Pacific Journal of Mathematics

MODULI SPACES OF STABLE PAIRS

YINBANG LIN

Volume 294 No. 1 May 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 294, No. 1, 2018
dx.doi.org/10.2140/pjm.2018.294.123

MODULI SPACES OF STABLE PAIRS

YINBANG LIN

We construct a moduli space of stable pairs over a smooth projective variety, parametrizing morphisms from a fixed coherent sheaf to a varying sheaf of fixed topological type, subject to a stability condition. This generalizes the notion used by Pandharipande and Thomas, following Le Potier, where the fixed sheaf is the structure sheaf of the variety. We then describe the relevant deformation and obstruction theories. We also show the existence of the virtual fundamental class in special cases.

1. Introduction

The past couple of decades of research have highlighted the importance of moduli spaces of *decorated* sheaves, which are sheaves with additional structure, such as one or more sections. Moduli spaces of rank two vector bundles with a section on a Riemann surface X,

$$E \to X \quad \text{and} \quad \alpha : \mathscr{O}_X \to E,$$

were used in [Thaddeus 1994] to deduce an important invariant of the moduli space of sheaves, the Verlinde number. More recently, Pandharipande and Thomas [2009; 2010] studied stable pairs (E, α), where E is a sheaf with dimension 1 support, on a Calabi–Yau threefold. They showed that invariants of this moduli space are closely related to the Gromov–Witten invariants of the Calabi–Yau threefold.

MSC2010: primary 14D20; secondary 14J60, 14N35.
Keywords: moduli space, stable pair, deformation, obstruction, virtual fundamental class.

We would like to broaden our perspective and replace the structure sheaf by a general coherent sheaf. Subject to a stability condition, we would like to parametrize morphisms of coherent sheaves,

$$\alpha : E_0 \to E,$$

where E_0 is a fixed coherent sheaf. We will denote such a morphism as a pair

$$(E, \alpha).$$

Let us set up the problem. We will work over an algebraically closed field k of characteristic 0. We denote by X a smooth projective variety of dimension n, with a fixed polarization $\mathscr{O}_X(1)$. We fix a coherent sheaf E_0 on X. Let P be a fixed polynomial of degree $d \le n$. Let $\delta \in \mathbb{Q}[m]$ be 0 or a polynomial with a positive leading coefficient; this will play the role of parameter for stability conditions.

When δ is large, i.e., $\deg \delta \ge \deg P$, a pair (E, α), such that the Hilbert polynomial of E equals P, is *stable* if E is pure and the support of $\operatorname{coker} \alpha$ has dimension strictly smaller than d. This is the most significant case geometrically. In this case, the moduli space of stable pairs is closely related to Grothendieck's Quot scheme. But intersection theory on the moduli space of stable pairs is expected to be more tractable than that on the Quot scheme. This is because we impose the purity condition on the sheaves underlying stable pairs, which allows us to avoid some large dimensional components.

The moduli space of stable pairs in the large δ case is expected to have interesting applications to the enumerative geometry of higher rank sheaves on a surface X. In particular, a potential application is towards the strange duality conjecture. The conjecture over curves was proved [Belkale 2008; Marian and Oprea 2007] by studying intersection theory on related Grassmannians and Quot schemes. It is reasonable to expect that a similar method using the moduli space of stable pairs will work for the surface case.

The study of stable pairs by Pandharipande and Thomas was built on Le Potier's work [1993] on coherent systems. The moduli space of coherent systems was also used to study the Donaldson numbers of the moduli space of sheaves [He 1998]. A *coherent system* on X is a pair (Γ, E), where E is a coherent sheaf and $\Gamma \subset H^0(X, E)$ is a subspace of global sections. A pair $(E, \alpha : \mathscr{O}_X \to E)$ can be viewed as a coherent system $(k\langle\alpha\rangle, E)$. However, when $\mathscr{O}_X$ is replaced by, for example, $\mathscr{O}_X^{\oplus 2}$, the pair can no longer be viewed as a coherent system, because the map

$$H^0(\alpha) : k^{\oplus 2} \to H^0(E)$$

may not be injective. Aside from this issue, there is yet another difference between pairs and coherent systems: while the morphism α is part of the data of the pair, the coherent system only remembers the image of $H^0(\alpha)$. Consequently, when one tries to parametrize $\alpha : E_0 \to E$ for general E_0, Le Potier's construction does not automatically apply. But the main ingredients of constructing the moduli space remain the same: Grothendieck's Quot scheme [1961b] and Mumford's geometric invariant theory [Mumford et al. 1994].

Theorem 1.1 (existence of moduli spaces). *For the moduli functor $\mathcal{S}_{E_0}(P, \delta)$ of S-equivalence classes of δ-semistable pairs, there exists a projective coarse moduli space $S_{E_0}(P, \delta)$. The moduli functor $\mathcal{S}^s_{E_0}(P, \delta)$ of equivalence classes of δ-stable pairs is represented by an open subscheme $S^s_{E_0}(P, \delta)$ of $S_{E_0}(P, \delta)$.*

Deformation-obstruction theory of stable pairs is very similar to that of the Quot scheme. For a quotient $q : E_0 \twoheadrightarrow F$, let $G = \ker q$, then we have a short exact sequence,

$$0 \to G \to E_0 \to F \to 0.$$

The deformation space and the obstruction space are $\mathrm{Hom}(G, F)$ and $\mathrm{Ext}^1(G, F)$. Notice that G is quasi-isomorphic to the cochain complex $J^\bullet = \{E_0 \to F\}$, and the deformation space and the obstruction space of this quotient are isomorphic to $\mathrm{Hom}(J^\bullet, F)$ and $\mathrm{Ext}^1(J^\bullet, F)$, respectively.

The deformation-obstruction problem of stable pairs has a similar answer. Let $\mathcal{A}rt_k$ be the category of local Artinian k-algebras with residue field k. Let $A, B \in \mathrm{Ob}\,\mathcal{A}rt_k$ and

$$0 \to K \to B \xrightarrow{\sigma} A \to 0$$

be a *small* extension, i.e., $\mathfrak{m}_B K = 0$. Suppose (E, α) is a stable pair. Let $I^\bullet$ denote the following cochain complex concentrating at degree 0 and 1:

$$I^\bullet = \{E_0 \xrightarrow{\alpha} E\}.$$

Theorem 1.2 (deformation-obstruction). *Suppose $\alpha_A : E_0 \otimes_k A \to E_A$ is a morphism over $X_A = X \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,A$ extending α, where E_A is a coherent sheaf flat over A. There is a class,*

$$\mathrm{ob}(\alpha_A, \sigma) \in \mathrm{Ext}^1(I^\bullet, E \otimes K),$$

such that there exists a flat extension of α_A over X_B if and only if $\mathrm{ob}(\alpha_A, \sigma) = 0$. If extensions exist, the space of extensions is a torsor under

$$\mathrm{Hom}(I^\bullet, E \otimes K).$$

In some special cases, $\mathrm{Ext}^i(I^\bullet, E) \neq 0$ only when $i = 0, 1$. In these cases, we will demonstrate the existence of the virtual fundamental class, which is important for the study of intersection theory on the moduli spaces.

Theorem 1.3 (virtual fundamental class). *Suppose X is a surface, E_0 is torsion-free, $\deg P = 1$, and $\deg \delta \geq 1$. Then the moduli space $S_{E_0}(P, \delta)$ of stable pairs admits a virtual fundamental class.*

The virtual fundamental class can be used to define invariants of the surface. Kool and Thomas [2014a; 2014b] studied stable pairs invariants with $E_0 \cong \mathscr{O}_X$ on surfaces, using the reduced obstruction theory, which is necessary. We will address the intersection theory of the moduli space of stable pairs on a surface in future work.

After this project was completed, we learned about the article [Wandel 2015] where the stability condition for pairs had been defined. When $\deg \delta < \deg P$, Theorem 1.1 of this paper had been stated as the main theorem, Theorem 3.8, in [Wandel 2015]. In the large δ case,

$$\deg \delta \geq \deg P,$$

the linearized ample line bundle needs to be chosen differently, as in (4-4), for the GIT construction. In this paper, the construction of the moduli space focuses on the large δ case, which is geometrically important but has not been treated in [Wandel 2015]. The construction is carried out from a basic level. For example, Lemma 3.5 is shown for characterizing stability in terms of global sections instead of Hilbert polynomials. As preparation, Section 2 introduces the notion of stable pair and states basic properties of pairs. Section 3 studies the boundedness of the family of stable pairs. Proofs of statements that have been proved in [Wandel 2015] are omitted. This paper also contains, in Section 5, the deformation-obstruction theory, captured by Theorem 1.2, which holds for all δ's, small or large. Section 6 shows the existence of the virtual fundamental class in special geometries (see Theorem 1.3). Section 7 gives examples of smooth moduli spaces and calculates their topological Euler characteristics.

We recently learned that the stable pair moduli space for $\deg \delta \geq \deg P$ was also previously studied in [Kollár 2008], where it appears as the moduli space of quotient husks. The author constructed it as a bounded proper separated algebraic space, and used it to study an analogue of the flattening decomposition theorem for reflexive hulls. The current paper settles affirmatively the question raised in [Kollár 2008] regarding the projectivity of the space.

We finally note that once it is constructed for $\deg \delta < \deg P$, the moduli space is available in an indirect way for $\deg \delta \geq \deg P$ as well. This follows from two facts: the set of critical values[1] is finite and the largest critical polynomial $\delta_{\max}$ has degree $< \deg P$. Let δ' be of degree $\deg P - 1$ and larger than $\delta_{\max}$. Then, for any δ with $\deg \delta \geq \deg P$, we have $S_{E_0}(P, \delta) \cong S_{E_0}(P, \delta')$. Although this observation is not made in [Wandel 2015], the author proves the set of critical δ's is finite.

This indirect argument does not, however, yield the linearized ample line bundle for $S_{E_0}(P, \delta)$ with $\deg \delta \geq \deg P$. For stability polynomials δ' with $\deg \delta' < \deg P$, the linearization depends directly on δ'; the highest critical polynomial $\delta_{\max}$ cannot

[1] A critical value is a value such that as δ crosses over it, the moduli space $S_{E_0}(P, \delta)$ changes.

be determined explicitly, however, since the boundedness which underlies the finiteness of the set of critical stability values is itself not explicit.

For some applications, it is nevertheless important to know the line bundle explicitly. A natural problem to study next is that of wall-crossing formulas, using Thaddeus' master space [Thaddeus 1996; Mochizuki 2009]. The construction of the master space requires the linearized ample line bundle. So, it is important to construct the moduli space directly via GIT and obtain the ample line bundle. We will address the problem of wall-crossing formulas in future work.

2. Basic properties of stable pairs

2A. ***Preliminaries on coherent sheaves.*** For a coherent sheaf E on $(X, \mathscr{O}_X(1))$, we denote by P_E its *Hilbert polynomial* . Recall that we can write the Hilbert polynomial in the form

$$P_E(m) = \sum_{i=0}^{d} a_i(E) \frac{m^i}{i!},$$

where d is the dimension of the support of E, which we simply write as $\dim E$, and $a_i(E) \in \mathbb{Q}$. We denote by

$$r(E) = a_d(E)$$

the *multiplicity* of E. The *reduced* Hilbert polynomial is

$$p_E = \frac{P_E}{r(E)}.$$

The *slope* of E is

$$\mu(E) = \frac{a_{d-1}(E)}{a_d(E)}.$$

A coherent sheaf E is *pure* if there is no subsheaf with lower dimensional support. It is *semistable* (respectively, *slope-semistable*) if it is pure and there is no subsheaf with larger reduced Hilbert polynomial (respectively, slope). For a pure sheaf, there is a *Harder–Narasimhan filtration* with respect to the slope

$$0 \subsetneqq E_1 \subsetneqq E_2 \subsetneqq \cdots \subsetneqq E_l = E,$$

where E_{t+1}/E_t is slope semistable and $\mu(E_t/E_{t-1}) > \mu(E_{t+1}/E_t)$, for $t \in [1, l-1]$. We shall denote $\mu_{\max}(E) = \mu(E_1)$ and $\mu_{\min}(E) = \mu(E_l/E_{l-1})$.

To construct the moduli space via GIT, the first step is to prove a boundedness result. For our convenience, we group a sequence of boundedness results here.

Theorem 2.1 (Grothendieck). *Suppose F is a pure coherent $\mathscr{O}_X$-module of dimension d. Then*:

(i) *The slopes of nonzero coherent subsheaves are bounded above.*

(ii) *The family of subsheaves $F' \subset F$ with slopes bounded below, such that the quotient F/F' is pure and of dimension d, is bounded.*

We can also make a statement similar to the second assertion about the boundedness of quotients. For the proof of this basic theorem, see [Grothendieck 1961b, Lemma 2.5].

Let Y be the scheme-theoretic support of a pure sheaf E of dimension d and multiplicity r. We include the following results discussed in [Le Potier 1993].

Lemma 2.2. *The degree of Y is no larger than r^2.*

Proof. This is clear from an equivalent definition of multiplicity [Le Potier 1993, Definition 2.1]. □

Lemma 2.3. *The minimum slope $\mu_{\min}(\mathscr{O}_Y)$ is bounded below by a constant determined by n, r, and d.*

Proof. See [Le Potier 1993, Lemma 2.12]. □

The following statement is crucial to our proof of boundedness.

Theorem 2.4 [Simpson 1994, Theorem 1.1]. *Let C be a rational constant. The family of pure coherent sheaves E with Hilbert polynomial $P_E = P$, such that $\mu_{\max}(E) \le C$, is bounded.*

Bounding $\mu_{\max}$ from above is equivalent to bounding $\mu_{\min}$ from below, because the Hilbert polynomial is additive in a short exact sequence.

We will also need the following statement.

Lemma 2.5 [Simpson 1994, Corollary 1.7]. *Suppose F is a slope semistable sheaf of dimension d, multiplicity r and slope μ. There is a constant C depending on r and d such that*[2]

$$\frac{h^0(F)}{r} \le \frac{1}{d!}\big([\mu + C]_+\big)^d.$$

2B. *Stable pairs.* Let E_0 be a coherent sheaf on X. Let P be a polynomial of degree d, and δ be 0 or a polynomial with a positive leading coefficient.

Definition 2.6. A *pair* (E, α) (of type P) on X consists of a coherent sheaf E with Hilbert polynomial P and a morphism $\alpha : E_0 \to E$. A *subpair* (E', α') consists of a coherent subsheaf $E' \subset E$ and a morphism $\alpha' : E_0 \to E'$, such that

$$\begin{cases} \iota \circ \alpha' = \alpha & \text{if } E' \supset \operatorname{im} \alpha, \\ \alpha' = 0 & \text{otherwise.} \end{cases}$$

Here, ι denotes the inclusion $E' \hookrightarrow E$. A *quotient pair* (E'', α'') consists of a coherent quotient sheaf $q : E \to E''$ and a morphism $\alpha'' = q \circ \alpha : E_0 \to E''$.

[2] $[x]_+ = \max\{0, x\}$.

We say a pair (E, α) has dimension d if $\dim E = d$.

A *morphism* $\phi : (E, \alpha) \to (F, \beta)$ of pairs is a morphism of sheaves $\phi : E \to F$ such that there is a constant $b \in k$, where $\phi \circ \alpha = b\beta$. By this definition, subpairs and quotient pairs can be viewed as morphisms. For simplicity, we shall use the notation ϕ for both the morphism of pairs and that of their underlying sheaves.

A *short exact sequence of pairs*,

$$0 \to (E', \alpha') \xrightarrow{\iota} (E, \alpha) \xrightarrow{q} (E'', \alpha'') \to 0,$$

consists of a short exact sequence of sheaves, $0 \to E' \to E \to E'' \to 0$, such that (E', α') is a subpair and (E'', α'') the corresponding quotient pair. More precisely, $\alpha'' = q \circ \alpha$ if $\alpha' = 0$, and $\alpha'' = 0$ if $\iota \circ \alpha' = \alpha$.

The *Hilbert polynomial* of a pair (E, α) is

$$P_{(E,\alpha)} = P_E + \epsilon(\alpha)\delta$$

and the *reduced Hilbert polynomial* of the pair is

$$p_{(E,\alpha)} = p_E + \frac{\epsilon(\alpha)\delta}{r(E)}.$$

Here,

$$\epsilon(\alpha) = \begin{cases} 1 & \text{if } \alpha \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the Hilbert polynomial is additive in a short exact sequence of pairs.

Definition 2.7. A pair (E, α) is δ-*stable* if

(i) E is pure;

(ii) $p_{(E',\alpha')} < p_{(E,\alpha)}$ for every proper subpair (E', α').

Semistability is defined similarly, replacing the strong inequality by the corresponding weak inequality.

Assuming purity, the second condition is equivalent to that for every proper quotient pair (E'', α'') of dimension d, $p_{(E'',\alpha'')} > p_{(E,\alpha)}$.

Convention. In the rest of this paper, if stability is characterized by a strong inequality, semistability can be characterized by the corresponding weak inequality. So, in such a case, we will only make the statement for stability.

When the context is clear, we will omit δ and only say a pair is stable or semistable.

Clearly, a pair $(E, 0)$ is (semi)stable if and only if E is (semi)stable as a coherent sheaf. We will call a pair (E, α) *nondegenerate* if $\alpha \neq 0$. We are primarily interested in nondegenerate semistable pairs, which we are going to parametrize.

A family of pairs parametrized by a scheme T is a morphism of sheaves

$$\alpha_T : \pi_X^* E_0 \to \mathscr{E}$$

over $T \times X$, such that $\mathscr{E}$ is flat over T. Here, π_X is the projection $T \times X \to X$. Two families $\alpha_T : \pi_X^* E_0 \to \mathscr{E}$ and $\beta_T : \pi_X^* E_0 \to \mathscr{F}$ are equivalent if there is an isomorphism

$$\psi : \mathscr{E} \to \mathscr{F} \quad \text{such that } \psi \circ \alpha_T = \beta_T.$$

In the large δ regime, semistable pairs have some special features.

Lemma 2.8. *When* $\deg \delta \geq \deg P$, *there is no nondegenerate strictly semistable pair*, *i.e.*, *every nondegenerate semistable pair is stable.*

Proof. Suppose (G, α') is a subpair of a semistable (E, α), such that $p_{(G,\alpha')} = p_{(E,\alpha)}$, that is,

$$p_G + \frac{\epsilon(\alpha')\delta}{r(G)} = p_E + \frac{\delta}{r(E)}.$$

Consider the leading coefficients. Because $\deg \delta \geq d$, we have $\epsilon(\alpha') = 1$ and $r(G) = r(E)$. Thus, $p_E = p_G$. Therefore, $P_E = P_G$, which implies that $G = E$. Hence, $(G, \alpha') = (E, \alpha)$. We have shown that (E, α) is not strictly semistable. □

We also have a reinterpretation of the stability condition.

Lemma 2.9. *Suppose E is a pure coherent sheaf with Hilbert polynomial $P_E = P$ and multiplicity $r(E) = r$. If* $\deg \delta \geq d = \deg P$, *then a pair (E, α) is stable if and only if for every proper subpair (G, α'),*

$$\frac{P_G}{2r(G) - \epsilon(\alpha')} < \frac{P}{2r - \epsilon(\alpha)}.$$

Proof. When $\deg \delta \geq d$, for any proper subpair (G, α'), the inequality

$$p_G + \epsilon(\alpha')\frac{\delta}{r(G)} < p_E + \epsilon(\alpha)\frac{\delta}{r}$$

is equivalent to

$$\text{(2-1)} \qquad \frac{\epsilon(\alpha')}{r(G)} \leq \frac{\epsilon(\alpha)}{r}, \quad \text{and in case of equality,} \quad p_G < p_E.$$

The latter can be easily seen to be equivalent to

$$\frac{r(G)}{2r(G) - \epsilon(\alpha')} \leq \frac{r}{2r - \epsilon(\alpha)}, \quad \text{and in case of equality,} \quad p_G < p_E.$$

This last condition is equivalent to the inequality in the statement. □

Moreover, there is a geometric characterization of stability.

Lemma 2.10. *If* $\deg \delta \geq \deg P$, *then* (E, α) *is stable if and only if* E *is pure and* $\dim \operatorname{coker} \alpha < \deg P$.

This is essentially [Wandel 2015, Proposition 1.12]. The author stated the result for the case where $\deg \delta \geq \dim X$ while his argument actually showed the same result under a weaker assumption $\deg \delta \geq \deg P$.

Pairs share some similar properties of sheaves.

Lemma 2.11. *Suppose* $\phi : (E, \alpha) \to (F, \beta)$ *is a nonzero morphism of pairs.*

(i) *Suppose* (E, α) *and* (F, β) *are* δ*-semistable pairs of dimension* d. *Then* $p_{(E,\alpha)} \leq p_{(F,\beta)}$.

(ii) *If* (E, α) *and* (F, β) *are* δ*-stable with the same reduced Hilbert polynomial,* ϕ *induces an isomorphism between* E *and* F. *In particular,* $\operatorname{End}((E, \alpha)) \cong k$ *for a stable pair* (E, α).

Proof. (i) Let α'' be $\phi \circ \alpha : E_0 \to \operatorname{im} \phi$. Then $(\operatorname{im} \phi, \alpha'')$ is a quotient pair of (E, α) and a subpair of (F, β). Thus,

$$p_{(E,\alpha)} \leq p_{(\operatorname{im} \phi, \alpha'')} \leq p_{(F,\beta)}. \tag{2-2}$$

(ii) Suppose not, then $\ker \phi \neq 0$ or $\operatorname{im} \phi \neq F$. We also have the inequalities (2-2). But two equalities do not hold simultaneously, which contradicts the fact that the two stable pairs have the same reduced Hilbert polynomial. Therefore, $\ker \phi = 0$ and $\operatorname{im} \phi = F$. Thus, ϕ is an isomorphism of coherent sheaves. Clearly, the inverse also provides an inverse of pairs. In particular, $\operatorname{End}((E, \alpha))$ is a finite-dimensional associative division algebra over the algebraically closed field k, and hence is k. □

The second part of the lemma is essentially [Wandel 2015, Lemma 1.6].

Proposition 2.12 (Harder–Narasimhan filtration). *Let* (E, α) *be a pair where* E *is pure of dimension* d. *Then there is a unique filtration by subpairs*

$$0 \subsetneqq (G_1, \alpha_1) \subsetneqq (G_2, \alpha_2) \subsetneqq \cdots \subsetneqq (G_l, \alpha_l) = (E, \alpha)$$

with $\operatorname{gr}_i = (G_i, \alpha_i)/(G_{i-1}, \alpha_{i-1})$ *such that*

(i) gr_i *is* δ*-semistable of dimension* d *for all* i;

(ii) $p_{\operatorname{gr}_i} > p_{\operatorname{gr}_{i+1}}$ *for all* i.

We call this filtration the ***Harder–Narasimhan filtration*** *of the pair.*

The proof is similar to the proof of the existence and uniqueness of the Harder–Narasimhan filtration of a pure sheaf [Shatz 1977, Theorem 1].

Evidently, in the filtration, there is only one nonzero α_i. In the case where $\deg \delta \geq d$, only α_1 is nonzero.

Proposition 2.13 (Jordan–Hölder filtration). *Let (E, α) be a semistable pair. There is a filtration*

$$0 \subsetneqq (F_1, \alpha_1) \subsetneqq (F_2, \alpha_2) \subsetneqq \cdots \subsetneqq (F_l, \alpha_l) = (E, \alpha),$$

such that each factor $\mathrm{gr}_i = (F_i, \alpha_i)/(F_{i-1}, \alpha_{i-1})$ *is stable with reduced Hilbert polynomial* $p_{(E,\alpha)}$. *Moreover*, $\mathrm{gr}(E, \alpha) = \oplus_i \mathrm{gr}_i$ *does not depend on the filtration.*

Proof. Since we have Lemma 2.11, the proof proceeds the same way as the argument for Jordan–Hölder filtrations of a semistable sheaf, see, e.g., [Huybrechts and Lehn 1997, Proposition 1.5.2]. □

Two semistable pairs are *S-equivalent*, if they have isomorphic Jordan–Hölder factors.

Let

$$\mathcal{S}_{E_0}(P, \delta) : \mathcal{S}ch_{/k} \to \mathcal{S}et$$

denote the moduli functor of S-equivalent nondegenerate semistable pairs of type P. Let

$$\mathcal{S}^s_{E_0}(P, \delta)$$

denote the moduli functor of equivalence classes of nondegenerate stable pairs.

3. Boundedness

In order to construct the moduli space via GIT, we first need to prove that the family of semistable pairs is bounded. As mentioned in the introduction, the case where $\deg \delta < \deg P$ has been treated in [Wandel 2015]. So, in this section and the next, we will focus on the case

$$\deg \delta \geq \deg P.$$

We will show boundedness using Theorem 2.4, by studying the $\mu_{\min}$'s of sheaves underlying semistable pairs.

Lemma 3.1. *Fix the Hilbert polynomial P. Assume* $\deg \delta \geq \deg P$. *Suppose (E, α) is a pair*, *which is semistable for some δ*, *with $P_E = P$. Then*, $\mu_{\min}(E)$ *is bounded below by a constant depending on P and X.*

Proof. Let (E, α) be a semistable pair. By Lemma 2.10,

$$\dim \operatorname{coker} \alpha < d. \tag{3-1}$$

Choose an m large enough such that $E_0(-m)$ is generated by global sections. Let Y be the scheme-theoretic support of E. The morphism α factors through $E_0|_Y$. We have the sequence of morphisms

$$H^0(E_0(m)) \otimes \mathcal{O}_Y(-m) \twoheadrightarrow E_0|_Y \to E \twoheadrightarrow \mathrm{gr}_s E,$$

where the last morphism is the surjection from E onto the last factor of the Harder–Narasimhan filtration with respect to the slope. By (3-1), the composition is nonzero. Therefore,

$$\begin{aligned}\mu_{\min}(E) = \mu(\mathrm{gr}_s E) &\geq \mu_{\min}(H^0(E_0(m)) \otimes \mathscr{O}_Y(-m)) \\ &= \mu_{\min}(\mathscr{O}_Y(-m)) = \mu_{\min}(\mathscr{O}_Y) - m,\end{aligned}$$

where the last term is bounded below, by Lemma 2.3. Thus, $\mu_{\min}(E)$ is bounded below by a constant, which depends on X and P. □

Remark 3.2. The lemma also holds for $\deg \delta < \deg P$. Moreover, the constant can be chosen to be independent of δ.

Combining Lemma 3.1 and Theorem 2.4, we obtain the following boundedness result.

Proposition 3.3. *Fix the Hilbert polynomial P. The family*

$$\{E \mid (E, \alpha) \text{ is a semistable pair of type } P \text{ with respect to some } \delta\}$$

of coherent sheaves on X is bounded.

For a bounded family of pure pairs, the family of factors of their Harder–Narasimhan filtrations is bounded:

Lemma 3.4. *Suppose $\Phi : \pi_X^* E_0 \to \mathscr{E}$ over $T \times X$ is a flat family of pure pairs over X parametrized by a finite type scheme T. For a closed point $t \in T$, let $\mathscr{E}(t) = \mathscr{E}|_{\operatorname{Spec} k(t) \times X}$ and $\Phi(t)$ be the corresponding morphism. Then, the family of the Harder–Narasimhan factors of $(\mathscr{E}(t), \Phi(t))$, for all $t \in T$, is bounded.*

The following proof is very similar to the proof of the corresponding statement about the boundedness of Harder–Narasimhan factors of pure sheaves [Huybrechts and Lehn 1997, Theorem 2.3.2]. We do not assume $\deg \delta \geq \deg P$ in this proof.

Proof. We can assume T to be integral. Define A as the set of 2-tuples (P'', ϵ''), such that there is a point $t \in T$ and a pure quotient $q : \mathscr{E}(t) \twoheadrightarrow E''$ with Hilbert polynomial $P_{E''} = P''$ and $\epsilon'' = \epsilon(q \circ \Phi(t))$, which destabilizes $(\mathscr{E}(t), \Phi(t))$:

$$p'' + \frac{\epsilon'' \delta}{r''} < p + \frac{\epsilon(\Phi(s))\delta}{r}.$$

Here, p and p'' denote the corresponding reduced Hilbert polynomials, and r and r'' denote the multiplicities. From this inequality, we know that $\mu(E'')$ is bounded above by a constant determined by P and δ. Therefore, A is a finite set by Theorem 2.1.

If this set is empty, then all pairs are semistable. Then, we are done. Otherwise, we define a total order $\preceq$ on A as:

$$(P_1, \epsilon_1) \preceq (P_2, \epsilon_2)$$

if $p_1 + \epsilon_1\delta/r_1 \le p_2 + \epsilon_2\delta/r_2$, and in the case of equality, $P_1 \ge P_2$. Let us consider whether there is a (P_-, ϵ_-), which is minimal with respect to the total order $\preceq$ and satisfies the condition that for a generic point $t \in T$, there is a pure quotient $q : \mathscr{E}(t) \to F$ with

$$P_F = P_- \quad \text{and} \quad \epsilon(q \circ \Phi(t)) = \epsilon_-. \tag{3-2}$$

If there is no such (P_-, ϵ_-), then generically, say over the open subscheme $U \subset T$, pairs are already semistable.

If there is such a (P_-, ϵ_-), let $U \subset T$ be the open family having quotients satisfying the condition (3-2). The minimal Harder–Narasimhan factors of pairs in U are parametrized by a subscheme of $\mathrm{Quot}^{P_-}(\mathscr{E})$. To parametrize all the Harder–Narasimhan factors of pairs parametrized by U, we can iterate the above process for the kernel, which is flat, of the universal quotient over $\mathrm{Quot}^{P_-}(\mathscr{E})$. This process will terminate due to multiplicity.

Then, we can run the same algorithm for pairs parametrized by irreducible components of the complement $T \setminus U$. Because T is noetherian, the process will terminate.

We have thus parametrized the Harder–Narasimhan factors by a finite sequence of Quot schemes. □

The following statement enables us to handle the semistability condition via spaces of global sections, instead of Hilbert polynomials.

Lemma 3.5. *Fix P and δ with $\deg\delta \ge \deg P$. Then there is an $m_0 \in \mathbb{Z}_{>0}$, such that for any integer $m \ge m_0$ and any pair (E, α), where E is pure with $P_E = P$ and multiplicity $r(E) = r$, the following assertions are equivalent.*

i) *The pair (E, α) is stable.*

ii) *$P_E(m) \le h^0(E(m))$, and for any proper subpair (G, α') where G is of multiplicity $r(G)$,*

$$\frac{h^0(G(m))}{2r(G) - \epsilon(\alpha')} < \frac{h^0(E(m))}{2r - \epsilon(\alpha)}.$$

iii) *For any proper quotient pair (F, α'') where F is of dimension d and multiplicity $r(F)$,*

$$\frac{h^0(F(m))}{2r(F) - \epsilon(\alpha'')} > \frac{P(m)}{2r - \epsilon(\alpha)}.$$

The proof is modified from that of a similar statement in [Le Potier 1993].

Proof. The proof will proceed as follows: i)⇒ ii)⇒ iii)⇒ i). The integer m_0 will be determined in the course of the proof, nonexplicitly.

i) $\Rightarrow$ ii) The family of sheaves underlying semistable pairs with a fixed Hilbert polynomial is bounded. Thus, there is $m_0 \in \mathbb{N}$ such that for any integer $m \geq m_0$, we have $H^i(E(m)) = 0$ for all $i > 0$. In particular, $P(m) = h^0(E(m))$.

In the course of proving the boundedness, we also proved that $\mu_{\max}(E)$ is bounded above, say $\mu_{\max}(E) \leq \mu$. For a proper subpair (G, α') of multiplicity $r(G)$, consider the Harder–Narasimhan filtration of G with respect to the slope. Let us denote the multiplicity and the slope of the i-th grading by r_i' and μ_i'. Then, we have $\mu_i' \leq \mu$. Notice that r_i' is positive and bounded above by r, which implies that there are only finitely many possible r_i''s and μ_i''s. Let $\nu = \mu_{\min}(G)$. By Lemma 2.5 and an easy calculation, we can find a constant B depending on r and d, such that[3]

$$\frac{h^0(G(m))}{r(G)} \leq \frac{1}{d!}\Big(\Big(1-\frac{1}{r}\Big)\big([\mu+m+B]_+\big)^d + \frac{1}{r}\big([\nu+m+B]_+\big)^d\Big). \tag{3-3}$$

Choose a constant $A > 0$, which is larger than all roots of P. Replace m_0 by $\max\{m_0, A\}$. Then

$$h^0(E(m)) = P(m) \geq \frac{r}{d!}(m-A)^d, \qquad \text{for all } m \geq m_0.$$

Suppose ν_0 is an integer such that

$$B + \mu\Big(1-\frac{1}{r}\Big) + \frac{\nu_0}{r} < -A.$$

Enlarging m_0 if necessary, we have

$$\frac{1}{d!}\Big(\Big(1-\frac{1}{r}\Big)\big([\mu+m+B]_+\big)^d + \frac{1}{r}\big([\nu_0+m+B]_+\big)^d\Big) < \frac{P(m)}{r}, \qquad \text{for all } m \geq m_0, \tag{3-4}$$

by considering the first and the second leading coefficients.

Thus, when $m \geq m_0$ and $\nu \leq \nu_0$, combining (3-3) and (3-4), we get

$$h^0(G(m)) < \frac{r(G)}{r} h^0(E(m)) \leq \frac{2r(G) - \epsilon(\alpha')}{2r - \epsilon(\alpha)} h^0(E(m)). \tag{3-5}$$

The last weak inequality is a consequence of (2-1).

We are left to consider the case where $\nu > \nu_0$. First, notice that we can assume E/G to be pure. If not, consider the saturation of G in E, namely, the smallest $\overline{G} \supset G$, such that $E/\overline{G}$ is pure. If we can prove the inequality in ii) for $\overline{G}$, then it's also true for G, since $r(G) = r(\overline{G})$ and $h^0(G(m)) \leq h^0(\overline{G}(m))$. Since $\mu(G) \geq \nu > \nu_0$, the family of such G is bounded, by Theorem 2.1. So, there are only finitely many Hilbert polynomials of the form P_G for such G. Moreover, we can enlarge m_0 again, if necessary, such that for $m \geq m_0$, $P_G(m) = h^0(G(m))$ and

$$\frac{P_G}{2r(G) - \epsilon(\alpha')} < \frac{P}{2r - \epsilon(\alpha)} \iff \frac{P_G(m)}{2r(G) - \epsilon(\alpha')} < \frac{P(m)}{2r - \epsilon(\alpha)}.$$

[3] To obtain this inequality, one can also see [Huybrechts and Lehn 1997, Corollary 3.3.8].

Therefore, by Lemma 2.9 and (3-5),

$$\frac{h^0(G(m))}{2r(G)-\epsilon(\alpha')} < \frac{h^0(E(m))}{2r-\epsilon(\alpha)}.$$

ii) ⇒ iii) From a proper quotient pair (F, α''), we can get a short exact sequence

$$0 \to (G, \alpha') \to (E, \alpha) \to (F, \alpha'') \to 0.$$

We thus obtain an exact sequence

$$0 \to H^0(G(m)) \to H^0(E(m)) \to H^0(F(m)). \tag{3-6}$$

Therefore, $h^0(F(m)) \geq h^0(E(m)) - h^0(G(m))$. Notice that $r(E) = r(G) + r(F)$ and $\epsilon(\alpha) = \epsilon(\alpha') + \epsilon(\alpha'')$. Thus,

$$\frac{h^0(F(m))}{2r(F)-\epsilon(\alpha'')} \geq \frac{h^0(E(m)) - h^0(G(m))}{(2r-\epsilon(\alpha)) - (2r(G)-\epsilon(\alpha'))} > \frac{h^0(E(m))}{2r-\epsilon(\alpha)} \geq \frac{P(m)}{2r-\epsilon(\alpha)}.$$

iii) ⇒ i) Take the Harder–Narasimhan filtration of E with respect to the slope. Suppose F is the last factor, then $\mu(F) = \mu_{\min}(E)$, denoted as μ''. By Lemma 2.5,

$$\frac{h^0(F(m))}{r(F)} \leq \frac{1}{d!}\big([\mu'' + m + C]_+\big)^d. \tag{3-7}$$

Let (F, α'') be the induced quotient pair. If $\epsilon(\alpha'') \neq 0$, then (E, α) is stable, since in the Harder–Narasimhan filtration, only the first morphism is nonzero. So, assume $\epsilon(\alpha'') = 0$. Then

$$\frac{P(m)}{r} < \frac{2P(m)}{2r-\epsilon(\alpha)} < \frac{h^0(F(m))}{2r(F)} \leq \frac{1}{d!}\big([\mu'' + m + C]_+\big)^d.$$

If $m \geq m_0$, the preceding inequality with $P(m)/r \geq (m - A)^d/d!$ implies that $m - A \leq \mu'' + m + C$. Therefore, $\mu_{\min}(E) = \mu'' \geq -A - C$. Thus, the family of coherent sheaves satisfying the third condition for some $m \geq m_0$ is bounded.

Let $\mathrm{gr}_s = (\mathrm{gr}_s E, \mathrm{gr}_s \alpha)$ denote the last Harder–Narasimhan factor of the pair (E, α). Then

$$\frac{h^0(\mathrm{gr}_s E(m))}{2r(\mathrm{gr}_s E) - \epsilon(\mathrm{gr}_s \alpha)} > \frac{P(m)}{2r-1}.$$

By Lemma 3.4, enlarging m_0 if necessary, we can assume that, for all $m \geq m_0$,

(i) $h^0(\mathrm{gr}_s E(m)) = P_{\mathrm{gr}_s E}(m)$;

(ii) $$\frac{P_{\mathrm{gr}_s E}(m)}{2r(\mathrm{gr}_s E) - \epsilon(\mathrm{gr}_s \alpha)} > \frac{P(m)}{2r-1} \iff \frac{P_{\mathrm{gr}_s}}{2r(\mathrm{gr}_s E) - \epsilon(\mathrm{gr}_s \alpha)} > \frac{P}{2r-1}.$$

Therefore, $\epsilon(\mathrm{gr}_i \alpha)/r(\mathrm{gr}_s E) \geq 1/r$, which implies $\epsilon(\mathrm{gr}_s \alpha) = 1$. Thus, $s = 1$, which means (E, α) is semistable, and thus stable. □

Replacing the strong inequalities by weak inequalities, we conclude that the lemma is also true.

4. Construction of the moduli space

Fix the smooth projective variety $(X, \mathcal{O}_X(1))$, the coherent sheaf E_0, the Hilbert polynomial P, and the stability condition δ.

By the boundedness results proven in the last section, there is an $N \in \mathbb{Z}$ such that for any integer $m > N$, the following conditions are satisfied:

(i) $E_0(m)$ is globally generated.

(ii) $E(m)$ is globally generated and has no higher cohomology for every E appearing in a δ-semistable pair (Proposition 3.3). Similar results hold for their Harder–Narasimhan factors (Lemma 3.4).

(iii) The three assertions in Lemma 3.5 are equivalent.

Fix such an m and let V be a vector space such that

$$\dim V = P(m).$$

Suppose (E, α) is a semistable pair, then E can be viewed as a quotient

$$q : V \otimes \mathcal{O}_X(-m) \twoheadrightarrow E.$$

Another datum of the pair is the morphism α. It gives rise to a linear map

$$\sigma : H^0(E_0(m)) \to H^0(E(m)) \cong V.$$

Thus, a semistable pair gives rise to the following diagram:

$$\begin{array}{ccccc} K_0 & \xhookrightarrow{\iota} & H^0(E_0(m)) \otimes \mathcal{O}_X(-m) & \xrightarrow{\text{ev}} & E_0 \\ & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\alpha} \\ & & V \otimes \mathcal{O}_X(-m) & \xrightarrow{q} & E \end{array}$$

Here, ι is the kernel of the evaluation map ev. Conversely, we can obtain a pair from a quotient q and a linear map σ as long as $q \circ \sigma \circ \iota = 0$. Also notice that $\sigma = 0$ if and only if $\alpha = 0$.

We will study the following spaces:

$$\mathbb{P} = \mathbb{P}(\mathrm{Hom}(H^0(E_0(m)), V)) = \mathrm{Proj}(H^0(E_0(m)) \otimes V^\vee),$$
$$Q = \mathrm{Quot}_X^P(V \otimes \mathcal{O}_X(-m)).$$

The second space is Grothendieck's Quot scheme, parametrizing quotients of $V \otimes \mathcal{O}_X(-m)$ with Hilbert polynomial P. This is motivated by a similar construction in [Huybrechts and Lehn 1995a; 1995b]. Spaces $\mathbb{P}$ and Q are fine moduli spaces, with the following universal families:

$$H^0(E_0(m)) \otimes \mathscr{O}_{\mathbb{P}} \to V \otimes \mathscr{O}_{\mathbb{P}}(1), \tag{4-1}$$

$$V \otimes \mathscr{O}_X(-m) \to \mathscr{E}. \tag{4-2}$$

Let

$$Z \subset \mathbb{P} \times Q$$

be the locally closed subscheme of points $\xi = ([\sigma], [q])$ such that

(i) $q \circ \sigma \circ \iota = 0$;

(ii) E is pure;

(iii) the quotient q induces an isomorphism of vector spaces $V \xrightarrow{\sim} H^0(E(m))$.

There is a natural $\mathrm{SL}(V)$-action on $\mathbb{P} \times Q$:

$$\big([\sigma], [q]\big).g = \big([g^{-1} \circ \sigma], [q \circ g]\big),$$

for $g \in \mathrm{SL}(V)$ and $([\sigma], [q]) \in \mathbb{P} \times Q$. It can be easily checked that this indeed defines a right action. It is clear that Z is invariant under this action. The closure $\overline{Z}$ of $Z \subset \mathbb{P} \times Q$ is invariant as well.

For a very large l, there is an $\mathrm{SL}(V)$-equivariant embedding,

$$Q = \mathrm{Quot}_X^P(V \otimes \mathscr{O}_X(-m)) \hookrightarrow \mathrm{Grass}(V \otimes H^0(\mathscr{O}_X(l-m)), P(l)),$$
$$[q : V \otimes \mathscr{O}_X(-m) \twoheadrightarrow E] \mapsto [H^0(q(l)) : V \otimes H^0(\mathscr{O}_X(l-m)) \twoheadrightarrow H^0(E(l))].$$

The standard very ample line bundle on the Grassmannian is $\mathrm{SL}(V)$-linearized. Let $\mathscr{O}_Q(1)$ be its pullback to Q. The line bundle $\mathscr{O}_{\mathbb{P}}(1)$ is also $\mathrm{SL}(V)$-linearized. Thus, for positive integers n_1 and n_2, the following line bundle is $\mathrm{SL}(V)$-linearized:

$$L = \mathscr{O}_{\mathbb{P}}(n_1) \boxtimes \mathscr{O}_Q(n_2).$$

We are going to construct the moduli space by taking the GIT quotient of $\overline{Z}$, eliminating the extra information coming from identifying V and $H^0(E(m))$. A key step is to relate the δ-stability condition to the GIT-stability condition with respect to L, which will occupy a large part of this section.

An application of the Hilbert–Mumford criterion shows the following lemma. It is very similar to [Wandel 2015, Proposition 4.3]. For the proof of the lemma, see [Lin 2016, Lemma 12].

Lemma 4.1. *For l very large, let $\xi = ([\sigma], [q]) \in \overline{Z}$ be a point with associated morphism $\alpha : E_0 \to E$. Then the following two conditions are equivalent*:

(i) *ξ is GIT-stable with respect to L.*

(ii) *For any nontrivial proper subspace* $W \subsetneq V$, *let* $G = q(W \otimes \mathscr{O}_X(-m))$. *Then*

$$P_G(l) > \frac{n_1}{n_2}\Big(\epsilon_W(\sigma) - \frac{\dim W}{\dim V}\Big) + P(l)\frac{\dim W}{\dim V}. \tag{4-3}$$

Here, $\epsilon_W(\sigma)$ *is either* 1 *or* 0 *depending on whether* W *contains* $\operatorname{im}\sigma$ *or not.*

GIT-semistability can also be characterized by the corresponding weak inequality. Now, let

$$\frac{n_1}{n_2} = \frac{P(l)}{2r}. \tag{4-4}$$

We fix an l such that

(i) Lemma 4.1 holds;

(ii) (4-3) holds if and only if it holds as an inequality of polynomials in l:

$$P_G > \frac{n_1}{n_2}\Big(\epsilon_W(\sigma) - \frac{\dim W}{\dim V}\Big) + P\frac{\dim W}{\dim V}. \tag{4-5}$$

We can ask for the second condition because the family of such G's is bounded.

In defining Z, we required the quotient to be pure. When we take the closure, we may include quotients which are not pure. But the following statement imposes restrictions.

Corollary 4.2. *If* $([\sigma], [q]) \in \overline{Z}$ *is GIT-semistable, then* $H^0(q(m)) : V \to H^0(E(m))$ *is injective and for any coherent subsheaf* $G \subset E$ *such that* $\dim G \le d-1$, $H^0(G(m)) = 0$.

Proof. Let W be the kernel of $H^0(q(m)) : V \to H^0(E(m))$, then for the image G we have

$$G = q(W \otimes \mathscr{O}_X(-m)) = 0.$$

The inequality (4-5) forces $\dim W$ to be zero, otherwise the right-hand side of the inequality is a positive polynomial while the left-hand side is 0.

Suppose $G \subset E$ such that $\dim G \le d-1$. If we let $W = H^0(G(m))$, then $q(W \otimes \mathscr{O}_X(-m)) \subset G$. By the inequality (4-5), we have $\dim W = 0$, otherwise the right-hand side will be a positive polynomial of degree no less than d, while the left hand side is of degree $\le d-1$. □

We are ready to relate the δ-stability condition to the GIT-stability condition.

Proposition 4.3. *Let* $([\sigma], [q])$ *be in* $\overline{Z}$ *and* (E, α) *be the corresponding pair. The following two assertions are equivalent*:

(i) $([\sigma], [q])$ *is GIT-(semi)stable with respect to* L.

(ii) (E, α) *is (semi)stable and* q *induces an isomorphism* $V \xrightarrow{\sim} H^0(E(m))$.

Recall that when $\deg \delta \ge \deg P$, there are no strictly semistable pairs.

Proof. First, assume that a point $([\sigma], [q]) \in \bar{Z}$ is GIT-semistable. Denote the quotient by

$$q : V \otimes \mathscr{O}(-m) \to E.$$

Then by Corollary 4.2, we know that the induced linear map $V \to H^0(E(m))$ is injective. The sheaf E can be deformed to a pure sheaf since $([\sigma], [q])$ is in the closure of Z. By [Huybrechts and Lehn 1997, Proposition 4.4.2], there is an exact sequence,

$$0 \to T_{d-1}(E) \to E \xrightarrow{\phi} F,$$

where $T_{d-1}(E)$ is the maximal dimension $d - 1$ subsheaf of E and such that $P_F = P_E = P$. According to Corollary 4.2, the exact sequence provides an injective linear map,

$$H^0(E(m)) \hookrightarrow H^0(F(m)).$$

For any dimension d quotient $\pi : F \twoheadrightarrow F''$, let G be the kernel of $\pi \circ \phi$,

$$0 \to G \to E \xrightarrow{\pi\circ\phi} F'' \to 0.$$

Let $W = V \cap H^0(G(m))$. Then we have

$$h^0(F''(m)) \geq h^0(E(m)) - h^0(G(m)) \geq \dim V - \dim W. \tag{4-6}$$

Let $r'' = r(F'')$. Let's consider the leading coefficients of the two sides of (4-3), viewed as polynomials in l. (This is where the argument diverges, depending on the degree of δ. Here, we focus on the case where $\deg \delta \geq d$.) Then

$$(2r(G) - \epsilon_W(\sigma)) \dim V \geq (2r - 1) \dim W. \tag{4-7}$$

Combining ((4-6), (4-7)), we have

$$\frac{h^0(F''(m))}{2r'' - \epsilon(\pi \circ \phi \circ \alpha)} \geq \frac{\dim V}{2r - 1} \cdot \frac{2r'' - (1 - \epsilon_W(\sigma))}{2r'' - \epsilon(\pi \circ \phi \circ \alpha)} \geq \frac{P(m)}{2r - 1}.$$

To prove the second inequality, notice that, when $\epsilon(\pi \circ \phi \circ \alpha) = 0$, $\operatorname{im} \alpha \subset G$. Therefore $\operatorname{im} \sigma \subset H^0(G(m))$. Thus, $\operatorname{im} \sigma \subset W$.

According to Lemma 3.5, the pair $(F, \phi \circ \alpha)$ is semistable. Therefore, by our choice of m, $h^0(F(m)) = P(m)$. We have the following commutative diagram:

$$\begin{array}{ccccc}
V \otimes \mathscr{O}_X(-m) & \xrightarrow{\sim} & H^0(E(m)) \otimes \mathscr{O}_X(-m)\,\mathrm{ev} & \xrightarrow{\sim} & H^0(F(m)) \otimes \mathscr{O}_X(-m)\,\mathrm{ev} \\
 & \searrow^{q} & \downarrow & & \downarrow\!\!\downarrow \\
 & & E & \xrightarrow{\phi} & F
\end{array}$$

So, ϕ is surjective. Since they have the same Hilbert polynomial, it is an isomorphism. Therefore, (E, α) is a semistable pair.

Next, we assume that (E,α) is semistable, thus stable, and $q(m)$ induces an isomorphism between global sections. For any nontrivial proper subspace $W \subsetneqq V$, let

$$G = q(W \otimes \mathscr{O}(-m))$$

and (G,α') the corresponding subpair. If $(G,\alpha') = (E,\alpha)$, the inequality in Lemma 4.1 holds. Assume that (G,α') is a proper subpair. According to Lemma 3.5, we have

$$\frac{h^0(G(m))}{2r(G)-\epsilon(\alpha')} < \frac{h^0(E(m))}{2r-1}.$$

From the commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & H^0(G(m)) \\ \Big\downarrow & & \Big\downarrow \\ V & \xrightarrow{\cong} & H^0(E(m)) \end{array}$$

we know that $\dim W \leq h^0(G(m))$. Thus,

$$\frac{\dim W}{2r(G)-\epsilon(\alpha')} < \frac{h^0(E(m))}{2r-1}.$$

Therefore,

$$r(G) > \frac{1}{2}\epsilon(\alpha') - \frac{1}{2}\cdot\frac{\dim W}{\dim V} + r\frac{\dim W}{\dim V},$$

which implies the inequality in Lemma 4.1, since $\epsilon(\alpha') \geq \epsilon_W(\sigma)$. Hence, $([\sigma],[q])$ is GIT-stable. □

We still need the following lemma, which will help us identify closed orbits. A pair is *polystable* if it is isomorphic to a direct sum of stable pairs, degenerate or not, with the same reduced Hilbert polynomial.

Lemma 4.4. *The closures of orbits of two points*, $([\sigma_1],[R_1])$ *and* $([\sigma_2],[R_2])$, *in* $\bar{Z}^{ss}$ *intersect if and only if their associated semistable pairs* (E_1,α_1) *and* (E_2,α_2) *have the same Jordan–Hölder factors. The orbit of a point* $([\sigma],[q])$ *is closed if and only if the associated pair* (E,α) *is polystable.*

The proof is similar to that of [Huybrechts and Lehn 1997, Theorem 4.3.3], using the following lemma on semicontinuity.

Lemma 4.5 (semicontinuity). *Suppose* $(\mathscr{F},\alpha)$ *and* $(\mathscr{G},\beta)$ *over* $X_T = T \times X$ *are two flat families of pairs, with Hilbert polynomials* $P_{\mathscr{F}}$ *and* $P_{\mathscr{G}}$, *parametrized by a scheme* T *of finite type over* k. *Then, the following function is semicontinuous*:

$$t \mapsto \dim_k \mathrm{Hom}_{\{t\}\times X}((\mathscr{F}_t,\alpha_t),(\mathscr{G}_t,\beta_t)).$$

The proof is modified from that of [Huybrechts and Lehn 1995a, Lemma 3.4].

Proof. The space $\operatorname{Hom}((\mathscr{F}_t, \alpha_t), (\mathscr{G}_t, \beta_t))$ is related to the pullback in the diagram

$$\begin{array}{ccc} C_t & \dashrightarrow & k \\ \downarrow & & \downarrow \cdot\beta_t \\ \operatorname{Hom}(\mathscr{F}_t, \mathscr{G}_t) & \xrightarrow{\circ\alpha_t} & \operatorname{Hom}(E_0, \mathscr{G}_t) \end{array}$$

in the sense that it satisfies the equality

$$\dim \operatorname{Hom}((\mathscr{F}_t, \alpha_t), (\mathscr{G}_t, \beta_t)) = \dim C_t - 1 + \epsilon(\beta_t).$$

By our flatness assumption, β_t is either always zero or never zero. Thus, it is enough to show that C_t is a fiber of a common coherent $\mathscr{O}_T$-module, as t varies. Since the question is local on T, assume $T = \operatorname{Spec} A$, where A is a k-algebra.

It is shown in the proof of [Huybrechts and Lehn 1995a, Lemma 3.4] that there is a bounded-above complex $M^\bullet_{E_0}$ of finite type free A-modules, such that for any A-module M,

(4-8) $$h^i(M^\bullet_{E_0} \otimes_A M) \cong \operatorname{Ext}^i_{X_T}(\pi_X^* E_0, \mathscr{G} \otimes_A M).$$

Similarly, there is such an $M^\bullet_{\mathscr{F}}$ that

(4-9) $$h^i(M^\bullet_{\mathscr{F}} \otimes_A M) \cong \operatorname{Ext}^i_{X_T}(\mathscr{F}, \mathscr{G} \otimes_A M).$$

The morphism α induces a morphism of complexes, which is still denoted as $\alpha : M^\bullet_{\mathscr{F}} \to M^\bullet_{E_0}$. The morphism β induces a morphism $\beta : A \to M^\bullet_{E_0}$. Thus, there is a morphism,

$$\psi = (\alpha, -\beta) : M^\bullet_{\mathscr{F}} \oplus A \to M^\bullet_{E_0}.$$

Then the mapping cone $C(\psi)$ fits in the distinguished triangle

$$C(\psi)[-1] \to M^\bullet_{\mathscr{F}} \oplus A \to M^\bullet_{E_0} \to C(\psi).$$

Taking the long exact sequence, we have

$$0 \to h^{-1}(C(\psi)) \to \operatorname{Hom}_{X_T}(\mathscr{F}, \mathscr{G}) \oplus A \to \operatorname{Hom}_{X_T}(\pi_X^* E_0, \mathscr{G}) \to \cdots .$$

Thus, we have the following fiber diagram:

$$\begin{array}{ccc} h^{-1}(C(\psi)) & \longrightarrow & A \\ \downarrow & & \downarrow \beta \\ \operatorname{Hom}_{X_T}(\mathscr{F}, \mathscr{G}) & \xrightarrow{\alpha} & \operatorname{Hom}_{X_T}(\pi_X^* E_0, \mathscr{G}) \end{array}$$

Therefore, together with (4-8) and (4-9) and the isomorphism $\operatorname{Ext}^i_{X_T}(\mathscr{F}, \mathscr{G} \otimes k(t)) \cong \operatorname{Ext}^i_{X_t}(\mathscr{F}_t, \mathscr{G}_t)$, we know $C_t \cong h^{-1}(C(\psi)) \otimes k(t)$. $\square$

We can now prove the existence of the moduli space.

Proof of Theorem 1.1. Let

$$S = S_{E_0}(P, \delta) = \bar{Z}^{ss} /\!\!/ SL(V)$$

be the GIT quotient. This is a projective scheme. We will show that this is the coarse moduli space of S-equivalence classes of semistable pairs.

Suppose we are given a family of semistable pairs parametrized by T:

$$\beta : \pi_X^* E_0 \to \mathscr{F}.$$

Let π be the projection from $T \times X$ onto T. Let m be chosen as before, then $\pi_*(\mathscr{F}(m))$ is locally free of rank $P(m) = \dim V$ and we obtain a morphism over T:

$$\pi_*(\beta(m)) : \pi_*(\pi_X^* E_0(m)) \to \pi_*(\mathscr{F}(m)).$$

Therefore, there is an open affine cover $T = \bigcup T_i$, such that $\pi_*(\mathscr{F}(m))|_{T_i}$ is free of rank $P(m)$ over each T_i. Choose an isomorphism over T_i:

$$\omega_i : V \otimes \mathscr{O}_{T_i} \to \pi_*(\mathscr{F}(m))|_{T_i}.$$

Then $\omega_i^{-1} \circ \pi_*(\beta(m))$ induces a morphism $T_i \to \mathbb{P}$. Also, the quotient

$$\mathrm{ev} \circ \pi^*(\omega_i) : V \otimes \mathscr{O}_X(-m) \xrightarrow{\cong} \pi^* \pi_*(\mathscr{F}(m)) \otimes \mathscr{O}_X(-m) \twoheadrightarrow \mathscr{F}$$

over $T_i \times X$ induces a morphism $T_i \to Q$. Thus, they induce a morphism $f_i : T_i \to \mathbb{P} \times Q$. By the definition of Z and Proposition 4.3, f_i factors through $\bar{Z}^{ss}$. Therefore, we obtain unambiguously a morphism,

$$f_\beta : T \to S.$$

Thus, we have a natural transformation,

$$\mathcal{S} = \mathcal{S}_{E_0}(P, \delta) \to \mathrm{Mor}(-, S).$$

Suppose there is a natural transformation,

$$\mathcal{S} \to \mathrm{Mor}(-, N). \tag{4-10}$$

Let $T = \bar{Z}^{ss}$. Universal families (4-1) and (4-2) induce

$$H^0(E_0(m)) \otimes \mathscr{O}_X(-m) \to V \otimes \mathscr{O}_{\mathbb{P}}(1) \otimes \mathscr{O}_X(-m) \twoheadrightarrow \mathscr{E} \otimes \mathscr{O}_{\mathbb{P}}(1).$$

Over T, the composition induces a family,

$$\pi_X^* E_0 \to \mathscr{E} \otimes \mathscr{O}_{\mathbb{P}}(1), \tag{4-11}$$

and thus an element in $\mathcal{S}(T)$. This in turn produces a map $T = \bar{Z}^{ss} \to N$. Because (4-10) is a natural transformation, this map is $\mathrm{SL}(V)$-equivariant, with the action on N being trivial. According to properties of a quotient, the map factors uniquely through S. Therefore, we have the following commutative diagram of functors:

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \mathrm{Mor}(-, S) \\ & \searrow & \downarrow \\ & & \mathrm{Mor}(-, N) \end{array}$$

Moreover, closed points in S are in bijection with S-equivalence classes of semistable pairs, according to Lemma 4.4. Thus, S is the coarse moduli space.

Let us consider the open set $\bar{Z}^s \subset \bar{Z}^{ss}$ of stable points. The geometric quotient

$$\bar{Z}^s \to \bar{Z}^s / \mathrm{SL}(V) = S^s_{E_0}(P, \delta) = S^s$$

provides a quasiprojective scheme parametrizing equivalence classes of stable pairs. We shall prove this quotient to be a principal $\mathrm{PGL}(V)$-bundle. It is enough to show that the stabilizers are products of the identity matrix and roots of unity.

Suppose a point $([\sigma], [q]) \in \bar{Z}^s$ gives rise to a stable pair $\alpha : E_0 \to E$ and $([\sigma], [q])$ is fixed by $g \in \mathrm{SL}(V)$, that is, $[\sigma] = [g^{-1} \circ \sigma]$ $[q] = [q \circ g]$. Then there is a scalar $a \in k^\times$, such that $g^{-1} \circ \sigma = a\sigma$, and there is an isomorphism $\phi : E \to E$, such that $\phi \circ q = q \circ g$. Therefore,

$$\phi \circ \alpha \circ \mathrm{ev} = a\alpha \circ \mathrm{ev} : H^0(E_0(m)) \otimes \mathscr{O}_X(-m) \to E.$$

So, $\phi \circ \alpha = a\alpha$. Thus, ϕ is a multiplication by a nonzero scalar, by Lemma 2.11. In the diagram

$$\begin{array}{ccc} V & \xrightarrow{H^0(q(m))} & H^0(E(m)) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle H^0(\phi(m))} \\ V & \xrightarrow{H^0(q(m))} & H^0(E(m)) \end{array}$$

the horizontal arrows are isomorphisms and the right vertical arrow is a multiplication by a nonzero scalar. Therefore, g is also a multiplication by a nonzero scalar. Because g lies in $\mathrm{SL}(V)$, it is the product of a root of unity and the identity matrix.

In the family (4-11), $\mathscr{E} \otimes \mathscr{O}_{\mathbb{P}}(1)$ is $\mathrm{SL}(V)$-equivariant. Although the actions of the center of $\mathrm{SL}(V)$ on $\mathscr{O}_{\mathbb{P}}(1)$ and $\mathscr{E}$ are not trivial, its action on $\mathscr{E} \otimes \mathscr{O}_{\mathbb{P}}(1)$ is. Thus, $\mathscr{E} \otimes \mathscr{O}_{\mathbb{P}}(1)$ is $\mathrm{PGL}(V)$-equivariant. Therefore, the restriction of (4-11) to $\bar{Z}^s \times X$ descends to $S^s \times X$ to give a universal family of pairs. Hence, S^s represents the functor $\mathcal{S}^s_{E_0}(P, \delta)$. □

Remark 4.6. The construction above can be carried out in the relative case. By [Grothendieck 1961b, Lemma 2.5], the boundedness result still holds. According to [Seshadri 1977], the GIT construction works in the relative setting. More concretely, let T be a k-scheme of finite type, $X \to T$ a flat projective morphism, and $\mathscr{E}_0$ a coherent $\mathscr{O}_X$-module flat over T. Then, there is a relative moduli space of δ-semistable pairs $\mathcal{S}_{\mathscr{E}_0}(P,\delta)$ which is projective over T. There is an open subscheme $\mathcal{S}^s_{\mathscr{E}_0}(P,\delta) \subset \mathcal{S}_{\mathscr{E}_0}(P,\delta)$ parametrizing stable pairs. Moreover, fibers $\mathcal{S}_{\mathscr{E}_0}(P,\delta)_t$ over closed points are moduli spaces of semistable pairs on X_t.

5. Deformation and obstruction theories

This section is devoted to the proof of Theorem 1.2, following [Huybrechts and Lehn 1997; Inaba 2002]. In Section 5A, we will outline the construction of the obstruction class and identify the deformation space. In Section 5B, we will fill in the proofs.

5A. ***Constructions.*** Suppose (E,α) is a stable pair and

$$0 \to K \to B \xrightarrow{\sigma} A \to 0$$

is a short exact sequence, where $A, B \in \mathcal{A}rt_k$ are local Artinian k-algebras with residue field k, such that $\mathfrak{m}_B K = 0$. Suppose

$$\alpha_A : E_0 \otimes A \to E_A$$

over

$$X_A = X \times \operatorname{Spec} A$$

is a (flat) extension of (E,α). Let

$$I^\bullet_A = \{E_0 \otimes A \to E_A\}$$

denote the complex positioned at 0 and 1. We would like to extend (E_A,α_A) to a pair (E_B,α_B) over X_B. This is similar to deforming a sheaf or a perfect complex. But we need to fix E_0.

We take two locally free resolutions $P^\bullet \xrightarrow{\sim} E_0$ and $Q^\bullet_A \xrightarrow{\sim} E_A$ and lift α_A to a morphism of complexes $\alpha^\bullet_A : P^\bullet \otimes A \to Q^\bullet_A$. Then, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_P^{-2}\otimes A} & P^{-1}\otimes A & \xrightarrow{d_P^{-1}\otimes A} & P^0\otimes A & \longrightarrow & E_0\otimes A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_A^{-1}} & & \downarrow{\scriptstyle \alpha_A^{0}} & & \downarrow{\scriptstyle \alpha_A} & & \\
\cdots & \xrightarrow{d_{Q_A}^{-2}} & Q_A^{-1} & \xrightarrow{d_{Q_A}^{-1}} & Q_A^0 & \longrightarrow & E_A & \longrightarrow & 0
\end{array}$$

where

$$P^i = V^i \otimes \mathscr{O}_X(-m_i) \quad \text{and} \quad Q^i_A = W^i \otimes \mathscr{O}_{X_A}(-n_i). \tag{5-1}$$

Here, V^i and W^i are vector spaces and $m_i, n_i \in \mathbb{N}$. Then, $Q^\bullet = Q^\bullet_A \otimes_A k$ is a resolution of E, because E_A is flat over A.

We can view the morphism α_A as a morphism between complexes concentrated at degree 0, then $I_A^\bullet$ can be viewed as a mapping cone $I_A^\bullet \cong C(\alpha_A)[-1] \cong C(\alpha_A^\bullet)[-1]$. For the sake of notation, we write down the mapping cone explicitly:

$$\cdots \to P^{-1} \otimes A \oplus Q_A^{-2} \xrightarrow{d_A^{-2}} P^0 \otimes A \oplus Q_A^{-1} \xrightarrow{d_A^{-1}} Q_A^0 \to 0,$$

where

$$d_A^i = \begin{pmatrix} -d_P^{i+1} \otimes A & 0 \\ \alpha_A^{i+1} & d_{Q_A}^i \end{pmatrix}. \tag{5-2}$$

We lift $d_{Q_A}^i$ to $d_{Q_B}^i$, getting a sequence $(Q_B^i, d_{Q_B}^i)_{i \le 0}$, where

$$Q_B^i = W^i \otimes \mathscr{O}_{X_B}(-n_i).$$

We also lift $\alpha_A^i : P^i \otimes A \to Q_A^i$ to $\alpha_B^i : P^i \otimes B \to Q_B^i$. We then obtain a sequence

$$(P^{i+1} \otimes B \oplus Q_B^i, d_B^i)_{i \le 0}, \tag{5-3}$$

where d_B^i is similar to d_A^i in (5-2). This is not necessarily a complex:

$$d_B^i \circ d_B^{i-1} = \begin{pmatrix} 0 & 0 \\ -\alpha_B^{i+1} \circ (d_P^i \otimes B) + d_{Q_B}^i \circ \alpha_B^i & d_{Q_B}^i \circ d_{Q_B}^{i-1} \end{pmatrix} \tag{5-4}$$

may not vanish. But when it is a complex, $(Q_B^\bullet, d_{Q_B}^\bullet)$ forms a complex and $\alpha_B^\bullet : P^\bullet \otimes B \to Q_B^\bullet$ is a morphism of complexes. Thus,

$$H^0(\alpha_B^\bullet) : E_0 \otimes B \to H^0(Q_B^\bullet, d_{Q_B}^\bullet)$$

provides a flat extension of α_A, according to Lemma 5.1, which will be stated and proved in the next subsection.

The lower row of (5-4) constitutes a map

$$P^\bullet[1] \otimes B \oplus Q_B^\bullet \to Q_B^\bullet[2]. \tag{5-5}$$

When restricted to X_A, it becomes zero. Moreover, $\mathfrak{m}_B K = 0$. The map above induces a map[4]

$$(\omega_P^\bullet, \omega_Q^\bullet) : C(\alpha^\bullet) \to Q_B^\bullet[2] \otimes_B K \cong Q^\bullet[2] \otimes_k K. \tag{5-6}$$

We claim that $(\omega_P^\bullet, \omega_Q^\bullet)$ is a morphism of complexes, which will be proven, see Lemma 5.3. This induces a class, which will be shown to be the obstruction class

$$\mathrm{ob}(\alpha_A, \sigma) = [(\omega_P^\bullet, \omega_Q^\bullet)] \in \mathrm{Hom}_{K(X)}(C(\alpha^\bullet), Q^\bullet[2] \otimes_k K). \tag{5-7}$$

[4]The argument used to deduce (5-6) from (5-5) will be applied repeatedly.

To identify $\mathrm{Hom}_{K(X)}(C(\alpha^\bullet), Q^\bullet[2] \otimes K)$ with $\mathrm{Ext}^1(I^\bullet, E \otimes K)$ in the theorem, we only need to take (5-1) to be very negative such that $H^i(X, E(m_j)) = 0$ and $H^i(X, E(n_j)) = 0$, for all $i > 0$ and $j \le 0$. Then

$$\mathrm{Ext}^1(I^\bullet, E \otimes K) \cong \mathrm{Hom}_{K(X)}(C(\alpha^\bullet), E[2] \otimes K) \cong \mathrm{Hom}_{K(X)}(C(\alpha^\bullet), Q^\bullet[2] \otimes K).$$

Suppose we have two extensions $\alpha_B : E_0 \otimes B \to E_B$ and $\beta_B : E_0 \otimes B \to F_B$, which arise from the following liftings:

$$\{d^i_{E_B} : Q^i_B \to Q^{i+1}_B, \alpha^i_B : P^i \otimes B \to Q^i_B\},$$
$$\{d^i_{F_B} : Q^i_B \to Q^{i+1}_B, \beta^i_B : P^i \otimes B \to Q^i_B\}.$$

The differences $d^i_{E_B} - d^i_{F_B}$ and $\alpha^i_B - \beta^i_B$ induce a morphism of complexes

$$(f^\bullet_P, f^\bullet_Q) : C(\alpha^\bullet) \to Q^\bullet[1] \otimes K. \tag{5-8}$$

This induces a class

$$v = [(f^\bullet_P, f^\bullet_Q)] \in \mathrm{Hom}_{K(X)}(C(\alpha^\bullet), Q^\bullet[1] \otimes K) \cong \mathrm{Ext}^1(I^\bullet, E \otimes K).$$

Conversely, given α_B and $(f^\bullet_P, f^\bullet_Q)$, we can produce another extension β_B.

Moreover, α_B and β_B are equivalent if and only if $v = 0$.

5B. *Proofs.* In this subsection, we fill in the proofs of several claims we made in Section 5A. We will assume the independence of choices in 5B1 and provide proofs of independence in 5B2. To simplify the notation, we will sometimes omit the superscripts in maps between complexes, such as $\alpha^\bullet$ and α^i.

5B1. *Obstruction classes.* We first show that $\mathrm{ob}(\alpha_A, \sigma)$ defined in (5-7) is an obstruction class.

Suppose an extension (E_B, α_B) exists. The definition of $\mathrm{ob}(\alpha_A, \sigma)$ does not depend on the choice of the resolution $Q^\bullet_A$. We can assume (E_B, α_B) arises by lifting $d^i_{Q_A}$ and α^i_A, making $Q^\bullet_B$ into a complex and $\alpha^\bullet_B$ a morphism of complexes. Then, $(\omega^\bullet_P, \omega^\bullet_Q) = 0$. Thus, $\mathrm{ob}(\alpha_A, \sigma) = 0$.

Conversely, suppose $\mathrm{ob}(\alpha_A, \sigma) = 0$. It is enough to show that $(\omega^\bullet_P, \omega^\bullet_Q) = 0$, after possible modifications of the liftings. The vanishing of $\mathrm{ob}(\alpha_A, \sigma)$ is equivalent to $(\omega^\bullet_P, \omega^\bullet_Q)$ being homotopic to 0. Let $(g^\bullet_P, g^\bullet_Q)$ be a homotopy. By abuse of notation, let ι denote inclusions

$$\iota : Q^i_B \otimes K \hookrightarrow Q^i_B.$$

Similarly, π denotes the corresponding quotients,

$$\pi : P^i \otimes B \twoheadrightarrow P^i \quad \text{and} \quad \pi : Q^i_B \twoheadrightarrow Q^i.$$

We can replace α_B and d_{Q_B} by

$$\alpha_B - \iota \circ g_P \circ \pi \quad \text{and} \quad d_{Q_B} - \iota \circ g_Q \circ \pi,$$

then the new $(\omega^\bullet_P, \omega^\bullet_Q)$ is zero.

The following well-known lemma is central to our argument. For completeness, we give a proof here.

Lemma 5.1. *Let* $(Q_A^\bullet, d_{Q_A}^\bullet)$ *be a sequence of the form* $Q_A^i \cong W^i \otimes \mathscr{O}_{X_A}(-n_i)$, $i \leq 0$, *such that*

$$(Q_A^\bullet, d_Q^\bullet) \otimes_A k \cong (Q^\bullet, d^\bullet)$$

is a resolution of E. *If* $(Q_A^\bullet, d_{Q_A}^\bullet)$ *is a complex, then it is exact except at the* 0*-th place and the cohomology* $H^0(Q_A^\bullet, d_{Q_A}^\bullet)$ *is an extension of* E *flat over* A.

Proof. There is a short exact sequence of complexes

$$0 \to Q_A^\bullet \otimes_A \mathfrak{m}_A \to Q_A^\bullet \to Q^\bullet \to 0.$$

First, let n be the least integer such that $\mathfrak{m}_A^n = 0$. We shall show that for $0 \leq i \leq n$, $Q_A^\bullet \otimes A/\mathfrak{m}_A^i$ is exact except at the 0-th place, by induction on i decreasingly. Tensor $Q_A^\bullet$ over A with the short exact sequence

$$0 \to \mathfrak{m}_A^{n-1} \to \mathfrak{m}_A^{n-2} \to \mathfrak{m}_A^{n-2}/\mathfrak{m}_A^{n-1} \to 0,$$

whose last term is a direct sum of copies of k. On the other hand, $Q_A^\bullet \otimes \mathfrak{m}_A^{n-1} \cong Q^\bullet \otimes_k \mathfrak{m}_A^{n-1}$. We deduce that the complexes $Q_A^\bullet \otimes \mathfrak{m}_A^{n-1}$ and $Q_A^\bullet \otimes \mathfrak{m}_A^{n-2}/\mathfrak{m}_A^{n-1}$ are exact except at the 0-th places. So, from the associated long exact sequence, $Q_A^\bullet \otimes \mathfrak{m}_A^{n-2}$ is also exact except at the 0-th place. Inductively, we can prove this for $Q_A^\bullet$.

Next, let $E_A = H^0(Q_A^\bullet, d_{Q_A}^\bullet)$. We shall show that $E_A \otimes A/\mathfrak{m}_A^i$ is flat for $1 \leq i \leq n$, by induction on i.

Of course $E_A \otimes_A A/\mathfrak{m}_A \cong E$ is flat over $A/\mathfrak{m}_A \cong k$. Tensor the short exact sequence

$$0 \to \mathfrak{m}_A/\mathfrak{m}_A^2 \to A/\mathfrak{m}_A^2 \to A/\mathfrak{m}_A \to 0 \tag{5-9}$$

by $Q_A^\bullet$ over A. Since the ideal $\mathfrak{m}_A/\mathfrak{m}_A^2$ is square-zero, we have the short exact sequence of complexes

$$0 \to Q^\bullet \otimes_k \mathfrak{m}_A/\mathfrak{m}_A^2 \to Q_A^\bullet \otimes_A A/\mathfrak{m}_A^2 \to Q^\bullet \to 0.$$

The associated long exact sequence degenerates to

$$0 \to E \otimes \mathfrak{m}_A/\mathfrak{m}_A^2 \to E_A \otimes A/\mathfrak{m}_A^2 \to E \to 0. \tag{5-10}$$

Therefore, $E_A \otimes_A A/\mathfrak{m}_A^2$ is flat over $A/\mathfrak{m}_A^2$, according to Lemma 5.2. Replacing (5-9) by

$$0 \to \mathfrak{m}_A^2/\mathfrak{m}_A^3 \to A/\mathfrak{m}_A^3 \to A/\mathfrak{m}_A^2 \to 0,$$

we can repeat this argument. Inductively, we can prove E_A is flat over A.

Similar to obtaining (5-10), we also have the short exact sequence

$$0 \to E_A \otimes \mathfrak{m}_A \to E_A \to E \to 0.$$

So, E_A is an extension of E flat over A. □

For the reader's convenience, we include the following basic lemma about flatness. For a proof, see [Hartshorne 2010, Proposition 2.2].

Lemma 5.2. *Let $B \to A$ be a surjective homomorphism of Noetherian rings whose kernel K is square zero. Then a B-module M' is flat over B if and only if $M = M' \otimes_B A$ is flat over A and the natural map $M \otimes_A K \to M'$ is injective.*

Lemma 5.3. *The map* (5-6) *is a morphism of complexes.*

Proof. We have two equalities

$$\text{(5-11)} \quad -\alpha_B \circ d_P \otimes B + d_{Q_B} \circ \alpha_B = \iota \circ \omega_P \circ \pi \quad \text{and} \quad d_{Q_B} \circ d_{Q_B} = \iota \circ \omega_Q \circ \pi.$$

The map (5-6) is indeed a morphism: one can show that

$$\iota \circ \left(d_Q \otimes K \circ (\omega_P, \omega_Q) - (\omega_P, \omega_Q) \begin{pmatrix} -d_P & 0 \\ \alpha & d_Q \end{pmatrix} \right) \circ \pi = 0.$$

Because ι is injective and π is surjective, (ω_P, ω_Q) commutes with differentials.[5] □

5B2. *Obstructions: independence of choices.* We now show that $\mathrm{ob}(\alpha_A, \sigma)$ is independent of various choices we have made: $\alpha_A^\bullet$, $\alpha_B^\bullet$, $d_{Q_B}^\bullet$, and $Q_A^\bullet$.

To start, if we choose a different lifting $\alpha_A^\bullet$ of α_A, then $(\omega_P^\bullet, \omega_Q^\bullet)$ only differs by a homotopy.

We next show that the morphism $(\omega_P^\bullet, \omega_Q^\bullet)$ is independent of liftings α_B and d_{Q_B}, modulo homotopy.

Let α'_B and d'_{Q_B} be different liftings, giving rise to $(\omega_P'^\bullet, \omega_Q'^\bullet)$. The differences $\alpha_B - \alpha'_B$ and $d_{Q_B} - d'_{Q_B}$ induce a map, which will be shown to be a homotopy,

$$(h_P^\bullet, h_Q^\bullet) : P^\bullet[1] \oplus Q^\bullet \to Q^\bullet[1] \otimes_k K.$$

We have the following equalities:

$$\text{(5-12)} \qquad \iota \circ h_P \circ \pi = \alpha_B - \alpha'_B \quad \text{and} \quad \iota \circ h_Q \circ \pi = d_{Q_B} - d'_{Q_B}.$$

Combining (5-11) and (5-12), we obtain

$$\omega_P - \omega'_P = -h_P \circ d_P + d_Q \otimes K \circ h_P + h_Q \circ \alpha,$$
$$\omega_Q - \omega'_Q = d_Q \otimes K \circ h_Q + h_Q \circ d_Q.$$

[5] The trick using ι and π will be applied repeatedly.

Therefore,

$$(\omega_P, \omega_Q) - (\omega'_P, \omega'_Q) = d_Q \otimes K \circ (h_P, h_Q) + (h_P, h_Q) \begin{pmatrix} -d_P & 0 \\ \alpha & d_Q \end{pmatrix},$$

which means $(\omega^\bullet_P, \omega^\bullet_Q)$ and $({\omega'}^\bullet_P, {\omega'}^\bullet_Q)$ are homotopic.

Finally, we show the independence of $Q^\bullet_A$.

Let $(R^\bullet_A, d^\bullet_{R_A})$ be another very negative resolution of the form:

$$R^i_A = W^{i'} \otimes \mathscr{O}_{X_A}(-n'_i).$$

Then, there is a lifting of the identity map $q^\bullet_A : Q^\bullet_A \to R^\bullet_A$, unique up to homotopy. Let

$$\beta^\bullet_A = q^\bullet_A \circ \alpha^\bullet_A : P^\bullet \otimes A \to R^\bullet_A.$$

Moreover, there is a morphism

$$\operatorname{diag}(\mathrm{id}, q^\bullet_A) : C(\alpha^\bullet_A) \to C(\beta^\bullet_A).$$

Lift $q^\bullet_A$ and $\beta^\bullet_A$ to $q^\bullet_B : Q^\bullet_B \to R^\bullet_B$ and $\beta^\bullet_B : P^\bullet \otimes B \to R^\bullet_B$. Then, we have a map of sequences

$$\operatorname{diag}(\mathrm{id}, q^\bullet_B) : P^\bullet[1] \otimes B \oplus Q^\bullet_B \to P^\bullet[1] \otimes B \oplus R^\bullet_B.$$

This fits in the following square, which is not necessarily commutative,

(5-13)
$$\begin{array}{ccc} P^\bullet[1] \otimes B \oplus Q^\bullet_B & \longrightarrow & Q^\bullet_B[2] \\ {\scriptstyle \operatorname{diag}(\mathrm{id}, q^\bullet_B)} \downarrow & & \downarrow {\scriptstyle q^\bullet_B} \\ P^\bullet[1] \otimes B \oplus R^\bullet_B & \longrightarrow & R^\bullet_B[2] \end{array}$$

Here, the two horizontal maps are defined as in (5-5). The square above induces

$$\begin{array}{ccc} P^\bullet[1] \oplus Q^\bullet & \xrightarrow{(\omega^\bullet_P, \omega^\bullet_Q)} & Q^\bullet[2] \otimes K \\ {\scriptstyle \operatorname{diag}(\mathrm{id}, q^\bullet)} \downarrow & & \downarrow {\scriptstyle q^\bullet} \\ P^\bullet[1] \oplus R^\bullet & \xrightarrow{(\bar\omega^\bullet_P, \bar\omega^\bullet_R)} & R^\bullet[2] \otimes K \end{array}$$

To show that $\operatorname{ob}(\alpha_A, \sigma)$ is independent of the resolution, it is enough to show that the two compositions in the square above differ by a homotopy. This is because, if they differ by a homotopy, two classes $[(\omega^\bullet_P, \omega^\bullet_Q)]$ and $[(\bar\omega^\bullet_P, \bar\omega^\bullet_R)]$ are identified via the isomorphism

$$\operatorname{Hom}_{K(X)}(C(\alpha^\bullet), Q^\bullet[2] \otimes K) \cong \operatorname{Hom}_{K(X)}(C(\beta^\bullet), R^\bullet[2] \otimes K).$$

Indeed, the difference $d_{R_B} \circ q_B - q_B \circ d_{Q_B}$ and $\beta_B - q_B \circ \alpha_B$ induce maps

$$\tau^\bullet : Q^\bullet \to R^\bullet[1] \otimes K \quad \text{and} \quad \upsilon^\bullet : P^\bullet \to Q^\bullet \otimes K.$$

There are the following equalities:

$$d_{R_B} \circ q_B - q_B \circ d_{Q_B} = \iota \circ \tau \circ \pi \quad \text{and} \quad \beta_B - q_B \circ \alpha_B = \iota \circ \upsilon \circ \pi. \tag{5-14}$$

Combining (5-11) and (5-14), we know that the difference of two compositions in (5-13) is

$$\iota \circ \big((\bar{\omega}_P, \bar{\omega}_R) \circ \operatorname{diag}(\mathrm{id}, q) - q \circ (\omega_P, \omega_Q)\big) \circ \pi$$
$$= \iota \circ \left((\upsilon, \tau) \circ \begin{pmatrix} -d_P & 0 \\ \alpha & d_Q \end{pmatrix} + d_R \otimes K \circ (\upsilon, \tau) \right) \circ \pi.$$

Thus, $(\upsilon^\bullet, \tau^\bullet)$ is a homotopy.

5B3. *Deformations.* Assume that the obstruction class $\mathrm{ob}(\alpha_A, \sigma)$ vanishes.

Suppose there are two extensions:

$$\alpha_B : E_0 \otimes B \to E_B \quad \text{and} \quad \beta_B : E_0 \otimes B \to F_B.$$

Resolve E_B and F_B by two very negative complex with identical terms but different differentials: $(Q^\bullet_B, d^\bullet_{E_B})$ and $(Q^\bullet_B, d^\bullet_{F_B})$. Then, lift α_B and β_B:

$$\begin{array}{ccc} P^\bullet \otimes B & \xrightarrow{\sim} & E_0 \otimes B \\ \downarrow{\scriptstyle \alpha^\bullet_B} & & \downarrow{\scriptstyle \alpha_B} \\ (Q^\bullet_B, d^\bullet_{E_B}) & \xrightarrow{\sim} & E_B \end{array} \quad \text{and} \quad \begin{array}{ccc} P^\bullet \otimes B & \xrightarrow{\sim} & E_0 \otimes B \\ \downarrow{\scriptstyle \beta^\bullet_B} & & \downarrow{\scriptstyle \beta_B} \\ (Q^\bullet_B, d^\bullet_{F_B}) & \xrightarrow{\sim} & F_B \end{array}$$

The differences $d^i_{E_B} - d^i_{F_B}$ and $\alpha^i_B - \beta^i_B$ induce maps

$$f^i_Q : Q^i \to Q^{i+1} \otimes K \quad \text{and} \quad f^i_P : P^i \to Q^i \otimes K.$$

One can show that these provide a morphism of complexes

$$(f^\bullet_P, f^\bullet_Q) : C(\alpha^\bullet) \to Q^\bullet[1] \otimes K. \tag{5-15}$$

Thus, they induce a class v defined by

$$v = [(f^\bullet_P, f^\bullet_Q)] \in \mathrm{Ext}^1(I^\bullet, E \otimes K).$$

Conversely, if we are given an extension (E_B, α_B) and a class v represented by (f_P, f_Q), then

$$\beta_B = \alpha_B - \iota \circ f_P \circ \pi \quad \text{and} \quad d_{F_B} = d_{E_B} - \iota \circ f_Q \circ \pi$$

produce a morphism of complexes $P^\bullet \otimes B \to (Q^\bullet_B, d^\bullet_{F_B})$. This induces an extension of (E_A, α_A):

$$(F_B, \beta_B) = (H^0(Q^\bullet_B, d^\bullet_{F_B}), H^0(\beta^\bullet_B)).$$

If we choose a different resolution $R^\bullet_B$ and define $(\bar{f}^\bullet_P, \bar{f}^\bullet_R)$ similarly as in (5-8), then $[(f^\bullet_P, f^\bullet_Q)]$ and $[(\bar{f}^\bullet_P, \bar{f}^\bullet_R)]$ are identified under the isomorphism

$$\mathrm{Hom}_{K(X)}(P^\bullet[1] \oplus Q^\bullet, Q^\bullet[1] \otimes K) \cong \mathrm{Hom}_{K(X)}(P^\bullet[1] \oplus R^\bullet, R^\bullet[1] \otimes K).$$

So, v is independent of the resolution $Q^\bullet_B$.

We next show that the difference of two equivalent extensions gives a zero class v. Indeed, suppose α_B and β_B are equivalent, then by Lemma 2.11, there is a constant $z \in B$ such that $\beta_B = z\alpha_B$. Denote the image of z in k as $\bar{z}$. We have proven that v is independent of resolutions. So, for our convenience, we take the same resolution $Q^\bullet_B$ for E_B and F_B, and take $\beta^\bullet = z\alpha^\bullet$. Then $f^\bullet_Q = 0$. Furthermore, $f^\bullet_P$ in (5-8) is homotopic to zero via homotopy

$$(0, 1 - \bar{z}) : P^{i+1} \oplus Q^i \to Q^i \otimes K.$$

Thus, the associated $v = 0$.

It remains to prove that if $(h^\bullet_P, h^\bullet_Q)$ is a homotopy between $(f^\bullet_P, f^\bullet_Q)$ and zero, then α_B and β_B are equivalent. One can actually check:

(i) $\mathrm{id} - \iota \circ h_Q \circ \pi : (Q^\bullet_B, d^\bullet_{E_B}) \to (Q^\bullet_B, d^\bullet_{F_B})$ is a morphism of complexes.

(ii) $(\mathrm{id} - \iota \circ h_Q \circ \pi) \circ \alpha_B = \beta_B - d_{F_B} \circ \iota \circ h_P \circ \pi - \iota \circ h_P \circ \pi \circ d_P \otimes B$.

Hence, there is a morphism ϕ commuting two families of stable pairs α_B and β_B. Therefore, by Lemma 2.11, this is an isomorphism.

6. Stable pairs on surfaces

In this section, we assume that $(X, \mathscr{O}_X(1))$ is a smooth projective surface, E_0 is torsion-free, P and δ are of degree 1. We shall demonstrate that in these cases, the moduli space of stable pairs admits a virtual fundamental class, proving Theorem 1.3.

To show the existence of the virtual fundamental class, it suffices to show that the obstruction theory is *perfect* [Behrend and Fantechi 1997; Li and Tian 1998]. That is, there is a two-term complex of locally free sheaves resolving the deformation and obstruction sheaves. In order to do this, we essentially need to show that there are no higher obstructions, which is guaranteed by the following lemma.

Lemma 6.1. *Fix a stable pair* (E, α). *Let* $I^\bullet$ *denote the complex* $\{E_0 \xrightarrow{\alpha} E\}$ *positioned at* 0 *and* 1. *Then*

$$\mathrm{Ext}^i(I^\bullet, E) = 0, \quad \textit{unless } i = 0, 1.$$

Proof. The stable pair fits into an exact sequence

$$0 \to K \to E_0 \to E \to Q \to 0,$$

which can be written as a distinguished triangle

$$K \to I^\bullet \to Q[-1] \to K[1].$$

Notice that K is torsion-free and Q is 0-dimensional. Apply the functor $\mathrm{Hom}(-, E)$ to this triangle. The associated long exact sequence is

$$0 \to \mathrm{Hom}(Q, E) \to \mathrm{Ext}^{-1}(I^\bullet, E) \to 0 \to \cdots \\ \to 0 \to \mathrm{Ext}^2(I^\bullet, E) \to \mathrm{Ext}^2(K, E) \to 0.$$

Since Q is 0-dimensional and E is pure, $\mathrm{Hom}(Q, E) = 0$. Thus, $\mathrm{Ext}^{-1}(I^\bullet, E) = 0$. The kernel K is torsion-free, so $\mathrm{Ext}^2(K, E) \cong \mathrm{Hom}(E, K \otimes \omega_X)^\vee = 0$. Therefore, $\mathrm{Ext}^2(I^\bullet, E) = 0$. □

Using this lemma, the expected dimension of the moduli space can be easily calculated via Hirzebruch–Riemann–Roch, knowing invariants of E_0.

Now, let

$$\mathbb{I}^\bullet = \{\pi_X^* E_0 \xrightarrow{\tilde{\alpha}} \mathbb{E}\}$$

be the universal pair, according to Theorem 1.1. By Theorem 1.2, the deformation sheaf and the obstruction sheaf are calculated by

$$R\pi_* R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{E}).$$

Take a finite complex $P^\bullet$ of locally free sheaves resolving $\mathbb{E}$ and a finite complex $Q^\bullet$ of very negative locally free sheaves resolving $\mathbb{I}^\bullet$. Take a finite, very negative locally free resolution $A^\bullet$ of $(Q^\bullet)^\vee \otimes P^\bullet$. Then

$$R\pi_* R\mathcal{H}om(\mathbb{I}^\bullet, \mathbb{E}) \cong R\pi_* R\mathcal{H}om(Q^\bullet, P^\bullet) \cong R\pi_* A^\bullet. \tag{6-1}$$

Denote this complex on the moduli space as $B^\bullet$. By Grothendieck–Verdier duality [Hartshorne 1966; Conrad 2000],

$$B^\bullet = R\pi_* A^\bullet \cong R\pi_* R\mathcal{H}om(A^{\bullet\vee} \otimes \omega_X, \omega_X) \\ \cong R\mathcal{H}om(R\pi_*(A^{\bullet\vee} \otimes \omega_X)[-2], \mathcal{O}).$$

Moreover, notice that

$$R\pi_*(A^{\bullet\vee} \otimes \omega_X) = \pi_*(A^{\bullet\vee} \otimes \omega_X)$$

is a complex of locally free sheaves, due to the negativity of A^j's. Thus, $B^\bullet$ is a complex of locally free sheaves as well. Denote the differentials as d^i's.

Next, we show that $B^\bullet$ can be truncated to degree 0 and 1. The cohomologies of $B^\bullet$ concentrate at degree 0 and 1, by Lemma 6.1. Suppose B^i for an $i \geq 2$ is the last term that is nonzero. Both B^i and B^{i-1} are locally free, then $\ker d^{i-1}$ is also locally free. Replace B^i by zero and B^{i-1} by $\ker d^{i-1}$. We get a new complex of locally free sheaves, which is quasi-isomorphic to $B^\bullet$. Inductively, we can trim $B^\bullet$ down to degree 1. On the other side, suppose B^j for a $j < 0$ is the first term that is nonzero. Then, d^j is injective fiberwise. Therefore, $\mathrm{coker}\, d^j$ is flat [Grothendieck 1961a, (10.2.4), Chapter 0], thus locally free. Hence, we can replace B^{j-1} by zero and B^j

by coker d^j to get a new complex of locally free sheaves. Inductively, $B^\bullet$ becomes a complex concentrated in degree 0 and 1, with cohomologies the deformation sheaf and the obstruction sheaf. Namely, we have the following exact sequence on $S_{E_0}(P,\delta)$:

$$0 \to \mathcal{D}ef \to B^0 \to B^1 \to \mathcal{O}bs \to 0,$$

where B^0 and B^1 are locally free.

Therefore, the moduli space admits a virtual fundamental class.

7. Examples

In this section, we study examples of moduli spaces of dimension 1 stable pairs over K3 surfaces. Let $(X, \mathscr{O}_X(1))$ be a polarized K3 surface, P be a Hilbert polynomial of degree 1, and δ be a positive polynomial of degree larger than 1. Let E_0 be a fixed coherent sheaf over X. Then a pair (E,α), such that $P_E = P$, is stable if E is pure and coker α has dimension 0, by Lemma 2.10.

Let $H = c_1(\mathscr{O}(1)) \in H_2(X,\mathbb{Z})$. Suppose the schematic support of E, which is a curve, has arithmetic genus h. There are two discrete invariants of E[6]:

$$\beta_h = c_1(E) \in H_2(X,\mathbb{Z}) \quad \text{and} \quad \chi(E) = 1-h+d. \tag{7-1}$$

They are related to the Hilbert polynomial by

$$P_E(m) = (\beta_h.H)m + 1 - h + d.$$

So, with the Hilbert polynomial fixed, there are only finitely many possible β_h's. The moduli space decomposes as a disjoint union:

$$S_{E_0}(P,\delta) = \coprod_{\beta_h} S_{E_0}(\beta_h, 1-h+d),$$

where $S_{E_0}(\beta_h, 1-h+d)$ denote the moduli space of stable pairs satisfying (7-1).

Let C_h be a representative in the class β_h; then the linear system $|C_h|$ is isomorphic to $\mathbb{P}^h$. Let

$$\mathcal{C}_h \subset |C_h| \times X$$

be the universal curve.

When $E_0 \cong \mathscr{O}_X$, by [Pandharipande and Thomas 2010, Proposition B.8],

$$S_{\mathscr{O}_X}(\beta_h, 1-h+d) \cong \mathcal{C}_h^{[d]},$$

where $\mathcal{C}_h^{[d]}$ is the relative Hilbert scheme of points. If there is an ample line bundle H such that

$$C_h.H = \min\{L.H \mid L \in \mathrm{Pic}(X),\ L.H > 0\}, \tag{7-2}$$

[6]There is a slight abuse of notation concerning β and d, but this is unlikely to cause confusion.

then $S_{\mathscr{O}_X}(\beta_h, 1-h+d)$ is a smooth scheme of dimension $h+d$, see [Kawai and Yoshioka 2000, Lemmas 5.117 and 5.175] or [Pandharipande and Thomas 2010, Proposition C.2].

The moduli space is not smooth in general for a higher rank E_0. For example, assume $E_0 \cong \mathscr{O}_X^{\oplus 2}$ and the stable pair $(E, \alpha : \mathscr{O}_X^{\oplus 2} \to E)$ maps a summand $\mathscr{O}_X$ to 0. Then, the deformation space of this stable pair is isomorphic to

$$\operatorname{Hom}(\mathscr{O}_X \to E, E) \oplus H^0(E).$$

The dimension of $\operatorname{Hom}(\mathscr{O}_X \to E, E)$ is $h+d$, while $h^0(E)$ may vary as E varies. But when d is large, we do expect the moduli space to be smooth for higher rank E_0.

Proposition 7.1. *Suppose β_h is irreducible, i.e., β_h is not a sum of two curve classes, and $d > 2h-2$. Then the moduli space $S_{\mathscr{O}_X^{\oplus r}}(\beta_h, 1-h+d)$ is smooth of dimension $rd+(r-2)(1-h)+1$.*

Proof. Apply the functor $\operatorname{Hom}(-, E)$ to

$$I^\bullet \to \mathscr{O}_X^{\oplus r} \to E \to I^\bullet[1].$$

According to Lemma 6.1, the associated long exact sequence is

$$0 \to \operatorname{Hom}(E, E) \to H^0(X, E)^{\oplus r} \to \operatorname{Hom}(I^\bullet, E) \to \\ \operatorname{Ext}^1(E, E) \to H^1(X, E)^{\oplus r} \to \operatorname{Ext}^1(I^\bullet, E) \to \operatorname{Ext}^2(E, E) \to 0.$$

Since β_h is irreducible, E is stable. Therefore, $\operatorname{ext}^2(E, E) = \operatorname{hom}(E, E) = 1$. When $d > 2h-2$, by Serre duality, $h^1(X, E) = h^1(C, E) = 0$ where C is the support of E. Thus, the tangent space $\operatorname{Hom}(I^\bullet, E)$ has constant dimension $\chi(I^\bullet, E)+1 = rd+(r-2)(1-h)+1$. □

For every $h \geq 0$, there exists a K3 surface X_h and a curve class $\beta_h \in H_2(X_h, \mathbb{Z})$, such that $\beta_h.\beta_h = 2h-2$ and (7-2) is satisfied, see [Kawai and Yoshioka 2000, Remark 5.110]. For each $h \geq 0$, we fix such X_h and β_h.

Kawai and Yoshioka [2000, Corollary 5.85] calculated the generating series of topological Euler characteristics of the moduli spaces.

Theorem 7.2 (Kawai–Yoshioka). *For $0 < |q| < |y| < 1$, the generating series of topological Euler characteristics is*

$$\sum_{h=0}^{\infty}\sum_{d=0}^{\infty} \chi_{\mathrm{top}}\big(S_{\mathscr{O}_{X_h}}(\beta_h, 1-h+d)\big) q^{h-1} y^{1-h+d} \\ = \left(\big(y^{-1/2}-y^{1/2}\big)^2 q \prod_{n=1}^{\infty}\big(1-q^n\big)^{20}\big(1-q^n y\big)^2\big(1-q^n y^{-1}\big)^2\right)^{-1}.$$

Next, we consider stable pairs over X_h of the form

$$\alpha : L_h \to E,$$

where L_h is a line bundle with the first Chern class $c_1(L_h) = l\beta_h$. Such a stable pair is equivalent to $\mathscr{O}_X \to E \otimes L_h^{-1}$. Notice that $c_1(E \otimes L_h^{-1}) = \beta_h$ and $\chi(E \otimes L_h^{-1}) = 1 - h + d - 2l(h-1)$. Therefore,

$$S_{L_h}(\beta_h, 1-h+d) \cong S_{\mathscr{O}_X}(\beta_h, 1-h+d-2l(h-1)).$$

If $\alpha \neq 0$, then $d \geq 2l(h-1)$. The generating series is

$$\begin{aligned}\sum_{h=0}^{\infty} \sum_{d=2l(h-1)}^{\infty} & \chi_{\text{top}}\big(S_{L_h}(\beta_h, 1-h+d)\big)q^{h-1}y^{d+1-h} \\ &= \sum_{h=0}^{\infty}\sum_{d=0}^{\infty} \chi_{\text{top}}\big(S_{\mathscr{O}_X}(\beta_h, 1-h+d)\big)(qy^{2l})^{h-1}y^{d+1-h} \\ &= \left(\big(y^{-\frac{1}{2}} - y^{\frac{1}{2}}\big)^2 qy^{2l} \prod_{n=1}^{\infty}\big(1-q^n y^{2nl}\big)^{20}\big(1-q^n y^{2nl+1}\big)^2\big(1-q^n y^{2nl-1}\big)^2\right)^{-1}.\end{aligned}$$

Now, we consider stable pairs over X_h of the form

$$\alpha : \bigoplus_i L_{i,h} \to E,$$

where $L_{i,h}$ is a line bundle with $c_1(L_{i,h}) = l_i\beta_h$. The proof of Proposition 7.1 can also show that the moduli space is smooth when d is large compared to l_i and h. Let $\mathbb{G}_m$ act on direct summands with distinct weights; then there is a natural $\mathbb{G}_m$-action on the moduli space $S_{X_h}^{\oplus L_{i,h}}(\beta_h, 1-h+d)$. A morphism $\oplus L_{i,h} \to E$ is fixed under the action if and only if exactly one summand $L_{i,h}$ is mapped to E nontrivially. Thus, we have the following the fixed loci:

$$S_{\oplus L_{i,h}}(\beta_h, 1-h+d)^{\mathbb{G}_m} \cong \coprod_i S_{L_{i,h}}(\beta_h, 1-h+d).$$

When $\alpha \neq 0$, $d \geq \min\{2l_i(h-1)\}$. To calculate the Euler characteristics, we can use the localization formula, even when the moduli space is not smooth [Lawson and Yau 1987]. Then,

$$\begin{aligned}\sum_h \sum_d & \chi_{\text{top}}\big(S_{\oplus L_{i,h}}(\beta_h, 1-h+d)\big)q^{h-1}y^{d+1-h} \\ &= \sum_i \left((y^{-\frac{1}{2}} - y^{\frac{1}{2}})^2 qy^{2l_i} \prod_{n=1}^{\infty}(1-q^n y^{2nl_i})^{20}(1-q^n y^{2nl_i+1})^2(1-q^n y^{2nl_i-1})^2\right)^{-1}.\end{aligned}$$

Acknowledgements

The author is grateful to Professor Alina Marian for introducing him to this problem and discussions throughout the process of studying this problem and preparing this paper. The author wants to thank Barbara Bolognese, Professor Daniel Huybrechts, Professor Anthony Iarrobino, Yaping Yang, and Gufang Zhao for the discussions and correspondences. The author also wants to thank the referees for valuable comments.

References

[Behrend and Fantechi 1997] K. Behrend and B. Fantechi, "The intrinsic normal cone", *Invent. Math.* **128**:1 (1997), 45–88. MR Zbl

[Belkale 2008] P. Belkale, "The strange duality conjecture for generic curves", *J. Amer. Math. Soc.* **21**:1 (2008), 235–258. MR Zbl

[Conrad 2000] B. Conrad, *Grothendieck duality and base change*, Lecture Notes in Math. **1750**, Springer, 2000. MR Zbl

[Grothendieck 1961a] A. Grothendieck, "Éléments de géométrie algébrique, III: Étude cohomologique des faisceaux cohérents, I", *Inst. Hautes Études Sci. Publ. Math.* **11** (1961), 5–167. MR Zbl

[Grothendieck 1961b] A. Grothendieck, "Techniques de construction et théorèmes d'existence en géométrie algébrique, IV: Les schémas de Hilbert", exposé 221 in *Séminaire Bourbaki, 1960/1961*, Secrétariat Mathématique, 1961. Reprinted as pp. 249–276 in *Séminaire Bourbaki* **6**, Soc. Math. France, Paris, 1995; correction in *Séminaire Bourbaki, 1961/1962* (1962), p. 302. Zbl

[Hartshorne 1966] R. Hartshorne, *Residues and duality*, Lecture Notes in Math. **20**, Springer, 1966. MR Zbl

[Hartshorne 2010] R. Hartshorne, *Deformation theory*, Graduate Texts in Math. **257**, Springer, 2010. MR Zbl

[He 1998] M. He, "Espaces de modules de systèmes cohérents", *Internat. J. Math.* **9**:5 (1998), 545–598. MR Zbl

[Huybrechts and Lehn 1995a] D. Huybrechts and M. Lehn, "Framed modules and their moduli", *Internat. J. Math.* **6**:2 (1995), 297–324. MR Zbl

[Huybrechts and Lehn 1995b] D. Huybrechts and M. Lehn, "Stable pairs on curves and surfaces", *J. Algebraic Geom.* **4**:1 (1995), 67–104. MR Zbl

[Huybrechts and Lehn 1997] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics **E31**, Vieweg, Braunschweig, 1997. MR Zbl

[Inaba 2002] M.-a. Inaba, "Toward a definition of moduli of complexes of coherent sheaves on a projective scheme", *J. Math. Kyoto Univ.* **42**:2 (2002), 317–329. MR Zbl

[Kawai and Yoshioka 2000] T. Kawai and K. Yoshioka, "String partition functions and infinite products", *Adv. Theor. Math. Phys.* **4**:2 (2000), 397–485. MR Zbl

[Kollár 2008] J. Kollár, "Hulls and husks", preprint, Princeton, 2008. arXiv

[Kool and Thomas 2014a] M. Kool and R. Thomas, "Reduced classes and curve counting on surfaces, I: Theory", *Algebr. Geom.* **1**:3 (2014), 334–383. MR Zbl

[Kool and Thomas 2014b] M. Kool and R. Thomas, "Reduced classes and curve counting on surfaces, II: Calculations", *Algebr. Geom.* **1**:3 (2014), 384–399. MR Zbl

[Lawson and Yau 1987] H. B. Lawson, Jr. and S. S.-T. Yau, "Holomorphic symmetries", *Ann. Sci. École Norm. Sup.* (4) **20**:4 (1987), 557–577. MR Zbl

[Le Potier 1993] J. Le Potier, *Systèmes cohérents et structures de niveau*, Astérisque **214**, Société Mathématique de France, Paris, 1993. MR Zbl

[Li and Tian 1998] J. Li and G. Tian, "Virtual moduli cycles and Gromov–Witten invariants of algebraic varieties", *J. Amer. Math. Soc.* **11**:1 (1998), 119–174. MR Zbl

[Lin 2016] Y. Lin, *Moduli spaces of stable pairs*, Ph.D. thesis, Northeastern University, 2016, Available at https://search.proquest.com/docview/1786926187?accountid=14496.

[Marian and Oprea 2007] A. Marian and D. Oprea, "The level-rank duality for non-abelian theta functions", *Invent. Math.* **168**:2 (2007), 225–247. MR Zbl

[Mochizuki 2009] T. Mochizuki, *Donaldson type invariants for algebraic surfaces*, Lecture Notes in Math. **1972**, Springer, 2009. MR Zbl

[Mumford et al. 1994] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse Mathematik (2) **34**, Springer, 1994. MR Zbl

[Pandharipande and Thomas 2009] R. Pandharipande and R. P. Thomas, "Curve counting via stable pairs in the derived category", *Invent. Math.* **178**:2 (2009), 407–447. MR Zbl

[Pandharipande and Thomas 2010] R. Pandharipande and R. P. Thomas, "Stable pairs and BPS invariants", *J. Amer. Math. Soc.* **23**:1 (2010), 267–297. MR Zbl

[Seshadri 1977] C. S. Seshadri, "Geometric reductivity over arbitrary base", *Advances in Math.* **26**:3 (1977), 225–274. MR Zbl

[Shatz 1977] S. S. Shatz, "The decomposition and specialization of algebraic families of vector bundles", *Compositio Math.* **35**:2 (1977), 163–187. MR Zbl

[Simpson 1994] C. T. Simpson, "Moduli of representations of the fundamental group of a smooth projective variety, I", *Inst. Hautes Études Sci. Publ. Math.* 79 (1994), 47–129. MR Zbl

[Thaddeus 1994] M. Thaddeus, "Stable pairs, linear systems and the Verlinde formula", *Invent. Math.* **117**:2 (1994), 317–353. MR Zbl

[Thaddeus 1996] M. Thaddeus, "Geometric invariant theory and flips", *J. Amer. Math. Soc.* **9**:3 (1996), 691–723. MR Zbl

[Wandel 2015] M. Wandel, "Moduli spaces of semistable pairs in Donaldson–Thomas theory", *Manuscripta Math.* **147**:3-4 (2015), 477–500. MR Zbl

Received June 3, 2016. Revised August 28, 2017.

YINBANG LIN
DEPARTMENT OF MATHEMATICS
NORTHEASTERN UNIVERSITY
BOSTON, MA
UNITED STATES

lin.yinb@husky.neu.edu

Current address:
YAU MATHEMATICAL SCIENCES CENTER
TSINGHUA UNIVERSITY
BEIJING
CHINA

yblin@math.tsinghua.edu.cn

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2018 is US $475/year for the electronic version, and $640/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 294 No. 1 May 2018

0030-8730(201805)294:1;1-T