Pacific Journal of Mathematics

A POSITIVE MASS THEOREM AND PENROSE INEQUALITY FOR GRAPHS WITH NONCOMPACT BOUNDARY

EZEQUIEL BARBOSA AND ADSON MEIRA

Volume 294 No. 2 June 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 294, No. 2, 2018
dx.doi.org/10.2140/pjm.2018.294.257

A POSITIVE MASS THEOREM AND PENROSE INEQUALITY FOR GRAPHS WITH NONCOMPACT BOUNDARY

EZEQUIEL BARBOSA AND ADSON MEIRA

We prove a version of the positive mass theorem for graph hypersurfaces with a noncompact boundary, in Euclidean space. We also prove a Penrose inequality for such hypersurfaces.

1. Introduction

Let (M^n, g) be an asymptotically flat Riemannian manifold. Suppose that the scalar curvature of g is nonnegative, $S_g \geq 0$. The Riemannian positive mass theorem states that, if either $3 \leq n \leq 7$ or $n \geq 3$ and the manifold is spin, the ADM-mass of g is nonnegative: $m_{\text{ADM}} \geq 0$. Moreover, $m_{\text{ADM}} = 0$ if and only if (M^n, g) is isometric to the Euclidean space $(\mathbb{R}^n, \delta)$. Recently, Almaraz, Barbosa and de Lima [Almaraz et al. 2016] defined a kind of ADM-mass for asymptotically flat manifolds with a noncompact boundary, and they proved that, if either $3 \leq n \leq 7$ or if $n \geq 3$ and the manifold is spin, then that ADM-mass is nonnegative, assuming the scalar curvature of the manifold and the mean curvature of the boundary are nonnegative. A similar positive mass theorem for all dimensions and with no spin condition has not been proved yet.

Although graphical hypersurfaces in Euclidean spaces are spin, Lam [2011] used an elementary method for such manifolds — asymptotically flat graphical hypersurfaces with an empty boundary — without invoking the spin structure, and proved the positive mass conjecture for graphical hypersurfaces with compact boundary. A version of the positive mass theorem for manifolds with compact boundary is known as the Penrose inequality. The main goal of this work is to provide an elementary proof for the positive mass theorem for graph hypersurfaces with noncompact boundary in Euclidean spaces, and a kind of Penrose inequality for such hypersurfaces. For more about the positive mass theorem and the Penrose inequality, see [Almaraz et al. 2016; Huang and Wu 2015; Lee and Sormani 2014; Mirandola and Vitório 2015].

Barbosa was supported by FAPEMIG and CNPq grants. Meira was supported by FAPEMIG and CAPES grants.
MSC2010: 53C24, 83C99.
Keywords: graphical hypersurface, ADM mass, scalar curvature, rigidity, asymptotically flat.

Let us be a little bit more precise with respect to the case where the manifold has a noncompact boundary. Let (M^n, g) be an oriented Riemannian manifold with a noncompact boundary Σ and dimension $n \geq 3$. We denote by S_g the scalar curvature of the manifold (M, g). We also assume that Σ is oriented by an outward pointing unit normal vector η, so that its mean curvature is $H_g = \operatorname{div}_g \eta$. We say that (M, g) is asymptotically flat with decay rate $\tau > 0$ if there exists a compact subset $K \subset M$ and a diffeomorphism $\Psi : M \setminus K \to \mathbb{R}^n_- \setminus \bar{B}_1^-(0)$ such that the following asymptotic expansion holds as $r \to +\infty$:

$$|g_{ij}(x) - \delta_{ij}| + r|g_{ij,k}(x)| + r^2|g_{ij,kl}(x)| = O(r^{-\tau}). \tag{1-1}$$

Here, $x = (x_1, \ldots, x_n)$ is the coordinate system induced by Ψ; $r = |x|$ and g_{ij} are the coefficients of g with respect to x; the comma denotes partial differentiation; $\mathbb{R}^n_- = \{x \in \mathbb{R}^n; x_n \leq 0\}$, and $\bar{B}_1^-(0) = \{x \in \mathbb{R}^n_-; |x| \leq 1\}$. The subset $M_\infty = M \backslash K$ is called the end of M. In this paper, we use the Einstein summation convention with the index ranges $i, j, \ldots = 1, \ldots, n$ and $\alpha, \beta, \ldots = 1, \ldots, n-1$. Observe that along Σ, $\{\partial_\alpha\}_\alpha$ spans $T\Sigma$ while ∂_n points inwards.

The most important example of a manifold in this class is the half-space $\mathbb{R}^n_-$ endowed with the standard flat metric δ; see Figure 1.

Definition 1.1. Suppose that $\tau > \frac{1}{2}(n-2)$ and S_g and H_g are integrable on M and Σ, respectively. In terms of asymptotically flat coordinates as above, the mass of (M, g) is given by

$$\mathfrak{m}_{(M,g)} = \lim_{r \to +\infty} \left\{ \int_{\mathcal{S}^{n-1}_{r,-}} (g_{ij,j} - g_{jj,i})\mu^i \, d\mathcal{S}^{n-1}_{r,-} + \int_{\mathcal{S}^{n-2}_r} g_{\alpha n} \vartheta^\alpha \, d\mathcal{S}^{n-2}_r \right\}, \tag{1-2}$$

where $\mathcal{S}^{n-1}_{r,-} \subset M$ is a large coordinate hemisphere of radius r with outward unit normal μ, and ϑ is the outward pointing unit conormal to $\mathcal{S}^{n-2}_r = \partial \mathcal{S}^{n-1}_{r,-}$, oriented as the boundary of the bounded region $\Sigma_r \subset \Sigma$.

Almaraz–Barbosa–de Lima [Almaraz et al. 2016] showed that the limit on the right-hand side of (1-2) exists and its value does not depend on the particular asymptotically flat coordinates chosen. Thus, $\mathfrak{m}_{(M,g)}$ is an invariant of the asymptotic geometry of (M, g). Moreover, they considered the following conjecture:

Conjecture 1.2. *If (M, g) is asymptotically flat with decay rate $\tau > \frac{1}{2}(n-2)$ as above and satisfies $S_g \geq 0$ and $H_g \geq 0$ then $\mathfrak{m}_{(M,g)} \geq 0$, with the equality occurring if and only if (M, g) is isometric to $(\mathbb{R}^n_-, \delta)$. Here, H_g is the mean curvature of the noncompact boundary, related to the outward pointing unit normal vector.*

This conjecture has been confirmed in some special cases in [Escobar 1992; Raulot 2011]. Finally, Almaraz–Barbosa–de Lima [Almaraz et al. 2016] showed that the following result holds.

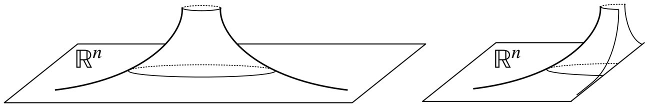

Figure 1. Asymptotically flat manifolds.

Theorem 1.3. *Conjecture 1.2 holds true if either* $3 \leq n \leq 7$ *or if* $n \geq 3$ *and* M *is spin.*

An immediate consequence of the rigidity statement in Theorem 1.3 is also worth noticing.

Corollary 1.4. *Let* (M, g) *be as in Theorem 1.3 and assume further that there exists a compact subset* $K \subset M$ *such that* $(M \setminus K, g)$ *is isometric to* $(\mathbb{R}^n_- \setminus \overline{B}^-_1(0), \delta)$. *Then* (M, g) *is isometric to* $(\mathbb{R}^n_-, \delta)$.

Now, we consider a graphical hypersurface with a noncompact boundary in the Euclidean space.

Definition 1.5. Let $\Omega \subset \mathbb{R}^n$ be a bounded subset. Let $F : \mathbb{R}^n_- \backslash \Omega \to \mathbb{R}^m$, $F(x) = (f^1(x), \dots, f^m(x))$ be a C^2 application. We will denote by $G(F)$ the graph of F. We say that F is asymptotically flat, with order $p > 0$, if the scalar curvature S of the graph of F with the metric of $\mathbb{R}^{n+m}$ is an integrable function over $G(F)$, and if there exists a compact subset $K \subset \mathbb{R}^n$ such that $\Omega \subset K$ and, over $\mathbb{R}^n_- \backslash K$, the partial derivatives $f^\alpha_i = \partial f^\alpha / \partial x_i$, $f^\alpha_{ij} = \partial^2 f^\alpha / (\partial x_i \partial x_j)$ satisfies

$$|f^\alpha_i(x)| = O(|x|^{-p/2}), \quad |f^\alpha_{ij}(x)| = O(|x|^{-p/2-1}), \quad |f^\alpha_{ijk}(x)| = O(|x|^{-p/2-2})$$

for all $\alpha = 1, \dots, m$ and $i, j, k = 1, \dots, n$.

From here, g_f will denote $\delta + df \otimes df$, where δ is the canonical metric of the Euclidean space. Our main result is the following:

Theorem 1.6 (positive mass theorem). *Let* $f : \mathbb{R}^n_- \to \mathbb{R}$ *be a* C^2 *function up to boundary, asymptotically flat over* $\mathbb{R}^n_- \backslash \Omega$, *with order* $p > \frac{1}{2}(n-2)$. *Let* $(\mathbb{R}^n_-, g_f)$ *be the graph of* f. *Suppose that* $f_n = 0$ *over* $\partial \mathbb{R}^n_-$, *that* $S \in L^1(\mathbb{R}^n_-)$, *and* $S \geq 0$. *We also assume that the mean curvature* $\overline{H}$ *of the boundary of* $(\mathbb{R}^n_-, g_f)$, *seen as a submanifold of* $(\mathbb{R}^n_-, g_f)$ *is such that* $\overline{H} \geq 0$ *and* $\overline{H} \in L^1(\partial \mathbb{R}^n_-)$. *Then, the mass of* $G(f)$ *is nonnegative. Moreover, it is null if and only if* $G(f)$ *is a half-plane.*

As a consequence of [Lee and Sormani 2014], we obtain the stability of the rigidity supposing that the graph is rotationally symmetric. We can also consider the Penrose inequality for such graphs.

Theorem 1.7 (Penrose inequality). *Let* $\Omega \subset \mathbb{R}^n_- \backslash \{x_n = 0\}$, $n \geq 3$, *be an open bounded set whose boundary is smooth and mean-convex. Suppose that* $\partial\Omega$ *is outer-minimizing or each connected component of* Ω *is star-shaped. Let* $f : \mathbb{R}^n_- \backslash \Omega \to \mathbb{R}$

be a function that is C^2 up to boundary, asymptotically flat, constant over each connected component of $\partial\Omega$ and such that $|Df| \to \infty$ when $x \to \partial\Omega$. We also suppose that $f_n = 0$ over $\partial\mathbb{R}^n_-$, the scalar curvature of the graph is nonnegative, the mean curvatures of the compact boundaries are nonnegative, and that the mean curvature of the noncompact boundary (viewed as a submanifold of the graph) is nonnegative. Then,

$$m_{(M,g)} \geq \frac{1}{2}\left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{(n-2)/(n-1)},$$

where $|\partial\Omega|$ is the $(n-1)$-volume of $\partial\Omega$.

2. Proof of the positive mass theorem

We start this section considering a very important proposition.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}$ be a C^2 function. Let $X : \Omega \to \mathbb{R}^n$ be the vector field given by*

$$X = \overline{U}(f_i f_{kk} - f_k f_{ik})e_i, \tag{2-1}$$

where $\overline{U} = 1/U$, $U = 1 + \langle Df, Df\rangle$, and $\langle\cdot,\cdot\rangle$ is the Euclidean metric. Consider the function $s : \Omega \to \mathbb{R}$ given by

$$s = \overline{U}(f_{ii} f_{kk} - f_{ik} f_{ik}) - \overline{U}^2 2 f_l f_{li}(f_i f_{kk} - f_k f_{ik}).$$

Then, $s = \operatorname{div} X$ and the scalar curvature of $(G(f), g)$ is s. Here, g is the induced metric and $G(f)$ is the graph of f.

Proof. See [Huang and Wu 2013; Lam 2011; Mirandola and Vitório 2015; Reilly 1973]. □

Now we can prove the following result.

Theorem 2.2. *Let $f : \mathbb{R}^n_- \to \mathbb{R}$ be an asymptotically flat function over $\mathbb{R}^n_- \backslash \Omega$ of class C^2 up to boundary, with order $p > \frac{1}{2}(n-2)$. Let $(\mathbb{R}^n_-, g_f)$ be the graph of f. We suppose the following items:*

- *$f_n = \partial f/\partial x_n \geq 0$ over $\partial\mathbb{R}^n_-$ and $\partial_n = (e_n, f_n)$ is normal to the noncompact boundary (this occurs when $f_n = 0$ over $\partial\mathbb{R}^n_-$, for example);*
- *$S \in L^1(\mathbb{R}^n_-)$ and $S \geq 0$;*
- *The mean curvature $\overline{H}$ of the boundary of $(\mathbb{R}^n_-, g_f)$, viewed as a submanifold of $(\mathbb{R}^n_-, g_f)$ be such that $\overline{H} \geq 0$ (with respect to the unit normal inward pointing vector field) and $\overline{H} \in L^1(\partial\mathbb{R}^n_-)$;*
- *The scalar second fundamental form $\tilde{h}$ (with respect to the unit normal upward pointing vector field) of the boundary of $(\mathbb{R}^n_-, g_f)$, viewed as a submanifold of $(\mathbb{R}^n, \delta) = (\partial\mathbb{R}^n_- \times \mathbb{R}, \delta)$, be such that $\sum_{i=1}^{n-1} \tilde{h}(\bar{\partial}_i, \bar{\partial}_i) \geq 0$ and $\sum_{i=1}^{n-1} \tilde{h}(\bar{\partial}_i, \bar{\partial}_i) \in$*

$L^1(\partial\mathbb{R}^n_-)$. Here, $\bar{e}_i \in \mathbb{R}^{n-1}$ is a canonical vector and $\bar{\partial}_i = (\bar{e}_i, f_i)$ is a tangent vector field.

Then,

$$c(n)m_{(M,g)} = \int_{\mathbb{R}^n_-} S\,\mathrm{d}x_\delta + \int_{\partial\mathbb{R}^n_-} \frac{\overline{H}}{\sqrt{1+(f_n)^2}}(2+|Df|^2)\,\mathrm{d}x_\delta + \int_{\partial\mathbb{R}^n_-} f_n\sqrt{1+|\overline{D}f|^2}\Bigl(\sum_{i=1}^{n-1}\tilde{h}(\bar{\partial}_i,\bar{\partial}_i)\Bigr)\frac{|Df|^2}{1+|Df|^2}\,\mathrm{d}x_\delta, \tag{2-2}$$

where $\overline{D}f = (f_1, \dots, f_{n-1})$. In particular, the mass $m_{(M,g)}$ is nonnegative.

Proof. Note that $\partial B_r^- = S_r^- \cup D_r$, where $S_r^- = \{x \in \mathbb{R}^n_- \mid \|x\| = r\}$, $D_r = \{x \in \partial\mathbb{R}^n_- \mid \|x\| \le r\}$ and $S_r^{n-2} = \partial D_r$. Remembering that $S = \operatorname{div} X$, we have

$$\begin{aligned}\int_{\mathbb{R}^n_-} S\,\mathrm{d}x_\delta &= \lim_{r\to\infty}\int_{B_r^-} S\,\mathrm{d}x_\delta = \lim_{r\to\infty}\int_{B_r^-}\operatorname{div} X\,\mathrm{d}x_\delta = \lim_{r\to\infty}\int_{\partial B_r^-}\langle X, N\rangle\,\mathrm{d}A_r\\ &= \lim_{r\to\infty}\int_{S_r^-}\Bigl\langle X, \frac{x}{|x|}\Bigr\rangle\,\mathrm{d}\sigma_r + \lim_{r\to\infty}\int_{D_r}\langle X, e_n\rangle\,\mathrm{d}x_\delta.\end{aligned}$$

By hypothesis, $f_i = O(|x|^{-p/2})$ and $f_{ik} = O(|x|^{-p/2-1})$ for all $i, k = 1, \dots, n$. Since $U - 1 = \langle Df, Df\rangle = O(|x|^{-p})$, we have $\lim_{|x|\to\infty} U = 1$, therefore we have $\lim_{|x|\to\infty}\overline{U} = 1$, where $\overline{U} = 1/U$. Therefore, $\overline{U} - 1 = -\overline{U}\langle Df, Df\rangle = O(|x|^{-p})$. With this, we conclude that

$$(\overline{U}-1)(f_i f_{kk} - f_k f_{ik}) = O(|x|^{-2p-1}).$$

Since $p > \frac{1}{2}(n-2)$, we have $2p+1 > n-1 = \dim S_r^-$. Thus,

$$\begin{aligned}\lim_{r\to\infty}\int_{S_r^-}\Bigl|(\overline{U}-1)(f_i f_{kk} - f_k f_{ik})\frac{x^i}{|x|}\Bigr|\,\mathrm{d}\sigma_r &\le \lim_{r\to\infty}\int_{S_r^-} C\cdot|x|^{-2p-1}\,\mathrm{d}\sigma_r\\ &\le C\lim_{r\to\infty} r^{-2p-1}|S_r^-| = 0.\end{aligned}$$

Then

$$\lim_{r\to\infty}\int_{S_r^-}(\overline{U}-1)(f_i f_{kk} - f_k f_{ik})\frac{x^i}{|x|}\,\mathrm{d}\sigma_r = 0.$$

Therefore,

$$\lim_{r\to\infty}\int_{S_r^-}\overline{U}(f_i f_{kk} - f_k f_{ik})\frac{x^i}{|x|}\,\mathrm{d}\sigma_r = \lim_{r\to\infty}\int_{S_r^-}(f_i f_{kk} - f_k f_{ik})\frac{x^i}{|x|}\,\mathrm{d}\sigma_r.$$

Thus,

$$\lim_{r\to\infty}\int_{S_r^-}\Bigl\langle X, \frac{x}{|x|}\Bigr\rangle\,\mathrm{d}\sigma_r = \lim_{r\to\infty}\int_{S_r^-}(f_i f_{kk} - f_k f_{ik})\frac{x_i}{|x|}\,\mathrm{d}\sigma_r.$$

Now, see that $\langle X, e_n\rangle = \overline{U}(f_n f_{kk} - f_k f_{nk}) = \overline{U}\sum_{k=1}^{n-1}(f_n f_{kk} - f_k f_{nk})$, because when $k = n$ the terms are canceled. On the other hand, since $\overline{U} = 1 - |Df|^2/(1+|Df|^2)$, we obtain

$$\langle X, e_n\rangle = \sum_{k=1}^{n-1}(f_n f_{kk} - f_k f_{nk}) - \frac{|Df|^2}{1+|Df|^2}\sum_{k=1}^{n-1}(f_n f_{kk} - f_k f_{nk}).$$

This implies that

$$\begin{aligned}\int_{\mathbb{R}^n_-} S\,\mathrm{d}x_\delta &= \lim_{r\to\infty}\int_{S_r^-}(f_i f_{kk} - f_k f_{ik})\frac{x_i}{|x|}\,\mathrm{d}\sigma_r + \lim_{r\to\infty}\int_{D_r}\sum_{k=1}^{n-1}(f_n f_{kk} - f_k f_{nk})\,\mathrm{d}x_\delta \\ &\qquad - \lim_{r\to\infty}\int_{D_r}\frac{|Df|^2}{1+|Df|^2}\sum_{k=1}^{n-1}(f_n f_{kk} - f_k f_{nk})\,\mathrm{d}x_\delta \\ &= \lim_{r\to\infty}\int_{S_r^-}(g_{ki,k} - g_{kk,i})(\nu_r)^i\,\mathrm{d}\sigma_r \\ &\qquad + \lim_{r\to\infty}\int_{D_r}\big(\mathrm{div}_{\mathbb{R}^{n-1}}(f_n\overline{D}f) - 2\langle\overline{D}f, \overline{D}f_n\rangle\big)\,\mathrm{d}x_\delta \\ &\qquad - \lim_{r\to\infty}\int_{D_r}\frac{|Df|^2}{1+|Df|^2}\sum_{k=1}^{n-1}(f_n f_{kk} - f_k f_{nk})\,\mathrm{d}x_\delta.\end{aligned}$$

Here, $\overline{D}f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_{n-1})$, ν_r is the normal vector field to S_r^-, and η_r is the normal vector field to S_r^{n-2}. Using that

$$\int_{D_r}\mathrm{div}_{\mathbb{R}^{n-1}}(f_n\overline{D}f)\,\mathrm{d}x_\delta = \int_{S_r^{n-2}} f_n\langle\overline{D}f, \eta_r\rangle\,\mathrm{d}\sigma_r,$$

we find

$$\begin{aligned}\int_{\mathbb{R}^n_-} S\,\mathrm{d}x_\delta &= \lim_{r\to\infty}\left\{\int_{S_r^-}(g_{ki,k} - g_{kk,i})(\nu_r)^i\,\mathrm{d}\sigma_r + \int_{S_r^{n-2}} f_n f_k(\eta_r)^k\,\mathrm{d}\sigma_r\right\} \\ &\qquad \lim_{r\to\infty}\left\{-2\int_{D_r}\langle\overline{D}f, \overline{D}f_n\rangle\,\mathrm{d}x_\delta - \int_{D_r}(f_n f_{kk} - f_k f_{nk})\frac{|Df|^2}{1+|Df|^2}\,\mathrm{d}x_\delta\right\} \\ &= \lim_{r\to\infty}\left\{\int_{S_r^-}(g_{ki,k} - g_{kk,i})(\nu_r)^i\,\mathrm{d}\sigma_r + \int_{S_r^{n-2}} g_{nk}(\eta_r)^k\,\mathrm{d}\sigma_r\right\} \\ &\qquad \lim_{r\to\infty}\left\{-2\int_{D_r}\langle\overline{D}f, \overline{D}f_n\rangle\,\mathrm{d}x_\delta - \int_{D_r}(f_n f_{kk} - f_k f_{nk})\frac{|Df|^2}{1+|Df|^2}\,\mathrm{d}x_\delta\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}c(n)m_{(M,g)} &= \int_{\mathbb{R}^n_-} S\,\mathrm{d}x_\delta + 2\lim_{r\to\infty}\int_{D_r}\sum_{i=1}^{n-1} f_{ni}f_i\,\mathrm{d}x_\delta \\ &\qquad + \lim_{r\to\infty}\int_{D_r}(f_n f_{kk} - f_k f_{nk})\frac{|Df|^2}{1+|Df|^2}\,\mathrm{d}x_\delta.\end{aligned}$$

Now we will calculate the second fundamental form of the boundary viewed as a submanifold of the graph of f. Note that $N = -\partial_n = -(e_n, f_n)$ is a normal field to the boundary, and moreover, inward pointing and tangent to the graph. We have that

$$\begin{aligned}\overline{\nabla}_{\partial_i}\partial_n &= (0, \dots, 0, \partial_i f_n) = (0, \dots, 0, \langle Df_n, \partial_i\rangle) \\ &= (0, \dots, 0, \langle (f_{n1}, \dots, f_{nn}, 0), (e_i, f_i)\rangle) = (0, \dots, 0, f_{ni}).\end{aligned}$$

Since $\partial_j = (e_j, f_j)$, we have that $\langle \overline{\nabla}_{\partial_i}\partial_n, \partial_j\rangle = f_{ni} f_j$. Thus,

$$\langle \overline{\nabla}_{\partial_i} N, \partial_j\rangle = -\langle \overline{\nabla}_{\partial_i}\partial_n, \partial_j\rangle = -f_{ni} f_j.$$

Here we used the fact that $\langle \overline{\nabla}_{\partial_i} N, \partial_j\rangle = -\langle N, II(\partial_i, \partial_j)\rangle$, where II is the second fundamental form of the boundary viewed as a submanifold of the graph, and we find that $\langle II(\partial_i, \partial_j), N)\rangle = f_{ni} f_j$. Therefore, denoting the scalar second fundamental form of the boundary viewed as a submanifold of the graph by $\bar{h}$, we find

$$\bar{h}(\partial_i, \partial_j) = \langle II(\partial_i, \partial_j), N/|N|\rangle = \frac{f_{ni} f_j}{\sqrt{1 + (f_n)^2}}.$$

With this, we see that the mean curvature of $\partial G(f)$ viewed as a submanifold of $G(f)$ is

$$\begin{aligned}\overline{H} = \sum_{i,j=1}^{n-1} g^{ij}\bar{h}(\partial_i, \partial_j) &= \sum_{i,j=1}^{n-1}\left(\delta_{ij} - \frac{f_i f_j}{1 + |Df|^2}\right)\frac{f_{ni} f_j}{\sqrt{1 + (f_n)^2}} \\ &= \sum_{i,j=1}^{n-1}\left(\frac{\delta_{ij} f_{ni} f_j}{\sqrt{1 + (f_n)^2}} - \frac{f_i f_j f_{ni} f_j}{(1 + |Df|^2)\sqrt{1 + (f_n)^2}}\right) \\ &= \sum_{i=1}^{n-1}\frac{f_{ni} f_i}{\sqrt{1 + (f_n)^2}} - \sum_{i,j=1}^{n-1}\frac{f_{ni} f_i (f_j)^2}{(1 + |Df|^2)\sqrt{1 + (f_n)^2}} \\ &= \sum_{i=1}^{n-1}\frac{f_{ni} f_i}{\sqrt{1 + (f_n)^2}} - \sum_{i=1}^{n-1}\frac{f_{ni} f_i}{\sqrt{1 + (f_n)^2}}\sum_{j=1}^{n-1}\frac{(f_j)^2}{1 + |Df|^2} \\ &= \sum_{i=1}^{n-1}\frac{f_{ni} f_i}{\sqrt{1 + (f_n)^2}}\left[1 - \sum_{j=1}^{n-1}\frac{(f_j)^2}{1 + |Df|^2}\right].\end{aligned}$$

Hence,

$$\sum_{i=1}^{n-1} f_{ni} f_i = \frac{\overline{H}\sqrt{1 + (f_n)^2}}{1 - \sum_{j=1}^{n-1}(f_j)^2/(1 + |Df|^2)}.$$

Therefore,

$$
\begin{aligned}
c(n)m_{(M,g)} \\
&= \int_{\mathbb{R}^n_-} S\,\mathrm{d}x_\delta + 2\lim_{r\to\infty}\int_{D_r} \frac{\overline{H}\sqrt{1+(f_n)^2}}{1-\sum_{j=1}^{n-1}(f_j)^2/(1+|Df|^2)}\,\mathrm{d}x_\delta \\
&\qquad + \lim_{r\to\infty}\int_{D_r}(f_n f_{kk} - f_k f_{nk})\frac{|Df|^2}{1+|Df|^2}\,\mathrm{d}x_\delta \\
&= \int_{\mathbb{R}^n_-} S\,\mathrm{d}x_\delta + \lim_{r\to\infty}\int_{D_r} \frac{\overline{H}\sqrt{1+(f_n)^2}}{1-\sum_{j=1}^{n-1}(f_j)^2/(1+|Df|^2)}\left\{2-\frac{|Df|^2}{1+|Df|^2}\right\}\mathrm{d}x_\delta \\
&\qquad + \lim_{r\to\infty}\int_{D_r} f_n f_{kk}\frac{|Df|^2}{1+|Df|^2}\,\mathrm{d}x_\delta.
\end{aligned}
$$

By hypothesis, $f_n \geq 0$ over $\partial\mathbb{R}^n_-$. Also, $\sum_{k=1}^{n-1} f_{kk} = \sqrt{1+|D\bar{f}|^2}\sum_{i=1}^{n-1}\tilde{h}(\bar{\partial}_i, \bar{\partial}_i)$. In this way,

$$
\begin{aligned}
c(n)m_{(M,g)} &= \int_{\mathbb{R}^n_-} S\,\mathrm{d}x_\delta + \int_{\partial\mathbb{R}^n_-} \frac{\overline{H}}{\sqrt{1+(f_n)^2}}(2+|Df|^2)\,\mathrm{d}x_\delta \\
&\qquad + \int_{\partial\mathbb{R}^n_-} f_n\sqrt{1+|D\bar{f}|^2}\left(\sum_{i=1}^{n-1}\tilde{h}(\bar{\partial}_i, \bar{\partial}_i)\right)\frac{|Df|^2}{1+|Df|^2}\,\mathrm{d}x_\delta \geq 0.
\end{aligned}
$$

In order to conclude, we will show that $\sum_{k=1}^{n-1} f_{kk} = \sqrt{1+|D\bar{f}|^2}\sum_{i=1}^{n-1}\tilde{h}(\bar{\partial}_i, \bar{\partial}_i)$. Viewed as a submanifold of $\partial\mathbb{R}^n_- \times \mathbb{R}$, the boundary is the graph of $\bar{f} = f|_{\partial\mathbb{R}^n_-}$. Thus, $\bar{\partial}_i = (\bar{e}_i, f_i)$, $i = 1, \ldots, n-1$, are tangent vector fields. Here $\bar{e}_i \in \mathbb{R}^{n-1}$. Moreover, $\bar{\eta} = (-D\bar{f}, 1)$ is a normal field. We have

$$\overline{\nabla}_{\bar{\partial}_i}\bar{\eta} = (-\bar{\partial}_i \bar{f}_1, \ldots, -\bar{\partial}_i \bar{f}_{n-1}, 0) = (-\langle D\bar{f}_1, \bar{\partial}_i\rangle, \ldots, -\langle D\bar{f}_{n-1}, \bar{\partial}_i\rangle, 0).$$

Using that

$$D\bar{f}_j = (\bar{f}_{j1}, \ldots, \bar{f}_{j(n-1)}) = (\bar{f}_{j1}, \ldots, \bar{f}_{j(n-1)}, 0)$$

and $\bar{\partial}_i = (\bar{e}_i, \bar{f}_i)$, we obtain that $-\langle D\bar{f}_j, \bar{\partial}_i\rangle = \bar{f}_{ji}$. Thus

$$\overline{\nabla}_{\bar{\partial}_i}\bar{\eta} = (-\bar{f}_{1i}, \ldots, -\bar{f}_{(n-1)i}, 0) = (-D\bar{f}_i, 0).$$

With this, $\langle\overline{\nabla}_{\bar{\partial}_i}\bar{\eta}, \bar{\partial}_j\rangle = -\bar{f}_{ji}$. Using the Weingarten equation, we find

$$\langle\widetilde{II}(\bar{\partial}_i, \bar{\partial}_j), \bar{\eta}\rangle = \bar{f}_{ji} = f_{ji}.$$

Therefore,

$$\sum_{i=1}^{n-1}\tilde{h}(\bar{\partial}_i, \bar{\partial}_i) = \sum_{k=1}^{n-1}\left\langle\widetilde{II}(\bar{\partial}_k, \bar{\partial}_k), \frac{\bar{\eta}}{|\bar{\eta}|}\right\rangle = \sum_{k=1}^{n-1}\frac{f_{kk}}{\sqrt{1+|D\bar{f}|^2}}. \qquad \square$$

In the next theorem, we will use a doubling argument to produce an asymptotically flat manifold without a compact boundary; more specifically, given $f : \mathbb{R}^n_- \to \mathbb{R}$, we will consider $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ given by $\tilde{f}(x_1, \dots, x_n) = f(x_1, \dots, x_n)$ if $x_n \le 0$, and $\tilde{f}(x_1, \dots, x_n) = f(x_1, \dots, -x_n)$ if $x_n > 0$. Then, using (2-2) and some results and arguments of [Huang and Wu 2013], we obtain the following result:

Theorem 2.3. *Let $f : \mathbb{R}^n_- \to \mathbb{R}$ be a C^{n+1} function up to boundary, asymptotically flat, over $\mathbb{R}^n_- \backslash \Omega$, with order $p > \frac{1}{2}(n-2)$ and such that $\tilde{f}$ defined as above is C^{n+1}. Let $(\mathbb{R}^n_-, g_f)$ be the graph of f. We suppose that $f_n = 0$ over $\partial\mathbb{R}^n_-$, that $S \in L^1(\mathbb{R}^n_-)$ and $S \ge 0$. We also suppose that the mean curvature $\overline{H}$ of the boundary of $(\mathbb{R}^n_-, g_f)$, viewed as a submanifold of $(\mathbb{R}^n_-, g_f)$ is such that $\overline{H} \ge 0$ and $\overline{H} \in L^1(\partial\mathbb{R}^n_-)$. If the mass of $G(f)$ is null, then $G(f)$ is a half-plane.*

Proof. We will assume that f is asymptotic to $\{x_{n+1} = 0, x_n \le 0\}$ and that $f \ne 0$; if not the result will be trivially true. Consider $\tilde{f}$ and note that $\tilde{f}$ is asymptotically flat, and its graph has integrable and nonnegative scalar curvature. By [Huang and Wu 2013, Theorem 4], we can suppose that the mean curvature of the graph of $\tilde{f}$ is nonnegative, $H(\tilde{f}) \ge 0$, with respect to ν, where ν is the vector field

$$\frac{(-D\tilde{f}, 1)}{\sqrt{1 + |D\tilde{f}|^2}}$$

(because we can reflect the graph over $\{x_{n+1} = 0\}$). Let B_r be an open ball in $\mathbb{R}^n$ centered at the origin of radius r; by [Huang and Wu 2013, Lemma 3.10],

$$\max_{\overline{B}_{r_2} \backslash B_{r_1}} \tilde{f} = \max_{\partial B_{r_2}} \tilde{f} \quad \forall\, r_2 > r_1 > 0.$$

Because $\max_{\partial B_{r_2}} \tilde{f} \to 0$ when $r_2 \to \infty$ (since $\tilde{f}$ is asymptotic to $\{x_{n+1} = 0\}$), we conclude that $\tilde{f} \le 0$ outside of B_{r_1}. Moreover, applying the strong maximum principle to $H(\tilde{f}) \ge 0$, we have $\tilde{f} < 0$ outside of B_{r_1}, unless $\tilde{f} \equiv 0$. In the latter case we can, moreover, conclude that $G(\tilde{f})$ is identical to $\{x_{n+1} = 0\}$, repeating the argument over $B_{r_2} \backslash B_{r_0}$, for $0 < r_0 < r_1$, and making $r_0 \to 0$. With this we conclude that if $\tilde{f} \ne 0$, then $\tilde{f} < 0$, i.e., $G(\tilde{f}) \subset \{x_{n+1} < 0\}$. Therefore, for $\epsilon > 0$ sufficiently small, some connected components of the level set $\{x \in \{x_{n+1} = 0\} \mid \tilde{f}(x) = -\epsilon\}$ lie over $G(\tilde{f})$ and have no boundary. We define $\Sigma_{-\epsilon}$ as being the connected component outermost, i.e., $\Sigma_{-\epsilon}$ is not enclosed by the others components. By Sard's theorem, $\Sigma_{-\epsilon}$ is smooth for almost all ϵ. Moreover, because $\tilde{f}$ tends to zero, for some small $\epsilon > 0$, we see that $\eta = -D\tilde{f}/|D\tilde{f}|$ is the unit normal vector on $\Sigma_{-\epsilon}$ pointing inward to the limited region in $\{x_{n+1} = 0\}$, which is delimited by $\Sigma_{-\epsilon}$. Let $H_{\Sigma_{-\epsilon}}$ be the mean curvature of $\Sigma_{-\epsilon}$ defined by η. Then, using that $H(\tilde{f}) \ge 0$ and $S(\tilde{f}) \ge 0$, by [Huang and Wu 2013, Theorem 2.2] we have $H_{\Sigma_{-\epsilon}} \ge 0$. Since $S(f) \ge 0$ and $c(n)m(g)(G(f)) = 0$, by (2-2) we conclude that $S(f) = 0$; then $S(\tilde{f}) = 0$ and $c(n)m_{(M,g)}(G(\tilde{f})) = 0$. This implies that $G(\tilde{f}) = \{x_{n+1} = 0\}$. If not, by [Huang and Wu 2013, Lemma 5.6] (or more generally by [Mirandola and Vitório 2015, Theorem 1.2]) we will have $H_{\Sigma_{-\epsilon}} = 0$ and therefore $\Sigma_{-\epsilon}$ will be a compact minimal hypersurface without boundary, and embedded in $\mathbb{R}^n$; this will be a contradiction. □

3. Penrose inequality

Let we start this section with a simple proposition that will be very useful in the next one.

Proposition 3.1. *Let $(M, \langle\cdot,\cdot\rangle)$ be an $(n+1)$-Riemannian manifold and $f : M \to \mathbb{R}$ a differentiable function. Let $\Sigma \subset M$ be an embedded hypersurface and ν a unitary normal field to Σ. Suppose that f is constant on Σ; then on Σ we have*

$$\Delta f = \operatorname{Hess} f(\nu,\nu) - H^{\Sigma}\langle Df, \nu\rangle.$$

Here, H^{Σ} is related to ν.

Proof. Given a point $x \in \Sigma$, we have

$$\begin{aligned}\Delta f &= \operatorname{div} Df = \operatorname{div}(\langle Df,\nu\rangle\nu) = \langle D\langle Df,\nu\rangle,\nu\rangle + \langle Df,\nu\rangle \operatorname{div}\nu \\ &= \langle \overline{\nabla}_{\nu} Df,\nu\rangle + \langle Df, \overline{\nabla}_{\nu}\nu\rangle + \langle Df,\nu\rangle \operatorname{div}\nu\end{aligned}$$

Using that Σ is embedded in M, we take a neighborhood U of x in Σ such that $U = g^{-1}(a)$, where $g : V \subset M \to \mathbb{R}$ is differentiable, $U \subset V$, and $a \in \mathbb{R}$ is a regular value of g. Take an orthonormal referential $E_1, \dots, E_{n+1}$ on V such that $E_{n+1} = Dg/|Dg| = \nu$. Denote by H the mean curvature and by A the second fundamental form of Σ; then

$$\begin{aligned}H &= \sum_{i=1}^{n}\langle A(E_i,E_i),\nu\rangle = \sum_{i=1}^{n}\langle \overline{\nabla}_{E_i}E_i,\nu\rangle \\ &= -\sum_{i=1}^{n}\langle \overline{\nabla}_{E_i}\nu, E_i\rangle - \langle\overline{\nabla}_{\nu}\nu,\nu\rangle = -\sum_{i=1}^{n+1}\langle \overline{\nabla}_{E_i}\nu, E_i\rangle = -\operatorname{div}\nu.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Delta f &= \langle \overline{\nabla}_{\nu} Df,\nu\rangle + \langle Df, \overline{\nabla}_{\nu}\nu\rangle - H^{\Sigma}\langle Df,\nu\rangle \\ &= \operatorname{Hess} f(\nu,\nu) + \langle Df,\nu\rangle\langle \nu, \overline{\nabla}_{\nu}\nu\rangle - H^{\Sigma}\langle Df,\nu\rangle \\ &= \operatorname{Hess} f(\nu,\nu) - H^{\Sigma}\langle Df,\nu\rangle.\end{aligned}$$ □

The next proposition will be useful in the proof of the Penrose inequality.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set such that the boundary of $\mathbb{R}^n_- \backslash \Omega$ is smooth. Let $f : \mathbb{R}^n_- \backslash \Omega \to \mathbb{R}$ be a C^2 function up to boundary, asymptotically flat, constant on each connected component of $\partial\Omega$, and such that $|Df| \to \infty$ when $x \to \partial\Omega$. We suppose that the graph of f has the induced metric from*

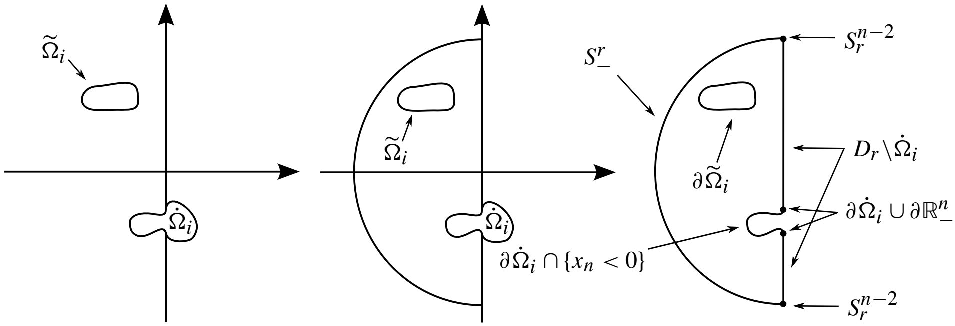

Figure 2. Sets of Proposition 3.2.

$\mathbb{R}^{n+1}$. *Denoting by* $\dot{\Omega}_i$, $i = 1, \dots, \dot{n}$, *the connected components of* Ω *such that* $\dot{\Omega}_i \cap \{x_n < 0\} \neq \varnothing$ *and* $\dot{\Omega}_i \cap \{x_n > 0\} \neq \varnothing$; *and by* $\widetilde{\Omega}_i$, $i = 1, \dots, \tilde{n}$, *the connected components of* Ω *such that* $\widetilde{\Omega}_i \subset \mathbb{R}^n_-$, *then*

$$\begin{aligned} c(n)m_{(M,g)} = &\int_{\mathbb{R}^n_- \backslash \Omega} S \,\mathrm{d}x_\delta + \int_{\partial\mathbb{R}^n_- \backslash \cup\dot{\Omega}_i} \frac{\overline{H}}{\sqrt{1+(f_n)^2}}(2+|Df|^2)\,\mathrm{d}x_\delta \\ &+ \int_{\partial\mathbb{R}^n_- \backslash \cup\dot{\Omega}_i} f_n\sqrt{1+|\overline{D}f|^2}\Big(\sum_{i=1}^{n-1}\tilde{h}(\bar{\partial}_i,\bar{\partial}_i)\Big)\frac{|Df|^2}{1+|Df|^2}\,\mathrm{d}x_\delta \\ &- \sum_{i=1}^{\dot{n}}\int_{\partial\dot{\Omega}_i\cap\partial\mathbb{R}^n_-} f_n\langle\overline{D}f,\dot{\eta}\rangle\,\mathrm{d}\dot{\sigma}_i + \int_{\cup\partial\dot{\Omega}_i\cap\{x_n<0\}} H^{\partial\Omega}\,\mathrm{d}\dot{\sigma} \\ &+ \int_{\cup\partial\widetilde{\Omega}_i} H^{\partial\Omega}\,\mathrm{d}\tilde{\sigma}. \end{aligned}$$

Here $c(n) = 2(n-1)\omega_{n-1}$ *and* $\dot{\eta}$ *is the unit normal vector field on* $\partial\dot{\Omega}$ *pointing inward to* $\dot{\Omega}$. *Here*, $\overline{H}$ *and* $\tilde{h}$ *are the mean curvature and the scalar second fundamental form, respectively defined in Theorem 2.2 and* $H^{\partial\Omega}$ *is the mean curvature of* $\partial\Omega$ *viewed as a submanifold of the hyperplanes containing them.*

Proof. We have

$$\begin{aligned} \text{(3-1)}\quad \int_{\mathbb{R}^n_-\backslash\Omega} S\,\mathrm{d}x_\delta &= \lim_{r\to\infty}\int_{B_r^-\backslash\Omega}\operatorname{div}X\,\mathrm{d}x_\delta \\ &= \lim_{r\to\infty}\int_{S_r^-}\Big\langle X,\frac{x}{|x|}\Big\rangle\mathrm{d}\sigma_r + \lim_{r\to\infty}\int_{D_r\backslash\cup\dot{\Omega}_i}\langle X,e_n\rangle\,\mathrm{d}x_\delta \\ &\qquad + \int_{\cup\partial\dot{\Omega}_i\cap\{x_n<0\}}\langle X,\dot{\eta}_i\rangle\,\mathrm{d}\dot{\sigma} + \int_{\cup\partial\widetilde{\Omega}_i}\langle X,\tilde{\eta}_i\rangle\,\mathrm{d}\tilde{\sigma}, \end{aligned}$$

where $X = 1/(1+|Df|^2)\,(f_i f_{kk} - f_k f_{ik})e_i$. Like in the Theorem 2.2, we find

$$\lim_{r\to\infty}\int_{S_r^-}\Big\langle X, \frac{x}{|x|}\Big\rangle \,\mathrm{d}\sigma_r = \lim_{r\to\infty}\int_{S_r^-}(f_i f_{kk} - f_k f_{ik})\frac{x_i}{|x|}\,\mathrm{d}\sigma_r. \tag{3-2}$$

On the other hand, since $\overline{U} = 1 - |Df|^2/(1+|Df|^2)$:

$$\langle X, e_n\rangle = \sum_{k=1}^{n-1}(f_n f_{kk} - f_k f_{nk}) - \frac{|Df|^2}{1+|Df|^2}\sum_{k=1}^{n-1}(f_n f_{kk} - f_k f_{nk}).$$

Thus,

$$\begin{aligned}\lim_{r\to\infty}\int_{D_r\setminus\cup\dot{\Omega}_i}\langle X, e_n\rangle\,\mathrm{d}x_\delta &= \lim_{r\to\infty}\int_{D_r\setminus\cup\dot{\Omega}_i}\sum_{k=1}^{n-1}(f_n f_{kk} - f_k f_{nk})\,\mathrm{d}x_\delta \\ &\qquad - \lim_{r\to\infty}\int_{D_r\setminus\cup\dot{\Omega}_i}\frac{|Df|^2}{1+|Df|^2}\sum_{k=1}^{n-1}(f_n f_{kk} - f_k f_{nk})\,\mathrm{d}x_\delta \\ &= \lim_{r\to\infty}\int_{D_r\setminus\cup\dot{\Omega}_i}(\mathrm{div}_{\mathbb{R}^{n-1}}(f_n\overline{D}f) - 2\langle\overline{D}f, \overline{D}f_n\rangle)\,\mathrm{d}x_\delta \\ &\qquad - \lim_{r\to\infty}\int_{D_r\setminus\cup\dot{\Omega}_i}\frac{|Df|^2}{1+|Df|^2}\sum_{k=1}^{n-1}(f_n f_{kk} - f_k f_{nk})\,\mathrm{d}x_\delta.\end{aligned}$$

Here, $\overline{D}f = (\partial f/\partial x_1, \dots, \partial f/\partial x_{n-1})$. Using that

$$\begin{aligned}&\lim_{r\to\infty}\int_{D_r\setminus\cup\dot{\Omega}_i}\mathrm{div}_{\mathbb{R}^{n-1}}(f_n\overline{D}f)\,\mathrm{d}x_\delta \\ &\qquad = \lim_{r\to\infty}\int_{S_r^{n-2}} f_n\langle\overline{D}f, \eta_r\rangle\,\mathrm{d}\sigma_r + \sum_{i=1}^{\dot{n}}\int_{\partial\dot{\Omega}_i\cap\partial\mathbb{R}^n_-} f_n\langle\overline{D}f, \dot{\eta}\rangle\,\mathrm{d}\dot{\sigma}_i,\end{aligned}$$

where η_r is the unit normal vector field on S_r^{n-2} pointing outward to D_r, we find

$$\begin{aligned}\lim_{r\to\infty}\int_{D_r\setminus\cup\dot{\Omega}_i}\langle X, e_n\rangle\,\mathrm{d}x_\delta = {}&\lim_{r\to\infty}\int_{S_r^{n-2}} f_n\langle\overline{D}f, \eta_r\rangle\,\mathrm{d}\sigma_r \\ &+\sum_{i=1}^{\dot{n}}\int_{\partial\dot{\Omega}_i\cap\partial\mathbb{R}^n_-} f_n\langle\overline{D}f, \dot{\eta}\rangle\,\mathrm{d}\dot{\sigma}_i \\ &-2\lim_{r\to\infty}\int_{D_r\setminus\cup\dot{\Omega}_i}\langle\overline{D}f, \overline{D}f_n\rangle\,\mathrm{d}x_\delta \\ &-\lim_{r\to\infty}\int_{D_r\setminus\cup\dot{\Omega}_i}\frac{|Df|^2}{1+|Df|^2}\sum_{k=1}^{n-1}(f_n f_{kk} - f_k f_{nk})\,\mathrm{d}x_\delta.\end{aligned} \tag{3-3}$$

Using (3-1), (3-2), and (3-3), we find

$$\begin{aligned}\int_{\mathbb{R}^n_-\setminus\Omega} S\,\mathrm{d}x_\delta &= \lim_{r\to\infty}\int_{S_r^-}(f_i f_{kk}-f_k f_{ik})\frac{x_i}{|x|}\,\mathrm{d}\sigma_r+\lim_{r\to\infty}\int_{S_r^{n-2}} f_n\langle \overline{D}f,\eta_r\rangle\,\mathrm{d}\sigma_r\\ &\quad+\sum_{i=1}^{\dot n}\int_{\partial\dot\Omega_i\cap\partial\mathbb{R}^n_-} f_n\langle\overline{D}f,\dot\eta\rangle\,\mathrm{d}\dot\sigma_i-2\lim_{r\to\infty}\int_{D_r\setminus\cup\dot\Omega_i}\langle\overline{D}f,\overline{D}f_n\rangle\,\mathrm{d}x_\delta\\ &\quad-\lim_{r\to\infty}\int_{D_r\setminus\cup\dot\Omega_i}\frac{|Df|^2}{1+|Df|^2}\sum_{k=1}^{n-1}(f_n f_{kk}-f_k f_{nk})\,\mathrm{d}x_\delta\\ &\quad+\int_{\cup\partial\dot\Omega_i\cap\{x_n<0\}}\langle X,\dot\eta_i\rangle\,\mathrm{d}\dot\sigma+\int_{\cup\partial\widetilde\Omega_i}\langle X,\tilde\eta_i\rangle\,\mathrm{d}\tilde\sigma.\end{aligned}$$

That is,

$$\begin{aligned}c(n)m_{(M,g)} &= \int_{\mathbb{R}^n_-\setminus\Omega} S\,\mathrm{d}x_\delta-\sum_{i=1}^{\dot n}\int_{\partial\dot\Omega_i\cap\partial\mathbb{R}^n_-} f_n\langle\overline{D}f,\dot\eta\rangle\,\mathrm{d}\dot\sigma_i\\ &\quad+2\lim_{r\to\infty}\int_{D_r\setminus\cup\dot\Omega_i}\langle\overline{D}f,\overline{D}f_n\rangle\,\mathrm{d}x_\delta\\ &\quad+\lim_{r\to\infty}\int_{D_r\setminus\cup\dot\Omega_i}\frac{|Df|^2}{1+|Df|^2}\sum_{k=1}^{n-1}(f_n f_{kk}-f_k f_{nk})\,\mathrm{d}x_\delta\\ &\quad-\int_{\cup\partial\dot\Omega_i\cap\{x_n<0\}}\langle X,\dot\eta_i\rangle\,\mathrm{d}\dot\sigma-\int_{\cup\partial\widetilde\Omega_i}\langle X,\tilde\eta_i\rangle\,\mathrm{d}\tilde\sigma.\end{aligned}\tag{3-4}$$

Like in the Theorem 2.2 we have

$$\begin{aligned}\lim_{r\to\infty}\int_{D_r\setminus\cup\dot\Omega_i} f_k f_{nk}\left\{2-\frac{|Df|^2}{1+|Df|^2}\right\}\mathrm{d}x_\delta\\ =\int_{\partial\mathbb{R}^n_-\setminus\cup\dot\Omega_i}\frac{\overline{H}}{\sqrt{1+(f_n)^2}}(2+|Df|^2)\,\mathrm{d}x_\delta,\end{aligned}\tag{3-5}$$

where $\overline{H}$ is the mean curvature of $\partial G(f)$, viewed as a submanifold of $G(f)$, and we also have

$$\begin{aligned}\lim_{r\to\infty}\int_{D_r\setminus\cup\dot\Omega_i} f_n f_{kk}\frac{|Df|^2}{1+|Df|^2}\,\mathrm{d}x_\delta\\ =\int_{\partial\mathbb{R}^n_-\setminus\cup\dot\Omega_i} f_n\sqrt{1+|\overline{D}f|^2}\left(\sum_{i=1}^{n-1}\tilde h(\bar\partial_i,\bar\partial_i)\right)\frac{|Df|^2}{1+|Df|^2}\,\mathrm{d}x_\delta,\end{aligned}\tag{3-6}$$

where $\tilde{h}$ is the scalar second fundamental form of $\partial G(f)$, viewed as a submanifold of $\partial\mathbb{R}^n_-$. Therefore,

$$
\begin{aligned}
\text{(3-7)}\quad c(n)m(g) = \int_{\mathbb{R}^n_-\setminus\Omega} S\,\mathrm{d}x_\delta + \int_{\partial\mathbb{R}^n_-\setminus\cup\dot{\Omega}_i} \frac{\overline{H}}{\sqrt{1+(f_n)^2}}(2+|Df|^2)\,\mathrm{d}x_\delta \\
+ \int_{\partial\mathbb{R}^n_-\setminus\cup\dot{\Omega}_i} f_n\sqrt{1+|\overline{D}f|^2}\biggl(\sum_{i=1}^{n-1}\tilde{h}(\bar{\partial}_i,\bar{\partial}_i)\biggr)\frac{|Df|^2}{1+|Df|^2}\,\mathrm{d}x_\delta \\
-\sum_{i=1}^{\dot{n}}\int_{\partial\dot{\Omega}_i\cap\partial\mathbb{R}^n_-} f_n\langle\overline{D}f,\dot{\eta}\rangle\,\mathrm{d}\dot{\sigma}_i \\
-\int_{\cup\partial\dot{\Omega}_i\cap\{x_n<0\}}\langle X,\dot{\eta}\rangle\,\mathrm{d}\dot{\sigma} - \int_{\cup\partial\widetilde{\Omega}_i}\langle X,\tilde{\eta}\rangle\,\mathrm{d}\tilde{\sigma}.
\end{aligned}
$$

The next computations of the mean curvature of the level set $\partial\Omega$ can also be found in [Lam 2011, Equation 5.3] or [Mirandola and Vitório 2015, Equation 33]. Since f is constant on $\partial\Omega$, we have that Df is normal to this set. Denote by Ω^c a component of Ω such that f increases when $x\to\partial\Omega^c$, thus $Df/|Df|$ is the unity normal vector field outward pointing to the graph on $\partial\Omega^c$ ($Df/|Df|$ points inward to Ω^c in the hyperplane containing Ω^c). Denote by Ω^d a component of Ω such that f decreases when $x\to\partial\Omega^d$, thus $-Df/|Df|$ is the unity normal vector field outward pointing to the graph on $\partial\Omega^d$ ($-Df/|Df|$ points inward to Ω^d in the hyperplane containing Ω^d). For an illustration, see Figure 3.

We have

$$
\begin{aligned}
\biggl\langle X,\frac{Df}{|Df|}\biggr\rangle &= \biggl\langle \overline{U}(f_i f_{kk}-f_k f_{ik})e_i,\frac{f_j}{|Df|}e_j\biggr\rangle = \frac{f_i\overline{U}}{|Df|}(f_i f_{kk}-f_k f_{ik}) \\
&= \frac{\overline{U}}{|Df|}(f_i^2 f_{kk}-f_k f_i f_{ik}) = \frac{\overline{U}}{|Df|}(|Df|^2\Delta f-\operatorname{Hess} f(Df,Df)) \\
&= \frac{|Df|^2}{|Df|}\overline{U}\biggl(\Delta f-\operatorname{Hess} f\biggl(\frac{Df}{|Df|},\frac{Df}{|Df|}\biggr)\biggr).
\end{aligned}
$$

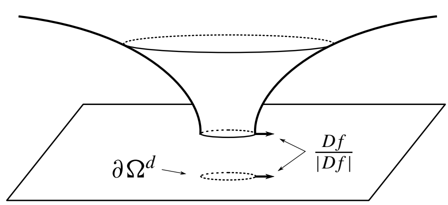

Figure 3. Illustration of the argument above.

Since $\nu = Df/|Df|$ is the unity normal vector field pointing inward to Ω^c, using Proposition 3.1 and that $\overline{U} = 1/(1+|Df|^2)$, we find

$$\begin{aligned}\left\langle X, \frac{Df}{|Df|}\right\rangle &= \frac{|Df|}{1+|Df|^2}\left(-H^{\partial\Omega^c}\left\langle Df, \frac{Df}{|Df|}\right\rangle\right)\\ &= -\frac{|Df|^2}{1+|Df|^2}H^{\partial\Omega^c}\left\langle \frac{Df}{|Df|}, \frac{Df}{|Df|}\right\rangle = -\frac{|Df|^2}{1+|Df|^2}H^{\partial\Omega^c}.\end{aligned}$$

Since $\nu = -Df/|Df|$ is the unity normal vector field pointing inward to Ω^d, using the Proposition 3.1 and that $\overline{U} = 1/(1+|Df|^2)$, we find

$$\begin{aligned}\left\langle X, -\frac{Df}{|Df|}\right\rangle &= -\frac{|Df|}{1+|Df|^2}\left(-H^{\partial\Omega^d}\left\langle Df, -\frac{Df}{|Df|}\right\rangle\right)\\ &= -\frac{|Df|^2}{1+|Df|^2}H^{\partial\Omega^d}\left\langle -\frac{Df}{|Df|}, -\frac{Df}{|Df|}\right\rangle = -\frac{|Df|^2}{1+|Df|^2}H^{\partial\Omega^d}.\end{aligned}$$

Here, $H^{\partial\Omega}$ is related with the vector field pointing inward to Ω. We know that $\lim_{x\to\partial\Omega}|Df(x)| = \infty$, thus $\lim_{x\to\partial\Omega}|Df|^2/(1+|Df|^2) = 1$, therefore, supposing that ν is the unity normal vector field pointing inward to Ω on $\partial\Omega$, on $\partial\Omega$ we have

$$\langle X, \nu\rangle = -H^{\partial\Omega}.$$

Using (3-7), we have

$$\begin{aligned}c(n)m_{(M,g)} = &\int_{\mathbb{R}^n_-\setminus\Omega} S\,\mathrm{d}x_\delta + \int_{\partial\mathbb{R}^n_-\setminus\cup\dot{\Omega}_i}\frac{\overline{H}}{\sqrt{1+(f_n)^2}}(2+|Df|^2)\,\mathrm{d}x_\delta\\ &+\int_{\partial\mathbb{R}^n_-\setminus\cup\dot{\Omega}_i} f_n\sqrt{1+|\overline{D}f|^2}\left(\sum_{i=1}^{n-1}\tilde{h}(\bar{\partial}_i,\bar{\partial}_i)\right)\frac{|Df|^2}{1+|Df|^2}\,\mathrm{d}x_\delta\\ &-\sum_{i=1}^{\dot{n}}\int_{\partial\dot{\Omega}_i\cap\partial\mathbb{R}^n_-} f_n\langle\overline{D}f, \dot{\eta}\rangle\,\mathrm{d}\dot{\sigma}_i + \int_{\cup\partial\dot{\Omega}_i\cap\{x_n<0\}} H^{\partial\Omega}\,\mathrm{d}\dot{\sigma}\\ &+\int_{\cup\partial\widetilde{\Omega}_i} H^{\partial\Omega}\,\mathrm{d}\tilde{\sigma}.\end{aligned}$$

□

Now we will enunciate some auxiliary results.

Proposition 3.3 [Guan and Li 2009, Theorem 2]. *Let $\Omega\subset\mathbb{R}^{n+1}$ be a limited and star-shaped set. We also suppose that $\partial\Omega$ is smooth and mean-convex. Denote by $H^{\partial\Omega}$ the mean curvature of $\partial\Omega$ with respect to the normal unit vector field inward pointing to Ω and by $B\subset\mathbb{R}^{n+1}$ a unit ball. Then,*

$$\frac{1}{2n\omega_n}\int_{\partial\Omega} H^{\partial\Omega}\,\mathrm{d}\mu_{\partial\Omega} \geq \frac{1}{2}\left(\frac{|\partial\Omega|}{\omega_n}\right)^{(n-1)/n}.$$

Moreover, the equality will occur if and only if Ω is a ball.

Proposition 3.4 [Freire and Schwartz 2014, Theorem 5, item (V)]. *Let $\Omega \subset \mathbb{R}^n$ be a set (not necessarily connected) limited, with a smooth mean-convex and outer-minimizing boundary. Denote by $H^{\partial\Omega}$ the mean curvature of $\partial\Omega$ with respect to the normal unit vector field inward pointing to Ω. Then,*

$$\frac{1}{2(n-1)\omega_{n-1}} \int_{\partial\Omega} H^{\partial\Omega}\, d\partial\Omega \geq \frac{1}{2}\left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{(n-2)/(n-1)}.$$

Moreover, the equality will occur if and only if each connected component of Ω is a rounded ball.

Lemma 3.5 [Huang and Wu 2015, Proposition 5.2]. *Let $a_1, \dots, a_k$ be nonnegative real numbers and $0 \leq \beta \leq 1$. Then,*

$$\sum_{i=1}^{k} a_i^{\beta} \geq \left(\sum_{i=1}^{k} a_i\right)^{\beta}.$$

If $0 \leq \beta < 1$, then the equality holds if and only if at most one element of $\{a_1, \dots, a_k\}$ is nonzero.

Using these results we obtain the following theorem:

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^n_- \setminus \{x_n = 0\}$, $n \geq 3$, be an open bounded set whose boundary is smooth and mean-convex. Suppose that $\partial\Omega$ is outer-minimizing or each connected component of Ω is star-shaped. Let $f : \mathbb{R}^n_- \setminus \Omega \to \mathbb{R}$ be a C^2 function up to boundary, asymptotically flat, constant on each connected component of $\partial\Omega$ and such that $|Df| \to \infty$ when $x \to \partial\Omega$. We also suppose that $f_n = 0$ on $\partial\mathbb{R}^n_-$, and the curvatures that appears at the Proposition 3.2 are nonnegative. Then,*

$$m_{(M,g)} \geq \frac{1}{2}\left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{(n-2)/(n-1)},$$

where $|\partial\Omega|$ denotes the area measure on $\partial\Omega$.

Proof. By the Proposition 3.2, we have

$$2(n-1)\omega_{n-1} m(g) = \int_{\mathbb{R}^n_- \setminus \Omega} S\, dx_\delta + \int_{\partial\mathbb{R}^n_-} \frac{\overline{H}}{\sqrt{1+(f_n)^2}}(2+|Df|^2)\, dx_\delta + \int_{\partial\Omega} H^{\partial\Omega}\, d\partial\Omega.$$

Denoting by Ω_i, $i = 1, \dots, k$, the connected components of Ω, we have

$$\begin{aligned} m_{(M,g)} &\geq \frac{1}{2(n-1)\omega_{n-1}} \sum_i \int_{\partial\Omega_i} H^{\partial\Omega_i}\, d\partial\Omega_i \geq \frac{1}{2} \sum_i \left(\frac{|\partial\Omega_i|}{\omega_{n-1}}\right)^{(n-2)/(n-1)} \\ &\geq \frac{1}{2}\left(\sum_i \frac{|\partial\Omega_i|}{\omega_{n-1}}\right)^{(n-2)/(n-1)} = \frac{1}{2}\left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{(n-2)/(n-1)}. \end{aligned}$$

□

References

[Almaraz et al. 2016] S. Almaraz, E. Barbosa, and L. L. de Lima, "A positive mass theorem for asymptotically flat manifolds with a non-compact boundary", *Comm. Anal. Geom.* **24**:4 (2016), 673–715. MR Zbl

[Escobar 1992] J. F. Escobar, "The Yamabe problem on manifolds with boundary", *J. Differential Geom.* **35**:1 (1992), 21–84. MR Zbl

[Freire and Schwartz 2014] A. Freire and F. Schwartz, "Mass-capacity inequalities for conformally flat manifolds with boundary", *Comm. Partial Differential Equations* **39**:1 (2014), 98–119. MR

[Guan and Li 2009] P. Guan and J. Li, "The quermassintegral inequalities for k-convex starshaped domains", *Adv. Math.* **221**:5 (2009), 1725–1732. MR Zbl

[Huang and Wu 2013] L.-H. Huang and D. Wu, "Hypersurfaces with nonnegative scalar curvature", *J. Differential Geom.* **95**:2 (2013), 249–278. MR Zbl

[Huang and Wu 2015] L.-H. Huang and D. Wu, "The equality case of the Penrose inequality for asymptotically flat graphs", *Trans. Amer. Math. Soc.* **367**:1 (2015), 31–47. MR Zbl

[Lam 2011] M.-K. G. Lam, *The Graph Cases of the Riemannian Positive Mass and Penrose Inequalities in All Dimensions*, Ph.D. thesis, Duke University, 2011, available at https://search.proquest.com/docview/868529318. MR

[Lee and Sormani 2014] D. A. Lee and C. Sormani, "Stability of the positive mass theorem for rotationally symmetric Riemannian manifolds", *J. Reine Angew. Math.* **686** (2014), 187–220. MR Zbl

[Mirandola and Vitório 2015] H. Mirandola and F. Vitório, "The positive mass theorem and Penrose inequality for graphical manifolds", *Comm. Anal. Geom.* **23**:2 (2015), 273–292. MR

[Raulot 2011] S. Raulot, "Green functions for the Dirac operator under local boundary conditions and applications", *Ann. Global Anal. Geom.* **39**:4 (2011), 337–359. MR Zbl

[Reilly 1973] R. C. Reilly, "On the Hessian of a function and the curvatures of its graph", *Michigan Math. J.* **20** (1973), 373–383. MR

Received September 8, 2016. Revised June 27, 2017.

EZEQUIEL BARBOSA
DEPARTAMENTO DE MATEMÁTICA
FEDERAL UNIVERSITY OF MINAS GERAIS
BELO HORIZONTE
BRAZIL
ezequiel@mat.ufmg.br

ADSON MEIRA
DEPARTAMENTO DE MATEMÁTICA
FEDERAL UNIVERSITY OF MINAS GERAIS
BELO HORIZONTE
BRAZIL
nosdariem@ufmg.br

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2018 is US $475/year for the electronic version, and $640/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 294 No. 2 June 2018

A positive mass theorem and Penrose inequality for graphs with noncompact boundary 257
EZEQUIEL BARBOSA and ADSON MEIRA

Diagrams for relative trisections 275
NICKOLAS A. CASTRO, DAVID T. GAY and JUANITA PINZÓN-CAICEDO

Linkage of modules with respect to a semidualizing module 307
MOHAMMAD T. DIBAEI and ARASH SADEGHI

Biharmonic hypersurfaces with constant scalar curvature in space forms 329
YU FU and MIN-CHUN HONG

Nonabelian Fourier transforms for spherical representations 351
JAYCE R. GETZ

Entropy of embedded surfaces in quasi-Fuchsian manifolds 375
OLIVIER GLORIEUX

Smooth Schubert varieties and generalized Schubert polynomials in algebraic cobordism of Grassmannians 401
JENS HORNBOSTEL and NICOLAS PERRIN

Sobolev inequalities on a weighted Riemannian manifold of positive Bakry–Émery curvature and convex boundary 423
SAÏD ILIAS and ABDOLHAKIM SHOUMAN

On the existence of closed geodesics on 2-orbifolds 453
CHRISTIAN LANGE

A Casselman–Shalika formula for the generalized Shalika model of SO_{4n} 473
MIYU SUZUKI

Nontautological bielliptic cycles 495
JASON VAN ZELM

Addendum: Singularities of flat fronts in hyperbolic space 505
MASATOSHI KOKUBU, WAYNE ROSSMAN, KENTARO SAJI, MASAAKI UMEHARA and KOTARO YAMADA