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LINKAGE OF MODULES WITH RESPECT TO A SEMIDUALIZING MODULE

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The notion of linkage with respect to a semidualizing module is introduced. This notion enables us to study the theory of linkage for modules in the Bass class with respect to a semidualizing module. It is shown that over a Cohen–Macaulay local ring with canonical module, every Cohen–Macaulay module of finite Gorenstein injective dimension is linked with respect to the canonical module. For a linked module M with respect to a semidualizing module, the connection between the Serre condition (S_n) on M and the vanishing of certain local cohomology modules of its linked module is discussed.

1. Introduction

The theory of linkage of ideals in commutative algebra was introduced by Peskine and Szpiro [1974]. Recall that two ideals I and J in a Cohen–Macaulay local ring R are said to be linked if there is a regular sequence α in their intersection such that $I = (\alpha : J)$ and $J = (\alpha : I)$. One of the main results in the theory of linkage, due to C. Peskine and L. Szpiro, indicates that the Cohen–Macaulay-ness property is preserved under linkage over Gorenstein local rings. They also give a counterexample to show that the above result is no longer true if the base ring is Cohen–Macaulay but non-Gorenstein. Attempts to generalize this theorem have led to several developments in linkage theory, especially by C. Huneke and B. Ulrich [Huneke 1982; Huneke and Ulrich 1987]. Schenzel [1982] used the theory of dualizing complexes to extend the basic properties of linkage to the linkage by Gorenstein ideals.

The classical linkage theory has been extended to modules by Martin [2000], Yoshino and Isogawa [2000], Martsinkovsky and Strooker [2004], and Nagel [2005], in different ways. Based on these generalizations, several works have been done on studying the linkage theory in the context of modules; see for example [Dibaei

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et al. 2011; Dibaei and Sadeghi 2013; 2015; Iima and Takahashi 2016; Sadeghi 2017; Celikbas et al. 2017]. In this paper, we introduce the notion of linkage with respect to a semidualizing module. This is a new notion of linkage for modules and includes the concept of linkage due to Martsinkovsky and Strooker.

To be more precise, let M and N be R-modules and let α be an ideal of R which is contained in $\operatorname{Ann}_R(M) \cap \operatorname{Ann}_R(N)$. Assume that K is a semidualizing R/α-module. We say that M is linked to N with respect to K if $M \cong \lambda_{R/\alpha}(K, N)$ and $N \cong \lambda_{R/\alpha}(K, M)$, where

$$\lambda_{R/\alpha}(K, -) := \Omega_K \operatorname{Tr}_K \operatorname{Hom}_{R/\alpha}(K, -),$$

where Ω_K, Tr_K are the syzygy and transpose operators, respectively, with respect to K. This notion enables us to study the theory of linkage for modules in the Bass class with respect to a semidualizing module. In the first main result of this paper, over a Cohen–Macaulay local ring with canonical module, it is proved that every Cohen–Macaulay module of finite Gorenstein injective dimension is linked with respect to the canonical module (see Theorem 3.12). More precisely:

Theorem A. *Let R be a Cohen–Macaulay local ring of dimension d with canonical module ω_R. Assume that $\mathfrak{a}$ is a Cohen–Macaulay quasi-Gorenstein ideal of grade n and that M is a Cohen–Macaulay R-module of grade n and of finite Gorenstein injective dimension (equivalently $M \in \mathscr{B}_{\omega_R}$). If $\mathfrak{a} \subseteq \operatorname{Ann}_R(M)$ and M is $\omega_{R/\mathfrak{a}}$-stable, then the following statements hold true:*

(i) *M is linked by ideal $\mathfrak{a}$ with respect to $\omega_{R/\mathfrak{a}}$.*

(ii) *$\lambda_{R/\mathfrak{a}}(\omega_{R/\mathfrak{a}}, M)$ has finite Gorenstein injective dimension.*

(iii) *$\lambda_{R/\mathfrak{a}}(\omega_{R/\mathfrak{a}}, M)$ is Cohen–Macaulay of grade n.*

Martsinkovsky and Strooker [2004, Corollary 2] proved that horizontal linkage preserves the maximal Cohen–Macaulay-ness property over Gorenstein rings, while it may not preserve this property over non-Gorenstein rings. Theorem A shows that, over a Cohen–Macaulay local ring with the canonical module, horizontal linkage with respect to canonical module preserves maximal Cohen–Macaulay-ness for every module of finite Gorenstein injective dimension. Note that over a Gorenstein ring, every module has finite Gorenstein injective dimension. Therefore, Theorem A can be viewed as a generalization of [Martsinkovsky and Strooker 2004, Corollary 2].

Recall that an R-module M is called G-perfect if $\operatorname{grade}_R(M) = \operatorname{G-dim}_R(M)$. If R is Cohen–Macaulay then M is G-perfect if and only if M is Cohen–Macaulay and $\operatorname{G-dim}_R(M) < \infty$. Let us denote the category of G-perfect R-modules by $\mathscr{X}$, and the category of Cohen–Macaulay R-modules of finite Gorenstein injective dimension is by $\mathscr{Y}$. Theorem A enables us to obtain the following adjoint equivalence (see Theorems 3.13 and 3.14).

Theorem B. *Let R be a Cohen–Macaulay local ring with canonical module ω_R and let $\mathfrak{a}$ be a Cohen–Macaulay quasi-Gorenstein ideal, $\bar{R} = R/\mathfrak{a}$. There is an adjoint equivalence*

$$\left\{ M \in \mathscr{X} \;\middle|\; \begin{matrix} M \text{ is linked by} \\ \text{the ideal } \mathfrak{a} \end{matrix} \right\} \underset{\mathrm{Hom}_{\bar{R}}(\omega_{\bar{R}},-)}{\overset{-\otimes_{\bar{R}}\omega_{\bar{R}}}{\rightleftarrows}} \left\{ N \in \mathscr{Y} \;\middle|\; \begin{matrix} N \text{ is linked by the ideal} \\ \mathfrak{a} \text{ with respect to } \omega_{\bar{R}} \end{matrix} \right\}.$$

Let R be a Cohen–Macaulay local ring with canonical module ω_R. For a linked R-module M, with respect to the canonical module, we study the connection between the Serre condition on M with vanishing of certain local cohomology modules of its linked module. We also establish a duality on local cohomology modules of a linked module which is a generalization of [Schenzel 1982, Theorem 4.1; Martsinkovsky and Strooker 2004, Theorem 10] (see Corollaries 4.9 and 4.12).

Theorem C. *Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring of dimension $d > 1$ with canonical module ω_R. Assume that an R-module M is horizontally linked to an R-module N with respect to ω_R and that M has finite Gorenstein injective dimension. Then the following statements hold true*:

(i) *M satisfies (S_n) if and only if $\mathrm{H}^i_{\mathfrak{m}}(N) = 0$ for $d - n < i < d$.*

(ii) *If M is generalized Cohen–Macaulay then*

$$\mathrm{H}^i_{\mathfrak{m}}(\mathrm{Hom}_R(\omega_R, M)) \cong \mathrm{Hom}_R(\mathrm{H}^{d-i}_{\mathfrak{m}}(N), \boldsymbol{E}_R(k)) \quad \text{for } 0 < i < d.$$

In particular, N is generalized Cohen–Macaulay.

The organization of the paper is as follows. In Section 2, we collect preliminary notions, definitions and some known results which will be used in this paper. In Section 3, the precise definition of linkage with respect to a semidualizing is given. We obtain some necessary conditions for an R-module to be linked with respect to a semidualizing (see Theorem 3.7). As a consequence, we prove Theorems A and B in this section. In Section 4, for a linked R-module M, with respect to a semidualizing, the relation between the Serre condition $\widetilde{S}_n$ on M with vanishing of certain relative cohomology modules of its linked module is studied. As a consequence, we prove Theorem C.

2. Preliminaries

Throughout the paper, R is a commutative Noetherian semiperfect ring and all R-modules are finitely generated. Note that a commutative ring R is semiperfect if and only if it is a finite direct product of commutative local rings [Lam 1991, Theorem 23.11]. Whenever, R is assumed to be local, its unique maximal ideal is denoted by $\mathfrak{m}$. The canonical module of R is denoted by ω_R.

Let M be an R-module. For a finite projective presentation $P_1 \xrightarrow{f} P_0 \to M \to 0$ of M, its transpose $\mathsf{Tr}M$ is defined as Coker f^*, where $(-)^* := \mathsf{Hom}_R(-, R)$, which satisfies the exact sequence

$$0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to \mathsf{Tr}M \to 0.$$

Moreover, $\mathsf{Tr}M$ is unique up to projective equivalence. Thus all minimal projective presentations of M represent isomorphic transposes of M. The syzygy module ΩM of M is the kernel of an epimorphism $P \xrightarrow{\alpha} M$, where P is a projective R-module which is unique up to projective equivalence. Thus ΩM is uniquely determined, up to isomorphism, by a projective cover of M.

Martsinkovsky and Strooker [2004] generalized the notion of linkage for modules over noncommutative semiperfect Noetherian rings (i.e., finitely generated modules over such rings have projective covers). In Proposition 1 of that paper, they introduced the operator $\lambda := \Omega\mathsf{Tr}$ and showed that ideals $\mathfrak{a}$ and $\mathfrak{b}$ are linked by zero ideal if and only if $R/\mathfrak{a} \cong \lambda(R/\mathfrak{b})$ and $R/\mathfrak{b} \cong \lambda(R/\mathfrak{a})$.

Definition 2.1 [Martsinkovsky and Strooker 2004, Definition 3]. Two R-modules M and N are said to be *horizontally linked* if $M \cong \lambda N$ and $N \cong \lambda M$. Equivalently, M is horizontally linked (to λM) if and only if $M \cong \lambda^2 M$.

A *stable* module is a module with no nonzero projective direct summands. An R-module M is called a *syzygy module* if it is embedded in a projective R-module. Let i be a positive integer, an R-module M is said to be an i-th syzygy if there exists an exact sequence

$$0 \to M \to P_{i-1} \to \cdots \to P_0,$$

where the $P_0, \ldots, P_{i-1}$ are projective. By convention, every module is a 0-th syzygy.

Here is a characterization of horizontally linked modules.

Theorem 2.2 [Martsinkovsky and Strooker 2004, Theorem 2 and Proposition 3]. *An R-module M is horizontally linked if and only if it is stable and* $\mathsf{Ext}_R^1(\mathsf{Tr}M, R) = 0$, *equivalently M is stable and is a syzygy module.*

Semidualizing modules were initially studied in [Foxby 1972; Golod 1984].

Definition 2.3. An R-module C is called a *semidualizing* module if the homothety morphism $R \to \mathsf{Hom}_R(C, C)$ is an isomorphism and $\mathsf{Ext}_R^i(C, C) = 0$ for all $i > 0$.

It is clear that R itself is a semidualizing R-module. Over a Cohen–Macaulay local ring R, a canonical module ω_R of R, if it exists, is a semidualizing module with finite injective dimension.

Conventions 2.4. Throughout let C denote a semidualizing R-module. We set $(-)^{\triangledown} = \mathsf{Hom}_R(-, C)$ and $(-)^{\curlyvee} = \mathsf{Hom}_R(C, -)$. The notation $(-)^*$ stands for the R-dual functor $\mathsf{Hom}_R(-, R)$. The canonical module of a Cohen–Macaulay local ring, if it exists, is denoted as ω_R; then we set $(-)^\dagger = \mathsf{Hom}_R(-, \omega_R)$.

Let $P_1 \xrightarrow{f} P_0 \to M \to 0$ be a projective presentation of an R-module M. The transpose of M with respect to C, denoted by $\mathsf{Tr}_C M$, is defined to be Coker $f^{\triangledown}$, which satisfies the exact sequence

$$0 \to M^{\triangledown} \to P_0^{\triangledown} \xrightarrow{f^{\triangledown}} P_1^{\triangledown} \to \mathsf{Tr}_C M \to 0. \tag{2.4.1}$$

By [Foxby 1972, Proposition 3.1], there exists the exact sequence

$$0 \to \mathsf{Ext}_R^1(\mathsf{Tr}_C M, C) \to M \to M^{\triangledown\triangledown} \to \mathsf{Ext}_R^2(\mathsf{Tr}_C M, C) \to 0. \tag{2.4.2}$$

The Gorenstein dimension has been extended to G_C-dimension in [Foxby 1972; Golod 1984].

Definition 2.5. An R-module M is said to have G_C*-dimension zero* if M is C-reflexive, i.e., the canonical map $M \to M^{\triangledown\triangledown}$ is bijective, and $\mathsf{Ext}_R^i(M, C) = 0 = \mathsf{Ext}_R^i(M^{\triangledown}, C)$ for all $i > 0$.

A G_C-resolution of an R-module M is a right acyclic complex of G_C-dimension zero modules whose 0-th homology is M. The module M is said to have finite G_C-dimension, denoted by $\mathsf{G}_C\text{-}\mathsf{dim}_R(M)$, if it has a G_C-resolution of finite length.

Note that, over a local ring R, a semidualizing R-module C is a canonical module if and only if $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) < \infty$ for all finitely generated R-modules M; see [Gerko 2001, Proposition 1.3].

In the following, we summarize some basic facts about G_C-dimension; see [Auslander and Bridger 1969; Golod 1984] for more details.

Theorem 2.6. *For an R-module M, the following statements hold true*:

(i) $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) = 0$ *if and only if* $\mathsf{Ext}_R^i(M, C) = 0 = \mathsf{Ext}_R^i(\mathsf{Tr}_C M, C)$ *for all* $i > 0$.

(ii) $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) = 0$ *if and only if* $\mathsf{G}_C\text{-}\mathsf{dim}_R(\mathsf{Tr}_C M) = 0$.

(iii) *If* $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) < \infty$ *then* $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) = \sup\{i \mid \mathsf{Ext}_R^i(M, C) \neq 0,\ i \geq 0\}$.

(iv) *If R is local and* $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) < \infty$, *then* $\mathsf{G}_C\text{-}\mathsf{dim}_R(M) = \mathsf{depth}\, R - \mathsf{depth}_R(M)$.

The Gorenstein injective dimension was introduced by Enochs and Jenda [1995].

Definition 2.7 [Enochs and Jenda 1995; Christensen 2000, Definition 6.2.2]. An R-module M is said to be *Gorenstein injective* if there is an exact sequence

$$I_\bullet = \cdots \to I_1 \xrightarrow{\partial_1} I_0 \xrightarrow{\partial_0} I_{-1} \to \cdots$$

of injective R-modules such that $M \cong \operatorname{Ker}(\partial_0)$ and $\mathsf{Hom}_R(E, I_\bullet)$ is exact for any injective R-module E. The Gorenstein injective dimension of M, denoted by $\mathsf{Gid}(M)$, is defined as the infimum of n for which there exists an exact sequence as $I_\bullet$ with $M \cong \operatorname{Ker}(I_0 \to I_{-1})$ and $I_i = 0$ for all $i < -n$. The Gorenstein injective dimension is a refinement of the classical injective dimension, $\mathsf{Gid}(M) \leq \operatorname{id}(M)$, with equality if $\mathrm{id}(M) < \infty$; see [Christensen 2000, Definition 6.2.6]. It follows that every module over a Gorenstein ring has finite Gorenstein injective dimension.

Definition 2.8. The *Auslander class with respect to C*, denoted by $\mathscr{A}_C$, consists of all R-modules M satisfying the following conditions:

(i) The natural map $\mu : M \to \mathrm{Hom}_R(C, M \otimes_R C)$ is an isomorphism.

(ii) $\mathrm{Tor}_i^R(M, C) = 0 = \mathrm{Ext}_R^i(C, M \otimes_R C)$ for all $i > 0$.

Dually, the *Bass class* with respect to C, denoted by $\mathscr{B}_C$, consists of all R-modules M satisfying the following conditions:

(i) The natural evaluation map $\mu : C \otimes_R \mathrm{Hom}_R(C, M) \to M$ is an isomorphism.

(ii) $\mathrm{Tor}_i^R(\mathrm{Hom}_R(C, M), C) = 0 = \mathrm{Ext}_R^i(C, M)$ for all $i > 0$.

In the following we collect some basic properties and examples of modules in the Auslander class, respectively in the Bass class, with respect to C, which will be used in the rest of this paper.

Fact 2.9. The following statements hold:

(i) If any two R-modules in a short exact sequence are in $\mathscr{A}_C$, respectively $\mathscr{B}_C$, then so is the third one [Foxby 1972, Lemma 1.3]. Hence, every module of finite projective dimension is in the Auslander class $\mathscr{A}_C$. Also the class $\mathscr{B}_C$ contains all modules of finite injective dimension.

(ii) Over a Cohen–Macaulay local ring R with canonical module ω_R, we have $M \in \mathscr{A}_{\omega_R}$ if and only if $\mathrm{G\text{-}dim}_R(M) < \infty$ [Foxby 1975, Theorem 1]. Similarly, $M \in \mathscr{B}_{\omega_R}$ if and only if $\mathrm{Gid}_R(M) < \infty$ [Christensen et al. 2006, Theorem 4.4].

(iii) The $\mathscr{P}_C$-projective dimension of M, denoted by $\mathscr{P}_C\text{-}\mathrm{pd}_R(M)$, is less than or equal to n if and only if there is an exact sequence

$$0 \to P_n \otimes_R C \to \cdots \to P_0 \otimes_R C \to M \to 0$$

such that each P_i is a projective R-module [Takahashi and White 2010, Corollary 2.10]. Note that if M has a finite $\mathscr{P}_C$-projective dimension, then $M \in \mathscr{B}_C$ by Corollary 2.9 of the same paper.

(iv) $M \in \mathscr{A}_C$ if and only if $M \otimes_R C \in \mathscr{B}_C$. Similarly, $M \in \mathscr{B}_C$ if and only if $M^\curlyvee \in \mathscr{A}_C$ [Takahashi and White 2010, Theorem 2.8].

Definition 2.10. Let M and N be R-modules. Denote by $\beta(M, N)$ the set of R-homomorphisms of M to N which pass through projective modules. That is, an R-homomorphism $f : M \to N$ lies in $\beta(M, N)$ if and only if it is factored as $M \to P \to N$, where P is projective. We denote the stable homomorphisms from M to N as the quotient module

$$\underline{\mathrm{Hom}}_R(M, N) = \mathrm{Hom}_R(M, N)/\beta(M, N).$$

By [Yoshino 1990, Lemma 3.9], there is a natural isomorphism

$$\underline{\mathsf{Hom}}_R(M, N) \cong \mathsf{Tor}_1^R(\mathsf{Tr}M, N). \tag{2.10.1}$$

The class of C-projective modules is defined as

$$\mathscr{P}_C = \{P \otimes_R C \mid P \text{ is projective}\}.$$

Two R-modules M and N are said to be *stably equivalent* with respect to C, denoted by $M \underset{C}{\approx} N$, if $C_1 \oplus M \cong C_2 \oplus N$ for some C-projective modules C_1 and C_2. We write $M \approx N$ when M and N are stably equivalent with respect to R. An R-module M is called *C-stable* if M does not have a direct summand isomorphic to a C-projective module. An R-module M is called a *C-syzygy module* if it is embedded in a C-projective R-module.

Remark 2.11. Let M be an R-module.

(i) Let $P_1 \xrightarrow{f} P_0 \to M \to 0$ be the minimal projective presentation of M. Then $\mathsf{Tr}M \otimes_R C \cong \mathsf{Tr}_C M$; see [Dibaei and Sadeghi 2015, Remark 2.1(i)].

(ii) Note that, by [Martsinkovsky 2010, Proposition 3(a)], $(P_1)^* \to \mathsf{Tr}M \to 0$ is minimal. Therefore, by (i), we get the exact sequence

$$0 \to \Omega_C \mathsf{Tr}_C M \to (P_1)^* \otimes_R C \to \mathsf{Tr}_C M \to 0,$$

where $\Omega_C \mathsf{Tr}_C M := \operatorname{Im} f^{\triangledown}$.

(iii) It follows, by (2.10.1), that if $\underline{\mathsf{Hom}}_R(M, C) = 0$, then $\Omega_C \mathsf{Tr}_C M \cong \lambda M \otimes_R C$.

Definition 2.12 [Maşek 2000]. An R-module M is said to satisfy the property $\widetilde{S}_k$ if $\mathsf{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geq \min\{k, \mathsf{depth}\, R_\mathfrak{p}\}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.

Note that, for a horizontally linked module M over a Cohen–Macaulay local ring R, the properties $\widetilde{S}_k$ and (S_k) are identical.

3. Horizontal linkage with respect to a semidualizing

In this section C stands for a semidualizing R-module and M is an R-module. Set $(M)^\curlyvee := \mathsf{Hom}_R(C, M)$ as in Conventions 2.4. In order to develop the notion of linkage with respect to C, we give the following definition.

Definition 3.1. The *linkage of M with respect to C* is defined as the module $\lambda_R(C, M) := \Omega_C \mathsf{Tr}_C(M^\curlyvee)$. The module M is said to be horizontally linked to an R-module N with respect to C if $\lambda_R(C, M) \cong N$ and $\lambda_R(C, N) \cong M$. Equivalently, M is horizontally linked (to $\lambda_R(C, M)$) with respect to C if and only if $M \cong \lambda_R^2(C, M) (= \lambda_R(C, \lambda_R(C, M)))$. In this situation M is called a *horizontally linked module with respect to C*.

Assume that $P_1 \xrightarrow{f} P_0 \to M^{\curlyvee} \to 0$ is the minimal projective presentation of $M^{\curlyvee}$. By Remark 2.11, $\lambda_R(C, M) = \operatorname{Im}(f^{\triangledown})$ and we obtain the exact sequence

$$0 \to \lambda_R(C, M) \to (P_1)^* \otimes_R C \to \mathsf{Tr}_C(M^{\curlyvee}) \to 0. \tag{3.1.1}$$

Therefore $\lambda_R(C, M)$ is unique, up to isomorphism. Having defined the horizontal linkage with respect to a semidualizing module C, the general linkage for modules is defined as follows.

Definition 3.2. Let $\mathfrak{a}$ be an ideal of R and let K be a semidualizing $R/\mathfrak{a}$-module. An R-module M is said to be *linked* to an R-module N by the ideal $\mathfrak{a}$, with respect to K, if $\mathfrak{a} \subseteq \operatorname{Ann}_R(M) \cap \operatorname{Ann}_R(N)$ and M and N are horizontally linked with respect to K as $R/\mathfrak{a}$-modules. In this situation we write $M \underset{\mathfrak{a}}{\overset{K}{\sim}} N$.

Lemma 3.3. *Assume that an R-module M satisfies the following conditions*:

(i) *M is a C-stable and C-syzygy.*

(ii) $\underline{\mathsf{Hom}}_R(M^{\curlyvee}, C) = 0 = \underline{\mathsf{Hom}}_R(\lambda(M^{\curlyvee}), C)$.

(iii) *$M \cong C \otimes_R M^{\curlyvee}$ and $\lambda(M^{\curlyvee}) \cong \mathsf{Hom}_R(C, C \otimes_R \lambda(M^{\curlyvee}))$.*

Then M is a horizontally linked R-module with respect to C.

Proof. As M is C-stable, by (iii), $M^{\curlyvee}$ is stable. By (i), we have the exact sequence $0 \to M \to P \otimes_R C$ for some projective R-module P. By applying the functor $(-)^{\curlyvee}$ to the above exact sequence, it is easy to see that $M^{\curlyvee}$ is a first syzygy. It follows from Theorem 2.2 that $M^{\curlyvee}$ is horizontally linked. In other words, $M^{\curlyvee} \cong \lambda^2(M^{\curlyvee})$. Therefore, we obtain the isomorphisms

$$\begin{aligned} M \cong C \otimes_R M^{\curlyvee} &\cong C \otimes_R \lambda^2(M^{\curlyvee}) \\ &\cong \Omega_C \mathsf{Tr}_C(\lambda(M^{\curlyvee})) \\ &\cong \Omega_C \mathsf{Tr}_C \, \mathsf{Hom}_R(C, C \otimes_R \lambda(M^{\curlyvee})) \\ &\cong \lambda_R(C, \lambda_R(C, M)), \end{aligned}$$

by Remark 2.11(iii) and our assumptions. □

For an integer n, set $X^n(R) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathsf{depth}\, R_{\mathfrak{p}} \leq n\}$.

Lemma 3.4. *Let M be an R-module. Consider the natural map*

$$\mu : M \to \mathsf{Hom}_R(C, M \otimes_R C).$$

Then the following statements hold true:

(i) *If M satisfies $\widetilde{S}_1$ and $\mu_{\mathfrak{p}}$ is a monomorphism for all $\mathfrak{p} \in X^0(R)$, then μ is a monomorphism.*

(ii) *If M satisfies $\widetilde{S}_2$, $M \otimes_R C$ satisfies $\widetilde{S}_1$ and $\mu_{\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p} \in X^1(R)$, then μ is an isomorphism.*

Proof. (i) Set $L = \operatorname{Ker}(\mu)$ and let $\mathfrak{p} \in \operatorname{Ass}_R(L)$. Therefore, $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 0$. As M satisfies $\widetilde{S}_1$, we know $\mathfrak{p} \in X^0(R)$ and so $L_\mathfrak{p} = 0$, which is a contradiction. Therefore, μ is a monomorphism.

(ii) By (i), μ is a monomorphism. Consider the exact sequence

$$0 \to M \xrightarrow{\mu} \operatorname{Hom}_R(C, M \otimes_R C) \to L' \to 0,$$

where $L' := \operatorname{Coker}(\mu)$. Let $\mathfrak{p} \in \operatorname{Ass}_R(L')$. If $\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{Hom}_R(C, M \otimes_R C)) \subseteq \operatorname{Ass}_R(M \otimes_R C)$, then $\operatorname{depth}_{R_\mathfrak{p}}(M \otimes_R C)_\mathfrak{p} = 0$. As $M \otimes_R C$ satisfies $\widetilde{S}_1$, one obtains $\mathfrak{p} \in X^0(R)$, which is a contradiction, because $\mu_\mathfrak{p}$ is an isomorphism for all $\mathfrak{p} \in X^0(R)$. Now let $\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Hom}_R(C, M \otimes_R C)_\mathfrak{p}) > 0$. It follows easily from the above exact sequence that $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 1$. As $M_\mathfrak{p}$ satisfies $\widetilde{S}_2$, we know $\mathfrak{p} \in X^1(R)$, which is a contradiction because $\mu_\mathfrak{p}$ is an isomorphism for all $\mathfrak{p} \in X^1(R)$. Therefore $L' = 0$ and μ is an isomorphism. □

The proof of the following lemma is dual to the proof of [Dibaei and Sadeghi 2015, Lemma 2.11].

Lemma 3.5. *Let R be a local ring, $n \geq 0$ an integer, and M an R-module. If $M \in \mathscr{B}_C$, then the following statements hold true*:

(i) $\operatorname{depth}_R(M) = \operatorname{depth}_R(M^\curlyvee)$ *and* $\dim_R(M) = \dim_R(M^\curlyvee)$.

(ii) *M satisfies $\widetilde{S}_n$ if and only if $M^\curlyvee$ does.*

(iii) *M is Cohen–Macaulay if and only if $M^\curlyvee$ is Cohen–Macaulay.*

Lemma 3.6 [Sather-Wagstaff et al. 2010, Lemma 2.8]. *Let M be an R-module that is in the Bass class $\mathscr{B}_C$. Then $\mathsf{G}_C\text{-}\dim_R(M) = 0$ if and only if $\mathsf{G}\text{-}\dim_R(M^\curlyvee) = 0$.*

In the following result, we give sufficient conditions for an element $M \in \mathscr{B}_C$ to be a horizontally linked module with respect to C.

Theorem 3.7. *Assume that $M \in \mathscr{B}_C$ is a C-syzygy and that $\operatorname{id}_{R_\mathfrak{p}}(C_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in X^1(R)$. If M is C-stable and $\underline{\operatorname{Hom}}_R(M^\curlyvee, C) = 0 = \operatorname{Ext}^1_R(M, C)$, then M is a horizontally linked module with respect to C.*

Proof. We shall prove that the conditions of Lemma 3.3 are satisfied. First note that

$$M \cong M^\curlyvee \otimes C, \tag{3.7.1}$$

because $M \in \mathscr{B}_C$. As seen in the proof of Lemma 3.3, $M^\curlyvee$ is horizontally linked. In other words, $M^\curlyvee \cong \lambda^2(M^\curlyvee)$ and so we obtain the exact sequence

$$0 \to M^\curlyvee \to P \to \operatorname{Tr}\lambda(M^\curlyvee) \to 0, \tag{3.7.2}$$

where P is a projective module. Applying $- \otimes_R C$ gives the exact sequence

$$0 \to \operatorname{Tor}_1^R(\operatorname{Tr}\lambda(M^\curlyvee), C) \to M^\curlyvee \otimes_R C \to P \otimes_R C \to \operatorname{Tr}\lambda(M^\curlyvee) \otimes_R C \to 0. \tag{3.7.3}$$

Let $\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{Tor}_1^R(\operatorname{Tr}\lambda(M^{\curlyvee}), C))$. It follows from (3.7.1) and the exact sequence (3.7.3) that $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{depth}_{R_\mathfrak{p}}((M^{\curlyvee} \otimes_R C)_\mathfrak{p}) = 0$. As M is a C-syzygy module, $\mathfrak{p} \in X^0(R)$. Note that, by Fact 2.9(iv), $M^{\curlyvee} \in \mathscr{A}_C$ and so, $\text{G-dim}_{R_\mathfrak{q}}((M^{\curlyvee})_\mathfrak{q}) = 0$ for all $\mathfrak{q} \in X^0(R)$ by Fact 2.9(ii) and Theorem 2.6(iv). As $\lambda(M^{\curlyvee})$ is a syzygy, one has

$$\operatorname{Ext}_R^1(\operatorname{Tr}\lambda(M^{\curlyvee}), R) = 0. \tag{3.7.4}$$

It follows from (3.7.4), Theorem 2.6 and the exact sequence (3.7.2) that

$$\text{G-dim}_{R_\mathfrak{p}}((\operatorname{Tr}\lambda(M^{\curlyvee}))_\mathfrak{p}) = 0.$$

In other words, by Fact 2.9(ii), $(\operatorname{Tr}\lambda(M^{\curlyvee}))_\mathfrak{p} \in \mathscr{A}_{C_\mathfrak{p}}$. Hence $\operatorname{Tor}_1^R(\operatorname{Tr}\lambda(M^{\curlyvee}), C)_\mathfrak{p} = 0$, which is a contradiction. Therefore, $\underline{\operatorname{Hom}}_R(\lambda(M^{\curlyvee}), C) \cong \operatorname{Tor}_1^R(\operatorname{Tr}\lambda(M^{\curlyvee}), C) = 0$ by (2.10.1).

Now we prove that the natural map $\mu : \lambda(M^{\curlyvee}) \to \operatorname{Hom}_R(C, C \otimes_R \lambda(M^{\curlyvee}))$ is an isomorphism. To this end, we concentrate on Lemma 3.4. As $M^{\curlyvee}$ is horizontally linked, we obtain the isomorphisms

$$\begin{aligned} \operatorname{Ext}_R^2(\operatorname{Tr}\lambda(M^{\curlyvee}), R) &\cong \operatorname{Ext}_R^1(\lambda^2(M^{\curlyvee}), R) \\ &\cong \operatorname{Ext}_R^1(M^{\curlyvee}, R) \\ &\cong \operatorname{Ext}_R^1(M^{\curlyvee}, C^{\curlyvee}) \\ &\cong \operatorname{Ext}_R^1(M, C) = 0, \end{aligned} \tag{3.7.5}$$

by [Takahashi and White 2010, Theorem 4.1 and Corollary 4.2]. It follows from (3.7.4) and (3.7.5) that $\lambda(M^{\curlyvee})$ is second syzygy and so it satisfies $\widetilde{S}_2$ by [Maşek 2000, Proposition 11]. By the exact sequence $0 \to \lambda(M^{\curlyvee}) \to P' \to \operatorname{Tr}(M^{\curlyvee}) \to 0$ and the fact that $\operatorname{Tor}_1^R(\operatorname{Tr}(M^{\curlyvee}), C) \cong \underline{\operatorname{Hom}}_R(M^{\curlyvee}, C) = 0$, it follows that $\lambda(M^{\curlyvee}) \otimes_R C$ satisfies $\widetilde{S}_1$. As M satisfies $\widetilde{S}_1$, by Fact 2.9(ii), (iv), Lemma 3.5 and Theorem 2.6(iv), $\text{G-dim}_{R_\mathfrak{p}}((M^{\curlyvee})_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in X^1(R)$. Therefore, $\text{G-dim}_{R_\mathfrak{p}}((\lambda(M^{\curlyvee}))_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in X^1(R)$ by [Martsinkovsky and Strooker 2004, Theorem 1] and so $(\lambda(M^{\curlyvee}))_\mathfrak{p} \in \mathscr{A}_{C_\mathfrak{p}}$ for all $\mathfrak{p} \in X^1(R)$ by Fact 2.9(ii). Hence μ is an isomorphism by Lemma 3.4. Now the assertion is clear by Lemma 3.3. □

Martsinkovsky and Strooker [2004, Corollary 2] proved that, over a Gorenstein ring, horizontal linkage preserves the property of a module to be maximal Cohen–Macaulay, while they showed in the example on page 601 of the same paper that over non-Gorenstein rings, being maximal Cohen–Macaulay need not be preserved under horizontal linkage. In the following, it is shown that, over a Cohen–Macaulay local ring with the canonical module, horizontal linkage with respect to canonical module preserves maximal Cohen–Macaulay-ness. Note that over a Gorenstein ring, every module has finite Gorenstein injective dimension. Therefore, the following result can be viewed as a generalization of [Martsinkovsky and Strooker 2004, Corollary 2].

Corollary 3.8. *Let R be a Cohen–Macaulay local ring with canonical module ω_R. Assume that M is a maximal Cohen–Macaulay R-module of finite Gorenstein injective dimension. If M is ω_R-stable then the following statements hold true*:

(i) *M is horizontally linked with respect to ω_R.*

(ii) *$\lambda_R(\omega_R, M)$ has finite Gorenstein injective dimension.*

(iii) *$\lambda_R(\omega_R, M)$ is maximal Cohen–Macaulay.*

Proof. (i) By Fact 2.9(ii), $M \in \mathscr{B}_{\omega_R}$. As M is maximal Cohen–Macaulay, it is a ω_R-syzygy and also $\mathsf{Ext}^1_R(M, \omega_R) = 0$. Therefore, by Theorem 3.7, it is enough to prove that $\underline{\mathsf{Hom}}_R(\mathsf{Hom}_R(\omega_R, M), \omega_R) = 0$. Note that $\mathsf{G\text{-}dim}_R(\mathsf{Hom}_R(\omega_R, M)) = 0$ by Theorem 2.6 and Lemma 3.6. Hence $\mathsf{G\text{-}dim}_R(\mathsf{Tr}\,\mathsf{Hom}_R(\omega_R, M)) = 0$ and $\mathsf{Tr}\,\mathsf{Hom}_R(\omega_R, M) \in \mathscr{A}_{\omega_R}$ by Fact 2.9(ii) so that $\mathsf{Tor}^R_i(\mathsf{Tr}\,\mathsf{Hom}_R(\omega_R, M), \omega_R) = 0$ for all $i > 0$. Indeed, by (2.10.1),

$$\underline{\mathsf{Hom}}_R(\mathsf{Hom}_R(\omega_R, M), \omega_R) \cong \mathsf{Tor}^R_1(\mathsf{Tr}\,\mathsf{Hom}_R(\omega_R, M), \omega_R) = 0.$$

Therefore, by Theorem 3.7, M is horizontally linked with respect to ω_R.

(ii) As we have seen in part (i), $\mathsf{Tr}(\mathsf{Hom}_R(\omega_R, M)) \in \mathscr{A}_{\omega_R}$. Hence by Fact 2.9(iv) and Remark 2.11(i), $\mathsf{Tr}_{\omega_R}(\mathsf{Hom}_R(\omega_R, M)) \in \mathscr{B}_{\omega_R}$. Therefore, $\mathsf{Gid}_R(\lambda_R(\omega_R, M)) < \infty$ by Fact 2.9(i) and the exact sequence (3.1.1).

(iii) By Lemma 3.5, $\mathsf{Hom}_R(\omega_R, M)$ is maximal Cohen–Macaulay. Therefore $\mathsf{Tr}_{\omega_R}(\mathsf{Hom}_R(\omega_R, M))$ is maximal Cohen–Macaulay by Theorem 2.6(ii). It follows from the exact sequence (3.1.1) that $\lambda_R(\omega_R, M)$ is maximal Cohen–Macaulay. □

To prove Theorem A, we first bring the following lemma and recall a definition.

Lemma 3.9. *Let R be a Cohen–Macaulay local ring and let I be an unmixed ideal of R. Assume that K is a semidualizing R/I-module and that M is an R-module which is linked by I with respect to K. Then* $\mathsf{grade}_R(M) = \mathsf{grade}_R(I)$.

Proof. First note that $\mathsf{grade}_R(M) = \inf\{\mathsf{depth}\, R_\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Supp}_R(M)\}$. Therefore, $\mathsf{grade}_R(M) = \mathsf{depth}\, R_\mathfrak{p}$ for some $\mathfrak{p} \in \mathsf{Min}_R(M)$ and so $\mathfrak{p}/I \in \mathsf{Min}_{R/I}(M)$. As M is linked by I with respect to K, it is a first K-syzygy module and so $\mathfrak{p}/I \in \mathrm{Ass}_{R/I}(R/I)$, because $\mathrm{Ass}_{R/I}(K) = \mathrm{Ass}_{R/I}(R/I)$. As I is unmixed, $\mathsf{grade}(I) = \mathsf{depth}\, R_\mathfrak{p}$. □

Let $(R, \mathfrak{m}, k)$ be a local ring and let M be an R-module. For every integer $n \geq 0$ the n-th Bass number $\mu^n_R(M)$ is the dimension of the k-vector space $\mathsf{Ext}^n_R(k, M)$.

Definition 3.10 [Avramov and Foxby 1997]. An ideal $\mathfrak{a}$ of a local ring R is called *quasi-Gorenstein* if $\mathsf{G\text{-}dim}_R(R/\mathfrak{a}) < \infty$ and for every $i \geq 0$ there is an equality of Bass numbers

$$\mu_R^{i+\mathsf{depth}\, R}(R) = \mu_{R/\mathfrak{a}}^{i+\mathsf{depth}\, R/\mathfrak{a}}(R/\mathfrak{a}).$$

Theorem 3.11 [Avramov and Foxby 1997, Corollary 7.9]. *Let R be a Cohen–Macaulay local ring with canonical module ω_R and let $\mathfrak{a}$ be a quasi-Gorenstein ideal of R. For an $R/\mathfrak{a}$-module M, we have $\mathsf{Gid}_R(M) < \infty$ if and only if $\mathsf{Gid}_{R/\mathfrak{a}}(M) < \infty$. Also, $\mathsf{G\text{-}dim}_R(M) < \infty$ if and only if $\mathsf{G\text{-}dim}_{R/\mathfrak{a}}(M) < \infty$.*

We now present Theorem A.

Theorem 3.12. *Let R be a Cohen–Macaulay local ring of dimension d with canonical module ω_R and let $\mathfrak{a}$ be a Cohen–Macaulay quasi-Gorenstein ideal of grade n, $\bar{R} = R/\mathfrak{a}$. Assume that M is a Cohen–Macaulay R-module of grade n and of finite Gorenstein injective dimension such that $\mathfrak{a} \subseteq \operatorname{Ann}_R(M)$. If M is $\omega_{\bar{R}}$-stable then the following statements hold true:*

(i) *M is linked by ideal $\mathfrak{a}$ with respect to $\omega_{\bar{R}}$.*

(ii) *$\lambda_{\bar{R}}(\omega_{\bar{R}}, M)$ has finite Gorenstein injective dimension.*

(iii) *$\lambda_{\bar{R}}(\omega_{\bar{R}}, M)$ is Cohen–Macaulay of grade n.*

Proof. (i) As R is Cohen–Macaulay,

$$d - n = d - \mathsf{grade}(\mathfrak{a}) = \dim(R/\mathfrak{a}).$$

On the other hand, as M is Cohen–Macaulay of grade n,

$$\mathsf{depth}_{\bar{R}}(M) = \mathsf{depth}_R(M) = \dim_R(M) = d - n.$$

Therefore, M is a maximal Cohen–Macaulay $\bar{R}$-module. By Theorem 3.11, $\mathsf{Gid}_{\bar{R}}(M)$ is finite and so M is horizontally linked with respect to $\omega_{\bar{R}}$ as an $\bar{R}$-module by Corollary 3.8.

(ii) By Corollary 3.8, $\mathsf{Gid}_{\bar{R}}(\lambda_{\bar{R}}(\omega_{\bar{R}}, M)) < \infty$, which by Theorem 3.11 is equivalent to $\mathsf{Gid}_R(\lambda_{\bar{R}}(\omega_{\bar{R}}, M)) < \infty$.

(iii) By Corollary 3.8, $\lambda_{\bar{R}}(\omega_{\bar{R}}, M)$ is a maximal Cohen–Macaulay $\bar{R}$-module. Hence

$$\mathsf{depth}_R(\lambda_{\bar{R}}(\omega_{\bar{R}}, M)) = \mathsf{depth}_{\bar{R}}(\lambda_{\bar{R}}(\omega_{\bar{R}}, M)) = \dim(R/\mathfrak{a}).$$

Also, by Lemma 3.9, $\mathsf{grade}_R(\lambda_{\bar{R}}(\omega_{\bar{R}}, M)) = n$. Hence,

$$\dim_R(\lambda_{\bar{R}}(\omega_{\bar{R}}, M)) = d - n = \dim R/\mathfrak{a}.$$

Therefore, $\lambda_{\bar{R}}(\omega_{\bar{R}}, M)$ is Cohen–Macaulay as an R-module. □

Let R be a Cohen–Macaulay local ring with canonical module ω_R. Set

$$\mathscr{X} := \mathrm{CM}(R) \cap \mathscr{A}_{\omega_R} \quad \text{and} \quad \mathscr{Y} := \mathrm{CM}(R) \cap \mathscr{B}_{\omega_R},$$

where $\mathrm{CM}(R)$ is the category of Cohen–Macaulay R-module. Now we prove Theorem B.

Theorem 3.13. *Let R be a Cohen–Macaulay local ring with canonical module ω_R and let $\mathfrak{a}$ be a Cohen–Macaulay quasi-Gorenstein ideal of grade n, $\overline{R} = R/\mathfrak{a}$. There is an adjoint equivalence*

$$\left\{ M \in \mathscr{X} \;\middle|\; \begin{array}{c} M \text{ is linked} \\ \text{by the ideal } \mathfrak{a} \end{array} \right\} \underset{\mathsf{Hom}_{\overline{R}}(\omega_{\overline{R}},-)}{\overset{-\otimes_{\overline{R}}\omega_{\overline{R}}}{\rightleftarrows}} \left\{ N \in \mathscr{Y} \;\middle|\; \begin{array}{c} N \text{ is linked by the ideal} \\ \mathfrak{a} \text{ with respect to } \omega_{\overline{R}} \end{array} \right\}.$$

Proof. Let $M \in \mathscr{X}$, which is linked by the ideal $\mathfrak{a}$. By Theorem 3.11, $M \in \mathscr{A}_{\omega_{\overline{R}}}$. Note that $\mathfrak{a}$ is a G-perfect ideal and so $\mathsf{grade}_R(M) = \mathsf{grade}_R(\mathfrak{a})$ by [Sadeghi 2017, Lemma 3.16]. Therefore

$$\mathsf{depth}_R(M) = \dim_R(M) = \dim R - \mathsf{grade}_R(M) = \dim R - \mathsf{grade}_R(\mathfrak{a}).$$

Hence M is a maximal Cohen–Macaulay $\overline{R}$-module. Set $N = M \otimes_{\overline{R}} \omega_{\overline{R}}$. By [Dibaei and Sadeghi 2015, Lemma 2.11], N is a maximal Cohen–Macaulay $\overline{R}$-module. Therefore $N \in \mathrm{CM}(R)$. Also, by Fact 2.9(iv) and Theorem 3.11, $N \in \mathscr{B}_{\omega_R}$. Hence $N \in \mathscr{Y}$. As $M \in \mathscr{A}_{\omega_{\overline{R}}}$,

$$M \cong \mathsf{Hom}_{\overline{R}}(\omega_{\overline{R}}, N). \tag{3.13.1}$$

Note that M is stable $\overline{R}$-module by Theorem 2.2. It follows from (3.13.1) that N is $\omega_{\overline{R}}$-stable. Hence, by Theorem 3.12, N is linked by the ideal $\mathfrak{a}$ with respect to $\omega_{\overline{R}}$.

Conversely, assume that $N \in \mathscr{Y}$, which is linked by the ideal $\mathfrak{a}$ with respect to $\omega_{\overline{R}}$. As N is Cohen–Macaulay, by Lemma 3.9,

$$\mathsf{depth}_R(N) = \dim_R(N) = \dim R - \mathsf{grade}_R(N) = \dim R - \mathsf{grade}_R(\mathfrak{a}).$$

Therefore N is a maximal Cohen–Macaulay $\overline{R}$-module. Set $M = \mathsf{Hom}_{\overline{R}}(\omega_{\overline{R}}, N)$. Note that by Theorem 3.11, $N \in \mathscr{B}_{\omega_{\overline{R}}}$. Hence $M \in \mathscr{A}_{\omega_R}$ by Fact 2.9(iv), and Theorem 3.11. Also, by Lemma 3.5, M is a maximal Cohen–Macaulay $\overline{R}$-module. Thus $M \in \mathscr{X}$. Set $X = \mathsf{Hom}_{\overline{R}}(\omega_{\overline{R}}, \lambda_{\overline{R}}(\omega_{\overline{R}}, N))$. It follows from Theorem 3.12(ii), Fact 2.9(ii), (iv), and Theorem 3.11 that $X \in \mathscr{A}_{\omega_{\overline{R}}}$. Also, by Theorem 3.12(iii) and Lemma 3.5, X is a maximal Cohen–Macaulay $\overline{R}$-module. Therefore, by Theorem 2.6(ii), (iv) and Fact 2.9(ii), $\mathsf{G\text{-}dim}_{\overline{R}}(\lambda_{\overline{R}} X) = 0$. In other words, $\lambda_{\overline{R}} X \in \mathscr{A}_{\omega_{\overline{R}}}$. Hence,

$$\lambda_{\overline{R}} X \cong \mathsf{Hom}_{\overline{R}}(\omega_{\overline{R}}, \lambda_{\overline{R}} X \otimes_{\overline{R}} \omega_{\overline{R}}). \tag{3.13.2}$$

As $\lambda_{\overline{R}} X$ is a first syzygy of $\mathsf{Tr}_{\overline{R}} X$, by Fact 2.9(i), $\mathsf{Tr}_{\overline{R}} X \in \mathscr{A}_{\omega_{\overline{R}}}$. Therefore $\underline{\mathsf{Hom}}_{\overline{R}}(X, \omega_{\overline{R}}) \cong \mathsf{Tor}_1^{\overline{R}}(\mathsf{Tr}_{\overline{R}} X, \omega_{\overline{R}}) = 0$. As N is linked by the ideal $\mathfrak{a}$ with respect to $\omega_{\overline{R}}$, it follows from Remark 2.11(iii) that

$$N \cong \Omega_{\omega_{\overline{R}}} \mathsf{Tr}_{\omega_{\overline{R}}} X \cong \lambda_{\overline{R}} X \otimes_{\overline{R}} \omega_{\overline{R}}. \tag{3.13.3}$$

It follows from (3.13.2) and (3.13.3) that $M \cong \lambda_{\overline{R}} X$. Hence, by [Avramov 1998, Corollary 1.2.5], M is a stable $\overline{R}$-module. By [Martsinkovsky and Strooker 2004, Theorem 1], M is linked by the ideal $\mathfrak{a}$. □

Let $\mathfrak{a}$ be an ideal of R an let M be an $R/\mathfrak{a}$-module. Recall that M is said to be *self-linked* by the ideal $\mathfrak{a}$ if $M \cong \lambda_{R/\mathfrak{a}} M$. Let K be a semidualizing $R/\mathfrak{a}$-module. An $R/\mathfrak{a}$-module N is called *self-linked* by the ideal $\mathfrak{a}$ with respect to K if $N \cong \lambda_{R/\mathfrak{a}}(K, N)$.

Theorem 3.14. *Let R be a Cohen–Macaulay local ring with canonical module ω_R and let $\mathfrak{a}$ be a Cohen–Macaulay quasi-Gorenstein ideal of grade n, $\bar{R} = R/\mathfrak{a}$. There is an adjoint equivalence*

$$\left\{ M \in \mathscr{A}_{\omega_R} \,\middle|\, \begin{array}{c} M \text{ is self-linked} \\ \text{by the ideal } \mathfrak{a} \end{array} \right\} \underset{\mathsf{Hom}_{\bar{R}}(\omega_{\bar{R}},-)}{\overset{-\otimes_{\bar{R}}\omega_{\bar{R}}}{\rightleftarrows}} \left\{ N \in \mathscr{B}_{\omega_R} \,\middle|\, \begin{array}{c} N \text{ is self-linked by the} \\ \text{ideal } \mathfrak{a} \text{ with respect to } \omega_{\bar{R}} \end{array} \right\}.$$

Proof. Let $M \in \mathscr{A}_{\omega_R}$ and let $M \cong \lambda_{\bar{R}} M$. It follows from Theorem 3.11 that $M \in \mathscr{A}_{\omega_{\bar{R}}}$. Set $N = M \otimes_{\bar{R}} \omega_{\bar{R}}$. Therefore,

$$M \cong \mathsf{Hom}_{\bar{R}}(\omega_{\bar{R}}, N). \tag{3.14.1}$$

As $M \cong \Omega_{\bar{R}} \mathsf{Tr}_{\bar{R}} M$, we have $\mathsf{Tr}_{\bar{R}} M \in \mathscr{A}_{\omega_{\bar{R}}}$. Hence,

$$\underline{\mathsf{Hom}}_{\bar{R}}(M, \omega_{\bar{R}}) \cong \mathsf{Tor}_1^{\bar{R}}(\mathsf{Tr}_{\bar{R}} M, \omega_{\bar{R}}) = 0.$$

It follows from (3.14.1) and Remark 2.11(iii) that

$$\begin{aligned} \lambda_{\bar{R}}(\omega_{\bar{R}}, N) &= \Omega_{\omega_{\bar{R}}} \mathsf{Tr}_{\omega_{\bar{R}}}(\mathsf{Hom}_{\bar{R}}(\omega_{\bar{R}}, N)) \\ &\cong \Omega_{\omega_{\bar{R}}} \mathsf{Tr}_{\omega_{\bar{R}}}(M) \\ &\cong \lambda_{\bar{R}} M \otimes_{\bar{R}} \omega_{\bar{R}} \\ &\cong M \otimes_{\bar{R}} \omega_{\bar{R}} = N. \end{aligned}$$

In other words, N is self-linked by the ideal $\mathfrak{a}$ with respect to $\omega_{\bar{R}}$. Also, by Fact 2.9(iv), Theorem 3.11, $N \in \mathscr{B}_{\omega_R}$.

Conversely, assume that $N \in \mathscr{B}_{\omega_R}$ which is self-linked by the ideal $\mathfrak{a}$ with respect to $\omega_{\bar{R}}$. Set $M = \mathsf{Hom}_{\bar{R}}(\omega_{\bar{R}}, N)$. It follows from Fact 2.9(iv), Theorem 3.11 that $M \in \mathscr{A}_{\omega_R}$. As $N \cong \lambda_{\bar{R}}(\omega_{\bar{R}}, N)$, we have $\mathsf{Tr}_{\omega_{\bar{R}}}(M) \in \mathscr{B}_{\omega_{\bar{R}}}$ by the exact sequence (3.1.1), Fact 2.9(i) and Theorem 3.11. It follows from Remark 2.11(i) and Fact 2.9(iv) that $\mathsf{Tr}_{\bar{R}}(M) \in \mathscr{A}_{\omega_{\bar{R}}}$. Therefore $\underline{\mathsf{Hom}}_{\bar{R}}(M, \omega_{\bar{R}}) \cong \mathsf{Tor}_1^{\bar{R}}(\mathsf{Tr}_{\bar{R}}(M)\omega_{\bar{R}}) = 0$. Hence, by Remark 2.11(iii),

$$N \cong \lambda_{\bar{R}}(\omega_{\bar{R}}, N) \cong \lambda_{\bar{R}}(M) \otimes_{\bar{R}} \omega_{\bar{R}}. \tag{3.14.2}$$

As $\mathsf{Tr}_{\bar{R}}(M) \in \mathscr{A}_{\omega_{\bar{R}}}$, we have $\lambda_{\bar{R}} M \in \mathscr{A}_{\omega_{\bar{R}}}$. Hence

$$\lambda_{\bar{R}} M \cong \mathsf{Hom}_{\bar{R}}(\omega_{\bar{R}}, \lambda_{\bar{R}} M \otimes_{\bar{R}} \omega_{\bar{R}}). \tag{3.14.3}$$

It follows from (3.14.2) and (3.14.3) that $M \cong \lambda_{\bar{R}} M$. □

4. Serre condition and vanishing of local cohomology

In this section, for a linked module, we study the relation between the Serre condition $\widetilde{S}_n$ and the vanishing of certain relative cohomology modules of its linked module. As a consequence, [Schenzel 1982, Theorem 4.1] is generalized. We start with the following lemma, which will be used in the proof of Theorem 4.2.

Lemma 4.1. *Let M be a C-syzygy module. Then $\mathrm{Ext}^1_R(\mathrm{Tr}_C(M^\curlyvee), C) = 0$. In particular, if M is horizontally linked with respect to C, then $\mathrm{Ext}^1_R(\mathrm{Tr}_C(M^\curlyvee), C) = 0$.*

Proof. Consider the exact sequence $0 \to M \to P \otimes_R C$, where P is a projective R-module. Applying the functor $(-)^\curlyvee$ to the above exact sequence, we get the exact sequence $0 \to M^\curlyvee \to P$. Therefore, $\mathrm{Ext}^1_R(\mathrm{Tr} M^\curlyvee, R) = 0$. By [Rotman 2009, Theorem 10.62], there is a third quadrant spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{Ext}^p_R(\mathrm{Tor}^R_q(\mathrm{Tr}(M^\curlyvee), C), C) \Rightarrow \mathrm{Ext}^{p+q}_R(\mathrm{Tr}(M^\curlyvee), R).$$

Hence we obtain the following exact sequence

$$0 \to \mathrm{Ext}^1_R(\mathrm{Tr}(M^\curlyvee) \otimes_R C, C) \to \mathrm{Ext}^1_R(\mathrm{Tr}(M^\curlyvee), R),$$

by [Rotman 2009, Theorem 10.33]. Hence, by Remark 2.11,

$$\mathrm{Ext}^1_R(\mathrm{Tr}_C(M^\curlyvee), C) \cong \mathrm{Ext}^1_R(\mathrm{Tr}(M^\curlyvee) \otimes_R C, C) = 0. \qquad \square$$

The following is a generalization of [Martsinkovsky and Strooker 2004, Theorem 1].

Theorem 4.2. *Let M be an R-module which is horizontally linked with respect to C. Assume $M \in \mathscr{B}_C$. Then $\mathrm{G}_C\text{-}\mathrm{dim}_R(M) = 0$ if and only if $\mathrm{G}\text{-}\mathrm{dim}_R((\lambda_R(C, M))^\curlyvee) = 0$.*

Proof. Set $N = \lambda_R(C, M)$. Consider the exact sequence

$$0 \to M \to P^* \otimes_R C \to \mathrm{Tr}_C(N^\curlyvee) \to 0, \tag{4.2.1}$$

where P is a projective R-module; see (3.1.1). As $M \in \mathscr{B}_C$, we know $\mathrm{Tr}_C(N^\curlyvee) \in \mathscr{B}_C$ by the exact sequence (4.2.1) and Fact 2.9(i). Hence $\mathrm{Tr}(N^\curlyvee) \in \mathscr{A}_C$ by Remark 2.11 and Fact 2.9(iv). In particular,

$$\mathrm{Tr}(N^\curlyvee) \cong \mathrm{Hom}_R(C, \mathrm{Tr}(N^\curlyvee) \otimes_R C) \cong \mathrm{Hom}_R(C, \mathrm{Tr}_C(N^\curlyvee)). \tag{4.2.2}$$

It follows from Theorem 2.6(ii), Lemma 3.6 and (4.2.2) that

$$\mathrm{G}\text{-}\mathrm{dim}_R(N^\curlyvee) = 0 \iff \mathrm{G}_C\text{-}\mathrm{dim}_R(\mathrm{Tr}_C(N^\curlyvee)) = 0. \tag{4.2.3}$$

On the other hand, by the exact sequence (4.2.1)

$$\mathrm{G}_C\text{-}\mathrm{dim}_R(M) = 0 \text{ and } \mathrm{Ext}^1_R(\mathrm{Tr}_C(N^\curlyvee), C) = 0 \iff \mathrm{G}_C\text{-}\mathrm{dim}_R(\mathrm{Tr}_C(N^\curlyvee)) = 0. \tag{4.2.4}$$

Now the assertion is clear by (4.2.3), (4.2.4) and Lemma 4.1. $\square$

The class $\mathscr{P}_C$ is precovering and then each R-module M has an augmented proper $\mathscr{P}_C$-resolution; that is, there is an R-complex

$$X^+ = \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to M \to 0$$

such that $\mathsf{Hom}_R(Y, X^+)$ is exact for all $Y \in \mathscr{P}_C$. The truncated complex

$$X = \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to 0$$

is called a proper $\mathscr{P}_C$-projective resolution of M. Proper $\mathscr{P}_C$-projective resolutions are unique up to homotopy equivalence.

Definition 4.3 [Takahashi and White 2010]. Let M and N be R-modules. The n-th relative cohomology module is defined as $\mathsf{Ext}^n_{\mathscr{P}_C}(M, N) = \mathrm{H}^n \,\mathsf{Hom}_R(X, N)$, where X is a proper $\mathscr{P}_C$-projective resolution of M.

Theorem 4.4 [Takahashi and White 2010, Theorem 4.1 and Corollary 4.2]. *Let M and N be R-modules. Then there exists an isomorphism*

$$\mathsf{Ext}^i_{\mathscr{P}_C}(M, N) \cong \mathsf{Ext}^i_R(M^{\curlyvee}, N^{\curlyvee})$$

for all $i \geq 0$. Moreover, if M and N are in $\mathscr{B}_C$ then $\mathsf{Ext}^i_{\mathscr{P}_C}(M, N) \cong \mathsf{Ext}^i_R(M, N)$ for all $i \geq 0$.

For a positive integer n, a module M is called an n-th C-syzygy module if there is an exact sequence $0 \to M \to C_1 \to C_2 \to \cdots \to C_n$, where $C_i \in \mathscr{P}_C$ for all i. The following results will be used in the proof of Theorem 4.7.

Lemma 4.5. *Let M be an R-module such that $\mathsf{G}_{C_p}\text{-}\mathsf{dim}_{\mathsf{R}_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in X^{n-2}(R)$. Then the following statements are equivalent*:

(i) *M is an n-th C-syzygy module.*

(ii) $\mathsf{Ext}^i_R(\mathsf{Tr}_C M, C) = 0$ *for* $0 < i < n$.

Proof. The proof is analogous to [Maşek 2000, Theorem 43]. □

Theorem 4.6 [Dibaei and Sadeghi 2015, Proposition 2.4]. *Let C be a semidualizing R-module and M an R-module. For a positive integer n, consider the following statements*:

(i) $\mathsf{Ext}^i_R(\mathsf{Tr}_C M, C) = 0$ *for all i, $1 \leq i \leq n$.*

(ii) *M is an n-th C-syzygy module.*

(iii) *M satisfies $\widetilde{S}_n$.*

Then the following implications hold true:

(a) (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

(b) *If M has finite G_C-dimension on $X^{n-1}(R)$, then* (iii) $\Longrightarrow$ (i).

The following is a generalization of [Schenzel 1982, Theorem 4.1].

Theorem 4.7. *Let M be an R-module which is horizontally linked with respect to C. Assume that $M \in \mathscr{B}_C$. For a positive integer n, consider the following statements*:

(i) $\mathsf{Ext}^i_{\mathscr{P}_C}(\lambda_R(C, M), C) = 0$ *for* $0 < i < n$.

(ii) *M is an n-th C-syzygy module.*

(iii) *M satisfies $\widetilde{S}_n$.*

Then the following implications hold true:

(a) (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

(b) *If M has finite G_C-dimension on $X^{n-2}(R)$, then the statements* (i) *and* (ii) *are equivalent.*

(c) *If M has finite G_C-dimension on $X^{n-1}(R)$, then all the statements* (i)–(iii) *are equivalent.*

Proof. Set $N = \lambda_R(C, M)$. Consider the exact sequence

$$0 \to M \to P \otimes_R C \to \mathsf{Tr}_C(N^{\curlyvee}) \to 0, \tag{4.7.1}$$

where P is a projective R-module. By Lemma 4.1,

$$\mathsf{Ext}^1_R(\mathsf{Tr}_C(N^{\curlyvee}), C) = 0. \tag{4.7.2}$$

Therefore, by [Dibaei and Sadeghi 2015, Lemma 2.2], the exact sequence (4.7.1) induces the exact sequence

$$0 \to \mathsf{Tr}_C\mathsf{Tr}_C(N^{\curlyvee}) \to Q \otimes_R C \to \mathsf{Tr}_C M \to 0, \tag{4.7.3}$$

where Q is a projective R-module. Moreover, by [Sadeghi 2017, Lemma 2.12], there exists the following exact sequence

$$0 \to N^{\curlyvee} \to \mathsf{Tr}_C\mathsf{Tr}_C(N^{\curlyvee}) \to X \to 0, \tag{4.7.4}$$

where $\mathsf{G}_C\text{-}\mathsf{dim}_R(X) = 0$. As M is horizontally linked with respect to C, it is a C-syzygy module and so $\mathsf{Ext}^1_R(\mathsf{Tr}_C M, C) = 0$. Therefore, by the exact sequences (4.7.3) and (4.7.4), we obtain

$$\mathsf{Ext}^i_R(\mathsf{Tr}_C M, C) = 0 \text{ for } 1 \le i \le n \iff \mathsf{Ext}^i_R(N^{\curlyvee}, C) = 0 \text{ for } 1 \le i \le n-1. \tag{4.7.5}$$

As $M \in \mathscr{B}_C$, by Fact 2.9(i) and the exact sequence (4.7.1), $\mathsf{Tr}_C(N^{\curlyvee}) \in \mathscr{B}_C$. Hence, by Fact 2.9(iv) and Remark 2.11(i), $\mathsf{Tr}(N^{\curlyvee}) \in \mathscr{A}_C$. It follows from [Sadeghi 2017, Theorem 4.1] that

$$\mathsf{Ext}^i_R(N^{\curlyvee}, C) = 0 \text{ for } 0 < i < n \iff \mathsf{Ext}^i_R(N^{\curlyvee}, R) = 0 \text{ for } 0 < i < n. \tag{4.7.6}$$

Note that, by Theorem 4.4, we have the isomorphism

$$\mathrm{Ext}^i_{\mathscr{P}_C}(N, C) \cong \mathrm{Ext}^i_R(N^{\curlyvee}, R) \quad \text{for all } i \geq 0. \tag{4.7.7}$$

Implications (a) and (c) follow from (4.7.5), (4.7.6), (4.7.7) and Theorem 4.6, and (b) follows from (4.7.5), (4.7.6), (4.7.7) and Lemma 4.5. □

Corollary 4.8. *Let C be a semidualizing R-module with $\mathrm{id}_{R_\mathfrak{p}}(C_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in X^{n-1}(R)$. Assume that M is an R-module which is horizontally linked with respect to C and that $M \in \mathscr{B}_C$. Then the following are equivalent*:

(i) *M satisfies $\widetilde{S}_n$.*

(ii) *M is an n-th C-syzygy module.*

(iii) $\mathrm{Ext}^i_R(\lambda_R(C, M), C) = 0$ *for* $0 < i < n$.

(iv) $\mathrm{Ext}^i_{\mathscr{P}_C}(\lambda_R(C, M), C) = 0$ *for* $0 < i < n$.

Proof. (i) $\Longrightarrow$ (iii): Set $N = \lambda_R(C, M)$. By Lemma 3.5,

$$M \text{ satisfies } \widetilde{S}_n \iff M^{\curlyvee} \text{ satisfies } \widetilde{S}_n. \tag{4.8.1}$$

By Lemma 4.1, $\mathrm{Ext}^1_R(\mathrm{Tr}_C(M^{\curlyvee}), C) = 0$. It follows from the exact sequence (3.1.1) that

$$\begin{aligned} &\mathrm{Ext}^i_R(N, C) = 0 \text{ for } 0 < i < n \\ &\qquad \iff \mathrm{Ext}^i_R(\mathrm{Tr}_C(M^{\curlyvee}), C) = 0 \text{ for } 0 < i < n+1. \end{aligned} \tag{4.8.2}$$

Now the assertion follows from (4.8.1), (4.8.2) and Theorem 4.6.

The equivalence of (i), (ii) and (iv) follows from Theorem 4.7. □

Now we are ready to present the first part of Theorem C.

Corollary 4.9. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d > 0$ with canonical module ω_R. Assume that M is an R-module of finite Gorenstein injective dimension which is horizontally linked with respect to ω_R. The following are equivalent*:

(i) *M satisfies (S_n).*

(ii) $\mathrm{H}^i_\mathfrak{m}(\lambda_R(\omega_R, M)) = 0$ *for* $d - n < i < d$.

In particular, M is maximal Cohen–Macaulay if and only if $\lambda_R(\omega_R, M)$ is maximal Cohen–Macaulay.

Proof. This is an immediate consequence of Corollary 4.8, Fact 2.9(ii) and the local duality theorem. □

One may translate Corollary 4.9 to a change-of-rings result.

Corollary 4.10. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with canonical module ω_R and let $\mathfrak{a}$ be a Cohen–Macaulay quasi-Gorenstein ideal of R of grade n, $\bar{R} = R/\mathfrak{a}$. Assume that M is an R-module of finite Gorenstein injective dimension which is linked by the ideal $\mathfrak{a}$ with respect to $\omega_{\bar{R}}$. The following are equivalent*:

(i) *M satisfies (S_n).*

(ii) $\mathrm{H}^i_{\mathfrak{m}}(\lambda_{\bar{R}}(\omega_{\bar{R}}, M)) = 0$ *for* $\dim R/\mathfrak{a} - n < i < \dim R/\mathfrak{a}$.

Proof. This is an immediate consequence of Corollary 4.9 and Theorem 3.11. □

Recall that an R-module M of dimension $d \geq 1$ is called a *generalized Cohen–Macaulay* module if $\ell(\mathrm{H}^i_{\mathfrak{m}}(M)) < \infty$ for all i, $0 \leq i \leq d-1$, where ℓ denotes the length. For an R-module M and positive integer n, set $\mathscr{T}_n^C M := \mathsf{Tr}_C \Omega^{n-1} M$.

Theorem 4.11. *Let R be a Cohen–Macaulay local ring of dimension $d > 1$ and let C be a semidualizing R-module with $\mathrm{id}_{R_\mathfrak{p}}(C_\mathfrak{p}) < \infty$ for all $\mathfrak{p} \in \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$. Assume that M is a generalized Cohen–Macaulay R-module which is horizontally linked with respect to C and that $M \in \mathscr{B}_C$. Then $\mathsf{Ext}^i_R(M^{\curlyvee}, C) \cong \mathrm{H}^i_{\mathfrak{m}}(\lambda_R(C, M))$ for $0 < i < d$. In particular, $\lambda_R(C, M)$ is generalized Cohen–Macaulay.*

Proof. Set $X = M^{\curlyvee}$ and $N = \lambda_R(C, M)$. As M is generalized Cohen–Macaulay, by [Trung 1986, Lemmas 1.2 and 1.4] and Theorem 2.6(iv), $\mathsf{G}_{C_p}\text{-}\mathsf{dim}_{\mathsf{R}_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{\mathfrak{m}\}$. Therefore $\mathsf{G}\text{-}\mathsf{dim}_{R_\mathfrak{p}}(X_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}\, R \setminus \{\mathfrak{m}\}$ by Lemma 3.6. Hence, $\mathsf{Ext}^i_R(X, C)$ has finite length for all $i > 0$. Consider the exact sequences

$$0 \to \mathsf{Ext}^i_R(X, C) \to \mathscr{T}_i^C X \to L_i \to 0, \tag{4.11.1}$$

$$0 \to L_i \to \oplus^{n_i} C \to \mathscr{T}_{i+1}^C X \to 0 \tag{4.11.2}$$

for all $i > 0$. By applying the functor $\Gamma_{\mathfrak{m}}(-)$ on the exact sequences (4.11.1) and (4.11.2), we get

$$\mathrm{H}^j_{\mathfrak{m}}(\mathscr{T}_{i-1}^C X) \cong \mathrm{H}^j_{\mathfrak{m}}(L_{i-1}) \quad \text{for all } i \text{ and } j, \text{ with } j \geq 1,\ i \geq 2, \tag{4.11.3}$$

$$\mathsf{Ext}^i_R(X, C) = \Gamma_{\mathfrak{m}}(\mathsf{Ext}^i_R(X, C)) \cong \Gamma_{\mathfrak{m}}(\mathscr{T}_i^C X) \quad \text{for all } i \geq 1, \tag{4.11.4}$$

and also

$$\mathrm{H}^j_{\mathfrak{m}}(\mathscr{T}_i^C X) \cong \mathrm{H}^{j+1}_{\mathfrak{m}}(L_{i-1}) \quad \text{for all } i \text{ and } j,\ 0 \leq j < d-1,\ i \geq 2. \tag{4.11.5}$$

As M is horizontally linked with respect to C, we have the exact sequence

$$0 \to N \to \oplus^n C \to \mathscr{T}_1^C X \to 0$$

for some integer $n > 0$. By applying the functor $\Gamma_{\mathfrak{m}}(-)$ to the above exact sequence, we get the isomorphism

$$\mathrm{H}^j_{\mathfrak{m}}(\mathscr{T}_1^C X) \cong \mathrm{H}^{j+1}_{\mathfrak{m}}(N) \quad \text{for all } j,\ 0 \leq j \leq d-2. \tag{4.11.6}$$

Now by using (4.11.3), (4.11.4), (4.11.5) and (4.11.6) we obtain the result. □

Now we give a proof for part (ii) of Theorem C as the following corollary.

Corollary 4.12. *Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring of dimension $d > 1$ with canonical module ω_R. Assume that M is an R-module of finite Gorenstein injective dimension which is horizontally linked with respect to ω_R. If M is generalized Cohen–Macaulay, then the following statements hold true*:

(i) $\mathrm{H}^i_{\mathfrak{m}}(\mathsf{Hom}_R(\omega_R, M)) \cong \mathsf{Hom}_R(\mathrm{H}^{d-i}_{\mathfrak{m}}(\lambda_R(\omega_R, M)), \boldsymbol{E}_R(k))$ *for* $0 < i < d$.

(ii) $\lambda_R(\omega_R, M)$ *is generalized Cohen–Macaulay.*

(iii) *If M is not maximal Cohen–Macaulay, then*

$$\mathsf{depth}_R(M) = \sup\{i < d \mid \mathrm{H}^i_{\mathfrak{m}}(\lambda_R(\omega_R, M)) \neq 0\}.$$

Proof. Parts (i) and (ii) follow immediately from Theorem 4.11 and the local duality theorem. Part (iii) follows from part (i) and Lemma 3.5. □

We end the paper with the following result, which is an immediate consequence of Corollary 4.12 and Theorem 3.11.

Corollary 4.13. *Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring with canonical module ω_R, let $\mathfrak{c}$ be a Cohen–Macaulay quasi-Gorenstein ideal of R, $\bar{R} = R/\mathfrak{c}$ and* $\dim \bar{R} = d > 1$. *Assume that M is an R-module of finite Gorenstein injective dimension which is linked by the ideal $\mathfrak{c}$ with respect to $\omega_{\bar{R}}$. If M is generalized Cohen–Macaulay, then*

$$\mathrm{H}^i_{\mathfrak{m}}(\mathsf{Hom}_{\bar{R}}(\omega_{\bar{R}}, M)) \cong \mathsf{Hom}_R(\mathrm{H}^{d-i}_{\mathfrak{m}}(\lambda_{\bar{R}}(\omega_{\bar{R}}, M)), \boldsymbol{E}_R(k))$$

for $0 < i < d$.

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