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MATT CLAY AND CAGLAR UYANIK

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Given isometric actions by a group G on finitely many δ-hyperbolic metric spaces, we provide a sufficient condition that guarantees the existence of a single element in G that is hyperbolic for each action. As an application we prove a conjecture of Handel and Mosher regarding relatively fully irreducible subgroups and elements in the outer automorphism group of a free group.

1. Introduction

A δ-*hyperbolic space* is a geodesic metric space where geodesic triangles are δ-*slim*: the δ-neighborhood of any two sides of a geodesic triangle contains the third side. Such spaces were introduced by Gromov [1987] as a coarse notion of negative curvature for geodesic metric spaces and since then have evolved into an indispensable tool in geometric group theory.

There is a classification of isometries of δ-hyperbolic metric spaces analogous to the classification of isometries of hyperbolic space $\mathbb{H}^n$ into elliptic, hyperbolic and parabolic. Of these, hyperbolic isometries have the best dynamical properties and are often the most desired. For example, typically they can be used to produce free subgroups in a group acting on a δ-hyperbolic space [Gromov 1987, 5.3B]; see also [Bridson and Haefliger 1999, III.Γ.3.20]. Another application is to show that a certain element does not have fixed points in its action on some set. Indeed, if the set naturally sits inside a δ-hyperbolic metric space and the given element acts as a hyperbolic isometry then it has no fixed points (in a strong sense). This strategy has been successfully employed for the curve complex of a surface and for the free factor complex of a free group by several authors [Clay et al. 2012; Clay and Pettet 2012; Dowdall and Taylor 2018; Fujiwara 2015; Gültepe 2017; Horbez 2016; Mangahas 2013; Taylor 2014].

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We consider the situation of a group acting on finitely many δ-hyperbolic spaces and produce a sufficient condition that guarantees the existence of a single element in the group that is a hyperbolic isometry for each of the spaces. Of course, a necessary condition is that for each of the spaces there is some element of the group that is a hyperbolic isometry. Thus we are concerned with when we may reverse the quantifiers: $\forall\exists \rightsquigarrow \exists\forall$. Our main result is the following theorem.

Theorem 5.1. *Suppose that $\{X_i\}_{i=1,\dots,n}$ is a collection of δ-hyperbolic spaces, G is a group and for each $i = 1, \dots, n$ there is a homomorphism $\rho_i : G \to \mathrm{Isom}(X_i)$ such that*

(1) *there is an element $f_i \in G$ such that $\rho_i(f_i)$ is hyperbolic; and*

(2) *for each $g \in G$, either $\rho_i(g)$ has a periodic orbit or is hyperbolic.*

Then there is an $f \in G$ such that $\rho_i(f)$ is hyperbolic for all $i = 1, \dots, n$.

Remark 1.1. Since the completion of this paper we have been alerted to the fact that Theorem 5.1 should follow from random walk techniques developed in [Björklund and Hartnick 2011; Maher and Tiozzo 2016]. Here we provide an elementary and constructive proof.

Essentially, we assume that there are no parabolic isometries and that elliptic isometries are relatively tame.

As an application of our main theorem we prove a conjecture of Handel and Mosher which involves exactly the same type of quantifier reversing: $\forall\exists \rightsquigarrow \exists\forall$. Consider a finitely generated subgroup $\mathcal{H} < \mathrm{IA}_N(\mathbb{Z}/3) < \mathrm{Out}(F_N)$ and a maximal $\mathcal{H}$-invariant filtration of F_N, the free group of rank N, by free factor systems,

$$\varnothing = \mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \cdots \sqsubset \mathcal{F}_m = \{[F_N]\}$$

(see Section 6). Handel and Mosher [2013a, Theorem D] prove that for each multi-edge extension $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ there exists some $\varphi_i \in \mathcal{H}$ that is irreducible with respect to $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$. They conjecture that there exists a single $\varphi \in \mathcal{H}$ that is irreducible with respect to each multi-edge extension $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$. We show that this is indeed the case.

Theorem 6.6. *For each finitely generated subgroup $\mathcal{H} < \mathrm{IA}_N(\mathbb{Z}/3) < \mathrm{Out}(F_N)$ and each maximal $\mathcal{H}$-invariant filtration by free factor systems,*

$$\varnothing = \mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \cdots \sqsubset \mathcal{F}_m = \{[F_N]\},$$

there is an element $\varphi \in \mathcal{H}$ such that for each $i = 1, \dots, m$ such that $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ is a multi-edge extension, φ is irreducible with respect to $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$.

Our paper is organized as follows. Section 2 contains background on δ-hyperbolic spaces and their isometries. In Section 3 we generalize a construction from [Clay and Pettet 2012] that is useful in constructing hyperbolic isometries. This result is Theorem 3.1. We examine certain cases that will arise in the proof of the main theorem to see how to apply Theorem 3.1 in Section 4. The proof of Theorem 5.1 constitutes Section 5. The application to $\mathrm{Out}(F_N)$ appears in Section 6.

2. Background on δ-hyperbolic spaces

In this section we recall basic notions and facts about δ-hyperbolic spaces, their isometries and their boundaries. The reader familiar with these topics can safely skip this section, with the exception of Definition 2.8. References for this section are [Alonso et al. 1991; Bridson and Haefliger 1999; Kapovich and Benakli 2002].

2A. *δ-hyperbolic spaces.* We recall the definition of a δ-hyperbolic space given in the Introduction.

Definition 2.1. Let (X, d) be a geodesic metric space. A geodesic triangle with sides α, β and γ is *δ-slim* if for each $x \in \alpha$, there is some $y \in \beta \cup \gamma$ such that $d(x, y) \le \delta$. The space X is said to be *δ-hyperbolic* if every geodesic triangle is δ-slim.

There are several equivalent definitions that we will use in the sequel. The first of these is insize. Let Δ be the geodesic triangle with vertices x, y and z and sides α from y to z, β from z to x and γ from x to y. There exist unique points $\hat{\alpha} \in \alpha$, $\hat{\beta} \in \beta$ and $\hat{\gamma} \in \gamma$, called the *internal points* of Δ, such that

$$d(x, \hat{\beta}) = d(x, \hat{\gamma}), \quad d(y, \hat{\gamma}) = d(y, \hat{\alpha}) \quad \text{and} \quad d(z, \hat{\alpha}) = d(z, \hat{\beta}).$$

The *insize* of Δ is the diameter of the set $\{\hat{\alpha}, \hat{\beta}, \hat{\gamma}\}$.

Another notion makes use of the so-called *Gromov product*:

$$(x \,.\, y)_w = \tfrac{1}{2}(d(x, w) + d(w, y) - d(x, y)). \tag{2-1}$$

The Gromov product is said to be *δ-hyperbolic* (*with respect to $w \in X$*) if for all $x, y, z \in X$,

$$(x \,.\, z)_w \ge \min\{(x \,.\, y)_w, (y \,.\, z)_w\} - \delta.$$

Proposition 2.2 [Alonso et al. 1991, Proposition 2.1; Bridson and Haefliger 1999, III.H.1.17 and III.H.1.22]. *The following are equivalent for a geodesic metric space X*:

(1) *There is a $\delta_1 \ge 0$ such that every geodesic triangle in X is δ_1-slim, i.e., X is δ_1-hyperbolic.*

(2) *There is a $\delta_2 \ge 0$ such that every geodesic triangle in X has insize at most δ_2.*

(3) *There is a $\delta_3 \ge 0$ such that for some* (*equivalently any*) *$w \in X$, the Gromov product is δ_3-hyperbolic.*

Henceforth, when we say X is a δ-hyperbolic space we assume that δ is large enough to satisfy each of the above conditions.

2B. *Boundaries.* There is a useful notion of a boundary for a δ-hyperbolic space that plays the role of the "sphere at infinity" for $\mathbb{H}^n$. This space is defined using equivalence classes of certain sequences of points in X and the Gromov product. Fix a basepoint $w \in X$.

Definition 2.3. We say a sequence $(x_n) \subseteq X$ *converges to infinity* if $(x_i \,.\, x_j)_w \to \infty$ as $i, j \to \infty$. Two such sequences (x_n), (y_n) are equivalent if $(x_i \,.\, y_j)_w \to \infty$ as $i, j \to \infty$. The *boundary of* X, denoted ∂X, is the set of equivalence classes of sequences $(x_n) \subseteq X$ that converge to infinity.

One can show that the notion of "converges to infinity" and the subsequent equivalence relation do not depend on the choice of basepoint $w \in X$ [Kapovich and Benakli 2002]. The definition of the Gromov product in (2-1) extends to boundary points $\hat{x}, \hat{y} \in \partial X$ by

$$(\hat{x} \,.\, \hat{y})_w = \inf\{\liminf_n (x_n \,.\, y_n)_w\},$$

where the infimum is over sequences $(x_n) \in \hat{x}$, $(y_n) \in \hat{y}$. If $y \in X$ then we set

$$(\hat{x} \,.\, y)_w = \inf\{\liminf_n (x_n \,.\, y)_w\},$$

where the infimum is over sequences $(x_n) \in \hat{x}$. For $x \in X$, the Gromov product $(x \,.\, \hat{y})_w$ is defined analogously. Let $\overline{X} = X \cup \partial X$.

We will make use of the following properties of the Gromov product on $\overline{X}$.

Proposition 2.4 [Alonso et al. 1991, Lemma 4.6; Bridson and Haefliger 1999, III.H.3.17]. *Let X be a δ-hyperbolic space.*

(1) *If $x, y \in \overline{X}$ then $(x \,.\, y)_w = \infty \iff x = y \in \partial X$.*

(2) *If $\hat{x} \in \partial X$ and $(x_n) \subseteq X$ then $(\hat{x} \,.\, x_n)_w \to \infty$ as $n \to \infty \iff (x_n) \in \hat{x}$.*

(3) *If $\hat{x}, \hat{y} \in \partial X$ and $(x_n) \in \hat{x}$, $(y_n) \in \hat{y}$ then*

$$(\hat{x} \,.\, \hat{y})_w \le \liminf_n (x_n \,.\, y_n)_w \le (\hat{x} \,.\, \hat{y})_w - 2\delta.$$

(4) *If $x, y, z \in \overline{X}$ then*

$$(x \,.\, z)_w \ge \min\{(x \,.\, y)_w, (y \,.\, z)_w\} - \delta.$$

Proposition 2.5 [Alonso et al. 1991, Proposition 4.8]. *The following collection of subsets of $\overline{X}$ forms a basis for a topology*:

(1) $B(x, r) = \{y \in X \mid d(x, y) < r\}$ *for each $x \in X$ and $r > 0$.*

(2) $N(\hat{x}, k) = \{y \in \overline{X} \mid (\hat{x} \,.\, y)_w > k\}$ *for each $\hat{x} \in \partial X$ and $k > 0$.*

2C. *Isometries.* As mentioned in the Introduction, there is a classification of isometries of a δ-hyperbolic space X into elliptic, parabolic and hyperbolic; see

[Gromov 1987, 8.1.B]. We will not make use of parabolic isometries and so do not give the definition here.

Definition 2.6. An isometry $f \in \mathrm{Isom}(X)$ is *elliptic* if for any $x \in X$, the set $\{f^n x \mid n \in \mathbb{Z}\}$ has bounded diameter.

An isometry $f \in \mathrm{Isom}(X)$ is *hyperbolic* if for any $x \in X$ there is a $t > 0$ such that

$$t|m-n| \leq d(f^m x, f^n x)$$

for all $m, n \in \mathbb{Z}$. In this case, one can show the sequence $(f^n x) \subseteq X$ converges to infinity and the equivalence class it defines in ∂X is independent of $x \in X$. This point in ∂X is called the *attracting fixed point* of f. The *repelling fixed point* of f is the attracting fixed point of f^{-1} and is represented by the sequence $(f^{-n}x) \subseteq X$.

The action of a hyperbolic isometry $f \in \mathrm{Isom}(X)$ on $\overline{X}$ has "North-South dynamics."

Proposition 2.7 [Gromov 1987, 8.1.G]. *Suppose that $f \in \mathrm{Isom}(X)$ is a hyperbolic isometry and that $U_+, U_- \subset \overline{X}$ are disjoint neighborhoods of the attracting and repelling fixed points of f respectively. There exists an $N \geq 1$ such that for $n \geq N$:*

$$f^n(\overline{X} - U_-) \subseteq U_+ \quad \textit{and} \quad f^{-n}(\overline{X} - U_+) \subseteq U_-.$$

We will make use of the following definition.

Definition 2.8. Suppose X is a δ-hyperbolic space and $f, g \in \mathrm{Isom}(X)$ are hyperbolic isometries. Let A_+, A_- be the attracting and repelling fixed points of f in ∂X and let B_+ and B_- be the attracting and repelling fixed points of g in ∂X. We say f and g are *independent* if

$$\{A_+, A_-\} \cap \{B_+, B_-\} = \varnothing.$$

Hyperbolic isometries that are not independent are said to be *dependent*.

3. A recipe for hyperbolic isometries

In this section we prove the principal tool used in the proof of the main result of this article, producing a single element in the given group that is hyperbolic for each action. The idea is to start with elements f and g that are hyperbolic for different actions and then combine them into a single element $f^a g^b$ that is hyperbolic for both actions. A theorem of Clay and Pettet shows that if g does not send the attracting fixed point of f to the repelling fixed point, then $f^a g$ is hyperbolic in the first action for large enough a. We can reverse the roles to get that fg^b is hyperbolic in the second action for large enough b. In order to simultaneously work with powers for both f and g, we need a uniform version of this result. That is the content of the next theorem, which generalizes [Clay and Pettet 2012, Theorem 4.1].

Theorem 3.1. *Suppose X is a δ-hyperbolic space and $f \in \mathrm{Isom}(X)$ is a hyperbolic isometry with attracting and repelling fixed points A_+ and A_- respectively. Fix disjoint neighborhoods U_+ and U_- in $\overline{X}$ for A_+ and A_- respectively. Then there is an $M \geq 1$ such that if $m \geq M$ and $g \in \mathrm{Isom}(X)$ then $f^m g$ is a hyperbolic isometry whenever $gU_+ \cap U_- = \varnothing$.*

The proof follows along the lines of [Clay and Pettet 2012, Theorem 4.1]. In the following two lemmas we assume the hypotheses of Theorem 3.1. The first lemma is obvious in the hypothesis of [Clay and Pettet 2012, Theorem 4.1] but requires a proof in this setting.

Lemma 3.2. *Given a point $x \in U_+ \cap X$, there are constants $t > 0$ and $C \geq 0$ such that if $g \in \mathrm{Isom}(X)$ is such that $gU_+ \cap U_- = \varnothing$ then $d(x, f^m gx) \geq mt - C$ for all $m \geq 0$.*

Proof. Let $A = \{f^n x \mid n \in \mathbb{Z}\}$ and for $z \in X$ let

$$d_z = \inf\{d(x', z) \mid x' \in A\}.$$

As f is a hyperbolic isometry, there is a constant $\tau \geq 1$ such that

$$\frac{1}{\tau}|m-n| \leq d(f^m x, f^n x) \leq \tau |m-n|.$$

This shows that for any $z \in X$ the set $\pi_z = \{x' \in A \mid d(x', z) = d_z\}$ is nonempty and finite.

Claim 1. *There is a constant $D \geq 0$ such that for any $z \in X$ and $x_z \in \pi_z$,*

$$d(x, z) \geq d(x, x_z) + d(x_z, z) - D.$$

Proof of Claim 1. Fix a point $x_z \in \pi_z$ and geodesics α from x_z to x, β from z to x_z and γ from z to x. Let Δ be the geodesic triangle formed with these segments and $\hat{\alpha} \in \alpha$, $\hat{\beta} \in \beta$ and $\hat{\gamma} \in \gamma$ be the internal points of Δ. These points satisfy the equalities

$$\begin{aligned} d(z, \hat{\beta}) &= d(z, \hat{\gamma}) = a, \\ d(x, \hat{\gamma}) &= d(x, \hat{\alpha}) = b, \\ d(x_z, \hat{\alpha}) &= d(x_z, \hat{\beta}) = c. \end{aligned}$$

As the insize of geodesic triangles is bounded by δ in a δ-hyperbolic space, we have that $d(\hat{\alpha}, \hat{\beta}), d(\hat{\beta}, \hat{\gamma}), d(\hat{\gamma}, \hat{\alpha}) \leq \delta$. By the Morse lemma [Bridson and Haefliger 1999, III.H.1.7], there is a constant R, only depending on τ and δ, and a point $y \in A$ such that $d(\hat{\alpha}, y) \leq R$. Thus we have

$$d(z, y) \leq d(z, \hat{\beta}) + d(\hat{\beta}, \hat{\alpha}) + d(\hat{\alpha}, y) \leq a + \delta + R.$$

As $x_z \in \pi_z$, we have

$$a + c = d(x_z, z) \leq d(z, y) \leq a + \delta + R,$$

and so $c \leq \delta + R$. Letting $D = 2\delta + 2R$ we compute

$$\begin{aligned} d(x, z) &= a + b = (b + c) + (a + c) - 2c \\ &\geq d(x, x_z) + d(x_z, z) - D. \end{aligned}$$

□

Claim 2. *There is a constant $M_0 \in \mathbb{Z}$ such that if $z \notin U_-$ and $f^m x \in \pi_z$ then $m \geq M_0$.*

Proof of Claim 2. Let $x_z = f^m x \in \pi_z$ and without loss of generality assume that $m \leq 0$. Using the constant D from Claim 1 we have:

$$\begin{aligned} (x_z \,.\, z)_x &= \tfrac{1}{2}(d(x, x_z) + d(x, z) - d(x_z, z)) \\ &\geq d(x, x_z) - D/2. \end{aligned}$$

Suppose that $i \leq m$ and let α be a geodesic from $f^i x$ to x. The Morse lemma implies that there is a $y \in \alpha$ such that $d(x_z, y) \leq R$. Therefore,

$$\begin{aligned} d(x, x_z) + d(x_z, f^i x) &\leq d(x, y) + d(y, f^i x) + 2R \\ &= d(x, f^i x) + 2R. \end{aligned}$$

Hence for such i we have:

$$\begin{aligned} (x_z \,.\, f^i x)_x &= \tfrac{1}{2}(d(x, x_z) + d(x, f^i x) - d(x_z, f^i x)) \\ &\geq d(x, x_z) - R. \end{aligned}$$

This shows that $(x_z \,.\, A_-)_x \geq d(x, x_z) - R - 2\delta$ and so for $K = \max\{D/2, R + 2\delta\}$ we have

$$(z \,.\, A_-)_x \geq \min\{(x_z \,.\, z)_x, (x_z \,.\, A_-)_x\} - \delta \geq d(x, x_z) - K - \delta.$$

As $z \notin U_-$, the Gromov product $(z \,.\, A_-)_x$ is bounded independently of z and hence $d(x, x_z)$ is also bounded. □

Now we will finish the proof of the lemma. Fix a point $x_g \in \pi_{gx}$. Clearly we have $f^m x_g \in \pi_{f^m gx}$ for $m \geq 0$. As $gx \notin U_-$, by Claim 2 we have $x_g = f^{M_0+n} x$ for some $n \geq 0$ and therefore,

$$\begin{aligned} d(x, f^m x_g) = d(x, f^{M_0+n+m} x) &\geq d(x, f^{m+n} x) - d(x, f^{M_0} x) \\ &\geq \tfrac{1}{\tau} m - \tau |M_0|. \end{aligned}$$

As $f^m x_g \in \pi_{f^m gx}$, Claim 1 implies

$$\begin{aligned} d(x, f^m gx) &\geq d(x, f^m x_g) + d(f^m x_g, f^m gx) - D \\ &\geq \tfrac{1}{\tau} m - (\tau |M_0| + D). \end{aligned}$$

Since the constants τ, D and M_0 only depend on f, x and the open neighborhoods U_+ and U_-, the lemma is proven. □

The next lemma replaces Lemma 4.3 in [Clay and Pettet 2012] and its proof is a small modification of the proof there.

Lemma 3.3. *Fix $x \in X \cap U_+$ and for $m \geq 0$ let α_m be a geodesic connecting x to $f^m gx$. Then there is an $\epsilon \geq 0$ and $M_1 \geq 0$ such that for $m \geq M_1$ the concatenation of the geodesics $\alpha_m \cdot f^m g\alpha_m$ is a $(1, \epsilon)$-quasigeodesic.*

Proof. Let $d_m = d(x, f^m gx)$.

As $gU_+ \cap U_- = \varnothing$ we have $U_+ \cap g^{-1}U_- = \varnothing$ and so the Gromov product $(g^{-1} f^{-m} x \,.\, f^m x)_x$ is bounded independent of g and $m \geq M_1$ for some constant M_1. Indeed, by Proposition 2.5 there is a $k \geq 0$ such that $N(A_+, k) \subseteq U_+$ and $M_1 \geq 0$ such that $f^{-m}x \in U_-$ and $f^m x \in N(A_+, k{+}2\delta)$ for $m \geq M_1$. Thus, $(A_+ \,.\, g^{-1} f^{-m} x)_x \leq k$ and so $(g^{-1} f^{-m} x \,.\, f^m x)_x \leq k + \delta$ as

$$\min\{(A_+ \,.\, f^m x)_x, (g^{-1} f^{-m} x \,.\, f^m x)_x\} - \delta \leq (A_+ \,.\, g^{-1} f^{-m} x)_x \leq k,$$

for $m \geq M_1$.

By making M_1 larger, we can assume that for $m \geq M_1$ we have

$$f^m(\overline{X} - U_-) \subseteq N(A_+, k + 4\delta),$$

by Proposition 2.7. Since $gx, x \notin U_-$, we have $f^m gx, f^m x \in N(A_+, k + 4\delta)$ and so $(f^m xg \,.\, f^m x)_x \geq k + 3\delta$. Hence $(g^{-1} f^{-m} x \,.\, f^m gx)_x \leq k + 2\delta$ as

$$\min\{(g^{-1} f^{-m} x \,.\, f^m gx)_x, (f^m gx \,.\, f^m x)_x\} - \delta \leq (g^{-1} f^{-m} x \,.\, f^m x)_x \leq k + \delta.$$

Therefore for $C = k + 2\delta$ and $m \geq M_1$ we have:

$$\begin{aligned} d(x, f^m g f^m gx) &= d(g^{-1} f^{-m} x, g f^m x) \\ &\geq d(g^{-1} f^{-m} x, x) + d(x, f^m gx) - 2C \\ &= 2d_m - 2C. \end{aligned}$$

The proof now proceeds exactly as that of Lemma 4.3 in [Clay and Pettet 2012]. □

Proof of Theorem 3.1. Using Lemmas 3.2 and 3.3, the proof of Theorem 3.1 proceeds exactly like that of Theorem 4.1 in [Clay and Pettet 2012]. We repeat the argument here.

Fix $x \in U_+ \cap X$, and let $t > 0$ and $C \geq 0$ be the constants from Lemma 3.2, and $\epsilon > 0$ and $M_1 \geq 0$ be the constants from Lemma 3.3. For $m \geq M_1$ we set $L_m = d(x, f^m gx) \geq mt - C$. As in Lemma 3.3, let $\alpha_m : [0, L_m] \to X$ be a geodesic connecting x to $f^m gx$, and let $\beta_m = \alpha_m \cdot f^m g\alpha_m$. Then define a path $\gamma : \mathbb{R} \to X$ by:

$$\gamma = \cdots (f^m g)^{-1}\beta_m \bigcup_{\alpha_m} \beta_m \bigcup_{f^m g\alpha_m} f^m g\beta_m \bigcup_{(f^m g)^2\alpha_m} (f^m g)^2 \beta_m \cdots$$

See Figure 1.

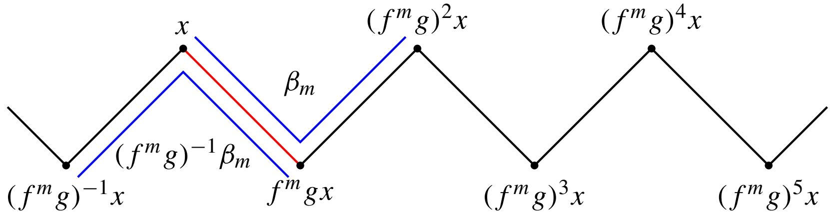

Figure 1. The path γ in the proof of Theorem 3.1.

By Lemma 3.3, γ is an L_m-local $(1, \epsilon)$-quasigeodesic and hence for m large enough, γ is a (λ', ϵ')-quasigeodesic from some $\lambda' \geq 1$ and $\epsilon' \geq 0$; see [Bridson and Haefliger 1999, III.H.1.7 and III.H.1.13] or [Clay and Pettet 2012, Theorem 4.4].

Let N be such that $t = \frac{1}{\lambda'} L_m N - \epsilon' > 0$. Then for any $k \neq \ell \in \mathbb{Z}$ we have

$$d((f^m g)^{Nk} x, (f^m g)^{N\ell} x) \geq \frac{1}{\lambda'} L_m N |k - \ell| - \epsilon' \geq t|k - \ell|.$$

Thus $(f^m g)^N$ is hyperbolic and therefore so is $f^m g$. □

We conclude this section with an application of Theorem 3.1 to dependent hyperbolic isometries; [Clay and Pettet 2012, Theorem 4.1] would suffice as well.

Proposition 3.4. *Suppose X is a δ-hyperbolic space and $f, g \in \mathrm{Isom}(X)$ are dependent hyperbolic isometries. There is an $N \geq 0$ such that if $n \geq N$ then fg^n is hyperbolic.*

Proof. Let $A_+,\ A_-,\ B_+,\ B_- \in \partial X$ be the attracting and repelling fixed points for f and g, respectively. Then $fB_+ \neq B_-$ as one of these points is fixed by f. Thus there are neighborhoods V_+ and V_- for B_+ and B_-, respectively, in $\overline{X}$ such that $fV_+ \cap V_- = \varnothing$. Let N be the constant from Theorem 3.1 applied to this setup after interchanging the roles of f and g. Hence $g^n f$, and therefore also its conjugate fg^n, are hyperbolic when $n \geq N$. □

4. Finding neighborhoods

We now need to understand when we can find neighborhoods satisfying the hypotheses of Theorem 3.1 for all powers (or, at least, many powers) of a given g. There are two cases that we examine: first when g has a fixed point and second when g is hyperbolic.

Proposition 4.1. *Suppose X is a δ-hyperbolic space and $f \in \mathrm{Isom}(X)$ is a hyperbolic isometry with attracting and repelling fixed points A_+ and A_- in ∂X. Suppose $g \in \mathrm{Isom}(X)$ has a fixed point and consider a sequence of elements $(g_k)_{k\in\mathbb{N}} \subseteq \langle g \rangle$. Then either*

(1) *there are disjoint neighborhoods U_+ and U_- of A_+ and A_-, respectively, and a constant $M \geq 1$ such that if $k \geq M$ then $g_k U_+ \cap U_- = \varnothing$; or*

(2) *there is a subsequence* (g_{k_n}) *so that* $g_{k_n}A_+ \to A_-$.

Further, if $gA_- = A_-$ *then* (1) holds.

Proof. Let $p \in X$ be such that $gp = p$. Thus $g_k p = p$ for all $k \in \mathbb{N}$.

Fix a system of decreasing disjoint neighborhoods U_-^k of A_- and U_+^k of A_+ indexed by the natural numbers so that:

$$\begin{aligned} (x \,.\, A_+)_p &\geq k + \delta, \quad \text{for } x \in U_+^k, \text{ and} \\ (x \,.\, A_-)_p &\geq k + \delta, \quad \text{for } x \in U_-^k. \end{aligned}$$

This implies that for any two points $x, x' \in U_+^k$ we have that

$$(x \,.\, x')_p \geq \min\{(x \,.\, A_+)_p, (x' \,.\, A_+)_p\} - \delta \geq k.$$

Likewise for any two points $y, y' \in U_-^k$ we have $(y \,.\, y')_p \geq k$.

For each $n \in \mathbb{N}$, define

$$I_n = \{k \in \mathbb{N} \mid g_k U_+^n \cap U_-^n \neq \varnothing\}.$$

If I_n is a finite set for some n, then (1) holds for the neighborhoods $U_- = U_-^n$ and $U_+ = U_+^n$ where $M = \max I_n + 1$.

Otherwise, there is a strictly increasing sequence $(k_n)_{n \in \mathbb{N}}$ such that $k_n \in I_n$. Hence, for each $n \in \mathbb{N}$, there is an element $x_n \in U_+^n$ such that $g_{k_n}x_n \in U_-^n$. In particular,

$$(g_{k_n}x_n \,.\, A_-)_p \geq n + \delta. \tag{4-1}$$

On the other hand, since $x_n \in U_+^n$ and g_{k_n} fixes the point p, we have

$$\begin{aligned} (g_{k_n}x_n \,.\, g_{k_n}A_+)_p &= (g_{k_n}x_n \,.\, g_{k_n}A_+)_{g_{k_n}p} \\ &= (x_n \,.\, A_+)_p \geq n + \delta. \end{aligned} \tag{4-2}$$

Combining (4-1) and (4-2), we get $(g_{k_n}A_+ \,.\, A_-)_p \geq n$ for any $n \in \mathbb{N}$. Hence (2) holds.

Now suppose that $gA_- = A_-$. As $A_+ \neq A_-$, there is a constant $D \geq 0$ such that $(f^{-k}p \,.\, f^k p)_p \leq D$ for all $k \in \mathbb{N}$. For any $n \in \mathbb{Z}$, we have $(f^{-k}p \,.\, g^n f^{-k}p)_p \to \infty$ as $k \to \infty$. In particular, for each $n \in \mathbb{Z}$, there is a constant $K_n \geq 0$ such that $(f^{-k}p \,.\, g^n f^{-k}p)_p \geq D + \delta$ for $k \geq K_n$. Therefore $(g^n f^{-k}p \,.\, f^k p)_p \leq D + \delta$ for $k \geq K_n$ as:

$$(f^{-k}p \,.\, f^k p)_p \geq \min\{(f^{-k}p \,.\, g^n f^{-k}p)_p, (g^n f^{-k}p \,.\, f^k p)_p\} - \delta.$$

As $gp = p$, we have $(f^{-k}p \,.\, g^n f^k p)_p = (g^{-n}f^{-k}p \,.\, f^k p)_p$ and so we see that $(f^{-k}p \,.\, g^n f^k p)_p \leq D + \delta$ for $k \geq K_{-n}$. This shows that (2) cannot hold if $gA_- = A_-$. $\square$

Proposition 4.2. *Suppose X is a δ-hyperbolic space and $f, g \in \mathrm{Isom}(X)$ are independent hyperbolic isometries. There are disjoint neighborhoods U_+ and U_- of A_+ and A_- and an $N \geq 1$ such that if $k \geq N$ then $g^k U_+ \cap U_- = \varnothing$.*

Proof. Let $A_+, A_-, B_+, B_- \in \partial X$ be the attracting and repelling fixed points for f and g, respectively. As f and g are independent, the set $\{A_-, A_+, B_-, B_+\}$ consists of four distinct points. Take mutually disjoint open neighborhoods U_-, U_+, V_-, V_+ of A_-, A_+, B_-, B_+, respectively. The North-South dynamics of the action of g on $\overline{X}$ implies that there exists an $N \geq 1$ such that $g^k(\overline{X} - V_-) \subset V_+$ for all $k \geq N$. In particular, $g^k U_+ \subseteq V_+$ and since $V_+ \cap U_- = \varnothing$ we see that $g^k U_+ \cap U_- = \varnothing$ for $k \geq N$. □

5. Simultaneously producing hyperbolic isometries

We can now apply the above propositions via a careful induction to prove the main result.

Theorem 5.1. *Suppose that $\{X_i\}_{i=1,\dots,n}$ is a collection of δ-hyperbolic spaces, G is a group and for each $i = 1, \dots, n$ there is a homomorphism $\rho_i : G \to \mathrm{Isom}(X_i)$ such that*

(1) *there is an element $f_i \in G$ such that $\rho_i(f_i)$ is hyperbolic; and*

(2) *for each $g \in G$, either $\rho_i(g)$ has a periodic orbit or is hyperbolic.*

Then there is an $f \in G$ such that $\rho_i(f)$ is hyperbolic for all $i = 1, \dots, n$.

Proof. We will prove this by induction. The case $n = 1$ obviously holds by hypothesis.

For $n \geq 2$, by induction there is an $f \in G$ such that for $i = 1, \dots, n-1$ the isometry $\rho_i(f) \in \mathrm{Isom}(X_i)$ is hyperbolic. For $i = 1, \dots, n-1$, let $A^i_+, A^i_- \in \partial X_i$ be the attracting and repelling fixed points of the hyperbolic isometry $\rho_i(f)$. By hypothesis, there is a $g \in G$ so that $\rho_n(g) \in \mathrm{Isom}(X_n)$ is hyperbolic. Let $B_+, B_- \in \partial X_n$ be the attracting and repelling fixed points of the hyperbolic isometry $\rho_n(g)$. Our goal is to find $a, b \in \mathbb{N}$ so that $\rho_i(f^a g^b)$ is hyperbolic for each $i = 1, \dots, n$.

We begin with some simplifications. If $\rho_n(f) \in \mathrm{Isom}(X_n)$ is hyperbolic then there is nothing to prove, so assume that $\rho_n(f)$ has a periodic orbit, and so after replacing f by a power we have that f has a fixed point. By replacing g with a power if necessary, we can assume that for $i = 1, \dots, n-1$ the isometry $\rho_i(g)$ is either the identity or has infinite order. In fact, we can assume that $\rho_i(g)$ has infinite order. Indeed, if $\rho_i(g)$ is the identity, then for all $a, b \in \mathbb{N}$ we have $\rho_i(f^a g^b) = \rho_i(f^a)$, which is hyperbolic by the inductive hypothesis. Hence any powers for f and g that work for all other indices between 1 and $n-1$ necessarily work for this index i as well. Again, by replacing g with a power if necessary, we can assume that for each $i = 1, \dots, n-1$ either $\rho_i(g)A^i_- = A^i_-$ or $\rho_i(g^b)A^i_- \neq A^i_-$ for each $b \in \mathbb{Z} - \{0\}$.

Finally, replacing g with a further power necessary, we can assume that for each $i = 1, \dots, n-1$ if $\rho_i(g)$ is not hyperbolic, then it has a fixed point. Analogously, by replacing f with a power if necessary, we can assume that the isometry $\rho_n(f)$ has infinite order and that either $\rho_n(f)B_- = B_-$ or $\rho_n(f^a)B_- \neq B_-$ for $a \in \mathbb{Z} - \{0\}$.

There are various scenarios depending on the dynamics of the isometries $\rho_i(g)$ and $\rho_n(f)$.

Let $E \subseteq \{1, \dots, n-1\}$ be the subset where the isometry $\rho_i(g)$ has a fixed point. Let $H = \{1, \dots, n-1\} - E$; this is of course the subset where $\rho_i(g)$ is hyperbolic. For $i \in H$, let $B^i_+, B^i_- \in \partial X_i$ be the attracting and repelling fixed points of the hyperbolic isometry $\rho_i(g)$. We further identify the subset $H' \subseteq H$ where $\rho_i(f)$ and $\rho_i(g)$ are independent.

We first deal with the spaces where $\rho_i(g)$ is hyperbolic. To this end, fix $i \in H$.

If $i \in H'$, then by Proposition 4.2 there are disjoint neighborhoods $U^i_+, U^i_- \subset \overline{X}_i$ of A^i_+ and A^i_-, respectively, and an N_i so that for $k \geq N_i$ we have $\rho_i(g^k)U^i_+ \cap U^i_- = \varnothing$. Applying Theorem 3.1 with the neighborhoods U_+ and U_-, there is an M_i so that for $a \geq M_i$ and $b \geq N_i$ the element $\rho_i(f^a g^b)$ is hyperbolic.

If $i \in H - H'$ then, by Proposition 3.4, for each $a \in \mathbb{N}$ there is a constant $C_i(a) \geq 0$ such that the isometry $\rho_i(f^a g^b)$ is hyperbolic if $b \geq C_i(a)$.

To create a uniform statement in the sequel, for $i \notin H'$ (including $i \in E$), set $C_i(a) = 0$ for all $a \in \mathbb{N}$. Also, set $M_i = N_i = 0$ for $i \in H - H'$.

Summarizing the situation so far, we let $\mathsf{M}_0 = \max\{M_i \mid i \in H\}$ and $\mathsf{N}_0 = \max\{N_i \mid i \in H\}$. Then, at this point, we know that if $i \in H$, $a \geq \mathsf{M}_0$ and $b \geq \mathsf{N}_0$ then the element $\rho_i(f^a g^b)$ is hyperbolic so long as $b \geq C_i(a)$.

Next we deal with the spaces where $\rho_i(g)$ has a fixed point. To this end, fix $i \in E$.

Let $E' \subseteq E$ be the subset where condition (1) of Proposition 4.1 holds using $\rho_i(g_k) = \rho_i(g^{\mathsf{N}_0 + k})$. The analysis here is similar to the case when $i \in H'$. By assumption, for $i \in E'$, there are disjoint neighborhoods $U^i_+, U^i_- \subset \overline{X}_i$ of A^i_+ and A^i_-, respectively, and an N_i so that for $k \geq N_i$ we have $\rho_i(g_k)U^i_+ \cap U^i_- = \varnothing$. Applying Theorem 3.1 with the neighborhoods U^i_+ and U^i_-, there is an M_i so that for $a \geq M_i$ the element $\rho_i(f^a g^b)$ is hyperbolic if $b \geq N_i$.

To summarize again, let $\mathsf{M}_1 = \max\{M_i \mid i \in H \cup E'\}$ and $\mathsf{N}_1 = \max\{N_i \mid i \in H \cup E'\}$. Then at this point, if $i \in H \cup E'$, $a \geq \mathsf{M}_1$ and $b \geq \mathsf{N}_1$ then the element $\rho_i(f^a g^b)$ is hyperbolic so long as $b \geq C_i(a)$.

It remains to deal with $E - E'$; enumerate this set by $\{i_1, \dots, i_\ell\}$. As condition (1) of Proposition 4.1 does not hold for $\rho_{i_1}(g_k) = \rho_{i_1}(g^{\mathsf{N}_0 + k})$ acting on X_{i_1}, there is a subsequence $(g^{k_n}) \subseteq (g^{\mathsf{N}_0 + k})$ such that $\rho_{i_1}(g^{k_n})A^{i_1}_+ \to A^{i_1}_-$. By iteratively passing to subsequences of (g^{k_n}), we can assume that for all $i \in E - E'$, either the sequence of points $(\rho_i(g^{k_n})A^i_+) \subseteq \partial X_i$ converges or is discrete.

Notice that for $i \in E - E'$, the final statement of Proposition 4.1 implies that $\rho_i(g)A^i_- \neq A^i_-$. Coupling this with one of our earlier simplifications, we have that $\rho_i(g^b)A^i_- \neq A^i_-$ for all $b \in \mathbb{Z} - \{0\}$. Hence, there is a $K \in \mathbb{N}$ such that for any $i \in E - E'$ the sequence (g^{K+k_n}) either satisfies $\rho_i(g^{K+k_n})A^i_+ \to p_i \neq A^i_-$, or $(\rho_i(g^{K+k_n})A^i_+) \subset \partial X_i$ is discrete. Indeed, suppose $\rho_i(g^{k_n})A^i_+ \to p_i$ (nothing new is being claimed in the discrete case). If p_i is not in $\{\rho_i(g^k)A^i_-\}_{k\in\mathbb{Z}}$, then neither is $\rho_i(g^K)p_i$ for any $K \in \mathbb{N}$ so $\rho_i(g^{K+k_n})A^i_+ \to \rho_i(g^K)p_i \neq A^i_-$. Else, if $p_i = \rho_i(g^{K_i})A^i_-$, then for $K \neq -K_i$ we have

$$\rho_i(g^{K+k_n})A^i_+ \to \rho_i(g^{K+K_i})A^i_- \neq A^i_-.$$

So by taking $K \in \mathbb{N}$ to avoid the finitely many such $-K_i$ we see that the claim holds. Without loss of generality, we can assume that $K \geq \mathsf{N}_1$.

Hence for each $i \in E - E'$, by Proposition 4.1, there are disjoint neighborhoods $U^i_+, U^i_- \subset \overline{X}$ of A^i_+ and A^i_-, respectively, and an N_i so that for $n \geq N_i$ we have $\rho_i(g^{K+k_n})U^i_+ \cap U^i_- = \varnothing$. Applying Theorem 3.1 with the neighborhoods U^i_+ and U^i_-, there is an M_i so that for $a \geq M_i$ the element $\rho_i(f^a g^{K+k_n})$ is hyperbolic if $n \geq N_i$.

Putting all of this together, let $\mathsf{M}_2 = \max\{M_i \mid 1 \leq i \leq n-1\}$ and let $\mathsf{N}_2 = \max\{N_i \mid i \in E - E'\}$. Thus for all $i = 1, \dots, n-1$, if $a \geq \mathsf{M}_2$, and $n \geq \mathsf{N}_2$ then $\rho_i(f^a g^{K+k_n})$ is hyperbolic so long as $K + k_n \geq C_i(a)$. (Notice $K + k_n \geq K \geq \mathsf{N}_1$ by assumption.)

We now work with the action on the space X_n. Interchanging the roles of f and g and arguing as above using Proposition 4.1 to the sequence of isometries $(\rho_n(f^\ell))$ we obtain a subsequence $(f^{\ell_m}) \subseteq (f^\ell)$ and constants M_3 and N_3 so that $\rho_n(f^{\ell_m} g^b)$ is hyperbolic if $m \geq \mathsf{M}_3$ and $b \geq \mathsf{N}_3$.

Fix some $m \geq \mathsf{M}_3$ large enough so that $a = \ell_m \geq \mathsf{M}_2$ and let

$$\mathsf{C} = \max\{C_i(a) \mid 1 \leq i \leq n-1\}.$$

Now for $n \geq \mathsf{N}_2$ large enough so that $b = K + k_n \geq \max\{\mathsf{C}, \mathsf{N}_3\}$ we have that $\rho_i(f^a g^b)$ is hyperbolic for $i = 1, \dots, n$ as desired. □

6. Application to Out(F_N)

Let F_N be a free group of rank $N \geq 2$. A *free factor system* of F_N is a finite collection $\mathcal{A} = \{[A_1], [A_2], \dots, [A_K]\}$ of conjugacy classes of subgroups of F_N, such that there exists a free factorization

$$F_N = A_1 * \cdots * A_K * B,$$

where B is a (possibly trivial) subgroup, called a *cofactor*. There is a natural partial ordering among the free factor systems: $\mathcal{A} \sqsubseteq \mathcal{B}$ if for each $[A] \in \mathcal{A}$ there is a $[B] \in \mathcal{B}$ such that $gAg^{-1} < B$ for some $g \in F_N$. In this case, we say that $\mathcal{A}$ is *contained* in $\mathcal{B}$ or $\mathcal{B}$ is an *extension* of $\mathcal{A}$.

Recall, the *reduced rank* of a subgroup $A < F_N$ is defined as

$$\underline{\mathrm{rk}}(A) = \min\{0, \mathrm{rk}(A) - 1\}.$$

We extend this to a free factor system by addition:

$$\underline{\mathrm{rk}}(\mathcal{A}) = \sum_{k=1}^{K} \underline{\mathrm{rk}}(A_k),$$

where $\mathcal{A} = \{[A_1], [A_2], \ldots, [A_K]\}$. An extension $\mathcal{A} \sqsubseteq \mathcal{B}$ is called a *multi-edge extension* if $\underline{\mathrm{rk}}(\mathcal{B}) \geq \underline{\mathrm{rk}}(\mathcal{A}) + 2$.

The group $\mathrm{Out}(F_N)$ naturally acts on the set of free factor systems as follows. Given $\mathcal{A} = \{[A_1], [A_2], \ldots, [A_K]\}$, and $\varphi \in \mathrm{Out}(F_N)$ choose a representative $\Phi \in \mathrm{Aut}(F_N)$ of φ, a realization $F_N = A_1 * \cdots * A_K * B$ of $\mathcal{A}$ and define $\varphi(\mathcal{A})$ to be the free factor system $\{[\Phi(A_1)], \ldots, [\Phi(A_K)]\}$. Given a free factor system $\mathcal{A}$ consider the subgroup $\mathrm{Out}(F_N; \mathcal{A})$ of $\mathrm{Out}(F_N)$ that stabilizes the free factor system $\mathcal{A}$. The group $\mathrm{Out}(F_N; \mathcal{A})$ is called the *outer automorphism group of F_N relative to $\mathcal{A}$*, or the *relative outer automorphism group* if the free factor system $\mathcal{A}$ is clear from context. If $\mathcal{A} = \{[A]\}$, there is a well-defined restriction homomorphism $\mathrm{Out}(F_N; \mathcal{A}) \to \mathrm{Out}(A)$, which we denote by $\varphi \mapsto \varphi|_A$ [Handel and Mosher 2013b, Fact 1.4].

For a subgroup $\mathcal{H} < \mathrm{Out}(F_N)$ and $\mathcal{H}$-invariant free factor systems $\mathcal{F}_1 \sqsubseteq \mathcal{F}_2$, we say that $\mathcal{H}$ is *irreducible with respect to the extension* $\mathcal{F}_1 \sqsubseteq \mathcal{F}_2$ if for any $\mathcal{H}$-invariant free factor system $\mathcal{F}$ such that $\mathcal{F}_1 \sqsubseteq \mathcal{F} \sqsubseteq \mathcal{F}_2$, it follows that either $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$. We sometimes say that $\mathcal{H}$ is *relatively irreducible* if the extension is clear from the context. The subgroup $\mathcal{H}$ is *relatively fully irreducible* if each finite index subgroup $\mathcal{H}' < \mathcal{H}$ is relatively irreducible. For an individual element $\varphi \in \mathrm{Out}(F_N)$, we say that φ is relatively (fully) irreducible if the cyclic subgroup $\langle \varphi \rangle$ is relatively (fully) irreducible.

In close analogy with Ivanov's classification [1992] of subgroups of mapping class groups, in a series of papers Handel and Mosher gave a classification of finitely generated subgroups of $\mathrm{Out}(F_N)$ [2013a; 2013b; 2013c; 2013d; 2013e].

Theorem 6.1 [Handel and Mosher 2013a, Theorem D]. *For each finitely generated subgroup $\mathcal{H} < \mathrm{IA}_N(\mathbb{Z}/3) < \mathrm{Out}(F_N)$, each maximal $\mathcal{H}$-invariant filtration by free factor systems*

$$\varnothing = \mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \cdots \sqsubset \mathcal{F}_m = \{[F_N]\},$$

and each $i = 1, \ldots, m$ such that $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ is a multi-edge extension, there exists $\varphi \in \mathcal{H}$ which is irreducible with respect to $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$.

Here, $\mathrm{IA}_N(\mathbb{Z}/3)$ is the finite index subgroup of $\mathrm{Out}(F_N)$ which is the kernel of the natural surjection

$$p : \mathrm{Out}(F_N) \to H^1(F_N, \mathbb{Z}/3) \cong GL(N, \mathbb{Z}/3).$$

For elements in $\mathrm{IA}_N(\mathbb{Z}/3)$, irreducibility is equivalent to full irreducibility hence in the above statement we can also conclude that φ is fully irreducible [Handel and Mosher 2013a, Theorem B].

Handel and Mosher conjecture that there is a single $\varphi \in \mathcal{H}$ which is (fully) irreducible for each multi-edge extension $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ [Handel and Mosher 2013a, Remark following Theorem D]. The goal of this section is to prove this conjecture. Invoking theorems of Handel–Mosher and Horbez–Guirardel, this is (essentially) an immediate application of Theorem 5.1. We state the setup and their theorems now.

Definition 6.2. Let $\mathcal{A}$ be a free factor system of F_N. The *complex of free factor systems of F_N relative to $\mathcal{A}$*, denoted $\mathcal{FF}(F_N; \mathcal{A})$, is the geometric realization of the partial ordering $\sqsubseteq$ restricted to proper free factor systems that properly contain $\mathcal{A}$.

If $\mathcal{A} = \{[A_1], [A_2], \dots, [A_K]\}$ is a free factor system for F_N, its *depth* is defined as:

$$\mathrm{D}_{\mathcal{FF}}(\mathcal{A}) = (2N-1) - \sum_{k=1}^{K} (2\,\mathrm{rk}(A_k) - 1).$$

The free factor system $\mathcal{A}$ is *nonexceptional* if $\mathrm{D}_{\mathcal{FF}}(\mathcal{A}) \geq 3$.

Theorem 6.3 [Handel and Mosher 2014, Theorem 1.2]. *For any nonexceptional free factor system $\mathcal{A}$ of F_N, the complex $\mathcal{FF}(F_N; \mathcal{A})$ is positive-dimensional, connected and δ-hyperbolic.*

Although the group $\mathrm{Out}(F_N)$ does not act on $\mathcal{FF}(F_N; \mathcal{A})$, the natural subgroup $\mathrm{Out}(F_N; \mathcal{A})$ associated to the free factor system $\mathcal{A}$ acts on $\mathcal{FF}(F_N; \mathcal{A})$ by simplicial isometries. In a companion paper Handel and Mosher characterize the elements of $\mathrm{Out}(F_N; \mathcal{A})$ that act as a hyperbolic isometry of $\mathcal{FF}(F_N; \mathcal{A})$:

Theorem 6.4 [Handel and Mosher $\geq$ 2018]. *For any nonexceptional free factor system $\mathcal{A}$ of F_N, $\varphi \in \mathrm{Out}(F_N; \mathcal{A})$ acts as a hyperbolic isometry on $\mathcal{FF}(F_N; \mathcal{A})$ if and only if φ is fully irreducible with respect to $\mathcal{A} \sqsubset \{[F_N]\}$.*

Remark 6.5. An alternative proof of Theorem 6.4 is given by Guirardel and Horbez [2017] using the description of the boundary of the relative free factor complex. Further, with a slight modification of the definition of the relative free factor complex, both Handel and Mosher and Guirardel and Horbez can additionally prove that the theorem holds for the only remaining multi-edge configuration which is when $\mathcal{A} = \{[A_1], [A_2], [A_3]\}$ and $F_N = A_1 * A_2 * A_3$. Yet another proof of Theorem 6.4 is given by Radhika Gupta [2016] using dynamics on relative outer space and relative currents.

We are now ready to prove our application:

Theorem 6.6. *For each finitely generated subgroup $\mathcal{H} < \mathrm{IA}_N(\mathbb{Z}/3) < \mathrm{Out}(F_N)$ and each maximal $\mathcal{H}$-invariant filtration by free factor systems*

$$\varnothing = \mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \cdots \sqsubset \mathcal{F}_m = \{[F_N]\},$$

there is an element $\varphi \in \mathcal{H}$ such that for each $i = 1, \dots, m$ such that $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ is a multi-edge extension, φ is irreducible with respect to $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$.

Proof. Let I be the subset of indices i such that $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ is a multi-edge extension.

Given $i \in I$, since $\mathcal{H} < \mathrm{IA}_N(\mathbb{Z}/3)$, each component of $\mathcal{F}_{i-1}$ and $\mathcal{F}_i$ is $\mathcal{H}$-invariant [Handel and Mosher 2013c, Lemma 4.2]. Moreover, by the argument at the beginning of Section 2.1 in [Handel and Mosher 2013e], since $\mathcal{H}$ is irreducible with respect to $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ (this follows from maximality of the filtration) there is precisely one component $[B_i] \in \mathcal{F}_i$ that is not a component of $\mathcal{F}_{i-1}$. Let $\widehat{\mathcal{A}}_i$ be the maximal subset of $\mathcal{F}_{i-1}$ such that $\widehat{\mathcal{A}}_i \sqsubset \{[B_i]\}$. Notice that this extension is again multi-edge, indeed $\underline{\mathrm{rk}}(B_i) - \underline{\mathrm{rk}}(\widehat{\mathcal{A}}_i) = \underline{\mathrm{rk}}(\mathcal{F}_i) - \underline{\mathrm{rk}}(\mathcal{F}_{i-1})$. The system $\widehat{\mathcal{A}}_i$ can be represented by $\{[A_{i,1}], \dots, [A_{i,K_i}]\}$ where $A_{i,k} < B_i$ for each k. Let $\mathcal{A}_i$ be the free factor system in the subgroup B_i consisting of the conjugacy classes in B_i of the subgroups $A_{i,k}$. Then a given $\varphi \in \mathcal{H}$ is irreducible with respect to $\widehat{\mathcal{A}}_i \sqsubset \{[B_i]\}$, equivalently $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ as the remaining components are the same, if and only if the restriction $\varphi|_{B_i} \in \mathrm{Out}(B_i; \mathcal{A}_i)$ is irreducible relative to $\mathcal{A}_i$.

For $i \in I$, let $X_i = \mathcal{FF}(B_i; \mathcal{A}_i)$ and consider the action homomorphism

$$\rho_i : \mathcal{H} \to \mathrm{Isom}(X_i)$$

defined by $\rho_i(\varphi) = \varphi|_{B_i}$. These spaces are δ-hyperbolic for some δ by Theorem 6.3, and by the above discussion and Theorem 6.4, $\rho_i(\varphi)$ is a hyperbolic isometry if $\varphi \in \mathcal{H}$ is irreducible with respect to $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$. If $\rho_i(\varphi)$ is not irreducible with respect to $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$, then $\rho_i(\varphi)$ fixes a point in X_i. By Theorem 6.1, for each $i \in I$, there exists some $\varphi_i \in \mathcal{H}$ that is irreducible with respect to $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ and hence $\rho_i(\varphi_i)$ is a hyperbolic isometry.

We are now in the model situation of Theorem 5.1. We conclude that there is a $\varphi \in \mathcal{H}$ such that $\rho_i(\varphi)$ is a hyperbolic isometry for all $i \in I$. By the above discussion, this means that φ is (fully) irreducible with respect to $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$ for each $i \in I$ as desired. $\square$

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References

[Alonso et al. 1991] J. M. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short, "Notes on word hyperbolic groups", pp. 3–63 in *Group theory from a geometrical viewpoint* (Trieste, 1990), edited by E. Ghys et al., World Scientific, River Edge, NJ, 1991. MR Zbl

[Björklund and Hartnick 2011] M. Björklund and T. Hartnick, "Biharmonic functions on groups and limit theorems for quasimorphisms along random walks", *Geom. Topol.* **15**:1 (2011), 123–143. MR Zbl

[Bridson and Haefliger 1999] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **319**, Springer, 1999. MR Zbl

[Clay and Pettet 2012] M. Clay and A. Pettet, "Current twisting and nonsingular matrices", *Comment. Math. Helv.* **87**:2 (2012), 385–407. MR Zbl

[Clay et al. 2012] M. T. Clay, C. J. Leininger, and J. Mangahas, "The geometry of right-angled Artin subgroups of mapping class groups", *Groups Geom. Dyn.* **6**:2 (2012), 249–278. MR Zbl

[Dowdall and Taylor 2018] S. Dowdall and S. Taylor, "Hyperbolic extensions of free groups", *Geom. Topol.* **22**:1 (2018), 517–570. MR Zbl

[Fujiwara 2015] K. Fujiwara, "Subgroups generated by two pseudo-Anosov elements in a mapping class group, II: Uniform bound on exponents", *Trans. Amer. Math. Soc.* **367**:6 (2015), 4377–4405. MR Zbl

[Gromov 1987] M. Gromov, "Hyperbolic groups", pp. 75–263 in *Essays in group theory*, edited by S. M. Gersten, Math. Sci. Res. Inst. Publ. **8**, Springer, 1987. MR Zbl

[Guirardel and Horbez 2017] V. Guirardel and C. Horbez, "Algebraic laminations for free products and arational trees", preprint, 2017. arXiv

[Gültepe 2017] F. Gültepe, "Fully irreducible automorphisms of the free group via Dehn twisting in $\sharp_k(S^2 \times S^1)$", *Algebr. Geom. Topol.* **17**:3 (2017), 1375–1405. MR Zbl

[Gupta 2016] R. Gupta, "Loxodromic elements for the relative free factor complex", preprint, 2016. arXiv

[Handel and Mosher 2013a] M. Handel and L. Mosher, "Subgroup decomposition in O$ut(F_n)$: introduction and research announcement", preprint, 2013. arXiv

[Handel and Mosher 2013b] M. Handel and L. Mosher, "Subgroup decomposition in O$ut(F_n)$, Part I: Geometric models", preprint, 2013. arXiv

[Handel and Mosher 2013c] M. Handel and L. Mosher, "Subgroup decomposition in O$ut(F_n)$, Part II: A relative Kolchin theorem", preprint, 2013. arXiv

[Handel and Mosher 2013d] M. Handel and L. Mosher, "Subgroup decomposition in $\mathrm{Out}(F_n)$, Part III: Weak attraction theory", preprint, 2013. arXiv

[Handel and Mosher 2013e] M. Handel and L. Mosher, "Subgroup decomposition in $\mathrm{Out}(F_n)$, Part IV: Relatively irreducible subgroups", preprint, 2013. arXiv

[Handel and Mosher 2014] M. Handel and L. Mosher, "Relative free splitting and free factor complexes, I: Hyperbolicity", preprint, 2014. arXiv

[Handel and Mosher $\geq$ 2018] M. Handel and L. Mosher, "Relative free splitting and free factor complexes, II: Loxodromic outer automorphisms", in preparation.

[Horbez 2016] C. Horbez, "A short proof of Handel and Mosher's alternative for subgroups of $\mathrm{Out}(F_N)$", *Groups Geom. Dyn.* **10**:2 (2016), 709–721. MR Zbl

[Ivanov 1992] N. V. Ivanov, *Subgroups of Teichmüller modular groups*, Translations of Mathematical Monographs **115**, Amer. Math. Soc., Providence, RI, 1992. MR Zbl

[Kapovich and Benakli 2002] I. Kapovich and N. Benakli, "Boundaries of hyperbolic groups", pp. 39–93 in *Combinatorial and geometric group theory* (New York, 2000/Hoboken, NJ, 2001), edited by S. Cleary et al., Contemp. Math. **296**, Amer. Math. Soc., Providence, RI, 2002. MR Zbl

[Maher and Tiozzo 2016] J. Maher and G. Tiozzo, "Random walks on weakly hyperbolic groups", *J. Reine Angew. Math.* (online publication January 2016).

[Mangahas 2013] J. Mangahas, "A recipe for short-word pseudo-Anosovs", *Amer. J. Math.* **135**:4 (2013), 1087–1116. MR Zbl

[Taylor 2014] S. J. Taylor, "A note on subfactor projections", *Algebr. Geom. Topol.* **14**:2 (2014), 805–821. MR Zbl

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MATT CLAY
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF ARKANSAS
FAYETTEVILLE, AR
UNITED STATES
mattclay@uark.edu

CAGLAR UYANIK
DEPARTMENT OF MATHEMATICS
VANDERBILT UNIVERSITY
NASHVILLE, TN
UNITED STATES
caglar.uyanik@vanderbilt.edu

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