Pacific Journal of Mathematics

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Volume 294 No. 1 May 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 294, No. 1, 2018
dx.doi.org/10.2140/pjm.2018.294.213

SOME AMBROSE- AND GALLOWAY-TYPE THEOREMS VIA BAKRY–ÉMERY AND MODIFIED RICCI CURVATURES

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We establish some compactness theorems of Ambrose- and Galloway-type for complete Riemannian manifolds in the context of the Bakry–Émery and modified Ricci curvatures. Our compactness theorems generalize previous ones obtained by Fernández-López and García-Río, Wei and Wylie, and Limoncu, Rimoldi, and Zhang.

1. Introduction

One of the most fundamental topics in Riemannian geometry is to investigate the relation between topology and geometric structure on Riemannian manifolds. To give nice compactness criteria for complete Riemannian manifolds is one of the most natural and interesting problems in Riemannian geometry. The celebrated theorem of Myers [1941] guarantees the compactness of complete Riemannian manifolds under some positive lower bounds on the Ricci curvature.

Theorem 1 [Myers 1941]. *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the Ricci curvature satisfies $\mathrm{Ric}_g \geqslant \lambda g$. Then (M, g) must be compact with finite fundamental group. Moreover, the diameter of (M, g) has the upper bound*

$$\operatorname{diam}(M, g) \leqslant \pi \sqrt{\frac{n-1}{\lambda}}.$$

The Myers theorem above has been widely generalized in various directions by many authors [Ambrose 1957; Calabi 1967; Fernández-López and García-Río 2008; Galloway 1979; 1982; Limoncu 2010; 2012; Lott 2003; Mastrolia et al. 2012; Morgan 2006; Qian 1997; Rimoldi 2011; Tadano 2016; 2017; Wei and Wylie 2009; Wraith 2006; Zhang 2014]. The first generalization was given by Ambrose [1957], where the positive lower bound on the Ricci curvature was replaced with an integral condition on the Ricci curvature along some geodesics.

MSC2010: primary 53C21; secondary 53C20.
Keywords: Myers-type theorem, Ambrose-type theorem, Galloway-type theorem, smooth metric measure space, Bakry–Émery Ricci curvature, modified Ricci curvature.

Theorem 2 [Ambrose 1957]. *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0, +\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_g(\dot\gamma(s), \dot\gamma(s))\, ds = +\infty.$$

Then (M, g) must be compact.

On the other hand, motivated by relativistic cosmology, Galloway [1979] proved the following compactness theorem by perturbing the positive lower bound on the Ricci curvature by the derivative in the radial direction of some bounded function:

Theorem 3 [Galloway 1979]. *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda > 0$ and $L \geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, the Ricci curvature satisfies*

$$\mathrm{Ric}_g(\dot\gamma, \dot\gamma)|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s),$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ. Then (M, g) must be compact. Moreover, the diameter of (M, g) has the upper bound

$$\mathrm{diam}(M, g) \leqslant \frac{\pi}{\lambda}\big(L + \sqrt{L^2 + (n-1)\lambda}\,\big).$$

One of the most important features of the two generalizations above is that the Ricci curvature is not required to be everywhere nonnegative.

In this paper, we shall establish some compactness theorems of Ambrose- and Galloway-type for complete Riemannian manifolds in the context of the Bakry–Émery and modified Ricci curvatures. To define the Bakry–Émery and modified Ricci curvatures, we first recall the definition of a smooth metric measure space.

Definition. A *smooth metric measure space* is a complete Riemannian manifold (M, g) with the weighted volume form $d\mu := e^{-f} d\,\mathrm{vol}_g$, where $f : M \to \mathbb{R}$ is a smooth function on M and vol_g denotes the Riemannian density with respect to the metric g. For a smooth metric measure space (M, g) and a positive constant $k \in (0, +\infty)$, we put

$$\mathrm{Ric}_f := \mathrm{Ric}_g + \mathrm{Hess}\, f \quad \text{and} \quad \mathrm{Ric}_f^k := \mathrm{Ric}_g + \mathrm{Hess}\, f - \frac{1}{k} df \otimes df \tag{1-1}$$

and call them a *Bakry–Émery Ricci curvature* and a *k-Bakry–Émery Ricci curvature*, respectively. We refer to f as a *potential function*. More generally, for a smooth vector field $V \in \mathfrak{X}(M)$ and a positive constant $k \in (0, +\infty)$, we define

$$\mathrm{Ric}_V := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g \quad \text{and} \quad \mathrm{Ric}_V^k := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g - \frac{1}{k} V^* \otimes V^*,$$

where V^* is the metric dual of V with respect to g. We call them a *modified Ricci curvature* and a *k-modified Ricci curvature*, respectively. We also put

$$\Delta_f := \Delta_g - \nabla f \cdot \nabla \quad \text{and} \quad \Delta_V := \Delta_g - V \cdot \nabla \tag{1-2}$$

and call them a *Witten–Laplacian* and a *V-Laplacian*, respectively. Here, Δ_g denotes the Laplacian with respect to g.

Note that if $f : M \to \mathbb{R}$ is constant in (1-1) and (1-2), then the Bakry–Émery Ricci curvature and the Witten–Laplacian are reduced to the Ricci curvature and the Laplacian, respectively. As in the classical case, for any smooth functions u, v on M with compact support, we have

$$\int_M g(\nabla u, \nabla v)\, d\mu = -\int_M (\Delta_f u) v\, d\mu = -\int_M u(\Delta_f v)\, d\mu.$$

Moreover, Bakry and Émery [1985] proved that for any smooth function u on M,

$$\tfrac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \mathrm{Ric}_f(\nabla u, \nabla u) + g(\nabla \Delta_f u, \nabla u), \tag{1-3}$$

which may be regarded as a natural extension of the Bochner–Weitzenböck formula

$$\tfrac{1}{2}\Delta_g |\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \mathrm{Ric}_g(\nabla u, \nabla u) + g(\nabla \Delta_g u, \nabla u). \tag{1-4}$$

Recently, the Bakry–Émery Ricci curvature and the Witten–Laplacian have received much attention in various areas of mathematics, since they are good substitutes for the Ricci curvature and the Laplacian respectively, allowing us to establish many interesting results in smooth metric measure spaces, such as eigenvalue estimates [Futaki et al. 2013], Li–Yau Harnack inequalities [Li 2005], and comparison theorems [Wei and Wylie 2009]. In particular, Wei and Wylie [2009] proved the following Myers-type theorem via Bakry–Émery Ricci curvature which extends Theorem 1 to the case of smooth metric measure spaces:

Theorem 4 [Wei and Wylie 2009]. *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the Bakry–Émery Ricci curvature satisfies $\mathrm{Ric}_f \geqslant \lambda g$. If the potential function satisfies $|f| \leqslant H$ for some nonnegative constant $H \geqslant 0$, then (M, g) must be compact. Moreover, the diameter of (M, g) has the upper bound*

$$\mathrm{diam}(M, g) \leqslant \pi \sqrt{\frac{n-1}{\lambda}} + \frac{4H}{\sqrt{(n-1)\lambda}}. \tag{1-5}$$

On the other hand, Fernández-López and García-Río [2008] proved that the compactness of a complete Riemannian manifold with a positive lower bound on the modified Ricci curvature may be characterized by an upper bound on the norm of the vector field appearing in the modified Ricci curvature.

Theorem 5 [Fernández-López and García-Río 2008]. *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the modified Ricci curvature satisfies $\mathrm{Ric}_V \geqslant \lambda g$. Then (M, g) is compact if and only if $|V|$ is bounded on M.*

In Theorem 5 above, no upper diameter estimate was given. By extending the proof of Theorem 1, Limoncu [2010] gave the following Myers-type theorem with an upper diameter estimate via modified Ricci curvature:

Theorem 6 [Limoncu 2010]. *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the modified Ricci curvature satisfies $\mathrm{Ric}_V \geqslant \lambda g$. If the vector field satisfies $|V| \leqslant K$ for some nonnegative constant $K \geqslant 0$, then (M, g) must be compact. Moreover, the diameter of (M, g) has the upper bound*

$$\mathrm{diam}(M, g) \leqslant \frac{\pi}{\lambda}\left(\frac{K}{\sqrt{2}} + \sqrt{\frac{K^2}{2} + (n-1)\lambda}\right). \tag{1-6}$$

An interesting problem in smooth metric measure spaces is to establish Ambrose- and Galloway-type theorems via Bakry–Émery Ricci curvature. An Ambrose-type theorem via Bakry–Émery Ricci curvature was first established by Zhang [2014] under the assumption that the potential function appearing in the Bakry–Émery Ricci curvature has at most linear growth in the distance function.

Theorem 7 [Zhang 2014]. *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0, +\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_f(\dot{\gamma}(s), \dot{\gamma}(s))\, ds = +\infty,$$

and the potential function satisfies $f(x) \leqslant \delta d(x, p) + \alpha$ for some constants δ and α, where $d(x, p)$ is the distance between x and p. Then (M, g) must be compact.

More generally, we shall prove the following Ambrose-type theorem via modified Ricci curvature which generalizes Theorem 5 above:

Theorem 8. *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0, +\infty) \to M$ emanating from p satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_V(\dot{\gamma}(s), \dot{\gamma}(s))\, ds = +\infty,$$

and the vector field satisfies $|V| \leqslant K$ for some nonnegative constant $K \geqslant 0$. Then (M, g) must be compact.

As to a Galloway-type theorem via Bakry–Émery Ricci curvature, we shall prove the following compactness theorem by modifying the alternative proof of Theorem 4 by Limoncu [2012] and its improvement by the author [Tadano 2016]:

Theorem 9. *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda > 0$ and $L \geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, the Bakry–Émery Ricci curvature satisfies*

$$\mathrm{Ric}_f(\dot{\gamma}, \dot{\gamma})|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s), \tag{1-7}$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ. If the potential function satisfies $|f| \leqslant H$ for some nonnegative constant $H \geqslant 0$, then (M, g) must be compact. Moreover, the diameter of (M, g) has the upper bound

$$\mathrm{diam}(M, g) \leqslant \frac{1}{\lambda}\bigl(2L + \sqrt{4L^2 + \{(n-1)\pi + 8H\}\lambda\pi}\,\bigr).$$

Remark. By taking $L = 0$, Theorem 9 above is reduced to the Myers-type theorem via Bakry–Émery Ricci curvature [Tadano 2016] with the diameter estimate

$$\mathrm{diam}(M, g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n - 1 + \frac{8H}{\pi}}. \tag{1-8}$$

Note that the estimate (1-8) above is sharper than (1-5) by Wei and Wylie [2009].

On the other hand, by modifying the proof of Theorem 6 above, we shall prove the following Galloway-type theorem via modified Ricci curvature:

Theorem 10. *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda > 0$ and $L \geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, the modified Ricci curvature satisfies*

$$\mathrm{Ric}_V(\dot{\gamma}, \dot{\gamma})|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s), \tag{1-9}$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ. If the vector field satisfies $|V| \leqslant K$ for some nonnegative constant $K \geqslant 0$, then (M, g) must be compact. Moreover, the diameter of (M, g) has the upper bound

$$\mathrm{diam}(M, g) \leqslant \frac{1}{\lambda}\bigl(2(L + K) + \sqrt{4(L + K)^2 + (n-1)\lambda\pi^2}\,\bigr).$$

Remark. By taking $L = 0$, Theorem 10 above is reduced to the Myers-type theorem via modified Ricci curvature [Tadano 2017] with the diameter estimate

$$\mathrm{diam}(M, g) \leqslant \frac{1}{\lambda}\bigl(2K + \sqrt{4K^2 + (n-1)\lambda\pi^2}\,\bigr). \tag{1-10}$$

Note that the estimate (1-10) above is sharper than (1-6) by Limoncu [2010].

Moreover, we shall prove the compactness of a complete Riemannian manifold with a lower bound on the modified Ricci curvature under the condition that the norm of the vector field appearing in the modified Ricci curvature has at most linear growth in the distance function.

Theorem 11. *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exist some constants $\lambda > 0$ and $L \geqslant 0$ such that for every pair of points in M and minimal geodesic γ joining those points, the modified Ricci curvature satisfies*

$$\mathrm{Ric}_V(\dot{\gamma}, \dot{\gamma})|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s), \tag{1-11}$$

where ϕ is some smooth function of the arc length satisfying $\phi \geqslant -L$ along γ. If the vector field satisfies $|V|(x) \leqslant \delta d(x, p) + \alpha$ for some constants $\delta < \lambda$ and α, where $d(x, p)$ is the distance between x and p, then (M, g) must be compact.

By taking $L = 0$ and $V = \nabla f$ for a smooth function $f : M \to \mathbb{R}$ in Theorem 11 above, we may recover the following compactness theorem due to Zhang [2014]:

Theorem 12 [Zhang 2014]. *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the Bakry–Émery Ricci curvature satisfies $\mathrm{Ric}_f \geqslant \lambda g$. If the potential function satisfies $f(x) \leqslant \delta(d(x, p) + \alpha)^2$ for some constants $\delta < \frac{1}{2}\lambda$ and α, where $d(x, p)$ is the distance between x and p, then (M, g) must be compact.*

Remark. A typical example of smooth metric measure spaces is a *Gaussian soliton* $(\mathbb{R}^n, g_0)$, where g_0 is the canonical flat metric on $\mathbb{R}^n$ and its potential function is given by the function $f(x) = \frac{1}{2}\lambda r^2(x)$. Here, $r = r(x)$ is the distance from the origin. The Gaussian soliton satisfies

$$\mathrm{Ric}_{g_0} + \mathrm{Hess}\, f = \lambda g_0.$$

The Gaussian soliton is an example to show that Theorem 11 is not true if $\delta = \lambda$, since the soliton is noncompact and satisfies $|\nabla f|(x) = \lambda r(x)$.

As in the case of the Bakry–Émery and modified Ricci curvatures, we may give some compactness theorems for complete Riemannian manifolds via k-Bakry–Émery and k-modified Ricci curvatures. Limoncu [2010] established the following Myers-type theorem via k-modified Ricci curvature without making any assumption on the vector field appearing in the k-modified Ricci curvature:

Theorem 13 [Limoncu 2010]. *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $\lambda > 0$ such that the*

k-modified Ricci curvature satisfies $\mathrm{Ric}_V^k \geqslant \lambda g$, *where* $k \in (0, +\infty)$. *Then* (M, g) *must be compact. Moreover, the diameter of* (M, g) *has the upper bound*

$$\mathrm{diam}(M, g) \leqslant \frac{\pi}{\sqrt{\lambda}}\sqrt{n+k-1}.$$

Remark. In the case where the vector field V is replaced with the gradient of some smooth function $f : M \to \mathbb{R}$, Theorem 13 above was already proved by Qian [1997].

As demonstrated by Wraith [2006], the key ingredient in proving Theorem 2 is the Riccati inequality for the Ricci curvature

$$\mathrm{Ric}_g(\partial_r, \partial_r) \leqslant -\dot{m} - \frac{1}{n-1}m^2,$$

which may be derived by applying the classical Bochner–Weitzenböck formula (1-4) to the distance function $r(x) = d(x, p)$. Here $m := \Delta_g r$. Recently, Li [2015] established the following Bochner–Weitzenböck formula via modified Ricci curvature:

$$\tfrac{1}{2}\Delta_V|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \mathrm{Ric}_V(\nabla u, \nabla u) + g(\nabla\Delta_V u, \nabla u). \tag{1-12}$$

By applying the Bochner–Weitzenböck formula (1-12) to the distance function $r(x) = d(x, p)$, we may derive the Riccati inequality for the k-modified Ricci curvature

$$\mathrm{Ric}_V^k(\partial_r, \partial_r) \leqslant -\dot{m}_V - \frac{(m_V)^2}{n+k-1},$$

where $m_V := \Delta_V r$. By using this Riccati inequality, we shall prove the following Ambrose-type theorem via k-modified Ricci curvature:

Theorem 14. *Let* (M, g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exists some point* $p \in M$ *for which every geodesic* $\gamma : [0, +\infty) \to M$ *emanating from* p *satisfies*

$$\int_0^{+\infty} \mathrm{Ric}_V^k(\dot{\gamma}(s), \dot{\gamma}(s))\, ds = +\infty,$$

where $k \in (0, +\infty)$. *Then* (M, g) *must be compact.*

As to a Galloway-type theorem via k-modified Ricci curvature, we shall prove the following compactness theorem by modifying the proof of Theorem 13 by Limoncu [2010].

Theorem 15. *Let* (M, g) *be an* n*-dimensional complete Riemannian manifold. Suppose that there exist some constants* $\lambda > 0$ *and* $L \geqslant 0$ *such that for every pair of points in* M *and minimal geodesic* γ *joining those points, the* k*-modified Ricci curvature satisfies*

$$\mathrm{Ric}_V^k(\dot{\gamma}, \dot{\gamma})|_{\gamma(s)} \geqslant \lambda + \frac{d\phi}{ds}(s), \tag{1-13}$$

where ϕ is some smooth function of the arc length satisfying $|\phi| \leqslant L$ along γ and $k \in (0, +\infty)$. Then (M, g) must be compact. Moreover, the diameter of (M, g) has the upper bound

$$\operatorname{diam}(M, g) \leqslant \frac{1}{\lambda}\bigl(2L + \sqrt{4L^2 + (n+k-1)\lambda\pi^2}\,\bigr).$$

Remark. In the case where the vector field V is replaced with the gradient of some smooth function $f : M \to \mathbb{R}$, Theorem 15 above was already proved by Rimoldi [2011].

This paper is organized as follows: In Section 2, after introducing our notation, we shall prove Theorems 8, 11, and 14. Ending with Section 3, we shall prove Theorems 9, 10, and 15.

2. Ambrose-type theorems

In this section, we shall prove Theorem 8, 11, and 14. Our proofs of these theorems are modifications of the alternative proof of Theorem 2 by Wraith [2006] and the proof of Theorem 7 by Zhang [2014]. Throughout this paper, we assume that (M, g) is an n-dimensional smooth connected oriented complete Riemannian manifold without boundary. Let $X, Y, Z \in \mathfrak{X}(M)$ be three smooth vector fields on M. For any smooth function $f \in \mathcal{C}^\infty(M)$, a *gradient vector field* and a *Hessian* of f are defined by

$$g(\nabla f, X) = df(X) \quad \text{and} \quad \operatorname{Hess} f(X, Y) = g(\nabla_X \nabla f, Y),$$

respectively. A *curvature* and a *Ricci curvature* are defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \quad \text{and} \quad \operatorname{Ric}_g(X,Y) = \sum_{i=1}^{n} g(R(e_i,X)Y,e_i),$$

respectively. Here, $\{e_i\}_{i=1}^n$ is an orthonormal frame of (M, g).

2.1. ***Proof of Theorem 8.*** We shall first prove Theorem 8. In order to prove Theorem 8, it is sufficient to show the following theorem:

Theorem 16. *Let (M, g) be an n-dimensional complete noncompact Riemannian manifold and $V \in \mathfrak{X}(M)$ be a smooth vector field on M satisfying $|V| \leqslant K$ for some nonnegative constant $K \geqslant 0$. Let $\gamma = \gamma(s)$, $s \geqslant 0$, be a geodesic in (M, g). If the limit*

$$\lim_{t \to +\infty} \int_0^t \operatorname{Ric}_V(\dot\gamma(s), \dot\gamma(s))\, ds$$

exists, then it must take a value less than infinity.

Proof. We shall prove this theorem by contradiction. Fix a point $p \in M$ and take a unit speed ray $\gamma = \gamma(s)$ emanating from p satisfying $\gamma(0) = p$. For any $s > 0$, let $m(s)$ be the mean curvature of the distance sphere of radius s about p at the point $\gamma(s)$. Note that $m(s)$ is smooth for $s > 0$. It is well-known that $m(s)$ satisfies the Riccati inequality

$$\mathrm{Ric}_g(\dot\gamma(s), \dot\gamma(s)) \leqslant -\dot m(s) - \frac{1}{n-1} m^2(s),$$

see [Cheeger 1991] for details. Hence, we have

$$\mathrm{Ric}_V(\dot\gamma(s), \dot\gamma(s)) \leqslant -\dot m(s) - \frac{1}{n-1} m^2(s) + \frac{1}{2}\mathcal{L}_V g(\dot\gamma(s), \dot\gamma(s)).$$

Since $\mathcal{L}_V g(\dot\gamma(s), \dot\gamma(s)) = 2\dot\gamma(s) g(V(\gamma(s)), \dot\gamma(s)) = 2(\partial/\partial s) g(V(\gamma(s)), \dot\gamma(s))$, by integrating both sides of the inequality just above, for all $t > 1$, we obtain

$$\text{(2-1)} \quad \int_1^t \mathrm{Ric}_V(\dot\gamma(s), \dot\gamma(s))\, ds \leqslant -m(t) + m(1) - \frac{1}{n-1}\int_1^t m^2(s)\, ds + g(V(\gamma(t)), \dot\gamma(t)) - g(V(\gamma(1)), \dot\gamma(1)).$$

Since $\gamma = \gamma(s)$ is a unit speed ray, the Cauchy–Schwarz inequality implies $|g(V, \dot\gamma)| \leqslant |V|$. By combining this inequality and the assumption $|V| \leqslant K$ in Theorem 16, we have $|g(V, \dot\gamma)| \leqslant K$. Hence, from (2-1) we obtain

$$\int_1^t \mathrm{Ric}_V(\dot\gamma(s), \dot\gamma(s))\, ds \leqslant -m(t) + m(1) - \frac{1}{n-1}\int_1^t m^2(s)\, ds + 2K.$$

Suppose, to derive a contradiction, that

$$\lim_{t\to+\infty} \int_0^t \mathrm{Ric}_V(\dot\gamma(s), \dot\gamma(s))\, ds = +\infty.$$

Then we have

$$\text{(2-2)} \qquad \lim_{t\to+\infty}\Big(-m(t) - \frac{1}{n-1}\int_1^t m^2(s)\, ds\Big) = +\infty.$$

In particular, we obtain

$$\lim_{t\to+\infty} -m(t) = +\infty.$$

Next, we shall show that there exists a finite number $T > 0$ such that

$$\text{(2-3)} \qquad \lim_{t\to T-0} -m(t) = +\infty,$$

which contradicts the smoothness of $m(t)$. First, it follows from (2-2) that there exists $t_1 > 1$ such that for all $t \geqslant t_1$, we have

$$\text{(2-4)} \qquad -m(t) - \frac{1}{n-1}\int_1^t m^2(s)\, ds > 2.$$

Define a sequence $\{t_i\}_{i=1}^{+\infty}$ inductively by

$$t_{i+1} = t_i + (n-1)\left(\tfrac{1}{2}\right)^{i-1}$$

for all $i \geqslant 1$. Note that $\{t_i\}_{i=1}^{+\infty}$ is an increasing sequence converging to

$$T := t_1 + 2(n-1).$$

Lemma 17. *For all $t \geqslant t_i,\ i \geqslant 1$, we have*

$$-m(t) > 2^i. \tag{2-5}$$

Proof of Lemma 17. By (2-4), the conclusion (2-5) is true for $i = 1$. Suppose that (2-5) holds for all $t \geqslant t_i$. Then, it follows from (2-4) and (2-5) that for all $t \geqslant t_{i+1}$,

$$\begin{aligned} -m(t) &> 2 + \frac{1}{n-1}\int_1^{t_i} m^2(t)\,dt + \frac{1}{n-1}\int_{t_i}^{t_{i+1}} m^2(t)\,dt \\ &> \frac{1}{n-1}\int_{t_i}^{t_{i+1}} m^2(t)\,dt \\ &> \frac{1}{n-1}\cdot (2^i)^2 \cdot (n-1)\cdot \left(\frac{1}{2}\right)^{i-1} = 2^{i+1}. \end{aligned}$$

Hence, (2-5) is true for all $t \geqslant t_{i+1}$. This proves Lemma 17. □

Thanks to Lemma 17, we have (2-3) which is the desired contradiction. The proof of Theorem 16 is completed. □

2.2. *Proof of Theorem 11.*

Next, we shall prove Theorem 11.

Proof of Theorem 11. We shall prove this theorem by contradiction. Assume that (M, g) is noncompact. Fix a point $p \in M$ and take a unit speed ray $\gamma = \gamma(s)$ emanating from p satisfying $\gamma(0) = p$. For any $s > 0$, let $m(s)$ be the mean curvature of the distance sphere of radius s about p at the point $\gamma(s)$. Note that $m(s)$ is smooth for $s > 0$. It follows from (2-1) and (1-11) that for all $t > 1$,

$$\begin{aligned} -m(t) - \frac{1}{n-1}\int_1^t m^2(s)\,ds + g(V(\gamma(t)), \dot\gamma(t)) &\geqslant \lambda t + \phi(t) + C_0 \\ &\geqslant \lambda t - L + C_0, \end{aligned} \tag{2-6}$$

where $C_0 := -m(1) + g(V(\gamma(1)), \dot\gamma(1)) - \phi(1) - \lambda$. It follows from the Cauchy–Schwarz inequality and the assumption $|V|(x) \leqslant \delta d(x, p) + \alpha$ in Theorem 11 that

$$g(V(\gamma(t)), \dot\gamma(t)) \leqslant |V(\gamma(t))| \leqslant \delta t + \alpha. \tag{2-7}$$

Hence, it follows from (2-6) and (2-7) that

$$-m(t) - \frac{1}{n-1}\int_1^t m^2(s)\,ds \geqslant (\lambda - \delta)t - L + C_0 - \alpha. \tag{2-8}$$

Since $\lambda > \delta$, (2-8) implies that there exists $t_1 > 1$ such that for all $t \geqslant t_1$, we have

$$-m(t) - \frac{1}{n-1}\int_1^t m^2(s)ds > 2.$$

Then, by using the same argument as in the proof of Theorem 16, we may derive the desired contradiction. The proof of Theorem 11 is completed. □

2.3. ***Proof of Theorem 14.*** Finally, we shall prove Theorem 14. In order to prove Theorem 14, it is sufficient to show the following theorem:

Theorem 18. *Let (M, g) be an n-dimensional complete noncompact Riemannian manifold, $V \in \mathfrak{X}(M)$ be a smooth vector field on M and $k \in (0, +\infty)$ be a positive constant. Let $\gamma = \gamma(s),\ s \geqslant 0$, be a geodesic in (M, g). If the limit*

$$\lim_{t\to+\infty}\int_0^t \mathrm{Ric}_V^k(\dot\gamma(s), \dot\gamma(s))ds$$

exists, then it must take a value less than infinity.

We shall prove Theorem 18 by using the following lemma which may be considered as an extension of the Bochner–Weitzenböck formula via modified Ricci curvature:

Lemma 19 [Li 2015]. *Let (M, g) be an n-dimensional Riemannian manifold. For any smooth vector field $V \in \mathfrak{X}(M)$ and smooth function $u : M \to \mathbb{R}$, we have*

$$\tfrac{1}{2}\Delta_V|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \mathrm{Ric}_V(\nabla u, \nabla u) + g(\nabla\Delta_V u, \nabla u). \tag{2-9}$$

In particular, for any positive constant $k \in (0, +\infty)$, we obtain

$$\frac{1}{2}\Delta_V|\nabla u|^2 \geqslant \frac{1}{n+k}(\Delta_V u)^2 + \mathrm{Ric}_V^k(\nabla u, \nabla u) + g(\nabla\Delta_V u, \nabla u). \tag{2-10}$$

Remark. If the vector field V is replaced with the gradient of some function $f : M \to \mathbb{R}$, then (2-9) is reduced to the Bochner–Weitzenböck formula (1-3) via Bakry–Émery Ricci curvature.

Proof of Lemma 19. For the reader's convenience, we recall the proof. This proof is based on the classical Bochner–Weitzenböck formula which asserts that for any smooth function $u : M \to \mathbb{R}$,

$$\tfrac{1}{2}\Delta_g|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \mathrm{Ric}_g(\nabla u, \nabla u) + g(\nabla\Delta_g u, \nabla u).$$

First, we shall prove (2-9). By definition of the V-Laplacian, we have

$$\begin{aligned}\tfrac{1}{2}\Delta_V|\nabla u|^2 &= \tfrac{1}{2}\Delta_g|\nabla u|^2 - \tfrac{1}{2}g(V, \nabla|\nabla u|^2)\\ &= |\mathrm{Hess} u|^2 + \mathrm{Ric}_g(\nabla u, \nabla u) + g(\nabla\Delta_g u, \nabla u) - \tfrac{1}{2}g(V, \nabla|\nabla u|^2).\end{aligned} \tag{2-11}$$

The last two terms of the right-hand side become

$$
\begin{aligned}
\text{(2-12)}\quad & g(\nabla\Delta_g u, \nabla u) - \tfrac{1}{2} g(V, \nabla|\nabla u|^2) \\
&= g(\nabla(\Delta_V u + g(V, \nabla u)), \nabla u) - V^i \nabla^j u \nabla_i \nabla_j u \\
&= g(\nabla\Delta_V u, \nabla u) + \nabla^i (V^j \nabla_j u) \nabla_i u - V^i \nabla^j u \nabla_i \nabla_j u \\
&= g(\nabla\Delta_V u, \nabla u) + \nabla^i V^j \nabla_i u \nabla_j u \\
&= g(\nabla\Delta_V u, \nabla u) + \nabla_i u \nabla_j u \big(\tfrac{1}{2}(\nabla^i V^j + \nabla^j V^i)\big) \\
&= g(\nabla\Delta_V u, \nabla u) + \tfrac{1}{2}\mathcal{L}_V g(\nabla u, \nabla u).
\end{aligned}
$$

By combining (2-11) and (2-12), we obtain (2-9). Next, we shall prove (2-10). By the Cauchy–Schwarz inequality, we have

$$
\text{(2-13)}\qquad |\mathrm{Hess}\, u|^2 \geqslant \frac{1}{n}(\Delta_g u)^2.
$$

Hence, it follows from (2-9) and (2-13) that

$$
\text{(2-14)}\quad \frac{1}{2}\Delta_V |\nabla u|^2 \geqslant \frac{1}{n}(\Delta_g u)^2 + \mathrm{Ric}_V^k(\nabla u, \nabla u) + g(\nabla\Delta_V u, \nabla u) + \frac{1}{k} g(V, \nabla u)^2.
$$

Recall the elementary inequality

$$
(a+b)^2 \geqslant \frac{1}{t}a^2 - \frac{1}{t-1}b^2, \qquad t > 1.
$$

By choosing $t = (n+k)/n$ in the inequality just above, we obtain

$$
\begin{aligned}
\text{(2-15)}\qquad \frac{1}{n}(\Delta_g u)^2 &= \frac{1}{n}(\Delta_V u + g(V, \nabla u))^2 \\
&\geqslant \frac{1}{n}\bigg(\frac{1}{\frac{n+k}{n}}(\Delta_V u)^2 - \frac{1}{\frac{n+k}{n}-1} g(V, \nabla u)^2\bigg) \\
&= \frac{1}{n+k}(\Delta_V u)^2 - \frac{1}{k} g(V, \nabla u)^2.
\end{aligned}
$$

By combining (2-14) and (2-15), we have (2-10). □

Now, we are in a position to prove Theorem 18.

Proof of Theorem 18. We shall prove this theorem by contradiction. Fix a point $p \in M$ and take a unit speed ray $\gamma = \gamma(s)$ emanating from p satisfying $\gamma(0) = p$. Let $r(x) = d(x, p)$ be the distance between x and p. By applying the inequality (2-10) to the distance function, we have

$$
\mathrm{Ric}_V^k(\dot\gamma(s), \dot\gamma(s)) \leqslant -\dot m_V(s) - \frac{1}{n+k-1} m_V^2(s),
$$

where $m_V(s) := (\Delta_V r)(\gamma(s))$. Note that $m_V(s)$ is smooth for $s > 0$. Suppose, to derive a contradiction, that

$$\lim_{t\to+\infty}\int_0^t \mathrm{Ric}_V^k(\dot\gamma(s),\dot\gamma(s))ds = +\infty.$$

Then we have

$$\lim_{t\to+\infty}\left(-m_V(t) - \frac{1}{n+k-1}\int_1^t m_V^2(s)\,ds\right) = +\infty. \tag{2-16}$$

In particular, we obtain

$$\lim_{t\to+\infty} -m_V(t) = +\infty.$$

Next, we shall show that there exists a finite number $T > 0$ such that

$$\lim_{t\to T-0} -m_V(t) = +\infty, \tag{2-17}$$

which contradicts the smoothness of $m_V(t)$. First, it follows from (2-16) that there exists $t_1 > 1$ such that for all $t \geqslant t_1$, we have

$$-m_V(t) - \frac{1}{n+k-1}\int_1^t m_V^2(s)\,ds > 2.$$

Define a sequence $\{t_i\}_{i=1}^{+\infty}$ inductively by

$$t_{i+1} = t_i + (n+k-1)\left(\tfrac{1}{2}\right)^{i-1}$$

for all $i \geqslant 1$. Note that $\{t_i\}_{i=1}^{+\infty}$ is an increasing sequence converging to

$$T := t_1 + 2(n+k-1).$$

Then, by using the same argument as in the proof of Lemma 17, we may prove the following lemma:

Lemma 20. *For all $t \geqslant t_i,\ i \geqslant 1$, we have*

$$-m_V(t) > 2^i.$$

Thanks to Lemma 20, we have (2-17) which is the desired contradiction. The proof of Theorem 18 is completed. □

3. Galloway-type theorems

In this section, we shall prove Theorems 9, 10, and 15. Our proofs of these theorems are based on modifications of the improvement of Theorem 4 by the author [Tadano 2016] and the proofs of Theorems 6 and 13 by Limoncu [2010; 2012]. In order to prove these theorems, we shall use the index form of a unit speed-minimizing geodesic segment. We refer the reader to the books [Lee 1997; Petersen 1998] for basic facts about this topic.

3.1. ***Proof of Theorem 9.*** We shall first prove Theorem 9.

Proof of Theorem 9. Take two arbitrary points $p, q \in M$. Since M is complete, there exists a unit speed-minimizing geodesic segment γ from p to q of length ℓ. Let $\{e_1 = \dot{\gamma}, e_2, \dots, e_n\}$ be a parallel orthonormal frame along γ. Recall that for any smooth function $h \in \mathcal{C}^\infty([0, \ell])$ satisfying $h(0) = h(\ell) = 0$, we have

$$\sum_{i=2}^{n} I(he_i, he_i) = \int_0^\ell ((n-1)\dot{h}^2 - h^2 \operatorname{Ric}_g(\dot{\gamma}, \dot{\gamma}))\, dt, \tag{3-1}$$

where $I(\,\cdot\,,\cdot\,)$ denotes the index form of γ. By using the assumption (1-7) in the integral expression (3-1), we obtain

$$\sum_{i=2}^{n} I(he_i, he_i) \leqslant \int_0^\ell \Big((n-1)\dot{h}^2 - \lambda h^2 + h^2 \operatorname{Hess} f(\dot{\gamma}, \dot{\gamma}) - h^2 \frac{d\phi}{dt}\Big)\, dt. \tag{3-2}$$

On the geodesic segment $\gamma(t)$, we have

$$\begin{aligned} h^2 \operatorname{Hess} f(\dot{\gamma}, \dot{\gamma}) &= h^2 g(\nabla_{\dot{\gamma}} \nabla f, \dot{\gamma}) = h^2 \dot{\gamma}(g(\nabla f, \dot{\gamma})) = h^2 \frac{d}{dt}(g(\nabla f, \dot{\gamma})) \\ &= -2h\dot{h} g(\nabla f, \dot{\gamma}) + \frac{d}{dt}(h^2 g(\nabla f, \dot{\gamma})) \\ &= 2f \frac{d}{dt}(h\dot{h}) - 2\frac{d}{dt}(f h\dot{h}) + \frac{d}{dt}(h^2 g(\nabla f, \dot{\gamma})), \end{aligned} \tag{3-3}$$

where in the last equality we have used $g(\nabla f, \dot{\gamma}) = df/dt(\gamma(t))$. Hence, by integrating both sides of (3-3), we obtain

$$\begin{aligned} \int_0^\ell h^2 \operatorname{Hess} f(\dot{\gamma}, \dot{\gamma})\, dt &= \int_0^\ell 2f \frac{d}{dt}(h\dot{h})\, dt - 2\Big[f h\dot{h}\Big]_0^\ell + \Big[h^2 g(\nabla f, \dot{\gamma})\Big]_0^\ell \\ &= 2\int_0^\ell f \frac{d}{dt}(h\dot{h})\, dt, \end{aligned} \tag{3-4}$$

where the last equality follows from $h(0) = h(\ell) = 0$. By (3-4) and the assumption $|f| \leqslant H$ in Theorem 9, we have

$$\int_0^\ell h^2 \operatorname{Hess} f(\dot{\gamma}, \dot{\gamma})\, dt \leqslant 2H \int_0^\ell \Big|\frac{d}{dt}(h\dot{h})\Big|\, dt. \tag{3-5}$$

On the other hand, from the assumption $|\phi| \leqslant L$ in Theorem 9, we obtain

$$\int_0^\ell h^2 \frac{d\phi}{dt}\, dt = [h^2\phi]_0^\ell - \int_0^\ell 2h\dot{h}\phi\, dt \geqslant -2L \int_0^\ell |h\dot{h}|\, dt. \tag{3-6}$$

From (3-2), (3-5), and (3-6), we have

$$\sum_{i=2}^{n} I(he_i, he_i) \leqslant \int_0^{\ell}\Big((n-1)\dot{h}^2 - \lambda h^2 + 2H\Big|\frac{d}{dt}(h\dot{h})\Big| + 2L|h\dot{h}|\Big)\,dt. \tag{3-7}$$

If the function h is taken to be $h(t) = \sin(\pi t/\ell)$, then we obtain

$$h\dot{h} = \frac{\pi}{\ell}\sin\Big(\frac{\pi t}{\ell}\Big)\cos\Big(\frac{\pi t}{\ell}\Big) = \frac{\pi}{2\ell}\sin\Big(\frac{2\pi t}{\ell}\Big).$$

Then (3-7) becomes

$$\begin{aligned}\sum_{i=2}^{n} I(he_i, he_i) \leqslant \int_0^{\ell}(n-1)\Big(\frac{\pi^2}{\ell^2}\cos^2\Big(\frac{\pi t}{\ell}\Big) - \lambda\sin^2\Big(\frac{\pi t}{\ell}\Big)\Big)\,dt \\ + 2H\Big(\frac{\pi}{\ell}\Big)^2\int_0^{\ell}\Big|\cos\Big(\frac{2\pi t}{\ell}\Big)\Big|\,dt + \frac{L\pi}{\ell}\int_0^{\ell}\Big|\sin\Big(\frac{2\pi t}{\ell}\Big)\Big|\,dt.\end{aligned} \tag{3-8}$$

Since

$$\begin{aligned}\int_0^{\ell}\dot{h}^2\,dt = \int_0^{\ell}\frac{\pi^2}{\ell^2}\cos^2\Big(\frac{\pi t}{\ell}\Big)\,dt &= \frac{\pi^2}{2\ell},\\ \int_0^{\ell}h^2\,dt = \int_0^{\ell}\sin^2\Big(\frac{\pi t}{\ell}\Big)\,dt &= \frac{\ell}{2},\\ \int_0^{\ell}\Big|\cos\Big(\frac{2\pi t}{\ell}\Big)\Big|\,dt &= \frac{2\ell}{\pi},\\ \int_0^{\ell}|h\dot{h}|\,dt = \int_0^{\ell}\frac{\pi}{2\ell}\Big|\sin\Big(\frac{2\pi t}{\ell}\Big)\Big|\,dt &= 1,\end{aligned} \tag{3-9}$$

it follows from (3-8) and (3-9) that

$$\sum_{i=2}^{n} I(he_i, he_i) \leqslant -\frac{1}{2\ell}\big(\lambda\ell^2 - 4L\ell - (n-1)\pi^2 - 8H\pi\big).$$

Since γ is a minimizing geodesic, we must obtain

$$\lambda\ell^2 - 4L\ell - (n-1)\pi^2 - 8H\pi \leqslant 0,$$

from where we have

$$\ell \leqslant \frac{1}{\lambda}\big(2L + \sqrt{4L^2 + \{(n-1)\pi + 8H\}\lambda\pi}\,\big).$$

This proves Theorem 9. □

3.2. *Proof of Theorem 10.* Next, we shall prove Theorem 10.

Proof of Theorem 10. By using the assumption (1-9) in the integral expression (3-1), we obtain

$$\sum_{i=2}^{n} I(he_i, he_i) \leqslant \int_0^{\ell}\Big((n-1)\dot{h}^2 - \lambda h^2 + \frac{1}{2}h^2(\mathcal{L}_V g)(\dot{\gamma}, \dot{\gamma}) - h^2\frac{d\phi}{dt}\Big)\,dt$$
$$= \int_0^{\ell}\Big((n-1)\dot{h}^2 - \lambda h^2 + h^2 g(\nabla_{\dot{\gamma}} V, \dot{\gamma}) - h^2\frac{d\phi}{dt}\Big)\,dt$$
$$= \int_0^{\ell}\Big((n-1)\dot{h}^2 - \lambda h^2 + h^2 \dot{\gamma}(g(V, \dot{\gamma})) - h^2\frac{d\phi}{dt}\Big)\,dt, \tag{3-10}$$

where the last equality follows from the parallelism of the metric g and $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. On the geodesic segment $\gamma(t)$, we have

$$h^2\dot{\gamma}(g(V, \dot{\gamma})) = h^2\frac{d}{dt}(g(V, \dot{\gamma})) = -2h\dot{h}g(V, \dot{\gamma}) + \frac{d}{dt}(h^2 g(V, \dot{\gamma})). \tag{3-11}$$

Hence, by integrating both sides of (3-11), we obtain

$$\int_0^{\ell} h^2\dot{\gamma}(g(V, \dot{\gamma}))\,dt = \int_0^{\ell} -2h\dot{h}g(V, \dot{\gamma})\,dt + [h^2 g(V, \dot{\gamma})]_0^{\ell}$$
$$\leqslant 2\int_0^{\ell} |h\dot{h}g(V, \dot{\gamma})|\,dt \tag{3-12}$$
$$\leqslant 2K\int_0^{\ell} |h\dot{h}|\,dt, \tag{3-13}$$

where the second inequality follows from $h(0) = h(\ell) = 0$. From (3-10), (3-13), and (3-6), we have

$$\sum_{i=2}^{n} I(he_i, he_i) \leqslant \int_0^{\ell}((n-1)\dot{h}^2 - \lambda h^2)\,dt + 2K\int_0^{\ell}|h\dot{h}|\,dt + 2L\int_0^{\ell}|h\dot{h}|\,dt. \tag{3-14}$$

If the function h is taken to be $h(t) = \sin(\pi t/\ell)$, then (3-14) becomes

$$\sum_{i=2}^{n} I(he_i, he_i) \leqslant \int_0^{\ell}\Big((n-1)\frac{\pi^2}{\ell^2}\cos^2\Big(\frac{\pi t}{\ell}\Big) - \lambda\sin^2\Big(\frac{\pi t}{\ell}\Big)\Big)\,dt$$
$$+ \frac{K\pi}{\ell}\int_0^{\ell}\Big|\sin\Big(\frac{2\pi t}{\ell}\Big)\Big|\,dt + \frac{L\pi}{\ell}\int_0^{\ell}\Big|\sin\Big(\frac{2\pi t}{\ell}\Big)\Big|\,dt. \tag{3-15}$$

It follows from (3-15) and (3-9) that

$$\sum_{i=2}^{n} I(he_i, he_i) \leqslant -\frac{1}{2\ell}\{\lambda\ell^2 - 4(L+K)\ell - (n-1)\pi^2\}.$$

Since γ is a minimizing geodesic, we must obtain

$$\lambda\ell^2 - 4(L+K)\ell - (n-1)\pi^2 \leqslant 0,$$

from where we have

$$\ell \leqslant \frac{1}{\lambda}\big(2(L+K) + \sqrt{4(L+K)^2 + (n-1)\lambda\pi^2}\,\big).$$

This proves Theorem 10. □

3.3. *Proof of Theorem 15.* Finally, we shall prove Theorem 15.

Proof of Theorem 15. By using the assumption (1-13) in the integral expression (3-1), we obtain

$$\begin{aligned}(3\text{-}16)\quad \sum_{i=2}^{n} I(he_i, he_i) &\leqslant \int_0^\ell \Big((n-1)\dot{h}^2 - \lambda h^2 + \frac{1}{2}h^2(\mathcal{L}_V g)(\dot\gamma,\dot\gamma) - h^2\frac{d\phi}{dt}\Big)\,dt \\ &\qquad - \frac{1}{k}\int_0^\ell h^2(g(V,\dot\gamma))^2\,dt \\ &\leqslant \int_0^\ell ((n-1)\dot{h}^2 - \lambda h^2 + 2|h\dot{h}g(V,\dot\gamma)| + 2L|h\dot{h}|)\,dt \\ &\qquad - \frac{1}{k}\int_0^\ell h^2(g(V,\dot\gamma))^2\,dt,\end{aligned}$$

where the last inequality follows from (3-12) and (3-6). Applying $P = |\dot{h}|$ and $Q = |hg(V,\dot\gamma)|$ to the Cauchy–Schwarz inequality

$$\int_0^\ell PQ\,dt \leqslant \sqrt{\int_0^\ell P^2\,dt}\sqrt{\int_0^\ell Q^2\,dt},$$

we have

$$(3\text{-}17)\qquad \int_0^\ell \big|h\dot{h}g(V,\dot\gamma)\big|\,dt \leqslant \sqrt{\int_0^\ell \dot{h}^2\,dt}\sqrt{\int_0^\ell h^2(g(V,\dot\gamma))^2\,dt}.$$

Applying $A = k\int_0^\ell \dot{h}^2\,dt \geqslant 0$ and $B = (1/k)\int_0^\ell h^2(g(V,\dot\gamma))^2\,dt \geqslant 0$ to the elementary inequality $2\sqrt{AB} \leqslant A + B$, we obtain

$$(3\text{-}18)\qquad 2\sqrt{\int_0^\ell \dot{h}^2\,dt}\sqrt{\int_0^\ell h^2(g(V,\dot\gamma))^2\,dt} \leqslant \int_0^\ell \Big(k\dot{h}^2 + \frac{1}{k}h^2(g(V,\dot\gamma))^2\Big)\,dt.$$

From (3-16), (3-17), and (3-18), we have

$$(3\text{-}19)\qquad \sum_{i=2}^{n} I(he_i, he_i) \leqslant \int_0^\ell ((n-1)\dot{h}^2 - \lambda h^2 + k\dot{h}^2 + 2L|h\dot{h}|)\,dt.$$

If the function h is taken to be $h(t)=\sin(\pi t/\ell)$, then (3-19) becomes

$$\sum_{i=2}^{n} I(he_i, he_i) \leqslant \int_0^\ell (n-1)\Big(\frac{\pi^2}{\ell^2}\cos^2\Big(\frac{\pi t}{\ell}\Big)-\lambda\sin^2\Big(\frac{\pi t}{\ell}\Big)\Big)\,dt + \frac{k\pi^2}{\ell^2}\int_0^\ell \cos^2\Big(\frac{\pi t}{\ell}\Big)\,dt + \frac{L\pi}{\ell}\int_0^\ell \Big|\sin\Big(\frac{2\pi t}{\ell}\Big)\Big|\,dt. \tag{3-20}$$

It follows from (3-20) and (3-9) that

$$\sum_{i=2}^{n} I(he_i, he_i) \leqslant -\frac{1}{2\ell}\{\lambda\ell^2-4L\ell-(n-1)\pi^2-k\pi^2\}.$$

Since γ is a minimizing geodesic, we must obtain

$$\lambda\ell^2-4L\ell-(n-1)\pi^2-k\pi^2\leqslant 0,$$

from where we have

$$\ell\leqslant\frac{1}{\lambda}\big(2L+\sqrt{4L^2+(n-1+k)\lambda\pi^2}\big).$$

This proves Theorem 15. □

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Received April 8, 2016. Revised June 8, 2017.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 294 No. 1 May 2018

0030-8730(201805)294:1;1-T