.

When  $r > 1$ ,  $\mathbb{Z}/N\mathbb{Z}$  is no longer a field, so the Jordan canonical form does not always exist, but as we shall see below, in these exceptional cases, the order of  $F$  is equal to  $p^m$  or  $2p^m$ , for some  $m \leq r$ , so we only need to enumerate the points in  $(\mathbb{Z}/N\mathbb{Z})^2$  that are fixed by  $F^{p^{m-1}}$  or  $F^{2p^{m-1}}$  accordingly, and as it turns out, this is an easy task.

<sup>2</sup>See Lemmata 7 and 8 in [Dang et al. 2013].

It would be convenient to consider an arbitrary lift of the matrix

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to a matrix in  $\tilde{F} \in \mathrm{SL}(2, \mathbb{Z}_p)$ ; since  $F \in \mathrm{SL}(2, \mathbb{Z}/N\mathbb{Z})$ , then at least one of the entries  $a, b, c, d$ , is not divisible by  $p$ , say  $a$ . Then, lift  $a, b, c$ , arbitrarily, to  $\tilde{a}, \tilde{b}, \tilde{c}$ , and put  $\tilde{d} = \tilde{a}^{-1}(1 + \tilde{b}\tilde{c})$ . We also put  $t = \mathrm{Tr}(F)$ ,  $\tilde{t} = \mathrm{Tr}(\tilde{F})$ ,  $\Delta = t^2 - 4$ ,  $\tilde{\Delta} = \tilde{t}^2 - 4$ , the discriminants of the characteristic polynomials of  $F, \tilde{F}$ , respectively. Finally, we put

$$\lambda = \frac{t + \sqrt{\Delta}}{2},$$

and the other root of the characteristic polynomial is  $\lambda^{-1}$ . We distinguish the following cases:

**Case I:**  $p \nmid \Delta$ . Here,  $\lambda \not\equiv \lambda^{-1} \pmod{p}$ ; otherwise, we would have

$$\Delta \equiv (\lambda + \lambda^{-1})^2 - 4 \equiv \lambda^2 + \lambda^{-2} - 2 \equiv 0 \pmod{p}.$$

We reduce the entries of  $F \pmod{p}$ . Since  $\lambda \not\equiv \lambda^{-1} \pmod{p}$ ,  $F$  is diagonalizable in  $\mathbb{Z}/p\mathbb{Z}$  when  $(\Delta/p) = 1$  or in a quadratic extension, namely  $\mathbb{F}_{p^2}$ , when  $(\Delta/p) = -1$ . In both cases, we consider the field  $K = \mathbb{Q}_p(\sqrt{\Delta})$ , whose ring of integers is  $\mathcal{O}_K = \mathbb{Z}_p[\sqrt{\Delta}]$  and the unique prime ideal is  $p\mathcal{O}_K = p\mathbb{Z}_p[\sqrt{\Delta}]$ . This extension is unramified, as  $p \nmid \Delta$ , hence the degree of the extension is equal to the degree of the extension of the residue fields. Therefore, the residue field of  $K$  is  $\mathbb{F}_p$  when  $(\Delta/p) = 1$  and  $\mathbb{F}_{p^2}$  otherwise.

So, there is a nonsingular matrix  $X$  with entries in the residue field of  $K$  such that

$$(4-1) \quad FX \equiv X \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \pmod{p\mathcal{O}_K},$$

the congruence meaning that we consider each entry  $\pmod{p\mathcal{O}_K}$ . We can lift  $X = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$  to a  $2 \times 2$  matrix with entries in  $\mathcal{O}_K$ , such that (4-1) becomes an equality in  $\mathcal{O}_K$  (and holds  $\pmod{N}$ , in particular). Indeed, if  $b$  is not divisible by  $p$ , then we lift  $x, z$  arbitrarily, and then put  $y = \tilde{b}^{-1}(\lambda - \tilde{a}x)$ ,  $w = \tilde{b}^{-1}(\lambda - \tilde{a}z)$ , and a similar lift is possible if  $c$  is not divisible by  $p$ . If both  $b$  and  $c$  are divisible by  $p$ , then  $F \pmod{p}$  is diagonal, therefore  $F \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$  or  $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \pmod{p}$ . Without loss of generality, we may assume that the first congruence holds. Lift  $x, w$ , arbitrarily, and then put  $y = (\lambda - \lambda^{-1})^{-1}\tilde{c}x$ , and  $z = (\lambda^{-1} - \lambda)^{-1}\tilde{b}w$ . We notice that since  $p \nmid \det(X)$ , then  $X^{-1} \in \mathrm{GL}(2, \mathcal{O}_K)$ ; we conclude that in all cases where  $p \nmid \Delta$ ,  $F$  is equivalent to a diagonal matrix, with entries perhaps in a larger ring. It is evident that in this case, the number of  $F$ -full points is  $N^2 - 1$ , since  $p \nmid \lambda$ , and  $\lambda \not\equiv 1 \pmod{p}$ .

**Case II:**  $p \mid \Delta$ . Reducing the matrix  $F \bmod p$ , we obtain a double eigenvalue, equal to  $\pm 1$ . Then, the Jordan canonical form of  $F$  is

$$\begin{pmatrix} \pm 1 & \beta \\ 0 & \pm 1 \end{pmatrix},$$

where  $\beta = 0$  or  $\beta = 1$ . It is clear that  $F^p \equiv \pm I \bmod p$  and  $F^{2p} \equiv I \bmod p$ , or  $F^{2p} \equiv I + pA \bmod p^2$ , for some matrix  $A$ . Raising both sides to the  $p$ -th power, we obtain  $F^{2p^2} \equiv I + p^2A \bmod p^3$ , and proceeding inductively we can show that

$$F^{2p^{r-1}} = I + \frac{N}{p}A,$$

hence  $F^{2N} = I$ . This shows that the order of  $F$  is either  $p^m$  or  $2p^m$ , for some  $m \leq r$ .

Suppose first that the order of  $F$  is  $p^m$ ,  $m \geq 1$ ; then, an element of  $(\mathbb{Z}/N\mathbb{Z})^2$  is  $F$ -full, if and only if it is not fixed by  $F^{p^{m-1}}$  (this follows from the fact that the cardinality of the orbit of any element under a group, divides the order of the group), and the latter is equivalent to the condition that this element is  $F^{p^{m-1}}$ -full. Therefore, we can reduce to the case where  $m = 1$ , that is, the order of  $F$  is  $p$ . Since the number of  $F$ -full points is the same in the conjugacy class of  $F$ , we may further assume that  $F$  reduced mod  $p$  is equal to

$$\begin{pmatrix} \pm 1 & \beta \\ 0 & \pm 1 \end{pmatrix}.$$

Now, let  $k$  be the smallest positive integer for which we have

$$F \equiv \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} + p^{k-1}D \bmod p^k$$

for some matrix  $D \not\equiv \mathbf{O} \bmod p$ , where  $\mathbf{O}$  is the zero matrix. We have  $2 \leq k \leq r+1$ . If  $\beta = 0$ , then  $k = r$ ; if  $k < r$ , then

$$F^p \equiv I + p^k D \bmod p^{k+1},$$

hence  $F^p \neq I$ , a contradiction. Similarly, if  $k = r+1$ , then  $F = I$ , which is also a contradiction. So,  $F = I + \frac{N}{p}D$ . A vector  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (\mathbb{Z}/N\mathbb{Z})^2$  is fixed by  $F$  if and only if

$$D\mathbf{x} \equiv \mathbf{0} \bmod p.$$

The set of such vectors reduced mod  $p$  form a proper vector subspace of  $(\mathbb{Z}/p\mathbb{Z})^2$ , so there are at most  $p$  of them. Then, the number of all the possible lifts of these vectors mod  $N$  is at most  $p^{2(r-1)} \cdot p = p^{2r-1}$ . Therefore, the number of  $F$ -full vectors in this case is at least  $p^{2r} - p^{2r-1} = N\varphi(N)$ .

If  $\beta = 1$ , then

$$F^p \equiv \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} + p^{k-1} \sum_{\kappa+\mu=p-1} \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix} D \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \pmod{p^k}.$$

We put

$$D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}$$

and compute the above sum mod  $p$ :

$$\begin{aligned} \sum_{\kappa+\mu=p-1} \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix} D \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} &= \sum_{\kappa+\mu=p-1} \begin{pmatrix} d_1 + \kappa d_3 & \mu d_1 + \kappa \mu d_3 + d_2 + \kappa d_4 \\ d_3 & \mu d_3 + d_4 \end{pmatrix} \\ &\equiv \mathbf{0} \pmod{p} \end{aligned}$$

since

$$\begin{aligned} \sum_{\kappa+\mu=p-1} 1 &= p, \\ \sum_{\kappa+\mu=p-1} \kappa &= \sum_{\kappa+\mu=p-1} \mu = p \cdot \frac{p-1}{2}, \\ \sum_{\kappa+\mu=p-1} \kappa \mu &= p \cdot \left( \frac{(p-1)^2}{2} - \frac{(p-1)(2p-1)}{6} \right). \end{aligned}$$

But then,  $F^p \not\equiv I \pmod{p^k}$ , a contradiction if  $k \leq r$ ; if  $k = r+1$ , then  $F^p = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \neq I$ . We conclude that if the order of  $F$  is  $p$  and  $r \geq 2$ , then  $\beta = 0$  (the case  $\beta \neq 0$  can only occur when  $r = 1$ , but this was treated in [Dang et al. 2013]).

Next, suppose that the order of  $F$  is  $2p^m$ . Then, a vector is  $F$ -full if and only if it is not fixed by  $F^{p^m}$  or  $F^{2p^{m-1}}$ . But  $F^{p^m} = -I$ , which only fixes the zero vector, so we only need to exclude the vectors fixed by  $F^{2p^{m-1}}$ ; however, this matrix has order  $p$ , so the above analysis applied to  $F^{2p^{m-1}}$  yields the fact that the number of  $F$ -full points is at least  $N\varphi(N)$ .

**4B. Proof of Lemma 4.4.** The trace  $\text{Tr}(U_F)$  is a quadratic Gauss sum [Appleby 2005]; we will use the following lemma by Turaev [1998, Lemma 1] which gives the absolute value of such a sum over an arbitrary finite abelian group  $G$ . Moreover, by (2-1) we may assume that  $N$  is a power of an odd prime,  $p$ .

Let's fix some notation first;  $q : G \rightarrow \mathbb{Q}/\mathbb{Z}$  denotes an arbitrary quadratic form on the finite abelian group  $G$ . Such a function is a quadratic form if the expression  $b^q(x, y) = q(x+y) - q(x) - q(y)$  is bilinear (we do not require homogeneity). The Gauss sum  $\Gamma(G, q)$  is defined to be

$$\frac{1}{|G|^{1/2}} \sum_{x \in G} e^{2\pi i q(x)}.$$



Lastly, for easy reference to the explicit formulae for the unitary matrices  $U_F$  given in [Appleby 2005], we decided to use the bra-ket notation; the set of (column) vectors

$$|0\rangle, |1\rangle, \dots, |N-1\rangle,$$

is the standard basis of  $\mathbb{C}^N$ , and  $\langle\varphi|$  is the conjugate transpose of  $|\varphi\rangle$ .

**Lemma 4.5** [Turaev 1998, Lemma 1]. *Let  $B$  be the kernel of the homomorphism  $G \rightarrow \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$  adjoint to the pairing  $b^q$ . If  $q(B) \neq 0$ , then  $\Gamma(G, q) = 0$ . If  $q(B) = 0$ , then  $|\Gamma(G, q)| = |B|^{1/2}$ .*

If  $p \nmid b$ , then the matrix  $F$  is called *prime*, and from the explicit formulae of [Appleby 2005, Lemmas 2 and 4], we get

$$U_F = \frac{e^{i\theta}}{\sqrt{N}} \sum_{r,s=0}^{N-1} \tau^{b^{-1}(as^2-2rs+dr^2)} |r\rangle\langle s|,$$

where  $\theta$  is an arbitrary phase, and  $b^{-1}$  the inverse of  $b \bmod N$ , hence

$$\text{Tr}(U_F) = \frac{e^{i\theta}}{\sqrt{N}} \sum_{r=0}^{N-1} \tau^{b^{-1}(t-2)r^2}.$$

where  $\tau = -e^{\pi i/N}$  and  $t = a + d = \text{Tr}(F)$ . Putting  $G = \mathbb{Z}/N\mathbb{Z}$  and

$$q(r) = \frac{b^{-1}(t-2)(N+1)}{2N} r^2,$$

we get  $\text{Tr}(U_F) = e^{i\theta} \Gamma(G, q)$ . The function  $q$  is a well-defined quadratic form on  $G$ ; indeed,  $r^2 \equiv r'^2 \bmod 2N$ , when  $r \equiv r' \bmod N$ , when  $N$  is odd. The associated bilinear pairing is

$$b^q(r, s) = \frac{b^{-1}(t-2)(N+1)}{N} rs,$$

and  $r \in B$  if and only if  $b^q(r, 1) = 0$ , or equivalently, if

$$b^{-1}(t-2)r \equiv 0 \bmod N.$$

So, if  $r \in B$  is arbitrary, then  $N$  divides  $b^{-1}(t-2)r^2$ , and therefore  $2N$  divides  $b^{-1}(t-2)(N+1)r^2$ , which shows that  $q(r) = 0$ . This proves that  $q(B) = 0$ , hence  $|\Gamma(G, q)| = |B|^{1/2} \geq 1$  and  $|\text{Tr}(U_F)| \geq 1$ .

Now, assume that  $p \mid b$ ; then  $p \nmid d$  (otherwise  $\det(F)$  would be divisible by  $p$ ) and we can write  $F$  as a product of two prime matrices, as follows:

$$F = F_1 F_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}$$

and by [Appleby 2005, Lemma 4], we have  $U_F = U_{F_1} U_{F_2}$ , where

$$U_{F_1} = \frac{e^{i\theta_1}}{\sqrt{N}} \sum_{u,v=0}^{N-1} \tau^{2uv} |u\rangle \langle v|$$

and

$$U_{F_2} = \frac{e^{i\theta_2}}{\sqrt{N}} \sum_{v,w=0}^{N-1} \tau^{d^{-1}(cw^2-2vw-bv^2)} |v\rangle \langle w|,$$

where  $\theta_1, \theta_2$  are arbitrary phases, hence

$$U_F = \frac{e^{i\theta}}{N} \sum_{u,w=0}^{N-1} \sum_{v=0}^{N-1} \tau^{2uv+d^{-1}(cw^2-2vw-bv^2)} |u\rangle \langle w|$$

and

$$\text{Tr}(U_F) = \frac{e^{i\theta}}{N} \sum_{u,v=0}^{N-1} \tau^{cd^{-1}u^2+2(1-d^{-1})uv-bd^{-1}v^2},$$

where  $\theta = \theta_1 + \theta_2$ . So, if we put  $G = (\mathbb{Z}/N\mathbb{Z})^2$  and  $q : G \rightarrow \mathbb{Q}/\mathbb{Z}$  the quadratic form

$$q(u, v) = \frac{N+1}{2N} (cd^{-1}u^2 + 2(1-d^{-1})uv - bd^{-1}v^2)$$

then  $\text{Tr}(U_F) = e^{i\theta} \Gamma(G, q)$ . The associated bilinear form is

$$b^q((u, v), (r, s)) = \frac{N+1}{N} (u \ v) A \begin{pmatrix} r \\ s \end{pmatrix}$$

where

$$A = \begin{pmatrix} cd^{-1} & 1-d^{-1} \\ 1-d^{-1} & -bd^{-1} \end{pmatrix}.$$

Now, let  $(u \ v) \in B$  be arbitrary. Then,

$$(u \ v) A \equiv (0 \ 0) \pmod{N},$$

otherwise we would have either  $b^q((u, v), (1, 0)) \neq 0$  or  $b^q((u, v), (0, 1)) \neq 0$ . In particular,  $N$  divides  $b^q((u, v), (u, v))$ , and since  $N$  is odd,  $2N$  divides

$$(N+1)(u \ v) A \begin{pmatrix} u \\ v \end{pmatrix},$$

which yields  $q(u, v) = 0$ . Thus,  $q(B) = 0$ , and

$$|\text{Tr}(U_F)| = |\Gamma(G, q)| = |B|^{1/2} \geq 1.$$

## 5. Uncertainty principles

The full spark property of (almost all) Gabor frames of windows defined over finite cyclic groups implies the following inequality for the short-time Fourier transform of  $f$ :

$$\|V_\varphi^* f\|_0 \geq N^2 - N + 1,$$

where  $N$  is the size of said group, for almost all  $\varphi \in \mathbb{C}^N$  and all nonzero  $f \in \mathbb{C}^N$  [Krahmer et al. 2008; Malikiosis 2015; Pfander 2013]. A possible connection between the set of pairs of the form  $(\|f\|_0, \|\hat{f}\|_0)$ , denoted by  $F$ , and the set  $F_\varphi$  of all pairs of the form

$$(\|f\|_0, \|V_\varphi^* f\|_0 - N^2 + N)$$

(for both sets we take  $f$  nonzero) was investigated in [Krahmer et al. 2008]. In particular, the following problem was proposed.

**Problem 5.1** [Krahmer et al. 2008]. Is it true that  $F = F_\varphi$  for almost all  $\varphi$ ?

When  $N = p$  a prime number, this problem was solved in the affirmative [Krahmer et al. 2008]. One has an exact characterization of the set  $F$  [Tao 2005] and the fact that all minors of the Gabor synthesis matrix are nonzero for all  $\varphi$  except for a set of measure zero [Lawrence et al. 2005, Theorem 4] leads to a characterization of the set  $F_\varphi$ , and equality between  $F$  and  $F_\varphi$  is easily confirmed. When  $N$  is composite, however, there is no exact characterization for the set  $F$ , so it is more difficult to obtain equality; this was confirmed numerically for dimensions up to 6 [Krahmer et al. 2008]. The question is whether we can prove equality between those two sets without using the characterization of  $F$ . We will show that one inclusion is possible, but the other one, namely  $F_\varphi \subseteq F$ , seems much harder to prove, if true.

As a final remark, we note that the spark deficiency of all Gabor frames of windows defined over abelian, noncyclic groups, implies that equality between  $F$  and  $F_\varphi$  can never be achieved, simply because there are  $f \in \mathbb{C}^G$  for which  $\|V_\varphi^* f\|_0 \leq N^2 - N$ , as shown in the proof of Theorem 3.3.

A useful identity is the following:

$$(5-1) \quad \|V_\varphi^* f\|_0 = \sum_{j=0}^{N-1} \|\widehat{T^j \varphi \cdot f}\|_0.$$

**Theorem 5.2.** *For almost all  $\varphi$  the inclusion  $F \subseteq F_\varphi$  holds. In addition, this  $\varphi$  can be taken to generate a full spark Gabor frame.*

*Proof.* First, we may restrict our attention to  $\varphi$  generating a full spark Gabor frame, as we already know that almost all  $\varphi$  satisfy this condition. This implies that all coordinates of  $\varphi$  are nonzero, otherwise the frequency translates of  $\varphi$  would form a

singular matrix. Next, for any pair  $(k, l) \in F$  we consider  $f_{k,l} \in \mathbb{C}^N$  with  $\|f_{k,l}\|_0 = k$  and  $\|\widehat{f_{k,l}}\|_0 = l$ . We may rewrite (5-1) as

$$(5-2) \quad \left\| V_\varphi^* \frac{f_{k,l}}{\varphi} \right\|_0 = \sum_{j=0}^{N-1} \left\| \widehat{\frac{T^j \varphi}{\varphi} \cdot f_{k,l}} \right\|_0 = \|\widehat{f_{k,l}}\|_0 + \sum_{j=1}^{N-1} \left\| \widehat{\frac{T^j \varphi}{\varphi} \cdot f_{k,l}} \right\|_0 \\ = l + \sum_{j=1}^{N-1} \left\| \widehat{\frac{T^j \varphi}{\varphi} \cdot f_{k,l}} \right\|_0.$$

It suffices to show that almost all  $\varphi$  satisfy

$$\left\| \widehat{\frac{T^j \varphi}{\varphi} \cdot f_{k,l}} \right\|_0 = N,$$

for all  $(k, l) \in F$  and  $1 \leq j \leq N-1$ , or equivalently, it suffices to show that

$$\Phi \sum_{g=0}^{N-1} \xi(g) f_{k,l}(g) \frac{\varphi(g-j)}{\varphi(g)} \neq 0,$$

for almost all  $\varphi \in \mathbb{C}^N$ , all characters  $\xi$ ,  $(k, l) \in F$ ,  $1 \leq j \leq N-1$ , where  $\Phi$  is the product of the coordinates of  $\varphi$ . But the left-hand side is a polynomial in the coordinates of  $\varphi$  with coefficients of the form  $\xi(g) f_{k,l}(g)$ , which shows that every such polynomial is nonzero, as the functions  $f_{k,l}$  are not identically zero. Therefore,  $\varphi$  has to avoid the zero set of finitely many nonzero polynomials, whose union is of measure zero. Thus, almost all  $\varphi$  satisfy

$$\left\| V_\varphi^* \frac{f_{k,l}}{\varphi} \right\|_0 = N^2 - N + l,$$

for every  $(k, l) \in F$ , as desired.  $\square$

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# ORDERED GROUPS AS A TENSOR CATEGORY

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**It is a classical theorem that the free product of ordered groups is orderable. In this note we show that, using a method of G. Bergman, an ordering of the free product can be constructed in a functorial manner, in the category of ordered groups and order-preserving homomorphisms. With this functor interpreted as a tensor product this category becomes a tensor (or monoidal) category. Moreover, if  $O(G)$  denotes the space of orderings of the group  $G$  with the natural topology, then for fixed groups  $F$  and  $G$  our construction can be considered a function  $O(F) \times O(G) \rightarrow O(F * G)$ . We show that this function is continuous and injective. Similar results hold for left-ordered groups.**

## 1. Introduction

An ordered group  $(G, <)$  is a group  $G$  together with a strict total ordering  $<$  of its elements such that  $x < y$  implies  $xz < yz$  and  $zx < zy$  for all  $x, y, z \in G$ . If such an ordering exists,  $G$  is said to be orderable. If  $(F, <_F)$  and  $(G, <_G)$  are ordered groups, a homomorphism  $\phi : F \rightarrow G$  is said to be order-preserving (relative to  $<_F, <_G$ ) if for all  $x, y \in F$  we have  $x <_F y \implies \phi(x) <_G \phi(y)$ . In this case the reverse implication follows, and  $\phi$  is necessarily injective.

A theorem of Vinogradov [1949] asserts that if  $F$  and  $G$  are orderable groups, then the free product  $F * G$  (sometimes called the coproduct, as in [Bergman 1990]) is orderable. Other proofs of this can be found in [Johnson 1968; Passman 1977; Bergman 1990], and a generalization in [Chiswell 2012]. A proof given in [Botto Mura and Rhemtulla 1977] was unfortunately found to have a gap, as discussed in [Holland and Medvedev 1994; Chiswell 2014]. Yet another proof, in [Révész 1987], was also shown to have a gap [Medvedev 1991].

Here we show that a version of the construction in [Bergman 1990] is functorial in the following sense. Suppose  $(F_i, <_{F_i})$ ,  $i = 0, 1$ , are ordered groups. We will

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construct an ordering  $<$  of  $F_0 * F_1$ , so that  $(F_0 * F_1, <)$  is an ordered group, and write

$$\mathfrak{F}((F_0, <_{F_0}), (F_1, <_{F_1})) := (F_0 * F_1, <).$$

Theorem 1 shows that  $\mathfrak{F}$  is a (bi)functor in the category  $\mathfrak{C}$  of ordered groups and order-preserving homomorphisms. We will show in Section 5 that this functor gives  $\mathfrak{C}$  the structure of a tensor, or monoidal, category.

**Theorem 1.** *Suppose that  $(F_i, <_{F_i})$ ,  $i = 0, 1$ , are ordered groups. Then the ordered group  $(F_0 * F_1, <_F) = \mathfrak{F}((F_0, <_{F_0}), (F_1, <_{F_1}))$  has the following properties:*

- (1)  $<_F$  extends the given orderings of  $F_i$  as subgroups of  $F_0 * F_1$ .
- (2) If  $(G_i, <_{G_i})$ ,  $i = 0, 1$ , are ordered groups and

$$(G_0 * G_1, <_G) = \mathfrak{F}((G_0, <_{G_0}), (G_1, <_{G_1}))$$

and if  $\phi_i : F_i \rightarrow G_i$ ,  $i = 0, 1$ , are homomorphisms which preserve the given orderings of  $F_i$  and  $G_i$ , then the homomorphism  $\phi_0 * \phi_1 : F_0 * F_1 \rightarrow G_0 * G_1$  is order-preserving, relative to  $<_F, <_G$ .

In Section 8, Theorem 1 will be extended to free products of an arbitrary, possibly infinite, collection of ordered groups. We will typically use multiplicative notation for groups and use 1 to denote the identity element, though additive groups are also considered, with 0 as identity element. We may also use 1 to denote the unit of a ring (all rings we consider are assumed to have a unit), as well as the natural number.

Many of our results could have been proven using the original construction of Vinogradov. Like Bergman's, his proof involves embedding a free product of groups into a ring of matrices. Vinogradov's matrices are infinite dimensional upper triangular matrices, whereas Bergman's are  $2 \times 2$  matrices with polynomial entries, a useful simplification.

## 2. Embedding free products in matrix rings

We use an observation of Bergman which generalizes the fact that the matrices  $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$  freely generate a free subgroup of the multiplicative group of invertible  $2 \times 2$  matrices with entries in the polynomial ring  $\mathbb{Z}[t]$ .

Consider a ring  $R$  without zero divisors and let  $F$  and  $G$  be multiplicative groups of nonzero elements of  $R$ . Let  $M_2(R[t])$  be the ring of  $2 \times 2$  matrices with entries in the polynomial ring  $R[t]$ . Then one can embed  $F$  in  $M_2(R[t])$  by  $f \mapsto \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}$ . But we can conjugate that by  $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$  to get a different embedding which has a highest degree in the upper right corner when  $f \neq 1$ :

$$\rho(f) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} f & (f-1)t \\ 0 & 1 \end{pmatrix}.$$



Similarly we embed  $G$  by

$$\rho(g) = \begin{pmatrix} 1 & 0 \\ (g-1)t & g \end{pmatrix}.$$

This then defines a multiplicative homomorphism  $\rho : F * G \rightarrow M_2(R[t])$ , which Bergman observes to be a faithful representation.

**Proposition 2** [Bergman 1990, Corollary 12]. *With the assumptions stated in the preceding paragraph,  $\rho : F * G \rightarrow M_2(R[t])$  is injective.*

*Proof.* Here is a sketch of a proof using a ping-pong argument. Let

$$f_k g_k f_{k-1} \cdots g_2 f_1 g_1 \neq 1$$

be a reduced word in  $F * G$ , with  $f_i \in F$ ,  $g_i \in G$  nonidentity elements (except possibly for  $i \in \{1, k\}$ ). Assume that  $g_1 \neq 1$ , the other case with  $g_1 = 1$ ,  $f_1 \neq 1$  being similar. We need to show that the product of matrices  $\rho(f_k)\rho(g_k) \cdots \rho(f_1)\rho(g_1)$  is not the identity matrix. Consider the set  $V$  of column vectors  $\begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$  with entries in  $R[t]$  and partition that set into three parts  $V = V_1 \sqcup V_2 \sqcup V_3$  according to their degrees as polynomials. Take  $V_1$  to be the set of such pairs with  $\deg A(t) > \deg B(t)$ ,  $V_2$  the set with  $\deg A(t) < \deg B(t)$  and  $V_3$  the set with equal degree.

Apply  $\rho(f_k)\rho(g_k) \cdots \rho(f_1)\rho(g_1)$  (on the left) to the vector  $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in V_3$  and note that  $\rho(g_1)$  sends  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  to  $\begin{pmatrix} 1 \\ g_1 + (g_1 - 1)t \end{pmatrix}$  which belongs to  $V_2$ . Then  $\rho(f_1)$  sends this result into  $V_1$ , which is then sent to  $V_2$  by  $\rho(g_2)$ , and so on. The end result, after multiplying all the matrices, will be in  $V_1$  or  $V_2$ , not  $V_3$ , and so the product cannot be the identity matrix.  $\square$

### 3. Constructing the ordering $\prec$

Suppose we are given two ordered groups,  $(F_0, <_{F_0})$  and  $(F_1, <_{F_1})$ . To embed them in a ring, we take  $R$  to be the integral group ring of their direct product:  $R = \mathbb{Z}(F_0 \times F_1)$ . It is well known that integral group rings of orderable groups have no zero divisors (see, for example, [Botto Mura and Rhemtulla 1977] p. 155), so  $R$  has no zero divisors. Define a multiplicative homomorphism  $\rho : F_0 * F_1 \rightarrow M_2(R[t])$  by

$$\rho(f_0) = \begin{pmatrix} f_0 & (f_0 - 1)t \\ 0 & 1 \end{pmatrix} \quad \rho(f_1) = \begin{pmatrix} 1 & 0 \\ (f_1 - 1)t & f_1 \end{pmatrix}, \quad f_i \in F_i.$$

By Proposition 2,  $\rho$  is faithful; it defines an isomorphism of  $F_0 * F_1$  onto a multiplicative subgroup of  $M_2(R[t])$ .

We now turn to the task of defining the ordering, choosing a specific recipe among many described in [Bergman 1990]. First we order  $F_0 \times F_1$  lexicographically, defining  $(f_0, f_1) < (f'_0, f'_1)$  if  $f_0 <_{F_0} f'_0$  or else  $f_0 = f'_0$  and  $f_1 <_{F_1} f'_1$ . Then the

group ring  $R = \mathbb{Z}(F_0 \times F_1)$  becomes an ordered ring<sup>1</sup> by declaring a nonzero element to be *positive* if the coefficient of the largest term (in the ordering  $<$  of  $F_0 \times F_1$ ) is a positive integer.

Note that as a ring element,  $f_0 \in F_0$ , which can be considered an abbreviation of  $1(f_0, 1) \in R$ , is considered positive even if  $f_0 <_{F_0} 1$  and it would be called “negative” as a group element. In particular, the diagonal elements of the matrices displayed above are all positive.

Bergman then orders  $M_2(R)$  as follows. Choose “an arbitrary order among the four ‘positions’ in a  $2 \times 2$  matrix, and call a nonzero element of this module ‘positive’ if in the first position in which a nonzero coefficient occurs, the coefficient is in fact positive.” To be definite, we will choose the 1, 1 position to be first, the 2, 2 position to be second, and the off-diagonal positions ordered third and fourth in some fixed way. Now to order the matrix ring of polynomials, call an element  $M$  of  $M_2(R[t])$  positive if it satisfies the following. Expand  $M = M_0 + M_1t + \cdots + M_k t^k$ , where each  $M_i$  belongs to  $M_2(R)$ . Let  $n \geq 0$  be the least integer such that  $t^n$  has nonzero coefficient and say  $M$  is positive if and only if the first nonzero entry of  $M_n$  is positive in the ordered ring  $R$ .

Bergman points out that “the orderings of the positions can be the same for all  $n$ , but need not — there is a lot of freedom here.” But we will use the same ordering of the positions, as described, throughout.

Finally, define an ordering of  $F_0 * F_1$  by declaring that  $x < y$  if and only if  $\rho(y) - \rho(x)$  is positive in  $M_2(R[t])$ .

#### 4. Proof of Theorem 1 and further properties of $<$

First we’ll argue that  $(F_0 * F_1, <)$  is an ordered group. Clearly  $<$  is a strict total ordering. To check invariance under multiplication, first note that every element of  $\rho(F_0 * F_1)$  in  $M_2(R[t])$ , when expanded in powers of  $t$ , has constant term a diagonal matrix with positive entries. (See the proof of Proposition 4 below to be more precise.) The product of such a matrix, on either side, with a positive matrix in  $M_2(R[t])$  will again be positive. Thus, if  $x, y, z \in F_0 * F_1$ , one has  $x < y \iff \rho(y) - \rho(x)$  is positive  $\iff \rho(z)(\rho(y) - \rho(x)) = \rho(z y) - \rho(z x)$  is positive  $\iff z x < z y$ . Right invariance is proved similarly.

Next we will show that the ordering  $<$  extends the given orderings  $<_{F_0}$  and  $<_{F_1}$ . Suppose  $f_0, f'_0 \in F_0$  and  $f_0 <_{F_0} f'_0$ . Then the difference between their images in  $M_2(R[t])$  is the matrix  $\begin{pmatrix} f'_0 - f_0 & 0 \\ 0 & 0 \end{pmatrix}$ , and noting that  $f'_0 - f_0$  is positive in  $R$  we conclude  $f_0 < f'_0$ . A similar argument shows that  $<$  also extends  $<_{F_1}$ .

<sup>1</sup>We understand an ordered ring  $(R, <)$  to be an ordered group as an additive group, for which the positive cone  $P = \{r \in R \mid 0 < r\}$  is also closed under multiplication.

This establishes the first part of Theorem 1. To prove the second part, note that  $\phi_0 \times \phi_1$  preserves the lexicographic orderings  $<_F, <_G$  of  $F_0 \times F_1$  and  $G_0 \times G_1$ , respectively. A homomorphism of groups naturally extends to a ring homomorphism of the integral group rings, and we see that if the group homomorphism preserves given orderings of the groups, then its extension takes “positive” elements of the group ring to positive elements. Then  $\phi_0 \times \phi_1$  defines a ring homomorphism  $R_F \rightarrow R_G$ , where  $R_F = \mathbb{Z}(F_0 \times F_1)$  and  $R_G = \mathbb{Z}(G_0 \times G_1)$ , which we will call  $\phi_0 \times \phi_1$  again. This extends to a ring homomorphism  $R_F[t] \rightarrow R_G[t]$ , and further induces an additive homomorphism  $M_2(R_F[t]) \rightarrow M_2(R_G[t])$ , which we will again call  $\phi_0 \times \phi_1$ .

The diagram

$$\begin{array}{ccc} F_0 * F_1 & \xrightarrow{\rho} & M_2(R_F[t]) \\ \phi_0 * \phi_1 \downarrow & & \phi_0 \times \phi_1 \downarrow \\ G_0 * G_1 & \xrightarrow{\rho} & M_2(R_G[t]) \end{array}$$

is commutative (we have used the same symbol  $\rho$  for different maps, but defined analogously), and as already mentioned,  $\phi_0 \times \phi_1$  takes positive matrix entries to positive matrix entries. We now argue that  $\phi_0 * \phi_1$  is order-preserving, relative to  $<_F, <_G$ . Suppose  $x, y \in F_0 * F_1$  and  $x <_F y$ . Then  $\rho(y) - \rho(x)$  is positive, and therefore  $\phi_0 \times \phi_1(\rho(y) - \rho(x))$  is positive in  $M_2(R_G[t])$ . But  $\phi_0 \times \phi_1(\rho(y) - \rho(x)) = \phi_0 \times \phi_1(\rho(y)) - \phi_0 \times \phi_1(\rho(x)) = \rho(\phi_0 * \phi_1(y)) - \rho(\phi_0 * \phi_1(x))$ , and since this is positive, we conclude that  $\phi_0 * \phi_1(x) <_G \phi_0 * \phi_1(y)$ .  $\square$

**Corollary 3.** *If  $(F, <_F)$  and  $(G, <_G)$  are ordered groups, then the ordered group  $(F * G, <) := \mathfrak{F}((F, <_F), (G, <_G))$  has the properties that  $<$  extends the orderings of  $F$  and  $G$ , and for any automorphisms  $\phi : F \rightarrow F$  and  $\psi : G \rightarrow G$  which preserve the given orderings, the automorphism  $\phi * \psi : F * G \rightarrow F * G$  preserves the ordering  $<$ .*

Following the terminology used in [Botto Mura and Rhemtulla 1977], we will call a homomorphism  $\phi : F \rightarrow G$  of ordered groups  $(F, <_F)$  and  $(G, <_G)$  an *order-homomorphism* (relative to the given orderings) if  $x \leq_F y$  implies  $\phi(x) \leq_G \phi(y)$  for all  $x, y \in F$ . Note that order-preserving homomorphisms are order-homomorphisms, and that order-homomorphisms need not be injective. Indeed, the order-preserving homomorphisms are exactly the order-homomorphisms which are injective. For example, using the lexicographic ordering of the direct product, the inclusions  $F \rightarrow F \times G$  and  $G \rightarrow F \times G$  are order-preserving, while the projection  $F \times G \rightarrow F$  is an order-homomorphism. But the projection  $F \times G \rightarrow G$  will not be an order-homomorphism, if the groups are nontrivial.

We’ll see that our construction of  $<$  has similar properties. First note that Theorem 1(1) implies that the natural inclusion homomorphisms  $F \rightarrow F * G$  and

$G \rightarrow F * G$  are order-preserving. There are also canonical maps  $F * G \rightarrow F$ , obtained by killing elements of  $G$ , and similarly  $F * G \rightarrow G$ . They combine to define a canonical homomorphism  $\alpha : F * G \rightarrow F \times G$ . Specifically, if  $f_1 g_1 f_2 \cdots f_k g_k$  is an element of  $F * G$ , with  $f_i \in F$  and  $g_i \in G$ , then  $\alpha(f_1 g_1 f_2 \cdots f_k g_k) = (f_1 \cdots f_k, g_1 \cdots g_k)$ .

**Proposition 4.** *Suppose that  $(F, <_F)$  and  $(G, <_G)$  are ordered groups. Then the canonical homomorphism  $\alpha : F * G \rightarrow F \times G$  is an order-homomorphism, relative to the lexicographic ordering of  $F \times G$  and the ordering  $<$  for  $F * G$ .*

*Proof.* If  $x \in F * G$  has image  $\alpha(x) = (f, g) \in F \times G$ , we observe that its image under the representation  $\rho : F * G \rightarrow M_2(R[t])$  may be written

$$\rho(x) = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} + \text{terms of positive degree.}$$

The conclusion follows from our convention for ordering  $M_2(R[t])$ .  $\square$

A subset  $C \subset G$  of an ordered group  $(G, <_G)$  is said to be *convex* if the inequalities  $c <_G g <_G c'$ , with  $c, c' \in C$  imply that  $g \in C$ . For example, it is easy to see that if  $(F, <_F)$  and  $(G, <_G)$  are ordered groups and  $\phi : F \rightarrow G$  is an order-homomorphism, then the kernel  $K$  of  $\phi$  is a convex subgroup of  $F$ .

**Corollary 5.** *The kernel of the homomorphism  $\alpha : F * G \rightarrow F \times G$  is convex, relative to the ordering  $<$  of  $F * G$ .*

The kernel of  $\alpha : F * G \rightarrow F \times G$  is known to be a free subgroup of  $F * G$ , freely generated by commutators of the form  $fgf^{-1}g^{-1}$ , where  $1 \neq f \in F$  and  $1 \neq g \in G$ .

**Corollary 6.** *If  $F * G$  is ordered by  $<$ , the canonical homomorphism  $F * G \rightarrow F$  is an order-homomorphism, but  $F * G \rightarrow G$  will not be an order-homomorphism, if the groups are nontrivial.*

Indeed, if  $f <_F f'$  in  $F$  while  $g' <_G g$  in  $G$ , we have, as elements of  $F * G$ , the inequality  $fg < f'g'$ . If the canonical map  $F * G \rightarrow G$  were an order-homomorphism, we'd conclude  $g <_G g'$ , a contradiction. The asymmetry exposed by this corollary cannot be corrected, as the following observation shows. We note that, by the same proof, the proposition also applies to direct products.

**Proposition 7.** *If  $F$  and  $G$  are nontrivial ordered groups, then there is no ordering of  $F * G$  for which both of the canonical homomorphisms  $F * G \rightarrow F$  and  $F * G \rightarrow G$  are order-homomorphisms.*

*Proof.* As above, choose  $f, f' \in F$  and  $g, g' \in G$  such that  $f <_F f'$  and  $g' <_G g$ . Suppose  $<$  is an ordering of  $F * G$  for which the canonical homomorphisms  $F * G \rightarrow F$  and  $F * G \rightarrow G$  are order-homomorphisms, and compare  $fg$  with  $f'g'$ . If  $fg < f'g'$ , then applying the map  $F * G \rightarrow F$  implies that  $f \leq_F f'$ , a contradiction. Similarly,  $fg > f'g'$  implies the contradiction  $g \geq_G g'$ .  $\square$

## 5. Structure as a tensor category

Recall that  $\mathfrak{C}$  denotes the category of ordered groups and order-preserving homomorphisms, and that  $\mathfrak{F} : \mathfrak{C} \times \mathfrak{C} \rightarrow \mathfrak{C}$  is a bifunctor. Let us rename  $\mathfrak{F}$  as follows, for ordered groups  $(F_0, <_{F_0})$  and  $(F_1, <_{F_1})$ :

$$(F_0, <_{F_0}) \otimes (F_1, <_{F_1}) := \mathfrak{F}((F_0, <_{F_0}), (F_1, <_{F_1})) = (F_0 * F_1, <)$$

It is well known that the category of groups under free product is a tensor category, with unit the trivial group; see [Mac Lane 1998, p. 161], or the definition given in [Wikipedia 2017]. I am grateful to Christian Kassel for suggesting the following to me.

**Theorem 8.** *With the bifunctor  $\otimes$  the category  $\mathfrak{C}$  is a tensor category, in other words, a monoidal category.*

For ordered groups  $(F_0, <_{F_0})$ ,  $(F_1, <_{F_1})$ ,  $(F_2, <_{F_2})$ , we have the isomorphism of groups

$$F_0 * (F_1 * F_2) \cong (F_0 * F_1) * F_2.$$

We need to check that the orderings constructed on both sides of this equivalence are the same under the isomorphism, in other words the isomorphism is order-preserving. But this follows from the observation that the lexicographic orderings on the direct products  $F_0 \times (F_1 \times F_2)$  and  $(F_0 \times F_1) \times F_2$ , used in the respective orderings of  $F_0 * (F_1 * F_2)$  and  $(F_0 * F_1) * F_2$ , both reduce to the lexicographic ordering of triples.

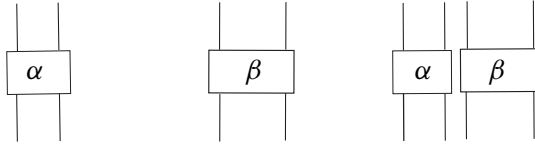
Similarly, the coherence relations involved in tensor categories follow from the observation that for ordered groups  $(F_i, <_{F_i})$ ,  $0 \leq i \leq 3$ , our orderings of the groups

$$\begin{aligned} (F_0 * F_1) * (F_2 * F_3), \quad (F_0 * (F_1 * F_2)) * F_3, \quad F_0 * ((F_1 * F_2) * F_3), \\ (F_0 * F_1) * (F_2 * F_3), \quad \text{and} \quad F_0 * (F_1 * (F_2 * F_3)) \end{aligned}$$

are identical (under their natural isomorphisms).

## 6. An application to braid groups

The original motivation for this study is the following application to the theory of braids. The braid group  $B_n$  acts by automorphisms on the free group  $\mathbb{F}_n$ , as observed by Artin [1925; 1947]. Free groups are orderable, and we may call a braid “order-preserving” if its image under the (faithful) Artin representation  $B_n \rightarrow \text{Aut}(\mathbb{F}_n)$  preserves *some* ordering of  $\mathbb{F}_n$ ; see [Kin and Rolfsen 2016]. In that paper it is noted that a braid is order-preserving if and only if the complement of the link in  $S^3$  consisting of the braid’s closure, plus the braid axis, has orderable fundamental group. It is used to show, for example, that of the two minimal volume orientable



**Figure 1.**  $\alpha \in B_m$  (left),  $\beta \in B_n$  (center),  $\alpha \otimes \beta \in B_{m+n}$  (right).

hyperbolic 2-cusped 3-manifolds, one has an orderable fundamental group, while the group of the other is not orderable (although it is left-orderable).

Multiplication of braids is by concatenation, and the product of two order-preserving braids need not be order-preserving, as observed in [Kin and Rolfsen 2016]. There is also a tensor product operation  $\otimes : B_m \times B_n \rightarrow B_{m+n}$  which forms an  $m+n$  strand braid  $\alpha \otimes \beta$  from an  $m$ -braid  $\alpha$  and an  $n$ -braid  $\beta$  by placing them side by side with no crossing between the strands of  $\alpha$  and those of  $\beta$ , as in Figure 1. See for example [Kassel and Turaev 2008, p. 69].

It is easy to see from the definition of Artin's representation that the automorphism of  $\mathbb{F}_{m+n} \cong \mathbb{F}_m * \mathbb{F}_n$  corresponding to  $\alpha \otimes \beta$  is just the free product of the automorphisms corresponding to  $\alpha$  and  $\beta$ .

**Corollary 9.** *The tensor product  $\alpha \otimes \beta$  of braids is order-preserving if and only if both  $\alpha$  and  $\beta$  are order-preserving braids.*

*Proof.* One direction follows from Corollary 3. For if  $\alpha$  and  $\beta$  preserve some orderings of  $\mathbb{F}_m$  and  $\mathbb{F}_n$  respectively, then  $\alpha \otimes \beta$  preserves the corresponding ordering  $<$  of  $\mathbb{F}_m * \mathbb{F}_n \cong \mathbb{F}_{m+n}$ . On the other hand, suppose  $\alpha \otimes \beta$  preserves an ordering of  $\mathbb{F}_{m+n} \cong \mathbb{F}_m * \mathbb{F}_n$ . Considering  $\mathbb{F}_m$  and  $\mathbb{F}_n$  as the natural subgroups of  $\mathbb{F}_m * \mathbb{F}_n$ , we see that the action of  $\alpha \otimes \beta$  leaves each of these subgroups invariant. Therefore the ordering of  $\mathbb{F}_{m+n}$  preserved by  $\alpha \otimes \beta$  restricts to each of the subgroups making the action of the braids  $\alpha$  and  $\beta$  order-preserving.  $\square$

We note the multiple use of the tensor product symbol. Indeed, let us say that the ordered free group  $(\mathbb{F}_n, <)$  *represents* the braid  $\beta \in B_n$  if the automorphism of  $\mathbb{F}_n$  corresponding to  $\beta$  under the Artin representation preserves the ordering  $<$ . We have observed the following.

**Proposition 10.** *If  $(\mathbb{F}_m, <)$  represents  $\alpha \in B_m$  and  $(\mathbb{F}_n, <')$  represents  $\beta \in B_n$ , then  $(\mathbb{F}_m, <) \otimes (\mathbb{F}_n, <')$  represents  $\alpha \otimes \beta \in B_{m+n}$ .*

## 7. Continuity

The goal of this section is to establish that our construction is continuous in an appropriate sense. If  $O(G)$  denotes the set of all (two-sided invariant) orderings of the group  $G$ , there is a natural topology on  $O(G)$ , defined below. Given orderable groups  $F$  and  $G$ , the construction defined in Section 3 can be considered a function

whose input is a pair of orderings  $<_F$  and  $<_G$  and the output is an ordering  $<$  of  $F * G$ , in other words a function

$$O(F) \times O(G) \rightarrow O(F * G).$$

We'll see that it is both continuous and injective.

**7.1. The space of orderings.** The set of orderings  $O(G)$  of the group  $G$  is endowed with a natural topology, as detailed by Sikora [2004]. See also [Dabkowska et al. 2007; Navas 2010]. Consider a specific ordering  $<_G$  of  $G$ , and choose a *finite* number of inequalities among elements of  $G$  which are satisfied using  $<_G$ . Then a basic neighborhood of  $<_G$  consists of all orderings of  $G$  for which all those inequalities remain true. Neighborhoods of this type form a basis for the topology we are considering. Equivalently, a neighborhood of  $<_G$  is defined by choosing some finite set of elements of  $G$  which are positive (greater than the identity) using  $<_G$ . Then take the neighborhood to consist of all orderings of  $G$  under which that finite set remains positive.

It is known, and not difficult to show, that  $O(G)$  is compact and totally disconnected. An isolated point of  $O(G)$  is an ordering which is “finitely determined” in the sense that it is the only ordering of  $G$  for which some finite set of inequalities holds. Sikora [2004] showed that for  $n \geq 2$ ,  $O(\mathbb{Z}^n)$  has no isolated points, and is homeomorphic with the Cantor set. Whether  $O(\mathbb{F}_n)$  has isolated points, for the free group  $\mathbb{F}_n$ ,  $n \geq 2$ , is an open question at the time of writing.

**7.2. Continuity of lexicographic ordering of direct products.** As a warmup to our main result, we consider the lexicographic ordering of direct products  $F \times G$  of ordered groups, as discussed in Section 3 (similar results would hold for the reverse lexicographic ordering). It may be considered a function

$$\mathfrak{L} : O(F) \times O(G) \rightarrow O(F \times G).$$

**Proposition 11.**  $\mathfrak{L}$  is continuous and injective.

*Proof.* We may assume both  $F$  and  $G$  are nontrivial groups; otherwise there is nothing to prove. For injectivity, suppose  $<_F$  and  $<'_F$  are orderings of  $F$  and that  $<_G$  and  $<'_G$  are orderings of  $G$ . Consider  $< = \mathfrak{L}(<_F, <_G)$  and  $<' = \mathfrak{L}(<'_F, <'_G)$ . If  $<_F$  and  $<'_F$  are distinct, there must be an element  $f \in F$  with  $1 <_F f$  but  $f <'_F 1$ . Then we have, for any  $g \in G$ , that  $1 < (f, g)$  and  $(f, g) <' 1$ . It follows that  $<$  and  $<'$  are distinct. Similarly, if  $<_G$  and  $<'_G$  are different, then one can find an element  $(1, g) \in F * G$  with  $(1, g)$  having different signs relative to the orderings  $<$  and  $<'$ . This establishes injectivity.

To establish continuity, note that a basic neighborhood  $\mathfrak{N}_<$  of  $<$  in  $O(F \times G)$

is defined by choosing some finite set of positive elements:

$$(f_1, g_1), \dots, (f_k, g_k), (1, g_{k+1}), \dots, (1, g_{k+l}).$$

Here we have

$$1 <_F f_1, \dots, 1 <_F f_k \quad \text{and} \quad 1 <_G g_{k+1}, \dots, 1 <_G g_{k+l},$$

whereas some of the list  $g_1, \dots, g_k$  may be negative in the ordering  $<_G$ . Possibly  $k = 0$  or  $l = 0$ .

Continuity will be established if we can find neighborhoods  $\mathfrak{N}_{<_F}$  of  $<_F$  in  $O(F)$  and  $\mathfrak{N}_{<_G}$  of  $<_G$  in  $O(G)$  so that  $\mathfrak{L}(\mathfrak{N}_{<_F} \times \mathfrak{N}_{<_G}) \subset \mathfrak{N}_{<}$ . But this is straightforward: take  $\mathfrak{N}_{<_F}$  to be the set of all orderings of  $F$  for which  $f_1, \dots, f_k$  are positive, and  $\mathfrak{N}_{<_G}$  the set of all orderings of  $G$  under which  $g_{k+1}, \dots, g_{k+l}$  are positive.  $\square$

**7.3. Continuity of the ordering of free products.** Recalling the construction in Section 3, we defined a function of ordered groups:

$$\mathfrak{F}((F, <_F), (G, <_G)) = (F * G, <).$$

By abuse of notation, if  $F$  and  $G$  are fixed, but orderings thereof are variable, we may write

$$\mathfrak{F}(<_F, <_G) = <.$$

Then we have a function of spaces of orderings:

$$\mathfrak{F} : O(F) \times O(G) \rightarrow O(F * G).$$

**Theorem 12.**  $\mathfrak{F}$  is continuous and injective.

*Proof.* One may prove injectivity as in Proposition 11; we leave the details to the reader. Note also that we proved continuity of the map  $\mathfrak{L}$  by showing that any finite set of inequalities in  $F \times G$  would be implied (under  $\mathfrak{L}$ ) by finitely many inequalities in  $F$  and in  $G$ .

We will argue similarly in this case; we'll try to avoid excessive notation and sketch the ideas. Suppose  $<_F$  and  $<_G$  are given orderings of  $F$  and  $G$ , respectively, and that  $< = \mathfrak{F}(<_F, <_G)$  is the corresponding ordering of the free product  $F * G$ . A neighborhood  $\mathfrak{N}_{<}$  of  $<$  in the space  $O(F * G)$  consists of all orderings of  $F * G$  for which all members of some finite set  $x_1, \dots, x_k$  of elements of  $F * G$  are positive, where  $1 < x_i$  for  $i = 1, \dots, k$ . But note that  $1 < x_i$  is equivalent to the matrix  $\rho(x_i) - \rho(1)$  being positive in  $M_2(\mathbb{Z}(F \times G)[t])$ , and this is positive if the first nonzero entry of that matrix, expanded in powers of  $t$ , is positive. That entry, an element of  $\mathbb{Z}(F \times G)$ , is positive if the coefficient of its greatest group element, say  $(f_i, g_i)$ , is a positive integer. But the condition that  $(f_i, g_i)$  is the greatest group element appearing in that entry is equivalent to a finite number of inequalities in  $F \times G$ , using the lexicographic ordering. This in turn, as in Proposition 11, is implied by a finite number of inequalities in  $F$  and  $G$  which are in particular



satisfied using the orderings  $<_F$  and  $<_G$ . Using the open neighborhoods  $\mathfrak{N}_{<_F}$  of  $<_F$  and  $\mathfrak{N}_{<_G}$  of  $<_G$  defined by those inequalities, we see  $\mathfrak{F}(\mathfrak{N}_{<_F}, \mathfrak{N}_{<_G}) \subset \mathfrak{N}_{<}$ , which establishes continuity of  $\mathfrak{F}$ .  $\square$

Suppose, in the procedure for defining  $<$  in Section 3, one used some ordering of  $F \times G$  other than the lexicographic one, but otherwise defined  $<$  in the same way. This then defines a function  $O(F \times G) \rightarrow O(F * G)$ , which we will call  $\mathfrak{M}$ , short for matrix construction. The proof of Theorem 12 actually shows that  $\mathfrak{M}$  is continuous. Our specific construction  $\mathfrak{F}$  may therefore be considered a composite

$$O(F) \times O(G) \xrightarrow{\mathfrak{L}} O(F \times G) \xrightarrow{\mathfrak{M}} O(F * G)$$

of two continuous functions, both injective.

## 8. Free product of arbitrarily many ordered groups

We now consider an arbitrary collection of ordered groups. For convenience, we assume the groups are indexed by an ordinal number  $\gamma$  and denote the collection by  $\{(F_\alpha, <_{F_\alpha})\}_{\alpha < \gamma}$ . So far we have been considering the case  $\gamma = 2$ .

**Theorem 13.** *Let  $\gamma \geq 2$  be an ordinal. Suppose  $\{(F_\alpha, <_{F_\alpha})\}_{\alpha < \gamma}$  is a collection of ordered groups and let  $F := *_{\alpha < \gamma} F_\alpha$  denote the free product. Then there is an ordering  $<_F$  of  $F$ , so that  $(F, <_F)$  is an ordered group, denoted*

$$\mathfrak{F}(\{(F_\alpha, <_{F_\alpha})\}_{\alpha < \gamma}) := (F, <_F),$$

and such that the following hold:

- (1) For each  $\alpha < \gamma$  the restriction of  $<_F$  to the natural subgroup  $F_\alpha$  of  $F$  equals  $<_{F_\alpha}$ .
- (2) If  $\{(G_\alpha, <_{G_\alpha})\}_{\alpha < \gamma}$  is another collection of ordered groups with  $G := *_{\alpha < \gamma} G_\alpha$  and

$$(G, <_G) = \mathfrak{F}(\{(G_\alpha, <_{G_\alpha})\}_{\alpha < \gamma}),$$

then for any collection  $\phi_\alpha : F_\alpha \rightarrow G_\alpha$  of homomorphisms defined for all  $\alpha < \gamma$  and which are order-preserving, relative to  $<_{F_\alpha}$  and  $<_{G_\alpha}$ , the free product homomorphism  $*_{\alpha < \gamma} \phi_\alpha : F \rightarrow G$  is order-preserving, relative to  $<_F$  and  $<_G$ .

*Proof.* We will define the ordering of  $F$  by induction, possibly transfinite. For that reason, we'll call the ordering  $<_\gamma$  and only later call it  $<_F$  also. The base for the induction, for  $\gamma = 2$ , is Theorem 1, taking  $<_2$  to be the ordering  $<$  defined there. For induction we may assume that orderings  $<_\beta$  have been defined for all the groups  $*_{\alpha < \beta} F_\alpha$  for all  $1 < \beta < \gamma$ , and that they satisfy (1) and (2) with  $\beta$  replacing  $\gamma$ . Note that  $*_{\alpha < \beta} F_\alpha$  is naturally a subgroup of  $*_{\alpha < \gamma} F_\alpha$ . To facilitate the induction, we'll prove that in addition to properties (1) and (2) of the theorem,  $<_\gamma$  further satisfies:

(3) If  $1 < \beta < \gamma$  the restriction of the ordering  $<_\gamma$  to  $*_{\alpha < \beta} F_\alpha$  coincides with  $<_\beta$ .

Again, by Theorem 1 this is satisfied for the base case  $\gamma = 2$ . To construct  $<_\gamma$  we consider two cases.

**Case 1:**  $\gamma$  is a successor ordinal:  $\gamma = \beta + 1$ . Since  $<_\beta$  is by hypothesis already defined, and noting that  $F$  can be naturally identified with  $(*_{\alpha < \beta} F_\alpha) * F_\beta$ , we use the functor  $\mathfrak{F}$  defined in the proof of Theorem 1 and take

$$(F, <_\gamma) \cong ((*_{\alpha < \beta} F_\alpha) * F_\beta, <_\gamma) := \mathfrak{F}((*_{\alpha < \beta} F_\alpha, <_\beta), (F_\beta, <_\beta)).$$

**Case 2:**  $\gamma$  is a limit ordinal. Then the group  $*_{\alpha < \gamma} F_\alpha$  is the union of its subgroups  $*_{\alpha < \beta} F_\alpha$  with  $\beta < \gamma$ . Thus to compare two group elements  $x, y$  in  $*_{\alpha < \gamma} F_\alpha$ , choose  $\beta < \gamma$  for which  $x, y \in *_{\alpha < \beta} F_\alpha$  and define  $x <_\gamma y$  if and only if  $x <_\beta y$ . By property (3), which may be assumed for ordinals less than  $\gamma$ , this does not depend on choice of  $\beta$ .

In either case, it is routine to verify that the ordering  $<_\gamma$  (also called  $<_F$ ) satisfies the conditions (1), (2) and (3).  $\square$

## 9. Left-ordered groups

An ordering  $<$  of the elements of a group  $G$  is a left-ordering if for all  $f, g, h \in G$  one has

$$g < h \implies fg < fh;$$

in this case we call  $(G, <)$  a left-ordered group. It is much easier than for the ordered case to see that the free product of left-ordered groups is left-orderable. For left-ordered groups  $(F, <_F)$  and  $(G, <_G)$  consider the short exact sequence

$$1 \rightarrow K \rightarrow F * G \rightarrow F \times G \rightarrow 1,$$

where  $F * G \rightarrow F \times G$  is the canonical homomorphism. The kernel  $K$  is a free group, which is orderable, and one can left-order  $F \times G$ , lexicographically. Since left-orderability (unlike orderability) is always preserved under extensions, we conclude that  $F * G$  is left-orderable.

On the other hand, our construction of the ordering  $<$  for the free product of ordered groups may be revised in a straightforward way to the left-ordered (or right-ordered) situation. One must be a bit careful. For a left-ordered group  $(G, <)$  the group ring  $\mathbb{Z}(G)$  is not, strictly speaking, an ordered ring by our definition. For example if we have  $g, g', h \in G$  with  $g < g'$  but  $gh > g'h$  then the ring elements  $g' - g$  and  $h$  are positive, whereas their product  $g'h - gh$  is not positive. However the product in the other order,  $hg' - hg$ , is necessarily positive, and more generally a positive element of  $\mathbb{Z}(G)$  multiplied on the left by a monomial with positive coefficient remains positive. This is enough to establish left-invariance of  $<$  in the proof of Theorem 1.

Therefore, we conclude that all the results above remain true if “ordered” is replaced by “left-ordered” throughout. In particular, the category of left-ordered groups and order-preserving homomorphisms is also a tensor category using our functorial construction.

## 10. Concluding remarks

The ordering we construct is by no means canonical; for example other choices of ordering the direct product, or the entries of matrices, can lead to a different ordering of the free product which satisfies the conditions of Theorem 1, and even defines a tensor category structure. Indeed, Corollary 6 reveals the asymmetry of the construction. In a real sense, the first group in the free product of two groups is treated preferentially in our construction. It could as well have been the reverse.

The argument given here does not extend to the larger category of ordered groups and order-homomorphisms (which are not necessarily injective) as some positive matrix entries may be mapped to zero under such a map. Extending our results to this category seems to be an open question.

As noted in [Bergman 1990], much of this can be done in the more general setting of ordered semigroups; see also [Johnson 1968]. We leave such generalization for the interested reader to contemplate.

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## MULTIPLICATION OF DISTRIBUTIONS AND A NONLINEAR MODEL IN ELASTODYNAMICS

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We consider the system  $u_t + (u^2/2)_x = \sigma_x$ ,  $\sigma_t + u\sigma_x = k^2 u_x$ , where  $k$  is a real number and the unknowns  $u(x, t)$  and  $\sigma(x, t)$  belong to convenient spaces of distributions. For this simplified model from elastodynamics, a rigorous solution concept defined in the setting of a distributional product is used. The explicit solution of a Riemann problem and the possible emergence of a  $\delta$  shock wave are established. For initial conditions containing a Dirac measure, a  $\delta'$  shock wave solution is also presented.

### 1. Introduction and main results

Let us consider the system

$$\begin{aligned} (1) \quad & u_t + \left(\frac{1}{2}u^2\right)_x = \sigma_x, \\ (2) \quad & \sigma_t + u\sigma_x = k^2 u_x, \end{aligned}$$

where  $t \in \mathbb{R}$  is the time variable,  $x \in \mathbb{R}$  is the one-dimensional space variable,  $u(x, t)$  and  $\sigma(x, t)$  are the unknowns state variables and  $k > 0$  is a real number.

This strictly hyperbolic system in nonconservative form arises in a simplified model from elastodynamics where  $u$  is the velocity,  $\sigma$  is the stress and  $k$  is the speed of propagation of the elastic waves. Several aspects of this model were already studied by J. J. Cauret, J. F. Colombeau, A. Y. Le Roux, and K. T. Joseph in the setting of Colombeau generalized functions. For details, see [Cauret et al. 1989; Colombeau and LeRoux 1988; Joseph 1997]. When initial data are smooth, it is well-known that global solutions do not exist because discontinuities in  $u$  and  $\sigma$  appear in finite time. Meanwhile, when  $u$  and  $\sigma$  are discontinuous, products of distributions arise which make no sense in the classic theory of distributions.

Different concepts of solution can be found in the literature: the weak asymptotic method [Danilov and Mitrovic 2008; Danilov and Shelkovich 2005a; 2005b], the measure theoretic method [Bouchut and James 1999; Brenier and Grenier 1998;

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Chen and Liu 2003; Huang 2005], the use of smooth function nets and weighted measures spaces [Keyfitz and Kranzer 1995], split delta functions [Nedeljkov 2002; Nedeljkov and Oberguggenberger 2008], Colombeau generalized functions [Cauret et al. 1989; Colombeau and LeRoux 1988; Joseph 1997; Nedeljkov 2004], and others. We will adopt a solution concept which is a consistent extension of the classical solution concept and is defined within the setting of a theory of distributional products. In our framework, the product of distributions is always a distribution that is not defined by approximation processes. Our products depend upon the choice of a certain function  $\alpha$  that encodes the indeterminacy inherent to such products. We stress that this indeterminacy is not in general avoidable and in many situations it also has a physical meaning. Concerning this point let us mention [Bressan and Rampazzo 1988; Colombeau and LeRoux 1988; Dal Maso et al. 1995; Sarrico 2003]. Naturally the existence and the solutions of differential equations or systems containing such products may depend (or not) on  $\alpha$ . We call such solutions  $\alpha$ -solutions. The possibility of its physical occurrence depends on the physical system. Sometimes we cannot previously know the behavior of the physical system, possibly due to features that were not considered in the formulation of the model with the goal of simplifying it. Thus, the mathematical indeterminacy sometimes observed may have this origin. In the present paper, however, the  $\alpha$ -solutions when they exist are independent of  $\alpha$ .

First, we consider for system (1)–(2) the initial conditions

$$(3) \quad u(x, 0) = a_1 + (a_2 - a_1)H(x),$$

$$(4) \quad \sigma(x, 0) = b_1 + (b_2 - b_1)H(x),$$

where  $a_1, a_2, b_1, b_2 \in \mathbb{R}$  and  $H$  stands for the Heaviside function. We will compute all  $\alpha$ -solutions of this problem within the space  $W$  of pairs of distributions  $(u, \sigma)$  of the form

$$(5) \quad u(x, t) = u_1 + (u_2 - u_1)H(x - Vt) + g(t)\delta(x - Vt),$$

$$(6) \quad \sigma(x, t) = \sigma_1 + (\sigma_2 - \sigma_1)H(x - Vt),$$

where  $\delta$  stands for the Dirac measure concentrated at the origin,  $u_1, u_2, \sigma_1, \sigma_2, V \in \mathbb{R}$ , and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^1$ -function. If  $b_1 = b_2$  there exists  $\alpha$ -solutions in  $W$  if and only if  $a_1 = a_2$  and we will see, in the space  $W$ , the arising of the  $\alpha$ -solution corresponding to the constant states,

$$(7) \quad u(x, t) = a_1,$$

$$(8) \quad \sigma(x, t) = b_1.$$

If  $b_1 \neq b_2$  there exists an  $\alpha$ -solution in  $W$  if and only if  $a_1 \neq a_2$  with the possible arising of the traveling wave

$$(9) \quad u(x, t) = a_1 + (a_2 - a_1)H(x - Vt),$$

$$(10) \quad \sigma(x, t) = b_1 + (b_2 - b_1)H(x - Vt),$$

which propagates with speed  $V = (a_1 + a_2)/2 - (b_2 - b_1)/(a_2 - a_1)$ . These  $\alpha$ -solutions depend neither on  $\alpha$  nor on the constant  $k > 0$ !

From a mathematical point of view, this situation leads us to consider the interesting case  $k = 0$  in which the eigenvalues of the system (1)–(2),  $\lambda_1 = u - k$  and  $\lambda_2 = u + k$  coincide and the system loses the strict hyperbolicity. In this case, assuming certain conditions to be specified later, we will see the possible emergence (in the same space  $W$ ) of a delta shock wave with the form (5). Thus, the space of functions is not sufficient to contain all possible  $\alpha$ -solutions of the Riemann problem (1)–(4) with  $k = 0$ .

Next, for the system (1)–(2), still with  $k = 0$ , we consider the initial conditions

$$(11) \quad u(x, 0) = a,$$

$$(12) \quad \sigma(x, 0) = b + m \delta(x),$$

where  $a, b, m \in \mathbb{R}$  and  $m \neq 0$ . We will see the possible emergence of the  $\alpha$ -solution,

$$(13) \quad u(x, t) = a + mt \delta'(x - at),$$

$$(14) \quad \sigma(x, t) = b + m \delta(x - at),$$

containing a  $\delta$  wave and a  $\delta'$  shock wave, both with speed  $a$ . This result is obtained within the space  $Z$  of pairs of distributions  $(u, \sigma)$  of the form

$$u(x, t) = u_1 + f(t) \delta'[x - \gamma(t)],$$

$$\sigma(x, t) = \sigma_1 + p \delta[x - \gamma(t)],$$

where  $u_1, \sigma_1, p \in \mathbb{R}$  and  $f, \gamma : \mathbb{R} \rightarrow \mathbb{R}$  are  $C^1$ -functions. Hence, the problem (1)–(2) with initial conditions (11) and (12) with  $k = 0$  also evolves to a situation more singular than the initial one and the measure space is no longer sufficient to contain all its possible  $\alpha$ -solutions. It is also a remarkable fact that, in the space  $Z$ , all those  $\alpha$ -solutions, when they exist, are independent of  $\alpha$ .

Regarding  $\delta'$ -waves, we must remember that they were first introduced by E. Yu. Panov and V. M. Shelkovich for certain systems of conservation laws [Panov and Shelkovich 2006; Shelkovich 2006]. The results show that these systems subjected to piecewise continuous initial data may develop not only  $\delta$ -waves, but also  $\delta'$ -waves [Sarrico 2012b; Shelkovich 2007; 2008].

Let us summarize the contents of this paper. In Section 2 a survey of the main ideas and formulas for multiplying distributions is presented. In Section 3 we define

the concept of  $\alpha$ -solution for the system (1)–(2). In Sections 4, 5 and 6 we justify rigorously all we have said in the beginning of this introduction.

## 2. Products of distributions

Let  $C^\infty$  be the space of indefinitely differentiable real or complex-valued functions defined on  $\mathbb{R}^N$ ,  $N \in \{1, 2, 3, \dots\}$ , and  $\mathcal{D}$  the subspace of  $C^\infty$  consisting of those functions with compact support. Let  $\mathcal{D}'$  be the space of Schwartz distributions and  $L(\mathcal{D})$  the space of continuous linear maps  $\phi : \mathcal{D} \rightarrow \mathcal{D}$ , where we suppose  $\mathcal{D}$  is endowed with the usual topology. We will sketch the main ideas of our distributional product (the reader can look at (18), (22) and (24) as definitions, if he prefers to skip this presentation). For proofs and other details concerning this product see [Sarrico 1988].

First, we define a product  $T\phi \in \mathcal{D}'$  for  $T \in \mathcal{D}'$  and  $\phi \in L(\mathcal{D})$  by

$$\langle T\phi, \xi \rangle = \langle T, \phi(\xi) \rangle,$$

for all  $\xi \in \mathcal{D}$ ; this makes  $\mathcal{D}'$  a right  $L(\mathcal{D})$ -module. Next, we define an epimorphism  $\tilde{\zeta} : L(\mathcal{D}) \rightarrow \mathcal{D}'$ , where the image of  $\phi$  is the distribution  $\tilde{\zeta}(\phi)$  given by

$$\langle \tilde{\zeta}(\phi), \xi \rangle = \int \phi(\xi),$$

for all  $\xi \in \mathcal{D}$  (when the domain of the integral is not specified we assume it to be  $\mathbb{R}^N$ ); given  $S \in \mathcal{D}'$ , we say that  $\phi$  is a representative operator of  $S$  if  $\tilde{\zeta}(\phi) = S$ . For instance, if  $\beta \in C^\infty$  is seen as a distribution, the operator  $\phi_\beta \in L(\mathcal{D})$  defined by  $\phi_\beta(\xi) = \beta\xi$ , for all  $\xi \in \mathcal{D}$ , is a representative operator of  $\beta$  because, for all  $\xi \in \mathcal{D}$ , we have

$$\langle \tilde{\zeta}(\phi_\beta), \xi \rangle = \int \phi_\beta(\xi) = \int \beta\xi = \langle \beta, \xi \rangle.$$

For this reason  $\tilde{\zeta}(\phi_\beta) = \beta$ . If  $T \in \mathcal{D}'$ , we also have

$$\langle T\phi_\beta, \xi \rangle = \langle T, \phi_\beta(\xi) \rangle = \langle T, \beta\xi \rangle = \langle T\beta, \xi \rangle,$$

for all  $\xi \in \mathcal{D}$ . Hence,

$$T\beta = T\phi_\beta.$$

Thus, given  $T, S \in \mathcal{D}'$ , we are tempted to define a natural product by setting  $TS := T\phi$ , where  $\phi \in L(\mathcal{D})$  is a representative operator of  $S$ , i.e.,  $\phi$  is such that  $\tilde{\zeta}(\phi) = S$ . Unfortunately, this product is not well defined, because  $TS$  depends on the representative  $\phi \in L(\mathcal{D})$  of  $S \in \mathcal{D}'$ .

This difficulty can be overcome, if we fix  $\alpha \in \mathcal{D}$  with  $\int \alpha = 1$  and define  $s_\alpha : L(\mathcal{D}) \rightarrow L(\mathcal{D})$  by

$$(15) \quad [(s_\alpha\phi)(\xi)](y) = \int \phi[(\tau_y\check{\alpha})\xi],$$



for all  $\xi \in \mathcal{D}$  and all  $y \in \mathbb{R}^N$ , where  $\tau_y \check{\alpha}$  is given by  $(\tau_y \check{\alpha})(x) = \check{\alpha}(x - y) = \alpha(y - x)$  for all  $x \in \mathbb{R}^N$ . It can be proved that for each  $\alpha \in \mathcal{D}$  with  $\int \alpha = 1$ ,  $s_\alpha(\phi) \in L(\mathcal{D})$ ,  $s_\alpha$  is linear,  $s_\alpha \circ s_\alpha = s_\alpha$  ( $s_\alpha$  is a projector of  $L(\mathcal{D})$ ),  $\ker s_\alpha = \ker \tilde{\zeta}$ , and  $\tilde{\zeta} \circ s_\alpha = \tilde{\zeta}$ .

Now, for each  $\alpha \in \mathcal{D}$ , we can define a general  $\alpha$ -product  $\odot_\alpha$  of  $T \in \mathcal{D}'$  with  $S \in \mathcal{D}'$  by setting

$$(16) \quad T \odot_\alpha S := T(s_\alpha \phi),$$

where  $\phi \in L(\mathcal{D})$  is a representative operator of  $S \in \mathcal{D}'$ . This  $\alpha$ -product is independent of the representative  $\phi$  of  $S$ , because if  $\phi$  and  $\psi$  are such that  $\tilde{\zeta}(\phi) = \tilde{\zeta}(\psi) = S$ , then  $\phi - \psi \in \ker \tilde{\zeta} = \ker s_\alpha$ . Hence,

$$T(s_\alpha \phi) - T(s_\alpha \psi) = T[s_\alpha(\phi - \psi)] = 0.$$

Since  $\phi$  in (16) satisfies  $\tilde{\zeta}(\phi) = S$ , we have  $\int \phi(\xi) = \langle S, \xi \rangle$  for all  $\xi \in \mathcal{D}$ , and by (15)

$$[(s_\alpha \phi)(\xi)](y) = \langle S, (\tau_y \check{\alpha})\xi \rangle = \langle S\xi, \tau_y \check{\alpha} \rangle = (S\xi * \alpha)(y),$$

for all  $y \in \mathbb{R}^N$ , which means that  $(s_\alpha \phi)(\xi) = S\xi * \alpha$ . Therefore, for all  $\xi \in \mathcal{D}$ ,

$$\begin{aligned} \langle T \odot_\alpha S, \xi \rangle &= \langle T(s_\alpha \phi), \xi \rangle = \langle T, (s_\alpha \phi)(\xi) \rangle = \langle T, S\xi * \alpha \rangle \\ &= [T * (S\xi * \alpha)](0) = [(S\xi) \check{*} (T * \check{\alpha})](0) = \langle (T * \check{\alpha})S, \xi \rangle, \end{aligned}$$

and we obtain an easier formula for the general product (16):

$$(17) \quad T \odot_\alpha S = (T * \check{\alpha})S.$$

In general, this  $\alpha$ -product is neither commutative nor associative but it is bilinear and satisfies the Leibniz rule written in the form

$$D_k(T \odot_\alpha S) = (D_k T) \odot_\alpha S + T \odot_\alpha (D_k S),$$

where  $D_k$  is the usual  $k$ -partial derivative operator in distributional sense ( $k = 1, 2, \dots, N$ ).

Recall that the usual Schwartz products of distributions are not associative and the commutative property is a convention inherent to the definition of such products (see the classical monograph of Schwartz [1966, pp. 117, 118 and 121] where these products are defined). Unfortunately, the  $\alpha$ -product (17), in general, is not consistent with the classical Schwartz products of distributions with functions.

In order to obtain consistency with the usual product of a distribution with a  $C^\infty$ -function, we are going to introduce some definitions and single out a certain subspace  $H_\alpha$  of  $L(\mathcal{D})$ .

An operator  $\phi \in L(\mathcal{D})$  is said to vanish on an open set  $\Omega \subset \mathbb{R}^N$ , if and only if  $\phi(\xi) = 0$  for all  $\xi \in \mathcal{D}$  with support contained in  $\Omega$ . The support of an operator  $\phi \in L(\mathcal{D})$  will be defined as the complement of the largest open set in which  $\phi$  vanishes.

Let  $\mathcal{N}$  be the set of operators  $\phi \in L(\mathcal{D})$  whose support has Lebesgue measure zero, and  $\rho(C^\infty)$  the set of operators  $\phi \in L(\mathcal{D})$  defined by  $\phi(\xi) = \beta\xi$  for all  $\xi \in \mathcal{D}$ , with  $\beta \in C^\infty$ . For each  $\alpha \in \mathcal{D}$ , with  $\int \alpha = 1$ , let us consider the space  $H_\alpha = \rho(C^\infty) \oplus s_\alpha(\mathcal{N}) \subset L(\mathcal{D})$ . It can be proved that  $\zeta_\alpha := \tilde{\zeta}|_{H_\alpha} : H_\alpha \rightarrow C^\infty \oplus \mathcal{D}'_\mu$  is an isomorphism ( $\mathcal{D}'_\mu$  stands for the space of distributions whose support has Lebesgue measure zero). Therefore, if  $T \in \mathcal{D}'$  and  $S = \beta + f \in C^\infty \oplus \mathcal{D}'_\mu$ , a new  $\alpha$ -product,  $\dot{\alpha}$ , can be defined by  $T_{\dot{\alpha}}S := T\phi_\alpha$ , where for each  $\alpha$ ,  $\phi_\alpha = \zeta_\alpha^{-1}(S) \in H_\alpha$ . Hence,

$$\begin{aligned} T_{\dot{\alpha}}S &= T\zeta_\alpha^{-1}(S) = T[\zeta_\alpha^{-1}(\beta + f)] \\ &= T[\zeta_\alpha^{-1}(\beta) + \zeta_\alpha^{-1}(f)] = T\beta + T \underset{\alpha}{\odot} f = T\beta + (T * \check{\alpha})f, \end{aligned}$$

and putting  $\alpha$  instead of  $\check{\alpha}$  (to simplify), we get

$$(18) \quad T_{\dot{\alpha}}S = T\beta + (T * \alpha)f.$$

Thus, the referred consistency is obtained when the  $C^\infty$ -function is placed at the right-hand side; if  $S \in C^\infty$ , then  $f = 0$ ,  $S = \beta$ , and  $T_{\dot{\alpha}}S = T\beta$ .

The  $\alpha$ -product (18) can be easily extended for  $T \in \mathcal{D}'^p$  and  $S = \beta + f \in C^p \oplus \mathcal{D}'_\mu$ , where  $p \in \{0, 1, 2, \dots, \infty\}$ ,  $\mathcal{D}'^p$  is the space of distributions of order  $\leq p$  in the sense of Schwartz ( $\mathcal{D}'^\infty$  means  $\mathcal{D}'$ ),  $T\beta$  is the Schwartz product of a  $\mathcal{D}'^p$ -distribution with a  $C^p$ -function, and  $(T * \alpha)f$  is the usual product of a  $C^\infty$ -function with a distribution. This extension is clearly consistent with all Schwartz products of  $\mathcal{D}'^p$ -distributions with  $C^p$ -functions, if the  $C^p$ -functions are placed at the right-hand side. It also keeps the bilinearity and satisfies the Leibniz rule written in the form

$$D_k(T_{\dot{\alpha}}S) = (D_kT)_{\dot{\alpha}}S + T_{\dot{\alpha}}(D_kS),$$

clearly under certain natural conditions; for  $T \in \mathcal{D}'^p$ , we must suppose  $S \in C^{p+1} \oplus \mathcal{D}'_\mu$ . Moreover, these products are invariant by translations, that is,

$$\tau_a(T_{\dot{\alpha}}S) = (\tau_aT)_{\dot{\alpha}}(\tau_aS),$$

where  $\tau_a$  stands for the usual translation operator in distributional sense. These products are also invariant for the action of any group of linear transformations  $h : \mathbb{R}^N \rightarrow \mathbb{R}^N$  with  $|\det h| = 1$ , that leave  $\alpha$  invariant.

Thus, for each  $\alpha \in \mathcal{D}$  with  $\int \alpha = 1$ , formula (18) allows us to evaluate the product of  $T \in \mathcal{D}'^p$  with  $S \in C^p \oplus \mathcal{D}'_\mu$ ; therefore, we have obtained a family of products, one for each  $\alpha$ .

From now on, we always consider the dimension  $N = 1$ . For instance, if  $\beta$  is a continuous function we have for each  $\alpha$ , by applying (18),

$$\begin{aligned} \delta_{\dot{\alpha}}\beta &= \delta_{\dot{\alpha}}(\beta + 0) = \delta\beta + (\delta * \alpha)0 = \beta(0)\delta, \\ \beta_{\dot{\alpha}}\delta &= \beta_{\dot{\alpha}}(0 + \delta) = \beta 0 + (\beta * \alpha)\delta = [(\beta * \alpha)(0)]\delta, \\ (19) \quad \delta_{\dot{\alpha}}\delta &= \delta_{\dot{\alpha}}(0 + \delta) = \delta 0 + (\delta * \alpha)\delta = \alpha\delta = \alpha(0)\delta, \end{aligned}$$

$$(20) \quad H_{\dot{\alpha}}\delta = (H * \alpha)\delta = \left( \int_{-\infty}^{+\infty} \alpha(-\tau)H(\tau) d\tau \right) \delta = \left( \int_{-\infty}^0 \alpha \right) \delta,$$

$$(21) \quad (D\delta)_{\dot{\alpha}}(D\delta) = [(D\delta) * \alpha]D\delta = \alpha'(0)D\delta - \alpha''(0)\delta.$$

For each  $\alpha$ , the support of the  $\alpha$ -product (18) satisfies  $\text{supp}(T_{\dot{\alpha}}S) \subset \text{supp } S$ , as for usual functions, but it may happen that  $\text{supp}(T_{\dot{\alpha}}S) \not\subset \text{supp } T$ .

It is also possible to multiply many other distributions preserving the consistency with all Schwartz products of distributions with functions. For instance, using the Leibniz formula to extend the  $\alpha$ -products, it is possible to write

$$(22) \quad T_{\dot{\alpha}}S = Tw + (T * \alpha)f,$$

with  $T \in \mathcal{D}'^{-1}$  and  $S = w + f \in L_{\text{loc}}^1 \oplus \mathcal{D}'_{\mu}$ , where  $\mathcal{D}'^{-1}$  stands for the space of distributions  $T \in \mathcal{D}'$  such that  $DT \in \mathcal{D}'^0$  and  $Tw$  is the usual pointwise product of  $T \in \mathcal{D}'^{-1}$  with  $w \in L_{\text{loc}}^1$ . Recall that, locally,  $T$  can be read as a function of bounded variation (see [Sarrico 2012a, §2] for details). For instance, since  $H \in \mathcal{D}'^{-1}$  and  $H = H + 0 \in L_{\text{loc}}^1 \oplus \mathcal{D}'_{\mu}$ , we have

$$(23) \quad H_{\dot{\alpha}}H = HH + (H * \alpha)0 = H.$$

because  $H \in \mathcal{D}'^{-1}$  and  $H = H + 0 \in L_{\text{loc}}^1 \oplus \mathcal{D}'_{\mu}$ . More generally, if  $T \in \mathcal{D}'^{-1}$  and  $S \in L_{\text{loc}}^1$ , then  $T_{\dot{\alpha}}S = TS$ ; actually, using (22) we can write

$$T_{\dot{\alpha}}S = T_{\dot{\alpha}}(S + 0) = TS + (T * \alpha)0 = TS.$$

Thus, in distributional sense, the  $\alpha$ -products of functions that, locally, are of bounded variation coincide with the usual pointwise product of these functions considered as a distribution. We stress that in (18) or (22) the convolution  $T * \alpha$  is not to be understood as an approximation of  $T$ . Those formulas are exact.

Another useful extension that will be applied is given by the formula

$$(24) \quad T_{\dot{\alpha}}S = D(Y_{\dot{\alpha}}S) - Y_{\dot{\alpha}}(DS),$$

for  $T \in \mathcal{D}'^0 \cap \mathcal{D}'_{\mu}$  and  $S, DS \in L_{\text{loc}}^1 \oplus \mathcal{D}'_c$ , where  $\mathcal{D}'_c \subset \mathcal{D}'_{\mu}$  is the space of distributions whose support is at most countable, and  $Y \in \mathcal{D}'^{-1}$  is such that  $DY = T$  (the products  $Y_{\dot{\alpha}}S$  and  $Y_{\dot{\alpha}}(DS)$  are supposed to be computed by (18) or (22)). The value of  $T_{\dot{\alpha}}S$  given by (24) is independent of the choice of  $Y \in \mathcal{D}'^{-1}$  such that  $DY = T$  (see

[Sarrico 2012a, p. 1004] for the proof). For instance, by (24) and (20) we have, for any  $\alpha$ ,

$$(25) \quad \delta_{\dot{\alpha}} H = D(H_{\dot{\alpha}} H) - H_{\dot{\alpha}}(DH) = DH - H_{\dot{\alpha}}\delta = \delta - \left( \int_{-\infty}^0 \alpha \right) \delta = \left( \int_0^{+\infty} \alpha \right) \delta,$$

so that

$$(26) \quad H_{\dot{\alpha}}\delta + \delta_{\dot{\alpha}}H = \delta$$

for any  $\alpha$ . The products (18), (22), and (24) are compatible; that is, if an  $\alpha$ -product can be computed by two of them, the result is the same.

### 3. The $\alpha$ -solution concept for the system (1)–(2)

Let  $I$  be an interval of  $\mathbb{R}$  with more than one point and let  $\mathcal{F}(I)$  be the space of continuously differentiable maps  $\tilde{u} : I \rightarrow \mathcal{D}'$  in the sense of the usual topology of  $\mathcal{D}'$ . For  $t \in I$  the notation  $[\tilde{u}(t)](x)$  is sometimes used to emphasize that the distribution  $\tilde{u}(t)$  acts on functions  $\xi \in \mathcal{D}$  which depend on  $x$ .

Let  $\Sigma(I)$  be the space of functions  $u : \mathbb{R} \times I \rightarrow \mathbb{C}$  such that

- (a) for each  $t \in I$ ,  $u(x, t) \in L^1_{\text{loc}}(\mathbb{R})$ ,
- (b)  $\tilde{u} : I \rightarrow \mathcal{D}'$ , defined by  $[\tilde{u}(t)](x) = u(x, t)$  is in  $\mathcal{F}(I)$ .

The natural injection  $u \mapsto \tilde{u}$  of  $\Sigma(I)$  into  $\mathcal{F}(I)$  allows us to identify any function of  $\Sigma(I)$  with a certain map in  $\mathcal{F}(I)$ . Since  $C^1(\mathbb{R} \times I) \subset \Sigma(I)$ , we can write the inclusions

$$C^1(\mathbb{R} \times I) \subset \Sigma(I) \subset \mathcal{F}(I).$$

Consequently, the identification  $u \mapsto \tilde{u}$  allows us to write the system (1)–(2) as follows

$$(27) \quad \frac{d\tilde{u}}{dt}(t) + \frac{1}{2}D[\tilde{u}(t)_{\dot{\alpha}}\tilde{u}(t)] = D\tilde{\sigma}(t),$$

$$(28) \quad \frac{d\tilde{\sigma}}{dt}(t) + \tilde{u}(t)_{\dot{\alpha}}D\tilde{\sigma}(t) = k^2D\tilde{u}(t).$$

**Definition 1.** Given  $\alpha$ , the pair  $(\tilde{u}, \tilde{\sigma}) \in \mathcal{F}(I) \times \mathcal{F}(I)$  will be called an  $\alpha$ -solution for the system (27)–(28) on  $I$ , if the  $\alpha$ -products that appear in this system are well defined and both equations are satisfied for all  $t \in I$ .

We have the following results:

**Theorem 2.** *If  $(u, \sigma)$  is a classical solution of (1)–(2) on  $\mathbb{R} \times I$  then, for any  $\alpha$ , the pair  $(\tilde{u}, \tilde{\sigma}) \in \mathcal{F}(I) \times \mathcal{F}(I)$  defined by  $[\tilde{u}(t)](x) = u(x, t)$ ,  $[\tilde{\sigma}(t)](x) = \sigma(x, t)$ , is an  $\alpha$ -solution of (27)–(28) on  $I$ .*

Note that by a classical solution of (1)–(2) on  $\mathbb{R} \times I$ , we mean a pair of  $C^1$ -functions  $(u(x, t), \sigma(x, t))$  that satisfies (1)–(2) on  $\mathbb{R} \times I$ .

**Theorem 3.** *If  $u, \sigma : \mathbb{R} \times I \rightarrow \mathbb{C}$  are  $C^1$ -functions and, for a certain  $\alpha$ , the pair  $(\tilde{u}, \tilde{\sigma}) \in \mathcal{F}(I) \times \mathcal{F}(I)$  defined by  $[\tilde{u}(t)](x) = u(x, t)$ ,  $[\tilde{\sigma}(t)](x) = \sigma(x, t)$  is an  $\alpha$ -solution of (27)–(28) on  $I$ , then the pair  $(u, \sigma)$  is a classical solution of (1)–(2) on  $\mathbb{R} \times I$ .*

For the proof it is enough to observe that any  $C^1$ -function  $u(x, t)$  can be read as a continuously differentiable function  $\tilde{u} \in \mathcal{F}(I)$  defined by  $[\tilde{u}(t)](x) = u(x, t)$  and to use the consistency of the  $\alpha$ -products with the classical Schwartz products.

Replacing  $\tilde{u}(t) \dot{\alpha} D\tilde{\sigma}(t)$  by  $D\tilde{\sigma}(t) \dot{\alpha} \tilde{u}(t)$  in (28), we get

$$(29) \quad \frac{d\tilde{\sigma}}{dt}(t) + D\tilde{\sigma}(t) \dot{\alpha} \tilde{u}(t) = k^2 D\tilde{u}(t),$$

which is not equivalent to (28) since our  $\alpha$ -products are not, in general, commutative. However, all we have said for the systems (1)–(2) and (27)–(28) is also valid for the systems formed by (1) and (2) and by (27) and (29). Taking advantage of this situation, we introduce a definition that further extends the concept of a classical solution:

**Definition 4.** Given  $\alpha$ , we define as an  $\alpha$ -solution for the system (1)–(2) on  $I$  any  $\alpha$ -solution of the system formed by (27) and (28) or by (27) and (29) on  $I$ .

As a consequence, an  $\alpha$ -solution  $(\tilde{u}, \tilde{\sigma})$  in this sense, read as an usual distributional solution  $(u, \sigma)$ , affords a consistent extension of the concept of a classical solution for the system (1)–(2). Thus, and for short, we also call  $(u, \sigma)$  an  $\alpha$ -solution of (1)–(2).

#### 4. The Riemann problem (1)–(4) with $k > 0$

Let us consider the system (1)–(2) with  $k > 0$ . We also consider  $(x, t) \in \mathbb{R} \times \mathbb{R}$  (we could also take  $\mathbb{R} \times [0, +\infty[$ ) and the unknowns  $u(x, t)$  and  $\sigma(x, t)$  submitted to the initial conditions (3) and (4). When we read this problem in  $\mathcal{F}(\mathbb{R})$  having in mind the identification  $u \mapsto \tilde{u}$ , we must replace the system (1)–(2) by the system (27)–(28) and the conditions (3)–(4) by the following ones:

$$(30) \quad \tilde{u}(0) = a_1 + (a_2 - a_1)H,$$

$$(31) \quad \tilde{\sigma}(0) = b_1 + (b_2 - b_1)H.$$

We will give, explicitly, all  $\alpha$ -solutions for this problem which belong to a set  $\tilde{W}$  defined as follows:  $(\tilde{u}, \tilde{\sigma}) \in \tilde{W}$  if and only if  $\tilde{u}, \tilde{\sigma} \in \mathcal{F}(\mathbb{R})$  and there exist real numbers  $u_1, u_2, \sigma_1, \sigma_2, V$  and a  $C^1$ -function  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$(32) \quad \tilde{u}(t) = u_1 + (u_2 - u_1)\tau_{Vt}H + g(t)\tau_{Vt}\delta,$$

$$(33) \quad \tilde{\sigma}(t) = \sigma_1 + (\sigma_2 - \sigma_1)\tau_{Vt}H.$$

**Theorem 5.** *Let us consider the problem (27)–(28) with the initial conditions (30)–(31) with  $k > 0$ .*

(I) *If  $b_1 = b_2$ , there exists an  $\alpha$ -solution in  $\tilde{W}$  if and only if  $a_1 = a_2$ ; moreover, for any  $\alpha$ , the  $\alpha$ -solution is unique in  $\tilde{W}$  and is given by*

$$(34) \quad \tilde{u}(t) = a_1,$$

$$(35) \quad \tilde{\sigma}(t) = b_1;$$

(II) *If  $b_1 \neq b_2$  there exists an  $\alpha$ -solution in  $\tilde{W}$  if and only if  $a_1 \neq a_2$  and we choose  $\alpha$  such that*

$$(36) \quad \int_{-\infty}^0 \alpha = \frac{1}{2} - \frac{b_2 - b_1}{(a_2 - a_1)^2} + \frac{k^2}{b_2 - b_1};$$

*moreover, for any  $\alpha$  satisfying this condition, the  $\alpha$ -solution is unique in  $\tilde{W}$  and is given by the traveling wave*

$$(37) \quad \tilde{u}(t) = a_1 + (a_2 - a_1)\tau_{Vt}H,$$

$$(38) \quad \tilde{\sigma}(t) = b_1 + (b_2 - b_1)\tau_{Vt}H,$$

*with speed*

$$V = \frac{a_1 + a_2}{2} - \frac{b_2 - b_1}{a_2 - a_1}.$$

*As we can see all of these  $\alpha$ -solutions, when they exist, are independent of  $\alpha$ .*

*Proof.* Let us suppose  $(\tilde{u}, \tilde{v}) \in \tilde{W}$ . Then we have (32) and (33), and by (30) and (31) we can write

$$u_1 + (u_2 - u_1)H + g(0)\delta = a_1 + (a_2 - a_1)H,$$

$$\sigma_1 + (\sigma_2 - \sigma_1)H = b_1 + (b_2 - b_1)H,$$

which implies  $g(0) = 0$ . Then, by restriction to the interval  $] -\infty, 0[$ , we have  $u_1 = a_1$  and  $\sigma_1 = b_1$ . As a consequence, we also have  $u_2 = a_2$  and  $\sigma_2 = b_2$ . Thus, from (32) and (33) it follows that

$$(39) \quad \tilde{u}(t) = a_1 + (a_2 - a_1)\tau_{Vt}H + g(t)\tau_{Vt}\delta,$$

$$(40) \quad \tilde{\sigma}(t) = b_1 + (b_2 - b_1)\tau_{Vt}H,$$

and so

$$\begin{aligned} \frac{d\tilde{u}}{dt}(t) &= -V(a_2 - a_1)\tau_{Vt}\delta + g'(t)\tau_{Vt}\delta - Vg(t)\tau_{Vt}D\delta, \\ \frac{d\tilde{\sigma}}{dt}(t) &= -V(b_2 - b_1)\tau_{Vt}\delta. \end{aligned}$$

By applying the bilinearity of the  $\alpha$ -products, the results (23), (20), (25), (19), and the already mentioned translation property, we have

$$\begin{aligned}
(41) \quad \tilde{u}(t)_{\dot{\alpha}} \tilde{u}(t) &= a_1^2 + 2a_1(a_2 - a_1)\tau_{V_t}H + 2a_1g(t)\tau_{V_t}\delta \\
&\quad + (a_2 - a_1)^2\tau_{V_t}H + (a_2 - a_1)g(t)\left(\int_{-\infty}^0 \alpha\right)\tau_{V_t}\delta \\
&\quad + (a_2 - a_1)g(t)\left(\int_0^{+\infty} \alpha\right)\tau_{V_t}\delta + g^2(t)\alpha(0)\tau_{V_t}\delta.
\end{aligned}$$

Since  $\int_{-\infty}^0 \alpha + \int_0^{+\infty} \alpha = 1$ , we also have

$$\begin{aligned}
D[\tilde{u}(t)_{\dot{\alpha}} \tilde{u}(t)] &= (a_2^2 - a_1^2)\tau_{V_t}\delta + [2a_1g(t) + (a_2 - a_1)g(t) + g^2(t)\alpha(0)]\tau_{V_t}D\delta, \\
D[\tilde{\sigma}(t)] &= (b_2 - b_1)\tau_{V_t}\delta, \\
D[\tilde{u}(t)] &= (a_2 - a_1)\tau_{V_t}\delta + g(t)\tau_{V_t}D\delta,
\end{aligned}$$

and

$$\begin{aligned}
(42) \quad \tilde{u}(t)_{\dot{\alpha}} D[\tilde{\sigma}(t)] &= [a_1(b_2 - b_1) + (a_2 - a_1)(b_2 - b_1)\int_{-\infty}^0 \alpha + (b_2 - b_1)g(t)\alpha(0)]\tau_{V_t}\delta.
\end{aligned}$$

Thus, (27)–(28) turn out to be

$$\begin{aligned}
0 &= [-V(a_2 - a_1) + g'(t) + \frac{1}{2}(a_2^2 - a_1^2) - (b_2 - b_1)]\tau_{V_t}\delta \\
&\quad + [-Vg(t) + 12(a_1 + a_2)g(t) + \frac{1}{2}(\alpha(0))g^2(t)]\tau_{V_t}D\delta, \\
0 &= [-V(b_2 - b_1) + a_1(b_2 - b_1) + (a_2 - a_1)(b_2 - b_1)\int_{-\infty}^0 \alpha \\
&\quad + (b_2 - b_1)g(t)\alpha(0) - k^2(a_2 - a_1)]\tau_{V_t}\delta - k^2g(t)\tau_{V_t}D\delta.
\end{aligned}$$

Hence, for all  $t \in \mathbb{R}$  we have

$$(43) \quad 0 = -V(a_2 - a_1) + g'(t) + \frac{1}{2}(a_2^2 - a_1^2) - (b_2 - b_1),$$

$$(44) \quad 0 = g(t)[-V + \frac{1}{2}(a_1 + a_2) + \frac{1}{2}(\alpha(0))g(t)],$$

$$\begin{aligned}
(45) \quad 0 &= -V(b_2 - b_1) + a_1(b_2 - b_1) + (a_2 - a_1)(b_2 - b_1)\int_{-\infty}^0 \alpha \\
&\quad + (b_2 - b_1)g(t)\alpha(0) - k^2(a_2 - a_1),
\end{aligned}$$

$$(46) \quad 0 = k^2g(t).$$

From (46) we conclude that  $g = 0$ , (44) is satisfied, and from (43) and (45) we have

$$(47) \quad 0 = -V(a_2 - a_1) + \frac{1}{2}(a_2^2 - a_1^2) - (b_2 - b_1),$$

$$(48) \quad 0 = -V(b_2 - b_1) + a_1(b_2 - b_1) + (a_2 - a_1)(b_2 - b_1)\int_{-\infty}^0 \alpha - k^2(a_2 - a_1).$$

Now, if  $b_1 = b_2$ , by (48) we have  $a_1 = a_2$ , (47) is satisfied and (I) follows from (39) and (40). If  $b_1 \neq b_2$ , from (48) we have

$$(49) \quad V = a_1 + (a_2 - a_1)\int_{-\infty}^0 \alpha - k^2\frac{a_2 - a_1}{b_2 - b_1},$$

and from (47) we can write

$$(50) \quad -a_1(a_2 - a_1) - (a_2 - a_1)^2 \int_{-\infty}^0 \alpha + k^2 \frac{(a_2 - a_1)^2}{b_2 - b_1} + \frac{a_2^2 - a_1^2}{2} - (b_2 - b_1) = 0.$$

Then it follows that  $a_1 \neq a_2$ , because if  $a_1 = a_2$  we would have  $b_1 = b_2$  which is a contradiction. As a consequence, from (50) we have

$$\int_{-\infty}^0 \alpha = \frac{1}{2} - \frac{b_2 - b_1}{(a_2 - a_1)^2} + \frac{k^2}{b_2 - b_1},$$

from (49) we have

$$V = \frac{a_1 + a_2}{2} - \frac{b_2 - b_1}{a_2 - a_1},$$

and (II) follows from (39) and (40).  $\square$

If in (28) we replace  $\tilde{u}(t)_{\tilde{\alpha}} D[\tilde{\sigma}(t)]$  by  $D[\tilde{\sigma}(t)]_{\tilde{\alpha}} \tilde{u}(t)$  we obtain for the value  $D[\tilde{\sigma}(t)]_{\tilde{\alpha}} \tilde{u}(t)$  the same value as  $\tilde{u}(t)_{\tilde{\alpha}} D[\tilde{\sigma}(t)]$ , with  $\int_0^{+\infty} \alpha$  instead of  $\int_{-\infty}^0 \alpha$  (now we must apply (25) instead of (20)). Hence, for the problem formed by the system (27) and (29) with initial conditions (30)–(31) we must replace Theorem 5 by another theorem where the only difference is at (36), where  $\int_{-\infty}^0 \alpha$  must be replaced by  $\int_0^{+\infty} \alpha$ !

As a consequence of Definition 4 these considerations allows us to conclude that the  $\alpha$ -solutions of the problem (1)–(4) with  $k > 0$ , which belong to  $W$ , can be read as stated in the introduction (see (7), (8), (9) and (10)).

## 5. The Riemann problem (1)–(4) with $k = 0$

In this extreme case we will see, in the same space of solutions  $W$ , the possible emergence of a  $\delta$  shock wave.

**Theorem 6.** *Let us consider the problem (27)–(28) with initial conditions (30)–(31) with  $k = 0$ .*

- (I) *If  $b_1 = b_2$  and  $a_1 = a_2$  there exists an  $\alpha$ -solution in  $\tilde{W}$  for any  $\alpha$ ; moreover, for any  $\alpha$ , this  $\alpha$ -solution is unique in  $W$  and is given by*

$$\tilde{u}(t) = a_1, \quad \tilde{\sigma}(t) = b_1.$$

- (II) *If  $b_1 = b_2$  and  $a_1 \neq a_2$  there exists an  $\alpha$ -solution in  $\tilde{W}$  for any  $\alpha$ ; moreover, for any  $\alpha$ , the  $\alpha$ -solution is unique in  $\tilde{W}$  and is given by*

$$(51) \quad \tilde{u}(t) = a_1 + (a_2 - a_1)\tau_{Vt}H,$$

$$(52) \quad \tilde{\sigma}(t) = b_1,$$

with  $V = \frac{1}{2}(a_1 + a_2)$ .



(III) If  $b_1 \neq b_2$  and  $a_1 = a_2$  there exists an  $\alpha$ -solution in  $\tilde{W}$  if and only if we choose  $\alpha$  such that  $\alpha(0) = 0$ ; moreover, for any  $\alpha$  satisfying this condition, the  $\alpha$ -solution is unique in  $\tilde{W}$  and is given by

$$(53) \quad \tilde{u}(t) = a_1 + (b_2 - b_1)t\tau_{a_1t}\delta,$$

$$(54) \quad \tilde{\sigma}(t) = b_1 + (b_2 - b_1)\tau_{a_1t}H.$$

(IV) If  $b_1 \neq b_2$  and  $a_1 \neq a_2$  there exists an  $\alpha$ -solution in  $\tilde{W}$  if and only if we choose  $\alpha$  such that

$$(55) \quad \int_{-\infty}^0 \alpha = \frac{1}{2} - \frac{b_2 - b_1}{(a_2 - a_1)^2},$$

or we choose  $\alpha$  such that

$$(56) \quad \alpha(0) = 0;$$

moreover, for any  $\alpha$  satisfying (55), the  $\alpha$ -solution is unique in  $\tilde{W}$  and is given by

$$(57) \quad \tilde{u}(t) = a_1 + (a_2 - a_1)\tau_{Vt}H,$$

$$(58) \quad \tilde{\sigma}(t) = b_1 + (b_2 - b_1)\tau_{Vt}H,$$

with  $V = (a_1 + a_2)/2 - (b_2 - b_1)/(a_2 - a_1)$ ; also for any  $\alpha$  satisfying (56), the  $\alpha$ -solution is unique in  $\tilde{W}$  and is given by

$$(59) \quad \tilde{u}(t) = a_1 + (a_2 - a_1)\tau_{Vt}H + (b_2 - b_1)t\tau_{Vt}\delta,$$

$$(60) \quad \tilde{\sigma}(t) = b_1 + (b_2 - b_1)\tau_{Vt}H,$$

with  $V = \frac{1}{2}(a_1 + a_2)$ . As we can see, all of these  $\alpha$ -solutions, when they exist, are independent of  $\alpha$ .

*Proof.* Let us suppose  $(\tilde{u}, \tilde{\sigma}) \in \tilde{W}$ . Then, we have (32), (33) and as we have seen in the proof of Theorem 5 we have  $g(0) = 0$ ,  $u_1 = a_1$ ,  $u_2 = a_2$ ,  $\sigma_1 = b_1$ ,  $\sigma_2 = b_2$  and also (39) and (40). From (27) and (28) we have (43)–(46) with  $k = 0$ , which means that for all  $t \in \mathbb{R}$  we can write

$$(61) \quad 0 = V(a_2 - a_1) + g'(t) + \frac{1}{2}(a_2^2 - a_1^2) - (b_2 - b_1),$$

$$(62) \quad 0 = g(t) \left[ -V + \frac{1}{2}(a_1 + a_2) + \frac{1}{2}\alpha(0)g(t) \right],$$

$$(63) \quad 0 = -V(b_2 - b_1) + a_1(b_2 - b_1) + (a_2 - a_1)(b_2 - b_1) \int_{-\infty}^0 \alpha + (b_2 - b_1)g(t)\alpha(0).$$

(I) Suppose  $b_1 = b_2$  and  $a_1 = a_2$ . Then (63) is satisfied and from (61) we have  $g'(t) = 0$ , which means that  $g(t) = 0$  and (62) is also satisfied. Then from (39) and (40) we have  $\tilde{u}(t) = a_1$  and  $\tilde{\sigma}(t) = b_1$  and (I) follows.

(II) Suppose  $b_1 = b_2$  and  $a_1 \neq a_2$ . Then (63) is satisfied and from (61) we have

$$g'(t) = V(a_2 - a_1) - \frac{1}{2}(a_2^2 - a_1^2),$$

which means that

$$(64) \quad g(t) = \left[ V(a_2 - a_1) - \frac{1}{2}(a_2^2 - a_1^2) \right] t.$$

Then, from (62) we conclude that  $V = \frac{1}{2}(a_1 + a_2)$  and by (64)  $g(t) = 0$  follows for all  $t$ . Then, from (39) and (40), we conclude that  $\tilde{u}(t) = a_1 + (a_2 - a_1)\tau_{Vt}H$  and  $\tilde{\sigma}(t) = b_1$  and (II) follows.

(III) Suppose  $b_1 \neq b_2$  and  $a_1 = a_2$ . Then by (61) we have  $g'(t) = b_2 - b_1$  which means that  $g(t) = (b_2 - b_1)t$ , and (62) turns out to be

$$t \left[ -V + a_1 + \frac{1}{2}\alpha(0)(b_2 - b_1)t \right] = 0,$$

which implies, for any  $t \neq 0$ ,

$$V = a_1 + \frac{1}{2}\alpha(0)(b_2 - b_1)t.$$

Thus, once  $V$  is constant, we have  $\alpha(0) = 0$ ,  $V = a_1$  and (63) is satisfied and (III) follows.

(IV) Suppose  $b_1 \neq b_2$  and  $a_1 \neq a_2$ . Then by (61) we have

$$V = \frac{g'(t)}{a_2 - a_1} + \frac{a_1 + a_2}{2} - \frac{b_2 - b_1}{a_2 - a_1},$$

and once  $V$  is constant we conclude that  $g'(t) = c$  (constant),  $g(t) = ct$ , and

$$(65) \quad V = \frac{c}{a_2 - a_1} + \frac{a_1 + a_2}{2} - \frac{b_2 - b_1}{a_2 - a_1}.$$

If  $c = 0$  we have  $g(t) = 0$  and

$$V = \frac{a_1 + a_2}{2} - \frac{b_2 - b_1}{a_2 - a_1}.$$

As a consequence, (62) is satisfied and (63) turns out to be

$$-(b_2 - b_1) \frac{a_2 + a_1}{2} + \frac{(b_2 - b_1)^2}{a_2 - a_1} + a_1(b_2 - b_1) + (a_2 - a_1)(b_2 - b_1) \int_{-\infty}^0 \alpha = 0,$$

which is possible if and only if

$$\int_{-\infty}^0 \alpha = \frac{1}{2} - \frac{b_2 - b_1}{(a_2 - a_1)^2}.$$

If  $c \neq 0$  we have from (65) and (62),

$$ct \left[ -\frac{c}{a_2 - a_1} - \frac{a_1 + a_2}{2} + \frac{b_2 - b_1}{a_2 - a_1} + \frac{a_1 + a_2}{2} + \frac{\alpha(0)}{2} ct \right] = 0,$$

and for all  $t \neq 0$  we will have

$$\frac{\alpha(0)}{2} ct - \frac{c}{a_2 - a_1} + \frac{b_2 - b_1}{a_2 - a_1} = 0,$$

which is possible if and only if  $\alpha(0) = 0$  and  $c = b_2 - b_1$  which, by (65), implies  $V = \frac{1}{2}(a_1 + a_2)$  and  $g(t) = (b_2 - b_1)t$ . Hence, (IV) follows.  $\square$

Thus, concerning the problem formed by the system (27) and (29) with initial conditions (30)–(31), Theorem 6 must be substituted with another theorem where the only difference is at (55), where  $\int_{-\infty}^0 \alpha$  must change to  $\int_0^{+\infty} \alpha$ !

As a consequence of Definition 4 we can conclude that the  $\alpha$ -solutions of the problem (1)–(4) with  $k = 0$ , can be described as in the introduction.

## 6. The arising of a $\delta'$ shock wave

For the system (1)–(2) with  $k = 0$  let us consider the initial conditions (11) and (12). Let us define the space  $\tilde{Z}$  by the condition  $(\tilde{u}, \tilde{\sigma}) \in \tilde{Z}$  if and only if  $\tilde{u}, \tilde{\sigma} \in \mathcal{F}(\mathbb{R})$  and there exist real numbers  $u_1, \sigma_1, p$  and  $C^1$ -functions  $f, \gamma : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$(66) \quad \tilde{u}(t) = u_1 + f(t)\tau_{\gamma(t)}D\delta,$$

$$(67) \quad \tilde{\sigma}(t) = \sigma_1 + p\tau_{\gamma(t)}\delta.$$

Now, the initial conditions (11) and (12) correspond in  $\mathcal{F}(\mathbb{R})$  to the conditions

$$(68) \quad \tilde{u}(0) = a,$$

$$(69) \quad \tilde{\sigma}(0) = b + m\delta,$$

with  $m \neq 0$ . We will see the possible emergence of a  $\delta'$  shock wave for problem (1)–(2) with initial conditions (11) and (12).

**Theorem 7.** *The problem (27)–(28) with  $k = 0$  and initial conditions (68) and (69) has  $\alpha$ -solutions in  $\tilde{Z}$  if and only if we choose  $\alpha$  such that  $\alpha'(0) = \alpha''(0) = 0$ ; moreover, for all  $\alpha$  satisfying this condition, the  $\alpha$ -solution is unique in  $\tilde{Z}$  and is given by*

$$(70) \quad \tilde{u}(t) = a + mt\tau_{at}D\delta,$$

$$(71) \quad \tilde{\sigma}(t) = b + m\tau_{at}\delta.$$

As we can see, when it exists, this  $\alpha$ -solution is also independent of  $\alpha$ .

*Proof.* Let us suppose  $(\tilde{u}, \tilde{\sigma}) \in \tilde{Z}$ . Then we have (66) and (67) and by (68) and (69) we have

$$(72) \quad u_1 + f(0)\tau_{\gamma(0)}D\delta = a,$$

$$(73) \quad \sigma_1 + p\tau_{\gamma(0)}\delta = b + m\delta.$$

From (72) we conclude that  $f(0) = 0$  and  $u_1 = a$ . From (73) we conclude that  $\sigma_1 = b$  and so, since  $m \neq 0$ , we have  $\gamma(0) = 0$  and  $p = m$ . Thus, we can write (66) and (67) in the form

$$(74) \quad \tilde{u}(t) = a + f(t)\tau_{\gamma(t)}D\delta,$$

$$(75) \quad \tilde{\sigma}(t) = b + m\tau_{\gamma(t)}\delta.$$

As a consequence, we have

$$\frac{d\tilde{u}}{dt}(t) = f'(t)\tau_{\gamma(t)}D\delta - \gamma'(t)f(t)\tau_{\gamma(t)}D^2\delta,$$

and using (21), we also have

$$\begin{aligned} \tilde{u}(t)_{\dot{\alpha}}\tilde{u}(t) &= a^2 + 2af(t)\tau_{\gamma(t)}D\delta + f^2(t)\tau_{\gamma(t)}[\alpha'(0)D\delta - \alpha''(0)\delta], \\ \frac{1}{2}D[\tilde{u}(t)_{\dot{\alpha}}\tilde{u}(t)] &= -\frac{1}{2}\alpha''(0)f^2(t)\tau_{\gamma(t)}D\delta + [af(t) + \frac{1}{2}\alpha'(0)f^2(t)]\tau_{\gamma(t)}D^2\delta, \\ D\tilde{\sigma}(t) &= m\tau_{\gamma(t)}D\delta, \\ \frac{d\tilde{\sigma}}{dt}(t) &= -m\gamma'(t)\tau_{\gamma(t)}D\delta, \\ \tilde{u}(t)_{\dot{\alpha}}D\tilde{\sigma}(t) &= -m\alpha''(0)f(t)\tau_{\gamma(t)}\delta + [ma + mf(t)\alpha'(0)]\tau_{\gamma(t)}D\delta. \end{aligned}$$

Then, (27)–(28) with  $k = 0$  turns out to be

$$\begin{aligned} 0 &= [f'(t) - \frac{1}{2}\alpha''(0)f^2(t) - m]\tau_{\gamma(t)}D\delta \\ &\quad + [-\gamma'(t)f(t) + af(t) + \frac{1}{2}\alpha'(0)f^2(t)]\tau_{\gamma(t)}D^2\delta, \\ 0 &= -m\alpha''(0)f(t)\tau_{\gamma(t)}\delta + [-m\gamma'(t) + ma + mf(t)\alpha'(0)]\tau_{\gamma(t)}D\delta. \end{aligned}$$

Hence, for all  $t \in \mathbb{R}$ , we have

$$(76) \quad 0 = f'(t) - \frac{1}{2}\alpha''(0)f^2(t) - m,$$

$$(77) \quad 0 = f(t)[- \gamma'(t) + a + \frac{1}{2}\alpha'(0)f(t)],$$

$$(78) \quad 0 = \alpha''(0)f(t),$$

$$(79) \quad 0 = -\gamma'(t) + a + f(t)\alpha'(0).$$

Now, we must note that  $\alpha''(0) = 0$  follows immediately because by (78) if  $\alpha''(0) \neq 0$ , we will have  $f = 0$  and by (76) we will also have  $m = 0$ , which is impossible. Thus, by (76) we have  $f(t) = mt$  and from (79) it follows that  $\gamma'(t) = a + \alpha'(0)mt$ . Then

by (77) we conclude that  $t^2\alpha'(0) = 0$  for all  $t \in \mathbb{R}$ , and  $\alpha'(0) = 0$  follows which means that  $\gamma'(t) = a$  and so,  $\gamma(t) = at$ . Finally (70) and (71) follow from (74) and (75). The theorem is proved.  $\square$

If in (28) we replace  $\tilde{u}(t)_{\dot{\alpha}} D\tilde{\sigma}(t)$  by  $D\tilde{\sigma}(t)_{\dot{\alpha}} \tilde{u}(t)$  we arrive exactly at the same theorem because in this case we simply have  $\tilde{u}(t)_{\dot{\alpha}} D\tilde{\sigma}(t) = D\tilde{\sigma}(t)_{\dot{\alpha}} \tilde{u}(t)$ . Hence, by Definition 4 we conclude that the  $\alpha$ -solutions of the problem (1)–(2) with  $k = 0$ , when subjected to the initial conditions (11) and (12) can be read as we said in the introduction (see (13) and (14)).

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# SOME AMBROSE- AND GALLOWAY-TYPE THEOREMS VIA BAKRY-ÉMERY AND MODIFIED RICCI CURVATURES

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**We establish some compactness theorems of Ambrose- and Galloway-type for complete Riemannian manifolds in the context of the Bakry–Émery and modified Ricci curvatures. Our compactness theorems generalize previous ones obtained by Fernández-López and García-Río, Wei and Wylie, and Limoncu, Rimoldi, and Zhang.**

## 1. Introduction

One of the most fundamental topics in Riemannian geometry is to investigate the relation between topology and geometric structure on Riemannian manifolds. To give nice compactness criteria for complete Riemannian manifolds is one of the most natural and interesting problems in Riemannian geometry. The celebrated theorem of Myers [1941] guarantees the compactness of complete Riemannian manifolds under some positive lower bounds on the Ricci curvature.

**Theorem 1** [Myers 1941]. *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exists some positive constant  $\lambda > 0$  such that the Ricci curvature satisfies  $\text{Ric}_g \geq \lambda g$ . Then  $(M, g)$  must be compact with finite fundamental group. Moreover, the diameter of  $(M, g)$  has the upper bound*

$$\text{diam}(M, g) \leq \pi \sqrt{\frac{n-1}{\lambda}}.$$

The Myers theorem above has been widely generalized in various directions by many authors [Ambrose 1957; Calabi 1967; Fernández-López and García-Río 2008; Galloway 1979; 1982; Limoncu 2010; 2012; Lott 2003; Mastrolia et al. 2012; Morgan 2006; Qian 1997; Rimoldi 2011; Tadano 2016; 2017; Wei and Wylie 2009; Wraith 2006; Zhang 2014]. The first generalization was given by Ambrose [1957], where the positive lower bound on the Ricci curvature was replaced with an integral condition on the Ricci curvature along some geodesics.

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**Theorem 2** [Ambrose 1957]. *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exists some point  $p \in M$  for which every geodesic  $\gamma : [0, +\infty) \rightarrow M$  emanating from  $p$  satisfies*

$$\int_0^{+\infty} \text{Ric}_g(\dot{\gamma}(s), \dot{\gamma}(s)) ds = +\infty.$$

*Then  $(M, g)$  must be compact.*

On the other hand, motivated by relativistic cosmology, Galloway [1979] proved the following compactness theorem by perturbing the positive lower bound on the Ricci curvature by the derivative in the radial direction of some bounded function:

**Theorem 3** [Galloway 1979]. *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exist some constants  $\lambda > 0$  and  $L \geq 0$  such that for every pair of points in  $M$  and minimal geodesic  $\gamma$  joining those points, the Ricci curvature satisfies*

$$\text{Ric}_g(\dot{\gamma}, \dot{\gamma})|_{\gamma(s)} \geq \lambda + \frac{d\phi}{ds}(s),$$

*where  $\phi$  is some smooth function of the arc length satisfying  $|\phi| \leq L$  along  $\gamma$ . Then  $(M, g)$  must be compact. Moreover, the diameter of  $(M, g)$  has the upper bound*

$$\text{diam}(M, g) \leq \frac{\pi}{\lambda} (L + \sqrt{L^2 + (n-1)\lambda}).$$

One of the most important features of the two generalizations above is that the Ricci curvature is not required to be everywhere nonnegative.

In this paper, we shall establish some compactness theorems of Ambrose- and Galloway-type for complete Riemannian manifolds in the context of the Bakry-Émery and modified Ricci curvatures. To define the Bakry-Émery and modified Ricci curvatures, we first recall the definition of a smooth metric measure space.

**Definition.** A *smooth metric measure space* is a complete Riemannian manifold  $(M, g)$  with the weighted volume form  $d\mu := e^{-f} d\text{vol}_g$ , where  $f : M \rightarrow \mathbb{R}$  is a smooth function on  $M$  and  $\text{vol}_g$  denotes the Riemannian density with respect to the metric  $g$ . For a smooth metric measure space  $(M, g)$  and a positive constant  $k \in (0, +\infty)$ , we put

$$(1-1) \quad \text{Ric}_f := \text{Ric}_g + \text{Hess } f \quad \text{and} \quad \text{Ric}_f^k := \text{Ric}_g + \text{Hess } f - \frac{1}{k} df \otimes df$$

and call them a *Bakry-Émery Ricci curvature* and a  *$k$ -Bakry-Émery Ricci curvature*, respectively. We refer to  $f$  as a *potential function*. More generally, for a smooth vector field  $V \in \mathfrak{X}(M)$  and a positive constant  $k \in (0, +\infty)$ , we define

$$\text{Ric}_V := \text{Ric}_g + \frac{1}{2} \mathcal{L}_V g \quad \text{and} \quad \text{Ric}_V^k := \text{Ric}_g + \frac{1}{2} \mathcal{L}_V g - \frac{1}{k} V^* \otimes V^*,$$



where  $V^*$  is the metric dual of  $V$  with respect to  $g$ . We call them a *modified Ricci curvature* and a *k-modified Ricci curvature*, respectively. We also put

$$(1-2) \quad \Delta_f := \Delta_g - \nabla f \cdot \nabla \quad \text{and} \quad \Delta_V := \Delta_g - V \cdot \nabla$$

and call them a *Witten–Laplacian* and a *V-Laplacian*, respectively. Here,  $\Delta_g$  denotes the Laplacian with respect to  $g$ .

Note that if  $f : M \rightarrow \mathbb{R}$  is constant in (1-1) and (1-2), then the Bakry–Émery Ricci curvature and the Witten–Laplacian are reduced to the Ricci curvature and the Laplacian, respectively. As in the classical case, for any smooth functions  $u, v$  on  $M$  with compact support, we have

$$\int_M g(\nabla u, \nabla v) d\mu = - \int_M (\Delta_f u) v d\mu = - \int_M u (\Delta_f v) d\mu.$$

Moreover, Bakry and Émery [1985] proved that for any smooth function  $u$  on  $M$ ,

$$(1-3) \quad \frac{1}{2} \Delta_f |\nabla u|^2 = |\text{Hess } u|^2 + \text{Ric}_f(\nabla u, \nabla u) + g(\nabla \Delta_f u, \nabla u),$$

which may be regarded as a natural extension of the Bochner–Weitzenböck formula

$$(1-4) \quad \frac{1}{2} \Delta_g |\nabla u|^2 = |\text{Hess } u|^2 + \text{Ric}_g(\nabla u, \nabla u) + g(\nabla \Delta_g u, \nabla u).$$

Recently, the Bakry–Émery Ricci curvature and the Witten–Laplacian have received much attention in various areas of mathematics, since they are good substitutes for the Ricci curvature and the Laplacian respectively, allowing us to establish many interesting results in smooth metric measure spaces, such as eigenvalue estimates [Futaki et al. 2013], Li–Yau Harnack inequalities [Li 2005], and comparison theorems [Wei and Wylie 2009]. In particular, Wei and Wylie [2009] proved the following Myers-type theorem via Bakry–Émery Ricci curvature which extends Theorem 1 to the case of smooth metric measure spaces:

**Theorem 4** [Wei and Wylie 2009]. *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exists some positive constant  $\lambda > 0$  such that the Bakry–Émery Ricci curvature satisfies  $\text{Ric}_f \geq \lambda g$ . If the potential function satisfies  $|f| \leq H$  for some nonnegative constant  $H \geq 0$ , then  $(M, g)$  must be compact. Moreover, the diameter of  $(M, g)$  has the upper bound*

$$(1-5) \quad \text{diam}(M, g) \leq \pi \sqrt{\frac{n-1}{\lambda}} + \frac{4H}{\sqrt{(n-1)\lambda}}.$$

On the other hand, Fernández-López and García-Río [2008] proved that the compactness of a complete Riemannian manifold with a positive lower bound on the modified Ricci curvature may be characterized by an upper bound on the norm of the vector field appearing in the modified Ricci curvature.

**Theorem 5** [Fernández-López and García-Río 2008]. *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exists some positive constant  $\lambda > 0$  such that the modified Ricci curvature satisfies  $\text{Ric}_V \geq \lambda g$ . Then  $(M, g)$  is compact if and only if  $|V|$  is bounded on  $M$ .*

In Theorem 5 above, no upper diameter estimate was given. By extending the proof of Theorem 1, Limoncu [2010] gave the following Myers-type theorem with an upper diameter estimate via modified Ricci curvature:

**Theorem 6** [Limoncu 2010]. *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exists some positive constant  $\lambda > 0$  such that the modified Ricci curvature satisfies  $\text{Ric}_V \geq \lambda g$ . If the vector field satisfies  $|V| \leq K$  for some nonnegative constant  $K \geq 0$ , then  $(M, g)$  must be compact. Moreover, the diameter of  $(M, g)$  has the upper bound*

$$(1-6) \quad \text{diam}(M, g) \leq \frac{\pi}{\lambda} \left( \frac{K}{\sqrt{2}} + \sqrt{\frac{K^2}{2} + (n-1)\lambda} \right).$$

An interesting problem in smooth metric measure spaces is to establish Ambrose- and Galloway-type theorems via Bakry-Émery Ricci curvature. An Ambrose-type theorem via Bakry-Émery Ricci curvature was first established by Zhang [2014] under the assumption that the potential function appearing in the Bakry-Émery Ricci curvature has at most linear growth in the distance function.

**Theorem 7** [Zhang 2014]. *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exists some point  $p \in M$  for which every geodesic  $\gamma : [0, +\infty) \rightarrow M$  emanating from  $p$  satisfies*

$$\int_0^{+\infty} \text{Ric}_f(\dot{\gamma}(s), \dot{\gamma}(s)) ds = +\infty,$$

*and the potential function satisfies  $f(x) \leq \delta d(x, p) + \alpha$  for some constants  $\delta$  and  $\alpha$ , where  $d(x, p)$  is the distance between  $x$  and  $p$ . Then  $(M, g)$  must be compact.*

More generally, we shall prove the following Ambrose-type theorem via modified Ricci curvature which generalizes Theorem 5 above:

**Theorem 8.** *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exists some point  $p \in M$  for which every geodesic  $\gamma : [0, +\infty) \rightarrow M$  emanating from  $p$  satisfies*

$$\int_0^{+\infty} \text{Ric}_V(\dot{\gamma}(s), \dot{\gamma}(s)) ds = +\infty,$$

*and the vector field satisfies  $|V| \leq K$  for some nonnegative constant  $K \geq 0$ . Then  $(M, g)$  must be compact.*

As to a Galloway-type theorem via Bakry–Émery Ricci curvature, we shall prove the following compactness theorem by modifying the alternative proof of Theorem 4 by Limoncu [2012] and its improvement by the author [Tadano 2016]:

**Theorem 9.** *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exist some constants  $\lambda > 0$  and  $L \geq 0$  such that for every pair of points in  $M$  and minimal geodesic  $\gamma$  joining those points, the Bakry–Émery Ricci curvature satisfies*

$$(1-7) \quad \text{Ric}_f(\dot{\gamma}, \dot{\gamma})|_{\gamma(s)} \geq \lambda + \frac{d\phi}{ds}(s),$$

where  $\phi$  is some smooth function of the arc length satisfying  $|\phi| \leq L$  along  $\gamma$ . If the potential function satisfies  $|f| \leq H$  for some nonnegative constant  $H \geq 0$ , then  $(M, g)$  must be compact. Moreover, the diameter of  $(M, g)$  has the upper bound

$$\text{diam}(M, g) \leq \frac{1}{\lambda} (2L + \sqrt{4L^2 + \{(n-1)\pi + 8H\}\lambda\pi}).$$

**Remark.** By taking  $L = 0$ , Theorem 9 above is reduced to the Myers-type theorem via Bakry–Émery Ricci curvature [Tadano 2016] with the diameter estimate

$$(1-8) \quad \text{diam}(M, g) \leq \frac{\pi}{\sqrt{\lambda}} \sqrt{n-1 + \frac{8H}{\pi}}.$$

Note that the estimate (1-8) above is sharper than (1-5) by Wei and Wylie [2009].

On the other hand, by modifying the proof of Theorem 6 above, we shall prove the following Galloway-type theorem via modified Ricci curvature:

**Theorem 10.** *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exist some constants  $\lambda > 0$  and  $L \geq 0$  such that for every pair of points in  $M$  and minimal geodesic  $\gamma$  joining those points, the modified Ricci curvature satisfies*

$$(1-9) \quad \text{Ric}_V(\dot{\gamma}, \dot{\gamma})|_{\gamma(s)} \geq \lambda + \frac{d\phi}{ds}(s),$$

where  $\phi$  is some smooth function of the arc length satisfying  $|\phi| \leq L$  along  $\gamma$ . If the vector field satisfies  $|V| \leq K$  for some nonnegative constant  $K \geq 0$ , then  $(M, g)$  must be compact. Moreover, the diameter of  $(M, g)$  has the upper bound

$$\text{diam}(M, g) \leq \frac{1}{\lambda} (2(L + K) + \sqrt{4(L + K)^2 + (n-1)\lambda\pi^2}).$$

**Remark.** By taking  $L = 0$ , Theorem 10 above is reduced to the Myers-type theorem via modified Ricci curvature [Tadano 2017] with the diameter estimate

$$(1-10) \quad \text{diam}(M, g) \leq \frac{1}{\lambda} (2K + \sqrt{4K^2 + (n-1)\lambda\pi^2}).$$

Note that the estimate (1-10) above is sharper than (1-6) by Limoncu [2010].

Moreover, we shall prove the compactness of a complete Riemannian manifold with a lower bound on the modified Ricci curvature under the condition that the norm of the vector field appearing in the modified Ricci curvature has at most linear growth in the distance function.

**Theorem 11.** *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exist some constants  $\lambda > 0$  and  $L \geq 0$  such that for every pair of points in  $M$  and minimal geodesic  $\gamma$  joining those points, the modified Ricci curvature satisfies*

$$(1-11) \quad \text{Ric}_V(\dot{\gamma}, \dot{\gamma})|_{\gamma(s)} \geq \lambda + \frac{d\phi}{ds}(s),$$

where  $\phi$  is some smooth function of the arc length satisfying  $\phi \geq -L$  along  $\gamma$ . If the vector field satisfies  $|V|(x) \leq \delta d(x, p) + \alpha$  for some constants  $\delta < \lambda$  and  $\alpha$ , where  $d(x, p)$  is the distance between  $x$  and  $p$ , then  $(M, g)$  must be compact.

By taking  $L = 0$  and  $V = \nabla f$  for a smooth function  $f : M \rightarrow \mathbb{R}$  in Theorem 11 above, we may recover the following compactness theorem due to Zhang [2014]:

**Theorem 12** [Zhang 2014]. *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exists some positive constant  $\lambda > 0$  such that the Bakry–Émery Ricci curvature satisfies  $\text{Ric}_f \geq \lambda g$ . If the potential function satisfies  $f(x) \leq \delta(d(x, p) + \alpha)^2$  for some constants  $\delta < \frac{1}{2}\lambda$  and  $\alpha$ , where  $d(x, p)$  is the distance between  $x$  and  $p$ , then  $(M, g)$  must be compact.*

**Remark.** A typical example of smooth metric measure spaces is a *Gaussian soliton*  $(\mathbb{R}^n, g_0)$ , where  $g_0$  is the canonical flat metric on  $\mathbb{R}^n$  and its potential function is given by the function  $f(x) = \frac{1}{2}\lambda r^2(x)$ . Here,  $r = r(x)$  is the distance from the origin. The Gaussian soliton satisfies

$$\text{Ric}_{g_0} + \text{Hess } f = \lambda g_0.$$

The Gaussian soliton is an example to show that Theorem 11 is not true if  $\delta = \lambda$ , since the soliton is noncompact and satisfies  $|\nabla f|(x) = \lambda r(x)$ .

As in the case of the Bakry–Émery and modified Ricci curvatures, we may give some compactness theorems for complete Riemannian manifolds via  $k$ -Bakry–Émery and  $k$ -modified Ricci curvatures. Limoncu [2010] established the following Myers-type theorem via  $k$ -modified Ricci curvature without making any assumption on the vector field appearing in the  $k$ -modified Ricci curvature:

**Theorem 13** [Limoncu 2010]. *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exists some positive constant  $\lambda > 0$  such that the*

$k$ -modified Ricci curvature satisfies  $\text{Ric}_V^k \geq \lambda g$ , where  $k \in (0, +\infty)$ . Then  $(M, g)$  must be compact. Moreover, the diameter of  $(M, g)$  has the upper bound

$$\text{diam}(M, g) \leq \frac{\pi}{\sqrt{\lambda}} \sqrt{n+k-1}.$$

**Remark.** In the case where the vector field  $V$  is replaced with the gradient of some smooth function  $f : M \rightarrow \mathbb{R}$ , Theorem 13 above was already proved by Qian [1997].

As demonstrated by Wraith [2006], the key ingredient in proving Theorem 2 is the Riccati inequality for the Ricci curvature

$$\text{Ric}_g(\partial_r, \partial_r) \leq -\dot{m} - \frac{1}{n-1}m^2,$$

which may be derived by applying the classical Bochner–Weitzenböck formula (1-4) to the distance function  $r(x) = d(x, p)$ . Here  $m := \Delta_g r$ . Recently, Li [2015] established the following Bochner–Weitzenböck formula via modified Ricci curvature:

$$(1-12) \quad \frac{1}{2} \Delta_V |\nabla u|^2 = |\text{Hess } u|^2 + \text{Ric}_V(\nabla u, \nabla u) + g(\nabla \Delta_V u, \nabla u).$$

By applying the Bochner–Weitzenböck formula (1-12) to the distance function  $r(x) = d(x, p)$ , we may derive the Riccati inequality for the  $k$ -modified Ricci curvature

$$\text{Ric}_V^k(\partial_r, \partial_r) \leq -\dot{m}_V - \frac{(m_V)^2}{n+k-1},$$

where  $m_V := \Delta_V r$ . By using this Riccati inequality, we shall prove the following Ambrose-type theorem via  $k$ -modified Ricci curvature:

**Theorem 14.** Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exists some point  $p \in M$  for which every geodesic  $\gamma : [0, +\infty) \rightarrow M$  emanating from  $p$  satisfies

$$\int_0^{+\infty} \text{Ric}_V^k(\dot{\gamma}(s), \dot{\gamma}(s)) ds = +\infty,$$

where  $k \in (0, +\infty)$ . Then  $(M, g)$  must be compact.

As to a Galloway-type theorem via  $k$ -modified Ricci curvature, we shall prove the following compactness theorem by modifying the proof of Theorem 13 by Limoncu [2010].

**Theorem 15.** Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold. Suppose that there exist some constants  $\lambda > 0$  and  $L \geq 0$  such that for every pair of points in  $M$  and minimal geodesic  $\gamma$  joining those points, the  $k$ -modified Ricci curvature satisfies

$$(1-13) \quad \text{Ric}_V^k(\dot{\gamma}, \dot{\gamma})|_{\gamma(s)} \geq \lambda + \frac{d\phi}{ds}(s),$$

where  $\phi$  is some smooth function of the arc length satisfying  $|\phi| \leq L$  along  $\gamma$  and  $k \in (0, +\infty)$ . Then  $(M, g)$  must be compact. Moreover, the diameter of  $(M, g)$  has the upper bound

$$\text{diam}(M, g) \leq \frac{1}{\lambda} (2L + \sqrt{4L^2 + (n+k-1)\lambda\pi^2}).$$

**Remark.** In the case where the vector field  $V$  is replaced with the gradient of some smooth function  $f : M \rightarrow \mathbb{R}$ , Theorem 15 above was already proved by Rimoldi [2011].

This paper is organized as follows: In Section 2, after introducing our notation, we shall prove Theorems 8, 11, and 14. Ending with Section 3, we shall prove Theorems 9, 10, and 15.

## 2. Ambrose-type theorems

In this section, we shall prove Theorem 8, 11, and 14. Our proofs of these theorems are modifications of the alternative proof of Theorem 2 by Wraith [2006] and the proof of Theorem 7 by Zhang [2014]. Throughout this paper, we assume that  $(M, g)$  is an  $n$ -dimensional smooth connected oriented complete Riemannian manifold without boundary. Let  $X, Y, Z \in \mathfrak{X}(M)$  be three smooth vector fields on  $M$ . For any smooth function  $f \in C^\infty(M)$ , a *gradient vector field* and a *Hessian* of  $f$  are defined by

$$g(\nabla f, X) = df(X) \quad \text{and} \quad \text{Hess } f(X, Y) = g(\nabla_X \nabla f, Y),$$

respectively. A *curvature* and a *Ricci curvature* are defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \quad \text{and} \quad \text{Ric}_g(X, Y) = \sum_{i=1}^n g(R(e_i, X)Y, e_i),$$

respectively. Here,  $\{e_i\}_{i=1}^n$  is an orthonormal frame of  $(M, g)$ .

**2.1. Proof of Theorem 8.** We shall first prove Theorem 8. In order to prove Theorem 8, it is sufficient to show the following theorem:

**Theorem 16.** *Let  $(M, g)$  be an  $n$ -dimensional complete noncompact Riemannian manifold and  $V \in \mathfrak{X}(M)$  be a smooth vector field on  $M$  satisfying  $|V| \leq K$  for some nonnegative constant  $K \geq 0$ . Let  $\gamma = \gamma(s)$ ,  $s \geq 0$ , be a geodesic in  $(M, g)$ . If the limit*

$$\lim_{t \rightarrow +\infty} \int_0^t \text{Ric}_V(\dot{\gamma}(s), \dot{\gamma}(s)) ds$$

*exists, then it must take a value less than infinity.*

*Proof.* We shall prove this theorem by contradiction. Fix a point  $p \in M$  and take a unit speed ray  $\gamma = \gamma(s)$  emanating from  $p$  satisfying  $\gamma(0) = p$ . For any  $s > 0$ , let  $m(s)$  be the mean curvature of the distance sphere of radius  $s$  about  $p$  at the point  $\gamma(s)$ . Note that  $m(s)$  is smooth for  $s > 0$ . It is well-known that  $m(s)$  satisfies the Riccati inequality

$$\text{Ric}_g(\dot{\gamma}(s), \dot{\gamma}(s)) \leq -\dot{m}(s) - \frac{1}{n-1}m^2(s),$$

see [Cheeger 1991] for details. Hence, we have

$$\text{Ric}_V(\dot{\gamma}(s), \dot{\gamma}(s)) \leq -\dot{m}(s) - \frac{1}{n-1}m^2(s) + \frac{1}{2}\mathcal{L}_V g(\dot{\gamma}(s), \dot{\gamma}(s)).$$

Since  $\mathcal{L}_V g(\dot{\gamma}(s), \dot{\gamma}(s)) = 2\dot{\gamma}(s)g(V(\gamma(s)), \dot{\gamma}(s)) = 2(\partial/\partial s)g(V(\gamma(s)), \dot{\gamma}(s))$ , by integrating both sides of the inequality just above, for all  $t > 1$ , we obtain

$$(2-1) \quad \int_1^t \text{Ric}_V(\dot{\gamma}(s), \dot{\gamma}(s)) ds \leq -m(t) + m(1) - \frac{1}{n-1} \int_1^t m^2(s) ds \\ + g(V(\gamma(t)), \dot{\gamma}(t)) - g(V(\gamma(1)), \dot{\gamma}(1)).$$

Since  $\gamma = \gamma(s)$  is a unit speed ray, the Cauchy–Schwarz inequality implies  $|g(V, \dot{\gamma})| \leq |V|$ . By combining this inequality and the assumption  $|V| \leq K$  in Theorem 16, we have  $|g(V, \dot{\gamma})| \leq K$ . Hence, from (2-1) we obtain

$$\int_1^t \text{Ric}_V(\dot{\gamma}(s), \dot{\gamma}(s)) ds \leq -m(t) + m(1) - \frac{1}{n-1} \int_1^t m^2(s) ds + 2K.$$

Suppose, to derive a contradiction, that

$$\lim_{t \rightarrow +\infty} \int_0^t \text{Ric}_V(\dot{\gamma}(s), \dot{\gamma}(s)) ds = +\infty.$$

Then we have

$$(2-2) \quad \lim_{t \rightarrow +\infty} \left( -m(t) - \frac{1}{n-1} \int_1^t m^2(s) ds \right) = +\infty.$$

In particular, we obtain

$$\lim_{t \rightarrow +\infty} -m(t) = +\infty.$$

Next, we shall show that there exists a finite number  $T > 0$  such that

$$(2-3) \quad \lim_{t \rightarrow T-0} -m(t) = +\infty,$$

which contradicts the smoothness of  $m(t)$ . First, it follows from (2-2) that there exists  $t_1 > 1$  such that for all  $t \geq t_1$ , we have

$$(2-4) \quad -m(t) - \frac{1}{n-1} \int_1^t m^2(s) ds > 2.$$

Define a sequence  $\{t_i\}_{i=1}^{+\infty}$  inductively by

$$t_{i+1} = t_i + (n-1)\left(\frac{1}{2}\right)^{i-1}$$

for all  $i \geq 1$ . Note that  $\{t_i\}_{i=1}^{+\infty}$  is an increasing sequence converging to

$$T := t_1 + 2(n-1).$$

**Lemma 17.** *For all  $t \geq t_i$ ,  $i \geq 1$ , we have*

$$(2-5) \quad -m(t) > 2^i.$$

*Proof of Lemma 17.* By (2-4), the conclusion (2-5) is true for  $i = 1$ . Suppose that (2-5) holds for all  $t \geq t_i$ . Then, it follows from (2-4) and (2-5) that for all  $t \geq t_{i+1}$ ,

$$\begin{aligned} -m(t) &> 2 + \frac{1}{n-1} \int_1^{t_i} m^2(t) dt + \frac{1}{n-1} \int_{t_i}^{t_{i+1}} m^2(t) dt \\ &> \frac{1}{n-1} \int_{t_i}^{t_{i+1}} m^2(t) dt \\ &> \frac{1}{n-1} \cdot (2^i)^2 \cdot (n-1) \cdot \left(\frac{1}{2}\right)^{i-1} = 2^{i+1}. \end{aligned}$$

Hence, (2-5) is true for all  $t \geq t_{i+1}$ . This proves Lemma 17.  $\square$

Thanks to Lemma 17, we have (2-3) which is the desired contradiction. The proof of Theorem 16 is completed.  $\square$

**2.2. Proof of Theorem 11.** Next, we shall prove Theorem 11.

*Proof of Theorem 11.* We shall prove this theorem by contradiction. Assume that  $(M, g)$  is noncompact. Fix a point  $p \in M$  and take a unit speed ray  $\gamma = \gamma(s)$  emanating from  $p$  satisfying  $\gamma(0) = p$ . For any  $s > 0$ , let  $m(s)$  be the mean curvature of the distance sphere of radius  $s$  about  $p$  at the point  $\gamma(s)$ . Note that  $m(s)$  is smooth for  $s > 0$ . It follows from (2-1) and (1-11) that for all  $t > 1$ ,

$$\begin{aligned} (2-6) \quad -m(t) - \frac{1}{n-1} \int_1^t m^2(s) ds + g(V(\gamma(t)), \dot{\gamma}(t)) &\geq \lambda t + \phi(t) + C_0 \\ &\geq \lambda t - L + C_0, \end{aligned}$$

where  $C_0 := -m(1) + g(V(\gamma(1)), \dot{\gamma}(1)) - \phi(1) - \lambda$ . It follows from the Cauchy-Schwarz inequality and the assumption  $|V|(x) \leq \delta d(x, p) + \alpha$  in Theorem 11 that

$$(2-7) \quad g(V(\gamma(t)), \dot{\gamma}(t)) \leq |V(\gamma(t))| \leq \delta t + \alpha.$$

Hence, it follows from (2-6) and (2-7) that

$$(2-8) \quad -m(t) - \frac{1}{n-1} \int_1^t m^2(s) ds \geq (\lambda - \delta)t - L + C_0 - \alpha.$$



Since  $\lambda > \delta$ , (2-8) implies that there exists  $t_1 > 1$  such that for all  $t \geq t_1$ , we have

$$-m(t) - \frac{1}{n-1} \int_1^t m^2(s) ds > 2.$$

Then, by using the same argument as in the proof of Theorem 16, we may derive the desired contradiction. The proof of Theorem 11 is completed.  $\square$

**2.3. Proof of Theorem 14.** Finally, we shall prove Theorem 14. In order to prove Theorem 14, it is sufficient to show the following theorem:

**Theorem 18.** *Let  $(M, g)$  be an  $n$ -dimensional complete noncompact Riemannian manifold,  $V \in \mathfrak{X}(M)$  be a smooth vector field on  $M$  and  $k \in (0, +\infty)$  be a positive constant. Let  $\gamma = \gamma(s)$ ,  $s \geq 0$ , be a geodesic in  $(M, g)$ . If the limit*

$$\lim_{t \rightarrow +\infty} \int_0^t \text{Ric}_V^k(\dot{\gamma}(s), \dot{\gamma}(s)) ds$$

*exists, then it must take a value less than infinity.*

We shall prove Theorem 18 by using the following lemma which may be considered as an extension of the Bochner–Weitzenböck formula via modified Ricci curvature:

**Lemma 19** [Li 2015]. *Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold. For any smooth vector field  $V \in \mathfrak{X}(M)$  and smooth function  $u : M \rightarrow \mathbb{R}$ , we have*

$$(2-9) \quad \frac{1}{2} \Delta_V |\nabla u|^2 = |\text{Hess } u|^2 + \text{Ric}_V(\nabla u, \nabla u) + g(\nabla \Delta_V u, \nabla u).$$

*In particular, for any positive constant  $k \in (0, +\infty)$ , we obtain*

$$(2-10) \quad \frac{1}{2} \Delta_V |\nabla u|^2 \geq \frac{1}{n+k} (\Delta_V u)^2 + \text{Ric}_V^k(\nabla u, \nabla u) + g(\nabla \Delta_V u, \nabla u).$$

**Remark.** If the vector field  $V$  is replaced with the gradient of some function  $f : M \rightarrow \mathbb{R}$ , then (2-9) is reduced to the Bochner–Weitzenböck formula (1-3) via Bakry–Émery Ricci curvature.

*Proof of Lemma 19.* For the reader's convenience, we recall the proof. This proof is based on the classical Bochner–Weitzenböck formula which asserts that for any smooth function  $u : M \rightarrow \mathbb{R}$ ,

$$\frac{1}{2} \Delta_g |\nabla u|^2 = |\text{Hess } u|^2 + \text{Ric}_g(\nabla u, \nabla u) + g(\nabla \Delta_g u, \nabla u).$$

First, we shall prove (2-9). By definition of the  $V$ -Laplacian, we have

$$(2-11) \quad \begin{aligned} \frac{1}{2} \Delta_V |\nabla u|^2 &= \frac{1}{2} \Delta_g |\nabla u|^2 - \frac{1}{2} g(V, \nabla |\nabla u|^2) \\ &= |\text{Hess } u|^2 + \text{Ric}_g(\nabla u, \nabla u) + g(\nabla \Delta_g u, \nabla u) - \frac{1}{2} g(V, \nabla |\nabla u|^2). \end{aligned}$$

The last two terms of the right-hand side become

$$\begin{aligned}
 (2-12) \quad g(\nabla \Delta_g u, \nabla u) - \frac{1}{2} g(V, \nabla |\nabla u|^2) \\
 &= g(\nabla(\Delta_V u + g(V, \nabla u)), \nabla u) - V^i \nabla^j u \nabla_i \nabla_j u \\
 &= g(\nabla \Delta_V u, \nabla u) + \nabla^i (V^j \nabla_j u) \nabla_i u - V^i \nabla^j u \nabla_i \nabla_j u \\
 &= g(\nabla \Delta_V u, \nabla u) + \nabla^i V^j \nabla_i u \nabla_j u \\
 &= g(\nabla \Delta_V u, \nabla u) + \nabla_i u \nabla_j u \left( \frac{1}{2} (\nabla^i V^j + \nabla^j V^i) \right) \\
 &= g(\nabla \Delta_V u, \nabla u) + \frac{1}{2} \mathcal{L}_V g(\nabla u, \nabla u).
 \end{aligned}$$

By combining (2-11) and (2-12), we obtain (2-9). Next, we shall prove (2-10). By the Cauchy–Schwarz inequality, we have

$$(2-13) \quad |\text{Hess } u|^2 \geq \frac{1}{n} (\Delta_g u)^2.$$

Hence, it follows from (2-9) and (2-13) that

$$(2-14) \quad \frac{1}{2} \Delta_V |\nabla u|^2 \geq \frac{1}{n} (\Delta_g u)^2 + \text{Ric}_V^k(\nabla u, \nabla u) + g(\nabla \Delta_V u, \nabla u) + \frac{1}{k} g(V, \nabla u)^2.$$

Recall the elementary inequality

$$(a + b)^2 \geq \frac{1}{t} a^2 - \frac{1}{t-1} b^2, \quad t > 1.$$

By choosing  $t = (n + k)/n$  in the inequality just above, we obtain

$$\begin{aligned}
 (2-15) \quad \frac{1}{n} (\Delta_g u)^2 &= \frac{1}{n} (\Delta_V u + g(V, \nabla u))^2 \\
 &\geq \frac{1}{n} \left( \frac{1}{\frac{n+k}{n}} (\Delta_V u)^2 - \frac{1}{\frac{n+k}{n} - 1} g(V, \nabla u)^2 \right) \\
 &= \frac{1}{n+k} (\Delta_V u)^2 - \frac{1}{k} g(V, \nabla u)^2.
 \end{aligned}$$

By combining (2-14) and (2-15), we have (2-10). □

Now, we are in a position to prove Theorem 18.

*Proof of Theorem 18.* We shall prove this theorem by contradiction. Fix a point  $p \in M$  and take a unit speed ray  $\gamma = \gamma(s)$  emanating from  $p$  satisfying  $\gamma(0) = p$ . Let  $r(x) = d(x, p)$  be the distance between  $x$  and  $p$ . By applying the inequality (2-10) to the distance function, we have

$$\text{Ric}_V^k(\dot{\gamma}(s), \dot{\gamma}(s)) \leq -\dot{m}_V(s) - \frac{1}{n+k-1} m_V^2(s),$$

where  $m_V(s) := (\Delta_V r)(\gamma(s))$ . Note that  $m_V(s)$  is smooth for  $s > 0$ . Suppose, to derive a contradiction, that

$$\lim_{t \rightarrow +\infty} \int_0^t \text{Ric}_V^k(\dot{\gamma}(s), \dot{\gamma}(s)) ds = +\infty.$$

Then we have

$$(2-16) \quad \lim_{t \rightarrow +\infty} \left( -m_V(t) - \frac{1}{n+k-1} \int_1^t m_V^2(s) ds \right) = +\infty.$$

In particular, we obtain

$$\lim_{t \rightarrow +\infty} -m_V(t) = +\infty.$$

Next, we shall show that there exists a finite number  $T > 0$  such that

$$(2-17) \quad \lim_{t \rightarrow T-0} -m_V(t) = +\infty,$$

which contradicts the smoothness of  $m_V(t)$ . First, it follows from (2-16) that there exists  $t_1 > 1$  such that for all  $t \geq t_1$ , we have

$$-m_V(t) - \frac{1}{n+k-1} \int_1^t m_V^2(s) ds > 2.$$

Define a sequence  $\{t_i\}_{i=1}^{+\infty}$  inductively by

$$t_{i+1} = t_i + (n+k-1) \left(\frac{1}{2}\right)^{i-1}$$

for all  $i \geq 1$ . Note that  $\{t_i\}_{i=1}^{+\infty}$  is an increasing sequence converging to

$$T := t_1 + 2(n+k-1).$$

Then, by using the same argument as in the proof of Lemma 17, we may prove the following lemma:

**Lemma 20.** *For all  $t \geq t_i$ ,  $i \geq 1$ , we have*

$$-m_V(t) > 2^i.$$

Thanks to Lemma 20, we have (2-17) which is the desired contradiction. The proof of Theorem 18 is completed.  $\square$

### 3. Galloway-type theorems

In this section, we shall prove Theorems 9, 10, and 15. Our proofs of these theorems are based on modifications of the improvement of Theorem 4 by the author [Tadano 2016] and the proofs of Theorems 6 and 13 by Limoncu [2010; 2012]. In order to prove these theorems, we shall use the index form of a unit speed-minimizing

geodesic segment. We refer the reader to the books [Lee 1997; Petersen 1998] for basic facts about this topic.

**3.1. Proof of Theorem 9.** We shall first prove Theorem 9.

*Proof of Theorem 9.* Take two arbitrary points  $p, q \in M$ . Since  $M$  is complete, there exists a unit speed-minimizing geodesic segment  $\gamma$  from  $p$  to  $q$  of length  $\ell$ . Let  $\{e_1 = \dot{\gamma}, e_2, \dots, e_n\}$  be a parallel orthonormal frame along  $\gamma$ . Recall that for any smooth function  $h \in C^\infty([0, \ell])$  satisfying  $h(0) = h(\ell) = 0$ , we have

$$(3-1) \quad \sum_{i=2}^n I(he_i, he_i) = \int_0^\ell ((n-1)\dot{h}^2 - h^2 \operatorname{Ric}_g(\dot{\gamma}, \dot{\gamma})) dt,$$

where  $I(\cdot, \cdot)$  denotes the index form of  $\gamma$ . By using the assumption (1-7) in the integral expression (3-1), we obtain

$$(3-2) \quad \sum_{i=2}^n I(he_i, he_i) \leq \int_0^\ell \left( (n-1)\dot{h}^2 - \lambda h^2 + h^2 \operatorname{Hess} f(\dot{\gamma}, \dot{\gamma}) - h^2 \frac{d\phi}{dt} \right) dt.$$

On the geodesic segment  $\gamma(t)$ , we have

$$\begin{aligned} (3-3) \quad h^2 \operatorname{Hess} f(\dot{\gamma}, \dot{\gamma}) &= h^2 g(\nabla_{\dot{\gamma}} \nabla f, \dot{\gamma}) = h^2 \dot{\gamma}(g(\nabla f, \dot{\gamma})) = h^2 \frac{d}{dt}(g(\nabla f, \dot{\gamma})) \\ &= -2h\dot{h}g(\nabla f, \dot{\gamma}) + \frac{d}{dt}(h^2 g(\nabla f, \dot{\gamma})) \\ &= 2f \frac{d}{dt}(h\dot{h}) - 2\frac{d}{dt}(fh\dot{h}) + \frac{d}{dt}(h^2 g(\nabla f, \dot{\gamma})), \end{aligned}$$

where in the last equality we have used  $g(\nabla f, \dot{\gamma}) = df/dt(\gamma(t))$ . Hence, by integrating both sides of (3-3), we obtain

$$\begin{aligned} (3-4) \quad \int_0^\ell h^2 \operatorname{Hess} f(\dot{\gamma}, \dot{\gamma}) dt &= \int_0^\ell 2f \frac{d}{dt}(h\dot{h}) dt - 2 \left[ fh\dot{h} \right]_0^\ell + \left[ h^2 g(\nabla f, \dot{\gamma}) \right]_0^\ell \\ &= 2 \int_0^\ell f \frac{d}{dt}(h\dot{h}) dt, \end{aligned}$$

where the last equality follows from  $h(0) = h(\ell) = 0$ . By (3-4) and the assumption  $|f| \leq H$  in Theorem 9, we have

$$(3-5) \quad \int_0^\ell h^2 \operatorname{Hess} f(\dot{\gamma}, \dot{\gamma}) dt \leq 2H \int_0^\ell \left| \frac{d}{dt}(h\dot{h}) \right| dt.$$

On the other hand, from the assumption  $|\phi| \leq L$  in Theorem 9, we obtain

$$(3-6) \quad \int_0^\ell h^2 \frac{d\phi}{dt} dt = [h^2 \phi]_0^\ell - \int_0^\ell 2h\dot{h}\phi dt \geq -2L \int_0^\ell |h\dot{h}| dt.$$

From (3-2), (3-5), and (3-6), we have

$$(3-7) \quad \sum_{i=2}^n I(he_i, he_i) \leq \int_0^\ell \left( (n-1)\dot{h}^2 - \lambda h^2 + 2H \left| \frac{d}{dt}(h\dot{h}) \right| + 2L|h\dot{h}| \right) dt.$$

If the function  $h$  is taken to be  $h(t) = \sin(\pi t/\ell)$ , then we obtain

$$h\dot{h} = \frac{\pi}{\ell} \sin\left(\frac{\pi t}{\ell}\right) \cos\left(\frac{\pi t}{\ell}\right) = \frac{\pi}{2\ell} \sin\left(\frac{2\pi t}{\ell}\right).$$

Then (3-7) becomes

$$(3-8) \quad \sum_{i=2}^n I(he_i, he_i) \leq \int_0^\ell (n-1) \left( \frac{\pi^2}{\ell^2} \cos^2\left(\frac{\pi t}{\ell}\right) - \lambda \sin^2\left(\frac{\pi t}{\ell}\right) \right) dt \\ + 2H \left( \frac{\pi}{\ell} \right)^2 \int_0^\ell \left| \cos\left(\frac{2\pi t}{\ell}\right) \right| dt + \frac{L\pi}{\ell} \int_0^\ell \left| \sin\left(\frac{2\pi t}{\ell}\right) \right| dt.$$

Since

$$(3-9) \quad \int_0^\ell \dot{h}^2 dt = \int_0^\ell \frac{\pi^2}{\ell^2} \cos^2\left(\frac{\pi t}{\ell}\right) dt = \frac{\pi^2}{2\ell}, \\ \int_0^\ell h^2 dt = \int_0^\ell \sin^2\left(\frac{\pi t}{\ell}\right) dt = \frac{\ell}{2}, \\ \int_0^\ell \left| \cos\left(\frac{2\pi t}{\ell}\right) \right| dt = \frac{2\ell}{\pi}, \\ \int_0^\ell |h\dot{h}| dt = \int_0^\ell \frac{\pi}{2\ell} \left| \sin\left(\frac{2\pi t}{\ell}\right) \right| dt = 1,$$

it follows from (3-8) and (3-9) that

$$\sum_{i=2}^n I(he_i, he_i) \leq -\frac{1}{2\ell} (\lambda \ell^2 - 4L\ell - (n-1)\pi^2 - 8H\pi).$$

Since  $\gamma$  is a minimizing geodesic, we must obtain

$$\lambda \ell^2 - 4L\ell - (n-1)\pi^2 - 8H\pi \leq 0,$$

from where we have

$$\ell \leq \frac{1}{\lambda} (2L + \sqrt{4L^2 + \{(n-1)\pi + 8H\}\lambda\pi}).$$

This proves Theorem 9. □

**3.2. Proof of Theorem 10.** Next, we shall prove Theorem 10.

*Proof of Theorem 10.* By using the assumption (1-9) in the integral expression (3-1), we obtain

$$\begin{aligned}
 (3-10) \quad \sum_{i=2}^n I(he_i, he_i) &\leq \int_0^\ell \left( (n-1)\dot{h}^2 - \lambda h^2 + \frac{1}{2}h^2(\mathcal{L}_V g)(\dot{\gamma}, \dot{\gamma}) - h^2 \frac{d\phi}{dt} \right) dt \\
 &= \int_0^\ell \left( (n-1)\dot{h}^2 - \lambda h^2 + h^2 g(\nabla_{\dot{\gamma}} V, \dot{\gamma}) - h^2 \frac{d\phi}{dt} \right) dt \\
 &= \int_0^\ell \left( (n-1)\dot{h}^2 - \lambda h^2 + h^2 \dot{\gamma}(g(V, \dot{\gamma})) - h^2 \frac{d\phi}{dt} \right) dt,
 \end{aligned}$$

where the last equality follows from the parallelism of the metric  $g$  and  $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ . On the geodesic segment  $\gamma(t)$ , we have

$$(3-11) \quad h^2 \dot{\gamma}(g(V, \dot{\gamma})) = h^2 \frac{d}{dt}(g(V, \dot{\gamma})) = -2h\dot{h}g(V, \dot{\gamma}) + \frac{d}{dt}(h^2 g(V, \dot{\gamma})).$$

Hence, by integrating both sides of (3-11), we obtain

$$\begin{aligned}
 (3-12) \quad \int_0^\ell h^2 \dot{\gamma}(g(V, \dot{\gamma})) dt &= \int_0^\ell -2h\dot{h}g(V, \dot{\gamma}) dt + [h^2 g(V, \dot{\gamma})]_0^\ell \\
 &\leq 2 \int_0^\ell |h\dot{h}g(V, \dot{\gamma})| dt
 \end{aligned}$$

$$(3-13) \quad \leq 2K \int_0^\ell |h\dot{h}| dt,$$

where the second inequality follows from  $h(0) = h(\ell) = 0$ . From (3-10), (3-13), and (3-6), we have

$$(3-14) \quad \sum_{i=2}^n I(he_i, he_i) \leq \int_0^\ell ((n-1)\dot{h}^2 - \lambda h^2) dt + 2K \int_0^\ell |h\dot{h}| dt + 2L \int_0^\ell |h\dot{h}| dt.$$

If the function  $h$  is taken to be  $h(t) = \sin(\pi t/\ell)$ , then (3-14) becomes

$$\begin{aligned}
 (3-15) \quad \sum_{i=2}^n I(he_i, he_i) &\leq \int_0^\ell \left( (n-1) \frac{\pi^2}{\ell^2} \cos^2\left(\frac{\pi t}{\ell}\right) - \lambda \sin^2\left(\frac{\pi t}{\ell}\right) \right) dt \\
 &\quad + \frac{K\pi}{\ell} \int_0^\ell \left| \sin\left(\frac{2\pi t}{\ell}\right) \right| dt + \frac{L\pi}{\ell} \int_0^\ell \left| \sin\left(\frac{2\pi t}{\ell}\right) \right| dt.
 \end{aligned}$$

It follows from (3-15) and (3-9) that

$$\sum_{i=2}^n I(he_i, he_i) \leq -\frac{1}{2\ell} \{ \lambda \ell^2 - 4(L+K)\ell - (n-1)\pi^2 \}.$$

Since  $\gamma$  is a minimizing geodesic, we must obtain

$$\lambda \ell^2 - 4(L + K)\ell - (n - 1)\pi^2 \leq 0,$$

from where we have

$$\ell \leq \frac{1}{\lambda} (2(L + K) + \sqrt{4(L + K)^2 + (n - 1)\lambda\pi^2}).$$

This proves Theorem 10. □

**3.3. Proof of Theorem 15.** Finally, we shall prove Theorem 15.

*Proof of Theorem 15.* By using the assumption (1-13) in the integral expression (3-1), we obtain

$$\begin{aligned} (3-16) \quad \sum_{i=2}^n I(he_i, he_i) &\leq \int_0^\ell \left( (n-1)\dot{h}^2 - \lambda h^2 + \frac{1}{2}h^2(\mathcal{L}_V g)(\dot{\gamma}, \dot{\gamma}) - h^2 \frac{d\phi}{dt} \right) dt \\ &\quad - \frac{1}{k} \int_0^\ell h^2(g(V, \dot{\gamma}))^2 dt \\ &\leq \int_0^\ell ((n-1)\dot{h}^2 - \lambda h^2 + 2|h\dot{h}g(V, \dot{\gamma})| + 2L|h\dot{h}|) dt \\ &\quad - \frac{1}{k} \int_0^\ell h^2(g(V, \dot{\gamma}))^2 dt, \end{aligned}$$

where the last inequality follows from (3-12) and (3-6). Applying  $P = |\dot{h}|$  and  $Q = |hg(V, \dot{\gamma})|$  to the Cauchy–Schwarz inequality

$$\int_0^\ell PQ \, dt \leq \sqrt{\int_0^\ell P^2 \, dt} \sqrt{\int_0^\ell Q^2 \, dt},$$

we have

$$(3-17) \quad \int_0^\ell |h\dot{h}g(V, \dot{\gamma})| \, dt \leq \sqrt{\int_0^\ell \dot{h}^2 \, dt} \sqrt{\int_0^\ell h^2(g(V, \dot{\gamma}))^2 \, dt}.$$

Applying  $A = k \int_0^\ell \dot{h}^2 \, dt \geq 0$  and  $B = (1/k) \int_0^\ell h^2(g(V, \dot{\gamma}))^2 \, dt \geq 0$  to the elementary inequality  $2\sqrt{AB} \leq A + B$ , we obtain

$$(3-18) \quad 2 \sqrt{\int_0^\ell \dot{h}^2 \, dt} \sqrt{\int_0^\ell h^2(g(V, \dot{\gamma}))^2 \, dt} \leq \int_0^\ell \left( k\dot{h}^2 + \frac{1}{k}h^2(g(V, \dot{\gamma}))^2 \right) dt.$$

From (3-16), (3-17), and (3-18), we have

$$(3-19) \quad \sum_{i=2}^n I(he_i, he_i) \leq \int_0^\ell ((n-1)\dot{h}^2 - \lambda h^2 + k\dot{h}^2 + 2L|h\dot{h}|) \, dt.$$

If the function  $h$  is taken to be  $h(t) = \sin(\pi t/\ell)$ , then (3-19) becomes

$$(3-20) \quad \sum_{i=2}^n I(he_i, he_i) \leq \int_0^\ell (n-1) \left( \frac{\pi^2}{\ell^2} \cos^2\left(\frac{\pi t}{\ell}\right) - \lambda \sin^2\left(\frac{\pi t}{\ell}\right) \right) dt \\ + \frac{k\pi^2}{\ell^2} \int_0^\ell \cos^2\left(\frac{\pi t}{\ell}\right) dt + \frac{L\pi}{\ell} \int_0^\ell \left| \sin\left(\frac{2\pi t}{\ell}\right) \right| dt.$$

It follows from (3-20) and (3-9) that

$$\sum_{i=2}^n I(he_i, he_i) \leq -\frac{1}{2\ell} \{ \lambda \ell^2 - 4L\ell - (n-1)\pi^2 - k\pi^2 \}.$$

Since  $\gamma$  is a minimizing geodesic, we must obtain

$$\lambda \ell^2 - 4L\ell - (n-1)\pi^2 - k\pi^2 \leq 0,$$

from where we have

$$\ell \leq \frac{1}{\lambda} (2L + \sqrt{4L^2 + (n-1+k)\lambda\pi^2}).$$

This proves Theorem 15. □

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# IRREDUCIBLE DECOMPOSITION FOR LOCAL REPRESENTATIONS OF QUANTUM TEICHMÜLLER SPACE

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**We give an irreducible decomposition of the so-called local representations (Bai, Bonahon and Liu, 2007) of the quantum Teichmüller space  $\mathcal{T}_q(\Sigma)$ , where  $\Sigma$  is a punctured surface of genus  $g > 0$  and  $q$  is an  $N$ -th root of unity with  $N$  odd. As an application, we construct a family of representations of the Kauffman bracket skein algebra of the closed surface  $\bar{\Sigma}$ .**

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## 1. Introduction

Let  $\Sigma$  be the surface obtained by removing  $s > 0$  points  $v_1, \dots, v_s$  from the closed oriented surface  $\bar{\Sigma}$  of genus  $g > 0$ . Denote by  $\mathcal{T}(\Sigma)$  the Teichmüller space of  $\Sigma$ , that is roughly speaking, the space of complete hyperbolic metrics on  $\Sigma$ . Given  $\lambda$  an ideal triangulation of  $\Sigma$  (that is a triangulation of the closed surface  $\bar{\Sigma}$  whose vertices are exactly the  $v_i$ ), Thurston [1986] constructed a parametrization of  $\mathcal{T}(\Sigma)$  by associating a strictly positive real number to each edge  $\lambda_i$  of the ideal triangulation,  $i \in \{1, \dots, n\}$ , where  $n = 6g - 6 + 3s$  is the number of edges of  $\lambda$ . These coordinates are called the *shear coordinates* associated to  $\lambda$ . In this coordinate system, the coefficients of the Weil–Petersson form on  $\mathcal{T}(\Sigma)$  depend only on the combinatorics of  $\lambda$  and are easy to compute.

For a parameter  $q \in \mathbb{C}^*$ , Chekhov and Fock [1999] defined the *quantum Teichmüller space*  $\mathcal{T}_q(\Sigma)$  of  $\Sigma$ , which is a deformation quantization of the Poisson algebra of a certain class of functions over  $\mathcal{T}(\Sigma)$ ; see also [Kashaev 1998] for a slightly different version and [Guo and Liu 2009] for a relation between the two.

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This algebraic object is obtained by gluing together a collection of noncommutative algebras  $\mathcal{T}_q(\lambda)$ , called *Chekhov–Fock algebras*, canonically associated to each ideal triangulation of  $\Sigma$ . A representation of  $\mathcal{T}_q(\Sigma)$  is then a family of representations  $\{\rho_\lambda : \mathcal{T}_q(\lambda) \rightarrow \text{End}(V)\}_{\lambda \in \Lambda(\Sigma)}$ , where  $\Lambda(\Sigma)$  is the space of all ideal triangulations of  $\Sigma$ , and  $\rho_\lambda$  and  $\rho_{\lambda'}$  satisfy compatibility conditions whenever  $\lambda \neq \lambda'$ . For  $\lambda \in \Lambda(\Sigma)$ , the representation  $\rho_\lambda$  is an avatar of the representation of  $\mathcal{T}_q(\Sigma)$  and carries almost all the information.

When  $q$  is a primitive  $N$ -th root of unity,  $\mathcal{T}_q(\lambda)$  admits finite-dimensional representations. In this paper, we will consider the case that  $N$  is odd. The irreducible representations of  $\mathcal{T}_q(\lambda)$  were studied in [Bonahon and Liu 2007], where it was shown that an irreducible representation of  $\mathcal{T}_q(\lambda)$  is classified (up to isomorphism) by a weight  $x_i \in \mathbb{C}^*$  assigned to each edge  $\lambda_i$ , a choice of  $N$ -th root  $p_j = (x_1^{k_{j1}} \cdots x_n^{k_{jn}})^{1/N}$  associated to each puncture  $v_j$  (where  $k_{ji}$  is the number of times a small simple loop around  $v_j$  intersects  $\lambda_i$ ) and an  $N$ -th root  $c = (x_1 \cdots x_n)^{1/N}$ . Such a representation has dimension  $N^{3g-3+s}$ .

Bai, Bonahon, and Liu [Bai et al. 2007] introduced another type of representations of  $\mathcal{T}_q(\lambda)$ , called *local representations*, which are well behaved under cut and paste. A local representation of  $\mathcal{T}_q(\lambda)$  is defined by an embedding into the tensorial product of *triangle algebras* (see definitions below). Isomorphism classes of local representations of  $\mathcal{T}_q(\lambda)$  are classified by a weight  $x_i \in \mathbb{C}^*$  associated to each edge  $\lambda_i$  and a choice of an  $N$ -th root  $c = (x_1 \cdots x_n)^{1/N}$ . Such a representation has dimension  $N^{4g-4+2s}$ .

It follows that a local representation of  $\mathcal{T}_q(\lambda)$  is not irreducible. In this paper, we address the question of the decomposition of a local representation into its irreducible components. We prove:

**Main Theorem.** *Let  $\lambda$  be an ideal triangulation of  $\Sigma$  and  $\rho$  be a local representation of  $\mathcal{T}_q(\lambda)$  classified by weight  $x_j \in \mathbb{C}^*$  associated to each edge  $\lambda_j$  and a choice of  $N$ -th root  $c = (x_1 \cdots x_n)^{1/N}$ . Then we have the decomposition*

$$\rho = \bigoplus_{i \in \mathcal{I}} \rho^{(i)}.$$

Here,  $\rho^{(i)}$  is an irreducible representation classified by the same  $x_j$ , an  $N$ -th root  $p_j^{(i)} = (x_1^{k_{j1}} \cdots x_n^{k_{jn}})^{1/N}$  associated to each puncture, and the same  $c$ . Moreover,  $\mathcal{I}$  is a finite set such that, given a choice of an  $N$ -th root  $p_j = (x_1^{k_{j1}} \cdots x_n^{k_{jn}})^{1/N}$  for each puncture, there exists exactly  $N^g$  elements  $i \in \mathcal{I}$  with  $p_j^{(i)} = p_j$  for all  $j \in \{1, \dots, s\}$ .

It is proved by Bai [2007] that if  $\lambda$  and  $\lambda'$  are two different triangulations of the square related by a diagonal switch, then the intertwining operators associated to two isomorphic representations  $\rho : \mathcal{T}_q(\lambda) \rightarrow \text{End}(V)$  and  $\rho' : \mathcal{T}_q(\lambda') \rightarrow \text{End}(V')$  correspond to the  $6j$ -symbols defined by Kashaev [1995]. These  $6j$ -symbols relate

hyperbolic geometry and quantum invariants and gave birth to the famous volume conjecture; see [Murakami 2011] for an overview.

In particular, Baseilhac and Benedetti [2005] used these  $6j$ -symbols to construct a  $(2 + 1)$ -dimensional topological quantum field theory (TQFT) on manifolds with  $\mathrm{PSL}(2, \mathbb{C})$ -character. Our result thus provides a decomposition of the vector spaces arising in the TQFT.

As an application, we adapt the construction of Bonahon and Wong [2015] to local representations of the balanced Chekhov–Fock algebra and obtain a family of representations of the Kauffman bracket skein algebra  $\mathcal{S}_A(\bar{\Sigma})$  of the closed surface  $\bar{\Sigma}$ . The vector space associated to this family of representations is canonically associated to an ideal triangulation  $\lambda$ . In particular, it makes the computations very explicit. It also behaves well under cut and paste.

In Section 2, we recall the definition of the Chekhov–Fock algebra, the quantum Teichmüller space, the triangle algebra and the local representations. In Section 3, we prove the Main Theorem. Finally, in Section 4, we explain the connections between quantum Teichmüller theory, skein theory and construct a new family of representations of  $\mathcal{S}_A(\bar{\Sigma})$ .

## 2. Chekhov–Fock algebra and representations of $\mathcal{T}_q(\Sigma)$

In this section, we define the Chekhov–Fock algebra  $\mathcal{T}_q(\lambda)$  associated to an ideal triangulation  $\lambda$ , describe its representations and recall the definition of the quantum Teichmüller space. Most results come from [Bonahon and Liu 2007; Bai et al. 2007].

In all this paper, for an integer  $n \in \mathbb{N}$ , set  $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$  and denote by  $\mathcal{U}(N)$  the group of  $N$ -th roots of unity.

**2.1. The Chekhov–Fock algebra.** In this subsection,  $q$  is a formal parameter and  $\Sigma$  is allowed to have boundary components with punctures on the boundary (and every boundary component has at least one puncture).

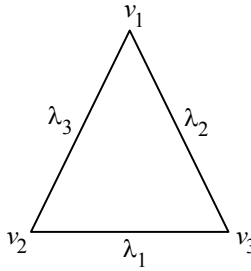
Let  $\lambda$  be an ideal triangulation of  $\Sigma$ . We denote by  $\lambda_1, \dots, \lambda_n$  the edges of  $\lambda$ . The *Fock’s matrix associated to  $\lambda$*  is the skew-symmetric  $n \times n$  matrix with integer coefficients  $\eta_\lambda = (\sigma_{ij})_{i,j=1,\dots,n}$  defined by

$$\sigma_{ij} = a_{ij} - a_{ji}$$

where  $a_{ij}$  is the number of angular sector delimited by  $\lambda_i$  and  $\lambda_j$  in the faces of  $\lambda$  with  $\lambda_i$  coming before  $\lambda_j$  counterclockwise.

**Definition 2.1.** The *Chekhov–Fock algebra of  $\lambda$*  is the algebra  $\mathcal{T}_q(\lambda)$  freely generated by the elements  $X_i^{\pm 1}$ ,  $i \in \{1, \dots, n\}$ , subject to the relations

$$X_i X_j = q^{2\sigma_{ij}} X_j X_i.$$



**Figure 1.** The triangle  $T$ .

The following example is of first importance.

**Example 2.2.** Let  $T$  be a disk with three punctures  $v_1, v_2, v_3$  on the boundary. The boundary arcs between the punctures provides a natural triangulation  $\lambda$  of  $T$  (see Figure 1).

The *triangle algebra* is  $\mathcal{T} := \mathcal{T}_q(\lambda)$ . It is generated by  $X_i^{\pm 1}$ ,  $i = 1, 2, 3$ , subject to the relations

$$X_i X_{i+1} = q^2 X_{i+1} X_i, \quad i \in \mathbb{Z}_3.$$

The algebraic structure of the Chekhov–Fock algebra is fairly simple. In particular, it is a quantum torus [Goodearl and Warfield 2004].

Given a monomial  $X = X_1^{k_1} \dots X_n^{k_n} \in \mathcal{T}_q(\lambda)$  for a multi-index  $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ , we define the *Weyl ordering* of  $X$  to be the monomial

$$[X] := q^{-\sum_{i < j} \sigma_{ij}} X_1^{k_1} \dots X_n^{k_n}.$$

The advantage of the Weyl ordering is its independence with respect to the order of the terms. In particular, for any permutation  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ , we have

$$[X_1^{k_1} \dots X_n^{k_n}] = [X_{\sigma(1)}^{k_{\sigma(1)}} \dots X_{\sigma(n)}^{k_{\sigma(n)}}].$$

For a multi-index  $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ , we define  $X_{\mathbf{k}} := [X_1^{k_1} \dots X_n^{k_n}] \in \mathcal{T}_q(\lambda)$ .

**2.2. Finite-dimensional representations of  $\mathcal{T}_q(\lambda)$ .** When the parameter  $q$  is a root of unity, the structure of the Chekhov–Fock algebra is drastically different. In particular,  $\mathcal{T}_q(\lambda)$  admits finite dimensional representations that we describe here.

In this subsection,  $q \in \mathbb{C}^*$  is a primitive  $N$ -th root of unity with  $N$  odd,  $\Sigma$  has no boundary component and  $\lambda$  is an ideal triangulation of  $\Sigma$  with edges labeled  $\lambda_1, \dots, \lambda_n$ .

**Definition 2.3.** For each puncture  $v_j$ , the *puncture invariant*  $P_j = [X_1^{k_{j1}} \dots X_n^{k_{jn}}] \in \mathcal{T}_q(\lambda)$  is the monomial associated to the multi-index  $\mathbf{k}_j = (k_{j1}, \dots, k_{jn}) \in \mathbb{N}^n$ , where  $k_{ji}$  is the minimum number of intersections between the edge  $\lambda_i$  and a closed simple loop around  $v_j$ .

The puncture invariants are of main importance in the representation theory of the Chekhov–Fock algebra. In particular:

**Proposition 2.4** [Bonahon and Liu 2007, Proposition 15]. *The center of  $\mathcal{T}_q(\lambda)$  is generated by:*

- $X_i^N$  for each  $i \in \{1, \dots, n\}$ .
- The puncture invariant  $P_j$  associated to each puncture  $v_j \in \{v_1, \dots, v_s\}$ .
- The element  $H := [X_1 \dots X_n]$ .

Note that  $[P_1 \dots P_s] = [H^2]$ .

A representation of  $\mathcal{T}_q(\lambda)$  is a morphism  $\rho : \mathcal{T}_q(\lambda) \rightarrow \text{End}(V)$  where  $V$  is a vector space. Such a representation is *finite-dimensional* when  $V$  is finite-dimensional and  $\rho$  is called *irreducible* when there is no proper linear subspace  $W \subset V$  preserved by  $\rho(\mathcal{T}_q(\lambda))$ . Two representations  $\rho : \mathcal{T}_q(\lambda) \rightarrow \text{End}(V)$  and  $\rho' : \mathcal{T}_q(\lambda) \rightarrow \text{End}(V')$  are *isomorphic* if there exists a linear isomorphism  $L : V \rightarrow V'$  such that

$$\rho'(X) = L \circ \rho(X) \circ L^{-1} \quad \text{for } X \in \mathcal{T}_q(\lambda).$$

**Theorem 2.5** [Bonahon and Liu 2007, Theorems 20 and 21]. *Up to isomorphism, any irreducible representation*

$$\rho : \mathcal{T}_q(\lambda) \rightarrow \text{End}(V)$$

*is determined by its restriction to the center of  $\mathcal{T}_q(\lambda)$  and is classified by a nonzero complex number  $x_i$  associated to each edges  $\lambda_i$ , a choice of an  $N$ -th root  $p_j = (x_1^{k_{j1}} \dots x_n^{k_{jn}})^{1/N}$  for each puncture  $v_j$  (where the  $k_{jk} \in \{1, 2\}$  are as in Definition 2.3) and a choice of a square root  $c = (p_0 \dots p_s)^{1/2}$ .*

*Such a representation is characterized by*

- $\rho(X_i^N) = x_i \text{Id}_V$ ,
- $\rho(P_j) = p_j \text{Id}_V$ ,
- $\rho(H) = c \text{Id}_V$ .

*Moreover, such a representation has dimension  $N^{3g-3+s}$ .*

Let us come back to Example 2.2. Recall that the triangle algebra  $\mathcal{T}$  is the algebra generated by  $X_i^{\pm 1}$ ,  $i \in \mathbb{Z}_3$ , with relations  $X_i X_{i+1} = q^2 X_{i+1} X_i$ .

The center of  $\mathcal{T}$  is generated by  $X_i^N$  and  $H = q^{-1} X_1 X_2 X_3$ . One easily checks that irreducible representations of  $\mathcal{T}$  have dimension  $N$  and are classified (up to isomorphism) by a choice of weight  $x_i \in \mathbb{C}^*$  associated to each edge  $\lambda_i$  and a central charge, that is a choice of an  $N$ -th root  $c = (x_1 x_2 x_3)^{1/N}$ ; see [Bai et al. 2007, Lemma 2] for more details.

More precisely, let  $V$  be the complex vector space generated by  $\{e_1, \dots, e_N\}$  and let  $\rho$  be an irreducible representation of  $\mathcal{T}$  classified by  $x_1, x_2, x_3 \in \mathbb{C}^*$  and  $c = (x_1 x_2 x_3)^{1/N}$ . Then, up to isomorphism, the action of  $\mathcal{T}$  on  $V$  is defined by

$$\rho(X_1)e_i = x'_1 q^{2i} e_i, \quad \rho(X_2)e_i = x'_2 e_{i+1}, \quad \rho(X_3)e_i = x'_3 q^{1-2i} e_{i-1},$$

where  $x'_i$  is an  $N$ -th root of  $x_i$  such that  $x'_1 x'_2 x'_3 = c$ . Note that, up to isomorphism,  $\rho$  is independent of the choice of the  $N$ -th root  $x'_i$  with  $x'_1 x'_2 x'_3 = c$ .

The following lemma will be useful in the next section. Recall that  $\mathcal{U}(N)$  is the group of  $N$ -th roots of unity.

**Lemma 2.6.** *Let  $\rho : \mathcal{T} \rightarrow \text{End}(V)$  be the representation of the triangle algebra classified by  $x_1 = x_2 = x_3 = 1$  and  $c \in \mathcal{U}(N)$ . For each  $i \in \mathbb{Z}_N$  and  $N$ -th root  $\zeta \in \mathcal{U}(N)$ , the eigenspace of  $\rho(X_i)$  of eigenvalue  $\zeta$  is one-dimensional.*

*Proof.* We use the explicit form of the representation  $\rho$  in  $V = \text{span}\{e_1, \dots, e_N\}$  described above. Set  $x'_1 = x'_2 = 1$ ,  $x'_3 = c$  and  $\zeta = q^{2k}$  for some  $k \in \{0, \dots, N-1\}$ .

For  $i = 1$ , one checks that the eigenspace of  $\rho(X_1)$  associated to  $\zeta$  is generated by  $e_k$ .

For  $i = 2$ , the vector  $\alpha_k := \sum_{i \in \mathbb{Z}_N} q^{-2ki} e_i$  satisfies  $\rho(X_2)\alpha_k = q^{2k}\alpha_k$  and  $\{\alpha_1, \dots, \alpha_k\}$  form a basis of  $V$ . So the eigenspace of  $\rho(X_2)$  associated to the eigenvalue  $\zeta$  is generated by  $\alpha_k$ .

For  $i = 3$ , we use the fact that  $\rho(q^{-1}X_1X_2X_3) = c \text{Id}_V$ , where  $c \in \mathcal{U}(N)$ .  $\square$

An ideal triangulation of  $\Sigma$  is composed by  $m$  faces  $T_1, \dots, T_m$ . Each face  $T_j$  determines a triangle algebra  $\mathcal{T}_j$  whose generators are associated to the three edges of  $T_j$ . It provides a canonical embedding  $\iota$  of  $\mathcal{T}_q(\lambda)$  into  $\mathcal{T}_1 \otimes \dots \otimes \mathcal{T}_m$  defined on the generators as follows:

- $\iota(X_i) = X_{j_i} \otimes X_{k_i}$  if  $\lambda_i$  belongs to two distinct triangles  $T_j$  and  $T_k$  and  $X_{j_i} \in \mathcal{T}_j$ ,  $X_{k_i} \in \mathcal{T}_k$  are the generators associated to the edge  $\lambda_i \in T_j$  and  $\lambda_i \in T_k$  respectively.
- $\iota(X_i) = [X_{j_{i_1}} X_{j_{i_2}}]$  if  $\lambda_i$  corresponds to two sides of the same face  $T_j$  and  $X_{j_{i_1}}, X_{j_{i_2}} \in \mathcal{T}_j$  are the associated generators.

**Definition 2.7.** A *local representation* of  $\mathcal{T}_q(\lambda)$  is a representation which factorizes as  $(\rho_1 \otimes \dots \otimes \rho_m) \circ \iota$  where  $\rho_i : \mathcal{T}_i \rightarrow V_i$  is an irreducible representation of the triangle algebra  $\mathcal{T}_i$  and  $\iota : \mathcal{T}_q(\lambda) \rightarrow \mathcal{T}_1 \otimes \dots \otimes \mathcal{T}_m$  is defined as above.

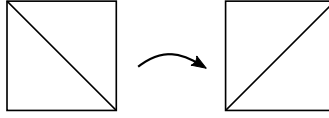
In particular, a local representation has dimension  $N^m$  where  $m = 4g - 4 + 2s$  is the number of faces of the triangulation.

Local representations were first introduced by Bai et al. [2007].

**Theorem 2.8** [Bai et al. 2007, Proposition 6]. *Up to isomorphism, a local representation*

$$\rho : \mathcal{T}_q(\lambda) \rightarrow \text{End}(V)$$





**Figure 2.** Flip of triangulation.

is classified by a nonzero complex number  $x_i$  associated to the edge  $\lambda_i$  and a choice of an  $N$ -th root  $c = (x_1 \dots x_n)^{1/N}$ . Such a representation satisfies

- $\rho(X_i^N) = x_i \text{Id}_V$ ,
- $\rho(H) = c \text{Id}_V$ .

Local representations have certain advantages over irreducible representations. First of all, these representations behave very well under cut and paste, so one can use them to construct invariant of 3-manifolds; see [Baseilhac and Benedetti 2005]. Also, the vector space associated to a local representation decomposes as a tensor product of vector spaces and each generator  $X_i \in \mathcal{T}_q(\lambda)$  associated to an edge  $\lambda_i$  only acts on the vector spaces associated to triangle adjacent to the edge  $\lambda_i$  (that is why these representations are called local).

**2.3. Quantum Teichmüller space and its representations.** The quantum Teichmüller space is obtained by gluing together a family of (division algebras of) Chekhov–Fock algebras indexed by the set of ideal triangulations of  $\Sigma$ .

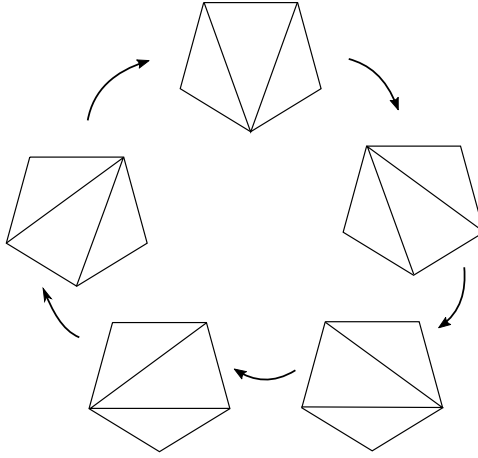
**The simplex of ideal triangulations.** Let  $\Lambda(\Sigma)$  be the set of ideal triangulations of  $\Sigma$ . We say that two triangulations  $\lambda, \lambda' \in \Lambda(\Sigma)$  differ by a flip if  $\lambda$  and  $\lambda'$  coincide everywhere except in a square made of two adjacent triangles where they differ as in Figure 2.

The *graph of ideal triangulations* is the graph whose set of vertices is  $\Lambda(\Sigma)$  and two vertices  $\lambda, \lambda' \in \Lambda(\Sigma)$  are connected by an edge if and only if  $\lambda$  and  $\lambda'$  differ by a flip.

The *simplex of ideal triangulations* is obtained from the graph of ideal triangulations by gluing a 2-simplex on each cycle corresponding to the pentagon relation (see Figure 3).

**Proposition 2.9** [Penner 1993]. *The simplex of ideal triangulations is connected and simply connected. Namely, any two different ideal triangulations are connected by a sequence of flips and two sequences between two ideal triangulations differ by a sequence of pentagon relations.*

**Coordinate change.** The Chekhov–Fock algebra  $\mathcal{T}_q(\lambda)$  associated to an ideal triangulation  $\lambda \in \Lambda(\Sigma)$  satisfies the Ore condition; see [Goodearl and Warfield 2004].



**Figure 3.** Pentagon relation.

In particular,  $\mathcal{T}_q(\lambda)$  has a well-defined division algebra  $\hat{\mathcal{T}}_q(\lambda)$  consisting of rational fractions satisfying some noncommutativity relations.

Let  $\lambda, \lambda' \in \Lambda(\Sigma)$  be two ideal triangulations related by a flip. Chekhov and Fock [1999] constructed coordinates change isomorphisms

$$\Psi_{\lambda\lambda'}^q : \hat{\mathcal{T}}_q(\lambda') \rightarrow \hat{\mathcal{T}}_q(\lambda).$$

These coordinates change satisfy the pentagon relation. In particular, using the result of Penner, they extend uniquely to coordinates change  $\Psi_{\lambda\lambda'}^q : \hat{\mathcal{T}}_q(\lambda') \rightarrow \hat{\mathcal{T}}_q(\lambda)$  for any two different ideal triangulations  $\lambda, \lambda' \in \Lambda(\Sigma)$ .

It was proved in [Liu 2009] that these coordinates change are the unique ones satisfying some natural relations, as for instance  $\Psi_{\lambda\lambda''}^q = \Psi_{\lambda\lambda'}^q \circ \Psi_{\lambda'\lambda''}^q$  for any  $\lambda, \lambda', \lambda'' \in \Lambda(\Sigma)$ . Moreover, when  $q = 1$ , these maps reduce to the classical coordinates change in Teichmüller theory; see [loc. cit.] for more details.

**Definition 2.10.** The *quantum Teichmüller space* of  $\Sigma$  is defined by

$$\mathcal{T}_q(\Sigma) := \bigsqcup_{\lambda \in \Lambda(\Sigma)} \hat{\mathcal{T}}_q(\lambda) / \sim,$$

where  $x_\lambda \in \hat{\mathcal{T}}_q(\lambda) \sim x_{\lambda'} \in \hat{\mathcal{T}}_q(\lambda')$  if and only if  $x_\lambda = \Psi_{\lambda\lambda'}^q(x_{\lambda'})$ .

Note that, as Since each coordinate change  $\Psi_{\lambda\lambda'}^q$  is an algebra isomorphism,  $\mathcal{T}_q(\Sigma)$  inherits an algebra structure, and the  $\hat{\mathcal{T}}_q(\lambda)$  can be thought as “global coordinates” on  $\mathcal{T}_q(\Sigma)$ .

**Representations.** A natural definition for a finite dimensional representation of  $\mathcal{T}_q(\Sigma)$  would be a family of finite dimensional representations

$$\{\rho_\lambda : \hat{\mathcal{T}}_q(\lambda) \rightarrow \text{End}(V_\lambda)\}_{\lambda \in \Lambda(\Sigma)}$$

such that for each pair  $\lambda, \lambda' \in \Lambda(\Sigma)$ , the representations  $\rho_{\lambda'}$  and  $\rho_\lambda \circ \Psi_{\lambda, \lambda'}^q$  of  $\hat{\mathcal{T}}_q(\lambda')$  are isomorphic (as representations).

However, as pointed out in [Bai et al. 2007, Section 4.2], when  $V_\lambda$  is finite-dimensional, there is no morphism  $\hat{\mathcal{T}}_q(\lambda) \rightarrow \text{End}(V_\lambda)$ . In fact,  $\hat{\mathcal{T}}_q(\lambda)$  is infinite-dimensional as a vector space while  $\text{End}(V_\lambda)$  is finite-dimensional and so, such a homomorphism  $\rho_\lambda$  would have nonzero kernel. In particular, there would exist elements  $x \in \hat{\mathcal{T}}_q(\lambda)$  such that  $\rho_\lambda(x) = 0$  and so,  $\rho_\lambda(x^{-1})$  would make no sense.

This motivates the following definition:

**Definition 2.11.** A *local representation* (respectively an *irreducible representation*) of  $\mathcal{T}_q(\Sigma)$  is a family of representations

$$\{\rho_\lambda : \mathcal{T}_q(\lambda) \rightarrow \text{End}(V_\lambda)\}_{\lambda \in \Lambda(\Sigma)}$$

such that for each  $\lambda, \lambda' \in \Lambda(\Sigma)$ ,  $\rho_\lambda$  is a local representation (respectively an irreducible representation) of  $\mathcal{T}_q(\lambda)$ , and  $\rho_{\lambda'}$  is isomorphic to  $\rho_\lambda \circ \Psi_{\lambda, \lambda'}^q$  whenever  $\rho_\lambda \circ \Psi_{\lambda, \lambda'}^q$  makes sense.

We say that  $\rho_\lambda \circ \Psi_{\lambda, \lambda'}^q$  makes sense, if for each Laurent polynomial  $X' \in \mathcal{T}_q(\lambda')$ ,

$$\Psi_{\lambda, \lambda'}(X') = P Q^{-1} = Q'^{-1} P' \in \hat{\mathcal{T}}_q(\lambda), \quad \text{for some } P, Q, P', Q' \in \mathcal{T}_q(\lambda).$$

In that case, we define

$$\rho_\lambda \circ \Psi_{\lambda, \lambda'}(X') := \rho_\lambda(P) \rho_\lambda(Q)^{-1} = \rho_\lambda(Q')^{-1} \rho_\lambda(P').$$

**Proposition 2.12** [Bai et al. 2007, Proposition 10]. *Let  $\lambda, \lambda' \in \Lambda(\Sigma)$  be two ideal triangulations of  $\Sigma$ . Then there exists a rational map*

$$\varphi_{\lambda, \lambda'} : \mathbb{C}^n \rightarrow \mathbb{C}^n$$

*such that a local representation  $\rho' : \mathcal{T}_q(\lambda') \rightarrow \text{End}(V_{\lambda'})$  classified by  $(x'_1, \dots, x'_n)$  and  $c' = (x'_1 \dots x'_n)^{1/N}$  is isomorphic to  $\rho_\lambda \circ \Psi_{\lambda, \lambda'}$  (whenever it makes sense) where  $\rho_\lambda : \mathcal{T}_q(\lambda) \rightarrow \text{End}(V_\lambda)$  is a local representation classified by  $(x_1, \dots, x_n)$  and  $c = (x_1 \dots x_n)^{1/N}$  if and only if  $c = c'$  and*

$$(x'_1, \dots, x'_n) = \varphi_{\lambda, \lambda'}(x_1, \dots, x_n).$$

**Remark 2.13.** The analogue is also proved in [Bonahon and Liu 2007] for irreducible representations. In particular, the rational maps  $\varphi_{\lambda, \lambda'}$  are the same.

It turns out that the rational maps  $\varphi_{\lambda\lambda'}$  correspond to the coordinates change of the shear-bend coordinates on the character variety  $\chi(\Sigma, \mathrm{SL}(2, \mathbb{C}))$ .

As a result, isomorphism classes of local (respectively irreducible) representations of  $\mathcal{T}_q(\Sigma)$  are classified, up to finitely many choices, by the character variety  $\chi(\Sigma, \mathrm{SL}(2, \mathbb{C}))$ ; see [loc. cit.] for more details.

### 3. Decomposition of local representations

In this section, we prove the Main Theorem. Let  $\rho : \mathcal{T}_q(\lambda) \rightarrow \mathrm{End}(V)$  be the local representation classified by the nonzero complex number  $x_i$  associated to each edge and the central charge  $c$ . Given a puncture invariant  $P_j = [X_1^{k_{j1}} \dots X_n^{k_{jn}}]$  (see Proposition 2.4) associated to the puncture  $v_j$ , the representation  $\rho$  satisfies

$$\rho(P_j^N) = x_1^{k_{j1}} \dots x_n^{k_{jn}} \mathrm{Id}_V.$$

It follows that if  $p_j$  is an eigenvalue of  $P_j$ , then  $p_j^N = x_1^{k_{j1}} \dots x_n^{k_{jn}}$ .

**Notation.**

- Given  $p_j \in \mathbb{C}^*$  so that  $p_j^N = x_1^{k_{j1}} \dots x_n^{k_{jn}}$ , we denote by

$$V_{p_j}(P_j) := \{x \in V : \rho(P_j)x = p_j x\}$$

the associated eigenspace.

- Given  $\mathbf{p} = (p_1, \dots, p_s)$  so that for each  $j$ ,  $p_j^N = x_1^{k_{j1}} \dots x_n^{k_{jn}}$ , set

$$V_{\mathbf{p}} := \{x \in V : \rho(P_j)x = p_j x, j = 1, \dots, s\} = \bigcap_{j=1}^s V_{p_j}(P_j).$$

The proof of the Main Theorem will follow the next proposition:

**Proposition 3.1.** *There exists an ideal triangulation  $\lambda_0 \in \Lambda(\Sigma)$  such that for each  $\mathbf{p}$  as above,  $V_{\mathbf{p}}$  has dimension  $N^{4g-3+s}$ .*

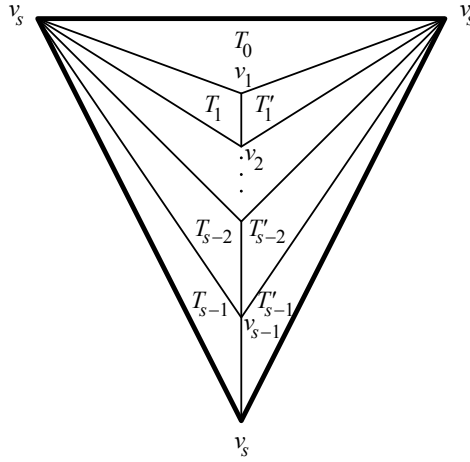
*Proof.* The dimension of  $V_{\mathbf{p}}$  does not depend on the numbers  $x_i \in \mathbb{C}^*$  characterizing  $\rho$ . In this proof, we will consider all the  $x_i$  equal to 1, which implies that the eigenvalues of  $\rho(P_i)$  are root of unity.

Consider the one punctured surface  $\Sigma' := \Sigma \cup \{v_1, \dots, v_{s-1}\}$ . As  $g > 0$ , there exists an ideal triangulation  $\lambda'$  of  $\Sigma'$ . Let  $T$  be a triangle of the triangulation  $\lambda'$  and consider the triangulation of  $T \setminus \{v_1, \dots, v_{s-1}\}$  as in Figure 4.

The union of the triangulation  $\lambda'$  and the one of  $T$  gives an ideal triangulation  $\lambda_0$  of  $\Sigma$ .

Consider a local representation  $\rho : \mathcal{T}_q(\lambda_0) \rightarrow \mathrm{End}(V)$ . The decomposition of the ideal triangulation  $\lambda_0$  gives the nice decomposition

$$V = W \otimes W',$$



**Figure 4.** Triangulation of  $T \cup \{v_1, \dots, v_s\}$ .

where  $W'$  is the vector space corresponding to the triangles of the triangulation  $\lambda'$  (except the triangle  $T$ ), and  $W$  corresponds to the triangles of  $T$ .

In particular, as the triangulation  $\lambda'$  contains  $4g - 2$  triangles,  $\dim(W') = N^{4g-3}$  (remember that we do not consider the vector space associated to  $T$ ), and  $\dim W = N^{2s-1}$ .

The interest of the triangulation  $\lambda_0$  is clear: the puncture invariant  $P_i$  associated to the puncture  $v_i \neq v_s$  acts as the identity on  $W'$ , so the eigenspaces of  $\rho(P_i)$  has the form  $E \otimes W'$  where  $E \subset W$  is an eigenspace of the restriction of  $\rho(P_i)$  on  $W$ . It motivates the following notation:

**Notation.**

- The vector space  $W$  decomposes as

$$W = W^0 \otimes \dots \otimes W^{s-1},$$

where  $W^0$  is associated to  $T_0$  and  $W^j$  to  $T_j$  and  $T'_j$  for  $j = 1, \dots, s-1$ .

- Given a root of unity  $p_k \in \mathcal{U}(N)$ , set

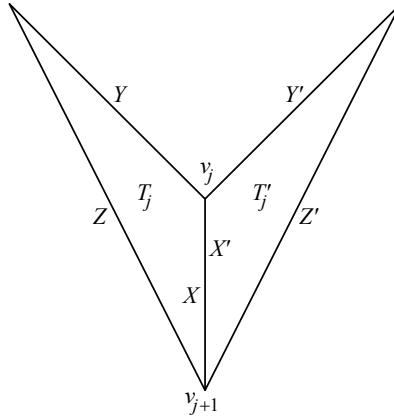
$$W_{p_k}^j(P_k) := \{x \in W^j : \rho(P_k)x = p_k x\}.$$

- For  $\mathbf{p} = (p_1, \dots, p_{s-1}) \in \mathcal{U}(N)^{s-1}$ , set

$$W_{\mathbf{p}}^j = \{x \in W^j : \rho(P_k)x = p_k x, k = 1, \dots, s-1\} = \bigcap_{k=1}^{s-1} W_{p_k}^j(P_k).$$

- Finally, set

$$W_{\mathbf{p}} = \{x \in W : \rho(P_k)x = p_k x, k = 1, \dots, s-1\}.$$



**Figure 5.** The generators of  $\mathcal{T}_j$  and  $\mathcal{T}'_j$ .

**Lemma 3.2.** *Using the above notation, and given  $\mathbf{p} \in \mathcal{U}(N)^{s-1}$ , we have the following:*

$$(1) \dim W_{\mathbf{p}}^0 = \begin{cases} 1 & \text{if } \mathbf{p} = (p_1, 1, \dots, 1), \\ 0 & \text{otherwise.} \end{cases}$$

(2) For  $j \in \{1, \dots, s-2\}$ ,

$$\dim W_{\mathbf{p}}^j = \begin{cases} 1 & \text{if } \mathbf{p} = (1, \dots, 1, p_j, p_{j+1}, 1, \dots, 1), \\ 0 & \text{otherwise.} \end{cases}$$

$$(3) \dim W_{\mathbf{p}}^{s-1} = \begin{cases} N & \text{if } \mathbf{p} = (1, \dots, 1, p_{s-1}), \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* (1) If  $k \neq 1$ ,  $v_k$  is not a vertex of  $T_0$ . It follows that  $P_k$  acts on  $W^0$  by the identity; so if  $p_k \neq 1$ ,  $W_{\mathbf{p}}^0 = \{0\}$ .

Now, if  $p_k = 1$  for all  $k \neq 1$ , then  $W_{\mathbf{p}}^0$  is the eigenspace of the action on  $W^0$  of the edge opposite to  $v_1$ . By Lemma 2.6, it is one-dimensional.

(2) Fix  $j \in \{1, \dots, s-2\}$ . For  $k \notin \{j, j+1\}$ ,  $v_k$  is neither a vertex of  $T_j$  nor of  $T'_j$ . Hence  $P_j$  acts on  $W^j$  as the identity, and if  $p_k \neq 1$ , then  $W_{\mathbf{p}}^j = \{0\}$ .

Take  $p_k = 1$  for all  $k \notin \{j, j+1\}$  and denote by  $X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}$  (respectively  $X'^{\pm 1}, Y'^{\pm 1}, Z'^{\pm 1}$ ) the generators of the triangle algebras  $\mathcal{T}_j$  (respectively  $\mathcal{T}'_j$ ) associated to the triangles  $T_j$  (respectively  $T'_j$ ) as in Figure 5. Set also  $W^j = V^j \otimes V'^j$  where  $V^j$  (respectively  $V'^j$ ) is the vector space associated to the representation of the triangle algebra  $\mathcal{T}_j$  (respectively  $\mathcal{T}'_j$ ).

Denote by  $c_j, c'_j \in \mathcal{U}(N)$  the central charges of the restriction of the representation to  $\mathcal{T}_j$  and  $\mathcal{T}'_j$  respectively. Then  $\rho(P_j)$  acts on  $V^j := \text{span}\{e_0, \dots, e_{N-1}\}$  like  $c_j Z^{-1}$  and on  $V'^j = \text{span}\{e'_0, \dots, e'_{N-1}\}$  like  $c'_j Z'^{-1}$ . In the same way,  $\rho(P_{j+1})$  acts on  $V_j$  like  $c_j Y^{-1}$  and on  $V'_j$  like  $c'_j Y'^{-1}$ .

Using the same action as in Example 2.2 and writing

$$c_j = q^p, \quad c'_j = q^{p'},$$

we get

$$\rho(P_j)e_k = q^{2k-1+p}e_{k+1}, \quad \rho(P_j)e'_l = q^{1-2l+p'}e_{l+1}$$

It follows that the action of  $P_j$  on  $W^j$  is given by

$$P_j \epsilon_{k,l} = q^{2(k-l)+p+p'} \epsilon_{k+1,l+1} \text{ where } \epsilon_{k,l} := e_k \otimes e'_l \in W^j.$$

In the same way, one sees that the action of  $P_{j+1}$  on  $W^j$  is given by

$$P_{j+1} \epsilon_{k,l} = q^{p+p'} \epsilon_{k-1,l-1}.$$

Now, for  $m, n \in \mathbb{Z}_N$ , set  $\alpha_{m,n} := \sum_{k=0}^{N-1} q^{2km} \epsilon_{k,k+n}$ , an easy calculation shows that

$$P_j \alpha_{m,n} = q^{-2(m+n)+p+p'} \alpha_{m,n}, \quad P_{j+1} \alpha_{m,n} = q^{2m+p+p'} \alpha_{m,n}.$$

It follows that  $\{\alpha_{n,m} : n, m \in \mathbb{Z}_N\}$  is a base of  $W^j$  and, for all  $p_j, p_{j+1} \in \mathcal{U}(N)$ , there exists a unique couple  $(m, n) \in \mathbb{Z}_N^2$  with  $p_j = q^{-2(m+n)+p+p'}$  and  $p_{j+1} = q^{2m+p+p'}$ .

Therefore,  $\dim W_p^j = 1$  if and only if  $p_k = 1$  for all  $k \neq j, j+1$ .

(3) If  $k \neq s-1$ , then  $v_k$  is neither a vertex of  $T_{s-1}$  nor a vertex of  $T'_{s-1}$ , so if  $p_k \neq 1$ , then  $W_h^s = \{0\}$ .

Suppose that  $p_k = 1$  for all  $k \in \{1, \dots, s-2\}$ , then

$$W_p^{s-1} \supset \bigoplus_{p_a p_b = p_{s-1}} V_{p_a}^{s-1}(P_{s-1}) \otimes V_{p_b}^{s-1}(P_{s-1}),$$

where  $V_{p_a}^s(P_{s-1})$  is the eigenspace associated to the eigenvalue  $p_a$  of the action of  $\rho(P_{s-1})$  on the vector space associated to the triangle  $T_{s-1}$ , and  $V_{p_b}^{s-1}(P_{s-1})$  is defined in an analogous way.

The direct sum contains  $N$  terms of dimension one, hence  $\dim(W_p^{s-1}) \geq N$ . On the other hand, we have

$$\dim(W^{s-1}) = N^2$$

and also

$$\dim(W^{s-1}) = \sum_{p \in \mathcal{U}(N)^{s-1}} \dim(W_p^{s-1}) \geq N \times N.$$

This implies that  $W_p^{s-1}$  has exactly dimension  $N$  for  $p = (1, \dots, 1, p_{s-1})$ .  $\square$

The proof of Proposition 3.1 follows from the following elementary remark:

**Remark 3.3.** For all  $j \in \{0, \dots, s-1\}$ , given  $\mathbf{p}_j \in \mathcal{U}(N)^{s-1}$  and a nonzero vector  $x_j \in W_{\mathbf{p}_j}^j$ , the vector  $x_0 \otimes \dots \otimes x_{s-1}$  is in  $W_{\mathbf{p}}$  where  $\mathbf{p} = \mathbf{p}_0 \mathbf{p}_1 \dots \mathbf{p}_{s-1}$  is obtained by taking the product on each component.

We thus have the inclusion

$$(1) \quad W_{\mathbf{p}} \supset \bigoplus_{\mathbf{p} = \mathbf{p}_0 \dots \mathbf{p}_{s-1}} W_{\mathbf{p}_0}^0 \otimes \dots \otimes W_{\mathbf{p}_{s-1}}^{s-1}.$$

Writing  $\mathbf{p}_j = (p_0^{(j)}, \dots, p_{s-1}^{(j)})$  and  $\mathbf{p} = (p_1, \dots, p_s)$ , one notes that from Lemma 3.2, the only nonzero terms in the direct sum of (1) are the  $\mathbf{p}_j$  satisfying

$$(2) \quad \begin{aligned} p_1^{(0)} p_1^{(1)} &= p_1, \\ p_2^{(1)} p_2^{(2)} &= p_2, \\ &\vdots \\ p_{s-1}^{(s-2)} p_{s-1}^{(s-1)} &= p_{s-1} \end{aligned}$$

There exists exactly  $N^{s-1}$  different choices for the  $\mathbf{p}_j$  satisfying relations (2).

Moreover, for each choice of  $\mathbf{p}_j$  satisfying (2), the vector space  $W_{\mathbf{p}_0}^0 \otimes \dots \otimes W_{\mathbf{p}_{s-1}}^{s-1}$  has dimension  $N$ . It follows that for each  $\mathbf{p} \in \mathcal{U}(N)^{s-1}$ ,

$$\dim W_{\mathbf{p}} \geq N^s.$$

On the other hand,

$$\dim W = N^{2s-1} = \sum_{\mathbf{p} \in \mathcal{U}(N)^{s-1}} \dim W_{\mathbf{p}},$$

hence each  $W_{\mathbf{p}}$  has exactly dimension  $N^s$ .

Finally, as the puncture invariants act as the identity on the vector space  $W'$ , the intersection of the eigenspaces of the  $\rho(P_j)$  for all  $j = 1, \dots, s-1$  has the form  $W_{\mathbf{p}} \otimes W'$  for some  $\mathbf{p} \in \mathcal{U}(N)^{s-1}$  and has dimension  $N^{4g-3+s}$ . As the representation  $\rho$  has fixed central charge  $c$ ,

$$\rho([P_1 P_2 \dots P_s]) = \rho([H^2]) = c^2 \text{Id}_V.$$

It follows that the action of  $\rho(P_s)$  on  $V$  can be easily expressed as a function of the action of the  $\rho(P_j)$  for  $j = 1, \dots, s-1$ , and we get the result.  $\square$

Proposition 3.1 implies the decomposition of the Main Theorem for the triangulation  $\lambda_0$ . Since the dimension of the eigenspaces depends continuously on the  $x_i$ , we get the decomposition for all value of  $x_i \in \mathbb{C}^*$ .

Indeed, consider the local representation

$$\rho : \mathcal{T}_q(\lambda_0) \rightarrow \text{End}(V)$$



classified by a nonzero complex number  $x_i$  associated to each edge and central charge  $c$ . Let  $\rho^{(i)} : \mathcal{T}_q(\lambda_0) \rightarrow \text{End}(V^{(i)})$  be an irreducible factor.

In particular,

$$\begin{aligned}\rho^{(i)}(X_i^N) &= \rho(X_i^N)|_{V^{(i)}} = x_i \text{Id}_{V^{(i)}}, \\ \rho^{(i)}(H) &= \rho(H)|_{V^{(i)}} = c \text{Id}_{V^{(i)}},\end{aligned}$$

so a necessary condition for  $\rho^{(i)}$  to be an irreducible factor is that it is classified by the same  $x_i$  and same central charge  $c$ .

For each puncture  $v_j$ , denote by  $p_j^{(i)}$  the  $N$ -th root of  $x_1^{k_{j1}} \dots x_n^{k_{jn}}$  so that

$$\rho^{(i)}(P_j) = p_j^{(i)} \text{Id}_{V^{(i)}}.$$

It follows that  $p_s^{(i)} = c^2(p_1^{(i)} \dots p_{s-1}^{(i)})^{-1}$  and  $V^{(i)} \subset V_{\mathbf{p}}$  where  $\mathbf{p} = (p_1^{(i)}, \dots, p_{s-1}^{(i)})$  with

$$V_{\mathbf{p}} = \{x \in V : \rho(P_j)x = p_j^{(i)}x, j = 1, \dots, s-1\}.$$

Finally, as  $\dim V_{\mathbf{p}} = N^{4g-3+s}$  and the dimension of an irreducible representation of  $\mathcal{T}_q(\lambda_0)$  has dimension  $N^{3g-3+s}$ ,  $V_{\mathbf{p}}$  contains exactly  $N^g$  irreducible factors classified by the same  $x_i$ , same central charge  $c$  and  $N$ -th root  $p_j^{(i)}$  associated to the puncture  $v_j$ .

**Proof in the general case.** Recall that, given another ideal triangulation  $\lambda \in \Lambda(\Sigma)$ , the “transition maps”  $\varphi_{\lambda_0\lambda} : \mathbb{C}^n \rightarrow \mathbb{C}^n$  defined in Section 2.3 are rational, hence defined on a Zariski open set of  $\mathbb{C}^n$ .

Now, consider a local representation

$$\rho : \mathcal{T}_q(\lambda) \rightarrow \text{End}(V_{\lambda}),$$

classified by a number  $x_i \in \mathbb{C}^*$  associated to each edge and central charge  $c$ .

If there exists  $(y_1, \dots, y_n) \in \mathbb{C}^n$  so that  $\varphi_{\lambda_0\lambda}(y_1, \dots, y_n) = (x_1, \dots, x_n)$  (which is a generic condition), then it follows from Section 2.3 that  $\rho_{\lambda}$  is isomorphic to  $\rho_{\lambda_0} : \mathcal{T}_q(\lambda_0) \rightarrow \text{End}(V_{\lambda_0})$ . It means that there exists an isomorphism

$$L_{\lambda_0\lambda} : V_{\lambda} \rightarrow V_{\lambda_0},$$

so that for each  $X \in \mathcal{T}_q(\lambda)$  we have

$$\rho_{\lambda_0}(\Psi_{\lambda_0\lambda}^q(X)) = L_{\lambda_0\lambda} \circ \rho_{\lambda}(X) \circ L_{\lambda_0\lambda}^{-1}.$$

Here  $\Psi_{\lambda_0\lambda}^q : \hat{\mathcal{T}}_q(\lambda) \rightarrow \hat{\mathcal{T}}_q(\lambda_0)$  are the coordinates change defined in Section 2.3.

As  $\rho_{\lambda_0}$  is a local representation of  $\mathcal{T}_q(\lambda_0)$ , there exists an irreducible decomposition of  $\rho_{\lambda_0}$  given by the decomposition

$$V_{\lambda_0} = \bigoplus_{i \in \mathcal{I}} V_{\lambda_0}^i.$$

In particular, each  $i \in \mathcal{I}$ ,  $V_{\lambda_0}^i$  is stable by  $\rho_{\lambda_0}$  and has dimension  $N^{3g-3+s}$ .

For each  $i \in \mathcal{I}$ , set  $V_{\lambda}^i := L_{\lambda_0 \lambda}^{(-1)}(V_{\lambda_0}^i)$ . One easily gets that each  $V_{\lambda}^i$  is stable by  $\rho_{\lambda}(\mathcal{T}_q(\lambda))$ , and has dimension  $N^{3g-3+s}$ . In this way we get a decomposition of  $\rho_{\lambda}$  into irreducible factors given by the decomposition

$$V_{\lambda} = \bigoplus_{i \in \mathcal{I}} V_{\lambda}^i.$$

Finally, if  $\rho_{\lambda}$  is classified by the parameters  $(x_1, \dots, x_n)$  which are not in the image of  $\varphi_{\lambda_0 \lambda}$ , one can deform continuously  $(x_1, \dots, x_n)$  to get the previous decomposition and, as the decomposition does not depend of the parameters, get the result for  $\rho_{\lambda}$ .

#### 4. Representations of the skein algebra

In this section, we use the Main Theorem to construct a nice family of representation of the Kauffman bracket skein algebra  $\mathcal{S}_A(\bar{\Sigma})$  of the closed surface  $\bar{\Sigma} = \Sigma \cup \{v_1, \dots, v_s\}$ . This is done by adapting the construction of Bonahon and Wong [2015] to the case of local representations.

In Section 4.1, we describe the balanced Chekhov–Fock algebra  $\mathcal{Z}_{\omega}(\lambda)$  associated to an ideal triangulation  $\lambda$  of  $\Sigma$  and characterize its irreducible representations. Then, in Section 4.2, we introduce the local representations of  $\mathcal{Z}_{\omega}(\lambda)$  and extend the Main Theorem to decompose these local representations into irreducible factors. Finally, in Section 4.3, we use the previous decomposition to construct a family of representations of  $\mathcal{S}_A(\bar{\Sigma})$ .

**4.1. Balanced Chekhov–Fock algebra.** Let  $q$  be a primitive  $N$ -th root of unity with  $N$  odd and let  $\omega$  be the unique fourth root of  $q$  which is also a primitive  $N$ -th root of unity. Namely, if  $q = e^{2i\pi \frac{k}{N}}$  with  $N$  and  $k$  coprime, then there is a unique  $l \in \mathbb{Z}_4$  so that  $k + lN \in 4\mathbb{Z}$ , and  $\omega = e^{2i\pi \frac{k}{4N}} e^{i \frac{l\pi}{2}}$ .

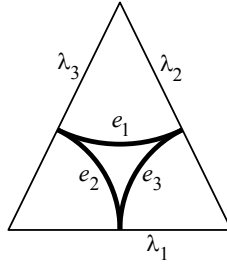
Let  $\lambda$  be an ideal triangulation of  $\Sigma$  whose edges are  $(\lambda_1, \dots, \lambda_n)$ . In order to avoid confusion, we will denote by  $X_i$  the generators of  $\mathcal{T}_q(\lambda)$  and by  $Z_i$  the generators of  $\mathcal{T}_{\omega}(\lambda)$ .

A multi-index  $\mathbf{k} \in \mathbb{Z}^n$  is called  $\lambda$ -balanced (or balanced) if for each triangle of the triangulation whose edges are  $j_1, j_2, j_3$  we have

$$k_{j_1} + k_{j_2} + k_{j_3} \in 2\mathbb{Z}.$$

A monomial  $Z \in \mathcal{T}_{\omega}(\lambda)$  is balanced if  $Z$  is a scalar multiple of  $Z_{\mathbf{k}}$  where  $\mathbf{k} \in \mathbb{Z}^n$  is balanced. (Here  $Z_{\mathbf{k}}$  is defined as in Section 2.1).

**Definition 4.1.** The balanced Chekhov–Fock algebra  $\mathcal{Z}_{\omega}(\lambda)$  is the subalgebra of  $\mathcal{T}_{\omega}(\lambda)$  generated (as a vector space) by balanced monomials.



**Figure 6.** Train track.

In particular, the image of the map

$$\begin{aligned} i : \mathcal{T}_q(\lambda) &\rightarrow \mathcal{T}_\omega(\lambda) \\ X_i &\mapsto Z_i^2 \end{aligned}$$

lies in  $\mathcal{Z}_\omega(\lambda)$  so we will consider  $\mathcal{T}_q(\lambda)$  as a subalgebra of  $\mathcal{Z}_\omega(\lambda)$ .

The ideal triangulation  $\lambda$  defines canonically a train track  $\tau_\lambda$  on  $\Sigma$  which looks like in Figure 6 on each triangle of the triangulation. Note that  $\tau_\lambda$  has a switch on each edge of  $\lambda$ .

We denote by  $\mathcal{W}(\tau_\lambda, \mathbb{Z})$  the abelian group of integer weight systems on  $\tau_\lambda$ . Namely, an element  $\alpha \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$  is a map that associates to each edge  $e$  of  $\tau_\lambda$  an integer  $\alpha(e)$  in such a way that at each switch, the sum of the weights of the incoming edges equals the sum of the weights of the outgoing edges.

A *weight system*  $\alpha \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$  on  $\tau_\lambda$  is a map that associates an integer to any edge of the train track in such a way that the sum of weights of the incoming edges equals the sum of weights of the outgoing edges. Given  $\alpha \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$  and an edge  $\lambda_i \in \lambda$ , the sum of the weights of the edges incoming to  $\lambda_i$  is an integer. It thus define a map

$$\varphi : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \rightarrow \mathbb{Z}^n$$

whose image is exactly the set of balanced multi-index. Thus, given an integer weight system  $\alpha \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$ , we define  $Z_\alpha \in \mathcal{Z}_\omega(\lambda)$  to be  $Z_{\varphi(\alpha)} = [Z_1^{\alpha_1} \dots Z_n^{\alpha_n}]$  where  $\varphi(\alpha) = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ . In particular, one gets the noncommutativity relations

$$Z_\alpha Z_\beta = \omega^{4\Omega(\alpha, \beta)} Z_\beta Z_\alpha,$$

where  $\Omega : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \times \mathcal{W}(\tau_\lambda, \mathbb{Z}) \rightarrow \mathbb{Z}$  is the Thurston intersection form; see [Bonahon and Wong 2012, Section 2] for more details.

**Definition 4.2.** A *twisted homomorphism* is a map  $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \rightarrow \mathbb{C}^*$  such that for every  $\alpha, \beta \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$ ,

$$\zeta(\alpha + \beta) = (-1)^{\Omega(\alpha, \beta)} \zeta(\alpha) \zeta(\beta).$$

Finally, note that each puncture  $v_j$  defines a integer weight system  $\eta_j \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$  as follow. The connected component  $D_j$  of  $\Sigma \setminus \tau_\lambda$  containing  $v_j$  is bounded by a finite number of edges of  $\tau_\lambda$ . For an edge  $e$  of  $\tau_\lambda$ , define  $\eta_j(e) \in \{0, 1, 2\}$  to be the number of times  $e$  lies in the boundary of  $D_j$ . In particular,

$$Z_{\eta_j}^2 = i(P_j)$$

where  $P_j \in \mathcal{T}_q(\lambda)$  is the puncture invariant associated to  $v_j$  and  $i : \mathcal{T}_q(\lambda) \rightarrow \mathcal{Z}_\omega(\lambda)$  is defined above.

Irreducible representations of  $\mathcal{Z}_\omega(\lambda)$  were classified in [Bonahon and Wong 2015]. They proved:

**Proposition 4.3** [Bonahon and Wong 2012, Proposition 14]. *Up to isomorphism, an irreducible representation  $\rho : \mathcal{Z}_\omega(\lambda) \rightarrow \text{End}(V)$  has dimension  $N^{3g-3+s}$  and is classified by a twisted homomorphism  $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \rightarrow \mathbb{C}^*$  and a choice of an  $N$ -th root  $h_j = \zeta(\eta_j)^{1/N}$  for each puncture  $v_j$ . Such a representation satisfies:*

- $\rho(Z_\alpha^N) = \zeta(\alpha) \text{Id}_V$  for all  $\alpha \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$ .
- $\rho(\eta_j) = \zeta(\eta_j) \text{Id}_V$  for all  $j \in \{1, \dots, s\}$ .

**4.2. Local representations of  $\mathcal{Z}_\omega(\lambda)$ .** Here, we introduce the notion of local representation of the balanced Chekhov–Fock algebra  $\mathcal{Z}_\omega(\lambda)$  associated to an ideal triangulation  $\lambda$ . We then extend the Main Theorem to give a decomposition of local representations of  $\mathcal{Z}_\omega(\lambda)$  into its irreducible components.

Since by our choice  $\omega$  is also a primitive  $N$ -th root of unity, there is a map

$$\mathcal{T}_\omega(\lambda) \rightarrow \bigotimes_{T_i \in F(\lambda)} \mathcal{T}_\omega(T_i)$$

as introduced in Section 2.2, where  $F(\lambda)$  is the set of faces of  $\lambda$  and  $\mathcal{T}_\omega(T_i)$  is the triangle algebra associated to the face  $T_i$ . It is clear that this map restricts to a morphism

$$\iota : \mathcal{Z}_\omega(\lambda) \rightarrow \bigotimes_{T_i \in F(\lambda)} \mathcal{Z}_\omega(T_i).$$

Here,  $\mathcal{Z}_\omega(T_i)$  is the *balanced triangle algebra* associated to the face  $T_i$ .

**Definition 4.4.** A *local representation of the balanced Chekhov–Fock algebra*  $\mathcal{Z}_\omega(\lambda)$  is a representation  $\rho : \mathcal{Z}_\omega(\lambda) \rightarrow \text{End}(V)$  that can be written as  $(\rho_1 \otimes \dots \otimes \rho_m) \circ \iota$  where each  $\rho_i$  is an irreducible representation of  $\mathcal{Z}_\omega(T_i)$ .

In order to classify local representations of  $\mathcal{Z}_\omega(\lambda)$ , we first have to understand the irreducible representations of the balanced triangle algebra  $\mathcal{Z}_\omega(T)$ . Let  $\tau$  be the train track in  $T$  with edges  $e_1, e_2, e_3$  as in Figure 6 and denote by  $\mathcal{W}(\tau, \mathbb{Z})$  The group of integer weight systems on  $\tau$ .

**Lemma 4.5.** *Up to isomorphism, an irreducible representation of the balanced triangle algebra  $\mathcal{Z}_\omega(T)$  has dimension  $N$  and is classified by a twisted homomorphism  $\zeta : \mathcal{W}(\tau, \mathbb{Z}) \rightarrow \mathbb{C}^*$  together with a choice of an  $N$ -th root  $C = (\zeta(\mu))^{1/N}$  where  $\mu \in \mathcal{W}(\tau, \mathbb{Z})$  is such that  $\mu(e_i) = 1$  for all  $i \in \mathbb{Z}_3$ . Such a representation satisfies*

- $\rho(Z_\alpha^N) = \zeta(\alpha) \text{Id}_V$ .
- $\rho(Z_\mu) = C \text{Id}_V$ .

*Proof.* The group  $\mathcal{W}(\tau, \mathbb{Z})$  is generated by  $\{\alpha_1, \alpha_2, \alpha_3\}$  where

$$\alpha_i(e_j) = \delta_{ij}, \quad i, j \in \mathbb{Z}_3.$$

It follows that the balanced triangle algebra  $\mathcal{Z}_\omega(T)$  is generated by  $Z_{\alpha_1}^{\pm 1}, Z_{\alpha_2}^{\pm 1}$  and  $Z_{\alpha_3}^{\pm 1}$  and the relations are

$$Z_{\alpha_1} Z_{\alpha_{i+1}} = \omega^{-2} Z_{\alpha_{i+1}} Z_{\alpha_i}, \quad i \in \mathbb{Z}_3.$$

If we denote by  $Z_i$  the generator of  $\mathcal{T}_\omega(T)$  associated to the edge  $\lambda_i$  (so for instance  $Z_{\alpha_1} = [Z_2 Z_3]$ ), the map

$$\Psi : \mathcal{Z}_\omega(T) \rightarrow \mathcal{T}_\omega(T), \quad Z_{\alpha_i} \mapsto Z_i^{-1}$$

is an isomorphism of algebras such that  $\Psi(Z_\mu) = H^{-1}$  where  $H = [Z_1 Z_2 Z_3]$ .

In particular, an irreducible representation  $\rho$  of  $\mathcal{Z}_\omega(\lambda)$  has the form  $\rho = \bar{\rho} \circ \Psi$  where  $\bar{\rho}$  is an irreducible representation of  $\mathcal{T}_\omega(T)$ .

Using the result of Section 2.2 and the fact that a twisted homomorphism of  $\mathcal{W}(\tau, \mathbb{Z})$  is fully determined by its value on the  $\alpha_i$ , we get the result.  $\square$

Let  $\tau_i$  be the restriction of the train track  $\tau_\lambda$  to the triangle  $T_i$  of the triangulation  $\lambda$ . A twisted homomorphism  $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \rightarrow \mathbb{C}^*$  induces a twisted homomorphism  $\zeta_i : \mathcal{W}(\tau_i, \mathbb{Z}) \rightarrow \mathbb{C}^*$  for each triangle  $T_i$  of the triangulation  $\lambda$ . In particular, the following proposition is a straightforward extension of [Bai et al. 2007, Proposition 6]:

**Proposition 4.6.** *A local representation  $\rho : \mathcal{Z}_\omega(\lambda) \rightarrow \text{End}(V)$  has dimension  $N^{4g-4+2s}$  and is classified (up to isomorphism) by a twisted homomorphism  $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \rightarrow \mathbb{C}^*$  and a choice of an  $N$ -th root  $C = \zeta(\mu)^{1/N}$  where  $\mu(e) = 1$  for all edge  $e$  of  $\tau_\lambda$ . Such a representation satisfies:*

- $\rho(Z_\alpha^N) = \zeta(\alpha) \text{Id}_V$ .
- $\rho(Z_\mu) = C \text{Id}_V$ .

Finally, the Main Theorem implies the following:

**Theorem 4.7.** *Let  $\rho : \mathcal{Z}_\omega(\lambda) \rightarrow \text{End}(V)$  be the (isomorphism class of) representation classified by the twisted homomorphism  $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \rightarrow \mathbb{C}^*$  and the choice of an  $N$ -th root  $C = \zeta(\mu)^{1/N}$  (where  $\mu$  is defined as above). Then  $\rho = \bigoplus_{i \in \mathcal{I}} \rho^{(i)}$  where each  $\rho^{(i)}$  is irreducible, classified by the same twisted homomorphism  $\zeta$  and*

$N$ -th root  $h_k^{(i)} = (\zeta(\eta_k))^{1/N}$  with  $h_1^{(i)} \dots h_s^{(i)} = C$  (here, the  $\eta_k$  are defined as in Section 4.1).

Moreover, for each choice of an  $N$ -th root  $h_k = (\zeta(\eta_k))^{1/N}$  for each  $k \in \{1, \dots, s\}$ , there are exactly  $N^g$  indices  $i \in \mathcal{I}$  such that  $h_k^{(i)} = h_k$  for all  $k$ .

*Proof.* Let  $\rho$  be a local representation of  $\mathcal{Z}_\omega(\lambda)$  classified by  $\zeta$  and  $C$ . In particular,  $\rho$  induces a local representation  $\bar{\rho} := \rho \circ i$  of  $\mathcal{T}_q(\lambda)$ , where  $i : \mathcal{T}_q(\lambda) \hookrightarrow \mathcal{Z}_\omega(\lambda)$ . The local representation  $\bar{\rho}$  is classified by the weight  $\zeta(\beta_i)$  for all edges  $\lambda_i$ , where  $\beta_i \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$  is defined by  $Z_{\beta_i} = Z_i^2 = i(X_i)$ .

Let  $P_j \in \mathcal{T}_q(\lambda)$  be the puncture invariant associated to the puncture  $v_j$ . The image of  $P_j$  in  $\mathcal{Z}_\omega(\lambda)$  is  $Z_{\eta_j}^2$ . We claim that the eigenspaces of  $\bar{\rho}(P_j)$  correspond to the eigenspaces of  $\rho(Z_{\eta_j})$ . In fact, if  $V_{h_j}(Z_{\eta_j})$  is the eigenspace of  $\rho(Z_{\eta_j})$  corresponding to the eigenvalue  $h_j = (\zeta(\eta_j))^{1/N}$ , then one has the inclusion

$$V_{h_j}(Z_{\eta_j}) \subset V_{p_j}(P_j),$$

where  $V_{p_j}(P_j)$  is the eigenspace of  $\bar{\rho}(P_j) = \rho(Z_{\eta_j}^2)$  corresponding to the eigenvalue  $p_j = h_j^2$ . Because there are only  $N$  different possible eigenvalues of  $\rho(Z_{\eta_j})$ , a dimension counting argument shows the equality.

Now, we apply the Main Theorem and get that, for each choice of  $(h_1, \dots, h_s)$  where  $h_j = (\zeta(Z_{\eta_j}))^{1/N}$ , the intersection  $V_{h_1}(Z_{\eta_1}) \cap \dots \cap V_{h_s}(Z_{\eta_s})$  has dimension  $N^{4g-3+s}$  and hence is made of  $N^g$  copies of the irreducible representation of  $\mathcal{Z}_\omega(\lambda)$  classified by  $\zeta$  and  $h_1, \dots, h_s$ .  $\square$

Bonahon and Wong [2012, Section 3] associate a character  $r_\zeta \in \chi(\Sigma, \mathrm{SL}(2, \mathbb{C}))$  to a twisted homomorphism  $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \rightarrow \mathbb{C}^*$  (here  $\chi(\Sigma, \mathrm{SL}(2, \mathbb{C}))$  is the algebraic quotient of  $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SL}_2(\mathbb{C}))$  by the action of  $\mathrm{SL}_2(\mathbb{C})$  by conjugation). In particular, the (irreducible or local) representations of the balanced Chekhov–Fock algebra associated to an ideal triangulation  $\lambda$  of  $\Sigma$  are classified, up to finitely many choice, by a Zariski open set in  $\chi(\Sigma, \mathrm{SL}(2, \mathbb{C}))$ .

Note that, if  $r_\zeta$  is the character associated to the twisted homomorphism  $\zeta$ , the holonomy of  $r_\zeta$  around a puncture  $v_j$  is parabolic exactly when  $\zeta(\eta_j) = 1$ .

**4.3. Representations of  $\mathcal{S}_A(\bar{\Sigma})$ .** We explain here how Theorem 4.7 gives rise to a new family of representations of the Kauffman bracket skein algebra of the closed surface  $\bar{\Sigma} = \Sigma \cup \{v_1, \dots, v_s\}$ .

**Skein algebra.** Given an oriented 3-manifold  $M$ , and a nonzero complex number  $A \in \mathbb{C}^*$ , consider the complex vector space  $V(M)$  freely generated by isotopy classes of framed links in  $M$ . The *skein module*  $\mathcal{S}_A(M)$  of  $M$  is the quotient of  $V(M)$  by the *Kauffman bracket skein relations* as defined in Figure 7.

Namely, we identify three different links when differ by the previous relation in an open ball and agree everywhere else.

$$\begin{aligned}
 & \text{Crossing} = A \cdot \text{Parallel} + A^{-1} \cdot \text{Parallel} \\
 & \text{Circle} = -A^2 - A^{-2}
 \end{aligned}$$

**Figure 7.** Kauffman bracket skein relations.

Given a framed link  $K \subset M$ , we denote by  $[K]$  its image in the skein module  $\mathcal{S}_A(M)$ .

When  $M = \Sigma \times [0, 1]$  for a surface  $\Sigma$ , the skein module  $\mathcal{S}_A(M) = \mathcal{S}_A(\Sigma)$  inherits an algebra structure given by superposition of links. Namely, given two framed links  $K_1$  and  $K_2$  in  $\Sigma \times [0, 1]$ , the product  $[K_1] \cdot [K_2]$  is defined to be the image of  $K_1 \cup K_2$  in  $\mathcal{S}_A(\Sigma)$ , where  $K_1 \cup K_2$  is given by the superposition of  $K_1$  on top of  $K_2$  where we rescaled so that  $K_1 \subset \Sigma \times [0, \frac{1}{2}]$  and  $K_2 \subset \Sigma \times [\frac{1}{2}, 1]$ . We call  $\mathcal{S}_A(\Sigma)$  with the product  $\cdot$  the *Kauffman bracket skein algebra of  $\Sigma$* .

Finite-dimensional representations of the skein algebra  $\mathcal{S}_A(\Sigma)$  are of main importance as they appear naturally in topological quantum field theory (TQFT). For example, the Witten–Reshetekin–Turaev TQFT [Blanchet et al. 1995; Turaev 1994].

**Classical shadow and quantum trace.** Let  $\mu : \mathcal{S}_A(\Sigma) \rightarrow \text{End}(V)$  be an irreducible representation of the Kauffman bracket skein algebra of  $\Sigma$ .

Bonahon and Wong [2016] (see also [Lê 2015a] for a simpler proof) proved that if  $A$  is a primitive  $N$ -th root of  $-1$ , the  $N$ -th Chebyshev polynomial  $T_N$  of the first kind of any skein  $[K] \in \mathcal{S}_A(\Sigma)$  is a central element in  $\mathcal{S}_A(\Sigma)$ . In particular, the precomposition of  $\rho$  by  $T_N$  maps each skein  $[K]$  to a multiple of the identity in  $\text{End}(V)$ . This multiple of the identity can be interpreted as an element  $r_\mu \in \chi(\Sigma, \text{SL}_2(\mathbb{C}))$  in the  $\text{SL}(2, \mathbb{C})$  character variety of  $\Sigma$ . This character is called the *classical shadow* of the representation  $\mu$ .

When  $A = \omega^{-2}$  (so  $A$  is a primitive  $N$ -th root of  $-1$ ) and  $\lambda$  is an ideal triangulation of  $\Sigma$ , Bonahon and Wong [2011] (see also [Lê 2015b] for a more conceptual proof) constructed a *quantum trace map*

$$\text{tr}_\omega^\lambda : \mathcal{S}_A(\Sigma) \rightarrow \mathcal{Z}_\omega(\lambda),$$

which turns out to be an injective algebra homomorphism.

In particular, by precomposing irreducible representations of  $\mathcal{Z}_\omega(\lambda)$  by the quantum trace, Bonahon and Wong [2012] obtained a family of irreducible representations of the Kauffman bracket skein algebra of  $S$  indexed by a Zariski open subset

of the character variety  $\chi(\Sigma, \mathrm{SL}_2(\mathbb{C}))$ . Moreover, taking the classical shadow of such an irreducible representation recovers the character.

**Representations of  $\mathcal{S}_A(\bar{\Sigma})$ .** The inclusion  $\Sigma \hookrightarrow \bar{\Sigma}$  gives an algebra homomorphism

$$\iota : \mathcal{S}_A(\bar{\Sigma}) \rightarrow \mathcal{S}_A(\Sigma).$$

Let  $r \in \chi(\Sigma, \mathrm{SL}(2, \mathbb{C}))$  be a character obtained from a character  $r' \in \chi(\bar{\Sigma}, \mathrm{SL}(2, \mathbb{C}))$  (namely, the holonomy of  $r$  around each puncture is trivial). If  $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \rightarrow \mathbb{C}^*$  is the twisted homomorphism associated to  $r$ , then  $\zeta(\eta_j) = 1$  for each puncture  $v_j$ .

Denote by

$$\rho : \mathcal{Z}_\omega(\lambda) \rightarrow \mathrm{End}(V)$$

the local representation of  $\mathcal{Z}_\omega(\lambda)$  classified by  $\zeta$  and the  $N$ -th root  $C = ((-\omega^4)^s)^{1/N}$ . Let  $E \subset V$  be the intersection of the eigenspaces of  $\rho(Z_{\eta_k})$  for  $k \in \{1, \dots, s\}$  corresponding to the eigenvalue  $-\omega^4$ .

By Theorem 4.7, the vector space  $E$  is stable by  $\rho(\mathcal{Z}_\omega(\lambda))$ , so we get an induced representation  $\rho' : \mathcal{Z}_\omega(\lambda) \rightarrow \mathrm{End}(E)$ . Note that  $\rho'$  is made of  $N^g$  copies of the irreducible representation of  $\mathcal{Z}_\omega(\lambda)$  classified by  $\zeta$  and puncture invariant  $-\omega^4$ .

**Proposition 4.8.** *There is a proper linear subspace  $F \subset E$  such that the composition*

$$\mu : \mathcal{S}_A(\bar{\Sigma}) \xrightarrow{\iota} \mathcal{S}_A(\Sigma) \xrightarrow{\bar{\rho}} \mathrm{End}(F)$$

*induces a representation of  $\mathcal{S}_A(\bar{\Sigma})$ . The classical shadow of each irreducible factor of  $\mu$  is same. Finally, the dimension of  $F$  is at least  $N^{4g-3}$  when  $g > 1$  and at least  $N^2$  when  $g = 1$ .*

*Proof.* This is a direct consequence of the construction of Bonahon and Wong [2015]. In fact, using the decomposition of  $\bar{\rho}$  into irreducible parts and considering the *total off-diagonal kernel* of each irreducible factor (see [op. cit., Section 4.2]), one gets the result.  $\square$

The vector space  $F$  is canonically associated to the triangulation  $\lambda$ , which makes the family of representations described above easier to handle for computations.

### Acknowledgment

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