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A LOCAL WEIGHTED AXLER–ZHENG THEOREM IN $\mathbb{C}^n$

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The well-known Axler–Zheng theorem characterizes compactness of finite sums of finite products of Toeplitz operators on the unit disk in terms of the Berezin transform of these operators. Subsequently this theorem was generalized to other domains and appeared in different forms, including domains in $\mathbb{C}^n$ on which the $\bar\partial$-Neumann operator N is compact. In this work we remove the assumption on N, and we study weighted Bergman spaces on smooth bounded pseudoconvex domains. We prove a local version of the Axler–Zheng theorem characterizing compactness of Toeplitz operators in the algebra generated by symbols continuous up to the boundary in terms of the behavior of the Berezin transform at strongly pseudoconvex points. We employ a Forelli–Rudin type inflation method to handle the weights.

1. Introduction

1.1. ***History.*** In the theory of Bergman space operators on the open unit disk $\mathbb{D}$, the Axler–Zheng theorem [1998] provides an important characterization of compactness of a large class of operators in terms of their Berezin transforms. Specifically this theorem states that if S is a finite sum of finite products of Toeplitz operators on the Bergman space $A^2(\mathbb{D})$ whose symbols are in $L^\infty(\mathbb{D})$, then S is compact if and only if $BS(z)$, the Berezin transform of S, tends to 0 as $|z| \to 1$. This theorem has been extended by Suárez [2007] to include all operators in the Toeplitz algebra in the unit ball in $\mathbb{C}^n$. Engliš [1999] extended the Axler–Zheng theorem to irreducible bounded symmetric domains and the unit polydisk. Mitkovski, Suárez and Wick [Mitkovski et al. 2013] proved a weighted version of Suárez's result on the unit ball in $\mathbb{C}^n$. Using the techniques of several complex variables, Čučković and Şahutoğlu [2013] proved a version of the Axler–Zheng theorem on smooth bounded pseudoconvex domains on which the $\bar\partial$-Neumann operator is compact. The use of the $\bar\partial$ techniques required that the operators in their theorem belong to the algebra $\mathscr{T}(\overline{\Omega})$ which is the norm closed algebra generated by $\{T_\phi : \phi \in C(\overline{\Omega})\}$. Recently, Kreutzer [2014] generalized Čučković and Şahutoğlu's result in a more abstract setting.

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Our aim is to extend the previous result of Čučković and Şahutoğlu in two ways: Firstly, we want to remove the hypothesis of the compactness of the $\bar{\partial}$-Neumann operator on Ω. We also want to consider weighted Bergman spaces. Our main theorem gives a local version of the Axler–Zheng theorem for a wide class of domains in $\mathbb{C}^n$. The novelty of our approach is to use the inflation of the domain argument pioneered by Forelli and Rudin [1974] and Ligocka [1989]. The second important ingredient is the B-regularity of the inflated domain which will give us the compactness of $\bar{\partial}$, thus replacing the assumption on the compactness of the $\bar{\partial}$-Neumann operator. As a corollary we obtain a weighted version of the Axler–Zheng theorem for strongly pseudoconvex domains, which itself is a new result.

1.2. ***Preliminaries.*** Let Ω be a C^1-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function ρ. We denote the boundary of Ω by $b\Omega$. Let $L^2(\Omega, (-\rho)^r)$ denote the square integrable functions on Ω with respect to the measure $(-\rho)^r dV$ where dV denotes the Lebesgue measure, $r \geq 0$, and

$$A^2(\Omega, (-\rho)^r) = \{f \in L^2(\Omega, (-\rho)^r) : f \text{ is holomorphic}\}.$$

Since $A^2(\Omega, (-\rho)^r)$ is a closed subspace of $L^2(\Omega, (-\rho)^r)$, a bounded orthogonal projection,

$$P_r : L^2(\Omega, (-\rho)^r) \to A^2(\Omega, (-\rho)^r),$$

(called Bergman projection) exists. P_r is an integral operator of the form

$$P_r(f)(z) = \int_\Omega K^r(z, \zeta) f(\xi)(-\rho)^r \, dV$$

for $f \in L^2(\Omega, (-\rho)^r)$. The integral kernel $K^r(z, \xi)$ is called the Bergman kernel and the normalized Bergman kernel $k_z^r(\xi)$ is defined as

$$k_z^r(\xi) = \frac{K^r(\xi, z)}{\sqrt{K^r(z, z)}}.$$

When $r = 0$ we drop the superscript r; that is, $K = K_\Omega$ denotes the unweighted Bergman kernel and k_z denotes the unweighted normalized Bergman kernel. For a bounded operator T on $A^2(\Omega, (-\rho)^r)$, the Berezin transform $B_r T$ of T is defined as

$$B_r T(z) = \langle T k_z^r, k_z^r \rangle_r,$$

where $\langle \cdot\, , \cdot \rangle_r$ is the inner product on $A^2(\Omega, (-\rho)^r)$.

For $\phi \in L^\infty(\Omega)$, the weighted Toeplitz operator T_ϕ^r and the weighted Hankel operator H_ϕ^r are defined as

$$T_\phi^r = P^r M_\phi,$$
$$H_\phi^r = (I - P^r) M_\phi,$$

where $M_\phi : A^2(\Omega, (-\rho)^r) \to L^2(\Omega, (-\rho)^r)$ denotes the multiplication by ϕ.

We use $\mathscr{T}(\overline{\Omega}, (-\rho)^r)$ to denote the norm closed subalgebra of bounded linear operators on $A^2(\Omega, (-\rho)^r)$ generated by the set of Toeplitz operators

$$\{T^r_\phi : \phi \in C(\overline{\Omega})\}.$$

For $\phi \in L^\infty$ we define $B_r\phi = B_r T_\phi$.

In this paper we look at weighted Hankel and Toeplitz operators on various domains and various weighted spaces. Whenever we need to clarify where these operators are defined, we will use appropriate subscripts and superscripts. In particular, when we need to emphasize the underlying domain we will write P^Ω, $K_\Omega(z, \xi)$, H^Ω_ϕ, and T^Ω_ϕ, where the Bergman spaces are unweighted. When we have weighted spaces and we need to indicate the domain and the weight we will write $P^{\Omega,r}$, $K^r_\Omega(z, \xi)$, $H^{\Omega,r}_\phi$, and $T^{\Omega,r}_\phi$.

1.3. ***Main result.*** We start with the following two definitions that capture the local structure of the main theorem. To motivate the following definition, if $f_j \to f$ weakly in $A^2(\Omega)$ then for any point $p \in b\Omega$ and $r > 0$ one can show that $f_j \to f$ weakly in $A^2(\Omega \cap B(p, r))$ where $B(p, r)$ is the open ball centered at p with radius r.

Definition 1. Let $r \geq 0$ and Ω be a C^2-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function ρ. Furthermore, let $\{f_j\} \subset A^2(\Omega, (-\rho)^r)$ be a sequence and $f \in A^2(\Omega, (-\rho)^r)$. We say that $\{f_j\}$ converges to f *weakly about strongly pseudoconvex points* if:

(i) $f_j \to f$ weakly in $A^2(\Omega, (-\rho)^r)$ as $j \to \infty$.

(ii) When Γ_Ω, the set of the weakly pseudoconvex points in $b\Omega$, is nonempty, there exists an open neighborhood U of Γ_Ω such that $\|f_j - f\|_{L^2(U\cap\Omega,(-\rho)^r)} \to 0$ as $j \to \infty$.

We note that on strongly pseudoconvex domains, sequences converging weakly about strongly pseudoconvex points and weakly convergent sequences coincide.

Definition 2. Let r, Ω, and ρ be as above. Furthermore, let $T : A^2(\Omega, (-\rho)^r) \to A^2(\Omega, (-\rho)^r)$ be a bounded linear operator. We say that T is *compact about strongly pseudoconvex points* if $Tf_j \to Tf$ in $A^2(\Omega, (-\rho)^r)$ whenever $f_j \to f$ weakly about strongly pseudoconvex points.

Remark 3. As shown in Proposition 13 below, it is interesting that any Hankel operator with a symbol continuous on the closure of the domain is compact about strongly pseudoconvex points.

With the help of these two definitions, we state our main result as follows.

Theorem 4. *Let r be a nonnegative real number, Ω be a C^2-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function ρ, and $T \in \mathscr{T}(\overline{\Omega}, (-\rho)^r)$.*

Then T is compact about strongly pseudoconvex points on $A^2(\Omega,(-\rho)^r)$ if and only if $\lim_{z\to p} B_rT(z)=0$ *for any strongly pseudoconvex point $p\in b\Omega$.*

If Ω is a strongly pseudoconvex domain then we have the following corollary.

Corollary 5. *Let r be a nonnegative real number, Ω be a C^2-smooth bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with a defining function ρ, and T be an element of $\mathscr{T}(\overline{\Omega},(-\rho)^r)$. Then T is compact on $A^2(\Omega,(-\rho)^r)$ if and only if* $\lim_{z\to p} B_rT(z)=0$ *for any $p\in b\Omega$.*

Remark 6. In the case of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ and $\rho(z)=|z|^2-1$, we partially recover [Mitkovski et al. 2013, Theorem 1.1]. Unlike the arguments on the unit ball, the proof of Corollary 5 does not require any explicit form for the weight or the weighted Bergman kernel.

2. Proof of Theorem 4

In this section, before we prove Theorem 4, we present some propositions and lemmas that encapsulate the technical details of the proof.

Proposition 7. *Let Ω be a C^2-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ and $\{T_j\}$ be a sequence of operators compact about strongly pseudoconvex points that converges to T in the operator norm. Then T is compact about strongly pseudoconvex points.*

Proof. Let $\{f_j\}$ be a sequence in $A^2(\Omega,(-\rho)^r)$ that converges to 0 weakly about strongly pseudoconvex points. Since $f_j\to 0$ weakly there exists $C>0$ such that

$$\sup\{\|f_j\| : j=1,2,3,\dots\}\le C.$$

Then for any k we have

$$\|Tf_j\|\le\|(T-T_k)f_j\|+\|T_kf_j\|\le C\|T-T_k\|+\|T_kf_j\|.$$

Let $\varepsilon>0$ be given. Since $T_j\to T$ in the operator norm, we choose k_ε such that $\|T-T_{k_\varepsilon}\|\le\varepsilon$. Then

$$\limsup_{j\to\infty}\|Tf_j\|\le C\varepsilon+\limsup_{j\to\infty}\|T_{k_\varepsilon}f_j\|\le C\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, we conclude that $Tf_j\to 0$. That is, T is compact about strongly pseudoconvex points. □

One of the key ideas in the proof is to use an inflated domain over Ω to understand the weighted Bergman spaces. For this purpose, unless stated otherwise, for the rest of the paper, Ω will be a bounded pseudoconvex domain in $\mathbb{C}^n$ with C^2-smooth boundary, ρ will be a defining function for Ω, and

(1) $$\Omega_r^p=\big\{(z,w)\in\mathbb{C}^n\times\mathbb{C}^p : z\in\Omega \text{ and } \rho(z)+|w_1|^{2p/r}+\dots+|w_p|^{2p/r}<0\big\},$$

where p is a positive integer and r is a real number such that $0 < r \le p$. For a function $f \in A^2(\Omega, (-\rho)^r)$, we let $F(z, w) = f(z)$ be the trivial extension of f to Ω_r^p. It easily follows from an iterated integral argument that $F \in A^2(\Omega_r^p)$.

The following proposition is interesting in its own right as it gives a relationship between the Bergman kernels of the inflated domain and base.

Proposition 8. *Using the notation above,*

$$K_\Omega^r(z, \xi) = c_{p,r} K_{\Omega_r^p}(z, 0; \xi, 0),$$

where $c_{p,r} = \int_{|\widetilde{w}_1|^{2p/r}+\cdots+|\widetilde{w}_p|^{2p/r}<1} dV(\widetilde{w})$ *and* $K_\Omega^r(z, \xi)$ *is the weighted Bergman kernel of* Ω *with weight* $(-\rho)^r$.

Proof. We will follow a standard inflation argument (see, for instance, [Forelli and Rudin 1974; Ligocka 1989]). Since Ω_r^p is a Hartogs domain with base Ω, the Bergman kernel of Ω_r^p can be written as

$$K_{\Omega_r^p}(z, w; \xi, \eta) = K_{\Omega_r^p}(z, 0; \xi, 0) + \sum_{|J|\ge 1} K_J(z, \xi) w^J \bar{\eta}^J,$$

where J is a multiindex with nonnegative entries. Then for $f \in A^2(\Omega, (-\rho)^r)$ and $z \in \Omega$ we have (F below is the trivial extension of f)

$$f(z) = \int_{\Omega_r^p} K_{\Omega_r^p}(z, 0; \xi, 0) F(\xi, \eta)\, dV(\xi, \eta) + \sum_{|J|\ge 1} \int_{\Omega_r^p} K_J(z, \xi) w^J \bar{\eta}^J F(\xi, \eta)\, dV(\xi, \eta). \tag{2}$$

However, the integrals under the sum on the right-hand side above all vanish.

Using the change of variables $\widetilde{w}_j = w_j/(-\rho(z))^{r/2p}$, one can compute that

$$\int_{|w_1|^{2p/r}+\cdots+|w_p|^{2p/r}<-\rho(z)} dV(w) = (-\rho(z))^r \int_{|\widetilde{w}_1|^{2p/r}+\cdots+|\widetilde{w}_p|^{2p/r}<1} dV(\widetilde{w}). \tag{3}$$

We denote

$$c_{p,r} = \int_{|\widetilde{w}_1|^{2p/r}+\cdots+|\widetilde{w}_p|^{2p/r}<1} dV(\widetilde{w}). \tag{4}$$

Then using (2), (3), and (4) we get

$$\begin{aligned} f(z) &= \int_{\Omega_r^p} K_{\Omega_r^p}(z, 0; \xi, 0) F(\xi, \eta)\, dV(\xi, \eta) \\ &= c_{p,r} \int_\Omega K_{\Omega_r^p}(z, 0; \xi, 0) f(\xi)(-\rho(\xi))^r\, dV(\xi). \end{aligned}$$

Therefore, $c_{p,r} K_{\Omega_r^p}(z, 0; \xi, 0) = K_\Omega^r(z, \xi)$. □

For a C^2-smooth function ρ around a point $P \in \mathbb{C}^n$, $X = (x_1, \dots, x_n) \in \mathbb{C}^n$, and $Y = (y_1, \dots, y_n) \in \mathbb{C}^n$, we define the complex Hessian of ρ at P as

$$H_\rho(P; X, Y) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho(P)}{\partial z_j \partial \bar{z}_k} x_j \bar{y}_k.$$

Furthermore, we use the notation $H_\rho(P; X) = H_\rho(P; X, X)$.

Lemma 9. *Let Ω be a C^2-smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $z_0 \in b\Omega$ be a strongly pseudoconvex point, and Ω_r^p be defined as in (1). Then there exists $s > 0$ such that $(z, w) \in b\Omega_r^p$ is strongly pseudoconvex for $|z - z_0| < s$ and $w_k \neq 0$ for all $1 \leq k \leq p$.*

Proof. Let $\tilde{\rho}(z, w) = \rho(z) + \lambda(w)$ where $\lambda(w) = |w_1|^{2p/r} + \dots + |w_p|^{2p/r}$ and $p \geq r$ an integer. Then $\tilde{\rho}$ is a C^2-smooth function. Assume that $Q = (z, w) \in b\Omega_r^p$ is near z_0 and X is a complex tangential vector to $b\Omega_r^p$ at Q. Then X can be written as $X = X_n + X_p$ where X_n and X_p are the components of X in the z and w variables, respectively. Then

$$H_{\tilde{\rho}}(Q; X) = H_\rho(z; X_n) + H_{\tilde{\rho}}(Q; X_n, X_p) + H_{\tilde{\rho}}(Q; X_p, X_n) + H_\lambda(w; X_p).$$

However, $H_{\tilde{\rho}}(Q; X_n, X_p) = H_{\tilde{\rho}}(Q; X_p, X_n) = 0$ as z and w are decoupled in $\tilde{\rho}$. Then

$$H_{\tilde{\rho}}(Q; X) = H_\rho(z; X_n) + H_\lambda(w; X_p).$$

Let π denote the projection from a neighborhood of $b\Omega$ in $\mathbb{C}^n$ onto $b\Omega$. Then $X_n = X_t + X_\nu$ where X_t is a tangential vector to $b\Omega$ at πz and X_ν is a vector complex normal to $b\Omega$ at πz. Then

$$H_\rho(z; X_n) = H_\rho(z; X_t) + H_\rho(z; X_t, X_\nu) + H_\rho(z; X_\nu, X_t) + H_\rho(z; X_\nu).$$

We note that the complex Hessian H_ρ changes continuously and $w \to 0$ as $z \to z_0$ (here we assume $(z, w) \in b\Omega_r^p$). Furthermore, $X_\nu \to 0$ as $z \to z_0$ (as the complex normal to $b\Omega$ at z_0 is parallel to the complex normal to $b\Omega_r^p$ at $(z_0, 0)$). Then, using the fact that z_0 is a strongly pseudoconvex point, we conclude that there exists $s > 0$ so that

$$H_\rho(z; X_n) \geq \frac{H_\rho(\pi z; X_t)}{2} > 0$$

for $|z - z_0| < s$ and $X_t \neq 0$. Also $H_\lambda(w; X_p) > 0$ whenever $X_p \neq 0$ and $w_k \neq 0$ for all k as λ is strictly plurisubharmonic whenever $w_k \neq 0$ for all k. Therefore, $H_{\tilde{\rho}}(Q; X) > 0$ for $Q = (z, w) \in b\Omega_r^p$ such that $|z - z_0| < s$ and $w_k \neq 0$ for all k. □

The following corollary follows from the previous lemma together with the fact that Ω_r^p has C^2-smooth boundary for $0 < r \leq p$.

Corollary 10. *Let Ω be a C^2-smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $z_0 \in b\Omega$ be a strongly pseudoconvex point, and Ω_r^p be defined as in (1). Then there exists $\varepsilon > 0$ such that $B((z_0, 0), \varepsilon) \cap \Omega_r^p$ is pseudoconvex.*

Next we will prove some statements about compactness of single Toeplitz and Hankel operators.

Lemma 11. *Let $\phi \in L^\infty(\Omega)$, $\{f_j\}$ be a bounded sequence in $A^2(\Omega, (-\rho)^r)$ and F_j be the trivial extension of f_j to Ω_r^p for each j where Ω_r^p is defined as in (1). Assume that $\{H_\phi^{\Omega_r^p} F_j\}$ is convergent in $L^2(\Omega_r^p)$. Then $\{H_\phi^{\Omega,r} f_j\}$ is convergent in $L^2(\Omega, (-\rho)^r)$.*

Proof. We will abuse the notation and denote the trivial extension of ϕ to Ω_r^p by ϕ. We assume that $\{H_\phi^{\Omega_r^p} F_j\}$ is convergent (and hence Cauchy). Let

$$G_j(z, w) = (H_\phi^{\Omega_r^p} F_j)(z, w)$$

and $g_j(z) = G_j(z, 0)$. Then G_j is holomorphic in w because

$$\frac{\partial G_j}{\partial \overline{w}_k} = \frac{\partial}{\partial \overline{w}_k}(I - P^{\Omega_r^p})(F_j\phi) = \frac{\partial (F_j\phi)}{\partial \overline{w}_k} = 0$$

for all j and $1 \le k \le p$. We note that $\partial(F_j\phi)/\partial\overline{w}_k = 0$ as $F_j\phi$ is independent of w_k. Then $|G_j(z, w) - G_k(z, w)|^2$ is subharmonic in w and using the mean value property for subharmonic functions together with (3) and (4) one can show that

$$|g_j(z) - g_k(z)|^2 \le \frac{1}{c_{p,r}(-\rho(z))^r} \int_{|w_1|^{2p/r}+\cdots+|w_p|^{2p/r}<-\rho(z)} |G_j(z, w) - G_k(z, w)|^2 \, dV(w)$$

for $j = 1, 2, \ldots$ and $z \in \Omega$. By integrating over Ω we get

$$c_{p,r}\|g_j - g_k\|^2_{L^2_{(0,1)}(\Omega,(-\rho)^r)} \le \|G_j - G_k\|^2_{L^2_{(0,1)}(\Omega_r^p)}$$

for $j, k = 1, 2, \ldots$. Then $\{g_j\}$ is a Cauchy sequence in $L^2_{(0,1)}(\Omega, (-\rho)^r)$ (and hence convergent) because $\|G_j - G_k\|_{L^2_{(0,1)}(\Omega_r^p)} \to 0$ as $j, k \to \infty$.

Let $h_j(z) = P^{\Omega_r^p}(\phi F_j)(z, 0)$. Then

$$c_{r,p}\|h_j\|^2_{L^2(\Omega,(-\rho)^r)} \le \|P^{\Omega_r^p}(\phi F_j)\|^2_{L^2(\Omega_r^p)} \le \|\phi F_j\|^2_{L^2(\Omega_r^p)} = c_{r,p}\|\phi f_j\|^2_{L^2(\Omega,(-\rho)^r)} < \infty$$

for each j. Hence, $h_j \in A^2(\Omega, (-\rho)^r)$ and $(I - P^{\Omega,r})h_j = 0$ for all j. We get equality between the last terms above because F_j and ϕ are independent of w. Now

$$\begin{aligned}(I - P^{\Omega,r})g_j &= (I - P^{\Omega,r})(\phi f_j - P^{\Omega_r^p}(\phi F_j)(\cdot, 0)) \\ &= (I - P^{\Omega,r})(\phi f_j) - (I - P^{\Omega,r})(h_j) = H_\phi^{\Omega,r} f_j.\end{aligned}$$

Therefore, the sequence $\{H_\phi^{\Omega,r} f_j\}$ is convergent in $L^2(\Omega, (-\rho)^r)$. □

Lemma 12. *Let r be a nonnegative real number and Ω be a C^2-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function ρ. Assume that $\phi \in C(\overline{\Omega})$ such that $\phi(z) = 0$ if z is a strongly pseudoconvex point in $b\Omega$. Then T^r_ϕ is compact about strongly pseudoconvex points on $A^2(\Omega, (-\rho)^r)$.*

Proof. Let $\{f_j\}$ be a sequence in $A^2(\Omega, (-\rho)^r)$ that (without loss of generality) converges to 0 weakly about strongly pseudoconvex points. Then $f_j \to 0$ weakly as $j \to \infty$ and there is a neighborhood U of weakly pseudoconvex points in $b\Omega$ such that

$$\|f_j\|_{L^2(U\cap\Omega,(-\rho)^r)} \to 0 \text{ as } j \to \infty.$$

Using the uniform boundedness principle and the fact that $f_j \to 0$ weakly we conclude that the sequence $\{f_j\}$ is bounded in $A^2(\Omega, (-\rho)^r)$. Furthermore, Cauchy estimates together with Montel's theorem (and the fact that $f_j \to 0$ weakly) imply that $\{f_j\}$ converges to zero uniformly on compact subsets of Ω. Using the fact that $\phi = 0$ on strongly pseudoconvex points, one can show that $\phi f_j \to 0$ in $A^2(\Omega, (-\rho)^r)$. Therefore, $T^r_\phi f_j \to 0$ in $A^2(\Omega, (-\rho)^r)$. That is, T^r_ϕ is compact about strongly pseudoconvex points on $A^2(\Omega, (-\rho)^r)$. □

Let Ω be a domain in $\mathbb{C}^n$. Then $z \in b\Omega$ is said to have a holomorphic (plurisubharmonic) peak function if there exists a holomorphic (plurisubharmonic) f that is continuous on $\overline{\Omega}$ such that $f(z) = 1$ and $|f(w)| < 1$ (or $f(w) < 1$ if f is plurisubharmonic) for $w \in \overline{\Omega} \setminus \{z\}$.

Next we show that any Hankel operator with a symbol continuous on the closure of the domain is compact about strongly pseudoconvex points.

Proposition 13. *Let r be a nonnegative real number, Ω be a C^2-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function ρ, and $\phi \in C(\overline{\Omega})$. Then*

$$H^r_\phi : A^2(\Omega, (-\rho)^r) \to L^2(\Omega, (-\rho)^r)$$

is compact about strongly pseudoconvex points.

Proof. We will prove more (see Corollary 14 below). First of all, for any $\phi \in C(\overline{\Omega})$ there exists $\{\phi_j\} \subset C^1(\overline{\Omega})$ such that $\phi_j \to \phi$ uniformly on $\overline{\Omega}$ as $j \to \infty$. Furthermore, $\{H^r_{\phi_j}\}$ converges to H^r_ϕ in the operator norm and, by Proposition 7, if $H^r_{\phi_j}$ is compact about strongly pseudoconvex points for every j then so is H^r_ϕ. Therefore, for the rest of the proof we will assume that $\phi \in C^1(\overline{\Omega})$. Secondly, the proof for $r = 0$ does not require the inflation argument in the next paragraph and hence it is easier than the case $r > 0$. Since both proofs are similar, except for the inflation argument, in the rest of the proof, we will assume that $r > 0$.

Let $z_0 \in b\Omega$ be a strongly pseudoconvex point. Then, by Corollary 10, the domain $B((z_0, 0), \varepsilon) \cap \Omega^p_r$ is pseudoconvex for small ε. Let $\varepsilon > 0$ be such that

$X_0 = b\Omega \cap \overline{B(z_0, \varepsilon)} \subset \mathbb{C}^n$ consists of strongly pseudoconvex points. Let us define

$$Y = b\Omega_r^p \cap \overline{B((z_0, 0), \varepsilon)} \cap \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^p : w_k = 0 \text{ for some } 1 \le k \le p\},$$
$$X_j = b\Omega_r^p \cap \overline{B((z_0, 0), \varepsilon)} \cap \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^p : |w_k| \ge 1/j \text{ for all } 1 \le k \le p\},$$

for $j = 1, 2, 3, \dots$. Then X_0 is B-regular as any point in X_0 has a holomorphic (hence plurisubharmonic) peak function on $\Omega \subset \mathbb{C}^n$. The same function (by extending it trivially) is also a plurisubharmonic peak function on $\Omega_r^p \subset \mathbb{C}^{n+p}$. Hence, $X_0 \times \{0\}$ is B-regular as a compact set in $\mathbb{C}^{n+p}$. Furthermore, Lemma 9 implies that we can shrink ε, if necessary, so that X_j's are composed of strongly pseudoconvex points for $j \ge 1$. Hence, X_j is B-regular for every $j = 0, 1, 2, \dots$.

Next we will apply a similar idea to Y in lower dimensions. Let us define $Y_1 = \bigcup_{m=1}^{p} Y_1^m$ where

$$Y_1^m = b\Omega_r^p \cap \overline{B((z_0, 0), \varepsilon)} \cap \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^p : w_k = 0 \text{ for } k \ne m\}.$$

We can write Y_1^m as the union of $X_0 \times \{0\}$ together with the compact sets

$$b\Omega_r^p \cap \overline{B((z_0, 0), \varepsilon)} \cap \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^p : |w_m| \ge 1/j,\ w_k = 0 \text{ for } k \ne m\}$$

for $j = 1, 2, 3, \dots$. However, we can think of the sets above as subsets in $\mathbb{C}^n \times \mathbb{C}$ and (by Lemma 9) they are composed of strongly pseudoconvex points. Hence, they are B-regular. Then [Sibony 1987, Proposition 1.9] implies that each Y_1^m is B-regular as it is a countable union of B-regular sets. Hence, applying Sibony's proposition again, we conclude that Y_1 is B-regular. Similarly, we can define $Y_2 \subset Y$ as a countable union of compact sets where all but at most two w_k are equal to 0. Using the same reasoning above adopted for Y_2 we can conclude that Y_2 is B-regular. In a similar fashion, we can define Y_l for $1 \le l \le p-1$ and prove that all of them are B-regular. Hence $Y = \left(\bigcup_{l=1}^{p} Y_l\right) \cup (X_0 \times \{0\})$ is B-regular. Then

$$b(\Omega_r^p \cap B((z_0, 0), \varepsilon)) \subset Y \cup \left(\bigcup_{j=1}^{\infty} X_j\right) \cup (X_0 \times \{0\}) \cup bB((z_0, 0), \varepsilon)$$

is B-regular (satisfies property (P) in Catlin's terminology) and, therefore, the $\bar{\partial}$-Neumann operator on $\Omega_r^p \cap B((z_0, 0), \varepsilon)$ is compact [Straube 2010, Theorem 4.8; Catlin 1984]. Then $H_\phi^{\Omega_r^p \cap B((z_0,0),\varepsilon)}$ is compact (see [Straube 2010, Proposition 4.1]) and Lemma 11 implies that $H_\phi^{\Omega \cap B((z_0,0),\varepsilon),r}$ is compact.

Next we will use local compact solution operators to show that H_ϕ^r is compact about strongly pseudoconvex points. Let $\{f_j\} \subset A^2(\Omega, (-\rho)^r)$ be a sequence weakly convergent about strongly pseudoconvex points. Then there exists an open neighborhood U of the set of weakly pseudoconvex points in $b\Omega$ such that

(i) $\{f_j\}$ is weakly convergent,

(ii) $\|f_j - f_k\|_{L^2(U \cap \Omega, (-\rho)^r)} \to 0$ as $j, k \to \infty$.

Let us choose $\{p_k : k = 1, \dots, m\} \subset b\Omega \setminus U$ and $\varepsilon_k > 0$ (for $k = 1, \dots, m$) such that

(i) $b\Omega \setminus U \subset \bigcup_{k=1}^m B(p_k, \varepsilon_k)$,

(ii) $H_\phi^{k,r} = H_\phi^{B(p_k,\varepsilon_k)\cap\Omega,r}$ is compact on $A^2(B(p_k, \varepsilon_k)\cap\Omega, (-\rho)^r)$ for $k = 1, \dots, m$.

Let us choose a strongly pseudoconvex domain $\Omega_{-1} \Subset \Omega$ and smooth cut-off functions $\chi_{-1} \in C_0^\infty(\Omega_{-1})$, $\chi_0 \in C_0^\infty(U)$, and $\chi_k \in C_0^\infty(B(p_k, \varepsilon))$ for $k = 1, \dots, m$ such that $\sum_{k=-1}^m \chi_k \equiv 1$ on $\overline{\Omega}$.

Let $H_\phi^{-1,r} = H_\phi^{\Omega_{-1},r}$, $H_\phi^{0,r} = H_\phi^{U\cap\Omega,r}$, and $g_j = \sum_{k=-1}^m \chi_k H_\phi^{k,r} f_j$. We note that $H_\phi^{-1,r}$ is compact as $\Omega_{-1} \Subset \Omega$ is strongly pseudoconvex (and $\rho < 0$ on the closure of Ω_{-1}); $\{H_\phi^{0,r} f_j\}$ is convergent as $\{f_j\}$ is convergent in $L^2(U \cap \Omega, (-\rho)^r)$; and by the previous part of this proof, $H_\phi^{k,r}$ is compact for each $k = 1, \dots, m$. Therefore, $\{g_j\}$ is convergent in $L^2(\Omega, (-\rho)^r)$. Furthermore,

$$\bar\partial g_j = f_j \bar\partial\phi + \sum_{k=-1}^m (\bar\partial\chi_k) H_\phi^{k,r} f_j.$$

Then $\left\{\sum_{k=-1}^m (\bar\partial\chi_k) H_\phi^{k,r} f_j\right\}$ is a convergent sequence of $\bar\partial$-closed $(0, 1)$-forms as both $\bar\partial g_j$ and $f_j \bar\partial\phi$ are $\bar\partial$-closed. Let $Z^r : L^2_{(0,1)}(\Omega, (-\rho)^r) \to L^2(\Omega, (-\rho)^r)$ be a bounded linear solution operator to $\bar\partial$ (see [Hörmander 1965]). Let

$$h_j = g_j - Z^r \sum_{k=-1}^m (\bar\partial\chi_k) H_\phi^{k,r} f_j.$$

Then $\{h_j\}$ is convergent and $\bar\partial h_j = f_j \bar\partial\phi$. So by taking projection on the orthogonal complement of $A^2(\Omega, (-\rho)^r)$ we get $(I - P^r)h_j = H_\phi^r f_j$. Therefore, $\{H_\phi^r f_j\}$ is convergent. □

Using the proof of the proposition above we get the following corollary.

Corollary 14. *Let r be a nonnegative real number and Ω be a C^2-smooth bounded pseudoconvex domain in $\mathbb{C}^n$ with a defining function ρ. Assume that Ω satisfies property* (*P*) *of Catlin* (*or B-regularity of Sibony*). *Then*

(i) *$\bar\partial$ has a compact solution operator on $K^2_{(0,1)}(\Omega, (-\rho)^r)$, the weighted $\bar\partial$-closed* $(0, 1)$*-forms,*

(ii) *$H_\phi^r : A^2(\Omega, (-\rho)^r) \to L^2(\Omega, (-\rho)^r)$ is compact for all $\phi \in C(\overline{\Omega})$.*

Proof. Since (ii) follows from (i), we will only prove (i). By a theorem of Diederich and Fornæss [1977], there exists a C^2-smooth defining function ρ_1 and $0 < \eta \le 1$ such that $-(-\rho_1)^\eta$ is a strictly plurisubharmonic exhaustion function for Ω. Since ρ_1 and ρ are comparable on $\overline{\Omega}$ it is enough to prove that $\bar\partial$ has a compact solution operator on $K^2_{(0,1)}(\Omega, (-\rho_1)^r)$.

Let $s = r/\eta \ge 0$ and q be an integer such that $s \le q$. We define

$$\Omega_s^q = \left\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^q : -(-\rho_1(z))^\eta + \lambda(w) < 0\right\},$$

where $\lambda(w) = |w_1|^{2q/s} + \cdots + |w_q|^{2q/s}$. Then $-(-\rho_1)^\eta + \lambda$ is a bounded C^2-smooth plurisubharmonic function and Ω_s^q is pseudoconvex. Furthermore, the first part of the proof of Proposition 13 shows that Ω_s^q satisfies property (P).

Let $\{f_j\}$ be a bounded sequence in $K^2_{(0,1)}(\Omega, (-\rho_1)^r)$. Then $\{F_j\}$ is a bounded sequence in $K^2_{(0,1)}(\Omega_s^q)$. As shown in the first part of this proof, Ω_s^q is a bounded (not necessarily C^2-smooth) pseudoconvex domain with property (P). Then $\{\bar\partial^* N^{\Omega_s^q} F_j\}$ has a convergent subsequence in $L^2(\Omega_s^q)$ where $N^{\Omega_s^q}$ is the $\bar\partial$-Neumann operator on $L^2_{(0,1)}(\Omega_s^q)$. By the proof of Proposition 8 and the fact that $\bar\partial^* N^{\Omega_s^q} F_j$ is holomorphic in w, we conclude that $\bar\partial^* N^{\Omega_s^q} F_j(\,\cdot\,, 0) \in L^2(\Omega, (-\rho_1)^r)$. Further, $\bar\partial\bar\partial^* N^{\Omega_s^q} F_j(\,\cdot\,, 0) = f_j$ for all j and $\{\bar\partial^* N^{\Omega_s^q} F_j(\,\cdot\,, 0)\}$ has a convergent subsequence in $L^2(\Omega, (-\rho_1)^r)$. Therefore, $\bar\partial$ has a compact solution operator $R\bar\partial^* N^{\Omega_s^q} E$ on $K^2_{(0,1)}(\Omega, (-\rho_1)^r)$ where E is the trivial extension operator and R is the restriction from Ω_s^q onto Ω. □

The following lemma is essentially contained in the proof of [Arazy and Engliš 2001, Proposition 1.3]. We present it here for the convenience of the reader.

Lemma 15. *Let r be a nonnegative real number, Ω be a bounded domain in $\mathbb{C}^n$, and $\phi \in C(\overline{\Omega})$. Assume that $z_0 \in b\Omega$ has a holomorphic peak function. Then*

$$\lim_{z \to z_0} B_r T^r_\phi(z) = \phi(z_0).$$

Proof. First, we prove that for any neighborhood U of z_0,

$$\int_{\Omega \setminus U} |k^r_z(w)|^2 (-\rho(w))^r \, dV(w) \to 0 \quad \text{as } z \to z_0. \tag{5}$$

Indeed, for given U and $\varepsilon > 0$ first we choose a holomorphic peak function g such that $|g(w)| \le \varepsilon$ for all $w \in \Omega \setminus U$. This can be simply done by taking a high enough power of the holomorphic peak function g. Then we choose $\delta > 0$ such that if $|z - z_0| < \delta$ and $z \in \Omega$ then $|g(z)| > 1 - \varepsilon$. In this case,

$$\begin{aligned}
\int_U |k^r_z(w)|^2 (-\rho(w))^r \, dV(w) \ge& \int_U |g(w)| |k^r_z(w)|^2 (-\rho(w))^r \, dV(w) \\
\ge& \left| \int_\Omega g(w) |k^r_z(w)|^2 (-\rho(w))^r \, dV(w) \right| \\
&\quad - \left| \int_{\Omega \setminus U} g(w) |k^r_z(w)|^2 (-\rho(w))^r \, dV(w) \right| \\
\ge& |g(z)| - \int_{\Omega \setminus U} |g(w)| |k^r_z(w)|^2 (-\rho(w))^r \, dV(w) \\
\ge& 1 - \varepsilon - \varepsilon \int_{\Omega \setminus U} |k^r_z(w)|^2 (-\rho(w))^r \, dV(w) \\
\ge& 1 - 2\varepsilon,
\end{aligned}$$

whenever $|z-z_0|<\delta$. This implies that for a given neighborhood U and $\varepsilon>0$, there exists $\delta>0$ such that if $|z-z_0|<\delta$ then

$$\int_{\Omega\setminus U}|k_z^r(w)|^2(-\rho(w))^r\,dV(w)\le\varepsilon.$$

This gives (5).

Now for $\varepsilon>0$, we choose a neighborhood U of z such that $|\phi(w)-\phi(z_0)|\le\varepsilon$ for all $w\in U$. Then for this neighborhood U and the same ε we choose $\delta>0$ such that if $|z-z_0|<\delta$ then $\int_{\Omega\setminus U}|k_z^r(w)|^2(-\rho(w))^r\,dV(w)\le\varepsilon/(1+2\|\phi\|_{L^\infty})$. In this case,

$$\begin{aligned}|B_rT_\phi^r(z)-\phi(z_0)|\le&\int_\Omega|\phi(w)-\phi(z_0)||k_z^r(w)|^2(-\rho(w))^r\,dV(w)\\ =&\int_U|\phi(w)-\phi(z_0)||k_z^r(w)|^2(-\rho(w))^r\,dV(w)\\ &+\int_{\Omega\setminus U}|\phi(w)-\phi(z_0)||k_z^r(w)|^2(-\rho(w))^r\,dV(w)\\ \le&\varepsilon\int_U|k_z^r(w)|^2(-\rho(w))^r\,dV(w)\\ &+2\|\phi\|_{L^\infty}\int_{\Omega\setminus U}|k_z^r(w)|^2(-\rho(w))^r\,dV(w)\\ \le&\varepsilon+\varepsilon=2\varepsilon.\end{aligned}$$

This indeed concludes $\lim_{z\to z_0}B_rT_\phi^r(z)=\phi(z_0)$. □

We note that on any bounded domain, we have (see [Čučković and Şahutoğlu 2014, Lemma 1])

$$T_{\phi_2}^rT_{\phi_1}^r=T_{\phi_2\phi_1}^r-H_{\bar\phi_2}^{r*}H_{\phi_1}^r.$$

Using the fact above inductively one can prove the following lemma.

Lemma 16. *Let r be a nonnegative real number and Ω be a C^1-smooth bounded domain in $\mathbb{C}^n$ with a defining function ρ. Suppose $\phi_1,\dots,\phi_m\in L^\infty(\Omega)$. Then*

$$\begin{aligned}T_{\phi_m}^rT_{\phi_{m-1}}^r\cdots T_{\phi_2}^rT_{\phi_1}^r=&T_{\phi_m\phi_{m-1}\cdots\phi_2\phi_1}^r-T_{\phi_m}^rT_{\phi_{m-1}}^r\cdots T_{\phi_3}^rH_{\bar\phi_2}^{r*}H_{\phi_1}^r\\ &-T_{\phi_m}^rT_{\phi_{m-1}}^r\cdots T_{\phi_4}^rH_{\bar\phi_3}^{r*}H_{\phi_2\phi_1}^r-\cdots-H_{\bar\phi_m}^{r*}H_{\phi_{m-1}\cdots\phi_2\phi_1}^r\\ =&T_{\phi_m\phi_{m-1}\cdots\phi_2\phi_1}^r+S^r,\end{aligned}$$

where S^r is a finite sum of finite products of operators and each product starts with a Hankel operator.

Therefore, if the symbols $\phi_1,\dots,\phi_m$ are continuous on $\overline\Omega$ we can write

$$T_\phi^r\cdots T_{\phi_m}^r=T_{\phi_1\cdots\phi_m}^r+S^r,\tag{6}$$

where S^r is a finite sum of finite products of operators such that each product starts with a Hankel operator with symbol continuous on $\overline{\Omega}$.

We state the lemma below for general weights $\mu(z)$ (not only the ones of the form $(-\rho)^k$) that are nonnegative (can vanish on the boundary) and continuous on Ω. The weights of this form are called admissible weights (see [Pasternak-Winiarski 1990]) and the corresponding weighted Bergman projections and kernels are well defined. We say two weights μ_1 and μ_2 are comparable if there exists $c > 0$ such that $c^{-1}\mu_1 < \mu_2 < c\mu_1$ on Ω.

Lemma 17. *Let Ω be a domain in $\mathbb{C}^n$ and μ_1 and μ_2 be comparable admissible weights. Let $k_z^{\mu_j}$ be the normalized Bergman kernel corresponding to μ_j for $j = 1, 2$, and $z_0 \in b\Omega$. Then $k_z^{\mu_1} \to 0$ weakly as $z \to z_0$ if and only if $k_z^{\mu_2} \to 0$ weakly as $z \to z_0$.*

Proof. It is enough to show one direction. So we will show that if $k_z^{\mu_1} \to 0$ weakly as $z \to z_0$ then $k_z^{\mu_2} \to 0$ weakly as $z \to z_0$. Since μ_1 and μ_2 are equivalent measures we have $A^2(\Omega, d\mu_1) = A^2(\Omega, d\mu_2)$ and there exists $C > 1$ such that

$$\frac{\|f\|_{\mu_1}}{C} \leq \|f\|_{\mu_2} \leq C\|f\|_{\mu_1}$$

for all $f \in A^2(\Omega, d\mu_1)$. We remind the reader that for $z \in \Omega$ we have

$$K_{\mu_j}(z, z) = \sup\{|f(z)|^2 : \|f\|_{\mu_j} \leq 1\},$$

where K_{μ_j} is the Bergman kernel corresponding to μ_j. Then K_{μ_1} and K_{μ_2} are equivalent on the diagonal in the sense that there exists $D = C^2 > 1$ such that

$$\frac{K_{\mu_1}(z, z)}{D} \leq K_{\mu_2}(z, z) \leq DK_{\mu_1}(z, z).$$

Now we assume that $k_z^{\mu_1} \to 0$ weakly as $z \to z_0$. Let us fix $f \in A^2(\Omega, d\mu_1)$. Then we have

$$\frac{f(z)}{\sqrt{K_{\mu_1}(z, z)}} = \langle f, k_z^{\mu_1}\rangle_{\mu_1} \to 0 \quad \text{as } z \to z_0.$$

Then

$$\langle f, k_z^{\mu_2}\rangle_{\mu_2} = \frac{f(z)}{\sqrt{K_{\mu_2}(z, z)}} \to 0 \quad \text{as } z \to z_0.$$

Therefore, we showed that if $k_z^{\mu_1} \to 0$ weakly as $z \to z_0$ then $k_z^{\mu_2} \to 0$ weakly as $z \to z_0$. □

Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and $z_0 \in b\Omega$. Then we call z_0 a *bumping point* if for any $\delta > 0$ there exists a pseudoconvex domain Ω_1 such that $\{z_0\} \cup \Omega \subset \Omega_1 \subset \Omega \cup B(z_0, \delta)$.

Lemma 18. *Let r be a nonnegative real number, Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with Lipschitz boundary, and $z_0 \in b\Omega$ be a bumping point. Then $k_z^r \to 0$ weakly as $z \to z_0$.*

Proof. By Lemma 17, without loss of generality, we assume that ρ denotes the negative distance to the boundary of Ω.

Let us fix $f \in A^2(\Omega, (-\rho)^r)$ and choose $r_1, r_2 > 0$ so that $0 < r_1 < r_2$ and the outward unit vector ν is transversal to $B(z_0, 2r_2) \cap b\Omega$. Since z_0 is a bumping point we choose a bounded pseudoconvex domain Ω_1 such that

$$\{z_0\} \cup \Omega \subset \Omega_1 \subset \Omega \cup B(z_0, r_1).$$

So even though Ω_1 contains a small neighborhood of z_0, we have

$$\Omega \setminus B(z_0, r_1) = \Omega_1 \setminus B(z_0, r_1).$$

Let us choose $\chi \in C_0^\infty(B(z_0, r_2))$ such that $\chi \equiv 1$ on a neighborhood of $\overline{B(z_0, r_1)}$. For $\varepsilon > 0$ small we define $f_\varepsilon(z) = f(z - \varepsilon\nu)$ and $g_\varepsilon = (1-\chi)f + \chi f_\varepsilon$. Then

(i) $f_\varepsilon \in A^2(\Omega \cap B(z_0, r_2), (-\rho)^r)$ and $f_\varepsilon \to f$ in $L^2(\Omega \cap B(z_0, r_2), (-\rho)^r)$,

(ii) $g_\varepsilon|_{\Omega \cap B(z_0, r_2)}$ is C^∞-smooth and $g_\varepsilon \to f$ in $L^2(\Omega, (-\rho)^r)$ as $\varepsilon \to 0$.

Let ρ_1 and $\mathrm{Supp}(\bar\partial\chi)$ denote the negative distance to the boundary of Ω_1 and the support of $\bar\partial\chi$, respectively. Then $\mathrm{Supp}(\bar\partial\chi) \cap \Omega = \mathrm{Supp}(\bar\partial\chi) \cap \Omega_1$ and $-\rho$ and $-\rho_1$ are equivalent on the support of $\bar\partial\chi$. We note that g_ε is well defined on Ω and not on Ω_1. However, $\bar\partial g_\varepsilon = 0$ on $\Omega \cap B(z_0, r_1)$ as $\bar\partial\chi = 0$ on $B(z_0, r_1)$ for all small $\varepsilon > 0$. Hence $\bar\partial g_\varepsilon$ can be extended trivially to be defined on Ω_1 as a $\bar\partial$-closed $(0, 1)$-form on Ω_1. Then there exists $C > 0$ such that

$$\|\bar\partial g_\varepsilon\|_{L^2(\Omega_1,(-\rho_1)^r)} \le C\|f - f_\varepsilon\|_{L^2(\Omega\cap B(z_0,r_2),(-\rho)^r)} \|\bar\partial\chi\|_{L^\infty(B(z_0,r_2))} \to 0 \quad \text{as } \varepsilon \to 0.$$

Next we will use Hörmander's theorem [1965] with the plurisubharmonic exponential weight $-r\log(-\rho_1)$. We note that $-\log(-\rho_1)$ is plurisubharmonic because Ω_1 is pseudoconvex. Then using Hörmander's theorem we get a constant $c_{\Omega_1} > 0$ (depending on Ω_1) and $h_\varepsilon \in L^2(\Omega_1)$ such that $\bar\partial h_\varepsilon = \bar\partial g_\varepsilon$ and

$$\|h_\varepsilon\|_{L^2(\Omega_1,(-\rho_1)^r)} \le c_{\Omega_1}\|\bar\partial g_\varepsilon\|_{L^2(\Omega_1,(-\rho_1)^r)}.$$

Furthermore, since $\bar\partial$ is elliptic on the interior and $\bar\partial g_\varepsilon$ is C^∞-smooth on Ω_1, we have $h_\varepsilon \in C^\infty(\Omega_1)$.

We define $\tilde f_n = g_{1/n} - h_{1/n}$ on Ω. We note that while $\bar\partial g_\varepsilon$ is even defined on all of Ω_1, g_ε and, hence, $\tilde f_{1/n}$ in general, are not. Then we have

(i) $\tilde f_n \in A^2(\Omega, (-\rho)^r)$ and $\tilde f_n \to f$ in $A^2(\Omega, (-\rho)^r)$,

(ii) $\tilde f_n|_{\Omega \cap B(z_0, r_1)} \in C^\infty(\overline{\Omega \cap B(z_0, r_1)})$.

So $\{\tilde{f}_n\}$ is a sequence converging to f and each member of the sequence is smooth up to the boundary of Ω on a neighborhood of z_0.

Finally, we will show weak convergence of k_z^r to 0 as $z \to z_0$.

$$\begin{aligned}|\langle f, k_z^r\rangle_{A^2(\Omega,(-\rho)^r)}| \leq &|\langle f-\tilde{f}_n, k_z^r\rangle_{A^2(\Omega,(-\rho)^r)}| + |\langle \tilde{f}_n, k_z^r\rangle_{A^2(\Omega,(-\rho)^r)}| \\ \leq &\|f-\tilde{f}_n\|_{L^2(\Omega,(-\rho)^r)} + \frac{|\tilde{f}_n(z)|}{\sqrt{K_r(z,z)}}.\end{aligned}$$

The first term on the right-hand side can be made arbitrarily small for large enough n, because $\|f-\tilde{f}_n\|_{L^2(\Omega,(-\rho)^r)} \to 0$ as $n \to \infty$. So for $\delta > 0$ given we choose n_δ so that $\|f-\tilde{f}_{n_\delta}\|_{L^2(\Omega,(-\rho)^r)} \leq \delta$. Then since $\tilde{f}_{n_\delta}$ is C^∞-smooth on $\overline{\Omega \cap B(z_0,r_1)}$ (and $K_r(z,z) \to \infty$ as $z \to z_0$), we conclude that $|\tilde{f}_{n_\delta}(z)|/\sqrt{K_r(z,z)} \to 0$ as $z \to z_0$. Hence, $\limsup_{z\to z_0}|\langle f, k_z^r\rangle| \leq \delta$ for arbitrary $\delta > 0$. Therefore, $k_z^r \to 0$ weakly as $z \to z_0$. □

Now we are ready to prove Theorem 4.

Proof of Theorem 4. In the case $r = 0$, the proof of the theorem simplifies greatly as inflation and the related techniques are unnecessary. So we will prove the more difficult case, $r > 0$.

First we assume that T is compact about strongly pseudoconvex points. Let Ω_r^p be defined as in (1) and $z_0 \in b\Omega$ be a strongly pseudoconvex point. Since small C^2-perturbations of strongly pseudoconvex points stay pseudoconvex, z_0 is a bumping point for Ω. Then Lemma 18 implies that $k_z^r \to 0$ weakly as $z \to z_0$. Furthermore, there exists an open neighborhood U of z_0 such that weakly pseudoconvex points are contained in $b\Omega \setminus \overline{U}$; and, as in the proof of (5), one can show that

$$\|k_z^r\|_{L^2(\Omega\setminus\overline{U},(-\rho)^r)} \to 0 \text{ as } z \to z_0.$$

Therefore, $\{k_z^r\}$ converges to 0 weakly about strongly pseudoconvex points as $z \to z_0$. Moreover, since T is compact about strongly pseudoconvex points (such operators map sequences of holomorphic functions weakly convergent about strongly pseudoconvex points to convergent sequences) we conclude that

$$B_rT(z) = \langle Tk_z^r, k_z^r\rangle_{A^2(\Omega,(-\rho)^r)} \to 0$$

as $z \to z_0$.

Next we prove the other direction. As a first step we assume that T is a finite sum of finite products of Toeplitz operators on $A^2(\Omega,(-\rho)^r)$ with symbols continuous on $\overline{\Omega}$. Furthermore, we assume that

$$\lim_{z\to z_0} B_rT(z) = 0$$

for any strongly pseudoconvex point $z_0 \in b\Omega$.

Lemma 16 implies that

$$T = T^r_\phi + S^r, \tag{7}$$

where $\phi \in C(\overline{\Omega})$ and S^r is a sum of operators that start with a Hankel operator with symbol continuous on $\overline{\Omega}$.

Lemma 15 implies that

$$\lim_{z \to z_0} B_r T^r_\phi(z) = \phi(z_0) \tag{8}$$

as strongly pseudoconvex points have holomorphic peak functions (see [Range 1986, Theorem 1.13 in Ch VI]).

By Proposition 13, the operator H^r_ψ is compact about strongly pseudoconvex points for any $\psi \in C(\overline{\Omega})$. Then $H^r_\psi k^r_z \to 0$ as $z \to z_0$ for any $\psi \in C(\overline{\Omega})$ because, as proven in the first part of this proof, $k^r_z \to 0$ weakly about strongly pseudoconvex points as $z \to z_0$. Hence, $B_r S^r(z) \to 0$ as $z \to z_0$. Combining this with (7) and (8) we can conclude that

$$\phi(z_0) = \lim_{z \to z_0} B_r T(z) = 0.$$

Since z_0 was an arbitrary strongly pseudoconvex point, we have $\phi = 0$ on all the strongly pseudoconvex boundary points. Then Lemma 12 and the fact that S^r is compact about strongly pseudoconvex points imply that T is compact about strongly pseudoconvex points.

Finally, we assume $T \in \mathscr{T}(\overline{\Omega}, (-\rho)^r)$. Then, using Lemma 16, for every $\varepsilon > 0$ there exists $\phi_\varepsilon \in C(\overline{\Omega})$ and an operator S^r_ε, compact about strongly pseudoconvex points, such that

$$\|T + T^r_{\phi_\varepsilon} + S^r_\varepsilon\| \leq \varepsilon.$$

Then for $z \in \Omega$ we have

$$\begin{aligned} |B_r T(z) + B_r T^r_{\phi_\varepsilon}(z) + B_r S^r_\varepsilon(z)| &= |\langle T k^r_z + T^r_{\phi_\varepsilon} k^r_z + S^r_\varepsilon k^r_z, k^r_z \rangle_r| \\ &\leq \|T + T^r_{\phi_\varepsilon} + S^r_\varepsilon\| \\ &\leq \varepsilon. \end{aligned}$$

Since $B_r S^r_\varepsilon(z) \to 0$ and $B_r T^r_{\phi_\varepsilon}(z) \to \phi_\varepsilon(z_0)$ as $z \to z_0$ (and we assume that $B_r T(z) \to 0$ as $z \to z_0$), we have $|\phi_\varepsilon(z_0)| \leq \varepsilon$. That is, $|\phi_\varepsilon|$ is less than or equal to ε on strongly pseudoconvex points of Ω. We choose $\psi_\varepsilon \in C(\overline{\Omega})$ such that $\psi_\varepsilon = 0$ on strongly pseudoconvex boundary points of Ω and

$$\sup\{|\psi_\varepsilon(z) - \phi_\varepsilon(z)| : z \in \overline{\Omega}\} \leq 2\varepsilon.$$

Then Lemma 12 implies that $T^r_{\psi_\varepsilon}$ is compact about strongly pseudoconvex points and

$$\|T^r_{\phi_\varepsilon} - T^r_{\psi_\varepsilon}\| \leq 2\varepsilon.$$

Hence

$$\|T + T^r_{\psi_\varepsilon} + S^r_\varepsilon\| \le \|T + T^r_{\phi_\varepsilon} + S^r_\varepsilon\| + \|T^r_{\psi_\varepsilon} - T^r_{\phi_\varepsilon}\| \le 3\varepsilon.$$

Therefore, T is in the norm closure of the compact about strongly pseudoconvex points operators. Finally, Proposition 7 implies that T is compact about strongly pseudoconvex points. □

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