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IRREDUCIBLE DECOMPOSITION FOR LOCAL REPRESENTATIONS OF QUANTUM TEICHMÜLLER SPACE

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We give an irreducible decomposition of the so-called local representations (Bai, Bonahon and Liu, 2007) of the quantum Teichmüller space $\mathcal{T}_q(\Sigma)$, where Σ is a punctured surface of genus $g > 0$ and q is an N-th root of unity with N odd. As an application, we construct a family of representations of the Kauffman bracket skein algebra of the closed surface $\overline{\Sigma}$.

1. Introduction

Let Σ be the surface obtained by removing $s > 0$ points $v_1, \dots, v_s$ from the closed oriented surface $\overline{\Sigma}$ of genus $g > 0$. Denote by $\mathcal{T}(\Sigma)$ the Teichmüller space of Σ, that is roughly speaking, the space of complete hyperbolic metrics on Σ. Given λ an ideal triangulation of Σ (that is a triangulation of the closed surface $\overline{\Sigma}$ whose vertices are exactly the v_i), Thurston [1986] constructed a parametrization of $\mathcal{T}(\Sigma)$ by associating a strictly positive real number to each edge λ_i of the ideal triangulation, $i \in \{1, \dots, n\}$, where $n = 6g - 6 + 3s$ is the number of edges of λ. These coordinates are called the *shear coordinates* associated to λ. In this coordinate system, the coefficients of the Weil–Petersson form on $\mathcal{T}(\Sigma)$ depend only on the combinatorics of λ and are easy to compute.

For a parameter $q \in \mathbb{C}^*$, Chekhov and Fock [1999] defined the *quantum Teichmüller space* $\mathcal{T}_q(\Sigma)$ of Σ, which is a deformation quantization of the Poisson algebra of a certain class of functions over $\mathcal{T}(\Sigma)$; see also [Kashaev 1998] for a slightly different version and [Guo and Liu 2009] for a relation between the two.

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This algebraic object is obtained by gluing together a collection of noncommutative algebras $\mathcal{T}_q(\lambda)$, called *Chekhov–Fock algebras*, canonically associated to each ideal triangulation of Σ. A representation of $\mathcal{T}_q(\Sigma)$ is then a family of representations $\{\rho_\lambda : \mathcal{T}_q(\lambda) \to \mathrm{End}(V)\}_{\lambda\in\Lambda(\Sigma)}$, where $\Lambda(\Sigma)$ is the space of all ideal triangulations of Σ, and ρ_λ and $\rho_{\lambda'}$ satisfy compatibility conditions whenever $\lambda \neq \lambda'$. For $\lambda \in \Lambda(\Sigma)$, the representation ρ_λ is an avatar of the representation of $\mathcal{T}_q(\Sigma)$ and carries almost all the information.

When q is a primitive N-th root of unity, $\mathcal{T}_q(\lambda)$ admits finite-dimensional representations. In this paper, we will consider the case that N is odd. The irreducible representations of $\mathcal{T}_q(\lambda)$ were studied in [Bonahon and Liu 2007], where it was shown that an irreducible representation of $\mathcal{T}_q(\lambda)$ is classified (up to isomorphism) by a weight $x_i \in \mathbb{C}^*$ assigned to each edge λ_i, a choice of N-th root $p_j = \left(x_1^{k_{j_1}} \cdots x_n^{k_{j_n}}\right)^{1/N}$ associated to each puncture v_j (where k_{j_i} is the number of times a small simple loop around v_j intersects λ_i) and an N-th root $c = (x_1 \dots x_n)^{1/N}$. Such a representation has dimension N^{3g-3+s}.

Bai, Bonahon, and Liu [Bai et al. 2007] introduced another type of representations of $\mathcal{T}_q(\lambda)$, called *local representations*, which are well behaved under cut and paste. A local representation of $\mathcal{T}_q(\lambda)$ is defined by an embedding into the tensorial product of *triangle algebras* (see definitions below). Isomorphism classes of local representations of $\mathcal{T}_q(\lambda)$ are classified by a weight $x_i \in \mathbb{C}^*$ associated to each edge λ_i and a choice of an N-th root $c = (x_1 \dots x_n)^{1/N}$. Such a representation has dimension $N^{4g-4+2s}$.

It follows that a local representation of $\mathcal{T}_q(\lambda)$ is not irreducible. In this paper, we address the question of the decomposition of a local representation into its irreducible components. We prove:

Main Theorem. *Let λ be an ideal triangulation of Σ and ρ be a local representation of $\mathcal{T}_q(\lambda)$ classified by weight $x_j \in \mathbb{C}^*$ associated to each edge λ_j and a choice of N-th root $c = (x_1 \dots x_n)^{1/N}$. Then we have the decomposition*

$$\rho = \bigoplus_{i\in\mathcal{I}} \rho^{(i)}.$$

Here, $\rho^{(i)}$ is an irreducible representation classified by the same x_j, an N-th root $p_j^{(i)} = (x_1^{k_{j_1}} \cdots x_n^{k_{j_n}})^{1/N}$ associated to each puncture, and the same c. Moreover, $\mathcal{I}$ is a finite set such that, given a choice of an N-th root $p_j = (x_1^{k_{j_1}} \cdots x_n^{k_{j_n}})^{1/N}$ for each puncture, there exists exactly N^g elements $i \in \mathcal{I}$ with $p_j^{(i)} = p_j$ for all $j \in \{1, \dots, s\}$.

It is proved by Bai [2007] that if λ and λ' are two different triangulations of the square related by a diagonal switch, then the intertwining operators associated to two isomorphic representations $\rho : \mathcal{T}_q(\lambda) \to \mathrm{End}(V)$ and $\rho' : \mathcal{T}_q(\lambda') \to \mathrm{End}(V')$ correspond to the $6j$-symbols defined by Kashaev [1995]. These $6j$-symbols relate hyperbolic geometry and quantum invariants and gave birth to the famous volume conjecture; see [Murakami 2011] for an overview.

In particular, Baseilhac and Benedetti [2005] used these $6j$-symbols to construct a $(2+1)$-dimensional topological quantum field theory (TQFT) on manifolds with $\mathrm{PSL}(2, \mathbb{C})$-character. Our result thus provides a decomposition of the vector spaces arising in the TQFT.

As an application, we adapt the construction of Bonahon and Wong [2015] to local representations of the balanced Chekhov–Fock algebra and obtain a family of representations of the Kauffman bracket skein algebra $\mathcal{S}_A(\overline{\Sigma})$ of the closed surface $\overline{\Sigma}$. The vector space associated to this family of representations is canonically associated to an ideal triangulation λ. In particular, it makes the computations very explicit. It also behaves well under cut and paste.

In Section 2, we recall the definition of the Chekhov–Fock algebra, the quantum Teichmüller space, the triangle algebra and the local representations. In Section 3, we prove the Main Theorem. Finally, in Section 4, we explain the connections between quantum Teichmüller theory, skein theory and construct a new family of representations of $\mathcal{S}_A(\overline{\Sigma})$.

2. Chekhov–Fock algebra and representations of $\mathcal{T}_q(\Sigma)$

In this section, we define the Chekhov–Fock algebra $\mathcal{T}_q(\lambda)$ associated to an ideal triangulation λ, describe its representations and recall the definition of the quantum Teichmüller space. Most results come from [Bonahon and Liu 2007; Bai et al. 2007].

In all this paper, for an integer $n \in \mathbb{N}$, set $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ and denote by $\mathcal{U}(N)$ the group of N-th roots of unity.

2.1. ***The Chekhov–Fock algebra.*** In this subsection, q is a formal parameter and Σ is allowed to have boundary components with punctures on the boundary (and every boundary component has at least one puncture).

Let λ be an ideal triangulation of Σ. We denote by $\lambda_1, \dots, \lambda_n$ the edges of λ. The *Fock's matrix associated to* λ is the skew-symmetric $n \times n$ matrix with integer coefficients $\eta_\lambda = (\sigma_{ij})_{i,j=1,\dots n}$ defined by

$$\sigma_{ij} = a_{ij} - a_{ji}$$

where a_{ij} is the number of angular sector delimited by λ_i and λ_j in the faces of λ with λ_i coming before λ_j counterclockwise.

Definition 2.1. The *Chekhov–Fock algebra of* λ is the algebra $\mathcal{T}_q(\lambda)$ freely generated by the elements $X_i^{\pm 1}$, $i \in \{1, \dots, n\}$, subject to the relations

$$X_i X_j = q^{2\sigma_{ij}} X_j X_i.$$

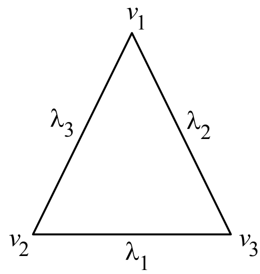

Figure 1. The triangle T.

The following example is of first importance.

Example 2.2. Let T be a disk with three punctures v_1, v_2, v_3 on the boundary. The boundary arcs between the punctures provides a natural triangulation λ of T (see Figure 1).

The *triangle algebra* is $\mathcal{T} := \mathcal{T}_q(\lambda)$. It is generated by $X_i^{\pm 1}$, $i = 1, 2, 3$, subject to the relations

$$X_i X_{i+1} = q^2 X_{i+1} X_i, \quad i \in \mathbb{Z}_3.$$

The algebraic structure of the Chekhov–Fock algebra is fairly simple. In particular, it is a quantum torus [Goodearl and Warfield 2004].

Given a monomial $X = X_1^{k_1} \dots X_n^{k_n} \in \mathcal{T}_q(\lambda)$ for a multi-index $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$, we define the *Weyl ordering* of X to be the monomial

$$[X] := q^{-\sum_{i<j} \sigma_{ij}} X_1^{k_1} \dots X_n^{k_n}.$$

The advantage of the Weyl ordering is its independence with respect to the order of the terms. In particular, for any permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$, we have

$$\big[X_1^{k_1} \dots X_n^{k_n}\big] = \big[X_{\sigma(1)}^{k_{\sigma(1)}} \dots X_{\sigma(n)}^{k_{\sigma(n)}}\big].$$

For a multi-index $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$, we define $X_{\mathbf{k}} := [X_1^{k_1} \dots X_n^{k_n}] \in \mathcal{T}_q(\lambda)$.

2.2. ***Finite-dimensional representations of $\mathcal{T}_q(\lambda)$.*** When the parameter q is a root of unity, the structure of the Chekhov–Fock algebra is drastically different. In particular, $\mathcal{T}_q(\lambda)$ admits finite dimensional representations that we describe here.

In this subsection, $q \in \mathbb{C}^*$ is a primitive N-th root of unity with N odd, Σ has no boundary component and λ is an ideal triangulation of Σ with edges labeled $\lambda_1, \dots, \lambda_n$.

Definition 2.3. For each puncture v_j, the *puncture invariant* $P_j = [X_1^{k_{j_1}} \dots X_n^{k_{j_n}}] \in \mathcal{T}_q(\lambda)$ is the monomial associated to the multi-index $\mathbf{k}_j = (k_{j_1}, \dots, k_{j_n}) \in \mathbb{N}^n$, where k_{j_i} is the minimum number of intersections between the edge λ_i and a closed simple loop around v_j.

The puncture invariants are of main importance in the representation theory of the Chekhov–Fock algebra. In particular:

Proposition 2.4 [Bonahon and Liu 2007, Proposition 15]. *The center of $\mathcal{T}_q(\lambda)$ is generated by*:

- X_i^N *for each* $i \in \{1, \dots, n\}$.
- *The* puncture invariant P_j *associated to each puncture* $v_j \in \{v_1, \dots, v_s\}$.
- *The element* $H := [X_1 \dots X_n]$.

Note that $[P_1 \dots P_s] = [H^2]$.

A *representation of* $\mathcal{T}_q(\lambda)$ is a morphism $\rho : \mathcal{T}_q(\lambda) \to \mathrm{End}(V)$ where V is a vector space. Such a representation is *finite-dimensional* when V is finite-dimensional and ρ is called *irreducible* when there is no proper linear subspace $W \subset V$ preserved by $\rho\big(\mathcal{T}_q(\lambda)\big)$. Two representations $\rho : \mathcal{T}_q(\lambda) \to \mathrm{End}(V)$ and $\rho' : \mathcal{T}_q(\lambda) \to \mathrm{End}(V')$ are *isomorphic* if there exists a linear isomorphism $L : V \to V'$ such that

$$\rho'(X) = L \circ \rho(X) \circ L^{-1} \quad \text{for } X \in \mathcal{T}_q(\lambda).$$

Theorem 2.5 [Bonahon and Liu 2007, Theorems 20 and 21]. *Up to isomorphism*, *any irreducible representation*

$$\rho : \mathcal{T}_q(\lambda) \to \mathrm{End}(V)$$

is determined by its restriction to the center of $\mathcal{T}_q(\lambda)$ *and is classified by a nonzero complex number* x_i *associated to each edges* λ_i, *a choice of an* N*-th root* $p_j = \big(x_1^{k_{j1}} \dots x_n^{k_{jn}}\big)^{1/N}$ *for each puncture* v_j (where the $k_{j_k} \in \{1, 2\}$ are as in Definition 2.3) *and a choice of a square root* $c = (p_0 \dots p_s)^{1/2}$.

Such a representation is characterized by

- $\rho(X_i^N) = x_i \,\mathrm{Id}_V$,
- $\rho(P_j) = p_j \,\mathrm{Id}_V$,
- $\rho(H) = c \,\mathrm{Id}_V$.

Moreover, *such a representation has dimension* N^{3g-3+s}.

Let us come back to Example 2.2. Recall that the triangle algebra $\mathcal{T}$ is the algebra generated by $X_i^{\pm 1}$, $i \in \mathbb{Z}_3$, with relations $X_i X_{i+1} = q^2 X_{i+1} X_i$.

The center of $\mathcal{T}$ is generated by X_i^N and $H = q^{-1} X_1 X_2 X_3$. One easily checks that irreducible representations of $\mathcal{T}$ have dimension N and are classified (up to isomorphism) by a choice of weight $x_i \in \mathbb{C}^*$ associated to each edge λ_i and a central charge, that is a choice of an N-th root $c = (x_1 x_2 x_3)^{1/N}$; see [Bai et al. 2007, Lemma 2] for more details.

More precisely, let V be the complex vector space generated by $\{e_1, \dots, e_N\}$ and let ρ be an irreducible representation of $\mathcal{T}$ classified by $x_1, x_2, x_3 \in \mathbb{C}^*$ and $c = (x_1x_2x_3)^{1/N}$. Then, up to isomorphism, the action of $\mathcal{T}$ on V is defined by

$$\rho(X_1)e_i = x'_1 q^{2i} e_i, \quad \rho(X_2)e_i = x'_2 e_{i+1}, \quad \rho(X_3)e_i = x'_3 q^{1-2i} e_{i-1},$$

where x'_i is an N-th root of x_i such that $x'_1x'_2x'_3 = c$. Note that, up to isomorphism, ρ is independent of the choice of the N-th root x'_i with $x'_1x'_2x'_3 = c$.

The following lemma will be useful in the next section. Recall that $\mathcal{U}(N)$ is the group of N-th roots of unity.

Lemma 2.6. *Let $\rho : \mathcal{T} \to \mathrm{End}(V)$ be the representation of the triangle algebra classified by $x_1 = x_2 = x_3 = 1$ and $c \in \mathcal{U}(N)$. For each $i \in \mathbb{Z}_3$ and N-th root $\zeta \in \mathcal{U}(N)$, the eigenspace of $\rho(X_i)$ of eigenvalue ζ is one-dimensional.*

Proof. We use the explicit form of the representation ρ in $V = \mathrm{span}\{e_1, \dots, e_N\}$ described above. Set $x'_1 = x'_2 = 1$, $x'_3 = c$ and $\zeta = q^{2k}$ for some $k \in \{0, \dots, N-1\}$.

For $i = 1$, one checks that the eigenspace of $\rho(X_1)$ associated to ζ is generated by e_k.

For $i = 2$, the vector $\alpha_k := \sum_{i \in \mathbb{Z}_N} q^{-2ki} e_i$ satisfies $\rho(X_2)\alpha_k = q^{2k}\alpha_k$ and $\{\alpha_1, \dots, \alpha_k\}$ form a basis of V. So the eigenspace of $\rho(X_2)$ associated to the eigenvalue ζ is generated by α_k.

For $i = 3$, we use the fact that $\rho(q^{-1}X_1X_2X_3) = c\,\mathrm{Id}_V$, where $c \in \mathcal{U}(N)$. □

An ideal triangulation of Σ is composed by m faces $T_1, \dots, T_m$. Each face T_j determines a triangle algebra $\mathcal{T}_j$ whose generators are associated to the three edges of T_j. It provides a canonical embedding ι of $\mathcal{T}_q(\lambda)$ into $\mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_m$ defined on the generators as follows:

- $\iota(X_i) = X_{j_i} \otimes X_{k_i}$ if λ_i belongs to two distinct triangles T_j and T_k and $X_{j_i} \in \mathcal{T}_j$, $X_{k_i} \in \mathcal{T}_k$ are the generators associated to the edge $\lambda_i \in T_j$ and $\lambda_i \in T_k$ respectively.
- $\iota(X_i) = [X_{j_{i_1}} X_{j_{i_2}}]$ if λ_i corresponds to two sides of the same face T_j and $X_{j_{i_1}}, X_{i_{j_2}} \in \mathcal{T}_j$ are the associated generators.

Definition 2.7. A *local representation* of $\mathcal{T}_q(\lambda)$ is a representation which factorizes as $(\rho_1 \otimes \cdots \otimes \rho_m) \circ \iota$ where $\rho_i : \mathcal{T}_i \to V_i$ is an irreducible representation of the triangle algebra $\mathcal{T}_i$ and $\iota : \mathcal{T}_q(\lambda) \to \mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_m$ is defined as above.

In particular, a local representation has dimension N^m where $m = 4g - 4 + 2s$ is the number of faces of the triangulation.

Local representations were first introduced by Bai et al. [2007].

Theorem 2.8 [Bai et al. 2007, Proposition 6]. *Up to isomorphism, a local representation*

$$\rho : \mathcal{T}_q(\lambda) \to \mathrm{End}(V)$$

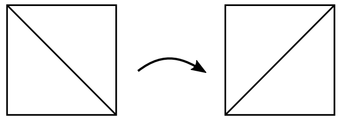

Figure 2. Flip of triangulation.

is classified by a nonzero complex number x_i associated to the edge λ_i and a choice of an N-th root $c = (x_1 \dots x_n)^{1/N}$. Such a representation satisfies

- $\rho(X_i^N) = x_i \,\mathrm{Id}_V$,
- $\rho(H) = c\,\mathrm{Id}_V$.

Local representations have certain advantages over irreducible representations. First of all, these representations behave very well under cut and paste, so one can use them to construct invariant of 3-manifolds; see [Baseilhac and Benedetti 2005]. Also, the vector space associated to a local representation decomposes as a tensor product of vector spaces and each generator $X_i \in \mathcal{T}_q(\lambda)$ associated to an edge λ_i only acts on the vector spaces associated to triangle adjacent to the edge λ_i (that is why these representations are called local).

2.3. ***Quantum Teichmüller space and its representations.*** The quantum Teichmüller space is obtained by gluing together a family of (division algebras of) Chekhov–Fock algebras indexed by the set of ideal triangulations of Σ.

The simplex of ideal triangulations. Let $\Lambda(\Sigma)$ be the set of ideal triangulations of Σ. We say that two triangulations $\lambda, \lambda' \in \Lambda(\Sigma)$ differ by a flip if λ and λ' coincide everywhere except in a square made of two adjacent triangles where they differ as in Figure 2.

The *graph of ideal triangulations* is the graph whose set of vertices is $\Lambda(\Sigma)$ and two vertices $\lambda, \lambda' \in \Lambda(\Sigma)$ are connected by an edge if and only if λ and λ' differ by a flip.

The *simplex of ideal triangulations* is obtained from the graph of ideal triangulations by gluing a 2-simplex on each cycle corresponding to the pentagon relation (see Figure 3).

Proposition 2.9 [Penner 1993]. *The simplex of ideal triangulations is connected and simply connected. Namely, any two different ideal triangulations are connected by a sequence of flips and two sequences between two ideal triangulations differ by a sequence of pentagon relations.*

Coordinate change. The Chekhov–Fock algebra $\mathcal{T}_q(\lambda)$ associated to an ideal triangulation $\lambda \in \Lambda(\Sigma)$ satisfies the Ore condition; see [Goodearl and Warfield 2004].

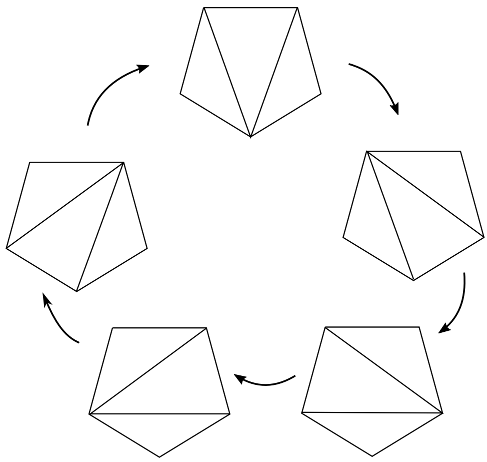

Figure 3. Pentagon relation.

In particular, $\mathcal{T}_q(\lambda)$ has a well-defined division algebra $\hat{\mathcal{T}}_q(\lambda)$ consisting of rational fractions satisfying some noncommutativity relations.

Let $\lambda, \lambda' \in \Lambda(\Sigma)$ be two ideal triangulations related by a flip. Chekhov and Fock [1999] constructed coordinates change isomorphisms

$$\Psi^q_{\lambda\lambda'} : \hat{\mathcal{T}}_q(\lambda') \to \hat{\mathcal{T}}_q(\lambda).$$

These coordinates change satisfy the pentagon relation. In particular, using the result of Penner, they extend uniquely to coordinates change $\Psi^q_{\lambda\lambda'} : \hat{\mathcal{T}}_q(\lambda') \to \hat{\mathcal{T}}_q(\lambda)$ for any two different ideal triangulations $\lambda, \lambda' \in \Lambda(\Sigma)$.

It was proved in [Liu 2009] that these coordinates change are the unique ones satisfying some natural relations, as for instance $\Psi^q_{\lambda\lambda''} = \Psi^q_{\lambda\lambda'} \circ \Psi^q_{\lambda'\lambda''}$ for any $\lambda, \lambda', \lambda'' \in \Lambda(\Sigma)$. Moreover, when $q = 1$, these maps reduce to the classical coordinates change in Teichmüller theory; see [loc. cit.] for more details.

Definition 2.10. The *quantum Teichmüller space of* Σ is defined by

$$\mathcal{T}_q(\Sigma) := \bigsqcup_{\lambda \in \Lambda(\Sigma)} \hat{\mathcal{T}}_q(\lambda) / \sim,$$

where $x_\lambda \in \hat{\mathcal{T}}_q(\lambda) \sim x_{\lambda'} \in \hat{\mathcal{T}}_q(\lambda')$ if and only if $x_\lambda = \Psi^q_{\lambda\lambda'}(x_{\lambda'})$.

Note that, as Since each coordinate change $\Psi^q_{\lambda\lambda'}$ is an algebra isomorphism, $\mathcal{T}_q(\Sigma)$ inherits an algebra structure, and the $\hat{\mathcal{T}}_q(\lambda)$ can be thought as "global coordinates" on $\mathcal{T}_q(\Sigma)$.

Representations. A natural definition for a finite dimensional representation of $\mathcal{T}_q(\Sigma)$ would be a family of finite dimensional representations

$$\{\rho_\lambda : \hat{\mathcal{T}}_q(\lambda) \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(\Sigma)}$$

such that for each pair $\lambda, \lambda' \in \Lambda(\Sigma)$, the representations $\rho_{\lambda'}$ and $\rho_\lambda \circ \Psi^q_{\lambda,\lambda'}$ of $\hat{\mathcal{T}}_q(\lambda')$ are isomorphic (as representations).

However, as pointed out in [Bai et al. 2007, Section 4.2], when V_λ is finite-dimensional, there is no morphism $\hat{\mathcal{T}}_q(\lambda) \to \mathrm{End}(V_\lambda)$. In fact, $\hat{\mathcal{T}}_q(\lambda)$ is infinite-dimensional as a vector space while $\mathrm{End}(V_\lambda)$ is finite-dimensional and so, such a homomorphism ρ_λ would have nonzero kernel. In particular, there would exists elements $x \in \hat{\mathcal{T}}_q(\lambda)$ such that $\rho_\lambda(x) = 0$ and so, $\rho_\lambda(x^{-1})$ would make no sense.

This motivates the following definition:

Definition 2.11. A *local representation* (respectively an *irreducible representation*) of $\mathcal{T}_q(\Sigma)$ is a family of representations

$$\{\rho_\lambda : \mathcal{T}_q(\lambda) \to \mathrm{End}(V_\lambda)\}_{\lambda \in \Lambda(\Sigma)}$$

such that for each $\lambda, \lambda' \in \Lambda(\Sigma)$, ρ_λ is a local representation (respectively an irreducible representation) of $\mathcal{T}_q(\lambda)$, and $\rho_{\lambda'}$ is isomorphic to $\rho_\lambda \circ \Psi^q_{\lambda\lambda'}$ whenever $\rho_\lambda \circ \Psi^q_{\lambda\lambda'}$ makes sense.

We say that $\rho_\lambda \circ \Psi^q_{\lambda\lambda'}$ makes sense, if for each Laurent polynomial $X' \in \mathcal{T}_q(\lambda')$,

$$\Psi_{\lambda\lambda'}(X') = PQ^{-1} = Q'^{-1}P' \in \hat{\mathcal{T}}_q(\lambda), \quad \text{for some } P, Q, P', Q' \in \mathcal{T}_q(\lambda).$$

In that case, we define

$$\rho_\lambda \circ \Psi_{\lambda\lambda'}(X') := \rho_\lambda(P)\rho_\lambda(Q)^{-1} = \rho_\lambda(Q')^{-1}\rho_\lambda(P').$$

Proposition 2.12 [Bai et al. 2007, Proposition 10]. *Let $\lambda, \lambda' \in \Lambda(\Sigma)$ be two ideal triangulations of Σ. Then there exists a rational map*

$$\varphi_{\lambda\lambda'} : \mathbb{C}^n \to \mathbb{C}^n$$

such that a local representation $\rho' : \mathcal{T}_q(\lambda') \to \mathrm{End}(V_{\lambda'})$ classified by $(x'_1, \dots, x'_n)$ and $c' = \left(x'_1 \dots x'_n\right)^{1/N}$ is isomorphic to $\rho_\lambda \circ \Psi_{\lambda\lambda'}$ (whenever it makes sense) where $\rho_\lambda : \mathcal{T}_q(\lambda) \to \mathrm{End}(V_\lambda)$ is a local representation classified by $(x_1, \dots, x_n)$ and $c = (x_1 \dots x_n)^{1/N}$ if and only if $c = c'$ and

$$(x'_1, \dots, x'_n) = \varphi_{\lambda\lambda'}(x_1, \dots, x_n).$$

Remark 2.13. The analogue is also proved in [Bonahon and Liu 2007] for irreducible representations. In particular, the rational maps $\varphi_{\lambda\lambda'}$ are the same.

It turns out that the rational maps $\varphi_{\lambda\lambda'}$ correspond to the coordinates change of the shear-bend coordinates on the character variety $\chi\big(\Sigma, \mathrm{SL}(2,\mathbb{C})\big)$.

As a result, isomorphism classes of local (respectively irreducible) representations of $\mathcal{T}_q(\Sigma)$ are classified, up to finitely many choices, by the character variety $\chi\big(\Sigma, \mathrm{SL}(2,\mathbb{C})\big)$; see [loc. cit.] for more details.

3. Decomposition of local representations

In this section, we prove the Main Theorem. Let $\rho : \mathcal{T}_q(\lambda) \to \mathrm{End}(V)$ be the local representation classified by the nonzero complex number x_i associated to each edge and the central charge c. Given a puncture invariant $P_j = [X_1^{k_{j1}} \dots X_n^{k_{jn}}]$ (see Proposition 2.4) associated to the puncture v_j, the representation ρ satisfies

$$\rho(P_j^N) = x_1^{k_{j1}} \dots x_n^{k_{jn}} \,\mathrm{Id}_V\,.$$

It follows that if p_j is an eigenvalue of P_j, then $p_j^N = x_1^{k_{j1}} \dots x_n^{k_{jn}}$.

Notation.

- Given $p_j \in \mathbb{C}^*$ so that $p_j^N = x_1^{k_{j1}} \dots x_n^{k_{jn}}$, we denote by

  $$V_{p_j}(P_j) := \{x \in V : \rho(P_j)x = p_j x\}$$

  the associated eigenspace.
- Given $\boldsymbol{p} = (p_1, \dots, p_s)$ so that for each j, $p_j^N = x_1^{k_{j1}} \dots x_n^{k_{jn}}$, set

  $$V_{\boldsymbol{p}} := \{x \in V : \rho(P_j)x = p_j x,\ j = 1, \dots, s\} = \bigcap_{j=1}^{s} V_{p_j}(P_j).$$

The proof of the Main Theorem will follow the next proposition:

Proposition 3.1. *There exists an ideal triangulation $\lambda_0 \in \Lambda(\Sigma)$ such that for each $\boldsymbol{p}$ as above, $V_{\boldsymbol{p}}$ has dimension N^{4g-3+s}.*

Proof. The dimension of $V_{\boldsymbol{p}}$ does not depend on the numbers $x_i \in \mathbb{C}^*$ characterizing ρ. In this proof, we will consider all the x_i equal to 1, which implies that the eigenvalues of $\rho(P_i)$ are root of unity.

Consider the one punctured surface $\Sigma' := \Sigma \cup \{v_1, \dots, v_{s-1}\}$. As $g > 0$, there exists an ideal triangulation λ' of Σ'. Let T be a triangle of the triangulation λ' and consider the triangulation of $T \setminus \{v_1, \dots, v_{s-1}\}$ as in Figure 4.

The union of the triangulation λ' and the one of T gives an ideal triangulation λ_0 of Σ.

Consider a local representation $\rho : \mathcal{T}_q(\lambda_0) \to \mathrm{End}(V)$. The decomposition of the ideal triangulation λ_0 gives the nice decomposition

$$V = W \otimes W',$$

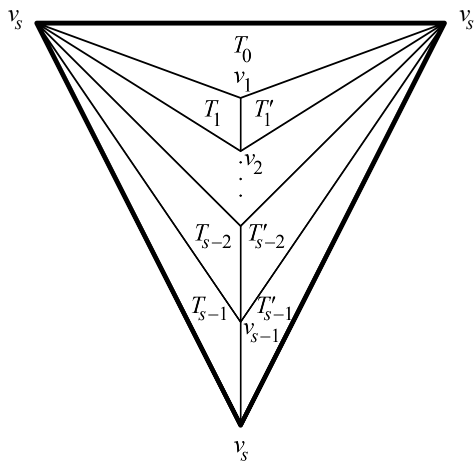

Figure 4. Triangulation of $T \cup \{v_1, \dots, v_s\}$.

where W' is the vector space corresponding to the triangles of the triangulation λ' (except the triangle T), and W corresponds to the triangles of T.

In particular, as the triangulation λ' contains $4g-2$ triangles, $\dim(W') = N^{4g-3}$ (remember that we do not consider the vector space associated to T), and $\dim W = N^{2s-1}$.

The interest of the triangulation λ_0 is clear: the puncture invariant P_i associated to the puncture $v_i \neq v_s$ acts as the identity on W', so the eigenspaces of $\rho(P_i)$ has the form $E \otimes W'$ where $E \subset W$ is an eigenspace of the restriction of $\rho(P_i)$ on W. It motivates the following notation:

Notation.

- The vector space W decomposes as

$$W = W^0 \otimes \cdots \otimes W^{s-1},$$

  where W^0 is associated to T_0 and W^j to T_j and T_j' for $j = 1, \dots, s-1$.

- Given a root of unity $p_k \in \mathcal{U}(N)$, set

$$W^j_{p_k}(P_k) := \{x \in W^j : \rho(P_k)x = p_k x\}.$$

- For $\boldsymbol{p} = (p_1, \dots, p_{s-1}) \in \mathcal{U}(N)^{s-1}$, set

$$W^j_{\boldsymbol{p}} = \{x \in W^j : \rho(P_k)x = p_k x,\ k = 1, \dots, s-1\} = \bigcap_{k=1}^{s-1} W^j_{p_k}(P_k).$$

- Finally, set

$$W_{\boldsymbol{p}} = \{x \in W : \rho(P_k)x = p_k x,\ k = 1, \dots, s-1\}.$$

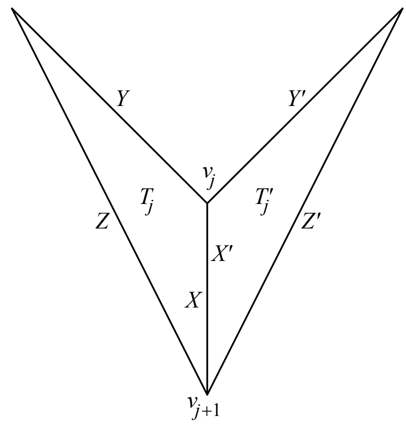

Figure 5. The generators of $\mathcal{T}_j$ and $\mathcal{T}_j'$.

Lemma 3.2. *Using the above notation, and given* $\boldsymbol{p} \in \mathcal{U}(N)^{s-1}$, *we have the following*:

(1) $\dim W^0_{\boldsymbol{p}} = \begin{cases} 1 & \textit{if } \boldsymbol{p} = (p_1, 1, \dots, 1), \\ 0 & \textit{otherwise.} \end{cases}$

(2) *For* $j \in \{1, \dots, s-2\}$,

$$\dim W^j_{\boldsymbol{p}} = \begin{cases} 1 & \textit{if } \boldsymbol{p} = (1, \dots, 1, p_j, p_{j+1}, 1, \dots, 1), \\ 0 & \textit{otherwise.} \end{cases}$$

(3) $\dim W^{s-1}_{\boldsymbol{p}} = \begin{cases} N & \textit{if } \boldsymbol{p} = (1, \dots, 1, p_{s-1}), \\ 0 & \textit{otherwise.} \end{cases}$

Proof. (1) If $k \neq 1$, v_k is not a vertex of T_0. It follows that P_k acts on W^0 by the identity; so if $p_k \neq 1$, $W^0_{\boldsymbol{p}} = \{0\}$.

Now, if $p_k = 1$ for all $k \neq 1$, then $W^0_{\boldsymbol{p}}$ is the eigenspace of the action on W^0 of the edge opposite to v_1. By Lemma 2.6, it is one-dimensional.

(2) Fix $j \in \{1, \dots, s-2\}$. For $k \notin \{j, j+1\}$, v_k is neither a vertex of T_j nor of T_j'. Hence P_j acts on W^j as the identity, and if $p_k \neq 1$, then $W^j_{\boldsymbol{p}} = \{0\}$.

Take $p_k = 1$ for all $k \notin \{j, j+1\}$ and denote by $X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}$ (respectively $X'^{\pm 1}, Y'^{\pm 1}, Z'^{\pm 1}$) the generators of the triangle algebras $\mathcal{T}_j$ (respectively $\mathcal{T}_j'$) associated to the triangles T_j (respectively $T'{}_j$) as in Figure 5. Set also $W^j = V^j \otimes V'^j$ where V^j (respectively V'^j) is the vector space associated to the representation of the triangle algebra $\mathcal{T}_j$ (respectively $\mathcal{T}_j'$).

Denote by $c_j, c_j' \in \mathcal{U}(N)$ the central charges of the restriction of the representation to $\mathcal{T}_j$ and $\mathcal{T}_j'$ respectively. Then $\rho(P_j)$ acts on $V^j := \operatorname{span}\{e_0, \dots, e_{N-1}\}$ like $c_j Z^{-1}$ and on $V'^j = \operatorname{span}\{e_0', \dots, e_{N-1}'\}$ like $c_j' Z'^{-1}$. In the same way, $\rho(P_{j+1})$ acts on V_j like $c_j Y^{-1}$ and on V_j' like $c_j' Y'^{-1}$.

Using the same action as in Example 2.2 and writing

$$c_j = q^p, \quad c'_j = q^{p'},$$

we get

$$\rho(P_j)e_k = q^{2k-1+p}e_{k+1}, \quad \rho(P_j)e'_l = q^{1-2l+p'}e_{l+1}$$

It follows that the action of P_j on W^j is given by

$$P_j\epsilon_{k,l} = q^{2(k-l)+p+p'}\epsilon_{k+1,l+1} \text{ where } \epsilon_{k,l} := e_k \otimes e'_l \in W^j.$$

In the same way, one sees that the action of P_{j+1} on W^j is given by

$$P_{j+1}\epsilon_{k,l} = q^{p+p'}\epsilon_{k-1,l-1}.$$

Now, for $m, n \in \mathbb{Z}_N$, set $\alpha_{m,n} := \sum_{k=0}^{N-1} q^{2km}\epsilon_{k,k+n}$, an easy calculation shows that

$$P_j\alpha_{m,n} = q^{-2(m+n)+p+p'}\alpha_{m,n}, \quad P_{j+1}\alpha_{m,n} = q^{2m+p+p'}\alpha_{m,n}.$$

It follows that $\{\alpha_{n,m} : n, m \in \mathbb{Z}_N\}$ is a base of W^j and, for all $p_j, p_{j+1} \in \mathcal{U}(N)$, there exists a unique couple $(m, n) \in \mathbb{Z}_N^2$ with $p_j = q^{-2(m+n)+p+p'}$ and $p_{j+1} = q^{2m+p+p'}$.

Therefore, $\dim W_{\boldsymbol{p}}^j = 1$ if and only if $p_k = 1$ for all $k \neq j, j+1$.

(3) If $k \neq s-1$, then v_k is neither a vertex of T_{s-1} nor a vertex of T'_{s-1}, so if $p_k \neq 1$, then $W_{\mathbf{h}}^s = \{0\}$.

Suppose that $p_k = 1$ for all $k \in \{1, \ldots, s-2\}$, then

$$W_{\boldsymbol{p}}^{s-1} \supset \bigoplus_{p_a p_b = p_{s-1}} V_{p_a}^{s-1}(P_{s-1}) \otimes V_{p_b}^{\prime s-1}(P_{s-1}),$$

where $V_{p_a}^s(P_{s-1})$ is the eigenspace associated to the eigenvalue p_a of the action of $\rho(P_{s-1})$ on the vector space associated to the triangle T_{s-1}, and $V_{p_b}^{\prime s-1}(P_{s-1})$ is defined in an analogous way.

The direct sum contains N terms of dimension one, hence $\dim(W_{\boldsymbol{p}}^{s-1}) \geq N$. On the other hand, we have

$$\dim(W^{s-1}) = N^2$$

and also

$$\dim(W^{s-1}) = \sum_{\boldsymbol{p} \in \mathcal{U}(N)^{s-1}} \dim(W_{\boldsymbol{p}}^{s-1}) \geq N \times N.$$

This implies that $W_{\boldsymbol{p}}^{s-1}$ has exactly dimension N for $\boldsymbol{p} = (1, \ldots, 1, p_{s-1})$. $\square$

The proof of Proposition 3.1 follows from the following elementary remark:

Remark 3.3. For all $j \in \{0, \dots, s-1\}$, given $\boldsymbol{p}_j \in \mathcal{U}(N)^{s-1}$ and a nonzero vector $x_j \in W^j_{\boldsymbol{p}_j}$, the vector $x_0 \otimes \cdots \otimes x_{s-1}$ is in $W_{\boldsymbol{p}}$ where $\boldsymbol{p} = \boldsymbol{p}_0 \boldsymbol{p}_1 \dots \boldsymbol{p}_{s-1}$ is obtained by taking the product on each component.

We thus have the inclusion

$$W_{\boldsymbol{p}} \supset \bigoplus_{\boldsymbol{p}=\boldsymbol{p}_0 \cdots \boldsymbol{p}_{s-1}} W^0_{\boldsymbol{p}_0} \otimes \cdots \otimes W^{s-1}_{\boldsymbol{p}_{s-1}}. \tag{1}$$

Writing $\boldsymbol{p}_j = \left(p_0^{(j)}, \dots, p_{s-1}^{(j)}\right)$ and $\boldsymbol{p} = (p_1, \dots, p_s)$, one notes that from Lemma 3.2, the only nonzero terms in the direct sum of (1) are the $\boldsymbol{p}_j$ satisfying

$$\begin{aligned} p_1^{(0)} p_1^{(1)} &= p_1, \\ p_2^{(1)} p_2^{(2)} &= p_2, \\ &\vdots \\ p_{s-1}^{(s-2)} p_{s-1}^{(s-1)} &= p_{s-1} \end{aligned} \tag{2}$$

There exists exactly N^{s-1} different choices for the $\boldsymbol{p}_j$ satisfying relations (2).

Moreover, for each choice of $\boldsymbol{p}_j$ satisfying (2), the vector space $W^0_{\boldsymbol{p}_0} \otimes \cdots \otimes W^{s-1}_{\boldsymbol{p}_{s-1}}$ has dimension N. It follows that for each $\boldsymbol{p} \in \mathcal{U}(N)^{s-1}$,

$$\dim W_{\boldsymbol{p}} \geq N^s.$$

On the other hand,

$$\dim W = N^{2s-1} = \sum_{\boldsymbol{p} \in \mathcal{U}(N)^{s-1}} \dim W_{\boldsymbol{p}},$$

hence each $W_{\boldsymbol{p}}$ has exactly dimension N^s.

Finally, as the puncture invariants act as the identity on the vector space W', the intersection of the eigenspaces of the $\rho(P_j)$ for all $j = 1, \dots, s-1$ has the form $W_{\boldsymbol{p}} \otimes W'$ for some $\boldsymbol{p} \in \mathcal{U}(N)^{s-1}$ and has dimension N^{4g-3+s}. As the representation ρ has fixed central charge c,

$$\rho([P_1 P_2 \dots P_s]) = \rho([H^2]) = c^2 \operatorname{Id}_V .$$

It follows that the action of $\rho(P_s)$ on V can be easily expressed as a function of the action of the $\rho(P_j)$ for $j = 1, \dots, s-1$, and we get the result. $\square$

Proposition 3.1 implies the decomposition of the Main Theorem for the triangulation λ_0. Since the dimension of the eigenspaces depends continuously on the x_i, we get the decomposition for all value of $x_i \in \mathbb{C}^*$.

Indeed, consider the local representation

$$\rho : \mathcal{T}_q(\lambda_0) \to \operatorname{End}(V)$$

classified by a nonzero complex number x_i associated to each edge and central charge c. Let $\rho^{(i)} : \mathcal{T}_q(\lambda_0) \to \mathrm{End}(V^{(i)})$ be an irreducible factor.

In particular,

$$\rho^{(i)}(X_i^N) = \rho(X_i^N)_{|V^{(i)}} = x_i \,\mathrm{Id}_{V^{(i)}},$$
$$\rho^{(i)}(H) = \rho(H)_{|V^{(i)}} \quad = c \,\mathrm{Id}_{V^{(i)}},$$

so a necessary condition for $\rho^{(i)}$ to be an irreducible factor is that it is classified by the same x_i and same central charge c.

For each puncture v_j, denote by $p_j^{(i)}$ the N-th root of $x_1^{k_{j1}} \dots x_n^{k_{jn}}$ so that

$$\rho^{(i)}(P_j) = p_j^{(i)} \,\mathrm{Id}_{V^{(i)}} .$$

It follows that $p_s^{(i)} = c^2 \big(p_1^{(i)} \dots p_{s-1}^{(i)}\big)^{-1}$ and $V^{(i)} \subset V_{\boldsymbol{p}}$ where $\boldsymbol{p} = \big(p_1^{(i)}, \dots, p_{s-1}^{(i)}\big)$ with

$$V_{\boldsymbol{p}} = \big\{x \in V : \rho(P_j)x = p_j^{(i)} x,\ j = 1, \dots, s-1\big\}.$$

Finally, as $\dim V_{\boldsymbol{p}} = N^{4g-3+s}$ and the dimension of an irreducible representation of $\mathcal{T}_q(\lambda_0)$ has dimension N^{3g-3+s}, $V_{\boldsymbol{p}}$ contains exactly N^g irreducible factors classified by the same x_i, same central charge c and N-the root $p_j^{(i)}$ associated to the puncture v_j.

Proof in the general case. Recall that, given another ideal triangulation $\lambda \in \Lambda(\Sigma)$, the "transition maps" $\varphi_{\lambda_0\lambda} : \mathbb{C}^n \to \mathbb{C}^n$ defined in Section 2.3 are rational, hence defined on a Zariski open set of $\mathbb{C}^n$.

Now, consider a local representation

$$\rho : \mathcal{T}_q(\lambda) \to \mathrm{End}(V_\lambda),$$

classified by a number $x_i \in \mathbb{C}^*$ associated to each edge and central charge c.

If there exists $(y_1, \dots, y_n) \in \mathbb{C}^n$ so that $\varphi_{\lambda_0\lambda}(y_1, \dots, y_n) = (x_1, \dots, x_n)$ (which is a generic condition), then it follows from Section 2.3 that ρ_λ is isomorphic to $\rho_{\lambda_0} : \mathcal{T}_q(\lambda_0) \to \mathrm{End}(V_{\lambda_0})$. It means that there exists an isomorphism

$$L_{\lambda_0\lambda} : V_\lambda \to V_{\lambda_0},$$

so that for each $X \in \mathcal{T}_q(\lambda)$ we have

$$\rho_{\lambda_0}(\Psi^q_{\lambda_0\lambda}(X)) = L_{\lambda_0\lambda} \circ \rho_\lambda(X) \circ L^{-1}_{\lambda_0\lambda}.$$

Here $\Psi^q_{\lambda_0\lambda} : \hat{\mathcal{T}}_q(\lambda) \to \hat{\mathcal{T}}_q(\lambda_0)$ are the coordinates change defined in Section 2.3.

As ρ_{λ_0} is a local representation of $\mathcal{T}_q(\lambda_0)$, there exists an irreducible decomposition of ρ_{λ_0} given by the decomposition

$$V_{\lambda_0} = \bigoplus_{i \in \mathcal{I}} V^i_{\lambda_0}.$$

In particular, each $i \in \mathcal{I}$, $V^i_{\lambda_0}$ is stable by ρ_{λ_0} and has dimension N^{3g-3+s}.

For each $i \in \mathcal{I}$, set $V^i_\lambda := L^{(-1)}_{\lambda_0\lambda}(V^i_{\lambda_0})$. One easily gets that each V^i_λ is stable by $\rho_\lambda(\mathcal{T}_q(\lambda))$, and has dimension N^{3g-3+s}. In this way we get a decomposition of ρ_λ into irreducible factors given by the decomposition

$$V_\lambda = \bigoplus_{i \in \mathcal{I}} V^i_\lambda.$$

Finally, if ρ_λ is classified by the parameters $(x_1, \dots, x_n)$ which are not in the image of $\varphi_{\lambda_0\lambda}$, one can deform continuously $(x_1, \dots, x_n)$ to get the previous decomposition and, as the decomposition does not depend of the parameters, get the result for ρ_λ.

4. Representations of the skein algebra

In this section, we use the Main Theorem to construct a nice family of representation of the Kauffman bracket skein algebra $\mathcal{S}_A(\overline{\Sigma})$ of the closed surface $\overline{\Sigma} = \Sigma \cup \{v_1, \dots, v_s\}$. This is done by adapting the construction of Bonahon and Wong [2015] to the case of local representations.

In Section 4.1, we describe the balanced Chekhov–Fock algebra $\mathcal{Z}_\omega(\lambda)$ associated to an ideal triagulation λ of Σ and characterize its irreducible representations. Then, in Section 4.2, we introduce the local representations of $\mathcal{Z}_\omega(\lambda)$ and extend the Main Theorem to decompose these local representations into irreducible factors. Finally, in Section 4.3, we use the previous decomposition to construct a family of representations of $\mathcal{S}_A(\overline{\Sigma})$.

4.1. ***Balanced Chekhov–Fock algebra.*** Let q be a primitive N-th root of unity with N odd and let ω be the unique fourth root of q which is also a primitive N-th root of unity. Namely, if $q = e^{2i\pi \frac{k}{N}}$ with N and k coprime, then there is a unique $l \in \mathbb{Z}_4$ so that $k + lN \in 4\mathbb{Z}$, and $\omega = e^{2i\pi \frac{k}{4N}} e^{i\frac{l\pi}{2}}$.

Let λ be an ideal triangulation of Σ whose edges are $(\lambda_1, \dots, \lambda_n)$. In order to avoid confusion, we will denote by X_i the generators of $\mathcal{T}_q(\lambda)$ and by Z_i the generators of $\mathcal{T}_\omega(\lambda)$.

A multi-index $\mathbf{k} \in \mathbb{Z}^n$ is called λ-*balanced* (or *balanced*) if for each triangle of the triangulation whose edges are j_1, j_2, j_3 we have

$$k_{j_1} + k_{j_2} + k_{j_3} \in 2\mathbb{Z}.$$

A monomial $Z \in \mathcal{T}_\omega(\lambda)$ is *balanced* if Z is a scalar multiple of $Z_{\mathbf{k}}$ where $\mathbf{k} \in \mathbb{Z}^n$ is balanced. (Here $Z_{\mathbf{k}}$ is defined as in Section 2.1).

Definition 4.1. The *balanced Chekhov–Fock algebra* $\mathcal{Z}_\omega(\lambda)$ is the subalgebra of $\mathcal{T}_\omega(\lambda)$ generated (as a vector space) by balanced monomials.

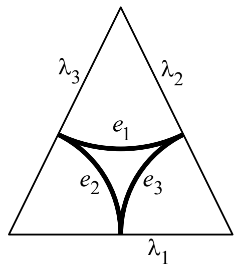

Figure 6. Train track.

In particular, the image of the map

$$\begin{array}{rcl} i: \mathcal{T}_q(\lambda) & \to & \mathcal{T}_\omega(\lambda) \\ X_i & \mapsto & Z_i^2 \end{array}$$

lies in $\mathcal{Z}_\omega(\lambda)$ so we will consider $\mathcal{T}_q(\lambda)$ as a subalgebra of $\mathcal{Z}_\omega(\lambda)$.

The ideal triangulation λ defines canonically a train track τ_λ on Σ which looks like in Figure 6 on each triangle of the triangulation. Note that τ_λ has a switch on each edge of λ.

We denote by $\mathcal{W}(\tau_\lambda, \mathbb{Z})$ the abelian group of integer weight systems on τ_λ. Namely, an element $\alpha \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$ is a map that associates to each edge e of τ_λ an integer $\alpha(e)$ in such a way that at each switch, the sum of the weights of the incoming edges equals the sum of the weights of the outgoing edges.

A *weight system* $\alpha \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$ on τ_λ is a map that associates an integer to any edge of the train track in such a way that the sum of weights of the incoming edges equals the sum of weights of the outgoing edges. Given $\alpha \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$ and an edge $\lambda_i \in \lambda$, the sum of the weights of the edges incoming to λ_i is an integer. It thus define a map

$$\varphi : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \to \mathbb{Z}^n$$

whose image is exactly the set of balanced multi-index. Thus, given an integer weight system $\alpha \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$, we define $Z_\alpha \in \mathcal{Z}_\omega(\lambda)$ to be $Z_{\varphi(\alpha)} = [Z_1^{\alpha_1} \dots Z_n^{\alpha_n}]$ where $\varphi(\alpha) = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$. In particular, one gets the noncommutativity relations

$$Z_\alpha Z_\beta = \omega^{4\Omega(\alpha,\beta)} Z_\beta Z_\alpha,$$

where $\Omega : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \times \mathcal{W}(\tau_\lambda, \mathbb{Z}) \to \mathbb{Z}$ is the Thurston intersection form; see [Bonahon and Wong 2012, Section 2] for more details.

Definition 4.2. A *twisted homomorphism* is a map $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \to \mathbb{C}^*$ such that for every $\alpha, \beta \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$,

$$\zeta(\alpha + \beta) = (-1)^{\Omega(\alpha,\beta)} \zeta(\alpha)\zeta(\beta).$$

Finally, note that each puncture v_j defines a integer weight system $\eta_j \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$ as follow. The connected component D_j of $\Sigma \setminus \tau_\lambda$ containing v_j is bounded by a finite number of edges of τ_λ. For an edge e of τ_λ, define $\eta_j(e) \in \{0, 1, 2\}$ to be the number of times e lies in the boundary of D_j. In particular,

$$Z_{\eta_j}^2 = i(P_j)$$

where $P_j \in \mathcal{T}_q(\lambda)$ is the puncture invariant associated to v_j and $i : \mathcal{T}_q(\lambda) \to \mathcal{Z}_\omega(\lambda)$ is defined above.

Irreducible representations of $\mathcal{Z}_\omega(\lambda)$ were classified in [Bonahon and Wong 2015]. They proved:

Proposition 4.3 [Bonahon and Wong 2012, Proposition 14]. *Up to isomorphism, an irreducible representation $\rho : \mathcal{Z}_\omega(\lambda) \to \mathrm{End}(V)$ has dimension N^{3g-3+s} and is classified by a twisted homomorphism $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \to \mathbb{C}^*$ and a choice of an N-th root $h_j = \zeta(\eta_j)^{1/N}$ for each puncture v_j. Such a representation satisfies*:

- $\rho(Z_\alpha^N) = \zeta(\alpha)\,\mathrm{Id}_V$ *for all* $\alpha \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$.
- $\rho(\eta_j) = \zeta(\eta_j)\,\mathrm{Id}_V$ *for all* $j \in \{1, \dots, s\}$.

4.2. ***Local representations of $\mathcal{Z}_\omega(\lambda)$.*** Here, we introduce the notion of local representation of the balanced Chekhov–Fock algebra $\mathcal{Z}_\omega(\lambda)$ associated to an ideal triangulation λ. We then extend the Main Theorem to give a decomposition of local representations of $\mathcal{Z}_\omega(\lambda)$ into its irreducible components.

Since by our choice ω is also a primitive N-th root of unity, there is a map

$$\mathcal{T}_\omega(\lambda) \to \bigotimes_{T_i \in F(\lambda)} \mathcal{T}_\omega(T_i)$$

as introduced in Section 2.2, where $F(\lambda)$ is the set of faces of λ and $\mathcal{T}_\omega(T_i)$ is the triangle algebra associated to the face T_i. It is clear that this map restricts to a morphism

$$\iota : \mathcal{Z}_\omega(\lambda) \to \bigotimes_{T_i \in F(\lambda)} \mathcal{Z}_\omega(T_i).$$

Here, $\mathcal{Z}_\omega(T_i)$ is the *balanced triangle algebra* associated to the face T_i.

Definition 4.4. A *local representation of the balanced Chekhov–Fock algebra* $\mathcal{Z}_\omega(\lambda)$ is a representation $\rho : \mathcal{Z}_\omega(\lambda) \to \mathrm{End}(V)$ that can be written as $(\rho_1 \otimes \cdots \otimes \rho_m) \circ \iota$ where each ρ_i is an irreducible representation of $\mathcal{Z}_\omega(T_i)$.

In order to classify local representations of $\mathcal{Z}_\omega(\lambda)$, we first have to understand the irreducible representations of the balanced triangle algebra $\mathcal{Z}_\omega(T)$. Let τ be the train track in T with edges e_1, e_2, e_3 as in Figure 6 and denote by $\mathcal{W}(\tau, \mathbb{Z})$ The group of integer weight systems on τ.

Lemma 4.5. *Up to isomorphism, an irreducible representation of the balanced triangle algebra $\mathcal{Z}_\omega(T)$ has dimension N and is classified by a twisted homomorphism $\zeta : \mathcal{W}(\tau, \mathbb{Z}) \to \mathbb{C}^*$ together with a choice of an N-th root $C = \big(\zeta(\mu)\big)^{1/N}$ where $\mu \in \mathcal{W}(\tau, \mathbb{Z})$ is such that $\mu(e_i) = 1$ for all $i \in \mathbb{Z}_3$. Such a representation satisfies*

- $\rho(Z_\alpha^N) = \zeta(\alpha)\,\mathrm{Id}_V$.
- $\rho(Z_\mu) = C\,\mathrm{Id}_V$.

Proof. The group $\mathcal{W}(\tau, \mathbb{Z})$ is generated by $\{\alpha_1, \alpha_2, \alpha_3\}$ where

$$\alpha_i(e_j) = \delta_{ij}, \quad i, j \in \mathbb{Z}_3.$$

It follows that the balanced triangle algebra $\mathcal{Z}_\omega(T)$ is generated by $Z_{\alpha_1}^{\pm 1}$, $Z_{\alpha_2}^{\pm 1}$ and $Z_{\alpha_3}^{\pm 1}$ and the relations are

$$Z_{\alpha_1} Z_{\alpha_{i+1}} = \omega^{-2} Z_{\alpha_{i+1}} Z_{\alpha_i}, \quad i \in \mathbb{Z}_3.$$

If we denote by Z_i the generator of $\mathcal{T}_\omega(T)$ associated to the edge λ_i (so for instance $Z_{\alpha_1} = [Z_2 Z_3]$), the map

$$\Psi : \mathcal{Z}_\omega(T) \to \mathcal{T}_\omega(T), \quad Z_{\alpha_i} \mapsto Z_i^{-1}$$

is an isomorphism of algebras such that $\Psi(Z_\mu) = H^{-1}$ where $H = [Z_1 Z_2 Z_3]$.

In particular, an irreducible representation ρ of $\mathcal{Z}_\omega(\lambda)$ has the form $\rho = \overline{\rho} \circ \Psi$ where $\overline{\rho}$ is an irreducible representation of $\mathcal{T}_\omega(T)$.

Using the result of Section 2.2 and the fact that a twisted homomorphism of $\mathcal{W}(\tau, \mathbb{Z})$ is fully determined by its value on the α_i, we get the result. □

Let τ_i be the restriction of the train track τ_λ to the triangle T_i of the triangulation λ. A twisted homomorphism $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \to \mathbb{C}^*$ induces a twisted homomorphism $\zeta_i : \mathcal{W}(\tau_i, \mathbb{Z}) \to \mathbb{C}^*$ for each triangle T_i of the triangulation λ. In particular, the following proposition is a straightforward extension of [Bai et al. 2007, Proposition 6]:

Proposition 4.6. *A local representation $\rho : \mathcal{Z}_\omega(\lambda) \to \mathrm{End}(V)$ has dimension $N^{4g-4+2s}$ and is classified (up to isomorphism) by a twisted homomorphism $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \to \mathbb{C}^*$ and a choice of an N-th root $C = \zeta(\mu)^{1/N}$ where $\mu(e) = 1$ for all edge e of τ_λ. Such a representation satisfies*:

- $\rho(Z_\alpha^N) = \zeta(\alpha)\,\mathrm{Id}_V$.
- $\rho(Z_\mu) = C\,\mathrm{Id}_V$.

Finally, the Main Theorem implies the following:

Theorem 4.7. *Let $\rho : \mathcal{Z}_\omega(\lambda) \to \mathrm{End}(V)$ be the (isomorphism class of) representation classified by the twisted homomorphism $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \to \mathbb{C}^*$ and the choice of an N-th root $C = \zeta(\mu)^{1/N}$ (where μ is defined as above). Then $\rho = \bigoplus_{i \in \mathcal{I}} \rho^{(i)}$ where each $\rho^{(i)}$ is irreducible, classified by the same twisted homomorphism ζ and*

N-th root $h_k^{(i)} = \big(\zeta(\eta_k)\big)^{1/N}$ with $h_1^{(i)} \dots h_s^{(i)} = C$ (here, the η_k are defined as in Section 4.1).

Moreover, for each choice of an N-th root $h_k = \big(\zeta(\eta_k)\big)^{1/N}$ for each $k \in \{1, \dots, s\}$, there are exactly N^g indices $i \in \mathcal{I}$ such that $h_k^{(i)} = h_k$ for all k.

Proof. Let ρ be a local representation of $\mathcal{Z}_\omega(\lambda)$ classified by ζ and C. In particular, ρ induces a local representation $\overline{\rho} := \rho \circ i$ of $\mathcal{T}_q(\lambda)$, where $i : \mathcal{T}_q(\lambda) \hookrightarrow \mathcal{Z}_\omega(\lambda)$. The local representation $\overline{\rho}$ is classified by the weight $\zeta(\beta_i)$ for all edges λ_i, where $\beta_i \in \mathcal{W}(\tau_\lambda, \mathbb{Z})$ is defined by $Z_{\beta_i} = Z_i^2 = i(X_i)$.

Let $P_j \in \mathcal{T}_q(\lambda)$ be the puncture invariant associated to the puncture v_j. The image of P_j in $\mathcal{Z}_\omega(\lambda)$ is $Z_{\eta_j}^2$. We claim that the eigenspaces of $\overline{\rho}(P_j)$ correspond to the eigenspaces of $\rho(Z_{\eta_j})$. In fact, if $V_{h_j}(Z_{\eta_j})$ is the eigenspace of $\rho(Z_{\eta_j})$ corresponding to the eigenvalue $h_j = \big(\zeta(\eta_j)\big)^{1/N}$, then one has the inclusion

$$V_{h_j}(Z_{\eta_j}) \subset V_{p_j}(P_j),$$

where $V_{p_j}(P_j)$ is the eigenspace of $\overline{\rho}(P_j) = \rho(Z_{\eta_j}^2)$ corresponding to the eigenvalue $p_j = h_j^2$. Because there are only N different possible eigenvalues of $\rho(Z_{\eta_j})$, a dimension counting argument shows the equality.

Now, we apply the Main Theorem and get that, for each choice of $(h_1, \dots, h_s)$ where $h_j = \big(\zeta(Z_{\eta_j})\big)^{1/N}$, the intersection $V_{h_1}(Z_{\eta_1}) \cap \dots \cap V_{h_s}(Z_{\eta_s})$ has dimension N^{4g-3+s} and hence is made of N^g copies of the irreducible representation of $\mathcal{Z}_\omega(\lambda)$ classified by ζ and $h_1, \dots, h_s$. $\square$

Bonahon and Wong [2012, Section 3] associate a character $r_\zeta \in \chi\big(\Sigma, \mathrm{SL}(2, \mathbb{C})\big)$ to a twisted homomorphism $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \to \mathbb{C}^*$ (here $\chi\big(\Sigma, \mathrm{SL}(2, \mathbb{C})\big)$ is the algebraic quotient of $\mathrm{Hom}\big(\pi_1(\Sigma), \mathrm{SL}_2(\mathbb{C})\big)$ by the action of $\mathrm{SL}_2(\mathbb{C})$ by conjugation). In particular, the (irreducible or local) representations of the balanced Chekhov–Fock algebra associated to an ideal triangulation λ of Σ are classified, up to finitely many choice, by a Zariski open set in $\chi\big(\Sigma, \mathrm{SL}(2, \mathbb{C})\big)$.

Note that, if r_ζ is the character associated to the twisted homomorphism ζ, the holonomy of r_ζ around a puncture v_j is parabolic exactly when $\zeta(\eta_j) = 1$.

4.3. ***Representations of $\mathcal{S}_A(\overline{\Sigma})$.*** We explain here how Theorem 4.7 gives rise to a new family of representations of the Kauffman bracket skein algebra of the closed surface $\overline{\Sigma} = \Sigma \cup \{v_1, \dots, v_s\}$.

Skein algebra. Given an oriented 3-manifold M, and a nonzero complex number $A \in \mathbb{C}^*$, consider the complex vector space $V(M)$ freely generated by isotopy classes of framed links in M. The *skein module* $\mathcal{S}_A(M)$ of M is the quotient of $V(M)$ by the *Kauffman bracket skein relations* as defined in Figure 7.

Namely, we identify three different links when differ by the previous relation in an open ball and agree everywhere else.

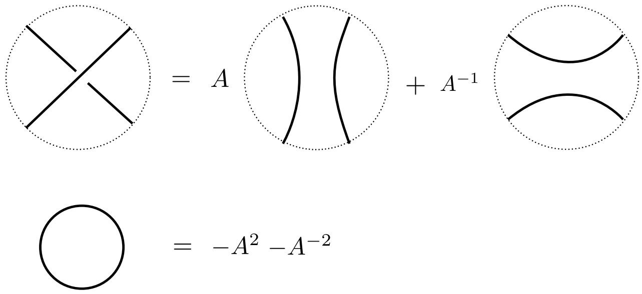

Figure 7. Kauffman bracket skein relations.

Given a framed link $K \subset M$, we denote by $[K]$ its image in the skein module $\mathcal{S}_A(M)$.

When $M = \Sigma \times [0, 1]$ for a surface Σ, the skein module $\mathcal{S}_A(M) = \mathcal{S}_A(\Sigma)$ inherits an algebra structure given by superposition of links. Namely, given two framed links K_1 and K_2 in $\Sigma \times [0, 1]$, the product $[K_1] \cdot [K_2]$ is defined to be the image of $K_1 \cup K_2$ in $\mathcal{S}_A(\Sigma)$, where $K_1 \cup K_2$ is given by the superposition of K_1 on top of K_2 where we rescaled so that $K_1 \subset \Sigma \times [0, \frac{1}{2}]$ and $K_2 \subset \Sigma \times [\frac{1}{2}, 1]$. We call $\mathcal{S}_A(\Sigma)$ with the product $\cdot$ the *Kauffman bracket skein algebra of S*.

Finite-dimensional representations of the skein algebra $\mathcal{S}_A(\Sigma)$ are of main importance as they appear naturally in topological quantum field theory (TQFT). For example, the Witten–Reshetekin–Turaev TQFT [Blanchet et al. 1995; Turaev 1994].

Classical shadow and quantum trace. Let $\mu : \mathcal{S}_A(\Sigma) \to \mathrm{End}(V)$ be an irreducible representation of the Kauffman bracket skein algebra of Σ.

Bonahon and Wong [2016] (see also [Lê 2015a] for a simpler proof) proved that if A is a primitive N-th root of -1, the N-th Chebyshev polynomial T_N of the first kind of any skein $[K] \in \mathcal{S}_A(\Sigma)$ is a central element in $\mathcal{S}_A(\Sigma)$. In particular, the precomposition of ρ by T_N maps each skein $[K]$ to a multiple of the identity in $\mathrm{End}(V)$. This multiple of the identity can be interpreted as an element $r_\mu \in \chi\big(\Sigma, \mathrm{SL}_2(\mathbb{C})\big)$ in the $\mathrm{SL}(2, \mathbb{C})$ character variety of Σ. This character is called *the classical shadow* of the representation μ.

When $A = \omega^{-2}$ (so A is a primitive N-th root of -1) and λ is an ideal triangulation of Σ, Bonahon and Wong [2011] (see also [Lê 2015b] for a more conceptual proof) constructed a *quantum trace map*

$$\mathrm{tr}^\lambda_\omega : \mathcal{S}_A(\Sigma) \to \mathcal{Z}_\omega(\lambda),$$

which turns out to be an injective algebra homomorphism.

In particular, by precomposing irreducible representations of $\mathcal{Z}_\omega(\lambda)$ by the quantum trace, Bonahon and Wong [2012] obtained a family of irreducible representations of the Kauffman bracket skein algebra of S indexed by a Zariski open subset of the character variety $\chi\big(\Sigma, \mathrm{SL}_2(\mathbb{C})\big)$. Moreover, taking the classical shadow of such an irreducible representation recovers the character.

Representations of $\mathcal{S}_A(\overline{\Sigma})$. The inclusion $\Sigma \hookrightarrow \overline{\Sigma}$ gives an algebra homomorphism

$$\iota : \mathcal{S}_A(\overline{\Sigma}) \to \mathcal{S}_A(\Sigma).$$

Let $r \in \chi\big(\Sigma, \mathrm{SL}(2, \mathbb{C})\big)$ be a character obtained from a character $r' \in \chi\big(\overline{\Sigma}, \mathrm{SL}(2, \mathbb{C})\big)$ (namely, the holonomy of r around each puncture is trivial). If $\zeta : \mathcal{W}(\tau_\lambda, \mathbb{Z}) \to \mathbb{C}^*$ is the twisted homomorphism associated to r, then $\zeta(\eta_j) = 1$ for each puncture v_j.

Denote by

$$\rho : \mathcal{Z}_\omega(\lambda) \to \mathrm{End}(V)$$

the local representation of $\mathcal{Z}_\omega(\lambda)$ classified by ζ and the N-th root $C = ((-\omega^4)^s)^{1/N}$. Let $E \subset V$ be the intersection of the eigenspaces of $\rho(Z_{\eta_k})$ for $k \in \{1, \ldots, s\}$ corresponding to the eigenvalue $-\omega^4$.

By Theorem 4.7, the vector space E is stable by $\rho\big(\mathcal{Z}_\omega(\lambda)\big)$, so we get an induced representation $\rho' : \mathcal{Z}_\omega(\lambda) \to \mathrm{End}(E)$. Note that ρ' is made of N^g copies of the irreducible representation of $\mathcal{Z}_\omega(\lambda)$ classified by ζ and puncture invariant $-\omega^4$.

Proposition 4.8. *There is a proper linear subspace $F \subset E$ such that the composition*

$$\mu : \mathcal{S}_A(\overline{\Sigma}) \xrightarrow{\iota} \mathcal{S}_A(\Sigma) \xrightarrow{\overline{\rho}} \mathrm{End}(F)$$

induces a representation of $\mathcal{S}_A(\overline{\Sigma})$. The classical shadow of each irreducible factor of μ is same. Finally, the dimension of F is at least N^{4g-3} when $g > 1$ and at least N^2 when $g = 1$.

Proof. This is a direct consequence of the construction of Bonahon and Wong [2015]. In fact, using the decomposition of $\overline{\rho}$ into irreducible parts and considering the *total off-diagonal kernel* of each irreducible factor (see [op. cit., Section 4.2]), one gets the result. □

The vector space F is canonically associated to the triangulation λ, which makes the family of representations described above easier to handle for computations.

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