Pacific Journal of Mathematics

MULTIPLICATION OF DISTRIBUTIONS AND A NONLINEAR MODEL IN ELASTODYNAMICS

C. O. R. SARRICO

Volume 294 No. 1 May 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 294, No. 1, 2018
dx.doi.org/10.2140/pjm.2018.294.195

MULTIPLICATION OF DISTRIBUTIONS AND A NONLINEAR MODEL IN ELASTODYNAMICS

C. O. R. SARRICO

We consider the system $u_t + (u^2/2)_x = \sigma_x$, $\sigma_t + u\sigma_x = k^2 u_x$, where k is a real number and the unknowns $u(x, t)$ and $\sigma(x, t)$ belong to convenient spaces of distributions. For this simplified model from elastodynamics, a rigorous solution concept defined in the setting of a distributional product is used. The explicit solution of a Riemann problem and the possible emergence of a δ shock wave are established. For initial conditions containing a Dirac measure, a δ' shock wave solution is also presented.

1. Introduction and main results

Let us consider the system

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = \sigma_x, \tag{1}$$

$$\sigma_t + u\sigma_x = k^2 u_x, \tag{2}$$

where $t \in \mathbb{R}$ is the time variable, $x \in \mathbb{R}$ is the one-dimensional space variable, $u(x, t)$ and $\sigma(x, t)$ are the unknowns state variables and $k > 0$ is a real number.

This strictly hyperbolic system in nonconservative form arises in a simplified model from elastodynamics where u is the velocity, σ is the stress and k is the speed of propagation of the elastic waves. Several aspects of this model were already studied by J. J. Cauret, J. F. Colombeau, A. Y. Le Roux, and K. T. Joseph in the setting of Colombeau generalized functions. For details, see [Cauret et al. 1989; Colombeau and LeRoux 1988; Joseph 1997]. When initial data are smooth, it is well-known that global solutions do not exist because discontinuities in u and σ appear in finite time. Meanwhile, when u and σ are discontinuous, products of distributions arise which make no sense in the classic theory of distributions.

Different concepts of solution can be found in the literature: the week asymptotic method [Danilov and Mitrovic 2008; Danilov and Shelkovich 2005a; 2005b], the measure theoretic method [Bouchut and James 1999; Brenier and Grenier 1998;

MSC2010: 35D99, 35L67, 46F10.
Keywords: products of distributions, nonconservative systems, nonstrictly hyperbolic systems, strictly hyperbolic systems, δ waves, δ' waves, Riemann problem.

Chen and Liu 2003; Huang 2005], the use of smooth function nets and weighted measures spaces [Keyfitz and Kranzer 1995], split delta functions [Nedeljkov 2002; Nedeljkov and Oberguggenberger 2008], Colombeau generalized functions [Cauret et al. 1989; Colombeau and LeRoux 1988; Joseph 1997; Nedeljkov 2004], and others. We will adopt a solution concept which is a consistent extension of the classical solution concept and is defined within the setting of a theory of distributional products. In our framework, the product of distributions is always a distribution that is not defined by approximation processes. Our products depend upon the choice of a certain function α that encodes the indeterminacy inherent to such products. We stress that this indeterminacy is not in general avoidable and in many situations it also has a physical meaning. Concerning this point let us mention [Bressan and Rampazzo 1988; Colombeau and LeRoux 1988; Dal Maso et al. 1995; Sarrico 2003]. Naturally the existence and the solutions of differential equations or systems containing such products may depend (or not) on α. We call such solutions α-solutions. The possibility of its physical occurrence depends on the physical system. Sometimes we cannot previously know the behavior of the physical system, possibly due to features that were not considered in the formulation of the model with the goal of simplifying it. Thus, the mathematical indeterminacy sometimes observed may have this origin. In the present paper, however, the α-solutions when they exist are independent of α.

First, we consider for system (1)–(2) the initial conditions

$$u(x,0) = a_1 + (a_2 - a_1)H(x), \tag{3}$$

$$\sigma(x,0) = b_1 + (b_2 - b_1)H(x), \tag{4}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and H stands for the Heaviside function. We will compute all α-solutions of this problem within the space W of pairs of distributions (u,σ) of the form

$$u(x,t) = u_1 + (u_2 - u_1)H(x - Vt) + g(t)\delta(x - Vt), \tag{5}$$

$$\sigma(x,t) = \sigma_1 + (\sigma_2 - \sigma_1)H(x - Vt), \tag{6}$$

where δ stands for the Dirac measure concentrated at the origin, $u_1, u_2, \sigma_1, \sigma_2, V \in \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ is a C^1-function. If $b_1 = b_2$ there exists α-solutions in W if and only if $a_1 = a_2$ and we will see, in the space W, the arising of the α-solution corresponding to the constant states,

$$u(x,t) = a_1, \tag{7}$$

$$\sigma(x,t) = b_1. \tag{8}$$

If $b_1 \neq b_2$ there exists an α-solution in W if and only if $a_1 \neq a_2$ with the possible arising of the traveling wave

$$u(x,t) = a_1 + (a_2 - a_1)H(x - Vt), \tag{9}$$

$$\sigma(x,t) = b_1 + (b_2 - b_1)H(x - Vt), \tag{10}$$

which propagates with speed $V = (a_1 + a_2)/2 - (b_2 - b_1)/(a_2 - a_1)$. These α-solutions depend neither on α nor on the constant $k > 0$!

From a mathematical point of view, this situation leads us to consider the interesting case $k = 0$ in which the eigenvalues of the system (1)–(2), $\lambda_1 = u - k$ and $\lambda_2 = u + k$ coincide and the system loses the strict hyperbolicity. In this case, assuming certain conditions to be specified later, we will see the possible emergence (in the same space W) of a delta shock wave with the form (5). Thus, the space of functions is not sufficient to contain all possible α-solutions of the Riemann problem (1)–(4) with $k = 0$.

Next, for the system (1)–(2), still with $k = 0$, we consider the initial conditions

$$u(x,0) = a, \tag{11}$$

$$\sigma(x,0) = b + m\,\delta(x), \tag{12}$$

where $a, b, m \in \mathbb{R}$ and $m \neq 0$. We will see the possible emergence of the α-solution,

$$u(x,t) = a + mt\,\delta'(x - at), \tag{13}$$

$$\sigma(x,t) = b + m\,\delta(x - at), \tag{14}$$

containing a δ wave and a δ' shock wave, both with speed a. This result is obtained within the space Z of pairs of distributions (u, σ) of the form

$$u(x,t) = u_1 + f(t)\,\delta'[x - \gamma(t)],$$
$$\sigma(x,t) = \sigma_1 + p\,\delta[x - \gamma(t)],$$

where $u_1, \sigma_1, p \in \mathbb{R}$ and $f, \gamma : \mathbb{R} \to \mathbb{R}$ are C^1-functions. Hence, the problem (1)–(2) with initial conditions (11) and (12) with $k = 0$ also evolves to a situation more singular then the initial one and the measure space is no longer sufficient to contain all its possible α-solutions. It is also a remarkable fact that, in the space Z, all those α-solutions, when they exist, are independent of α.

Regarding δ'-waves, we must remember that they were first introduced by E. Yu. Panov and V. M. Shelkovich for certain systems of conservation laws [Panov and Shelkovich 2006; Shelkovich 2006]. The results show that these systems subjected to piecewise continuous initial data may develop not only δ-waves, but also δ'-waves [Sarrico 2012b; Shelkovich 2007; 2008].

Let us summarize the contents of this paper. In Section 2 a survey of the main ideas and formulas for multiplying distributions is presented. In Section 3 we define the concept of α-solution for the system (1)–(2). In Sections 4, 5 and 6 we justify rigorously all we have said in the beginning of this introduction.

2. Products of distributions

Let $\mathcal{C}^\infty$ be the space of indefinitely differentiable real or complex-valued functions defined on $\mathbb{R}^N$, $N \in \{1, 2, 3, \ldots\}$, and $\mathcal{D}$ the subspace of C^∞ consisting of those functions with compact support. Let $\mathcal{D}'$ be the space of Schwartz distributions and $L(\mathcal{D})$ the space of continuous linear maps $\phi : \mathcal{D} \to \mathcal{D}$, where we suppose $\mathcal{D}$ is endowed with the usual topology. We will sketch the main ideas of our distributional product (the reader can look at (18), (22) and (24) as definitions, if he prefers to skip this presentation). For proofs and other details concerning this product see [Sarrico 1988].

First, we define a product $T\phi \in \mathcal{D}'$ for $T \in \mathcal{D}'$ and $\phi \in L(\mathcal{D})$ by

$$\langle T\phi, \xi\rangle = \langle T, \phi(\xi)\rangle,$$

for all $\xi \in \mathcal{D}$; this makes $\mathcal{D}'$ a right $L(\mathcal{D})$-module. Next, we define an epimorphism $\tilde{\zeta} : L(\mathcal{D}) \to \mathcal{D}'$, where the image of ϕ is the distribution $\tilde{\zeta}(\phi)$ given by

$$\langle \tilde{\zeta}(\phi), \xi\rangle = \int \phi(\xi),$$

for all $\xi \in \mathcal{D}$ (when the domain of the integral is not specified we assume it to be $\mathbb{R}^N$); given $S \in \mathcal{D}'$, we say that ϕ is a representative operator of S if $\tilde{\zeta}(\phi) = S$. For instance, if $\beta \in C^\infty$ is seen as a distribution, the operator $\phi_\beta \in L(\mathcal{D})$ defined by $\phi_\beta(\xi) = \beta\xi$, for all $\xi \in \mathcal{D}$, is a representative operator of β because, for all $\xi \in \mathcal{D}$, we have

$$\langle \tilde{\zeta}(\phi_\beta), \xi\rangle = \int \phi_\beta(\xi) = \int \beta\xi = \langle \beta, \xi\rangle.$$

For this reason $\tilde{\zeta}(\phi_\beta) = \beta$. If $T \in \mathcal{D}'$, we also have

$$\langle T\phi_\beta, \xi\rangle = \langle T, \phi_\beta(\xi)\rangle = \langle T, \beta\xi\rangle = \langle T\beta, \xi\rangle,$$

for all $\xi \in \mathcal{D}$. Hence,

$$T\beta = T\phi_\beta.$$

Thus, given $T, S \in \mathcal{D}'$, we are tempted to define a natural product by setting $TS := T\phi$, where $\phi \in L(\mathcal{D})$ is a representative operator of S, i.e., ϕ is such that $\tilde{\zeta}(\phi) = S$. Unfortunately, this product is not well defined, because TS depends on the representative $\phi \in L(\mathcal{D})$ of $S \in \mathcal{D}'$.

This difficulty can be overcome, if we fix $\alpha \in \mathcal{D}$ with $\int \alpha = 1$ and define $s_\alpha : L(\mathcal{D}) \to L(\mathcal{D})$ by

$$[(s_\alpha\phi)(\xi)](y) = \int \phi[(\tau_y\check{\alpha})\xi], \tag{15}$$

for all $\xi \in \mathcal{D}$ and all $y \in \mathbb{R}^N$, where $\tau_y \check{\alpha}$ is given by $(\tau_y \check{\alpha})(x) = \check{\alpha}(x-y) = \alpha(y-x)$ for all $x \in \mathbb{R}^N$. It can be proved that for each $\alpha \in \mathcal{D}$ with $\int \alpha = 1$, $s_\alpha(\phi) \in L(\mathcal{D})$, s_α is linear, $s_\alpha \circ s_\alpha = s_\alpha$ (s_α is a projector of $L(\mathcal{D})$), $\ker s_\alpha = \ker \tilde{\zeta}$, and $\tilde{\zeta} \circ s_\alpha = \tilde{\zeta}$.

Now, for each $\alpha \in \mathcal{D}$, we can define a general α-product $\underset{\alpha}{\odot}$ of $T \in \mathcal{D}'$ with $S \in \mathcal{D}'$ by setting

$$T \underset{\alpha}{\odot} S := T(s_\alpha \phi), \tag{16}$$

where $\phi \in L(\mathcal{D})$ is a representative operator of $S \in \mathcal{D}'$. This α-product is independent of the representative ϕ of S, because if ϕ and ψ are such that $\tilde{\zeta}(\phi) = \tilde{\zeta}(\psi) = S$, then $\phi - \psi \in \ker \tilde{\zeta} = \ker s_\alpha$. Hence,

$$T(s_\alpha \phi) - T(s_\alpha \psi) = T[s_\alpha(\phi - \psi)] = 0.$$

Since ϕ in (16) satisfies $\tilde{\zeta}(\phi) = S$, we have $\int \phi(\xi) = \langle S, \xi \rangle$ for all $\xi \in \mathcal{D}$, and by (15)

$$[(s_\alpha \phi)(\xi)](y) = \langle S, (\tau_y \check{\alpha})\xi \rangle = \langle S\xi, \tau_y \check{\alpha} \rangle = (S\xi * \alpha)(y),$$

for all $y \in \mathbb{R}^N$, which means that $(s_\alpha \phi)(\xi) = S\xi * \alpha$. Therefore, for all $\xi \in \mathcal{D}$,

$$\begin{aligned} \langle T \underset{\alpha}{\odot} S, \xi \rangle &= \langle T(s_\alpha \phi), \xi \rangle = \langle T, (s_\alpha \phi)(\xi) \rangle = \langle T, S\xi * \alpha \rangle \\ &= [T * (S\xi * \alpha)\check{\ }](0) = [(S\xi)\check{\ } * (T * \check{\alpha})](0) = \langle (T * \check{\alpha})S, \xi \rangle, \end{aligned}$$

and we obtain an easier formula for the general product (16):

$$T \underset{\alpha}{\odot} S = (T * \check{\alpha})S. \tag{17}$$

In general, this α-product is neither commutative nor associative but it is bilinear and satisfies the Leibniz rule written in the form

$$D_k(T \underset{\alpha}{\odot} S) = (D_k T) \underset{\alpha}{\odot} S + T \underset{\alpha}{\odot} (D_k S),$$

where D_k is the usual k-partial derivative operator in distributional sense ($k = 1, 2, \ldots, N$).

Recall that the usual Schwartz products of distributions are not associative and the commutative property is a convention inherent to the definition of such products (see the classical monograph of Schwartz [1966, pp. 117, 118 and 121] where these products are defined). Unfortunately, the α-product (17), in general, is not consistent with the classical Schwartz products of distributions with functions.

In order to obtain consistency with the usual product of a distribution with a C^∞-function, we are going to introduce some definitions and single out a certain subspace H_α of $L(\mathcal{D})$.

An operator $\phi \in L(\mathcal{D})$ is said to vanish on an open set $\Omega \subset \mathbb{R}^N$, if and only if $\phi(\xi) = 0$ for all $\xi \in \mathcal{D}$ with support contained in Ω. The support of an operator $\phi \in L(\mathcal{D})$ will be defined as the complement of the largest open set in which ϕ vanishes.

Let $\mathcal{N}$ be the set of operators $\phi \in L(\mathcal{D})$ whose support has Lebesgue measure zero, and $\rho(C^\infty)$ the set of operators $\phi \in L(\mathcal{D})$ defined by $\phi(\xi) = \beta\xi$ for all $\xi \in \mathcal{D}$, with $\beta \in C^\infty$. For each $\alpha \in \mathcal{D}$, with $\int \alpha = 1$, let us consider the space $H_\alpha = \rho(C^\infty) \oplus s_\alpha(\mathcal{N}) \subset L(\mathcal{D})$. It can be proved that $\zeta_\alpha := \tilde{\zeta}|_{H_\alpha} : H_\alpha \to C^\infty \oplus \mathcal{D}'_\mu$ is an isomorphism ($\mathcal{D}'_\mu$ stands for the space of distributions whose support has Lebesgue measure zero). Therefore, if $T \in \mathcal{D}'$ and $S = \beta + f \in C^\infty \oplus \mathcal{D}'_\mu$, a new α-product, $\dot{\alpha}$, can be defined by $T \dot{\alpha} S := T\phi_\alpha$, where for each α, $\phi_\alpha = \zeta_\alpha^{-1}(S) \in H_\alpha$. Hence,

$$\begin{aligned} T \dot{\alpha} S &= T\zeta_\alpha^{-1}(S) = T[\zeta_\alpha^{-1}(\beta + f)] \\ &= T[\zeta_\alpha^{-1}(\beta) + \zeta_\alpha^{-1}(f)] = T\beta + T \underset{\alpha}{\odot} f = T\beta + (T * \check{\alpha})f, \end{aligned}$$

and putting α instead of $\check{\alpha}$ (to simplify), we get

$$T \dot{\alpha} S = T\beta + (T * \alpha)f. \tag{18}$$

Thus, the referred consistency is obtained when the C^∞-function is placed at the right-hand side; if $S \in C^\infty$, then $f = 0$, $S = \beta$, and $T \dot{\alpha} S = T\beta$.

The α-product (18) can be easily extended for $T \in \mathcal{D}'^p$ and $S = \beta + f \in C^p \oplus \mathcal{D}'_\mu$, where $p \in \{0, 1, 2, \dots, \infty\}$, $\mathcal{D}'^p$ is the space of distributions of order $\leq p$ in the sense of Schwartz ($\mathcal{D}'^\infty$ means $\mathcal{D}'$), $T\beta$ is the Schwartz product of a $\mathcal{D}'^p$-distribution with a C^p-function, and $(T * \alpha)f$ is the usual product of a C^∞-function with a distribution. This extension is clearly consistent with all Schwartz products of $\mathcal{D}'^p$-distributions with C^p-functions, if the C^p-functions are placed at the right-hand side. It also keeps the bilinearity and satisfies the Leibniz rule written in the form

$$D_k(T \dot{\alpha} S) = (D_k T) \dot{\alpha} S + T \dot{\alpha} (D_k S),$$

clearly under certain natural conditions; for $T \in \mathcal{D}'^p$, we must suppose $S \in C^{p+1} \oplus \mathcal{D}'_\mu$. Moreover, these products are invariant by translations, that is,

$$\tau_a(T \dot{\alpha} S) = (\tau_a T) \dot{\alpha} (\tau_a S),$$

where τ_a stands for the usual translation operator in distributional sense. These products are also invariant for the action of any group of linear transformations $h : \mathbb{R}^N \to \mathbb{R}^N$ with $|\det h| = 1$, that leave α invariant.

Thus, for each $\alpha \in \mathcal{D}$ with $\int \alpha = 1$, formula (18) allows us to evaluate the product of $T \in \mathcal{D}'^p$ with $S \in C^p \oplus \mathcal{D}'_\mu$; therefore, we have obtained a family of products, one for each α.

From now on, we always consider the dimension $N = 1$. For instance, if β is a continuous function we have for each α, by applying (18),

$$\delta_{\dot\alpha}\beta = \delta_{\dot\alpha}(\beta + 0) = \delta\beta + (\delta * \alpha)0 = \beta(0)\delta,$$
$$\beta_{\dot\alpha}\delta = \beta_{\dot\alpha}(0 + \delta) = \beta 0 + (\beta * \alpha)\delta = [(\beta * \alpha)(0)]\delta,$$
$$\delta_{\dot\alpha}\delta = \delta_{\dot\alpha}(0 + \delta) = \delta 0 + (\delta * \alpha)\delta = \alpha\delta = \alpha(0)\delta, \tag{19}$$

$$H_{\dot\alpha}\delta = (H * \alpha)\delta = \left(\int_{-\infty}^{+\infty} \alpha(-\tau)H(\tau)\,d\tau\right)\delta = \left(\int_{-\infty}^{0} \alpha\right)\delta, \tag{20}$$

$$(D\delta)_{\dot\alpha}(D\delta) = [(D\delta) * \alpha]D\delta = \alpha'(0)D\delta - \alpha''(0)\delta. \tag{21}$$

For each α, the support of the α-product (18) satisfies $\operatorname{supp}(T_{\dot\alpha}S) \subset \operatorname{supp} S$, as for usual functions, but it may happen that $\operatorname{supp}(T_{\dot\alpha}S) \not\subset \operatorname{supp} T$.

It is also possible to multiply many other distributions preserving the consistency with all Schwartz products of distributions with functions. For instance, using the Leibniz formula to extend the α-products, it is possible to write

$$T_{\dot\alpha}S = Tw + (T * \alpha)f, \tag{22}$$

with $T \in \mathcal{D}'^{-1}$ and $S = w + f \in L^1_{\text{loc}} \oplus \mathcal{D}'_\mu$, where $\mathcal{D}'^{-1}$ stands for the space of distributions $T \in \mathcal{D}'$ such that $DT \in \mathcal{D}'^0$ and Tw is the usual pointwise product of $T \in \mathcal{D}'^{-1}$ with $w \in L^1_{\text{loc}}$. Recall that, locally, T can be read as a function of bounded variation (see [Sarrico 2012a, §2] for details). For instance, since $H \in \mathcal{D}'^{-1}$ and $H = H + 0 \in L^1_{\text{loc}} \oplus \mathcal{D}'_\mu$, we have

$$H_{\dot\alpha}H = HH + (H * \alpha)0 = H. \tag{23}$$

because $H \in \mathcal{D}'^{-1}$ and $H = H + 0 \in L^1_{\text{loc}} \oplus \mathcal{D}'_\mu$. More generally, if $T \in \mathcal{D}'^{-1}$ and $S \in L^1_{\text{loc}}$, then $T_{\dot\alpha}S = TS$; actually, using (22) we can write

$$T_{\dot\alpha}S = T_{\dot\alpha}(S + 0) = TS + (T * \alpha)0 = TS.$$

Thus, in distributional sense, the α-products of functions that, locally, are of bounded variation coincide with the usual pointwise product of these functions considered as a distribution. We stress that in (18) or (22) the convolution $T * \alpha$ is not to be understood as an approximation of T. Those formulas are exact.

Another useful extension that will be applied is given by the formula

$$T_{\dot\alpha}S = D(Y_{\dot\alpha}S) - Y_{\dot\alpha}(DS), \tag{24}$$

for $T \in \mathcal{D}'^0 \cap \mathcal{D}'_\mu$ and $S, DS \in L^1_{\text{loc}} \oplus \mathcal{D}'_c$, where $\mathcal{D}'_c \subset \mathcal{D}'_\mu$ is the space of distributions whose support is at most countable, and $Y \in \mathcal{D}'^{-1}$ is such that $DY = T$ (the products $Y_{\dot\alpha}S$ and $Y_{\dot\alpha}(DS)$ are supposed to be computed by (18) or (22)). The value of $T_{\dot\alpha}S$ given by (24) is independent of the choice of $Y \in \mathcal{D}'^{-1}$ such that $DY = T$ (see

[Sarrico 2012a, p. 1004] for the proof). For instance, by (24) and (20) we have, for any α,

$$\text{(25)}\quad \delta \dot{\alpha} H = D(H \dot{\alpha} H) - H \dot{\alpha}(DH) = DH - H \dot{\alpha} \delta = \delta - \Big(\int_{-\infty}^{0} \alpha \Big) \delta = \Big(\int_{0}^{+\infty} \alpha \Big) \delta,$$

so that

$$\text{(26)}\qquad H \dot{\alpha} \delta + \delta \dot{\alpha} H = \delta$$

for any α. The products (18), (22), and (24) are compatible; that is, if an α-product can be computed by two of them, the result is the same.

3. The α-solution concept for the system (1)–(2)

Let I be an interval of $\mathbb{R}$ with more than one point and let $\mathcal{F}(I)$ be the space of continuously differentiable maps $\tilde{u} : I \to \mathcal{D}'$ in the sense of the usual topology of $\mathcal{D}'$. For $t \in I$ the notation $[\tilde{u}(t)](x)$ is sometimes used to emphasize that the distribution $\tilde{u}(t)$ acts on functions $\xi \in \mathcal{D}$ which depend on x.

Let $\Sigma(I)$ be the space of functions $u : \mathbb{R} \times I \to \mathbb{C}$ such that

(a) for each $t \in I$, $u(x, t) \in L^1_{\text{loc}}(\mathbb{R})$,

(b) $\tilde{u} : I \to \mathcal{D}'$, defined by $[\tilde{u}(t)](x) = u(x, t)$ is in $\mathcal{F}(I)$.

The natural injection $u \mapsto \tilde{u}$ of $\Sigma(I)$ into $\mathcal{F}(I)$ allows us to identify any function of $\Sigma(I)$ with a certain map in $\mathcal{F}(I)$. Since $C^1(\mathbb{R} \times I) \subset \Sigma(I)$, we can write the inclusions

$$C^1(\mathbb{R} \times I) \subset \Sigma(I) \subset \mathcal{F}(I).$$

Consequently, the identification $u \mapsto \tilde{u}$ allows us to write the system (1)–(2) as follows

$$\text{(27)}\qquad \frac{d\tilde{u}}{dt}(t) + \tfrac{1}{2} D[\tilde{u}(t) \dot{\alpha} \tilde{u}(t)] = D\tilde{\sigma}(t),$$

$$\text{(28)}\qquad \frac{d\tilde{\sigma}}{dt}(t) + \tilde{u}(t) \dot{\alpha} D\tilde{\sigma}(t) = k^2 D\tilde{u}(t).$$

Definition 1. Given α, the pair $(\tilde{u}, \tilde{\sigma}) \in \mathcal{F}(I) \times \mathcal{F}(I)$ will be called an α-solution for the system (27)–(28) on I, if the α-products that appear in this system are well defined and both equations are satisfied for all $t \in I$.

We have the following results:

Theorem 2. *If (u, σ) is a classical solution of (1)–(2) on $\mathbb{R} \times I$ then, for any α, the pair $(\tilde{u}, \tilde{\sigma}) \in \mathcal{F}(I) \times \mathcal{F}(I)$ defined by $[\tilde{u}(t)](x) = u(x, t)$, $[\tilde{\sigma}(t)](x) = \sigma(x, t)$, is an α-solution of (27)–(28) on I.*

Note that by a classical solution of (1)–(2) on $\mathbb{R}\times I$, we mean a pair of C^1-functions $(u(x,t),\sigma(x,t))$ that satisfies (1)–(2) on $\mathbb{R}\times I$.

Theorem 3. *If $u,\sigma:\mathbb{R}\times I\to\mathbb{C}$ are C^1-functions and, for a certain α, the pair $(\tilde u,\tilde\sigma)\in\mathcal{F}(I)\times\mathcal{F}(I)$ defined by $[\tilde u(t)](x)=u(x,t)$, $[\tilde\sigma(t)](x)=\sigma(x,t)$ is an α-solution of* (27)–(28) *on I, then the pair (u,σ) is a classical solution of* (1)–(2) *on $\mathbb{R}\times I$.*

For the proof it is enough to observe that any C^1-function $u(x,t)$ can be read as a continuously differentiable function $\tilde u\in\mathcal{F}(I)$ defined by $[\tilde u(t)](x)=u(x,t)$ and to use the consistency of the α-products with the classical Schwartz products.

Replacing $\tilde u(t)\dot{_\alpha}D\tilde\sigma(t)$ by $D\tilde\sigma(t)\dot{_\alpha}\tilde u(t)$ in (28), we get

$$\frac{d\tilde\sigma}{dt}(t)+D\tilde\sigma(t)\dot{_\alpha}\tilde u(t)=k^2D\tilde u(t), \tag{29}$$

which is not equivalent to (28) since our α-products are not, in general, commutative. However, all we have said for the systems (1)–(2) and (27)–(28) is also valid for the systems formed by (1) and (2) and by (27) and (29). Taking advantage of this situation, we introduce a definition that further extends the concept of a classical solution:

Definition 4. Given α, we define as an α-solution for the system (1)–(2) on I any α-solution of the system formed by (27) and (28) or by (27) and (29) on I.

As a consequence, an α-solution $(\tilde u,\tilde\sigma)$ in this sense, read as an usual distributional solution (u,σ), affords a consistent extension of the concept of a classical solution for the system (1)–(2). Thus, and for short, we also call (u,σ) an α-solution of (1)–(2).

4. The Riemann problem (1)–(4) with $k>0$

Let us consider the system (1)–(2) with $k>0$. We also consider $(x,t)\in\mathbb{R}\times\mathbb{R}$ (we could also take $\mathbb{R}\times[0,+\infty[$) and the unknowns $u(x,t)$ and $\sigma(x,t)$ submitted to the initial conditions (3) and (4). When we read this problem in $\mathcal{F}(\mathbb{R})$ having in mind the identification $u\mapsto\tilde u$, we must replace the system (1)–(2) by the system (27)–(28) and the conditions (3)–(4) by the following ones:

$$\tilde u(0)=a_1+(a_2-a_1)H, \tag{30}$$

$$\tilde\sigma(0)=b_1+(b_2-b_1)H. \tag{31}$$

We will give, explicitly, all α-solutions for this problem which belong to a set $\widetilde W$ defined as follows: $(\tilde u,\tilde\sigma)\in\widetilde W$ if and only if $\tilde u,\tilde\sigma\in\mathcal{F}(\mathbb{R})$ and there exist real numbers $u_1,u_2,\sigma_1,\sigma_2$, V and a C^1-function $g:\mathbb{R}\to\mathbb{R}$ such that

$$\tilde u(t)=u_1+(u_2-u_1)\tau_{Vt}H+g(t)\tau_{Vt}\delta, \tag{32}$$

$$\tilde\sigma(t)=\sigma_1+(\sigma_2-\sigma_1)\tau_{Vt}H. \tag{33}$$

Theorem 5. *Let us consider the problem* (27)–(28) *with the initial conditions* (30)–(31) *with* $k > 0$.

(I) *If* $b_1 = b_2$, *there exists an* α*-solution in* $\widetilde{W}$ *if and only if* $a_1 = a_2$; *moreover, for any* α, *the* α*-solution is unique in* $\widetilde{W}$ *and is given by*

$$\tilde{u}(t) = a_1, \tag{34}$$

$$\tilde{\sigma}(t) = b_1; \tag{35}$$

(II) *If* $b_1 \neq b_2$ *there exists an* α*-solution in* $\widetilde{W}$ *if and only if* $a_1 \neq a_2$ *and we choose* α *such that*

$$\int_{-\infty}^{0} \alpha = \frac{1}{2} - \frac{b_2 - b_1}{(a_2 - a_1)^2} + \frac{k^2}{b_2 - b_1}\,; \tag{36}$$

moreover, for any α *satisfying this condition, the* α*-solution is unique in* $\widetilde{W}$ *and is given by the traveling wave*

$$\tilde{u}(t) = a_1 + (a_2 - a_1)\tau_{Vt} H, \tag{37}$$

$$\tilde{\sigma}(t) = b_1 + (b_2 - b_1)\tau_{Vt} H, \tag{38}$$

with speed

$$V = \frac{a_1 + a_2}{2} - \frac{b_2 - b_1}{a_2 - a_1}.$$

As we can see all of these α*-solutions, when they exist, are independent of* α.

Proof. Let us suppose $(\tilde{u}, \tilde{v}) \in \widetilde{W}$. Then we have (32) and (33), and by (30) and (31) we can write

$$u_1 + (u_2 - u_1)H + g(0)\delta = a_1 + (a_2 - a_1)H,$$
$$\sigma_1 + (\sigma_2 - \sigma_1)H = b_1 + (b_2 - b_1)H,$$

which implies $g(0) = 0$. Then, by restriction to the interval $]-\infty, 0[$, we have $u_1 = a_1$ and $\sigma_1 = b_1$. As a consequence, we also have $u_2 = a_2$ and $\sigma_2 = b_2$. Thus, from (32) and (33) it follows that

$$\tilde{u}(t) = a_1 + (a_2 - a_1)\tau_{Vt} H + g(t)\tau_{Vt}\delta, \tag{39}$$

$$\tilde{\sigma}(t) = b_1 + (b_2 - b_1)\tau_{Vt} H, \tag{40}$$

and so

$$\frac{d\tilde{u}}{dt}(t) = -V(a_2 - a_1)\tau_{Vt}\,\delta + g'(t)\tau_{Vt}\,\delta - Vg(t)\tau_{Vt} D\delta,$$
$$\frac{d\tilde{\sigma}}{dt}(t) = -V(b_2 - b_1)\tau_{Vt}\,\delta.$$

By applying the bilinearity of the α-products, the results (23), (20), (25), (19), and the already mentioned translation property, we have

$$\begin{aligned}(41)\qquad \tilde{u}(t)\dot{_\alpha}\tilde{u}(t) &= a_1^2 + 2a_1(a_2-a_1)\tau_{Vt}H + 2a_1 g(t)\tau_{Vt}\delta \\ &\quad + (a_2-a_1)^2\tau_{Vt}H + (a_2-a_1)g(t)\big(\textstyle\int_{-\infty}^0 \alpha\big)\tau_{Vt}\delta \\ &\quad + (a_2-a_1)g(t)\big(\textstyle\int_0^{+\infty}\alpha\big)\tau_{Vt}\delta + g^2(t)\alpha(0)\tau_{Vt}\delta.\end{aligned}$$

Since $\int_{-\infty}^0 \alpha + \int_0^{+\infty}\alpha = 1$, we also have

$$\begin{aligned} D[\tilde{u}(t)\dot{_\alpha}\tilde{u}(t)] &= (a_2^2-a_1^2)\tau_{Vt}\delta + [2a_1 g(t) + (a_2-a_1)g(t) + g^2(t)\alpha(0)]\tau_{Vt}D\delta, \\ D[\tilde{\sigma}(t)] &= (b_2-b_1)\tau_{Vt}\delta, \\ D[\tilde{u}(t)] &= (a_2-a_1)\tau_{Vt}\delta + g(t)\tau_{Vt}D\delta, \end{aligned}$$

and

$$\begin{aligned}(42)\quad & \tilde{u}(t)\dot{_\alpha}D[\tilde{\sigma}(t)] \\ &= \big[a_1(b_2-b_1) + (a_2-a_1)(b_2-b_1)\textstyle\int_{-\infty}^0\alpha + (b_2-b_1)g(t)\alpha(0)\big]\tau_{Vt}\delta.\end{aligned}$$

Thus, (27)–(28) turn out to be

$$\begin{aligned} 0 &= \big[-V(a_2-a_1) + g'(t) + \tfrac{1}{2}(a_2^2-a_1^2) - (b_2-b_1)\big]\tau_{Vt}\delta \\ &\quad + \big[-Vg(t) + 12(a_1+a_2)g(t) + \tfrac{1}{2}(\alpha(0))g^2(t)\big]\tau_{Vt}D\delta, \\ 0 &= \big[-V(b_2-b_1) + a_1(b_2-b_1) + (a_2-a_1)(b_2-b_1)\textstyle\int_{-\infty}^0\alpha \\ &\quad + (b_2-b_1)g(t)\alpha(0) - k^2(a_2-a_1)\big]\tau_{Vt}\delta - k^2 g(t)\tau_{Vt}D\delta. \end{aligned}$$

Hence, for all $t \in \mathbb{R}$ we have

$$(43)\quad 0 = -V(a_2-a_1) + g'(t) + \tfrac{1}{2}(a_2^2-a_1^2) - (b_2-b_1),$$

$$(44)\quad 0 = g(t)\big[-V + \tfrac{1}{2}(a_1+a_2) + \tfrac{1}{2}(\alpha(0))g(t)\big],$$

$$\begin{aligned}(45)\quad 0 &= -V(b_2-b_1) + a_1(b_2-b_1) + (a_2-a_1)(b_2-b_1)\textstyle\int_{-\infty}^0\alpha \\ &\quad + (b_2-b_1)g(t)\alpha(0) - k^2(a_2-a_1),\end{aligned}$$

$$(46)\quad 0 = k^2 g(t).$$

From (46) we conclude that $g = 0$, (44) is satisfied, and from (43) and (45) we have

$$(47)\quad 0 = -V(a_2-a_1) + \tfrac{1}{2}(a_2^2-a_1^2) - (b_2-b_1),$$

$$(48)\quad 0 = -V(b_2-b_1) + a_1(b_2-b_1) + (a_2-a_1)(b_2-b_1)\textstyle\int_{-\infty}^0\alpha - k^2(a_2-a_1).$$

Now, if $b_1 = b_2$, by (48) we have $a_1 = a_2$, (47) is satisfied and (I) follows from (39) and (40). If $b_1 \neq b_2$, from (48) we have

$$(49)\qquad V = a_1 + (a_2-a_1)\int_{-\infty}^0 \alpha - k^2\frac{a_2-a_1}{b_2-b_1},$$

and from (47) we can write

$$(50)\quad -a_1(a_2-a_1)-(a_2-a_1)^2\int_{-\infty}^{0}\alpha+k^2\frac{(a_2-a_1)^2}{b_2-b_1}+\frac{a_2^2-a_1^2}{2}-(b_2-b_1)=0.$$

Then it follows that $a_1 \neq a_2$, because if $a_1 = a_2$ we would have $b_1 = b_2$ which is a contradiction. As a consequence, from (50) we have

$$\int_{-\infty}^{0}\alpha=\frac{1}{2}-\frac{b_2-b_1}{(a_2-a_1)^2}+\frac{k^2}{b_2-b_1},$$

from (49) we have

$$V=\frac{a_1+a_2}{2}-\frac{b_2-b_1}{a_2-a_1},$$

and (II) follows from (39) and (40). □

If in (28) we replace $\tilde{u}(t)\dot{_\alpha}D[\tilde{\sigma}(t)]$ by $D[\tilde{\sigma}(t)]\dot{_\alpha}\tilde{u}(t)$ we obtain for the value $D[\tilde{\sigma}(t)]\dot{_\alpha}\tilde{u}(t)$ the same value as $\tilde{u}(t)\dot{_\alpha}D[\tilde{\sigma}(t)]$, with $\int_0^{+\infty}\alpha$ instead of $\int_{-\infty}^{0}\alpha$ (now we must apply (25) instead of (20)). Hence, for the problem formed by the system (27) and (29) with initial conditions (30)–(31) we must replace Theorem 5 by another theorem where the only difference is at (36), where $\int_{-\infty}^{0}\alpha$ must be replaced by $\int_0^{+\infty}\alpha$!

As a consequence of Definition 4 these considerations allows us to conclude that the α-solutions of the problem (1)–(4) with $k>0$, which belong to W, can be read as stated in the introduction (see (7), (8), (9) and (10)).

5. The Riemann problem (1)–(4) with $k = 0$

In this extreme case we will see, in the same space of solutions W, the possible emergence of a δ shock wave.

Theorem 6. *Let us consider the problem* (27)–(28) *with initial conditions* (30)–(31) *with* $k = 0$.

(I) *If* $b_1 = b_2$ *and* $a_1 = a_2$ *there exists an* α*-solution in* $\widetilde{W}$ *for any* α; *moreover, for any* α, *this* α*-solution is unique in* W *and is given by*

$$\tilde{u}(t)=a_1,\quad \tilde{\sigma}(t)=b_1.$$

(II) *If* $b_1 = b_2$ *and* $a_1 \neq a_2$ *there exists an* α*-solution in* $\widetilde{W}$ *for any* α; *moreover, for any* α, *the* α*-solution is unique in* $\widetilde{W}$ *and is given by*

$$(51)\qquad \tilde{u}(t)=a_1+(a_2-a_1)\tau_{Vt}H,$$

$$(52)\qquad \tilde{\sigma}(t)=b_1,$$

with $V=\frac{1}{2}(a_1+a_2)$.

(III) *If* $b_1 \neq b_2$ *and* $a_1 = a_2$ *there exists an* α*-solution in* $\widetilde{W}$ *if and only if we choose* α *such that* $\alpha(0) = 0$; *moreover, for any* α *satisfying this condition, the* α*-solution is unique in* $\widetilde{W}$ *and is given by*

$$\tilde{u}(t) = a_1 + (b_2 - b_1)t\tau_{a_1 t}\delta, \tag{53}$$

$$\tilde{\sigma}(t) = b_1 + (b_2 - b_1)\tau_{a_1 t}H. \tag{54}$$

(IV) *If* $b_1 \neq b_2$ *and* $a_1 \neq a_2$ *there exists an* α*-solution in* $\widetilde{W}$ *if and only if we choose* α *such that*

$$\int_{-\infty}^{0} \alpha = \frac{1}{2} - \frac{b_2 - b_1}{(a_2 - a_1)^2}, \tag{55}$$

or we choose α *such that*

$$\alpha(0) = 0; \tag{56}$$

moreover, for any α *satisfying* (55), *the* α*-solution is unique in* $\widetilde{W}$ *and is given by*

$$\tilde{u}(t) = a_1 + (a_2 - a_1)\tau_{Vt}H, \tag{57}$$

$$\tilde{\sigma}(t) = b_1 + (b_2 - b_1)\tau_{Vt}H, \tag{58}$$

with $V = (a_1 + a_2)/2 - (b_2 - b_1)/(a_2 - a_1)$; *also for any* α *satisfying* (56), *the* α*-solution is unique in* $\widetilde{W}$ *and is given by*

$$\tilde{u}(t) = a_1 + (a_2 - a_1)\tau_{Vt}H + (b_2 - b_1)t\tau_{Vt}\delta, \tag{59}$$

$$\tilde{\sigma}(t) = b_1 + (b_2 - b_1)\tau_{Vt}H, \tag{60}$$

with $V = \frac{1}{2}(a_1 + a_2)$. *As we can see, all of these* α*-solutions, when they exist, are independent of* α.

Proof. Let us suppose $(\tilde{u}, \tilde{\sigma}) \in \widetilde{W}$. Then, we have (32), (33) and as we have seen in the proof of Theorem 5 we have $g(0) = 0$, $u_1 = a_1$, $u_2 = a_2$, $\sigma_1 = b_1$, $\sigma_2 = b_2$ and also (39) and (40). From (27) and (28) we have (43)–(46) with $k = 0$, which means that for all $t \in \mathbb{R}$ we can write

$$0 = V(a_2 - a_1) + g'(t) + \tfrac{1}{2}(a_2^2 - a_1^2) - (b_2 - b_1), \tag{61}$$

$$0 = g(t)\big[-V + \tfrac{1}{2}(a_1 + a_2) + \tfrac{1}{2}\alpha(0)g(t)\big], \tag{62}$$

$$0 = -V(b_2 - b_1) + a_1(b_2 - b_1) + (a_2 - a_1)(b_2 - b_1)\int_{-\infty}^{0}\alpha + (b_2 - b_1)g(t)\alpha(0). \tag{63}$$

(I) Suppose $b_1 = b_2$ and $a_1 = a_2$. Then (63) is satisfied and from (61) we have $g'(t) = 0$, which means that $g(t) = 0$ and (62) is also satisfied. Then from (39) and (40) we have $\tilde{u}(t) = a_1$ and $\tilde{\sigma}(t) = b_1$ and (I) follows.

(II) Suppose $b_1 = b_2$ and $a_1 \neq a_2$. Then (63) is satisfied and from (61) we have

$$g'(t) = V(a_2 - a_1) - \tfrac{1}{2}(a_2^2 - a_1^2),$$

which means that

$$g(t) = \big[V(a_2 - a_1) - \tfrac{1}{2}(a_2^2 - a_1^2)\big]t. \tag{64}$$

Then, from (62) we conclude that $V = \frac{1}{2}(a_1 + a_2)$ and by (64) $g(t) = 0$ follows for all t. Then, from (39) and (40), we conclude that $\tilde{u}(t) = a_1 + (a_2 - a_1)\tau_{Vt}H$ and $\tilde{\sigma}(t) = b_1$ and (II) follows.

(III) Suppose $b_1 \neq b_2$ and $a_1 = a_2$. Then by (61) we have $g'(t) = b_2 - b_1$ which means that $g(t) = (b_2 - b_1)t$, and (62) turns out to be

$$t\big[-V + a_1 + \tfrac{1}{2}\alpha(0)(b_2 - b_1)t\big] = 0,$$

which implies, for any $t \neq 0$,

$$V = a_1 + \tfrac{1}{2}\alpha(0)(b_2 - b_1)t.$$

Thus, once V is constant, we have $\alpha(0) = 0$, $V = a_1$ and (63) is satisfied and (III) follows.

(IV) Suppose $b_1 \neq b_2$ and $a_1 \neq a_2$. Then by (61) we have

$$V = \frac{g'(t)}{a_2 - a_1} + \frac{a_1 + a_2}{2} - \frac{b_2 - b_1}{a_2 - a_1},$$

and once V is constant we conclude that $g'(t) = c$ (constant), $g(t) = ct$, and

$$V = \frac{c}{a_2 - a_1} + \frac{a_1 + a_2}{2} - \frac{b_2 - b_1}{a_2 - a_1}. \tag{65}$$

If $c = 0$ we have $g(t) = 0$ and

$$V = \frac{a_1 + a_2}{2} - \frac{b_2 - b_1}{a_2 - a_1}.$$

As a consequence, (62) is satisfied and (63) turns out to be

$$-(b_2 - b_1)\frac{a_2 + a_1}{2} + \frac{(b_2 - b_1)^2}{a_2 - a_1} + a_1(b_2 - b_1) + (a_2 - a_1)(b_2 - b_1)\int_{-\infty}^{0}\alpha = 0,$$

which is possible if and only if

$$\int_{-\infty}^{0}\alpha = \frac{1}{2} - \frac{b_2 - b_1}{(a_2 - a_1)^2}.$$

If $c \neq 0$ we have from (65) and (62),

$$ct\left[-\frac{c}{a_2-a_1}-\frac{a_1+a_2}{2}+\frac{b_2-b_1}{a_2-a_1}+\frac{a_1+a_2}{2}+\frac{\alpha(0)}{2}ct\right]=0,$$

and for all $t \neq 0$ we will have

$$\frac{\alpha(0)}{2}ct-\frac{c}{a_2-a_1}+\frac{b_2-b_1}{a_2-a_1}=0,$$

which is possible if and only if $\alpha(0)=0$ and $c=b_2-b_1$ which, by (65), implies $V=\frac{1}{2}(a_1+a_2)$ and $g(t)=(b_2-b_1)t$. Hence, (IV) follows. □

Thus, concerning the problem formed by the system (27) and (29) with initial conditions (30)–(31), Theorem 6 must be substituted with another theorem where the only difference is at (55), where $\int_{-\infty}^{0}\alpha$ must change to $\int_{0}^{+\infty}\alpha$!

As a consequence of Definition 4 we can conclude that the α-solutions of the problem (1)–(4) with $k=0$, can be described as in the introduction.

6. The arising of a δ' shock wave

For the system (1)–(2) with $k=0$ let us consider the initial conditions (11) and (12). Let us define the space $\tilde{Z}$ by the condition $(\tilde{u},\tilde{\sigma}) \in \tilde{Z}$ if and only if $\tilde{u}, \tilde{\sigma} \in \mathcal{F}(\mathbb{R})$ and there exist real numbers u_1, σ_1, p and C^1-functions $f, \gamma : \mathbb{R} \to \mathbb{R}$ such that

$$\tilde{u}(t)=u_1+f(t)\tau_{\gamma(t)}D\delta, \tag{66}$$

$$\tilde{\sigma}(t)=\sigma_1+p\tau_{\gamma(t)}\delta. \tag{67}$$

Now, the initial conditions (11) and (12) correspond in $\mathcal{F}(\mathbb{R})$ to the conditions

$$\tilde{u}(0)=a, \tag{68}$$

$$\tilde{\sigma}(0)=b+m\delta, \tag{69}$$

with $m \neq 0$. We will see the possible emergence of a δ' shock wave for problem (1)–(2) with initial conditions (11) and (12).

Theorem 7. *The problem* (27)–(28) *with $k=0$ and initial conditions* (68) *and* (69) *has α-solutions in $\tilde{Z}$ if and only if we choose α such that $\alpha'(0)=\alpha''(0)=0$; moreover, for all α satisfying this condition, the α-solution is unique in $\tilde{Z}$ and is given by*

$$\tilde{u}(t)=a+mt\tau_{at}D\delta, \tag{70}$$

$$\tilde{\sigma}(t)=b+m\tau_{at}\delta. \tag{71}$$

As we can see, when it exists, this α-solution is also independent of α.

Proof. Let us suppose $(\tilde{u}, \tilde{\sigma}) \in \tilde{Z}$. Then we have (66) and (67) and by (68) and (69) we have

$$u_1 + f(0)\tau_{\gamma(0)} D\delta = a, \tag{72}$$

$$\sigma_1 + p\tau_{\gamma(0)}\delta = b + m\delta. \tag{73}$$

From (72) we conclude that $f(0) = 0$ and $u_1 = a$. From (73) we conclude that $\sigma_1 = b$ and so, since $m \neq 0$, we have $\gamma(0) = 0$ and $p = m$. Thus, we can write (66) and (67) in the form

$$\tilde{u}(t) = a + f(t)\tau_{\gamma(t)} D\delta, \tag{74}$$

$$\tilde{\sigma}(t) = b + m\tau_{\gamma(t)}\delta. \tag{75}$$

As a consequence, we have

$$\frac{d\tilde{u}}{dt}(t) = f'(t)\tau_{\gamma(t)} D\delta - \gamma'(t) f(t)\tau_{\gamma(t)} D^2\delta,$$

and using (21), we also have

$$\begin{aligned}
\tilde{u}(t)\dot{_\alpha}\tilde{u}(t) &= a^2 + 2af(t)\tau_{\gamma(t)} D\delta + f^2(t)\tau_{\gamma(t)}[\alpha'(0)D\delta - \alpha''(0)\delta], \\
\tfrac{1}{2}D[\tilde{u}(t)\dot{_\alpha}\tilde{u}(t)] &= -\tfrac{1}{2}\alpha''(0) f^2(t)\tau_{\gamma(t)} D\delta + \big[af(t) + \tfrac{1}{2}\alpha'(0) f^2(t)\big]\tau_{\gamma(t)} D^2\delta, \\
D\tilde{\sigma}(t) &= m\tau_{\gamma(t)} D\delta, \\
\frac{d\tilde{\sigma}}{dt}(t) &= -m\gamma'(t)\tau_{\gamma(t)} D\delta, \\
\tilde{u}(t)\dot{_\alpha} D\tilde{\sigma}(t) &= -m\alpha''(0) f(t)\tau_{\gamma(t)}\delta + [ma + mf(t)\alpha'(0)]\tau_{\gamma(t)} D\delta.
\end{aligned}$$

Then, (27)–(28) with $k = 0$ turns out to be

$$\begin{aligned}
0 &= \big[f'(t) - \tfrac{1}{2}\alpha''(0) f^2(t) - m\big]\tau_{\gamma(t)} D\delta \\
&\qquad + \big[-\gamma'(t) f(t) + af(t) + \tfrac{1}{2}\alpha'(0) f^2(t)\big]\tau_{\gamma(t)} D^2\delta, \\
0 &= -m\alpha''(0) f(t)\tau_{\gamma(t)}\delta + [-m\gamma'(t) + ma + mf(t)\alpha'(0)]\tau_{\gamma(t)} D\delta.
\end{aligned}$$

Hence, for all $t \in \mathbb{R}$, we have

$$0 = f'(t) - \tfrac{1}{2}\alpha''(0) f^2(t) - m, \tag{76}$$

$$0 = f(t)\big[-\gamma'(t) + a + \tfrac{1}{2}\alpha'(0) f(t)\big], \tag{77}$$

$$0 = \alpha''(0) f(t), \tag{78}$$

$$0 = -\gamma'(t) + a + f(t)\alpha'(0). \tag{79}$$

Now, we must note that $\alpha''(0) = 0$ follows immediately because by (78) if $\alpha''(0) \neq 0$, we will have $f = 0$ and by (76) we will also have $m = 0$, which is impossible. Thus, by (76) we have $f(t) = mt$ and from (79) it follows that $\gamma'(t) = a + \alpha'(0)mt$. Then by (77) we conclude that $t^2\alpha'(0) = 0$ for all $t \in \mathbb{R}$, and $\alpha'(0) = 0$ follows which means that $\gamma'(t) = a$ and so, $\gamma(t) = at$. Finally (70) and (71) follow from (74) and (75). The theorem is proved. □

If in (28) we replace $\tilde{u}(t)\dot{_\alpha} D\tilde{\sigma}(t)$ by $D\tilde{\sigma}(t)\dot{_\alpha}\tilde{u}(t)$ we arrive exactly at the same theorem because in this case we simply have $\tilde{u}(t)\dot{_\alpha} D\tilde{\sigma}(t) = D\tilde{\sigma}(t)\dot{_\alpha}\tilde{u}(t)$. Hence, by Definition 4 we conclude that the α-solutions of the problem (1)–(2) with $k = 0$, when subjected to the initial conditions (11) and (12) can be read as we said in the introduction (see (13) and (14)).

Acknowledgments

The author thanks Professor Carlos J. Luz of CIDMA/Universidade de Aveiro, for the help in the preparation of this paper. The present research was supported by Fundação para a Ciência e a Tecnologia, UID/MAT/04561/2013.

References

[Bouchut and James 1999] F. Bouchut and F. James, "Duality solutions for pressureless gases, monotone scalar conservation laws, and uniqueness", *Comm. Partial Differential Equations* **24**:11-12 (1999), 2173–2189. MR Zbl

[Brenier and Grenier 1998] Y. Brenier and E. Grenier, "Sticky particles and scalar conservation laws", *SIAM J. Numer. Anal.* **35**:6 (1998), 2317–2328. MR Zbl

[Bressan and Rampazzo 1988] A. Bressan and F. Rampazzo, "On differential systems with vector-valued impulsive controls", *Boll. Un. Mat. Ital. B* (7) **2**:3 (1988), 641–656. MR Zbl

[Cauret et al. 1989] J.-J. Cauret, J.-F. Colombeau, and A. Y. LeRoux, "Discontinuous generalized solutions of nonlinear nonconservative hyperbolic equations", *J. Math. Anal. Appl.* **139**:2 (1989), 552–573. MR Zbl

[Chen and Liu 2003] G.-Q. Chen and H. Liu, "Formation of δ-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids", *SIAM J. Math. Anal.* **34**:4 (2003), 925–938. MR Zbl

[Colombeau and LeRoux 1988] J.-F. Colombeau and A. Y. LeRoux, "Multiplications of distributions in elasticity and hydrodynamics", *J. Math. Phys.* **29**:2 (1988), 315–319. MR Zbl

[Dal Maso et al. 1995] G. Dal Maso, P. G. Lefloch, and F. Murat, "Definition and weak stability of nonconservative products", *J. Math. Pures Appl.* (9) **74**:6 (1995), 483–548. MR Zbl

[Danilov and Mitrovic 2008] V. G. Danilov and D. Mitrovic, "Delta shock wave formation in the case of triangular hyperbolic system of conservation laws", *J. Differential Equations* **245**:12 (2008), 3704–3734. MR Zbl

[Danilov and Shelkovich 2005a] V. G. Danilov and V. M. Shelkovich, "Delta-shock wave type solution of hyperbolic systems of conservation laws", *Quart. Appl. Math.* **63**:3 (2005), 401–427. MR

[Danilov and Shelkovich 2005b] V. G. Danilov and V. M. Shelkovich, "Dynamics of propagation and interaction of δ-shock waves in conservation law systems", *J. Differential Equations* **211**:2 (2005), 333–381. MR Zbl

[Huang 2005] F. Huang, "Weak solution to pressureless type system", *Comm. Partial Differential Equations* **30**:1-3 (2005), 283–304. MR Zbl

[Joseph 1997] K. T. Joseph, "Generalized solutions to a Cauchy problem for a nonconservative hyperbolic system", *J. Math. Anal. Appl.* **207**:2 (1997), 361–387. MR Zbl

[Keyfitz and Kranzer 1995] B. L. Keyfitz and H. C. Kranzer, "Spaces of weighted measures for conservation laws with singular shock solutions", *J. Differential Equations* **118**:2 (1995), 420–451. MR Zbl

[Nedeljkov 2002] M. Nedeljkov, "Unbounded solutions to some systems of conservation laws—split delta shock waves", *Mat. Vesnik* **54**:3-4 (2002), 145–149. MR Zbl

[Nedeljkov 2004] M. Nedeljkov, "Delta and singular delta locus for one-dimensional systems of conservation laws", *Math. Methods Appl. Sci.* **27**:8 (2004), 931–955. MR Zbl

[Nedeljkov and Oberguggenberger 2008] M. Nedeljkov and M. Oberguggenberger, "Interactions of delta shock waves in a strictly hyperbolic system of conservation laws", *J. Math. Anal. Appl.* **344**:2 (2008), 1143–1157. MR Zbl

[Panov and Shelkovich 2006] E. Y. Panov and V. M. Shelkovich, "δ'-shock waves as a new type of solutions to systems of conservation laws", *J. Differential Equations* **228**:1 (2006), 49–86. MR Zbl

[Sarrico 1988] C. O. R. Sarrico, "About a family of distributional products important in the applications", *Portugal. Math.* **45**:3 (1988), 295–316. MR Zbl

[Sarrico 2003] C. O. R. Sarrico, "Distributional products and global solutions for nonconservative inviscid Burgers equation", *J. Math. Anal. Appl.* **281**:2 (2003), 641–656. MR Zbl

[Sarrico 2012a] C. O. R. Sarrico, "The multiplication of distributions and the Tsodyks model of synapses dynamics", *Int. J. Math. Anal.* (*Ruse*) **6**:21-24 (2012), 999–1014. MR Zbl

[Sarrico 2012b] C. O. R. Sarrico, "Products of distributions, conservation laws and the propagation of δ'-shock waves", *Chin. Ann. Math. Ser. B* **33**:3 (2012), 367–384. MR Zbl

[Schwartz 1966] L. Schwartz, *Théorie des distributions*, Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée, Hermann, Paris, 1966. MR Zbl

[Shelkovich 2006] V. M. Shelkovich, "The Riemann problem admitting δ-, $\delta\prime$-shocks, and vacuum states (the vanishing viscosity approach)", *J. Differential Equations* **231**:2 (2006), 459–500. MR Zbl

[Shelkovich 2007] V. M. Shelkovich, "δ - and δ'-shocks and the transportation and concentration processes", *PAMM* **7**:1 (2007), 2040039–2040040.

[Shelkovich 2008] V. M. Shelkovich, "Singular solutions of δ- and δ'-shock wave type of systems of conservation laws, and transport and concentration processes", *Uspekhi Mat. Nauk* **63**:3(381) (2008), 73–146. In Russian; translated in *Russ. Math. Surv.* **63**:3 (2008), 473–546. MR Zbl

Received March 1, 2017.

C. O. R. SARRICO
CMAFCIO
UNIVERSIDADE DE LISBOA
CAMPO GRANDE
LISBOA
PORTUGAL
corsarrico@gmail.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2018 is US $475/year for the electronic version, and $640/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 294 No. 1 May 2018

0030-8730(201805)294:1;1-T