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YU FU AND MIN-CHUN HONG

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Let M^n be a biharmonic hypersurface with constant scalar curvature in a space form $\mathbb{M}^{n+1}(c)$. We show that M^n has constant mean curvature if $c > 0$ and M^n is minimal if $c \leq 0$, provided that the number of distinct principal curvatures is no more than 6. This partially confirms Chen's conjecture and the generalized Chen's conjecture. As a consequence, we prove that there exist no proper biharmonic hypersurfaces with constant scalar curvature in Euclidean space $\mathbb{E}^{n+1}$ or hyperbolic space $\mathbb{H}^{n+1}$ for $n < 7$.

1. Introduction

In 1983, Eells and Lemaire [1983] introduced the concept of *biharmonic maps* in order to generalize classical theory of harmonic maps. A biharmonic map ϕ between an n-dimensional Riemannian manifold (M^n, g) and an m-dimensional Riemannian manifold (N^m, h) is a critical point of the bienergy functional

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 \, dv_g,$$

where $\tau(\phi) = \operatorname{trace} \nabla d\phi$ is the tension field of ϕ that vanishes for a harmonic map. More clearly, the Euler–Lagrange equation associated to the bienergy is given by

$$\tau_2(\phi) = -\Delta\tau(\phi) - \operatorname{trace} R^N(d\phi, \tau(\phi))d\phi = 0,$$

where R^N is the curvature tensor of N^m (see, e.g., [Jiang 1987]). We call ϕ to be a biharmonic map if its bitension field $\tau_2(\phi)$ vanishes.

Biharmonic maps between Riemannian manifolds have been extensively studied by geometers. In particular, many authors investigated a special class of biharmonic maps named *biharmonic immersions*. An immersion $\phi : (M^n, g) \to (N^m, h)$ is biharmonic if and only if its mean curvature vector field $\vec{H}$ fulfills the fourth-order semilinear elliptic equations (see, e.g., [Caddeo et al. 2001]):

$$\Delta\vec{H} + \operatorname{trace} R^N(d\phi, \vec{H})d\phi = 0. \tag{1-1}$$

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It is well known that any minimal immersion (satisfying $\vec{H}=0$) is harmonic. The nonharmonic biharmonic immersions are called proper biharmonic.

We should mention that *biharmonic submanifolds* in a Euclidean space $\mathbb{E}^m$ were independently defined by B. Y. Chen in the middle of 1980s (see [Chen 1991]) with the geometric condition $\Delta\vec{H}=0$, or equivalently $\Delta^2\phi=0$. Interestingly, both biharmonic submanifolds and biharmonic immersions in Euclidean spaces coincide with each other.

In recent years, the classification problem of biharmonic submanifolds has attracted great attention in geometry. In particular, there is a longstanding conjecture on biharmonic submanifolds due to Chen:

Chen's conjecture [1991]. *Every biharmonic submanifold in Euclidean space $\mathbb{E}^m$ is minimal.*

Chen's conjecture still remains open, even for hypersurfaces. In three decades, only partial answers to Chen's conjecture have been obtained, e.g., [Akutagawa and Maeta 2013; Alías et al. 2013; Chen 2015; Ou 2012]. In the case of hypersurfaces, Chen's conjecture is true for the following special cases:

- Surfaces in $\mathbb{E}^3$ [Chen 1991; Jiang 1987].
- Hypersurfaces with at most two distinct principal curvatures in $\mathbb{E}^m$ [Dimitrić 1992].
- Hypersurfaces in $\mathbb{E}^4$ [Hasanis and Vlachos 1995] (see also [Defever 1998]).
- $\delta(2)$-ideal and $\delta(3)$-ideal hypersurfaces in $\mathbb{E}^m$ [Chen and Munteanu 2013].
- Weakly convex hypersurfaces in $\mathbb{E}^m$ [Luo 2014].
- Hypersurfaces with at most three distinct principal curvatures in $\mathbb{E}^m$ [Fu 2015a].
- Generic hypersufaces with irreducible principal curvature vector fields in $\mathbb{E}^m$ [Koiso and Urakawa 2014].
- Invariant hypersurfaces of cohomogeneity one in $\mathbb{E}^m$ [Montaldo et al. 2016].

In 2001, Caddeo, Montaldo and Oniciuc [Caddeo et al. 2001] proposed the following generalized Chen's conjecture:

Generalized Chen's conjecture. *Every biharmonic submanifold in a Riemannian manifold with nonpositive sectional curvature is minimal.*

Recently, Ou and Tang [2012] constructed a family of counterexamples, where the generalized Chen's conjecture is false when the ambient space has nonconstant negative sectional curvature. However, the generalized Chen's conjecture remains open when the ambient spaces have constant sectional curvature. For more recent developments of the generalized Chen's conjecture, we refer to [Chen 2014; 2015; Montaldo and Oniciuc 2006; Oniciuc 2012; Nakauchi and Urakawa 2011; Ou 2016].

The classification of proper biharmonic submanifolds in Euclidean spheres is rather rich and interesting. The first example of proper biharmonic hypersurfaces is a generalized Clifford torus $S^p(1/\sqrt{2}) \times S^q(1/\sqrt{2}) \hookrightarrow \mathbb{S}^{n+1}$ with $p \neq q$ and $p+q=n$, given by Jiang [1986]. The complete classifications of biharmonic hypersurfaces in $\mathbb{S}^3$ and $\mathbb{S}^4$ were obtained in [Caddeo et al. 2001; Balmuş et al. 2010]. Moreover, biharmonic hypersurfaces with at most three distinct principal curvatures in $\mathbb{S}^n$ were classified in [Balmuş et al. 2010; Fu 2015b]. For more details, we refer the readers to [Balmuş et al. 2013; Loubeau and Oniciuc 2014; Oniciuc 2002; 2012; Ichiyama et al. 2010].

In general, the classification problem of proper biharmonic hypersurfaces in space forms becomes more complicated when the number of distinct principal curvatures is 4 or more.

In view of the above aspects, it is reasonable to study biharmonic submanifolds with some geometric conditions. In geometry, hypersurfaces with constant scalar curvature have been intensively studied by many geometers for the rigidity problem and classification problem, for instance, see [Cheng and Yau 1977]. Some estimate for scalar curvature of compact proper biharmonic hypersurfaces with constant scalar curvature in spheres was obtained in [Balmuş et al. 2008]. Recently, it was proved in [Fu 2015c] that a biharmonic hypersurface with constant scalar curvature in the 5-dimensional space forms $\mathbb{M}^5(c)$ necessarily has constant mean curvature.

Motivated by above results, in this paper we consider biharmonic hypersurfaces M^n with constant scalar curvatures in a space form $\mathbb{M}^n(c)$. More precisely, we get:

Theorem 1.1. *Let M^n be an orientable biharmonic hypersurface with at most six distinct principal curvatures in $\mathbb{M}^{n+1}(c)$. If the scalar curvature R is constant, then M^n has constant mean curvature.*

In general, it is difficult to deal with the biharmonic immersion equation (1-1) due to its high nonlinearity. In order to prove Theorem 1.1, we use some new ideas to overcome the difficulty of treating the equation of a biharmonic hypersurface. More precisely, we transfer the problem into a system of algebraic equations (see Lemma 3.3), so we can determine the behavior of the principal curvature functions by investigating the solution of the system of algebraic equations (see Lemma 3.4). Then, we are able to prove that a biharmonic hypersurface with constant scalar curvatures in a space form $\mathbb{M}^n(c)$ must have constant mean curvature, provided that the number of distinct principal curvature is no more than 6. We would like to point out that our approach in this paper is different from those in [Fu 2015b; 2015c; Defever 1998; Balmuş et al. 2010].

Remark 1.2. Balmuş, Montaldo and Oniciuc in [Balmuş et al. 2008] conjectured that the proper biharmonic hypersurfaces in $\mathbb{S}^{n+1}$ must have constant mean curvature. Theorem 1.1 with $c=1$ gives a partial answer to this conjecture.

We should point out that the complete classification of proper biharmonic hypersurfaces with constant mean curvature in a sphere is still open in the case where the number of distinct principal curvatures is more than 3 (see [Oniciuc 2012]).

Moreover, combining these results with the biharmonic equations in Section 2, we have:

Corollary 1.3. *Any biharmonic hypersurface with constant scalar curvature and with at most six distinct principal curvatures in Euclidean space* $\mathbb{E}^{n+1}$ *or hyperbolic space* $\mathbb{H}^{n+1}$ *is minimal.*

Thus, this result gives a partial answer to Chen's conjecture and the generalized Chen's conjecture.

Further, as a direct consequence, we get the following characterization result:

Corollary 1.4. *Any biharmonic hypersurface with constant scalar curvature in Euclidean space* $\mathbb{E}^{n+1}$ *or hyperbolic space* $\mathbb{H}^{n+1}$ *for* $n < 7$ *has to be minimal.*

Remark 1.5. We could replace or weaken the *constant scalar curvature* condition in Theorem 1.1 by *constant length of the second fundamental form* or *linear Weingarten type*, i.e., the scalar curvature R satisfying $R = aH + b$ for some constants a and b. In fact, the discussion is extremely similar to the proof of Theorem 1.1 and the same conclusion holds true as well.

The paper is organized as follows. In Section 2, we recall some necessary background theory for hypersurfaces and equivalent conditions for biharmonic hypersurfaces. In Section 3, we prove some useful lemmas (Lemmas 3.1–3.6), which are crucial to prove the main theorem. Finally, in Section 4, we give a proof of Theorem 1.1.

2. Preliminaries

In this section, we recall some basic material for the theory of hypersurfaces immersed in a Riemannian space form.

Let $\phi : M^n \to \mathbb{M}^{n+1}(c)$ be an isometric immersion of a hypersurface M^n into a space form $\mathbb{M}^{n+1}(c)$ with constant sectional curvature c. Denote the Levi-Civita connections of M^n and $\mathbb{M}^{n+1}(c)$ by ∇ and $\tilde{\nabla}$, respectively. Let X and Y denote the vector fields tangent to M^n and let ξ be a unit normal vector field. Then the Gauss and Weingarten formulas (see [Chen 2015]) are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2-1}$$

$$\tilde{\nabla}_X \xi = -AX, \tag{2-2}$$

where h is the second fundamental form and A is the Weingarten operator. Note that the second fundamental form h and the Weingarten operator A are related by

$$\langle h(X, Y), \xi \rangle = \langle AX, Y \rangle. \tag{2-3}$$

The mean curvature vector field $\vec{H}$ is defined by

$$\vec{H} = \frac{1}{n}\operatorname{trace} h. \tag{2-4}$$

Moreover, the Gauss and Codazzi equations are given, respectively, by

$$R(X, Y)Z = c\big(\langle Y, Z\rangle X - \langle X, Z\rangle Y\big) + \langle AY, Z\rangle AX - \langle AX, Z\rangle AY,$$
$$(\nabla_X A)Y = (\nabla_Y A)X,$$

where R is the curvature tensor of M^n and $(\nabla_X A)Y$ is given by

$$(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y) \tag{2-5}$$

for all X, Y, Z tangent to M^n.

Assume that $\vec{H} = H\xi$ and H denotes the mean curvature.

By identifying the tangent and the normal parts of the biharmonic condition (1-1) for hypersurfaces in a space form $\mathbb{M}^{n+1}(c)$, the following characterization result for M^n to be biharmonic was obtained (see also [Caddeo et al. 2002; Balmuş et al. 2010]):

Proposition 2.1. *The immersion $\phi : M^n \to \mathbb{M}^{n+1}(c)$ of a hypersurface M^n in an $n+1$-dimensional space form $\mathbb{M}^{n+1}(c)$ is biharmonic if and only if*

$$\begin{cases} \Delta H + H \operatorname{trace} A^2 = ncH, \\ 2A \operatorname{grad} H + nH \operatorname{grad} H = 0. \end{cases} \tag{2-6}$$

The Laplacian operator Δ on M^n acting on a smooth function f is given by

$$\Delta f = -\operatorname{div}(\nabla f) = -\sum_{i=1}^{n} < \nabla_{e_i}(\nabla f), e_i > = -\sum_{i=1}^{n}(e_i e_i - \nabla_{e_i} e_i) f. \tag{2-7}$$

The following result was obtained in [Fu 2015b]:

Theorem 2.2. *Let M^n be an orientable proper biharmonic hypersurface with at most three distinct principal curvatures in $\mathbb{M}^{n+1}(c)$. Then M^n has constant mean curvature.*

3. Some lemmas

We now consider an orientable biharmonic hypersurface M^n $(n > 3)$ in a space form $\mathbb{M}^{n+1}(c)$.

In general, the set M_A of all points of M^n, at which the number of distinct eigenvalues of the Weingarten operator A (i.e., the principal curvatures) is locally constant, is open and dense in M^n. Since M^n with at most three distinct principal curvatures everywhere in a space form $\mathbb{M}^{n+1}(c)$ is CMC, i.e., the mean curvature is constant (Theorem 2.2), one can work only on the connected component of M_A consisting of points where the number of principal curvatures is more than 3 (by passing to the limit, H will be constant on the whole M^n). On that connected component, the principal curvature functions of A are always smooth.

Suppose that, on the component, the mean curvature H is not constant. Thus, there is a point p where $\operatorname{grad} H(p) \neq 0$. In the following, we will work on a neighborhood of p where $\operatorname{grad} H(p) \neq 0$ at any point of M^n.

The second equation of (2-6) shows that $\operatorname{grad} H$ is an eigenvector of the Weingarten operator A with the corresponding principal curvature $-nH/2$. We may choose e_1 such that e_1 is parallel to $\operatorname{grad} H$, and with respect to some suitable orthonormal frame $\{e_1, \dots, e_n\}$, the Weingarten operator A of M takes the form

$$A = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n), \tag{3-1}$$

where λ_i are the principal curvatures and $\lambda_1 = -nH/2$. Therefore, it follows from (2-4) that $\sum_{i=1}^n \lambda_i = nH$, and hence

$$\sum_{i=2}^n \lambda_i = -3\lambda_1. \tag{3-2}$$

Denote by R the scalar curvature and by B the squared length of the second fundamental form h of M. It follows from (3-1) that B is given by

$$B = \operatorname{trace} A^2 = \sum_{i=1}^n \lambda_i^2 = \sum_{i=2}^n \lambda_i^2 + \lambda_1^2. \tag{3-3}$$

From the Gauss equation, the scalar curvature R is given by

$$R = n(n-1)c + n^2H^2 - B = n(n-1)c + 3\lambda_1^2 - \sum_{i=2}^n \lambda_i^2. \tag{3-4}$$

Hence

$$\sum_{i=2}^n \lambda_i^2 = n(n-1)c - R + 3\lambda_1^2. \tag{3-5}$$

Since $\operatorname{grad} H = \sum_{i=1}^n e_i(H)e_i$ and e_1 is parallel to $\operatorname{grad} H$, it follows that

$$e_1(H) \neq 0, \qquad e_i(H) = 0, \quad 2 \leq i \leq n,$$

and hence

$$e_1(\lambda_1) \neq 0, \qquad e_i(\lambda_1) = 0, \quad 2 \leq i \leq n. \tag{3-6}$$

Put $\nabla_{e_i} e_j = \sum_{k=1}^n \omega_{ij}^k e_k$ $(1 \leq i, j \leq n)$. A direct computation concerning the compatibility conditions $\nabla_{e_k}\langle e_i, e_i\rangle = 0$ and $\nabla_{e_k}\langle e_i, e_j\rangle = 0$ $(i \neq j)$ yields, respectively, that

$$\omega_{ki}^i = 0, \qquad \omega_{ki}^j + \omega_{kj}^i = 0, \quad i \neq j. \tag{3-7}$$

The Codazzi equation yields

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega^j_{ji}, \tag{3-8}$$

$$(\lambda_i - \lambda_j)\omega^j_{ki} = (\lambda_k - \lambda_j)\omega^j_{ik}, \tag{3-9}$$

for distinct i, j, k.

Moreover, from (3-6) we have

$$[e_i, e_j](\lambda_1) = 0,$$

which yields directly

$$\omega^1_{ij} = \omega^1_{ji}, \qquad 2 \le i, j \le n \quad \text{and} \quad i \ne j. \tag{3-10}$$

Lemma 3.1. *Let M^n be an orientable biharmonic hypersurface with nonconstant mean curvature in $\mathbb{M}^{n+1}(c)$. Then the multiplicity of the principal curvature λ_1 (which equals $-nH/2$) is* 1, *i.e.*, $\lambda_j \ne \lambda_1$ *for* $2 \le j \le n$.

Proof. If $\lambda_j = \lambda_1$ for $j \ne 1$, by putting $i = 1$ in (3-8), we get

$$0 = (\lambda_1 - \lambda_j)\omega^j_{j1} = e_1(\lambda_j) = e_1(\lambda_1),$$

which contradicts (3-6). □

Lemma 3.2. *The smooth real-valued functions λ_i and ω^1_{ii} $(2 \le i \le n)$ satisfy the following differential equations*:

$$e_1e_1(\lambda_1) = e_1(\lambda_1)\Big(\sum\nolimits_{i=2}^n \omega^1_{ii}\Big) + \lambda_1(n(n-2)c - R + 4\lambda_1^2), \tag{3-11}$$

$$e_1(\lambda_i) = \lambda_i\omega^1_{ii} - \lambda_1\omega^1_{ii}, \tag{3-12}$$

$$e_1(\omega^1_{ii}) = (\omega^1_{ii})^2 + \lambda_1\lambda_i + c. \tag{3-13}$$

Proof. Substituting $H = -2\lambda_1/n$ into the first equation of (2-6), and using (2-7), (3-6), (3-3) and (3-5), we get (3-11). By putting $i = 1$ in (3-8), combining this with (3-9) gives (3-12).

Next, we will prove (3-13).

For $j = 1$ and $i \ne 1$ in (3-8), by (3-6) we have $\omega^1_{1i} = 0$ $(i \ne 1)$. Combining this with (3-7), we have

$$\omega^i_{11} = 0, \quad \text{for } 1 \le i \le n. \tag{3-14}$$

For $j = 1$, and $k, i \ne 1$ in (3-9) we have

$$(\lambda_i - \lambda_1)\omega^1_{ki} = (\lambda_k - \lambda_1)\omega^1_{ik},$$

which together with (3-10) yields

$$\omega^1_{ki} = 0, \quad k \ne i, \quad \text{if } \lambda_k \ne \lambda_i. \tag{3-15}$$

For $i \neq j$ and $2 \leq i, j \leq n$, if $\lambda_i = \lambda_j$, then by putting $k = 1$ in (3-9) we have

$$(\lambda_1 - \lambda_i)\omega_{i1}^j = 0,$$

which together with Lemma 3.1, (3-15) and (3-7) yields

$$\omega_{i1}^j = 0, \quad i \neq j, \quad \text{and} \quad 2 \leq i, j \leq n. \tag{3-16}$$

From the Gauss equation and (3-1), we have $\langle R(e_1, e_i)e_1, e_i\rangle = -\lambda_1\lambda_i - c$. On the other hand, the Gauss curvature tensor R is defined by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Using (3-14), (3-16) and (3-7), a direct computation gives

$$\langle R(e_1, e_i)e_1, e_i\rangle = -e_1(\omega_{ii}^1) + (\omega_{ii}^1)^2.$$

Thus, we obtain differential equation (3-13), completing the proof of Lemma 3.2. □

Consider an integral curve of e_1 passing through $p = \gamma(t_0)$ as $\gamma(t)$, $t \in I$. Since $e_i(\lambda_1) = 0$ for $2 \leq i \leq n$ and $e_1(\lambda_1) \neq 0$, it is easy to show that there exists a local chart $(U; t = x^1, x^2, \ldots, x^m)$ around p, such that $\lambda_1(t, x^2, \ldots, x^m) = \lambda_1(t)$ on the whole neighborhood of p.

In the following, we begin our arguments under the assumption that the scalar curvature R is always constant. The following system of algebraic equations is important for us to proceed further.

Lemma 3.3. *Assume that R is constant. We have*

$$\sum_{i=2}^{n}(\omega_{ii}^1)^k = f_k(t), \quad \text{for } k = 1, \ldots, 5, \tag{3-17}$$

where $f_k(t)$ are some smooth real-valued functions with respect to t.

Proof. Since $e_1(\lambda_1) \neq 0$, $\lambda_1 = \lambda_1(t)$ and R is constant, (3-11) becomes

$$\sum_{i=2}^{n}\omega_{ii}^1 = f_1(t), \tag{3-18}$$

where

$$f_1(t) = \frac{e_1e_1(\lambda_1) - \lambda_1\big(n(n-2)c + 4\lambda_1^2 - R\big)}{e_1(\lambda_1)}.$$

Taking the sum of (3-13) and (3-12) for i and taking into account (3-2) and (3-18), respectively, we have

$$\sum_{i=2}^{n}\big(\omega_{ii}^1\big)^2 = f_2(t), \tag{3-19}$$

$$\sum_{i=2}^{n}\lambda_i\omega_{ii}^1 = g_1(t), \tag{3-20}$$

where $f_2 = 3\lambda_1^2 - (n-1)c + e_1(f_1)$ and $g_1(t) = \lambda_1 f_1 - 3e_1(\lambda_1)$.

Multiplying by ω_{ii}^1 on both sides of (3-13), we have

$$\tfrac{1}{2}e_1((\omega_{ii}^1)^2) = (\omega_{ii}^1)^3 + \lambda_1\lambda_i\omega_{ii}^1 + c\omega_{ii}^1.$$

Using this and (3-18)–(3-20), we obtain

$$\sum_{i=2}^{n}(\omega_{ii}^1)^3 = f_3(t), \tag{3-21}$$

where $f_3 = \frac{1}{2}e_1(f_2) - \lambda_1 g_1 - cf_1$.

Differentiating (3-20) with respect to e_1 and using (3-12) and (3-13), we have

$$e_1(g_1) = 2\sum_{i=2}^{n}\lambda_i(\omega_{ii}^1)^2 + \lambda_1\sum_{i=2}^{n}\lambda_i^2 + c\sum_{i=2}^{n}\lambda_i - \lambda_1\sum_{i=2}^{n}(\omega_{ii}^1)^2. \tag{3-22}$$

From (3-2), (3-5) and (3-19), this yields

$$\sum_{i=2}^{n}\lambda_i(\omega_{ii}^1)^2 = g_2(t), \tag{3-23}$$

where $g_2 = \frac{1}{2}\{e_1(g_1) - \lambda_1\big(n(n-1)c - R + 3\lambda_1^2\big) + 3c\lambda_1 + \lambda_1 f_2\}$.

Multiplying by $(\omega_{ii}^1)^2$ on both sides of (3-13), we have

$$\tfrac{1}{3}e_1((\omega_{ii}^1)^3) = (\omega_{ii}^1)^4 + \lambda_1\lambda_i(\omega_{ii}^1)^2 + c(\omega_{ii}^1)^2.$$

Applying (3-19), (3-21) and (3-23) to this, we obtain

$$\sum_{i=2}^{n}(\omega_{ii}^1)^4 = f_4(t), \tag{3-24}$$

where $f_4 = \frac{1}{3}e_1(f_3) - \lambda_1 g_2 - cf_2$.

Multiplying by λ_i on both sides of (3-12) gives

$$\lambda_i^2\omega_{ii}^1 = \tfrac{1}{2}e_1(\lambda_i^2) + \lambda_1\lambda_i\omega_{ii}^1,$$

which together with (3-5) and (3-20) yields

$$\sum_{i=2}^{n}\lambda_i^2\omega_{ii}^1 = g_3(t), \tag{3-25}$$

where $g_3 = 3\lambda_1 e_1(\lambda_1) + \lambda_1 g_1$.

Differentiating (3-23) with respect to e_1 and using (3-12) and (3-13), we have

$$e_1(g_2) = 3\sum_{i=2}^{n}\lambda_i(\omega_{ii}^1)^3 - \lambda_1\sum_{i=2}^{n}(\omega_{ii}^1)^3 + 2\lambda_1\sum_{i=2}^{n}\lambda_i^2\omega_{ii}^1 + 2c\sum_{i=2}^{n}\lambda_i\omega_{ii}^1. \tag{3-26}$$

Substituting (3-20), (3-21) and (3-25) into (3-26) gives

$$\sum_{i=2}^{n} \lambda_i (\omega_{ii}^1)^3 = g_4(t), \tag{3-27}$$

where

$$g_4 = \tfrac{1}{3}(e_1(g_2) + \lambda_1 f_3 - 2\lambda_1 g_3 - 2cg_1).$$

Multiplying by $(\omega_{ii}^1)^3$ on both sides of (3-13), we have

$$\tfrac{1}{4} e_1((\omega_{ii}^1)^4) = (\omega_{ii}^1)^5 + \lambda_1 \lambda_i (\omega_{ii}^1)^3 + c(\omega_{ii}^1)^3.$$

Applying (3-21), (3-24) and (3-27) to this, we have

$$\sum_{i=2}^{n} \bigl(\omega_{ii}^1\bigr)^5 = f_5(t), \tag{3-28}$$

where $f_5 = \frac{1}{4} e_1(f_4) - \lambda_1 g_4 - cf_3$. □

Lemma 3.4. *Assume that R is constant. If the number m of distinct principal curvatures satisfies $m \le 6$, then $e_i(\lambda_j) = 0$ for $2 \le i, j \le n$, i.e., all principal curvature λ_i depend only on one variable t.*

Proof. Since the number m of distinct principal curvatures satisfies $m \le 6$, there are at most five distinct principal curvatures for λ_i $(2 \le i \le n)$ except λ_1. It follows easily from (3-12) and (3-13) that

$$\lambda_i \ne \lambda_j \quad \Leftrightarrow \quad \omega_{ii}^1 \ne \omega_{jj}^1.$$

We now distinguish the following two cases:

Case A: Suppose that $m = 6$. We denote by $\tilde{\lambda}_i$ the five distinct principal curvatures with the corresponding multiplicities n_i for $1 \le i \le 5$. Note that here n_i are positive integers and $\sum_{i=1}^{5} n_i = n - 1$ (see Lemma 3.1). According to (3-12), let

$$u_i := \frac{e_1(\tilde{\lambda}_i)}{\tilde{\lambda}_i - \lambda_1}.$$

Thus, the u_i are mutually different for $1 \le i \le 5$.

In this case, the system of polynomial equations (3-17) becomes

$$\begin{aligned} n_1u_1 + n_2u_2 + n_3u_3 + n_4u_4 + n_5u_5 &= f_1, \\ n_1u_1^2 + n_2u_2^2 + n_3u_3^2 + n_4u_4^2 + n_5u_5^2 &= f_2, \\ n_1u_1^3 + n_2u_2^3 + n_3u_3^3 + n_4u_4^3 + n_5u_5^3 &= f_3, \\ n_1u_1^4 + n_2u_2^4 + n_3u_3^4 + n_4u_4^4 + n_5u_5^4 &= f_4, \\ n_1u_1^5 + n_2u_2^5 + n_3u_3^5 + n_4u_4^5 + n_5u_5^5 &= f_5. \end{aligned} \tag{3-29}$$

Since $e_i(f_1) = 0$ for $2 \le i \le n$, differentiating both sides of the equations in (3-29) with respect to e_i $(2 \le i \le n)$, we obtain

$$\begin{aligned} n_1e_i(u_1)+n_2e_i(u_2)+n_3e_i(u_3)+n_4e_i(u_4)+n_5e_i(u_5)&=0,\\ n_1u_1e_i(u_1)+n_2u_2e_i(u_2)+n_3u_3e_i(u_3)+n_4u_4e_i((u_4)+n_5u_5e_i(u_5)&=0,\\ n_1u_1^2e_i(u_1)+n_2u_2^2e_i(u_2)+n_3u_3^2e_i(u_3)+n_4u_4^2e_i(u_4)+n_5u_5^2e_i(u_5)&=0,\\ n_1u_1^3e_i(u_1)+n_2u_2^3e_i(u_2)+n_3u_3^3e_i(u_3)+n_4u_4^3e_i(u_4)+n_5u_5^3e_i(u_5)&=0,\\ n_1u_1^4e_i(u_1)+n_2u_2^4e_i(u_2)+n_3u_3^4e_i(u_3)+n_4u_4^4e_i(u_4)+n_5u_5^4e_i(u_5)&=0. \end{aligned} \tag{3-30}$$

Now consider this system of five linear equations with five unknowns $e_i(u_k)$ for $1 \le k \le 5$.

According to Cramer's rule in linear algebra, for any k, $e_i(u_k) \equiv 0$ holds true if and only if the determinant of the coefficient matrix of (3-30) is not vanishing, i.e.,

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1\\ u_1 & u_2 & u_3 & u_4 & u_5\\ u_1^2 & u_2^2 & u_3^2 & u_4^2 & u_5^2\\ u_1^3 & u_2^3 & u_3^3 & u_4^3 & u_5^3\\ u_1^4 & u_2^4 & u_3^4 & u_4^4 & u_5^4 \end{vmatrix} \neq 0. \tag{3-31}$$

We note that the determinant in (3-31) is the famous Vandermonde determinant with order 5 and hence

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1\\ u_1 & u_2 & u_3 & u_4 & u_5\\ u_1^2 & u_2^2 & u_3^2 & u_4^2 & u_5^2\\ u_1^3 & u_2^3 & u_3^3 & u_4^3 & u_5^3\\ u_1^4 & u_2^4 & u_3^4 & u_4^4 & u_5^4 \end{vmatrix} = \prod_{1\le j<i\le 5}(u_i-u_j). \tag{3-32}$$

Since the u_i are mutually different for $1 \le i \le 5$, (3-32) implies that (3-31) holds true identically. Hence, we have $e_i(u_k) = 0$ for any $1 \le k \le 5$ and $2 \le i \le n$.

Therefore, by using $e_i(u_k) = 0$ and

$$e_ie_1(u_k) - e_1e_i(u_k) = [e_i, e_1](u_k) = \sum_{j=2}^{n}(\omega_{i1}^j - \omega_{1i}^j)e_j(u_k),$$

we get

$$e_ie_1(u_k) = 0.$$

Noting that with the notation u_k, (3-13) becomes

$$e_1(u_k) = (u_k)^2 + \lambda_1\lambda_k + c.$$

Differentiating the above equation with respect to e_i, by taking into account $e_i(u_k) = 0$ and $e_i e_1(u_k) = 0$ we derive

$$e_i(\lambda_k) = 0$$

for any $1 \le k \le 5$ and $2 \le i \le n$.

Case B: Suppose $m \le 5$. Denote by $\tilde{\lambda}_i$ the distinct principal curvatures with the corresponding multiplicities n_i for $1 \le i \le 4$. Then the number of different u_i is less than or equal to 4. In the case that four of the u_i are mutually different, it is only necessary to consider the system (3-17) for $k = 1, 2, 3, 4$. A similar discussion to the one in Case A yields the conclusion. If less than four of the u_i are mutually different, then the conclusion follows by some arguments similar to the above.

Thus, we conclude Lemma 3.4. □

Lemma 3.5. *For three arbitrary distinct principal curvatures λ_i, λ_j and λ_k, where $2 \le i, j, k \le n$, we have the following relations*:

$$\omega_{ij}^k(\lambda_j - \lambda_k) = \omega_{ji}^k(\lambda_i - \lambda_k) = \omega_{kj}^i(\lambda_j - \lambda_i), \tag{3-33}$$

$$\omega_{ij}^k\omega_{ji}^k + \omega_{jk}^i\omega_{kj}^i + \omega_{ik}^j\omega_{ki}^j = 0, \tag{3-34}$$

$$\omega_{ij}^k(\omega_{jj}^1 - \omega_{kk}^1) = \omega_{ji}^k(\omega_{ii}^1 - \omega_{kk}^1) = \omega_{kj}^i(\omega_{jj}^1 - \omega_{ii}^1). \tag{3-35}$$

Proof. We recall from the beginning of this section that the number m of distinct principal curvatures satisfies $m \ge 4$. Hence, by taking into account the second expression of (3-7) and (3-9) for three distinct principal curvatures λ_i, λ_j and λ_k $(2 \le i, j, k \le n)$, we obtain (3-33) and (3-34) immediately.

Let us consider (3-35). It follows from the Gauss equation that

$$\langle R(e_i, e_j)e_k, e_1\rangle = 0.$$

Moreover, since $\omega_{ij}^1 = 0$ for $i \ne j$ from (3-7) and (3-16), from the definition of the curvature tensor we have

$$\omega_{ij}^k(\omega_{jj}^1 - \omega_{kk}^1) = \omega_{ji}^k(\omega_{ii}^1 - \omega_{kk}^1). \tag{3-36}$$

Similarly, by considering $\langle R(e_j, e_k)e_i, e_1\rangle = 0$ one also has

$$\omega_{jk}^i(\omega_{kk}^1 - \omega_{ii}^1) = \omega_{kj}^i(\omega_{jj}^1 - \omega_{ii}^1),$$

which together with (3-7) and (3-36) gives (3-35). □

Lemma 3.6. *Under the assumptions as above, we have*

$$\omega_{ii}^1\omega_{jj}^1 - \sum_{k=2,\, k\ne l_{(i,j)}}^{n} 2\omega_{ij}^k\omega_{ji}^k = -\lambda_i\lambda_j - c, \quad \textit{for } \lambda_i \ne \lambda_j, \tag{3-37}$$

where $l_{(i,j)}$ stands for the indexes satisfying $\lambda_{l_{(i,j)}} = \lambda_i$ or λ_j.

Proof. In the following, we consider the case that the number m of distinct principal curvatures is 6.

Without loss of generality, except in the case of λ_1, we assume $\lambda_p, \lambda_q, \lambda_r, \lambda_u, \lambda_v$ are the five distinct principal curvatures in sequence with the corresponding multiplicities n_1, n_2, n_3, n_4, n_5, respectively, i.e.,

$$\lambda_1,\ \underbrace{\lambda_p, \dots, \lambda_p}_{n_1},\ \underbrace{\lambda_q, \dots, \lambda_q}_{n_2},\ \underbrace{\lambda_r, \dots, \lambda_r}_{n_3},\ \underbrace{\lambda_u, \dots, \lambda_u}_{n_4},\ \underbrace{\lambda_v, \dots, \lambda_v}_{n_5}.$$

We now compute $\langle R(e_p, e_q)e_p, e_q\rangle$. On one hand, it follows from the Gauss equation and (3-1) that

$$\langle R(e_p, e_q)e_p, e_q\rangle = -\lambda_p\lambda_q - c. \tag{3-38}$$

On the other hand, since

$$\begin{aligned}
\nabla_{e_p}\nabla_{e_q}e_p &= \sum_{k=1}^{n} e_p\big(\omega_{qp}^k\big)e_k + \sum_{k=1}^{n}\omega_{qp}^k\sum_{l=1}^{n}\omega_{pk}^l e_l,\\
\nabla_{e_q}\nabla_{e_p}e_p &= \sum_{k=1}^{n} e_q\big(\omega_{pp}^k\big)e_k + \sum_{k=1}^{n}\omega_{pp}^k\sum_{l=1}^{n}\omega_{qk}^l e_l,\\
\nabla_{[e_p,e_q]}e_p &= \sum_{k=1}^{n}\big(\omega_{pq}^k - \omega_{qp}^k\big)\sum_{l=1}^{n}\omega_{kp}^l e_l,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\langle R(e_p, e_q)e_p, e_q\rangle = \\
&\quad e_p(\omega_{qp}^q) + \sum_{k=1}^{n}\omega_{qp}^k\omega_{pk}^q - e_q(\omega_{pp}^q) - \sum_{k=1}^{n}\omega_{pp}^k\omega_{qk}^q - \sum_{k=1}^{n}(\omega_{pq}^k - \omega_{qp}^k)\omega_{kp}^q.
\end{aligned} \tag{3-39}$$

Since $\lambda_p \neq \lambda_q$, from (3-8), (3-7) and Lemma 3.4 we have

$$\omega_{qp}^q = \omega_{qq}^p = \omega_{pp}^q = 0 \quad \text{and} \quad \sum_{k=2}^{n}\omega_{pp}^k\omega_{qk}^q = 0. \tag{3-40}$$

Moreover, if $2 \le k \le n_1 + 1$, then $\lambda_k = \lambda_p$, by the second expression of (3-7) and (3-9) we get

$$(\lambda_p - \lambda_k)\omega_{qp}^k = (\lambda_q - \lambda_k)\omega_{pq}^k \quad \text{and} \quad (\lambda_k - \lambda_q)\omega_{pk}^q = (\lambda_p - \lambda_q)\omega_{kp}^q,$$

which imply that

$$\omega_{pq}^k = \omega_{pk}^q = \omega_{kp}^q = 0. \tag{3-41}$$

Similarly, if $n_1+2\le k\le n_1+n_2+1$, we also have

$$\omega_{pq}^k=\omega_{pk}^q=\omega_{kp}^q=0. \tag{3-42}$$

Hence, by taking (3-40)–(3-42) into account, (3-39) becomes

$$\langle R(e_p,e_q)e_p,e_q\rangle=\omega_{pp}^1\omega_{qq}^1+\sum_{k=n_1+n_2+2}^{n}\bigl\{\omega_{qp}^k\omega_{pk}^q-(\omega_{pq}^k-\omega_{qp}^k)\omega_{kp}^q\bigr\},$$

which together with (3-38), (3-7) and (3-34) gives

$$\omega_{pp}^1\omega_{qq}^1-\sum_{k=n_1+n_2+2}^{n}2\omega_{pq}^k\omega_{qp}^k=-\lambda_p\lambda_q-c. \tag{3-43}$$

Similarly, we can deduce other equations for different pairs $\omega_{pp}^1\omega_{rr}^1$, $\omega_{pp}^1\omega_{uu}^1$, ... Hence we get (3-37).

In the case that the number m of distinct principal curvatures is equal to 4 or 5, a very similar argument gives (3-37) as well. □

4. Proof of Theorem 1.1

Assume that the mean curvature H is not constant.

Differentiating (3-2) with respect to e_1 and using (3-12) and (3-13), we obtain

$$3e_1(\lambda_1)=\sum_{i=2}^{n}(\lambda_1-\lambda_i)\omega_{ii}^1. \tag{4-1}$$

Following the previous section, we only deal with the case where the number of distinct principal curvatures is 6, i.e., $m=6$. In fact, the proofs for the cases $m=4,5$ are very similar, so we omit them here without loss of generality.

According to Lemma 3.5, we consider the following cases:

Case A: $\omega_{pq}^r\ne 0$, $\omega_{pq}^u\ne 0$, and $\omega_{pq}^v\ne 0$. Since $\lambda_p,\lambda_q,\lambda_r,\lambda_u,\lambda_v$ are mutually different, equations (3-33) and (3-35) reduce to

$$\begin{aligned}\frac{\omega_{pp}^1-\omega_{qq}^1}{\lambda_p-\lambda_q}=\frac{\omega_{pp}^1-\omega_{rr}^1}{\lambda_p-\lambda_r}&=\frac{\omega_{qq}^1-\omega_{rr}^1}{\lambda_q-\lambda_r}\\ &=\frac{\omega_{pp}^1-\omega_{uu}^1}{\lambda_p-\lambda_u}=\frac{\omega_{qq}^1-\omega_{uu}^1}{\lambda_q-\lambda_u}\\ &=\frac{\omega_{pp}^1-\omega_{vv}^1}{\lambda_p-\lambda_v}=\frac{\omega_{qq}^1-\omega_{vv}^1}{\lambda_q-\lambda_v}.\end{aligned}$$

Thus, there exist two smooth functions φ and ψ depending on t such that

$$\omega_{ii}^1=\varphi\lambda_i+\psi. \tag{4-2}$$

Differentiating with respect to e_1 on both sides of (4-2), and using (3-12) and (3-13) we get

$$e_1(\varphi) = \lambda_1(\varphi^2 + 1) + \varphi\psi, \tag{4-3}$$

$$e_1(\psi) = \psi(\lambda_1\varphi + \psi) + c. \tag{4-4}$$

Taking into account (4-2), and using (3-2), (3-5) one has

$$\sum_{i=2}^{n} \omega_{ii}^1 = -3\lambda_1\varphi + (n-1)\psi,$$

and (4-1) and (3-11), respectively, become

$$3e_1(\lambda_1) = \big(R - n(n-1)c - 6\lambda_1^2\big)\varphi + (n+2)\lambda_1\psi, \tag{4-5}$$

$$e_1e_1(\lambda_1) = e_1(\lambda_1)(-3\lambda_1\varphi + (n-1)\psi) + \lambda_1\big(n(n-2)c - R + 4\lambda_1^2\big). \tag{4-6}$$

Differentiating (4-5) with respect to e_1, we may eliminate $e_1e_1(\lambda_1)$ by (4-6). Using (4-3), (4-4) and (4-6) we have

$$3(n-4)e_1(\lambda_1)\psi = \lambda_1\big(6R - (4n^2 - 12n - 3)c - 27\lambda_1^2\big). \tag{4-7}$$

Note here, $n > 4$ since the number of distinct principal curvatures is 6.

Eliminating $e_1(\lambda_1)$ between (4-5) and (4-7) gives

$$(n-4)\big\{\big(R - n(n-1)c - 6\lambda_1^2\big)\varphi\psi + (n+2)\lambda_1\psi^2\big\} = \lambda_1\big(6R - (4n^2 - 12n - 3)c - 27\lambda_1^2\big). \tag{4-8}$$

Further, differentiating (4-7) with respect to e_1, by (4-4), (4-6), (4-7), (4-5) we have

$$\big(432\lambda_1^4 + a_1\lambda_1^2 + a_2\big)\varphi + \big\{-54(n+3)\lambda_1^3 + a_3\lambda_1\big\}\psi = 12(n-4)\lambda_1^3 + a_4\lambda_1, \tag{4-9}$$

where

$$\begin{aligned} a_1 &= (97n^2 - 111n + 60)c - 105R, \\ a_2 &= ((4n^2 - 9n + 9)c - 6R)(n(n-1)c - R), \\ a_3 &= 12R - (4n^2 - 6n + 21)c, \\ a_4 &= 3n(n-4)(n-2)c. \end{aligned}$$

Differentiating (4-9) with respect to e_1 and using (4-3) and (4-4), we get

$$\begin{aligned} &(1728\lambda_1^3 + 2a_1\lambda_1)\varphi e_1(\lambda_1) + (432\lambda_1^4 + a_1\lambda_1^2 + a_2)\{\lambda_1(\varphi^2 + 1) + \varphi\psi\} \\ &\quad + \{-162(n+3)\lambda_1^2 + a_3\}\psi e_1(\lambda_1) + \{-54(n+3)\lambda_1^3 + a_3\lambda_1\}\{\psi(\lambda_1\varphi + \psi) + c\} \\ &\qquad = (36(n-4)\lambda_1^2 + a_4)e_1(\lambda_1). \end{aligned}$$

Multiplying by $3(n-4)$ on both sides of the above equation and using (4-5) and (4-7) we have

$$\begin{aligned}(n-4)\big(1728\lambda_1^3+2a_1\lambda_1\big)\varphi\big\{\big(R-n(n-1)c-6\lambda_1^2\big)\varphi+(n+2)\lambda_1\psi\big\}\\ +3(n-4)\big(432\lambda_1^4+a_1\lambda_1^2+a_2\big)\big\{\lambda_1(\varphi^2+1)+\varphi\psi\big\}\\ +\lambda_1\big\{-162(n+3)\lambda_1^2+a_3\big\}\big\{6R-(4n^2-12n-3)c-27\lambda_1^2\big\}\\ +3(n-4)\big\{-54(n+3)\lambda_1^3+a_3\lambda_1\big\}\big\{\psi(\lambda_1\varphi+\psi)+c\big\}\\ =(n-4)\big(36(n-4)\lambda_1^2+a_4\big)\big\{\big(R-n(n-1)c-6\lambda_1^2\big)\varphi+(n+2)\lambda_1\psi\big\}.\end{aligned}\tag{4-10}$$

Note that Equation (4-10) could be rewritten as

$$q_1(\lambda_1)\varphi^2+q_2(\lambda_1)\varphi\psi+q_3(\lambda_1)\psi^2+q_4(\lambda_1)\varphi+q_5(\lambda_1)\psi+q_6(\lambda_1)=0,\tag{4-11}$$

where q_i are nontrivial polynomials concerning the function λ_1 and given by:

$$\begin{aligned}q_1&=(n-4)\big(1728\lambda_1^3+2a_1\lambda_1\big)\big(R-n(n-1)c-6\lambda_1^2\big)\\&\qquad+3(n-4)\big(432\lambda_1^4+a_1\lambda_1^2+a_2\big)\lambda_1,\\ q_2&=(n-4)(n+2)\lambda_1\big(1728\lambda_1^3+2a_1\lambda_1\big)\\&\qquad+3(n-4)\big(432\lambda_1^4+a_1\lambda_1^2+a_2\big)+3(n-4)\big\{-54(n+3)\lambda_1^3+a_3\lambda_1\big\}\lambda_1,\\ q_3&=3(n-4)\big\{-54(n+3)\lambda_1^3+a_3\lambda_1\big\},\\ q_4&=(n-4)\big(36(n-4)\lambda_1^2+a_4\big)\big(R-n(n-1)c-6\lambda_1^2\big),\\ q_5&=-(n-4)(n+2)\big(36(n-4)\lambda_1^2+a_4\big)\lambda_1,\\ q_6&=-3(n-4)\big(432\lambda_1^4+a_1\lambda_1^2+a_2\big)\lambda_1\\&\qquad+\lambda_1\big(-162(n+3)\lambda_1^2+a_3\big)\big\{6R-(4n^2-12n-3)c-27\lambda_1^2\big\}\\&\qquad+3c(n-4)\big\{-54(n+3)\lambda_1^3+a_3\lambda_1\big\}.\end{aligned}\tag{4-12}$$

In the same manner, (4-8) and (4-9) could be also rewritten, respectively, as:

$$p_1(\lambda_1)\varphi\psi+p_2(\lambda_1)\psi^2=p_3(\lambda_1),\tag{4-13}$$

$$h_1(\lambda_1)\varphi+h_2(\lambda_1)\psi=h_3(\lambda_1),\tag{4-14}$$

where p_i, h_i $(i=1,2)$ are polynomials concerning the function λ_1 and given by

$$\begin{aligned}p_1&=(n-4)\big(R-n(n-1)c-6\lambda_1^2\big),\\ p_2&=(n-4)(n+2)\lambda_1,\\ p_3&=\lambda_1\big(6R-(4n^2-12n-3)c-27\lambda_1^2\big),\\ h_1&=432\lambda_1^4+a_1\lambda_1^2+a_2,\\ h_2&=-54(n+3)\lambda_1^3+a_3\lambda_1,\\ h_3&=12(n-4)\lambda_1^3+a_4\lambda_1.\end{aligned}\tag{4-15}$$

Multiplying by h_1^2 on both sides of (4-11), by taking into account (4-14) we may eliminate φ and get

$$P_1\psi^2 + P_2\psi = P_3, \tag{4-16}$$

where

$$\begin{aligned} P_1 &= q_1h_2^2 - q_2h_1h_2 + q_3h_1^2, \\ P_2 &= -2q_1h_2h_3 + q_2h_1h_3 - q_4h_1h_2 + q_5h_1^2, \\ P_3 &= -q_1h_3^2 - q_4h_1h_3 - q_6h_1^2. \end{aligned} \tag{4-17}$$

Similarly, eliminating φ in (4-13) by using (4-14) yields

$$Q_1\psi^2 + Q_2\psi = Q_3, \tag{4-18}$$

where

$$\begin{aligned} Q_1 &= p_2h_1 - p_1h_2, \\ Q_2 &= p_1h_3, \\ Q_3 &= p_3h_1. \end{aligned} \tag{4-19}$$

Moreover, multiplying by Q_1 and P_1 on both sides of the equations (4-16) and (4-18), respectively, after eliminating the 'ψ^2' part we obtain

$$(P_2Q_1 - P_1Q_2)\psi = P_3Q_1 - P_1Q_3. \tag{4-20}$$

Multiplying (4-20) by $P_1\psi$ and then combining this with (4-16) gives

$$\big\{P_1(P_3Q_1 - P_1Q_3) + P_2(P_2Q_1 - P_1Q_2)\big\}\psi = P_3(P_2Q_1 - P_1Q_2). \tag{4-21}$$

At last, after eliminating ψ between (4-20) and (4-21) we get

$$\begin{aligned} P_1(P_3Q_1 - P_1Q_3)^2 + P_2(P_2Q_1 - P_1Q_2)(P_3Q_1 - P_1Q_3) \\ = P_3(P_2Q_1 - P_1Q_2)^2. \end{aligned} \tag{4-22}$$

We observe from (4-12), (4-15), (4-17) and (4-19) that both P_i and Q_i $(1 \le i \le 3)$ are polynomials concerning λ_1 with constant coefficients. Hence, it follows that

$$\begin{aligned} P_1 &= -10077696(n-4)(n+3)(n-1)\lambda_1^{11} + \cdots, \\ P_2 &= -839808(n-4)^2(11n+5)\lambda_1^{11} + \cdots, \\ P_3 &= -69984(19n+113)\lambda_1^{13} + \cdots, \\ Q_1 &= 108(n-4)(n-1)\lambda_1^5 + \cdots, \\ Q_2 &= -72(n-4)^2\lambda_1^5 + \cdots, \\ Q_3 &= -11664\lambda_1^7 + \cdots, \end{aligned}$$

where we only need to write the highest order terms of λ_1.

By substituting P_i and Q_i into (4-22), we get a polynomial equation concerning λ_1 with constant coefficients $c_i = c_i(n, c, R)$:

$$\sum_{i=0}^{47} c_i \lambda_1^i = 0, \tag{4-23}$$

where the coefficient c_{47} of the highest order term satisfies

$$c_{47} = -10077696(n-4)^2(n+3)(n-1)^2\big[69984 \times 108(19n+113) \\ +10077696 \times 11664(n+3)\big]^2 \neq 0.$$

Therefore, λ_1 has to be constant and $H = -2\lambda_1/n$ is a constant, which is a contradiction.

Case B: $\omega_{pq}^r \neq 0$, $\omega_{pq}^u \neq 0$, and $\omega_{ij}^k = 0$ for all other distinct triplets $\{i, j, k\}$ and distinct principal curvatures $\lambda_i, \lambda_j, \lambda_k$. Then, (3-37) implies that

$$\omega_{pp}^1 \omega_{vv}^1 = -\lambda_p \lambda_v - c, \tag{4-24}$$

$$\omega_{qq}^1 \omega_{vv}^1 = -\lambda_q \lambda_v - c, \tag{4-25}$$
$$\omega_{rr}^1 \omega_{vv}^1 = -\lambda_r \lambda_v - c,$$
$$\omega_{uu}^1 \omega_{vv}^1 = -\lambda_u \lambda_v - c.$$

Similar to Case A, since $\omega_{pq}^r \neq 0$ and $\omega_{pq}^u \neq 0$, (3-33) and (3-35) imply that

$$\omega_{ii}^1 = \varphi \lambda_i + \psi, \quad \text{for } i = p, q, r, u, \tag{4-26}$$

where φ and ψ satisfy the differential equations (4-3) and (4-4).

Substituting (4-26) into (4-24) and (4-25), we obtain

$$\omega_{vv}^1 = -\frac{1}{\varphi}\lambda_v, \tag{4-27}$$

$$\lambda_v \psi = c\varphi, \tag{4-28}$$

which means that ω_{vv}^1 and λ_v are determined completely by φ and ψ.

Substitute (4-26)–(4-28) into (4-1), and then differentiate it with respect to e_1. By using (4-3), (4-4) and (3-11), a similar discussion as in Case A gives a polynomial concerning the function λ_1 with constant coefficients. Hence, λ_1 has to be constant, which yields a contradiction as well.

Case C: $\omega_{pq}^r \neq 0$ (or $\omega_{pq}^r = 0$), and all the $\omega_{ij}^k = 0$ for distinct triplets $\{i, j, k\}$ and distinct principal curvatures $\lambda_i, \lambda_j, \lambda_k$. Then, (3-37) implies that

$$\omega_{pp}^1 \omega_{uu}^1 = -\lambda_p \lambda_u - c, \quad \omega_{pp}^1 \omega_{vv}^1 = -\lambda_p \lambda_v - c, \tag{4-29}$$

$$\omega_{qq}^1 \omega_{uu}^1 = -\lambda_q \lambda_u - c, \quad \omega_{qq}^1 \omega_{vv}^1 = -\lambda_q \lambda_v - c, \tag{4-30}$$

$$\omega_{rr}^1 \omega_{uu}^1 = -\lambda_r \lambda_u - c, \quad \omega_{rr}^1 \omega_{vv}^1 = -\lambda_r \lambda_v - c, \tag{4-31}$$

$$\omega_{uu}^1 \omega_{vv}^1 = -\lambda_u \lambda_v - c. \tag{4-32}$$

We first consider $\lambda_i \neq 0$ for $i = p, q, r, u, v$. Consequently, (4-29)–(4-32) reduce to

$$\frac{\omega_{pp}^1}{\lambda_p} = \frac{\omega_{qq}^1}{\lambda_q} = \frac{\omega_{rr}^1}{\lambda_r} = -\frac{\lambda_u - \lambda_v}{\omega_{uu}^1 - \omega_{vv}^1},$$
$$\frac{\omega_{uu}^1}{\lambda_u} = \frac{\omega_{vv}^1}{\lambda_v} = -\frac{\lambda_p - \lambda_q}{\omega_{pp}^1 - \omega_{qq}^1},$$

and hence

$$\frac{\omega_{pp}^1}{\lambda_p} = \frac{\omega_{qq}^1}{\lambda_q} = \frac{\omega_{rr}^1}{\lambda_r} = \varphi, \tag{4-33}$$

$$\frac{\omega_{uu}^1}{\lambda_u} = \frac{\omega_{vv}^1}{\lambda_v} = \psi \tag{4-34}$$

for two functions φ and ψ.

Substituting (4-33) and (4-34) back to (4-29) gives

$$(1 + \varphi\psi)\lambda_p\lambda_u = -c,$$
$$(1 + \varphi\psi)\lambda_p\lambda_v = -c,$$

which imply that $\lambda_u = \lambda_v$. This is impossible.

If $\lambda_p = 0$, then (3-12) and (4-29) imply that $\omega_{pp}^1 = 0$ and $c = 0$. Then (4-30) and (4-31) yield

$$\frac{\omega_{uu}^1}{\lambda_u} = \frac{\omega_{vv}^1}{\lambda_v} = \gamma \tag{4-35}$$

for some function γ. However, combining (4-35) with (4-32) gives $\gamma^2 = -1$. Hence it is a contradiction.

Lastly, we consider $\lambda_u = 0$. Then (3-12) and (4-29) reduce to $\omega_{uu}^1 = c = 0$. The second equations of (4-29)–(4-31) show that

$$\frac{\omega_{pp}^1}{\lambda_p} = \frac{\omega_{qq}^1}{\lambda_q} = \frac{\omega_{rr}^1}{\lambda_r} = \varphi, \tag{4-36}$$

$$\frac{\omega_{vv}^1}{\lambda_v} = -\frac{1}{\varphi}. \tag{4-37}$$

By taking into account (4-36) and (4-37) together with (3-11) and (4-1), a very similar and direct computation as in Case A also gives a polynomial concerning the function λ_1 with constant coefficients. Hence, this is a contradiction and the mean curvature H must be constant.

In conclusion, the proof of Theorem 1.1 is completed.

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YU FU
SCHOOL OF MATHEMATICS
DONGBEI UNIVERSITY OF FINANCE AND ECONOMICS
DALIAN
CHINA
yufu@dufe.edu.cn

MIN-CHUN HONG
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF QUEENSLAND
BRISBANE
AUSTRALIA
hong@maths.uq.edu.au

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