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We construct a symplectic flow on a surface of genus $g \geq 2$, $\Sigma_{g\geq 2}$, with exactly $2g-2$ hyperbolic fixed points and no other periodic orbits. Moreover, we prove that a (strongly nondegenerate) symplectomorphism of $\Sigma_{g\geq 2}$ isotopic to the identity has infinitely many periodic points if there exists a fixed point with nonzero mean index. From this result, we obtain two corollaries, namely that such a symplectomorphism of $\Sigma_{g\geq 2}$ with an elliptic fixed point or with strictly more than $2g-2$ fixed points has infinitely many periodic points provided that the flux of the isotopy is "irrational".

1. Introduction and main results

In this paper, we construct a symplectic flow ψ^t on a closed surface with genus $g \geq 2$, $\Sigma_{g\geq 2}$, having exactly $2g-2$ hyperbolic fixed points and no other periodic points. This is a genuine flow and it satisfies an "irrationality" condition on its flux; see property (1-1). This construction yields the computation of the Floer–Novikov homology when (1-1) holds. With this information and assuming (1-1), we prove that a (strongly nondegenerate) symplectomorphism ϕ on $\Sigma_{g\geq 2}$ (connected to the identity by an isotopy ϕ_t) possessing a fixed point with nonzero mean index has infinitely many periodic points. As a consequence of this result, we see that the presence of an elliptic fixed point or of strictly more than $2g-2$ fixed points implies the existence of infinitely many periodic points.

We are interested in symplectomorphisms which are not Hamiltonian. However our results fit in the context of a conjecture of B. Z. Gürel [2013; 2014] which suggests that the presence of an *unnecessary* fixed point of a Hamiltonian diffeomorphism guarantees the existence of infinitely many periodic points. There, *unnecessary* is viewed from a homological or geometrical perspective. The results in [Gürel 2013; 2014] support the conjecture when the fixed point is unnecessary from a homological viewpoint. From the geometrical perspective, the conjecture is supported, e.g., by the result in [Ginzburg and Gürel 2014] where V. L. Ginzburg and

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B. Z. Gürel prove that, for a vast class of symplectic manifolds (which includes the complex projective spaces $\mathbb{CP}^n$), a Hamiltonian diffeomorphism with a hyperbolic fixed point has infinitely many periodic points.

Furthermore, the conjecture by Gürel is a variant of a conjecture by H. Hofer and E. Zehnder [1994, page 263] claiming that "every Hamiltonian map on a compact symplectic manifold (M, ω) possessing more fixed points than necessarily required by the V. Arnold conjecture possesses always infinitely many periodic points". For instance, the conjecture in [Hofer and Zehnder 1994] on $\mathbb{CP}^n$ claims that a nondegenerate Hamiltonian diffeomorphism has infinitely many periodic points if it fixes more than $n+1$ points. This was motivated by the result of J. Franks [1988] stating that an area-preserving diffeomorphism on S^2 with more than two fixed points has infinitely many periodic points (see also [Franks 1992; 1996; Le Calvez 1999; Bramham and Hofer 2012; Collier et al. 2012; Kerman 2012] for symplectic topological proofs).

Recall that a Hamiltonian diffeomorphism on a closed surface with genus $g \geq 1$ always has infinitely many periodic points. This statement was conjectured to hold on the torus by C. Conley in a lecture given on April 6th 1984, in the University of Wisconsin. This was later proved in [Hingston 2009] and it has been generalized to a vast class of symplectic manifolds; see Ginzburg's proof [2010] and, e.g., [Ginzburg and Gürel 2012; Hein 2012; Ginzburg et al. 2015] for more contributions.

The background discussed so far concerns Hamiltonian diffeomorphisms. For symplectomorphisms which need not be Hamiltonian, H. V. Lê and K. Ono [1995] proved a version of Arnold's conjecture for nondegenerate symplectomorphisms. A lower bound for the number of fixed points of a symplectomorphism is given by the sum of the Betti numbers of the Novikov homology of a closed 1-form representing the cohomology class given by the flux of an isotopy connecting the identity to the symplectomorphism. Observe that this lower bound may be zero as in the case of the 2-torus. Moreover, when the flux of the isotopy is zero, the Novikov homology associated to the flux is the ordinary homology of M and, in this case, i.e., when the symplectomorphism is Hamiltonian, this is the statement of Arnold's conjecture.

There is also an analogue of the result by Ginzburg and Gürel [2014] which claims that if a symplectomorphism (satisfying some conditions on its flux) has a hyperbolic fixed point, then there are infinitely many periodic points. If the hyperbolic fixed point corresponds to a contractible periodic orbit, the result is proved in [Batoréo 2015] for some class of manifolds which includes, for instance, the product of $\mathbb{CP}^n$ with a $2m$-dimensional torus, $\mathbb{CP}^n \times \mathbb{T}^{2m}$, with $m \leq n$ (or $\mathbb{CP}^n \times P^{2m}$, with P^{2m} a symplectically aspherical $2m$-manifold). The case when the hyperbolic periodic orbit is noncontractible was proved in [Batoréo 2017] and it holds, for example, on the product spaces $\mathbb{CP}^n \times \Sigma_{g\geq 2}$. We point out that the existence of infinitely many periodic points is guaranteed by the presence of a hyperbolic fixed point on $\mathbb{CP}^n$ and on $\mathbb{CP}^n \times \Sigma_{g\geq 2}$. However, such a result does not hold on $\Sigma_{g\geq 2}$.

In fact, Theorem 1.1 and the construction of Section 3A give a symplectomorphism with finitely many hyperbolic fixed points and no other periodic points; see [Katok and Hasselblatt 1995, Exercise 14.6.1]. The number of fixed points of this symplectomorphism is exactly $2g-2$, which is the lower bound for the number of fixed points of a diffeomorphism given by the Lefschetz fixed point theorem. We prove the presence of infinitely many periodic points of a (strongly nondegenerate) symplectomorphism on $\Sigma_{g\geq 2}$ (with an "irrationality" assumption on its flux) provided the existence of a fixed point with nonzero mean index (see Theorem 1.4). Such a condition is satisfied if the fixed point is elliptic (see Theorem 1.3) or if the number of fixed points is strictly greater than $2g-2$ (see Theorem 1.5).

In Sections 1A and 1B, we state the main theorems of this paper. The theorems in Section 1A refer to the existence of the symplectomorphism with exactly $2g-2$ fixed points and no other periodic points (Theorem 1.1) and to the computation of the Floer–Novikov homology of symplectomorphisms satisfying condition (1-1) (Theorem 1.2). In Section 1B, we state the theorems which give sufficient conditions for the existence of infinitely many periodic points of symplectomorphisms with flux as in (1-1) (Theorems 1.3–1.5). The remaining sections are organized as follows: in Section 2, we present the definitions and known results used in the statements and proofs of our theorems, in Section 3, we prove the results stated in Section 1A and, in Section 4, we prove the theorems stated in Section 1B.

1A. ***Existence of a symplectomorphism with exactly 2g − 2 hyperbolic fixed points and no other periodic points.*** Consider a closed surface Σ with genus g greater than or equal to 2 and a symplectic form ω on Σ. The first cohomology group $H^1(\Sigma;\mathbb{R})$ of the surface Σ can be identified with $\mathbb{R}^{2g}$ and hence the image of $[\phi_t]\in\widetilde{\mathrm{Symp}}_0(\Sigma,\omega)$ under the flux homomorphism (see Section 2B) can be viewed as a $2g$-tuple,

$$(u_1, v_1, \dots, u_g, v_g)\in\mathbb{R}^{2g},$$

where $\widetilde{\mathrm{Symp}}_0(\Sigma,\omega)$ is the universal covering of the identity component of the group of symplectomorphisms on Σ. Moreover, the kernel of the flux homomorphism is given by the universal covering of the group of Hamiltonian diffeomorphisms, $\widetilde{\mathrm{Ham}}(\Sigma,\omega)$. We recall that the flux homomorphism

$$\mathrm{Flux}:\widetilde{\mathrm{Symp}}_0(\Sigma,\omega)\to H^1(\Sigma,\mathbb{R});\quad [\phi_t]\mapsto\left[\int_0^1\omega(X_t,\cdot)\,dt\right]$$

descends to a homomorphism

$$\mathrm{Flux}:\mathrm{Symp}_0(\Sigma,\omega)\to H^1(\Sigma,\mathbb{R});\quad \phi\mapsto\left[\int_0^1\omega(X_t,\cdot)\,dt\right]$$

since Σ is atoroidal (see Section 2B). If a symplectomorphism ϕ satisfies

$$\text{(1-1)}\quad \mathrm{Flux}(\phi) = (u_1, v_1, \dots, u_g, v_g) \quad \text{with } u_i \neq 0 \text{ and } \frac{v_i}{u_i} \in \mathbb{R} \setminus \mathbb{Q}, \quad \text{for all } i = 1, \dots, g,$$

we say that it satisfies the *flux condition*.

Remark 1. If ϕ satisfies the flux condition (1-1), then ϕ^k also satisfies the flux condition (for all $k \in \mathbb{N}$).

Our first main result is the following:

Theorem 1.1. *Given* $(u_1, v_1, \dots, u_g, v_g) \in \mathbb{R}^{2g}$ *such that*

$$u_i \neq 0 \quad \textit{and} \quad v_i/u_i \in \mathbb{R} \setminus \mathbb{Q}, \quad i = 1, \dots, g,$$

there exists a symplectic flow

$$\psi^t_{(u_1,v_1,\dots,u_g,v_g)} : \Sigma \to \Sigma$$

with no periodic orbits other than (*exactly*) $2g-2$ *hyperbolic fixed points and*

$$\mathrm{Flux}(\psi^t_{(u_1,v_1,\dots,u_g,v_g)}|_{t\in[0,1]}) = (u_1, v_1, \dots, u_g, v_g).$$

Denote by $\mathrm{HFN}_*(\phi)$ the Floer–Novikov homology of a symplectomorphism ϕ of Σ isotopic to the identity (see Section 2D for the definition). Using the construction of the (genuine) flow $\psi^t_{(u_1,v_1,\dots,u_g,v_g)}$ given by the previous theorem, we compute $\mathrm{HFN}_*(\phi)$ for nondegenerate symplectomorphisms ϕ satisfying (1-1) (see Theorem 1.2). In the following theorem, one can take any ring (e.g., $\mathbb{Z}$ or $\mathbb{Q}$) as the ground ring $\mathbb{F}$. In this paper, for the sake of simplicity, all complexes and homology groups are defined over the ground field $\mathbb{F} = \mathbb{Z}_2$.

Theorem 1.2. *Let* $\phi \in \mathrm{Symp}_0(\Sigma, \omega)$ *be nondegenerate and satisfying the flux condition* (1-1). *Then the Floer–Novikov homology of* ϕ *is given by*

$$\text{(1-2)}\qquad \mathrm{HFN}_r(\phi) = \begin{cases} \mathbb{F}^{2g-2} \otimes \Lambda_\theta & \textit{if } r = 0, \\ 0 & \textit{if } r \neq 0. \end{cases}$$

We point out that Lê and Ono [1995, Theorem 8.1] proved that, for a certain class of symplectic manifolds, if the flux of the isotopy is sufficiently small, then the Floer–Novikov homology of the isotopy may be computed by the Novikov homology of a closed 1-form representing the flux of the isotopy. Namely, on Σ, [Lê and Ono 1995, Theorem 8.1] states that there exists $\varepsilon > 0$ such that if $\|\theta\|_{C^1} < \varepsilon$, then

$$\mathrm{HFN}_*(\phi) \simeq HN_{*+1}(\theta),$$

where $[\theta] = \mathrm{Flux}(\phi)$. In Theorem 1.2, in contrast, the flux of ϕ is not assumed to be small.

We will now compute the Novikov homology of θ when $[\theta] = \mathrm{Flux}(\phi) = (u_1, v_1, \ldots, u_g, v_g)$ with $u_1, v_1, \ldots, u_g, v_g \in \mathbb{R}$ rationally independent. Consider the homomorphism $\pi_1(\Sigma) \to \mathbb{R}$ defined by

$$\gamma \mapsto \int_\gamma \theta \tag{1-3}$$

which we also denote by $[\theta]$. Since $u_1, v_1, \ldots, u_g, v_g$ are rationally independent, the kernel $\ker([\theta])$ is the commutator $[\pi_1(\Sigma), \pi_1(\Sigma)]$ of the fundamental group $\pi_1(\Sigma)$. Let $\pi : \widetilde{\Sigma} \to \Sigma$ be the covering space associated to the homomorphism $[\theta]$, i.e., $\widetilde{\Sigma}$ is the maximal free abelian covering of Σ. Then there exists a function $\overline{f} : \widetilde{\Sigma} \to \mathbb{R}$ such that $\pi^*\theta = d\overline{f}$. We recall that the Novikov complex of θ is defined in the same way as the Morse complex of $\overline{f}$; see, e.g., [Lê and Ono 1995], namely Section 6 and Appendix C, and [Ono 2006].

As mentioned in the example of [Lê and Ono 1995, Section 7], the Betti numbers of $\widetilde{\Sigma}$ are 0, $2g-2$ and 0. Hence, by [Lê and Ono 1995, Theorem 8.1], for $\|\theta\|_{C^1}$ sufficiently small,

$$\mathrm{HFN}_*(\phi) \simeq \begin{cases} \mathbb{F}^{2g-2} \otimes \Lambda_\theta & \text{if } * = 0, \\ 0 & \text{if } * \neq 0, \end{cases}$$

which coincides with the computations in Theorem 1.2.

Notice that, when $u_1, v_1, \ldots, u_g, v_g \in \mathbb{R}$ are rationally independent, the sum of the Betti numbers of the Novikov homology of θ, with $[\theta] = (u_1, v_1, \ldots, u_g, v_g)$, is $2g-2$ (regardless of whether $\|\theta\|$ is sufficiently small or not) and hence the lower bound given by the main theorem in [Lê and Ono 1995, page 156] is attained by the symplectic flow given by Theorem 1.1.

Remark 2. We observe that:

(1) Due to conventions on the indices, the Floer–Novikov homology in this paper is the Floer–Novikov homology considered in [Lê and Ono 1995] with the degree shifted by $n = 1$.

(2) On Σ, the Novikov rings $\Lambda_{\theta,\omega}$ and Λ_θ in [Lê and Ono 1995] are isomorphic and hence

$$\mathrm{Nov}_*(\theta) \otimes_{\Lambda_\theta} \Lambda_{\theta,\omega} \simeq \mathrm{Nov}_*(\theta).$$

(3) $\varepsilon > 0$ is taken small enough so that the conditions in [Ono 2006, Definition 3.9] are satisfied. See also [Ono 2006, Theorem 3.12].

Remark 3 (noncontractible orbits). In this paper, the Floer–Novikov homology is defined for contractible periodic orbits (as in [Ono 2006]), unless explicitly stated otherwise. If the fixed points of the symplectomorphisms correspond to noncontractible periodic orbits, take the Floer–Novikov homology for noncontractible periodic orbits defined in [Burghelea and Haller 2001]. In that case, the

Floer–Novikov homology of a nondegenerate $\phi \in \mathrm{Symp}_0(\Sigma, \omega)$ satisfying (1-1) is $\mathrm{HFN}_*(\phi, \zeta) = 0$, where ζ is a nontrivial free homotopy class of loops in Σ. See Remark 17.

1B. ***Existence of infinitely many periodic points.*** Consider a strongly nondegenerate symplectomorphism ϕ (see page 27 for the definition) on a closed surface Σ (with genus $g \geq 2$) satisfying the flux condition (1-1). The following theorem gives a condition under which ϕ has infinitely many periodic points.

Theorem 1.3. *Let $\phi \in \mathrm{Symp}_0(\Sigma, \omega)$ be strongly nondegenerate. Suppose ϕ satisfies the flux condition (1-1) and that ϕ has an elliptic fixed point. Then ϕ has infinitely many periodic points.*

Remark 4. If x_0 corresponds to a noncontractible periodic orbit, Theorem 1.3 remains valid. See Remark 18.

Theorem 1.3 follows from a more general result:

Theorem 1.4. *Let $\phi \in \mathrm{Symp}_0(\Sigma, \omega)$ be strongly nondegenerate. Suppose ϕ satisfies the flux condition (1-1) and that ϕ has a fixed point x_0 such that its mean index $\Delta(x_0)$ is not zero. Then ϕ has infinitely many periodic points.*

In Section 4, we prove, more precisely, that if ϕ has finitely many fixed points, then every large prime is a simple period, i.e., a period of a simple (noniterated) orbit. (In particular, the number of simple periods less than or equal to k is of order at least $k/\log(k)$.) One of the main tools used in the proof of this theorem is Floer–Novikov homology and the proof relies on Theorem 1.2. Another consequence of Theorem 1.4 is Theorem 1.5, which gives a sufficient condition on the number of fixed points of ϕ for the existence of infinitely many periodic points of ϕ.

Theorem 1.5. *Let $\phi \in \mathrm{Symp}_0(\Sigma, \omega)$ be strongly nondegenerate and suppose it satisfies the flux condition (1-1). If the number of fixed points of ϕ is (strictly) greater than $2g-2$, then ϕ has infinitely many periodic points.*

2. Preliminaries

Consider a closed surface Σ with genus $g \geq 2$ and a symplectic structure ω on Σ. In this section, we follow [Burghelea and Haller 2001; Ginzburg and Gürel 2015; Lê and Ono 1995; Ono 2006; Salamon and Zehnder 1992].

2A. ***A covering space of the space of contractible loops.*** Let $\mathcal{L}\Sigma$ be the space of contractible loops in Σ and $\Omega\Sigma$ be the space of based contractible loops in Σ. The map $ev : \mathcal{L}\Sigma \to \Sigma$ defined by $x \mapsto x(0)$ is a fibration with fiber $\Omega\Sigma$ (see, e.g., [Hu 1959, page 83] for the details). It induces a long exact sequence on the homotopy groups and part of it is given by

$$\pi_1(\Omega\Sigma) \to \pi_1(\mathcal{L}\Sigma) \to \pi_1(\Sigma).$$

Since this fibration admits a section consisting of constant loops,

$$\pi_1(\mathcal{L}\Sigma) \cong \pi_1(\Omega\Sigma) \oplus \pi_1(\Sigma).$$

With the identification $\pi_1(\Omega\Sigma) \equiv \pi_2(\Sigma)$ (see, e.g., [Adams 1978, pages 5–7] for the details) and since $\pi_2(\Sigma) = 0$, we have

$$\pi_1(\mathcal{L}\Sigma) \cong \pi_1(\Sigma).$$

Let θ be a closed 1-form on Σ and consider the homomorphism

$$[\bar{\theta}] : \pi_1(\mathcal{L}\Sigma) \to \mathbb{R} \tag{2-1}$$

induced by the homomorphism $[\theta] : \pi_1(\Sigma) \to \mathbb{R}$ defined by (1-3). Moreover, take the covering $\pi : \widetilde{\Sigma} \to \Sigma$ associated with $\ker([\theta]) \leqslant \pi_1(\Sigma)$. When $\ker([\theta]) = 0$, $\widetilde{\Sigma}$ is the universal covering of Σ. Choose a function $\bar{f} : \widetilde{\Sigma} \to \mathbb{R}$ such that $d\bar{f} = \pi^*\theta$.

Denote by $\widetilde{\mathcal{L}\Sigma}$ the covering space of $\mathcal{L}\Sigma$ associated with $\ker([\bar{\theta}]) \leqslant \pi_1(\mathcal{L}\Sigma)$. The deck transformation group of $p : \widetilde{\mathcal{L}\Sigma} \to \mathcal{L}\Sigma$ is

$$\Gamma := \frac{\pi_1(\mathcal{L}\Sigma)}{\ker([\bar{\theta}])} \cong \frac{\pi_1(\Sigma)}{\ker([\theta])}. \tag{2-2}$$

Following [Ono 2006], an element of the covering space $\widetilde{\mathcal{L}\Sigma}$ can be described as an equivalence class (for a relation $\sim$) of a loop $\tilde{x}$ in $\widetilde{\Sigma}$ where the relation $\sim$ is defined by $\tilde{x} \sim \tilde{y}$ if

$$\pi \circ \tilde{x} = \pi \circ \tilde{y} \tag{2-3}$$

and

$$\bar{f}(\tilde{x}(o)) = \bar{f}(\tilde{y}(o)) \tag{2-4}$$

where o is the base point of S^1, i.e., $1 \in \partial D^2 \subset \mathbb{C}$. We observe that conditions (2-3) and (2-4) are equivalent to $\tilde{x} = \tilde{y}$ and, hence, $\widetilde{\mathcal{L}\Sigma}$ is in fact the space $\mathcal{L}\widetilde{\Sigma}$ of contractible loops in $\widetilde{\Sigma}$.

Remark 5. The homomorphisms $\mathcal{I}_\omega$ and $\mathcal{I}_{c_1}$ defined by [Ono 2006] are identically zero when $M = \Sigma$, since $\pi_2(\Sigma) = 0$. Moreover, the homomorphism $\mathcal{I}_\eta$ from that paper is the map $[\bar{\theta}]$ in (2-1).

2B. ***Symplectomorphisms and periodic orbits.*** We denote by $\mathrm{Symp}(\Sigma, \omega)$ the group of symplectomorphisms of (Σ, ω) and by $\mathrm{Symp}_0(\Sigma, \omega)$ the component of the identity in $\mathrm{Symp}(\Sigma, \omega)$.

Let $\phi \in \mathrm{Symp}_0(\Sigma, \omega)$ and consider ϕ_t a symplectic isotopy connecting the identity $\phi_0 = \mathrm{id}$ to $\phi_1 = \phi$ and define a vector field X_t by

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t.$$

The flux homomorphism is defined on the universal covering $\widetilde{\mathrm{Symp}}_0(\Sigma,\omega)$ of $\mathrm{Symp}_0(\Sigma,\omega)$ by

$$\mathrm{Flux}:\widetilde{\mathrm{Symp}}_0(\Sigma,\omega)\to H^1(\Sigma;\mathbb{R});\quad [\phi_t]\mapsto\left[\int_0^1\omega(X_t,\cdot)\,dt\right].$$

This homomorphism is surjective, its kernel is given by $\widetilde{\mathrm{Ham}}(\Sigma,\omega)$, i.e., the universal covering of the group of Hamiltonian diffeomorphisms (see [McDuff and Salamon 1995]) and, when $g\geq 2$, (see [Kędra 2000]) it descends to a homomorphism

$$\mathrm{Flux}:\mathrm{Symp}_0(\Sigma,\omega)\to H^1(\Sigma,\mathbb{R});\quad \phi\mapsto\left[\int_0^1\omega(X_t,\cdot)\,dt\right].$$

Remark 6 [McDuff and Salamon 1995, pages 316–317]. Under the usual identification of $H^1(\Sigma;\mathbb{R})$ with $\mathrm{Hom}(\pi_1(\Sigma),\mathbb{R})$, the cohomology class $\mathrm{Flux}([\phi_t])$ corresponds to the homomorphism

$$\pi_1(\Sigma)\to\mathbb{R};\quad \gamma\mapsto\int_0^1\int_0^1\omega(X_t(\gamma(s)),\dot\gamma(s))\,ds\,dt,$$

for $\gamma:S^1=\mathbb{R}/\mathbb{Z}\to\Sigma$. Geometrically, the value of $\mathrm{Flux}([\phi_t])$ on the loop γ is the symplectic area swept by the path γ under the isotopy ϕ_t.

Denote by θ a closed 1-form such that $\mathrm{Flux}([\phi])=[\theta]\in H^1(\Sigma;\mathbb{R})$.

Lê and Ono [1995, Lemma 2.1] proved that $\{\phi_t\}$ can be deformed through symplectic isotopies (keeping the end points fixed) so that the cohomology classes $[\omega(X'_t,\cdot)]$, for all $t\in[0,1]$, and $\mathrm{Flux}([\phi'_t])=[\theta]$ are the same (where X'_t is the vector field associated with the deformed symplectic isotopies ϕ'_t). Namely, each element in $\widetilde{\mathrm{Symp}}_0(\Sigma,\omega)$ admits a representative symplectic isotopy generated by a smooth path of closed 1-forms θ_t on Σ whose cohomology class is identically equal to the flux, i.e.,

$$-\omega(X'_t,\cdot)=\theta+dh_t=:\theta_t$$

for some Hamiltonian $h_t:\Sigma\to\mathbb{R},\ t\in S^1$, that is 1-periodic in time.

The fixed points of $\phi=\phi_1$ are in one-to-one correspondence with 1-periodic solutions of the differential equation

$$\dot x(t)=X_{\theta_t}(t,x(t)),\tag{2-5}$$

where X_{θ_t} is defined by $\omega(X_{\theta_t},\cdot)=-\theta_t$. From now on we denote the vector field X_{θ_t} also by X_t.

A 1-periodic solution x of (2-5) is called *nondegenerate* if 1 is not an eigenvalue of the linearized return map $d\phi_{x(0)}:T_{x(0)}\Sigma\to T_{x(0)}\Sigma$. If all 1-periodic orbits of X_t are nondegenerate, then the associated symplectomorphism ϕ is called nondegenerate and if all periodic orbits of X_t are nondegenerate then ϕ is called *strongly* *nondegenerate* . Moreover, if all periodic orbits of X_t are nondegenerate, then the set $\mathcal{P}(\theta_t)$ of 1-periodic solutions of (2-5) is finite.

The set $\mathcal{P}(\theta_t)$ coincides with the zero set of the closed 1-form defined on the space of contractible loops on Σ, $\mathcal{L}\Sigma$, by

$$\begin{aligned}\alpha_{\{\phi_t\}}(x,\xi) &= \int_0^1 \omega(\dot{x} - X_t, \xi)\,dt \\ &= \int_0^1 \omega(\dot{x},\xi) + \theta_t(x(t))(\xi)\,dt \\ &= \int_0^1 \omega(\dot{x},\xi)\,dt + \int_0^1 (\theta + dh_t)(\xi)\,dt\end{aligned}$$

where $x \in \mathcal{L}\Sigma$ and $\xi \in T_x\mathcal{L}\Sigma$ (i.e., ξ is a tangent vector field along the loop x or, equivalently, $\xi(t) \in T_{x(t)}\Sigma$).

A primitive function of the pull-back of the 1-form $\alpha_{\{\phi_t\}}$ to the covering space $\widetilde{\mathcal{L}}\Sigma$ (defined in Section 2A) is given by

$$\mathcal{A}_{\{\phi_t\}}(\tilde{x}) := -\int_{D^2} v^*\omega + \int_0^1 (\bar{f} + h_t \circ \pi)(\tilde{x}(t))\,dt$$

where $v : D^2 \to \Sigma$ is some disc in Σ with $\pi \circ \tilde{x} = v|_{\partial D^2}$. Notice that the right-hand side is independent of the choice of the disc v.

2C. ***The mean index and the Conley–Zehnder index.*** For every continuous path $\Phi : [0, 1] \to \mathrm{Sp}(2)$ of 2×2 symplectic matrices such that $\Phi(0) = \mathrm{Id}$, the mean index $\Delta(\Phi)$ measures, roughly speaking, the total rotation angle swept by certain eigenvalues on the unit circle. We describe this index (and the Conley–Zehnder index) explicitly.

Let A be a symplectic matrix in Sp(2). Then it has two eigenvalues λ_1 and λ_2 such that either $\lambda_i \in S^1 \subset \mathbb{C}$ or $\lambda_i \in \mathbb{R} \setminus \{-1, 1\}$, where $i = 1, 2$, and $\lambda_1\lambda_2 = 1$. We denote the spectrum of A, i.e., the set of eigenvalues of A, by $\sigma(A)$.

If $1 \notin \sigma(A)$, we say A is *nondegenerate*. We distinguish two cases of nondegenerate matrices:

- The eigenvalues are real ($\sigma(A) \subset \mathbb{R} \setminus \{-1, +1\}$). Then $0 < \lambda_1 < 1 < \lambda_2 = \lambda_1^{-1}$ or $\lambda_1 < -1 < \lambda_2 = \lambda_1^{-1} < 0$. In this case, A is called *hyperbolic*.
- The eigenvalues are on the unit circle ($\sigma(A) \subset S^1 \setminus \{1\}$) in which case A is called *elliptic*.

Set

$$\rho(A) = \begin{cases} e^{i\nu} & \text{if } A \text{ is conjugate to a rotation by an angle } \nu \in (-\pi, \pi), \\ 1 & \text{if } \sigma(A) \subset \mathbb{R}_{>0}, \\ -1 & \text{if } \sigma(A) \subset \mathbb{R}_{<0}. \end{cases}$$

This function $\rho : \mathrm{Sp}(2) \to S^1$ is continuous, invariant by conjugation and equal to $\det_{\mathbb{C}} : U(1) \to S^1$ on $U(1)$. When

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

we have $\rho(A) = -1$. Then, given a path $\Phi : [0, 1] \to \mathrm{Sp}(2)$, there is a continuous function $\eta(\cdot)$ such that $\rho(\Phi(t)) = e^{i\eta(t)}$ measuring the rotation of certain unit eigenvalues and the *mean index* of Φ is defined by

$$\Delta(\Phi) := \frac{\eta(1) - \eta(0)}{\pi}.$$

Denote the set of nondegenerate matrices in Sp(2) by $\mathrm{Sp}(2)^*$. This set has two connected components

$$\mathrm{Sp}(2)^+ := \{A \in \mathrm{Sp}(2)^* : \det(A - I) > 0\}$$

and

$$\mathrm{Sp}(2)^- := \{A \in \mathrm{Sp}(2)^* : \det(A - I) < 0\}.$$

Remark 7. The set $\mathrm{Sp}(2)^+$ consists of matrices in $\mathrm{Sp}(2)^*$ which are elliptic or hyperbolic with negative eigenvalues and $\mathrm{Sp}(2)^-$ is the set of matrices in $\mathrm{Sp}(2)^*$ which are hyperbolic with positive eigenvalues.

Define the matrices

$$W^+ := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathrm{Sp}(2)^+ \quad \text{and} \quad W^- := \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \in \mathrm{Sp}(2)^-.$$

For $A \in \mathrm{Sp}(2)^*$, consider a path $\Psi_A : [0, 1] \to \mathrm{Sp}(2)^*$ connecting $A \in \mathrm{Sp}(2)^\pm$ to $W^\pm$. Then, the *Conley–Zehnder index* of Φ with $\Phi(1) \in \mathrm{Sp}(2)^*$ is, by definition,

$$\mu_{\mathrm{CZ}}(\Phi) := \Delta(\Phi \# \Psi_{\Phi(1)}) \in \mathbb{Z},$$

where $\Phi \# \Psi_{\Phi(1)}$ is the concatenation of the paths Φ and $\Psi_{\Phi(1)}$ in Sp(2).

The mean index and the Conley–Zehnder index of Φ satisfy the relation

$$0 \neq |\Delta_{\{\phi_t\}}(\Phi) - \mu_{\mathrm{CZ}}(\Phi)| < 1$$

when $\Phi(1)$ is nondegenerate. We recall some properties of the indices where we assume $\Phi(1) \in \mathrm{Sp}(2)^*$ and $-1 \notin \sigma(\Phi(1))$; see Remark 8.

Result 1.
- If $\Phi(1)$ is elliptic, then $\Delta(\Phi) \neq 0$.
- If $\Phi(1)$ is hyperbolic then $\Delta(\Phi) \in \mathbb{Z}$. Equivalently, if $\Delta(\Phi) \in \mathbb{R} \setminus \mathbb{Z}$, then $\Phi(1)$ is elliptic.

Result 2. If $\Phi(1)$ is elliptic, then $\mu_{\mathrm{CZ}}(\Phi)$ is an odd integer. Equivalently, if $\mu_{\mathrm{CZ}}(\Phi)$ is an even integer, then $\Phi(1)$ is hyperbolic.

Result 3. If $\Phi(1)$ is hyperbolic, then $\Delta(\Phi) = \mu_{\mathrm{CZ}}(\Phi)$. Moreover, the eigenvalues of $\Phi(1)$ are positive if and only if $\mu_{\mathrm{CZ}}(\Phi)$ is even.

Remark 8. In the main theorems of this paper, we assume that $\Phi(1)$ is strongly nondegenerate and, hence, $-1 \notin \sigma(\Phi(1))$.

For every $x \in \mathcal{P}(\theta_t)$, there is a well-defined mean index and, when x is nondegenerate, the Conley–Zehnder index of x is also well-defined. In fact, for $\tilde{x} \in \widetilde{\mathcal{L}}\Sigma$, there is a well-defined, up to homotopy, $\mathbb{C}$-vector bundle trivialization of $x^*T\Sigma$, and the linearized flow along $x \in \mathcal{P}(\theta_t)$,

$$d\phi_t : T_{x(0)}\Sigma \to T_{x(t)}\Sigma,$$

can be viewed as a symplectic path,

$$\Phi : [0, 1] \to \mathrm{Sp}(2). \tag{2-6}$$

Then the mean index Δ_{ϕ_t} is defined by

$$\Delta_{\{\phi_t\}}(\tilde{x}) := \Delta(\Phi)$$

and the Conley–Zehnder index μ_{CZ} is defined, for nondegenerate orbits x, by

$$\mu_{\mathrm{CZ}}(\tilde{x}) := \mu_{\mathrm{CZ}}(\Phi).$$

Since Σ is aspherical, the indices are independent of the lift $\tilde{x}$ of x and we write

$$\Delta_{\{\phi_t\}}(x) \quad \text{and} \quad \mu_{\mathrm{CZ}}(x)$$

for the mean index and the Conley–Zehnder index of x, respectively.

These indices satisfy the properties

$$\Delta_{\{\phi_t^k\}}(x^k) = k\Delta_{\{\phi_t\}}(x) \tag{2-7}$$

and

$$|\Delta_{\{\phi_t\}}(x) - \mu_{\mathrm{CZ}}(x)| < 1 \quad \text{(when } x \text{ is nondegenerate).} \tag{2-8}$$

Furthermore, we say that a nondegenerate periodic orbit $x \in \mathcal{P}(\theta_t)$ is elliptic, or hyperbolic, if the endpoint of the associated symplectic path as in (2-6) is elliptic, or hyperbolic, respectively. Moreover, the stated results hold for a periodic orbit x if they are satisfied by the corresponding symplectic path Φ, as in (2-6). For instance, the claim for orbits corresponding to the first part of Result 1 enunciates that if x is an elliptic orbit for ϕ, then its mean index is not zero.

Remark 9 (noncontractible orbits). Let ζ be a free homotopy class of maps $S^1 \to \Sigma$. Fix a reference loop z in ζ and a trivialization of TM along z. They give rise to a well defined, up to homotopy, $\mathbb{C}$-vector bundle trivialization of x^*TM for every $x \in \mathcal{L}_\zeta M$ and, for a 1-periodic orbit of ϕ, the linearized flow along x,

$$d\phi_t : T_{x(0)}M \to T_{x(t)}M,$$

can be viewed as a symplectic path $\Phi : [0, 1] \to \mathrm{Sp}(2n)$. Consider the abelian principal covering $\widetilde{\mathcal{L}}_\zeta \Sigma$ with structure group

$$\Gamma_\zeta := \frac{\pi_1(\mathcal{L}_\zeta \Sigma)}{\ker([\bar{\theta}])},$$

where $[\bar{\theta}] : \pi_1(\mathcal{L}_\zeta \Sigma) \to \mathbb{R}$. The mean index and the Conley–Zehnder index are defined as above and, since Σ is atoroidal, in this case the indices are also independent of the lifts.

2D. ***The Floer–Novikov homology.*** In this section, we revisit the definition of the Floer–Novikov homology for contractible nondegenerate periodic orbits.

Consider a smooth almost complex structure J on Σ compatible with ω, i.e., such that

$$g(X, Y) := \omega(X, JY)$$

defines a Riemannian metric on Σ. We will denote by $\mathcal{J}$ the set of almost complex structures compatible with ω. Choose $J \in \mathcal{J}$ and let $\tilde{g}$ denote the induced weak Riemannian metric on $\mathcal{L}\Sigma$ given by

$$\tilde{g}(X_x, Y_x) = \int_{S^1} g(X_x(t), Y_x(t))\, dt,$$

where X_x and Y_x are vector fields along x. A gradient flow line is a mapping $u : \mathbb{R} \times S^1 \to \Sigma$ satisfying

$$\partial_s u(s, t) + J\big(\partial_t u(s, t) - X_t(u(s, t))\big) = 0. \tag{2-9}$$

The maps $u : \mathbb{R} \to \mathcal{L}\Sigma$ which satisfy (2-9) with boundary conditions

$$\lim_{s \to \pm\infty} \tilde{u}(s, t) = \tilde{x}_\pm(t), \tag{2-10}$$

for some lift $\tilde{u} : \mathbb{R} \to \widetilde{\mathcal{L}}\Sigma$ of u, can be seen as connecting orbits between $\tilde{x}_-$ and $\tilde{x}_+$. We denote by $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ the space of finite energy solutions of (2-9) and (2-10). The energy of a connecting orbit in this space is given by

$$E(u) := \int_{\mathbb{R}\times S^1} |\partial_s u|_g^2 ds dt = \mathcal{A}_{\{\phi_t\}}(\tilde{x}_+) - \mathcal{A}_{\{\phi_t\}}(\tilde{x}_-)$$

when x_- and x_+ are nondegenerate. The space $\mathcal{M}(\tilde{x}_-, \tilde{x}_+)$ is a smooth manifold of dimension $\mu_{\mathrm{CZ}}(x_+) - \mu_{\mathrm{CZ}}(x_-)$. It admits a natural $\mathbb{R}$-action given by reparametrization. For nondegenerate $x, y \in \mathcal{P}(\theta_t)$ such that

$$\mu_{\mathrm{CZ}}(x) - \mu_{\mathrm{CZ}}(y) = 1,$$

we have that $\mathcal{M}(\tilde{x}, \tilde{y})/\mathbb{R}$ is finite and set

$$n_2(\tilde{x}, \tilde{y}) := \#\mathcal{M}(\tilde{x}, \tilde{y})/\mathbb{R} \quad \text{modulo } 2.$$

Denote by $\mathcal{P}_k$ the set of elements $\tilde{x} \in \widetilde{\mathcal{L}}\Sigma$ such that $x \in \mathcal{P}(\theta_t)$ and $\mu_{\mathrm{CZ}}(x) = k$. Consider the chain complex where the k-th chain group C_k consists of all formal sums

$$\sum \xi_{\tilde{x}} \cdot \tilde{x}$$

with $\tilde{x} \in \mathcal{P}_k$, $\xi_{\tilde{x}} \in \mathbb{Z}_2$ and such that, for all $c \in \mathbb{R}$, the set

$$\{\tilde{x} \mid \xi_{\tilde{x}} \neq 0,\ \mathcal{A}_{\{\phi_t\}}(\tilde{x}) > c\}$$

is finite. Denote by

$$\Lambda_\theta = \Lambda(\Gamma, [\bar{\theta}], \mathbb{F})$$

the Novikov ring associated with the group Γ (defined in (2-2)) and the weighting homomorphism $[\bar{\theta}]$ (defined in (2-1)) with values in the field $\mathbb{F} = \mathbb{Z}_2$; see [Hofer and Salamon 1995, Section 4]. The chain group C_k is a torsion-free module over the algebra Λ_θ. The rank of this module is the number of elements of $\mathcal{P}_k$; see [Lê and Ono 1995, Lemma 4.2]. For a generator $\tilde{x}$ in C_k, the boundary operator ∂_k is defined as

$$\partial_k(\tilde{x}) = \sum_{\mu_{\mathrm{CZ}}(\tilde{y}) = k-1} n_2(\tilde{x}, \tilde{y})\tilde{y}.$$

Since ∂_k is invariant under the action of Γ, we extend ∂_k as a Λ_θ-linear map from C_k to C_{k-1}. The boundary operator ∂ satisfies $\partial^2 = 0$. The homology groups

$$\mathrm{HFN}_k(\{\phi_t\}, J) = \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}$$

are called the Floer–Novikov homology groups and they are graded Λ_θ-modules.

Moreover, this homology is invariant under exact deformations of the closed form θ_t (see [Lê and Ono 1995, Theorem 4.3]) and hence two paths with the same flux have isomorphic associated Floer–Novikov homology groups.

Remark 10 (Floer–Novikov homology for noncontractible orbits). As mentioned in the introduction, the Floer–Novikov homology is defined for orbits which lie in some free homotopy class ζ. Here, we refer the reader to [Burghelea and Haller 2001] for the details and point out that the Conley–Zehnder index defined in that paper when $\zeta = 0$ may result in a shift of the standard grading of the Floer–Novikov homology by an even integer; see [Burghelea and Haller 2001, Remark 3.4].

3. Proofs of Theorems 1.1 and 1.2

In this section, we construct a flow ψ^t with exactly $2g-2$ hyperbolic fixed points and no other periodic orbits on a surface Σ with genus $g \geq 2$. This proves Theorem 1.1, and yields the Floer–Novikov homology of a symplectomorphism satisfying property (1-1) and hence also establishes Theorem 1.2.

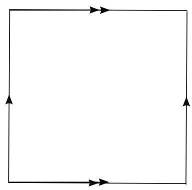

Figure 1. Torus: $[0, 1] \times [0, 1]$.

3A. ***Construction of a symplectic flow with exactly $2g-2$ hyperbolic fixed points and no other periodic orbits.*** We start with the case when Σ is a surface of genus $g = 2$. The construction has three steps.

In the first step, take two 2-tori $\mathbb{T}_1$ and $\mathbb{T}_2$ and the linear flow ϕ_i^t on each torus $\mathbb{T}_i$ $(i = 1, 2)$:

$$\phi_i^t(x_i, y_i) = (tu_i x_i, tv_i y_i) \quad \text{with} \quad u_i \neq 0 \quad \text{and} \quad \frac{v_i}{u_i} \in \mathbb{R} \setminus \mathbb{Q}, \quad i = 1, 2.$$

Here x_i, y_i are the coordinates on $\mathbb{T}_i = \mathbb{R}^2/\mathbb{Z}^2$, $i = 1, 2$.

Representing each torus by a square $[0, 1] \times [0, 1]$, where the sides $\{0\} \times [0, 1]$ and $[0, 1] \times \{0\}$ are identified with $\{1\} \times [0, 1]$ and $[0, 1] \times \{1\}$, respectively (see Figure 1), consider a square R_1 in $\mathbb{T}_1$ such that two parallel sides are segments of a linear flow line (of ϕ_1^t) with length $\varepsilon > 0$ and a square L_2 in $\mathbb{T}_2$ where two parallel sides are segments of a linear flow line (of ϕ_2^t) with length $\varepsilon > 0$ (see Remark 11). In Figure 2, there are three pictures. The two on the left refer to torus $\mathbb{T}_1$. The first one represents a flow line of ϕ_1^t (with slope v_1/u_1) and the second one shows the square R_1 where two of its sides are segments of the represented flow line. The picture on the right refers to the torus $\mathbb{T}_2$ where a flow line of ϕ_2^t (with slope v_2/u_2) is represented together with the square L_2.

Remark 11. In the current case, where $g = 2$, ε is small enough so that the squares R_1 and L_2 are inside the square $[0, 1] \times [0, 1]$. See Remark 16 for the general case.

Remark 12. In order to distinguish the boundaries of the squares from the interiors of the squares, we denote by R_1 and L_2 their boundaries and by $\mathring{R}_1$ and $\mathring{L}_2$ their interiors.

In the second step, consider a surface P obtained by a homotopy between a circle (of radius $\varepsilon/4$) and a square (with side length equal to ε) and a surface U

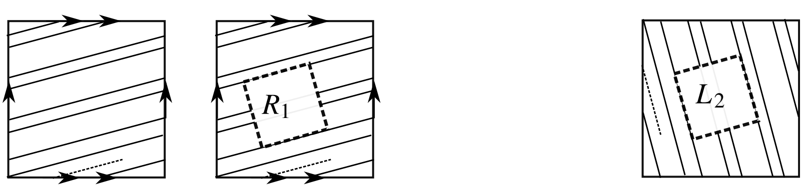

Figure 2. Tori $\mathbb{T}_1$ (left) and $\mathbb{T}_2$ (right) and linear flow lines.

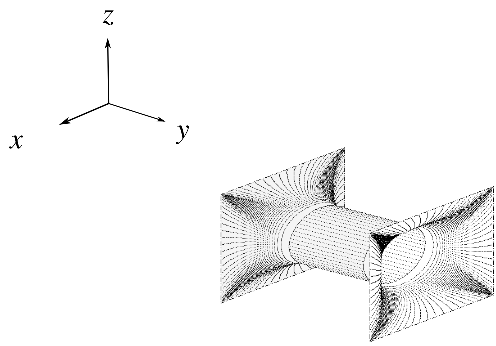

Figure 3. Surface U.

defined piecewise, in the middle, by a (horizontal) cylinder with radius $\varepsilon/4$ together with a surface P at each end (with circles identified) as shown in Figure 3. For $(x, y, z) \in U$, we have $-\varepsilon/2 \le x, z \le \varepsilon/2$ and $-1 \le y \le 1$. The boundary of U is the disjoint union of two squares S^L and S^R which lie in the planes $\{y = -1\}$ and $\{y = 1\}$, respectively. Let $H : U \to [-\varepsilon/2, \varepsilon/2] \subset \mathbb{R}$ be a smooth function defined by

$$H(x, y, z) = (1 - \beta(y))yz + \beta(y)z, \quad \text{for } (x, y, z) \in U, \tag{3-1}$$

where $\beta : [-1, 1] \to [0, 1]$ is a smooth function which is 0 when y is in $(-c, c)$, 1 when y is in $[-1, -1+d) \cup (1-d, 1]$ and strictly monotone in $(-1+d, -c) \cup (c, 1-d)$ with $0 < c < 1-d$, $d < 0$. (See Figure 4 and Remark 14 for the choice of the real numbers c and d.)

The Hamiltonian flow lines of H are depicted in Figure 5. The picture on the left shows the Hamiltonian flow lines in U when y is *near* -1, in the middle are the Hamiltonian flow lines in U when y is *near* 0 and on the right are the Hamiltonian flow lines in U when y is *near* 1.

Remark 13. Here, "y is near -1" means that $y \in [-1, -1+d)$. Similarly, "y is near 0" means $y \in (-c, c)$ and "y is near 1" means $y \in (1-d, 1]$.

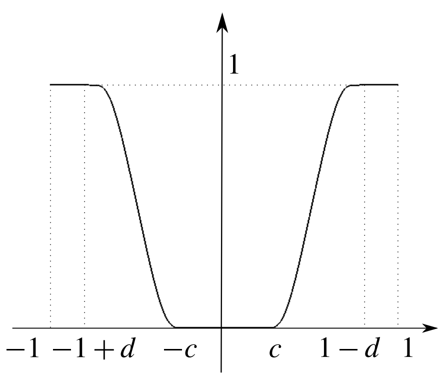

Figure 4. Function β.

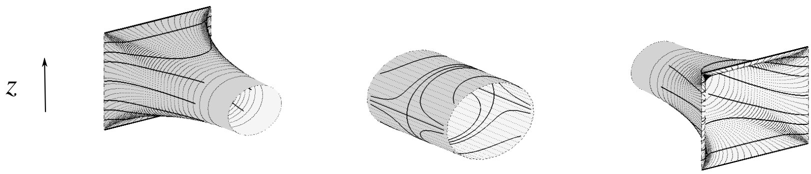

Figure 5. Flow lines of the Hamiltonian H on the surface U.

In the last step,

- cut off $\mathring{R}_1$ from $\mathbb{T}_1$ and $\mathring{L}_2$ from $\mathbb{T}_2$,
- identify R_1 with S^L so that the sides of R_1 given by segments of a flow line correspond to the sides of S^L determined by $z = \pm\varepsilon/2$ (see Figure 6), and
- identify L_2 with S^R so that the sides of L_2 given by segments of a flow line correspond to the sides of S^R determined by $z = \pm\varepsilon/2$.

This construction yields a closed surface Σ of genus 2 with a symplectic flow $\psi^t : \Sigma \to \Sigma$ which coincides with

- the linear flow ϕ_1^t on $\mathbb{T}_1 \setminus \mathring{R}_1$,
- the linear flow ϕ_2^t on $\mathbb{T}_2 \setminus \mathring{L}_2$,
- the Hamiltonian flow of H on U.

Each flow line of ψ^t lies entirely either

(1) on the circle $U \cap \{y = 0\}$,
(2) on $\mathbb{T}_1 \setminus \mathring{R}_1 \cup (U \cap \{y < 0\}) =: V^-$, or
(3) on $\mathbb{T}_2 \setminus \mathring{L}_2 \cup (U \cap \{y > 0\}) =: V^+$.

We observe that a flow line of ψ^t does not intersect both V^- and V^+. In (1), ψ^t has two hyperbolic fixed points and no other periodic orbits. In (2), ψ^t has no periodic orbits. In fact, by construction, when a flow line of ψ^t given by ϕ_1^t reaches R_1, it will either

- stay on U and converge to one of the hyperbolic fixed points, or
- cross R_1 again after some time and continue in the same flow line of ϕ_1^t when exiting $\mathbb{T}_1 \setminus \mathring{R}_1$ (since at S^L the Hamiltonian is given by the height function).

This property together with the fact that ϕ_1^t is an irrational linear flow imply the nonexistence of (long) periodic orbits of ψ^t on V^-. Case (3) is similar to (2) and there are no periodic orbits of ψ^t on V^+.

Remark 14. In the function β, the real numbers c and d are selected so that $c < 0.5 < 1 - d$ and c and $1 - d$ are close enough so that the flow ψ^t has the above properties. For instance, we may choose $c = d = 0.4$.

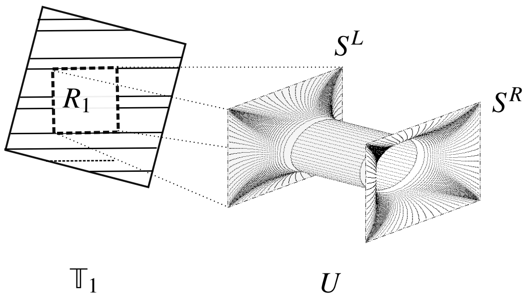

Figure 6. Identification of R_1 with S^L.

Therefore, we have obtained a symplectic flow on Σ with exactly two hyperbolic fixed points, no other periodic orbits. Let us see that the flux of this symplectic flow is given by (u_1, v_1, u_2, v_2).

Recall that the fundamental group $\pi_1(\Sigma)$ of a surface of genus 2 is given by the group

$$\langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1\rangle$$

with generators a_1, b_1, a_2, b_2 and relation $[a_1, b_1][a_2, b_2] = 1$, where $[a, b] = aba^{-1}b^{-1}$ is the commutator of a and b. Consider the following loops in Σ:

- γ_1, such that $[\gamma_1] = [a_1]$ in $\pi_1(\Sigma)$ and it corresponds to a vertical line in $\mathbb{T}_1$ such that $\psi^t \circ \gamma_1$ does not intersect $\mathring{R}_1 \cup R_1$ for all $t \in [0, 1]$,
- γ_2, such that $[\gamma_2] = [b_1]$ in $\pi_1(\Sigma)$ and it corresponds to a horizontal line in $\mathbb{T}_1$ such that $\psi^t \circ \gamma_2$ does not intersect $\mathring{R}_1 \cup R_1$ for all $t \in [0, 1]$,
- γ_3, such that $[\gamma_3] = [a_2]$ in $\pi_1(\Sigma)$ and it corresponds to a vertical line in $\mathbb{T}_2$ such that $\psi^t \circ \gamma_3$ does not intersect $\mathring{L}_2 \cup L_2$ for all $t \in [0, 1]$, and
- γ_4, such that $[\gamma_4] = [b_2]$ in $\pi_1(\Sigma)$ and it corresponds to a horizontal line in $\mathbb{T}_2$ such that $\psi^t \circ \gamma_4$ does not intersect $\mathring{L}_2 \cup L_2$ for all $t \in [0, 1]$.

Remark 15. We may have to take $\varepsilon > 0$ (in the definitions of the squares R_1 and L_2) sufficiently small so that the above conditions on the loops γ_i are satisfied.

The area swept by γ_i $(i = 1, 2)$ under $\psi^1_{(u_1,v_1,u_2,v_2)}$ is the area swept by γ_i under ϕ^t_1 and hence it is u_1 when $i = 1$ and v_1 when $i = 2$. The area swept by γ_i $(i = 3, 4)$ under $\psi^1_{(u_1,v_1,u_2,v_2)}$ is the area swept by γ_i under ϕ^t_2 and hence it is u_2 when $i = 3$ and v_2 when $i = 4$. Therefore, the flux of the symplectic flow $\psi^t_{(u_1,v_1,u_2,v_2)}$ $(t \in [0, 1])$ is (u_1, v_1, u_2, v_2); recall Remark 6.

The general case, where Σ is a surface of genus $g \geq 2$, is similar to the case where $g = 2$. Take g copies of 2-tori, $\mathbb{T}_1, \ldots, \mathbb{T}_g$, and the linear flow on each $\mathbb{T}_i$:

$$\phi^t_i(x_i, y_i) = (tu_ix_i, tv_iy_i) \quad \text{with} \quad u_i \neq 0 \quad \text{and} \quad \frac{v_i}{u_i} \in \mathbb{R} \setminus \mathbb{Q}, \quad i = 1, \ldots, g.$$

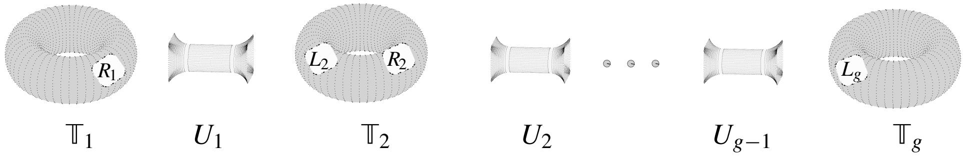

Figure 7. Construction of the surface with genus g.

On each torus $\mathbb{T}_i$ (viewed as a square, as above), consider two squares R_i and L_i such that

- $\mathring{R}_g \cup R_g = \varnothing$ and $\mathring{L}_1 \cup L_1 = \varnothing$,
- $\mathring{R}_i \cup R_i$ and $\mathring{L}_i \cup L_i$ are disjoint,
- two parallel sides of R_i are segments of a flow line of ϕ_i^t in $\mathbb{T}_i$ $(i \neq g)$,
- two parallel sides of L_i are segments of a flow line of ϕ_i^t in $\mathbb{T}_i$ $(i \neq 1)$, and
- the length of the sides of each square is ε.

Remark 16. In the general case, where $g \geq 2$, ε is small enough so that

$$(\mathring{R}_i \cup R_i)\dot{\cup}(\mathring{L}_i \cup L_i)$$

is inside the square $[0, 1] \times [0, 1]$.

Let U_i, with $i = 1, \ldots, g-1$ be $g-1$ copies of the surface U and the corresponding functions $H_i : U_i \to [-\varepsilon/2, \varepsilon/2] \subset \mathbb{R}$ defined as in (3-1). Much as in the case where $g = 2$, we denote the boundary components of U_i by S_i^L and S_i^R. For each $i = 1, \ldots, g$ (see Figure 7),

- cut off $\mathring{R}_i$ and $\mathring{L}_i$ from $\mathbb{T}_i$,
- identify R_i with S_i^L so that the sides of R_i given by segments of a flow line correspond to the sides of S_i^L determined by $z = \pm\varepsilon/2$, and
- identify L_i with S_i^R so that the sides of L_i given by segments of a flow line correspond to the sides of S_i^R determined by $z = \pm\varepsilon/2$.

We have thus obtained a closed surface Σ with genus $g \geq 2$ and a symplectic flow on Σ

$$\psi^t_{(u_1, v_1, \ldots, u_g, v_g)} : \Sigma \to \Sigma$$

which coincides with

- the linear flow ϕ_i^t on $\mathbb{T}_i \setminus (\mathring{R}_i \cup \mathring{L}_i)$, $i = 1, \ldots, g$,
- the Hamiltonian flow of H_i on U_i, $i = 1, \ldots, g-1$.

Arguing as in the case $g = 2$, we obtain Theorem 1.1.

3B. ***The Floer–Novikov homology of symplectomorphisms satisfying the flux condition (1-1).*** Consider $\phi \in \mathrm{Symp}_0(\Sigma, \omega)$ such that

$$\mathrm{Flux}(\phi) = (u_1, v_1, \ldots, u_g, v_g) \text{ with } u_i \neq 0 \text{ and } \frac{v_i}{u_i} \notin \mathbb{Q} \text{ for all } i = 1, \ldots, g.$$

Then $\mathrm{Flux}(\phi) = \mathrm{Flux}(\psi^t_{(u_1,v_1,\ldots,u_g,v_g)}|_{t\in[0,1]})$, where $\psi^t_{(u_1,v_1,\ldots,u_g,v_g)}$ is the symplectic flow constructed in Section 3A with flux equal to $(u_1, v_1, \ldots, u_g, v_g)$.

The symplectic flow $\psi^t_{(u_1,v_1,\ldots,u_g,v_g)}$ has $2g-2$ hyperbolic fixed points. Then the mean index and the Conley–Zehnder index of the fixed points are 0. Since there are no other periodic orbits, we have that C_0 is the only nontrivial group of the (Floer–Novikov) chain complex and it is generated by $2g-2$ fixed points. Hence, the Floer–Novikov homology of

$$\psi = \psi^1_{(u_1,v_1,\ldots,u_g,v_g)}$$

is given by

$$\mathrm{HFN}_r(\psi) = \begin{cases} \mathbb{F}^{2g-2} \otimes \Lambda_\theta & \text{if } r = 0, \\ 0 & \text{if } r \neq 0. \end{cases}$$

Since $\mathrm{Flux}(\phi) = \mathrm{Flux}(\psi^t_{(u_1,v_1,\ldots,u_g,v_g)}|_{t\in[0,1]})$, Theorem 1.2 follows by the comment on page 31 after the definition of the Floer–Novikov homology.

Remark 17 (the noncontractible case of Theorem 1.2). Since $\psi^t_{(u_1,v_1,\ldots,u_g,v_g)}$ has no noncontractible periodic orbits, the Floer–Novikov homology for noncontractible orbits of a strongly nondegenerate ϕ is $\mathrm{HFN}_*(\phi, \zeta) = 0$ for any nontrivial free homotopy class of loops ζ.

4. Proofs of Theorems 1.3–1.5

4A. ***Proofs of Theorems 1.3 and 1.4.*** Theorem 1.3 follows from Theorem 1.4 and the first case of Result 1. Let us then prove Theorem 1.4.

Assume ϕ has finitely many fixed points. Let S be the finite set of fixed points y of ϕ such that $\Delta_{\{\phi_t\}}(y) \neq \Delta_{\{\phi_t\}}(x_0)$. If $S \neq \varnothing$, then define

$$\tau_0 := \min\{k > 1 : k|\Delta_{\{\phi_t\}}(x_0) - \Delta_{\{\phi_t\}}(y)| > 3 \text{ for all } y \in S\},$$

otherwise take $\tau_0 := 2$.

The proof goes by contradiction. Let τ be a prime integer greater than τ_0 such that all τ-periodic points are iterations of fixed points. We show that, with these assumptions, x_0^τ, the τ-th iteration of x_0, contributes nontrivially to the Floer–Novikov homology in degree $\mu := \mu_{\mathrm{CZ}}(x_0^\tau) \neq 0$ which contradicts Theorem 1.2.

If x_0^τ connects to y^τ, some τ-th iteration of a fixed point y of ϕ, by a solution of the Floer–Novikov Equation (2-9), then

$$|\mu_{\mathrm{CZ}}(x_0^\tau) - \mu_{\mathrm{CZ}}(y^\tau)| = 1. \tag{4-1}$$

If $y \in S$, then

$$\tau |\Delta_{\{\phi_t\}}(x_0) - \Delta_{\{\phi_t\}}(y)| > 3 \tag{4-2}$$

and we obtain the following contradiction:

$$1 = |\mu_{\mathrm{CZ}}(x_0^\tau) - \mu_{\mathrm{CZ}}(y^\tau)| \geq \tau |\Delta_{\{\phi_t\}}(x_0) - \Delta_{\{\phi_t\}}(y)| - 2 > 1,$$

where the first inequality follows from (2-7) and (2-8) and the last inequality follows from (4-2). If $y \notin S$, then $\Delta_{\{\phi_t\}}(x_0^\tau) = \Delta_{\{\phi_t\}}(y^\tau)$ by (2-7) and $\mu_{\mathrm{CZ}}(x_0^\tau) = \mu_{\mathrm{CZ}}(y^\tau)$ by (2-8) which contradicts (4-1). Hence, x_0^τ is not connected to any y^τ which implies that $\mathrm{HFN}_\mu(\phi^\tau) \neq 0$ where $\mu := \mu_{\mathrm{CZ}}(x_0^\tau)$.

If μ were 0, then x_0^τ would be hyperbolic (by Result 2). Then we would have that $\Delta_{\{\phi_t^\tau\}}(x_0^\tau) = \mu_{\mathrm{CZ}}(x_0^\tau) = 0$ (by Result 3) which implies, by (2-7), that $\Delta_{\{\phi_t\}}(x_0) = 0$. This contradicts our assumption on x_0. Therefore, $\mu \neq 0$ and we obtained the wanted contradiction. □

Remark 18 (the noncontractible cases of Theorems 1.3 and 1.4).

• In Theorem 1.3, the result still holds true if the elliptic periodic orbit corresponding to x_0 is noncontractible. In this case, we choose τ as above, fix the free homotopy class $\tau\zeta$, where ζ is the free homotopy class of the loop corresponding to x_0, consider x_0^τ as the reference loop in $\tau\zeta$ and work with the (noncontractible) Floer–Novikov homology $\mathrm{HFN}(\phi^\tau, \tau\zeta)$. (Recall Remarks 9 and 10.) By Theorem 1.2 and Remark 10, the Floer–Novikov homology $\mathrm{HFN}_*(\phi^\tau, \tau\zeta)$ is 0 when $*$ is an odd integer. Since x_0 is elliptic, its Conley–Zehnder index $\mu_{\mathrm{CZ}}(x_0)$ is odd (by Result 2). Moreover, using the above argument, x_0^τ is not connected to any y^τ which implies that x_0^τ contributes nontrivially to the Floer–Novikov homology in some odd degree. If the Conley–Zehnder index $\mu_{\mathrm{CZ}}(x_0^\tau)$ were even, then x_0^τ would be hyperbolic and $\mu_{\mathrm{CZ}}(x_0^\tau) = \Delta(x_0^\tau) = \tau\Delta(x_0)$ would be even. Since τ is odd, the mean index $\Delta(x_0)$ would also be even and, by (2-8), the Conley–Zehnder index $\mu_{\mathrm{CZ}}(x_0) = \Delta(x_0)$ would be even. Hence x_0 would be hyperbolic contradicting the hypothesis on x_0. The result then follows.

• In Theorem 1.4, if the fixed point x_0 with nonzero mean index corresponds to a noncontractible periodic orbit with nontrivial homotopy class ζ and its τ-th iterations, with τ a prime integer, lie in nontrivial homotopy classes $\tau\zeta$, then ϕ has infinitely many periodic points. These points correspond to periodic orbits which lie in the free homotopy classes formed by iterations of the orbit corresponding to x_0. In this case, the proof is essentially the same as in the contractible case. (The last paragraph is not needed.) Recall Remark 17.

4B. *Proof of Theorem 1.5*. Suppose the number of fixed points of ϕ is greater than $2g-2$. By (1-2), there exist $2g-2$ fixed points $x_1, \dots, x_{2g-2}$ of ϕ which contribute nontrivially to the Floer–Novikov homology of ϕ. If there exists $j \in \{1, \dots, 2g-2\}$

such that $\Delta_{\{\phi_t\}}(x_j) \neq 0$, then, by Theorem 1.4, the result follows. If not, then $\Delta_{\{\phi_t\}}(x_i) = 0$ for all $i = 1, \dots, 2g-2$. Take a fixed point x such that $x \neq x_i$ $(i = 1, \dots, 2g-2)$. Either $\mu_{CZ}(x) = 0$, $\mu_{CZ}(x) = 1$ or $\mu_{CZ}(x) = 2$.

Let us first consider the case $\mu_{CZ}(x) = 0$. By (1-2), there exists $y \in C_1$ such that y is connected to x by a solution of the Floer–Novikov equation (2-9). Then, either y is elliptic or y is hyperbolic. If y is elliptic, the result follows by Theorem 1.3. If y is hyperbolic, then $\Delta_{\{\phi_t\}}(y) = \mu_{CZ}(y) = 1 \neq 0$ and the result follows by Theorem 1.4.

Assume now $\mu_{CZ}(x) = 1$. Then, the result follows by the same argument used for y in the previous step.

Finally, assume $\mu_{CZ}(x) = 2$. Then, by (2-8), we have that $\Delta_{\{\phi_t\}}(x) \neq 0$ and the result follows by Theorem 1.4. □

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