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We give a topological description of the quotient space $\Omega(G)/G$, in the case when $G \subset \mathrm{PSL}(3, \mathbb{C})$ is a discrete subgroup acting on $\mathbb{P}^2_{\mathbb{C}}$ and the maximum number of complex projective lines in general position contained in the Kulkarni limit set $\Lambda(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \Omega(G)$ is equal to 4. Moreover, we give a topological description of the quotient space $\Omega(G)/G$ in the case when G is a lattice of the Heisenberg group.

1. Introduction

Complex Kleinian groups were introduced by José Seade and Alberto Verjovsky [2001]. A complex Kleinian group G is a subgroup of $\mathrm{PSL}(n+1, \mathbb{C})$ acting properly and discontinuously on a nonempty G-invariant open subset of $\mathbb{P}^n_{\mathbb{C}}$. We remark that complex Kleinian groups are discrete subgroups of $\mathrm{PSL}(n+1, \mathbb{C})$ but the converse is not necessarily true; for example, the group $\mathrm{PSL}(3, \mathbb{Z})$ is a discrete subgroup of $\mathrm{PSL}(3, \mathbb{C})$ which is not a complex Kleinian group. See [Barrera et al. 2014].

There is no standard definition of limit set for a complex Kleinian group, we use the following three notions: Kulkarni limit set, Myrberg limit set, or the complement of a maximal region of discontinuity which are discussed in detail in [Barrera et al. 2016]. However by some additional hypotheses on the action of G on the projective plane, all these concepts of limit set are equivalent; see [Barrera et al. 2011a]. In the classical theory of Kleinian groups, there is a theorem which states that discrete infinite subgroups contain one, two, or infinity points in its limit set. On the other hand, Angel Cano and José Seade show that every infinite discrete subgroup of $\mathrm{PSL}(3, \mathbb{C})$ has a complex projective line contained in its limit set (see [Cano and Seade 2014]), in consequence, the limit set of infinite subgroups of $\mathrm{PSL}(3, \mathbb{C})$ is an uncountable subset of $\mathbb{P}^2_{\mathbb{C}}$.

Thus, it is natural to say that $G \subset \mathrm{PSL}(3, \mathbb{C})$ is an *elementary complex Kleinian group* whenever its limit set contains a finite number of complex projective lines; see [Cano et al. 2013]. There is another kind of group whose limit set contains

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infinitely many complex projective lines but only finitely many in general position. We call these groups *elementary complex Kleinian groups of type II*.

In [Barrera et al. 2011b], the authors give an algebraic characterization of those complex Kleinian groups such that the maximum number of complex projective lines in general position contained in its Kulkarni limit set is equal to 4. In this article we describe the topology of the quotient space of these groups. In fact, we prove the following theorem:

Theorem 1.1. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a torsion-free complex Kleinian group, such that the maximum number of complex projective lines in general position contained in its Kulkarni limit set is equal to* 4, *then*:

(i) *The group G is isomorphic to a lattice of the group* $\mathbb{S}\mathrm{ol}$ (*see Section 2F*).

(ii) *If Ω_0 is a G-invariant connected component of the Kulkarni discontinuity region of G, then Ω_0/G is diffeomorphic to* $(\mathbb{S}\mathrm{ol}/G) \times \mathbb{R}$.

Corollary 1.2. *There is a countable number of nonisomorphic complex Kleinian groups such that the maximum number of complex projective lines in general position contained in its Kulkarni limit set is equal to* 4.

Corollary 1.3. *Under the hypotheses of Theorem 1.1*, Ω_0/G *is a fiber bundle with base $\mathbb{S}^1$ and fiber $\mathbb{T}^2 \times \mathbb{R}$.*

Theorem 1.4. *If G is a lattice on the three-dimensional real Heisenberg group $\mathcal{H}$, then there exists a G-invariant open set $\Omega \subset \mathbb{P}^2_{\mathbb{C}}$ such that Ω/G is diffeomorphic to* $(\mathcal{H}/G) \times \mathbb{R}$.

This article is organized in the following way: In Section 2 we include some basic preliminaries about complex Kleinian groups, and a brief survey on complex Kleinian groups, such that the maximum number of complex lines contained in their Kulkarni limit set is equal to 4 [Barrera et al. 2011b]. In Section 3 we give an explicit smooth foliation of the bidisc $\mathbb{H} \times \mathbb{H}$ where the leaves are diffeomorphic copies of $\mathbb{S}\mathrm{ol}$. In Section 4, we study the geometry of the leaves and we show that the bidisc $\mathbb{H} \times \mathbb{H}$ is diffeomorphic to $\mathbb{S}\mathrm{ol} \times \mathbb{R}$. Moreover this diffeomorphism is G-equivariant, where G denotes an hyperbolic toral group. In Section 5 we do some explicit computations to determine the Riemannian metrics of the leaves. For each leaf we obtain an isometric embedding of the group $\mathbb{S}\mathrm{ol}$ to $\mathbb{H} \times \mathbb{H}$. Finally, we give a proof of Theorem 1.1.

Corollaries 1.2 and 1.3 are a consequence of Theorem 1.1 and [de la Harpe 2000, Proposition 30]. In Section 6 we give a proof of Theorem 1.4, the procedure is similar to the proof of Theorem 1.1, except that we have not a G-equivariant diffeomorphism between $\mathbb{C} \times \mathbb{H}$ and $\mathcal{H} \times \mathbb{R}$. However the proof can be done because the natural action of G on $\mathbb{C} \times \mathbb{H}$ translated to $\mathcal{H} \times \mathbb{R}$ is the classical action on the first factor of G on $\mathcal{H}$, and it is the trivial action on the second factor.

2. Preliminaries

The purpose of this section is to provide some definitions and results about complex Kleinian groups that will be helpful to the reader. For more details see [Cano et al. 2013; Barrera et al. 2011b; 2011a].

2A. *Projective geometry.* We recall that the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ is defined as the orbit space of the usual scalar multiplication action of the Lie group $\mathbb{C}^*$ in $\mathbb{C}^3 \setminus \{\mathbf{0}\}$ and it is denoted by

$$\mathbb{P}^2_{\mathbb{C}} := (\mathbb{C}^3 \setminus \{\mathbf{0}\})/\mathbb{C}^*.$$

This is a compact connected complex 2-dimensional manifold. Let $[\,]:\mathbb{C}^3\setminus\{\mathbf{0}\} \to \mathbb{P}^2_{\mathbb{C}}$ be the quotient map. If $\beta = \{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{C}^3$, we write $[e_j] = e_j$, for $j = 1, 2, 3$, and if $\boldsymbol{z} = (z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$ then we write $[\boldsymbol{z}] = [z_1 : z_2 : z_3]$. Also, $\ell \subset \mathbb{P}^2_{\mathbb{C}}$ is said to be a complex line if $[\ell]^{-1} \cup \{\mathbf{0}\}$ is a complex linear subspace of dimension 2. Given two distinct points $[\boldsymbol{z}], [\boldsymbol{w}] \in \mathbb{P}^2_{\mathbb{C}}$, there is a unique complex projective line passing through $[\boldsymbol{z}]$ and $[\boldsymbol{w}]$. This kind of complex projective line is called a *line*, for short, and it is denoted by $\overleftrightarrow{[\boldsymbol{z}], [\boldsymbol{w}]}$. Consider the action of $\mathbb{C}^*$ on $\mathrm{GL}(3, \mathbb{C})$ given by the usual scalar multiplication, then

$$\mathrm{PGL}(3, \mathbb{C}) = \mathrm{GL}(3, \mathbb{C})/\mathbb{C}^*$$

is a Lie group whose elements are called projective transformations. Letting $[[\,]] : \mathrm{GL}(3, \mathbb{C}) \to \mathrm{PGL}(3, \mathbb{C})$ be the quotient map, $g \in \mathrm{PGL}(3, \mathbb{C})$ and $\boldsymbol{g} \in \mathrm{GL}(3, \mathbb{C})$, we say that $\boldsymbol{g}$ is a lift of g if $[[\boldsymbol{g}]] = g$. One can show that $\mathrm{PGL}(3, \mathbb{C})$ is a Lie group which acts transitively, effectively and by biholomorphisms on $\mathbb{P}^2_{\mathbb{C}}$ by $[[\boldsymbol{g}]]([\boldsymbol{w}]) = [\boldsymbol{g}(\boldsymbol{w})]$, where $\boldsymbol{w} \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$ and $\boldsymbol{g} \in \mathrm{GL}(3, \mathbb{C})$.

We could have considered the action of the cube roots of unity $\{1, \omega, \omega^2\} \subset \mathbb{C}^*$ on $\mathrm{SL}(3, \mathbb{C})$ given by the usual scalar multiplication, then

$$\mathrm{PSL}(3, \mathbb{C}) = \mathrm{SL}(3, \mathbb{C})/\{1, \omega, \omega^2\} \cong \mathrm{PGL}(3, \mathbb{C}).$$

We denote by $\mathrm{M}_{3\times3}(\mathbb{C})$ the space of all 3×3 matrices with entries in $\mathbb{C}$ equipped with the standard topology. The quotient space

$$\mathrm{End}(3, \mathbb{C}) := (\mathrm{M}_{3\times3}(\mathbb{C}) \setminus \{\mathbf{0}\})/\mathbb{C}^*$$

is called the space of *pseudoprojective maps of* $\mathbb{P}^2_{\mathbb{C}}$ and it is naturally identified with the projective space $\mathbb{P}^8_{\mathbb{C}}$. Since $\mathrm{GL}(3, \mathbb{C})$ is an open, dense, $\mathbb{C}^*$-invariant set of $\mathrm{M}_{3\times3}(\mathbb{C}) \setminus \{\mathbf{0}\}$, we get that the space of pseudoprojective maps of $\mathbb{P}^2_{\mathbb{C}}$ is a compactification of $\mathrm{PGL}(3, \mathbb{C})$ (or $\mathrm{PSL}(3, \mathbb{C})$). As in the case of projective maps, if $\boldsymbol{s}$ is an element in $\mathrm{M}_{3\times3}(\mathbb{C}) \setminus \{\mathbf{0}\}$, then $[\boldsymbol{s}]$ denotes the equivalence class of the matrix $\boldsymbol{s}$ in the space of pseudoprojective maps of $\mathbb{P}^2_{\mathbb{C}}$. Also, we say that $\boldsymbol{s} \in \mathrm{M}_{3\times3}(\mathbb{C}) \setminus \{\mathbf{0}\}$ is a lift of the pseudoprojective map S whenever $[\boldsymbol{s}] = S$.

Let S be an element in $(\mathrm{M}_{3\times 3}(\mathbb{C})\setminus\{\mathbf{0}\})/\mathbb{C}^*$ and $\boldsymbol{s}$ a lift to $\mathrm{M}_{3\times 3}(\mathbb{C})\setminus\{\mathbf{0}\}$ of S. The matrix $\boldsymbol{s}$ induces a nonzero linear transformation $s:\mathbb{C}^3\to\mathbb{C}^3$, which is not necessarily invertible. Let $\mathrm{Ker}(s)\subsetneq\mathbb{C}^3$ be its kernel and let $\mathrm{Ker}(S)$ denote its projectivization to $\mathbb{P}^2_{\mathbb{C}}$, taking into account that $\mathrm{Ker}(S):=\varnothing$ whenever $\mathrm{Ker}(s)=\{(0,0,0)\}$.

2B. *Discontinuous actions on* $\mathbb{P}^2_{\mathbb{C}}$.

Definition 2.1. Let $G\subset\mathrm{PSL}(3,\mathbb{C})$ be a group. We say that G is a *complex Kleinian group* if it acts properly and discontinuously on an open nonempty G-invariant set $U\subset\mathbb{P}^2_{\mathbb{C}}$, meaning that for each pair of compact subsets $C, D\subset U$, the set

$$\{g\in G : g(C)\cap D\neq\varnothing\}$$

is *finite*.

One of the main difficulties in deciding whether a group G is Kleinian complex is to find an open set verifying the definition above. In order to give an answer to this problem we study two mathematical concepts: the equicontinuity set of G and the Kulkarni discontinuity region of G. Now, we discuss each of these concepts.

2C. *The equicontinuity set.* The concept of equicontinuity has long been studied in mathematics. For convenience to the reader, we include the definition and notation that we use in this work.

Definition 2.2. The *equicontinuity set* for a family $\mathcal{F}$ of endomorphisms of $\mathbb{P}^2_{\mathbb{C}}$, denoted $\mathrm{Eq}(\mathcal{F})$, is defined as the set of points $z\in\mathbb{P}^2_{\mathbb{C}}$ for which there is an open neighborhood U of z such that $\{f|_U : f\in\mathcal{F}\}$ is a normal family.

This modern approach and ideas on this concept were studied by Angel Cano in his Ph.D thesis. However, thanks to a reference by Ravi Kulkarni to works of Myrberg, we found that some of these results had already been discovered, in an arcane mathematical language. However, it is fair to acknowledge Angel Cano for rediscovering these results and applying them successfully to the theory of complex Kleinian groups.

Definition 2.3. Let $G\subset\mathrm{PSL}(3,\mathbb{C})$ be a discrete group. If

$$G'=\{S \text{ is a pseudoprojective map of } \mathbb{P}^2_{\mathbb{C}} : S \text{ is a cluster point of } G\},$$

then the *Myrberg limit set* (see [Myrberg 1925]) is defined as the set

$$\Lambda_{\mathrm{Myr}}(G)=\bigcup_{S\in G'}\mathrm{Ker}(S).$$

Myrberg [1925] shows that G acts properly and discontinuously on $\mathbb{P}^2_{\mathbb{C}}\setminus\Lambda_{\mathrm{Myr}}(G)$.

Theorem 2.4 [Barrera et al. 2011a]. *If* $G\subset\mathrm{PSL}(3,\mathbb{C})$ *is a discrete group*, *then*:

(i) *The group* G *acts properly and discontinuously on* $\mathrm{Eq}(G)$.

(ii) *The equicontinuity set of G satisfies*:

$$\mathrm{Eq}(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \Lambda_{\mathrm{Myr}}(G)$$

(iii) *If U is an open G-invariant subset such that $\mathbb{P}^2_{\mathbb{C}} \setminus U$ contains at least three complex lines in general position*, *then* $U \subset \mathrm{Eq}(G)$.

2D. *Kulkarni discontinuity region.* Kulkarni [1978] defined a limit set for groups of homeomorphisms acting on a locally compact Hausdorff space. For the reader's convenience, we explain this construction in the context of projective spaces.

Definition 2.5. Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a subgroup.

- The set $L_0(G)$ is defined as the closure of the set of points in $\mathbb{P}^2_{\mathbb{C}}$ with infinite isotropy group.
- The set $L_1(G)$ is defined as the closure of the set of cluster points of the orbit Gz, where z runs over $\mathbb{P}^2_{\mathbb{C}} \setminus L_0(G)$.
- The set $L_2(G)$ is defined as the closure of the set of cluster points of the family of compact sets $\{g(K) : g \in G\}$, where K runs over all the compact subsets of

$$\mathbb{P}^2_{\mathbb{C}} \setminus (L_0(G) \cup L_1(G)).$$

The *Kulkarni limit set* of G is defined as

$$\Lambda_{\mathrm{Kul}}(G) = L_0(G) \cup L_1(G) \cup L_2(G).$$

The *Kulkarni discontinuity region* of G is defined as:

$$\Omega(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \Lambda_{\mathrm{Kul}}(G).$$

Kulkarni [1978] proves that G acts properly and discontinuously on the set $\Omega(G)$. However, $\Omega(G)$ is not necessarily the maximal open subset of $\mathbb{P}^2_{\mathbb{C}}$ where G acts properly and discontinuously.

We notice that the Kulkarni limit set is a generalization of the classical limit set for a discrete subgroup of hyperbolic isometries acting on the sphere at infinity of hyperbolic space. In general, it is very hard to give an explicit computation of the Kulkarni limit set. In [Navarrete 2006; 2008], we can find these computations for the cyclic subgroups of $\mathrm{PSL}(3, \mathbb{C})$ and for discrete subgroups of $\mathrm{PU}(2, 1)$ acting on the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$.

We could define the limit set as the complement of a maximal open set where the group acts properly and discontinuously, but in general there is no canonical way to build this G-invariant open set. On the other hand, when we ensure the existence of this maximal open set, this notion of limit set has good properties. See [Barrera et al. 2014].

2E. *Four lines complex Kleinian groups.* This section is devoted to complex Kleinian groups of $\mathrm{PSL}(3,\mathbb{C})$ such that the maximum number of complex projective lines in general position contained in its Kulkarni limit set is equal to 4. For simplicity we call groups of this kind *four lines complex Kleinian groups*. In [Barrera et al. 2011b], the authors give an algebraic characterization of four lines complex Kleinian groups. For the reader's convenience, we reproduce briefly the main ideas and the notation used there.

Letting $A \in \mathrm{SL}(2,\mathbb{Z})$, with $|tr(A)| > 2$, we define the following discrete subgroup of $\mathrm{PSL}(3,\mathbb{C})$, called *hyperbolic toral group*.

$$G_A = \left\{ \begin{pmatrix} A^k & \boldsymbol{b} \\ \boldsymbol{0} & 1 \end{pmatrix} : \boldsymbol{b} \in M(2\times 1,\mathbb{Z}),\ k \in \mathbb{Z} \right\},$$

The group G_A is a four lines complex Kleinian group and moreover if G is a four lines complex Kleinian group, then there exists a hyperbolic toral group G_A such that $[G : G_A] \le 8$.

It is possible to conjugate the group G_A to a group, still denoted by G_A, where each element is of the form

$$\begin{pmatrix} \lambda^k & 0 & ny_0 + mx_0 \\ 0 & \lambda^{-k} & nx_0 + mz_0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $k,\ m$ and n run over $\mathbb{Z}$ and λ is one of the eigenvalues of A. At this point it is not hard to see that the Kulkarni discontinuity region consists of four disjoint copies of $\mathbb{H}^{\pm} \times \mathbb{H}^{\pm}$, where $\mathbb{H}^{+}$ is the upper half plane and $\mathbb{H}^{-}$ is the lower half plane.

2F. *The* $\mathbb{S}$ol *geometries.* $\mathbb{S}$ol is one of the eight geometries defined by William Thurston in his famous program of geometrization of compact three manifolds. The group $\mathbb{S}$ol is defined as the space $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ equipped with the group operation

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, t_1 \right) \cdot \left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, t_2 \right) = \left(\begin{pmatrix} x_1 + e^{t_1} x_2 \\ y_1 + e^{-t_1} y_2 \end{pmatrix}, t_1 + t_2 \right).$$

In fact, it is a Lie group and it is equipped with the left-invariant Riemannian metric: $ds^2 = e^{2t}dx^2 + e^{-2t}dy^2 + dt^2$. An interesting fact about the $\mathbb{S}$ol geometries is given by the following theorem of [de la Harpe 2000], which we state for convenience:

Proposition 2.6. *Let $A,\ B$ in $\mathrm{GL}(2,\mathbb{Z})$ be two matrices with traces of absolute value strictly larger than 2. The semidirect products $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$, and $\mathbb{Z}^2 \rtimes_B \mathbb{Z}$ considered as the matrix groups*

$$\left\{ \begin{pmatrix} A^k & \boldsymbol{b} \\ \boldsymbol{0} & 1 \end{pmatrix} : \boldsymbol{b} \in M(2\times 1,\mathbb{Z}),\ k \in \mathbb{Z} \right\},$$

$$\left\{ \begin{pmatrix} B^k & \boldsymbol{b} \\ \boldsymbol{0} & 1 \end{pmatrix} : \boldsymbol{b} \in M(2 \times 1, \mathbb{Z}),\ k \in \mathbb{Z} \right\}$$

are isomorphic if and only if A is conjugate in $\mathrm{GL}(2, \mathbb{Z})$ *to B or B^{-1}, and they are quasi-isometric in all cases.*

The quotient spaces $\mathbb{S}\mathrm{ol}/(\mathbb{Z}^2 \rtimes_A \mathbb{Z})$ are examples of compact three manifolds where the topological type is determined by the fundamental group. For more details about this subject see [Scott 1983; Thurston 1997; de la Harpe 2000].

3. Foliation of $\mathbb{H} \times \mathbb{H}$ by $\mathbb{S}\mathrm{ol}$

In [Barrera et al. 2011b], the authors introduce the concept of hyperbolic toral groups. These groups are matrix groups where the elements are given by

$$\begin{pmatrix} \lambda^k & 0 & ny_0 + mx_0 \\ 0 & \lambda^{-k} & nx_0 + mz_0 \\ 0 & 0 & 1 \end{pmatrix},$$

where λ is a fixed real number, $|\lambda| \neq 1$ and $k,\ n,\ m$ run over $\mathbb{Z}$. It is not hard to check this group is isomorphic to $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$. Moreover, a continuous version of this group is given in the following way. Let $A \in \mathrm{SL}(2, \mathbb{Z})$ be a hyperbolic automorphism with Jordan form

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

and consider the set of matrices of the form

$$\begin{pmatrix} \lambda^t & 0 & x \\ 0 & \lambda^{-t} & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $t,\ x,\ y$ run over $\mathbb{R}$. It is not hard to check that this group is isomorphic to $\mathbb{S}\mathrm{ol} = \mathbb{R}^2 \rtimes \mathbb{R}$. For convenience for our computations we use this representation of the Lie group $\mathbb{S}\mathrm{ol}$. In the sequel, we will use the product metric in $\mathbb{H} \times \mathbb{H}$, where we endow each copy of $\mathbb{H}$ with a metric homothetic to the hyperbolic metric by a factor of $\frac{1}{2}$:

$$\frac{dx_1^2 + dy_1^2}{2y_1^2} + \frac{dx_2^2 + dy_2^2}{2y_2^2}.$$

Proposition 3.1. *Let $z_1, z_2 \in \mathbb{H}$. We define a natural action of* $\mathbb{S}\mathrm{ol}$ *in $\mathbb{H} \times \mathbb{H}$ by*

$$\begin{pmatrix} \lambda^t & 0 & x \\ 0 & \lambda^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda^t z_1 + x \\ \lambda^{-t} z_2 + y \\ 1 \end{pmatrix}.$$

The natural action of the group $\mathbb{S}\mathrm{ol}$ *on* $\mathbb{H}\times\mathbb{H}$ *satisfies the following*:

(i) *The action is free.*

(ii) *For each* $z=(z_1,z_2)\in\mathbb{H}\times\mathbb{H}$ *the function* $f_z:\mathbb{S}\mathrm{ol}\to\mathbb{H}\times\mathbb{H}$ *defined by* $f_z(g)=gz$ *is a smooth embedding.*

(iii) *If* $\{e_1,e_2,e_3,e_4\}$ *is the canonical basis of* $\mathbb{R}^4$, *then*

$$X=y_1e_2+y_2e_4$$

is one of the two normal unitary vector fields to the embedding $f_z(\mathbb{S}\mathrm{ol})$ *in* $\mathbb{H}\times\mathbb{H}$, *which therefore is a smooth vector field globally defined.*

Proof. (i) Let $z=(z_1,z_2)\in\mathbb{H}\times\mathbb{H}$, and assume that $\gamma\in\mathbb{S}\mathrm{ol}$ is such that $\gamma\cdot z=z$. Let $z_k=x_k+iy_k$, where $k=1,2$. Taking imaginary parts in the action, we get

$$\lambda^t y_1=y_1,\qquad\qquad \lambda^{-t}y_2=y_2,$$

so $\lambda^t=\lambda^{-t}=1$, because imaginary parts can not be null. Taking real parts, we obtain $x_1+x=x_1$ and $x_2+y=x_2$, then $x=y=0$.

(ii) From the definition of the action, it is clear that f_z is smooth in t,x,y, which parametrizes $\mathbb{S}\mathrm{ol}$. By straightforward computations, we have that df_z has the Jacobian matrix given by

$$[df_z]=\begin{pmatrix}\ln(\lambda)\,\lambda^t x_1 & 1 & 0\\ \ln(\lambda)\,\lambda^t y_1 & 0 & 0\\ -\ln(\lambda)\,\lambda^{-t}x_2 & 0 & 1\\ -\ln(\lambda)\,\lambda^{-t}y_2 & 0 & 0\end{pmatrix}.$$

Since $y_1>0$, the Jacobian matrix has rank 3. Therefore, f_z in an immersion.

If $z_k=x_y+iy_k$, and $z'=(x_1',y_1',x_2',y_2')\in\mathbb{H}\times\mathbb{H}$, define t such that $\lambda^t=y_1'/y_1$, and (x',y') such that

$$\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}x_1'-\frac{y_1'x_1}{y_1}\\ x_2'-\frac{y_2'y_2}{y_1'}\end{pmatrix}.$$

These values for (t,x',y') define a mapping F, from $\mathbb{H}\times\mathbb{H}$ to $\mathbb{R}^3\cong\mathbb{S}\mathrm{ol}$, such that $F\circ f_z=\mathrm{Id}$. Note that F is a left continuous inverse for f_z, and hence, f_z is an homeomorphism.

(iii) The given formula for the product metric implies that X is unitary. By the form of the Jacobian matrix, the tangent space to the leaf passing through $z=(z_1,z_2)\in\mathbb{H}\times\mathbb{H}$ is spanned by the vectors, $e_1,\ e_3,\ \lambda^t x_1e_1+\lambda^t y_1e_2-\lambda^{-t}x_2e_3-\lambda^{-t}y_2e_4$.

A straightforward computation shows that $X=\lambda^t y_1e_2+\lambda^{-t}y_2e_4$ is orthogonal to the spanning tangent vectors. Finally, the result is obtained by taking $t=0$. $\square$

By this theorem, if we vary z, we obtain a foliation $f_z(\mathbb{S}\mathrm{ol})$ of $\mathbb{H}\times\mathbb{H}$ by copies of $\mathbb{S}\mathrm{ol}$. We proceed to show that this foliation is globally rectifiable, in the sense that it induces a diffeomorphism to $\mathbb{R}^3\times\mathbb{R}$, such that the hyperplanes $\mathbb{R}^3\times\{t\}$ correspond to the leaves, and are diffeomorphic to $\mathbb{S}\mathrm{ol}$. For details on the theory of foliations, the reader can consult [Candel and Conlon 2000].

4. Geometry of the leaves

In the previous section, we described how $\mathbb{S}\mathrm{ol}$ induces a foliation in the space $\mathbb{H}\times\mathbb{H}$ and gave an explicit formula for a smooth vector field X, normal to any leaf of the foliation in a product metric, which is homothetic to the canonical metric in each hyperbolic factor. In this section, we study the dynamics of the integral curves for this normal field. Let

$$\psi(t)=(z_1(t),z_2(t))$$

be an integral curve of the field, where $z_k=x_k+iy_k$ is as before. From the definition of X, it follows that the integral curves satisfy the set of equations

$$\dot{x}_k=0,\qquad\qquad \dot{y}_k=y_k.$$

These equations can be readily solved to get constant solutions in the real part of each copy of the hyperbolic, and exponentials in the imaginary parts. The flow of the normal field defines a one-parameter family of diffeomorphisms in $\mathbb{H}\times\mathbb{H}$, denoted by $\psi_t(z_1,z_2)$, where

$$\psi_t(z_1,z_2)=(x_1,\ e^t y_1,\ x_2,\ e^t y_2),\quad t\in\mathbb{R},$$

and it satisfies the following:

Proposition 4.1. (i) *The flow ψ_t rules $\mathbb{H}\times\mathbb{H}$ by geodesics.*

(ii) *The action of f_z is* ***equivariant*** *with the action induced by the flow, that is,*

$$\psi_s\circ f_z=f_{\psi_s(z)}.$$

Proof. (i) Both curves

$$(x_1,e^t y_1)\quad\text{and}\quad(x_2,e^t y_2)$$

correspond to a parametrization of a vertical geodesic in $\mathbb{H}$ with respect to the hyperbolic metric. Since the metric we consider is homothetic to the standard hyperbolic metric, with a constant factor, these parametrizations correspond to geodesics with respect to this metric as well. Since the metric in $\mathbb{H}\times\mathbb{H}$ is a product, the result follows (see [Gallot et al. 2004]).

(ii) $$\psi_s\circ f_z(t,x,y)=(\lambda^t x_1+x,\ \lambda^t e^s y_1,\ \lambda^{-t}x_2+y,\ \lambda^{-t}e^s y_2),$$

which is the same expression obtained calculating $f_{\psi_s(z)}$. □

Observe that if we reparametrize the presentation of $\mathbb{S}\mathrm{ol}$ we have used so far by the change of coordinates

$$(t', x', y') \mapsto \left(\frac{t'}{\ln(\lambda)}, x', y'\right),$$

we recover the original description of the group as given in [Scott 1983], and that under this reparametrization, we can analyze the geometry of the foliation in simpler terms, i.e., we can and will assume that

$$\psi_s \circ f_z(t, x, y) = \left(e^t x_1 + x,\ e^{t+s} y_1, e^{-t} x_2 + y,\ e^{-t+s} y_2\right)$$

in order to analyze the metric properties of the foliation.

Proposition 4.2. *Let $z = (iy_1, iy_2)$. Consider the leaf $f_z : \mathbb{S}\mathrm{ol} \to \mathbb{H} \times \mathbb{H}$, then the pullback metric is*

$$dt^2 + \frac{e^{-2t}}{2y_1^2} dx^2 + \frac{e^{2t}}{2y_2^2} dy^2.$$

In particular, if $y_1 = y_2 = 1/\sqrt{2}$, f_z is an isometric embedding of $\mathbb{S}\mathrm{ol}$ into $\mathbb{H} \times \mathbb{H}$.

Proof. We have

$$f_z(t, x, y) = (x,\ e^t y_1,\ y,\ e^{-t} y_2).$$

Therefore, the Jacobian matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ e^t y_1 & 0 & 0 \\ 0 & 0 & 1 \\ -e^{-t} y_2 & 0 & 0 \end{pmatrix}.$$

Applying the product metric to the basis vectors $e^t y_1 e_2 - e^{-t} y 2 e_4,\ e_1,\ e_3$ we get the result. □

In the sequel, unless otherwise established, z_0 will denote the special point $1/\sqrt{2}\,(i, i)$.

Corollary 4.3. *The leaves $f_{\psi_s(z_0)} : \mathbb{S}\mathrm{ol} \to \mathbb{H} \times \mathbb{H}$ can be identified with $\mathbb{S}\mathrm{ol}$, up to a homothety in the direction spanned by the x, y coordinates.*

Proof. We have

$$\psi_s \circ f_{z_0}(t, x, y) = \left(x,\ \frac{1}{\sqrt{2}} e^{t+s},\ y,\ \frac{1}{\sqrt{2}} e^{-t+s}\right).$$

If we pullback the induced metric to $\mathbb{S}\mathrm{ol}$, we get

$$dt^2 + e^{-2(t+s)} dx^2 + e^{2(t-s)} dy^2.$$

Define $F_s : \mathbb{S}\mathrm{ol} \to \mathbb{S}\mathrm{ol}$ by $F_s(t, x, y) = (t, e^s x, e^s y)$. Another pullback with F_s turns the induced metric into the standard metric in $\mathbb{S}\mathrm{ol}$. □

Proposition 4.4. *The foliation is globally rectifiable*: *there is a diffeomorphism* $\Psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{H} \times \mathbb{H}$, *such that each hyperplane* $\mathbb{R}^3 \times \{c\}$ *is diffeomorphic to a leaf.*

Proof. $\mathbb{S}$ol is diffeomorphic to $\mathbb{R}^3$ in a natural way. Any $\gamma \in \mathbb{S}$ol is uniquely determined by a triplet (t, x, y). Define $\Psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{H} \times \mathbb{H}$ by $\Psi(t, x, y, s) = \psi_s \circ f_{z_0}(\gamma)$. The function Ψ is injective because the action is free. Given $z' \in \mathbb{H} \times \mathbb{H}$, there is a leaf going through it, and since $\psi_s(z_0)$ traverses all the leaves, there exists a number s, such that $\psi_{-s}(z')$ is in the leaf passing through z_0. Let $\gamma \in \mathbb{S}$ol be such that $\psi_{-s}(z') = f_{z_0}(\gamma)$. Therefore,

$$z' = \psi_s \circ f_{z_0}(\gamma),$$

which implies that Ψ is also surjective. Finally, the Jacobian of Ψ is

$$[d\Psi] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ e^{t+s}/\sqrt{2} & 0 & 0 & e^{t+s}/\sqrt{2} \\ 0 & 0 & 1 & 0 \\ -e^{-t+s}/\sqrt{2} & 0 & 0 & e^{-t+s}/\sqrt{2} \end{pmatrix},$$

which is nondegenerated. By the inverse function theorem, Ψ is a diffeomorphism. The last claim follows from the fact that ψ_s maps leaves onto leaves. □

Corollary 4.5. *The previous diffeomorphism can be modified*, *such that it not only maps the foliation to a Cartesian product globally*, *but also maps each leaf in the foliation isometrically to* $\mathbb{S}$ol.

Proof. The pullback of the metric in $\mathbb{H} \times \mathbb{H}$ with the previous diffeomorphism is

$$dt^2 + e^{-2(t+s)}dx^2 + e^{2(t-s)}dy^2 + ds^2,$$

which is analogous to the expression in Corollary 4.3. Let

$$\tilde{\Psi}(t, x, y, s) = \Psi(t, e^s x, e^s y, s).$$

$\tilde{\Psi}$ is a leaf-preserving diffeomorphism such that, for fixed s, it isometrically maps $\mathbb{S}$ol into the leaf $\mathbb{R}^3 \times \{s\}$. □

5. Extrinsic geometry

Proposition 5.1. *Integral curves of the normal field X are geodesics.*

Proof. We previously found that the integral curves of the field are given by $\gamma(t) = (x_1, e^t y_1, x_2, e^t y_2)$. Let $\phi(t)$ be a smooth curve in $\mathbb{H}$ with the homothetic metric. Then,

$$\|\dot{\phi}(t)\|^2 = \frac{\dot{x}^2 + \dot{y}^2}{2y^2},$$

which is half the standard hyperbolic square length [Gallot et al. 2004]. Therefore, a curve minimizes hyperbolic arc length if and only if it minimizes the homothetic metric arc length, i.e., geodesics in both cases are the same. It is a well known fact that the vertical curves $(x_k, e^t y_k)$ are geodesics in hyperbolic space. Finally, since γ can be projected in two geodesics and the metric is a product, γ is a geodesic in $\mathbb{H}\times\mathbb{H}$ (see 3.15 in [Gallot et al. 2004]). $\square$

Proposition 5.2. *There are isometries in $\mathbb{H}\times\mathbb{H}$ acting transitively and sending leaves onto leaves.*

Proof. We work in the $\mathbb{R}^3\times\mathbb{R}$ picture with the $\tilde{\Psi}$ isometry. By straightforward calculations we have that the mappings

$$(t, x, y, s) \mapsto (t+t',\ e^{t'+s'}x+x',\ e^{-t'+s'}y+y',\ s+s')$$

are isometries. The first claim comes from the fact that given a pair of points (t_k, x_k, y_k, s_k), there exists exactly one such isometry sending one onto another. That this isometry sends leaves onto leaves is obvious, since under this diffeomorphism, they correspond to hypersurfaces where s is constant. $\square$

We aim to calculate the distance between any pair of leaves. Recall that in any metric space, the distance from a point p to a set $S\neq\varnothing$ is given by the expression

$$d(p, S) = \inf\{d(p, x) : x \in S\}.$$

See [Munkres 2000] for details.

Proposition 5.3. *The separation between two leaves in $\mathbb{H}\times\mathbb{H}$ is constant. Moreover, if leaves are parametrized with the normal field affine parameter, then leaves' separation is given by the difference $|s-s'|$ between the parameters corresponding to any leaf.*

Proof. A point in a leaf can be parametrized as

$$\left(x,\ \frac{e^{s+t}}{\sqrt{2}},\ y,\ \frac{e^{s-t}}{\sqrt{2}}\right),$$

where x, y, t are arbitrary, and s is the parameter corresponding to the leaf. Given a second point in another leaf, say,

$$\left(x',\ e^{s'+t'}/\sqrt{2},\ y',\ e^{s'-t'}/\sqrt{2}\right),$$

and since the metric is a product, we can find a geodesic minimizing the arc length in $\mathbb{H}\times\mathbb{H}$, such that, in each factor $\mathbb{H}$, the distance is also minimized [Gallot et al. 2004]. On the other hand, the metric we use in each factor of $\mathbb{H}\times\mathbb{H}$ is half the hyperbolic distance, for which a well-known formula gives us the distance [Anderson 2005]. Let ρ_k denote the distance in each factor with our metric, then,

$$\cosh\big(\sqrt{2}\rho_1\big) = 1 + \frac{2(x-x')^2 + (e^{s+t} - e^{s'+t'})^2}{2e^{s+t}e^{s'+t'}},$$

$$\cosh\big(\sqrt{2}\rho_2\big) = 1 + \frac{2(y-y')^2 + (e^{s-t} - e^{s'-t'})^2}{2e^{s-t}e^{s'-t'}},$$

where the $\sqrt{2}$ factor within the hyperbolic cosine is due to the factor relating the standard hyperbolic metric with ours. The previous expression shows that, in order to get the minimum distance, x' must be equal to x and y' to y. Simplifying the previous expressions for such values of x' and y', we find

$$\cosh\big(\sqrt{2}\rho_1\big) = \cosh(s - s' + t - t'),$$
$$\cosh\big(\sqrt{2}\rho_2\big) = \cosh(s - s' + t' - t).$$

Therefore,

$$\rho_1 = \frac{|s - s' + t - t'|}{\sqrt{2}}, \qquad \rho_2 = \frac{|s - s' + t' - t|}{\sqrt{2}},$$

and the distance in the product metric is given by $\sqrt{\rho_1^2 + \rho_2^2}$. In order for this distance to be a minimum, a short analysis shows that one must take $t' = t$, and the statement follows. □

Proposition 5.4. *The principal curvatures of each leaf are* -1 *with multiplicity two, and* 0*. The principal directions are determined by the integral curves of the vectors* $\partial_x,\ \partial_y,\ \partial_t$ *respectively.*

Proof. Recall the principal curvatures and directions for an orientable submanifold M are determined by the *shape operator*, S, which, in codimension 1, can be regarded as the mapping $TM \to TM$ given by $v_x \mapsto \nabla_{v_x} X$, where X is the normal field to the manifold, compatible with orientation (see [Spivak 1979, Chapter 1]). Here, the principal directions and curvatures are the shape operator eigenvectors, and eigenvalues. Consider a leaf embedded in $\mathbb{H} \times \mathbb{H}$,

$$\left(x,\ \frac{e^{-t-s}}{\sqrt{2}},\ y,\ \frac{e^{t-s}}{\sqrt{2}}\right),$$

with normal field $X = x_2\partial_2 + x_4\partial_4$, where $x_2 = e^{-t-s}/\sqrt{2}$ and $x_4 = e^{t-s}/\sqrt{2}$. A calculation shows that

$$\nabla X = -dx_1 \otimes \partial_1 - dx_3 \otimes \partial_3,$$

i.e., the shape operator is diagonal, once expressed in the base for the tangent space to the leaf, spanned by the coordinate vectors $\partial_1,\ \partial_3$, and the vector $-x_2\partial_2 + x_4\partial_4$, with eigenvalues $\{-1, -1, 0\}$ counted with multiplicity. □

5A. ***Proof of Theorem 1.1.*** Let G be a complex Kleinian group with a maximum of four lines in general position contained in its limit set, then G acts properly and discontinuously in four disjoint copies of $\mathbb{H} \times \mathbb{H}$. Without loss of generality we can assume that $\mathbb{H} \times \mathbb{H}$ is G-invariant. By Proposition 3.1, if

$$\psi : \mathbb{S}\mathrm{ol} \times \mathbb{R} \to \mathbb{H} \times \mathbb{H}$$

is a G-equivariant diffeomorphism, then $(\mathbb{H}\times\mathbb{H})/G$ is diffeomorphic to $(\mathbb{S}\mathrm{ol}/G)\times\mathbb{R}$. We notice the topological type is perfectly determined by the group G. In fact, the group G is the fundamental group of the manifold $(\mathbb{H} \times \mathbb{H})/G$. We remember the Kulkarni discontinuity region is equal to four disjoint copies of $\mathbb{H} \times \mathbb{H}$, hence Ω/G is equal to four disjoint copies of $(\mathbb{H} \times \mathbb{H})/G$. We remark that if G represents a lattice of the Lie group $\mathbb{S}\mathrm{ol}$, then $\mathbb{S}\mathrm{ol}/G$ is a compact 3 manifold. This last statement implies in some sense that $\mathbb{S}\mathrm{ol}/G$ is the compact heart of $(\mathbb{H} \times \mathbb{H})/G$.

6. The Heisenberg group

Given a symplectic vector space, V, with symplectic form ω, recall the *Heisenberg* group, $\mathcal{H}$, is the space $V \times \mathbb{R}$, with the product operation given by

$$(v, t) * (w, s) = (v + w,\ t + s + \omega(v, w)).$$

An account of this group in the context of complex hyperbolic geometry can be found in [Cano et al. 2013]. If V is of dimension 2, and $\{\partial_p, \partial_q\}$ is a symplectic base for V, that is, $\omega(\partial_p, \partial_q) = 1$, a well-known fact from Lie group theory is that there is a faithful representation $\mathcal{H} \to \mathrm{SL}(3, \mathbb{R})$ [Binz and Pods 2008], given by

$$(p\partial_p + q\partial_q t) \to \begin{pmatrix} 1 & p & t + \frac{1}{2}pq \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}.$$

We will use this representation and identify $\mathcal{H}$ with a subgroup of $\mathrm{SL}(3, \mathbb{R})$. Therefore, we will identify $\mathcal{H}$ with $\mathbb{R}^3$, with group structure,

$$(a, b, c) * (a', b', c') = (a + a', b + b', c + c' + ab'),$$

which corresponds to the matrix product

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a' & c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix}.$$

With these identifications, there is a natural left action $\mathcal{H} \circlearrowleft \mathbb{C} \times \mathbb{H}$:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \\ 1 \end{pmatrix} = \begin{pmatrix} z + aw + c \\ w + b \\ 1 \end{pmatrix},$$

which we will denote by $(a, b, c) * (z, w)$.

Proposition 6.1. *The action of $\mathcal{H}$ in $\mathbb{C}\times\mathbb{H}$ is free.*

Proof. If $(a,b,c)*(z,w)=(z,w)$, then

$$\begin{aligned} z+aw+c&=z,\\ w+b&=w. \end{aligned}$$

From this linear system, one deduces that $a=b=c=0$. □

Proposition 6.2. *For fixed $(z,w)\in\mathbb{C}\times\mathbb{H}$, the orbit $h\in\mathcal{H}\mapsto h*(z,w)$ defines a differentiable embedding $\mathcal{H}\hookrightarrow\mathbb{C}\times\mathbb{H}$.*

Proof. The map is injective, since the action is free. Let $w=p+qi$, the Jacobian matrix of the mapping in $(a,b,c)\in\mathcal{H}$ is given by

$$\begin{pmatrix} p & 0 & 1\\ q & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{pmatrix}.$$

Since the Jacobian has rank 3, the action defines a local diffeomorphism, and hence an embedding. □

Therefore, the action of $\mathcal{H}$ defines a foliation of $\mathbb{C}\times\mathbb{H}$, in analogy with the foliation of $\mathbb{H}\times\mathbb{H}$ generated by $\mathbb{S}$ol.

Proposition 6.3. *Consider $\mathbb{C}\times\mathbb{H}$ as a subset of $\mathbb{R}^4$, but with the product metric of the euclidean metric in $\mathbb{C}$ and the hyperbolic metric in $\mathbb{H}$. If $e_1,\dots,e_4$ denote the canonical coordinates in $\mathbb{R}^4$, and (p,q) denotes the coordinates in $\mathbb{H}$, then the vector field $X=qe_4$ is unitary and orthogonal to any leaf of the foliation generated by $\mathcal{H}$.*

Proof. From Theorem 2.4, the vector fields $pe_1+qe_2,\ e_3,\ e_1$ generate the tangent space to the orbit of $(z,w)\in\mathbb{C}\times\mathbb{H}$, where $w=p+qi$. Since the metric is a product, X is orthogonal to pe_1+qe_2 and e_1. Moreover, the metric in $\mathbb{H}$ is conformal to the euclidean, and therefore X is orthogonal to e_3. Finally, qe_4 is unitary in the hyperbolic metric. □

Corollary 6.4. *Let $(z,w)\in\mathbb{C}\times\mathbb{H}$, $z=x+yi$ and $w=p+qi$. The integral curves of X are geodesics.*

Proof. The integral curves of X are constant in the first factor, and vertical straight lines in the hyperbolic factor. □

Although in this case, the action induced by the normal field X *is not* equivariant, we can describe in a precise way the quotients $(\mathbb{C}\times\mathbb{H})/\Gamma$, where Γ is a discrete subgroup of $\mathcal{H}$. Moreover, if Γ acts properly discontinuously in $\mathbb{C}\times\mathbb{H}$, it has to act in the same way in Heisenberg, because the slices $\mathcal{H}\times\{qi\}$ are preserved. This is a general property of Lie groups that we prove in the following lemma.

Lemma 6.5. *Let X and Y be two locally compact spaces. If $\Gamma \circlearrowleft X \times Y$, and the action of $g \in \Gamma$ can be decomposed as $g \cdot (x, y) = (g \cdot x, y)$ then Γ acts properly discontinuously in X if and only if it acts properly discontinuously in $X \times Y$.*

Proof. Let $K \subset X$ be a compact set. Fix $y \in Y$. With the product topology, $K \times \{y\}$ is a compact set in $X \times Y$. One can easily verify the equality

$$\{g \in \Gamma : g \cdot K \cap K \neq \varnothing\} = \left\{g \in \Gamma : g \times 1 \cdot K \times \{y\} \cap K \times \{y\} \neq \varnothing\right\}.$$

If Γ acts properly discontinuously in $X \times Y$, the previous equality implies that it acts properly discontinuously in X. On the other hand, if $K \subset X \times Y$ is compact, the product topology together with the local compacity implies that we can find an open set $U \times V$, with $U \in X$ and $V \in Y$, such that $\overline{U}$ is compact in X, $\overline{V}$ is compact in Y, and $K \subset U \times V$. We have the contention

$$\{g \in \Gamma : gK \cap K \neq \varnothing\} \subset \left\{g \in \Gamma : g \cdot l(\overline{U} \times \overline{V}) \cap \overline{U} \times \overline{V} \neq \varnothing\right\}$$

Take $g \in \Gamma$ and $(x, y) \in \overline{U} \times \overline{V}$, such that $g \cdot (x, y) \in \overline{U} \times \overline{V}$. Since $g \cdot (x, y) = (g \cdot x, y)$, it follows that $g \cdot x \in \overline{U}$. Therefore, the second set in the previous contention is at the same time contained in

$$\{g \in \Gamma : g \cdot \overline{U} \cap \overline{U} \neq \varnothing\}.$$

If Γ acts properly discontinuously in X, this set has to be finite, and the same must be true for the set of intersections in $X \times Y$, that is, Γ acts properly discontinuously in X. □

Proposition 6.6. *$\mathbb{C} \times \mathbb{H}$ is diffeomorphic to $\mathcal{H} \times \mathbb{R}$, where, up to diffeomorphism, $\mathcal{H}$ acts on the first factor only.*

Proof. Let $\gamma = (a, b, c) \in \mathcal{H}$ and take $(0, qi) \in \mathbb{C} \times \mathbb{H}$. We can describe the orbits $\gamma \cdot (0, qi)$ explicitly:

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ qi \\ 1 \end{pmatrix} = \begin{pmatrix} aqi + c \\ qi + b \\ 1 \end{pmatrix}.$$

Therefore, there is exactly one $(0, qi)$ in each orbit of the group action. Define $\Psi : \mathcal{H} \times \mathbb{R} \to \mathbb{C} \times \mathbb{H}$ as

$$\Psi(\gamma, q) = \gamma \cdot (0, qi).$$

It can be shown that Ψ is bijective. It is a diffeomorphism, since an explicit computation shows that $d\Psi$ maps the canonical vectors $T_{(\gamma,q)}\mathcal{H} \times \mathbb{R} \cong \mathbb{R}^4 \to T_{\gamma\cdot(0,qi)}\mathbb{C} \times \mathbb{H} \cong \mathbb{R}^4$:

$$\{\partial_1, \ldots, \partial_4\} \mapsto \{q\partial_2, \partial_3, \partial_1, a\partial_2 + \partial_4\}.$$

The last assertion follows since the action is associative, i.e., $\gamma' \cdot (\gamma \cdot (0, qi)) = (\gamma' \cdot \gamma) \cdot (0, qi)$, and therefore, preserves the imaginary part on the second factor. □

6A. ***Proof of Theorem 1.4.*** The proof is analogous to that of Theorem 1.1, the only difference is that we need the technical Lemma 6.5. A consequence of Theorem 1.4 is the following corollary:

Corollary 6.7. *If $\Gamma < \mathcal{H}$ is a discrete subgroup acting properly and discontinuously in $\mathbb{C} \times \mathbb{H}$, up to diffeomorphism, $(\mathbb{C} \times \mathbb{H})/\Gamma \cong (\mathcal{H}/\Gamma) \times \mathbb{R}$, and the quotient $\mathcal{H}/\Gamma$ is a manifold whose fundamental group is $\pi_1(\mathcal{H}/\Gamma) \cong \Gamma$.*

Example 6.8. Let $\mathcal{H}_{\mathbb{Z}} < \mathcal{H}$ be the discrete subgroup of Heisenberg matrices with integer coefficients. It can be shown that the unit cube $K_C = [0, 1]^3 \subset \mathcal{H}$ is a fundamental region for the action of $\mathcal{H}_{\mathbb{Z}}$ [Lukyanenko 2014]. The quotient $\mathcal{H}_{\mathbb{Z}} \setminus \mathcal{H}$ is an example of a nilmanifold, whose fundamental group is

$$\mathcal{H}_{\mathbb{Z}} \cong \langle m, n, k : [m, n] = k^4 \rangle;$$

see [Lukyanenko 2014]. In view of the previous results, $\mathcal{H}_{\mathbb{Z}}$ acts properly and discontinuously in $\mathbb{C} \times \mathbb{H}$, and the quotient $(\mathbb{C} \times \mathbb{H})/\mathcal{H}_{\mathbb{Z}}$ is a product of a nilmanifold times $\mathbb{R}$, whose fundamental group has the previous presentation.

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