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ENTROPY OF EMBEDDED SURFACES IN QUASI-FUCHSIAN MANIFOLDS

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We compare critical exponents for quasi-Fuchsian groups acting on the hyperbolic 3-space and entropy of invariant disks embedded in $\mathbb{H}^3$. We give a rigidity theorem for all embedded surfaces when the action is Fuchsian and a rigidity theorem for negatively curved surfaces when the action is quasi-Fuchsian.

1. Introduction

The aim of this paper is to compare two geometric invariants of Riemannian manifolds: critical exponent and volume entropy. The first one is defined through the action of the fundamental group on the universal cover, the second one is defined for compact manifolds as the exponential growth rate of the volume of balls in the universal cover. These two invariants have been studied in many cases; we pursue this study for quasi-Fuchsian manifolds.

Let Γ be a group acting on a simply connected Riemannian manifold (X, g). If the action on X is discrete we define the *critical exponent* by

$$\delta(\Gamma) := \limsup_{R\to\infty} \frac{1}{R} \operatorname{Card}\{\gamma \in \Gamma \mid d(\gamma \cdot o, o) \leq R\}, \tag{1}$$

where o is any point in X. It does not depend on this particular base point thanks to triangle inequality. If we want to insist on the space on which Γ acts we will write $\delta(\Gamma, X)$.

The volume entropy $h(g)$ of a Riemannian compact manifold (Σ, g) is defined by

$$h(g) := \lim_{R\to\infty} \frac{\log \operatorname{Vol}_g(B_g(o, R))}{R}, \tag{2}$$

where $B_g(o, R)$ is the ball of radius R and center o in the universal cover of Σ. We will also use the notation $h(X)$ for the exponential growth rate of ball volumes in a a simply connected manifold X.

It is a classical fact, using a simple volume argument that the volume entropy coincides with the critical exponent of $\pi_1(\Sigma)$ acting on $\widetilde{\Sigma}$. Moreover, a famous

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theorem of G. Besson, G. Courtois and S. Gallot [Besson et al. 1995] said that the entropy allows us to distinguish the hyperbolic metric in the set of all metrics, $\mathrm{Met}(\Sigma)$. Note that entropy is sensitive to homothetic transformations: for any $\lambda > 0$ we have $h(\lambda^2 g) = \frac{1}{\lambda} h(g)$. Assume that Σ admits a hyperbolic metric g_0 and let $\mathrm{Met}_0(\Sigma)$ be the set of metrics on Σ whose volume is equal to $\mathrm{Vol}(\Sigma, g_0)$, then the theorem of Besson, Courtois, and Gallot says that for all $g \in \mathrm{Met}_0(\Sigma)$

$$h(g) \geq h(g_0), \tag{3}$$

with equality if and only if $g = g_0$.

Our aim is to study the behavior of the volume entropy for a subset of all the metrics on a surface. This subset is the metrics induced by an incompressible embedding into quasi-Fuchsian manifolds. It has not the cone structure of $\mathrm{Met}(\Sigma)$: it is not invariant by all homothetic transformations. Hence we will look at the behavior of $h(g)$ without normalization by the volume.

Let S be a compact surface of genus $g \geq 2$ and $\Gamma = \pi_1(S)$ its fundamental group. A Fuchsian representation of Γ is a faithful and discrete representation in $\mathrm{PSL}_2(\mathbb{R})$. A quasi-Fuchsian representation is a perturbation of Fuchsian representation in $\mathrm{PSL}_2(\mathbb{C})$. More precisely it is a discrete and faithful representation of Γ into $\mathrm{Isom}(\mathbb{H}^3)$ such that the limit set on $\partial\mathbb{H}^3$ is a Jordan curve. A celebrated theorem of R. Bowen [1979] asserts that for quasi-Fuchsian representations, the critical exponent is minimal and equal to 1 if and only if the representation is Fuchsian.

We choose an isometric, totally geodesic embedding of $\mathbb{H}^2$ in $\mathbb{H}^3$ (the equatorial plane in the ball model for example). This embedding gives an inclusion i : $\mathrm{Isom}(\mathbb{H}^2) \to \mathrm{Isom}(\mathbb{H}^3)$.

Let ρ be a Fuchsian representation of Γ. The group Γ acts naturally on $\mathbb{H}^2$ and $\mathbb{H}^3$ by ρ and $i \circ \rho$, respectively. For every point $o \in \mathbb{H}^2$ we have

$$d_{\mathbb{H}^3}(i \circ \rho(\gamma)o, o) = d_{\mathbb{H}^2}(\rho(\gamma)o, o),$$

since $\mathbb{H}^2$ is totally geodesic in $\mathbb{H}^3$. The critical exponents for these two actions of Γ are then equal

$$\delta(\Gamma, \mathbb{H}^3) = \delta(\Gamma, \mathbb{H}^2) = 1.$$

In light of this trivial example, two questions rise up. What is the entropy of a Γ-invariant disk which is not totally geodesic? What happens when we modify the Fuchsian representation in $\mathrm{PSL}_2(\mathbb{C})$?

We will answer the first question. Since ρ is a Fuchsian representation, the critical exponent of Γ acting on $\mathbb{H}^3$ through $i \circ \rho$ is 1, and we have the following:

Theorem 1.1. *Suppose Γ is Fuchsian. Let Σ be a Γ-invariant disk embedded in $\mathbb{H}^3$. We have*

$$h(\Sigma) \leq \delta(\Gamma, \mathbb{H}^3), \tag{4}$$

with equality if and only if Σ *is the totally geodesic hyperbolic plane preserved by* Γ.

Note that $\delta(\Gamma, \mathbb{H}^3) = h(\Sigma, g_0)$, hence the last theorem can be rewritten as follows:

Theorem 1.2. *For all metrics g obtained as induced metrics by an incompressible embedding in a Fuchsian manifold we have*

$$h(g) \leq h(g_0) \tag{5}$$

with equality if and only if $g = g_0$.

We *did not* renormalize by the volume; this explains the dichotomy between (3) and (5).

We will prove this theorem in the next section. The inequality is trivial since the induced distance between two points is always greater than the distance in $\mathbb{H}^3$: $d_\Sigma \geq d_{\mathbb{H}^3}$, but the rigidity is not. We have no geometrical (curvature) hypothesis on Σ, therefore it is not obvious at all to show that the inequality is strict as soon as Σ is not totally geodesic. Indeed we cannot use the "usual" techniques of negative curvature like Bowen–Margulis measure, or even the uniqueness of geodesic between two points.

We obtain an answer to the second question under a geometrical hypothesis on the curvature:

Theorem 1.3. *Let* Γ *be a quasi-Fuchsian group and* $\Sigma \subset \mathbb{H}^3$ *a* Γ*-invariant embedded disk. We suppose that* Σ *endowed with the induced metric has negative curvature. We then have*

$$h(\Sigma) \leq I(\Sigma, \mathbb{H}^3)\delta(\Gamma, \mathbb{H}^3),$$

where $I(\Sigma, \mathbb{H}^3)$ *is the geodesic intersection between* Σ *and* $\mathbb{H}^3$. *Moreover, equality occurs if and only if the length spectrum of* Σ/Γ *is proportional to that of* $\mathbb{H}^3/\Gamma$.

The *geodesic intersection* will be defined in Section 3A. Roughly, it is the average ratio of the length between two points of Σ for the extrinsic and intrinsic distance. We need the curvature assumption to define and use this invariant.

This theorem implies Theorem 1.1 only for negatively curved embedded disks but not in its full generality. Indeed, when Γ is Fuchsian, and Σ/Γ has the same length spectrum as $\mathbb{H}^3/\Gamma$ it follows directly by the work of J-P. Otal [1990] that $\Sigma = \mathbb{H}^2/\Gamma$. However, using the fact that Σ is embedded in $\mathbb{H}^3$ we will be able to prove without the Fuchsian hypothesis that if the two marked length spectra are equal then Σ is totally geodesic, and therefore we obtain the following corollary of Theorem 1.3:

Corollary 1.4. *Under the assumptions of Theorem 1.3 we have*

$$h(\Sigma) \leq \delta(\Gamma, \mathbb{H}^3),$$

with equality if and only if Γ *is fuchsian and* Σ *is the totally geodesic hyperbolic plane, preserved by* Γ.

The proof of this corollary raises the following question generalizing this result: if a quasi-Fuchsian manifold has the same length spectrum as a negatively curved surface, does it imply that it is in fact Fuchsian? We answer this question using a well-known result of Y. Benoist, showing the following theorem:

Theorem 1.5. *Let M be a quasi-Fuchsian manifold and Σ a hyperbolic* (*in the sense that it has constant curvature* -1) *surface. Suppose that M and Σ have proportional length spectrum* (*i.e.*, *there exists $k \in \mathbb{R}^+$ such that for all $\gamma \in \Gamma$, $\ell_M(\gamma) = k\ell_\Sigma(\gamma)$*), *then M is Fuchsian and Σ is isometric to the totally geodesic surface in M.*

Theorem 1.3 has to be compared to results obtained by G. Knieper who compared entropy for two different metrics on the same manifolds, and our proof of Theorem 1.3 follows his paper [Knieper 1995]. As in his paper, we obtain that the intersection is larger than 1 as soon as Γ is not Fuchsian.

The theorem is also related to the work of M. Bridgeman and E. Taylor [2000]; indeed, we answer in the negative Question 2 of their paper. And finally, we can see our work as an extension of U. Hamenstadt's [2002], where she compared the geodesic intersection between the boundary of convex hulls and $\mathbb{H}^3$ for quasi-Fuchsian manifolds.

As we said, the two proofs are very different from one another. For the Fuchsian case, we give precise estimates for the length of some paths of the hyperbolic plane. We show that in some sense the length between two points on Σ is much greater than the extrinsic distance between those two points. For quasi-Fuchsian manifolds, we use well-known techniques of negative curvature geometry: we compare the Patterson–Sullivan measures for $\mathbb{H}^3$ and for Σ.

2. Fuchsian case

In this section we are going to prove Theorem 1.1. This theorem has a strong condition on Γ, i.e., it is conjugate to a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ but we make no geometrical assumptions on Σ. As we said, there could be more than one geodesic between two points on Σ.

We already remarked that the inequality is trivial, as is the equality when Σ is totally geodesic. Therefore, the only thing left to prove is the strict inequality when Σ is not totally geodesic or in other words if $\Sigma \neq \mathbb{H}^2$ then $h(\Sigma) < 1$.

The proof of the theorem is based on the comparison between the distances on equidistant surfaces of the totally geodesic Γ-invariant hyperbolic plane. We are going to prove several lemmas which together give Theorem 1.1. The strict inequality follows directly from Lemmas 2.2 and 2.8. We denote by $\mathbb{D}$ the totally geodesic, Γ-invariant plane. The induced metric on $\mathbb{D}$ is the usual hyperbolic metric, and we will denote it by $\mathbb{H}^2$. We are first going to see that between all the equidistant surfaces, $\mathbb{H}^2$ has the biggest entropy. Then we will make this argument work when only one part of the surface is "above" $\mathbb{D}$. The idea to prove it, is to consider another distance d_m on $\mathbb{D}$, which will be used as an intermediary between Σ and $\mathbb{H}^2$. We will explain, after the definition of d_m, how the two comparisons will be proved.

Let us begin to parametrize $\mathbb{H}^3$ by $\mathbb{H}^2 \times \mathbb{R}$ as follows: take an orientation for the unit normal tangent space of $\mathbb{H}^2$, then to a point $x \in \mathbb{H}^3$ we associate $s(x)$ the orthogonal projection from $\mathbb{H}^3$ to $\mathbb{H}^2$. This is the first parameter of the parametrization. The oriented distance along this geodesic gives the second one. Hence the parametrization, called Fermi coordinates, is defined by

$$\mathbb{H}^3 \mapsto \mathbb{H}^2 \times \mathbb{R}, \qquad z \to (s(z), \hat{d}(z, s(z))),$$

where $\hat{d}$ is the oriented distance defined by the choice of the orientation on the unit normal tangent of $\mathbb{H}^2$. With this parametrization, the metric on $\mathbb{H}^3$ is

$$g_{\mathbb{H}^3} = \cosh^2(r) g_0 + dr^2.$$

Look at $S(r)$ the equidistant disk at distance r of $\mathbb{H}^2$; its metric, induced by the one on $\mathbb{H}^3$, is $g_r = \cosh^2(r) g_0$. It is isometric to a hyperbolic plane of curvature $1/\cosh(r)$, and its volume entropy is $h(S(r)) = h(0)/\cosh(r) = 1/\cosh(r)$, hence the entropy is maximal if and only if $r = 0$. For the general case, we are going to refine this argument showing that it is sufficient that a small part of Σ is over $\mathbb{H}^2$ for the entropy to be strictly less than 1.

Let Σ be a embedded Γ-invariant disk in $\mathbb{H}^3$. We assume that $\Sigma \neq \mathbb{D}$, and we endow Σ with its induced metric. Let x, y be two points on Σ. Let c_Σ be a geodesic on Σ linking x to y. We parametrize c_Σ by its Fermi coordinates, (c, r). We then have

$$\begin{aligned} d_\Sigma(x, y) &= \int_0^L \|c'_\Sigma(t)\|_\Sigma \, dt \\ &= \int_0^L \sqrt{r'(t)^2 + \cosh^2(r(t)) \|c'(t)\|_{g_0}^2} \, dt. \\ &\geq \int_0^L \cosh(r(t)) \|c'(t)\|_{g_0} \, dt. \end{aligned} \tag{6}$$

We now endow $\mathbb{D}$ with a different distance to the one coming from hyperbolic metric. It will play the role of intermediary to compare $d_\Sigma(x, y)$ on Σ with $d_{g_0}(s(x), s(y))$ on $\mathbb{H}^2$.

We call σ the restriction of s on Σ. Since $\Sigma \neq \mathbb{D}$, there exist $x_0 \in \mathbb{D} \setminus \Sigma$, $\varepsilon > 0$ and $\eta > 0$ such that

$$d_{\mathbb{H}^3}(\sigma^{-1} B(x_0, 2\varepsilon), \mathbb{D}) > \eta.$$

This means that *all* the points in the pre-image of $B(x_0, 2\varepsilon)$ by σ are at distance greater than η from $\mathbb{D}$. We will assume that 2ε is smaller than the injectivity radius of $\mathbb{H}^2/\Gamma$ so that the translations of $B(x_0, 2\varepsilon)$ by Γ are disjoint. We have taken 2ε in order to simplify the proof of Lemma 2.4.

We now consider on $\mathbb{D}$ the metric g_m defined by putting weight on the translations of $B(x_0, 2\varepsilon)$ by Γ.

Definition 2.1. We define g_m by

$$g_m := \begin{cases} \cosh(\eta)^2 g_0 & \text{on } \Gamma \cdot B(x_0, 2\varepsilon), \\ g_0 & \text{elsewhere.} \end{cases}$$

We will index by m objects which depend on this metric. Note that this metric is not continuous but it still defines a length space. Let $c : [0, 1] \to \mathbb{D}$ be a C^1 path, we then have

$$\ell_m(c) = \int_0^1 \|\dot{c}(t)\|_{g_m}\, dt.$$

This gives a distance d_m on $\mathbb{D}$ by choosing

$$d_m(x, y) := \inf_c \{\ell_m(c) \mid c(0) = x,\ c(1) = y\}.$$

In order to prove Theorem 1.1 we will compare the entropy of $(\mathbb{D}, d_m)$ with the one of Σ and the one of $\mathbb{H}^2$. The comparison with the entropy of Σ is quite easy and follows quickly from the definition of d_m and the inequality (6). The comparison with the entropy of $\mathbb{H}^2$ is more subtle. Indeed, there exist geodesics of $\mathbb{H}^2$ which are geodesics for $(\mathbb{D}, d_m)$ (any lift of a closed geodesic which does not cross the ball $B(x_0, 2\varepsilon)/\Gamma$) on $\mathbb{H}^2/\Gamma$). We will first prove that two points of $\mathbb{D}$ which are joined by a geodesic of $\mathbb{H}^2$ which often crosses $\Gamma \cdot B(x_0, 2\varepsilon)$ are much farther away from each other for d_m distance, see Lemma 2.4. Then, we will use a large deviation theorem for the geodesic flow (Theorem 2.6), to show that there are few geodesics which do not cross $\Gamma \cdot B(x_0, 2\varepsilon)$ (Lemma 2.7). It will follow from these two results that the balls of radius R for d_m are almost completely included in balls of radius R/C of $\mathbb{H}^2$ for $C > 1$ (Lemma 2.8). The two comparisons give the proof of Theorem 1.1.

The comparison between $h(\Sigma)$ and the critical exponent of $(\mathbb{D}, d_m)$ follows from the inequality (6) and the definition of d_m.

Lemma 2.2. *We have*

$$h(\Sigma) \le \delta((\mathbb{D}, d_m)).$$

Proof. Let $x \in \Sigma$ and $o = \sigma(x) \in \mathbb{D}$. Since Σ/Γ is compact, we have

$$h(\Sigma) = \lim_{R\to\infty} \frac{1}{R} \log \operatorname{Card}\{\gamma \in \Gamma \mid d_\Sigma(\gamma x, x) \le R\}.$$

And by definition

$$\delta((\mathbb{D}, d_m)) = \lim_{R\to\infty} \frac{1}{R} \log \operatorname{Card}\{\gamma \in \Gamma \mid d_m(\gamma o, o) \le R\}.$$

It is sufficient to prove that $d_\Sigma(x, y) \geq d_m(s(x), s(y))$, for all $x, y \in \Sigma$. Let $c_\Sigma = (c, r)$ be a geodesic on Σ joining x to y. Recall that we have

$$d_\Sigma(x, y) \geq \int_0^L \cosh(r(t))\|c'(t)\|_{g_0}\, dt.$$

If $c(t) \notin \Gamma \cdot B(x_0, 2\varepsilon)$, then $\|c'(t)\|_{g_m} = \|c'(t)\|_{g_0}$. In particular,

$$\|c'(t)\|_{g_m} \leq \cosh(r(t))\|c'(t)\|_{g_0}.$$

If $c(t) \in \Gamma \cdot B(x_0, 2\varepsilon)$, then by definition of g_m, $\|c'(t)\|_{g_m} = \cosh(\eta)\|c'(t)\|_{g_0}$ and since Σ is "far" from $\mathbb{D}$, $r(t) > \eta$. In particular,

$$\|c'(t)\|_{g_m} \leq \cosh(r(t))\|c'(t)\|_{g_0}.$$

Finally,

$$\begin{aligned} d_\Sigma(x, y) &\geq \int_0^L \|c'(t)\|_{g_m}\, dt \\ &\geq l_m(c) \\ &\geq d_m(s(x), s(y)). \end{aligned}$$

□

Our next aim is to compare the distances d_m and $d_{\mathbb{H}^2}$. Let us fix some notations before stating the first lemma. For all $v \in T^1\mathbb{H}^2$, let ζ_R^v be the probability measure on $T^1\mathbb{H}^2$, defined for all Borel sets $E \subset T^1\mathbb{H}^2$ by

$$\zeta_R^v(E) = \frac{1}{R}\int_0^R \chi_E(\phi_t^{\mathbb{H}^2}(v))\, dt,$$

where χ_E is the indicator function of E. For a Borel set E which is a unitary tangent bundle of a subset of $\mathbb{D}$, $E := T^1 A$, we have

$$\zeta_R^v(E) = \frac{1}{R}\,\mathrm{Leb}\{t \in [0, R] \mid c_v(t) \in A\}$$

since $\phi_t^{\mathbb{H}^2}(v) \in E$ is equivalent to $c_v(t) = \pi\phi_t^{\mathbb{H}^2}(v) \in A$.

Let L be the Liouville measure on the unitary tangent bundle of the quotient surface $T^1\mathbb{H}^2/\Gamma$. Recall that the metric g_m is given by $g_m = \cosh^2(\eta)g_0$ on $T^1\Gamma B(x_0, 2\varepsilon)$. We fix $K := T^1(\Gamma \cdot B(x_0, \varepsilon))$.[1]

Definition 2.3. Let $\kappa > 0$ be such $L(K/\Gamma) - 2\kappa > 0$. We define the sets

$$\mathcal{E}(R) := \{v \in T^1\mathbb{H}^2 \mid \zeta_R^v(K) > L(K/\Gamma) - \kappa\},$$

and for all points $o \in \mathbb{H}^2$, we note

$$\mathcal{E}_o(R) := \{v \in T_o^1\mathbb{H}^2 \mid \zeta_R^v(K) > L(K/\Gamma) - \kappa\}.$$

[1]We use a ball of half the size, for a technical reason that appears at the beginning of the proof of Lemma 2.4.

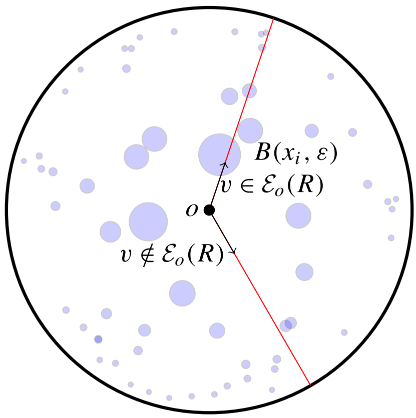

Figure 1. $\Gamma \cdot B(x_0, \varepsilon)$, $\mathcal{E}_o(R)$ and $\mathcal{E}_o^c(R)$.

A geodesic of length R whose direction is given by a vector $v \in \mathcal{E}(R)$ crosses πK "often", that is, at least a number of times proportional to R; see Figure 1. Indeed, if $v \in \mathcal{E}(R)$ we have

$$\frac{1}{R} \operatorname{Leb}\{t \in [0, R] \mid c_0(t) \cap \pi K \neq \varnothing\} > L(K/\Gamma) - \kappa > \kappa > 0,$$

since $\dot{c}_0(t) \in K$ is equivalent to $c_0(t) \in \pi K$ by definition of K.

The next argument is the key in the proof of Theorem 1.1. It shows that we can compare the length of a geodesic in $\mathbb{H}^2$ which often crosses πK with its d_m-length.

Lemma 2.4. *There exists $C > 1$, such that for all $R > 0$, for all $v \in \mathcal{E}_o(R)$ and for all $x \in \{\exp(tv) \mid t \in [R, 2R]\}$, we have*

$$d_m(o, x) \geq C d_{\mathbb{H}^2}(o, x). \tag{7}$$

Proof. Let c_0 be the geodesic for g_0 and c_m be a minimizing geodesic for g_m between o and x. Let d be the hyperbolic distance between o and x, $d = d_{\mathbb{H}^2}(o, x)$, and we parametrize c_0 by unit speed; we thus have $c_0(d) = x$. Let $N(R)$ be the number of intersections between πK and $c_0([0, R])$, that is N is the number of connected components of $c_0([0, R]) \cap \pi K$. On one hand, all components of $c_0([0, R]) \cap \pi K$ are inside balls of radius ε, hence c_0 "stays" at most 2ε in each components. On the other hand, the hypothesis $v \in \mathcal{E}_o(R)$, implies

$$\frac{1}{R} \operatorname{Leb}\{t \in [0, R] \mid c_0(t) \cap \pi K \neq \varnothing\} > L(K/\Gamma) - \kappa = \kappa > 0.$$

These two facts imply that $2\varepsilon N(R) \geq \kappa R$, that is to say,

$$N(R) \geq \frac{\kappa}{2\varepsilon} R. \tag{8}$$

For $i \leq N(R)$, let $t_i \in [0, d]$ such that $c_0(t_i) \in \pi K$ and $c_0[t_{i-1}, t_i] \setminus \pi K$ is connected: we just have chosen a point $x_i = c_0(t_i)$ in each ball of πK crossing c_0.

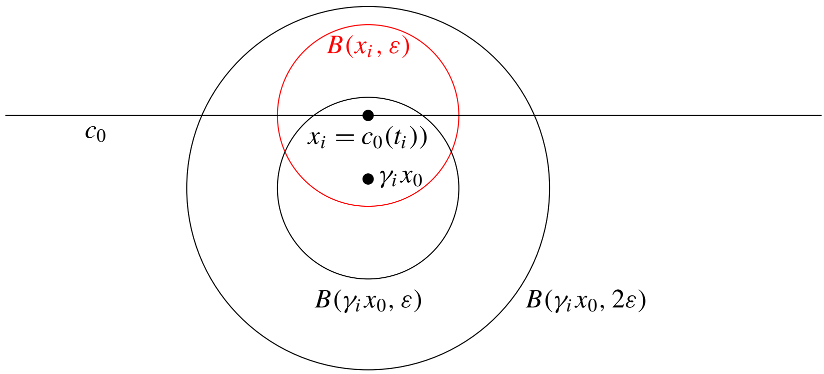

Figure 2. c_0 meets $B(\gamma_i x_0, \varepsilon)$. $B(x_i, \varepsilon) \subset B(\gamma_i x_0, 2\varepsilon)$.

There exists $\gamma_i \in \Gamma$ such that $x_i \in B(\gamma_i x_0, \varepsilon)$, hence $B(x_i, \varepsilon) \subset B(\gamma_i x_0, 2\varepsilon)$ on which the metric g_m is $g_m = \cosh^2(\eta) g_0$. See Figure 2. Therefore the geodesic c_0 is divided into $N(R)$ segments: $[x_i, x_{i+1}]$, such that for every i we know that on the ball $B(x_i, \varepsilon)$ the metric g_m is given by $g_m = \cosh^2(\eta) g_0$. We want a lower bound on $d_m(o, x)$, therefore we can estimate the length of c_m with the metric given by $\cosh^2(\eta) g_0$ on the smaller balls $B(x_i, \varepsilon) \subset B(\gamma_i x_0, 2\varepsilon)$ and g_0 on the rest of the plane.

We call y_i the middle of $[x_i, x_{i+1}]$. We now restrict our attention to one segment $[y_i, y_{i+1}]$. Let $0 < a < 1$ whose dependence on η will be made clear in the rest of the proof. We are going to analyze two different cases.

Case 1: c_m crosses $B(x_i, a\varepsilon)$. Let Δ_i be the lines (geodesics in $\mathbb{H}^2$) orthogonal to c_0 and passing through y_i. Let z_i^1 and z_i^2 be the end points of the diameter of $B(x_i, \varepsilon)$ defined by $z_i^1 = c_0(t_i - \varepsilon)$ and $z_i^2 = c_0(t_i + \varepsilon)$, and call D_i^1 and D_i^2 the lines orthogonal to c_0 and passing through z_i^1 and z_i^2. See Figure 3.

We want to consider the intersections between c_m and the lines Δ_i, D_i^1 and D_i^2. There might be many intersections. We will call the first intersection of c_m with a line D the point $c_m(t_f)$ where $t_f := \inf\{t \mid c_m(t) \in D\}$, and the last intersection of c_m with D the point $c_m(t_l)$, where $t_l := \sup\{t \mid c_m(t) \in D\}$.

Let A_i', B_i' and C_i' be the last intersections of c_m with Δ_i, D_i^1 and D_i^2, respectively. Let B_i, C_i and A_{i+1} be the first intersections of c_m with D_i^1, D_i^2 and Δ_{i+1}, respectively. This divides c_m into five connected components:

$$[A_i', B_i], \quad [B_i, B_i'], \quad [B_i', C_i], \quad [C_i, C_i'], \quad [C_i', A_{i+1}].$$

Our work will be to give a lower bound for the length of each component; see Figure 3. Since it might happen that $B_i = B_i'$ and $C_i = C_i'$ the bound on the length of those two components will be trivial: $d_m(B_i, B_i') \geq 0$ and $d_m(C_i, C_i') \geq 0$.

The g_m-length of c_m from A_i' to B_i is equal to (or larger than) its g_0-length since the metric g_m is equal to the metric g_0 outside K. Moreover the g_0-length of c_m

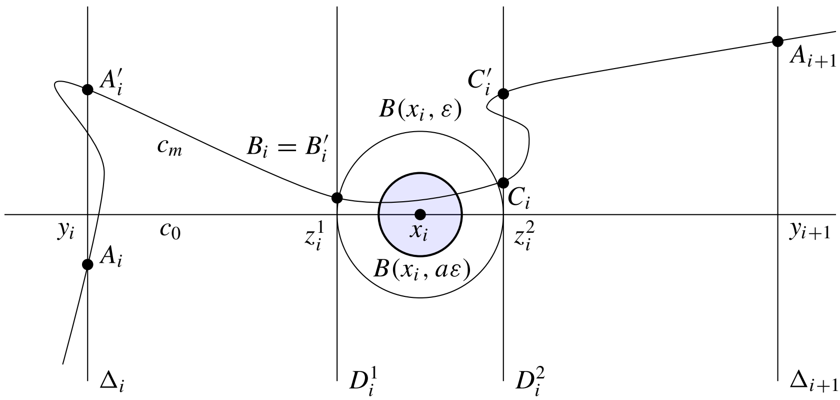

Figure 3. c_m crosses $B(x_i, a\varepsilon)$.

from A'_i to B_i is greater than $d_{g_0}(y_i, z_i^1)$ since the orthogonal projection decreases lengths. We then have

$$d_m(A'_i, B_i) \geq d_{g_0}(y_i, z_i^1).$$

For the same reasons we have

$$d_m(C'_i, A_{i+1}) \geq d_{g_0}(z_i^2, y_{i+1}).$$

We want to give a lower bound for the g_m-length of c_m between B'_i and C_i. We made the assumption that c_m crosses the ball $B(x_i, a\varepsilon)$ hence c_m stays at least $2\varepsilon - 2a\varepsilon$ in the ball $B(x_i, \varepsilon)$. In other words if c_m is unitary for g_0 we have $\mathrm{Leb}\{t \mid c_m(t) \cap B(x_i, \varepsilon) \neq \varnothing\} \geq 2\varepsilon - 2a\varepsilon$. In the ball $B(x_i, \varepsilon)$, the metric g_m is equal to $\cosh(\eta)^2 g_0$ hence the g_m-length satisfies

$$\begin{aligned} d_m(B'_i, C_i) &\geq \int_{\{t \mid c_m(t)\cap B(x_i,\varepsilon)\neq\varnothing\}} \|\dot{c}_m(t)\|_m \, dt = \int_{\{t \mid c_m(t)\cap B(x_i,\varepsilon)\neq\varnothing\}} \cosh(\eta) \\ &\geq \varepsilon \cosh(\eta)(2 - 2a). \end{aligned}$$

Choose $a > 0$ such that $\cosh(\eta)(2\varepsilon - 2a\varepsilon) > 2\varepsilon$, that is to say $a \leq 1 - 1/\cosh(\eta)$. In order to fix the idea we set $a := \frac{1}{2}(1 - 1/\cosh(\eta))$. This implies

$$\begin{aligned} d_m(B'_i, C_i) &\geq \varepsilon \cosh(\eta)(2 - 2a) \\ &= \varepsilon \cosh(\eta)\Big(2 - \Big(1 - \frac{1}{\cosh(\eta)}\Big)\Big) \\ &= (\cosh(\eta) + 1)\varepsilon \\ &= 2\varepsilon + \varepsilon[\cosh(\eta) - 1)] \\ &= d_{g_0}(z_i^1, z_i^2) + \varepsilon[\cosh(\eta) - 1)]. \end{aligned}$$

Thus, we have proven

$$d_m(A_i, A_{i+1}) \geq d_m(A'_i, A_{i+1}) \geq d_{g_0}(y_i, y_{i+1}) + \varepsilon[\cosh(\eta) - 1]. \tag{9}$$

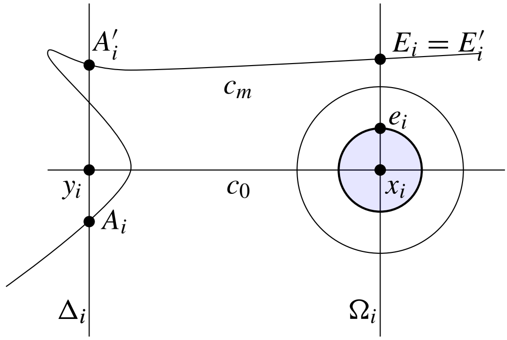

Figure 4. c_m does not cross $B(x_i, a\varepsilon)$.

Case 2: c_m does not cross $B(x_i, a\varepsilon)$. Let Δ_i be the line orthogonal to c_0 and passing through y_i, and Ω_i the one through x_i. Call A_i' the last intersection of c_m and Δ_i and E_i the first intersection of c_m with Ω_i. Since c_m does not cross $B(x_i a\varepsilon)$, E_i is in one of the connected components of $\Omega_i \setminus B(x_i, a\varepsilon)$. Named e_i the intersection of $S(x_i, a\varepsilon)$ (the sphere of center x_i and diameter $a\varepsilon$) and Ω_i in the same connected component as E_i, this is also the orthogonal projection of E_i on $B(x_i, a\varepsilon)$. See Figure 4.

We parametrize the geodesic Ω_i by $\mathbb{R}$; we give $\omega : \mathbb{R} \to \mathbb{H}^2$ such that $\omega(\mathbb{R}) = \Omega_i$. We suppose that $\omega(0) = x_i$ and the orientation is chosen in order to have $\omega(a\varepsilon) = e_i$. The function $t \to d_{g_0}(\omega(t), \Delta_i)$ is convex, and has a minimum at 0; it is hence increasing on $\mathbb{R}^+$. Therefore, $d_{g_0}(\Delta_i, E_i) \geq d_{\mathbb{H}^2}(\Delta_i, e_i)$. It follows that

$$d_m(A_i', E_i) \geq d_{\mathbb{H}^2}(A_i', E_i) \geq d_{g_0}(\Delta_i, E_i) \geq d_{g_0}(\Delta_i, e_i).$$

Let us compute $d_{g_0}(\Delta_i, e_i)$. We fix some notation:

$$L = d_{g_0}(\Delta_i, e_i), \qquad l = d_{g_0}(y_i, x_i), \qquad H = d_{g_0}(y_i, e_i).$$

Now Pythagoras' theorem in hyperbolic geometry for the triangle $(y_i x_i e_i)$ gives

$$\cosh(l)\cosh(a\varepsilon) = \cosh(H).$$

Let θ be the angle $\widehat{x_i y_i e_i}$. We have

$$\cos(\theta) = \frac{\tanh(l)}{\tanh(H)}$$

and

$$\sin(\pi/2 - \theta) = \frac{\sinh(L)}{\sinh(H)}.$$

Hence

$$\sinh(L) = \sinh(H)\frac{\tanh(l)}{\tanh(H)} = \cosh(H)\tanh(l) = \cosh(a\varepsilon)\sinh(l).$$

From this equation, we *cannot* conclude that $L > l + u$ for some $u > 0$. Indeed if L goes to 0 so does l. To avoid this problem we are going to assume that l is greater than the injectivity radius of S.

Note the following property of sinh which is a consequence of easy calculus. For all $x_0 > 0$ and $\varpi > 1$, there exists $u > 0$, such that for all $x > x_0$, we have $\varpi \sinh(x) \geq \sinh(x + u)$. Now we can choose y_i on c_0 in order to have

$$d_{g_0}(x_i, y_i) \geq s/2,$$

where s is the injectivity radius of $\mathbb{H}^2/\Gamma$. Consequently, applying the previous property with $\varpi = \cosh(a\varepsilon)$ and $x_0 = s/2$, there exists $u > 0$ such that

$$\cosh(a\varepsilon)\sinh(l) \geq \sinh(l + u).$$

Since sinh is increasing we deduce that

$$L \geq l + u.$$

Altogether, we show that there exists $u > 0$ such that

$$d_m(A_i', E_i) \geq d_{g_0}(y_i, x_i) + u.$$

By the same arguments we can show that

$$d_m(E_i', A_{i+1}) \geq d_{g_0}(x_i, y_{i+1}) + u.$$

(E_i' is the last intersection of c_m with Ω_i). Hence, if c_m does not meet $B(x_i, a\varepsilon)$, the g_m-length of c_m between A_i and A_{i+1} satisfies, (taking trivial bounds for first and last intersections)

$$d_m(A_i, A_{i+1}) \geq d_{g_0}(y_i, y_{i+1}) + 2u. \tag{10}$$

Now, let $\alpha := \min\{\varepsilon[\cosh(\eta) - 1]; 2u\}$. From (9) and (10) we have

$$d_m(A_i, A_{i+1}) \geq d_{g_0}(y_i, y_{i+1}) + \alpha.$$

Summing on i we get

$$d_m(o, x) \geq d_{g_0}(o, x) + N(R)\alpha.$$

Equation (8) and the fact that $d_{g_0}(o, x) \leq 2R$[2] imply that

$$N(R) \geq \frac{\kappa}{2\varepsilon} R \geq \frac{\kappa}{4\varepsilon} d_{g_0}(o, x).$$

Consequently,

$$d_m(o, x) \geq \left(1 + \frac{\alpha\kappa}{4\varepsilon}\right) d_{g_0}(o, x).$$

This proves the lemma with $C = \left(1 + \frac{\alpha\kappa}{4\varepsilon}\right)$. □

[2]This is where we use the upper bound on $d_{g_0}(o, x)$.

We now compare the entropy of $(\mathbb{D}, d_m)$ with that of $\mathbb{H}^2$. Let us define

$$\mathcal{F}_o(R) = \{\exp(tv) \mid t \in \mathbb{R}^+,\, v \in \mathcal{E}_o(R)\}.$$

We denote by $B_m(o, 2R)$ the ball of radius $2R$ for the d_m distance.

Lemma 2.5. *Let $C' := \min(2, C)$ where C satisfies Lemma 2.4. For all $o \in \mathbb{D}$, and all $R > 0$,*

$$B_m(o, 2R) \subset B_{\mathbb{H}^2}(o, 2R/C') \cup \big(B_{\mathbb{H}^2}(o, 2R) \cap \mathcal{F}_o^c(R)\big).$$

Proof. We have $B_m(o, 2R) = (B_m(o, 2R) \cap \mathcal{F}_o(R)) \cup (B_m(o, 2R) \cap \mathcal{F}_o^c(R))$. Let $x \in B_m(o, 2R) \cap \mathcal{F}_o(R)$. Since $d_{\mathbb{H}^2}(o, x) \le d_m(o, x)$, it follows that $d_{\mathbb{H}^2}(o, x) \le 2R$. There are only two possibilities. If $d_{\mathbb{H}^2}(o, x) \le R$, we have in particular $d_{\mathbb{H}^2}(o, x) \le 2R/C'$. However, if $d_{\mathbb{H}^2}(o, x) \ge R$, we apply Lemma 2.4 and we get $d_{\mathbb{H}^2}(o, x) \le 2R/C \le 2R/C'$. Therefore,

$$B_m(o, 2R) \cap \mathcal{F}_o(R) \subset B_{\mathbb{H}^2}\Big(o, \frac{2R}{C'}\Big) \cap \mathcal{F}_o(R) \subset B_{\mathbb{H}^2}\Big(o, \frac{2R}{C'}\Big).$$

Since we also have for $R > 0$, $B_m(o, 2R) \subset B_{\mathbb{H}^2}(o, 2R)$, this gives

$$B_m(o, 2R) \cap \mathcal{F}_o^c(R) \subset B_{\mathbb{H}^2}(o, 2R) \cap \mathcal{F}_o^c(R),$$

and proves the lemma. □

The Liouville measure on $T^1\mathbb{H}^2$ is the product of the riemannian measure of $\mathbb{H}^2$ with the angular measure on every fiber. We denote this product by $L = d\mu(x) \times d\theta(x)$. Our aim is to show that the set $\mathcal{E}_o^c(R)$ is small and the volume of $(B_{\mathbb{H}^2}(o, 2R) \cap \mathcal{F}_o^c(R))$ is small compared to the one of $B_{\mathbb{H}^2}(o, 2R)$. For this we are going to use a large deviation theorem of Y. Kifer [1990] which gives an upper bound on the mass of the vectors which do not behave as the Liouville measure.

Let $\mathcal{P}$ be the set of probability measures on $T^1\mathbb{H}^2/\Gamma$ endowed with the weak topology. Let $\mathcal{P}^t$ be the subset of $\mathcal{P}$ of probability measures invariant by the geodesic flow. We also denote by L the Liouville measure on the quotient $T^1\mathbb{H}^2/\Gamma$. Recall that for a vector $v \in T^1\mathbb{H}^2/\Gamma$ we denote by ζ_v^R the probability measure given for all Borel subsets $E \subset T^1\mathbb{H}^2/\Gamma$ by

$$\zeta_R^v(E) = \frac{1}{R}\int_0^R \chi_E(\phi_t^{\mathbb{H}^2/\Gamma}(v))\, dt.$$

Theorem 2.6 [Kifer 1990, Theorem 3.4]. *Let $\overline{A}$ be a compact subset of $\mathcal{P}$,*

$$\limsup_{T\to\infty} \frac{1}{T} \log L\big\{v \in T^1\mathbb{H}^2/\Gamma \mid \zeta_v^T \in \overline{A}\big\} \le -\inf_{\mu\in\overline{A}\cap\mathcal{P}^t} f(\mu),$$

where $f(\mu) = 1 - h_\mu(\phi_t^{\mathbb{H}^2/\Gamma})$ and $h_\mu(\phi_t^{\mathbb{H}^2/\Gamma})$ is the entropy of the geodesic flow $\phi_t^{\mathbb{H}^2/\Gamma}$ with respect to μ.

The fact that the theorem can be applied in this setting is explained after the Theorem 3.4 in [Kifer 1990]. In this reference the function f is given by a formula which seems different. One can look at [Paulin et al. 2015, Chapter 7], where the authors explain in detail why the geodesic flow of negatively curved surfaces satisfies the hypothesis of Kifer's theorem, and that one can take $f(\mu) = 1 - h_\mu(\phi_t^{\mathbb{H}^2/\Gamma})$.

Lemma 2.7. *There exist $o \in \mathbb{H}^2$, $\alpha > 0$ and $R_0 > 0$ such that for all $R > R_0$,*

$$\theta_o(\mathcal{E}_o^c(R)) \leq e^{-\alpha R}.$$

Proof. Let us keep the notations of Lemma 2.4. $K = T^1\Gamma \cdot B(x, \varepsilon)$ and we consider the following subset of $\mathcal{P}$:

$$A := \{\mu \in \mathcal{P} \mid \mu(K/\Gamma) \leq L(K/\Gamma) - \kappa\}.$$

This set is *not* closed for the weak topology. Its closure satisfies

$$\overline{A} \subset \{\mu \in \mathcal{P} \mid \mu(T^1\Gamma \cdot B^\circ(x, \varepsilon)/\Gamma) \leq L(K/\Gamma) - \kappa\},$$

where $B^\circ(x, \varepsilon)$ is the open ball. There might be equality between the two sets, but we won't use it.

However, since the unitary tangent bundle of the sphere $S(x, \varepsilon)$ is transverse to the flow, we have

$$\{v \in T^1\mathbb{H}^2/\Gamma \mid \zeta_v^R \in A\} = \{v \in T^1\mathbb{H}^2/\Gamma \mid \zeta_v^R \in \overline{A}\}.$$

Since $L \notin \overline{A}$ and L is the unique measure of maximal entropy satisfying $h(L) = 1$, we have

$$-\inf_{\mu \in \overline{A}} f(\mu) = -\alpha < 0.$$

Besides, it is clear that the set $\mathcal{E}^c(R) = \{v \in T^1\mathbb{H}^2 \mid \zeta_R^v(K) \leq L(K/\Gamma) - \kappa\}$ is Γ-invariant from the Γ invariance of K. By definition and the previous remark we get

$$\begin{aligned}\mathcal{E}^c(R)/\Gamma &= \{v \in T^1\mathbb{H}^2/\Gamma \mid \zeta_v^R \in A\}\\ &= \{v \in T^1\mathbb{H}^2/\Gamma \mid \zeta_v^R \in \overline{A}\}.\end{aligned}$$

Theorem 2.6 says that there exists $R_0 > 0$ such that for all $R > R_0$ we have

$$L(\mathcal{E}^c(R)/\Gamma) \leq e^{-\alpha R}.$$

The product structure of L implies the existence of a point $o \in \mathbb{H}^2/\Gamma$ such that

$$\theta_o(\mathcal{E}_o^c(R)/\Gamma) \leq e^{-\alpha R}.$$

The lemma follows, choosing any lift of o in $\mathbb{H}^2$. □

We finish the proof of Theorem 1.1 with Lemma 2.8, which compares the critical exponent between d_m and hyperbolic distance. Lemmas 2.2 and 2.8 conclude the proof.

Lemma 2.8. *There exists $u > 0$ such that*

$$\delta((\mathbb{D}, d_m)) \leq 1 - u.$$

Proof. We are going to show that the volume entropy of $(\mathbb{D}, d_m)$ satisfies the inequality, which implies a similar result on the critical exponent.

Let $o \in \mathbb{D}$ be a point satisfying Lemma 2.7. From Lemma 2.5, we have

$$B_m(o, 2R) \subset B_{\mathbb{H}^2}\Big(o, \frac{2R}{C'}\Big) \cup (B_{\mathbb{H}^2}(o, 2R) \cap \mathcal{F}_o^c(R)).$$

On one hand we have the classical upper bound $\mathrm{Vol}(B_{\mathbb{H}^2}(o, 2R/C')) = O(e^{2R/C'})$. On the other hand the volume form on $\mathbb{H}^2$ can be written in polar coordinates as $\sinh(r) dr d\theta$, hence for all $R > R_0$ we get

$$\begin{aligned}\mathrm{Vol}\Big(B_{\mathbb{H}^2}(o, 2R) \cap \mathcal{F}_o^c(R)\Big) = \int_0^{2R}\int_{\mathcal{E}_o^c(R)} \sinh(r)\, d\theta\, dr \leq \int_0^{2R} e^{-\alpha R} e^r\, dr \\ \leq e^{(2-\alpha)R}.\end{aligned}$$

Let $u > 0$, defined by $1 - u = \max(1/C', (1 - \alpha/2)) < 1$. The last two upper bounds give

$$\mathrm{Vol}(B_m(o, 2R)) = O(e^{2R/C'}) + O(e^{(2-\alpha)R}) = O(e^{2(1-u)R}).$$

We finish by taking the log and the limit. □

3. Quasi-Fuchsian case

3A. *Geodesic intersection.* Let Σ be an incompressible surface in M. We designate by $\phi_t^{\mathbb{H}^3}$, ϕ_t^{Σ} the geodesic flows on the unitary tangent spaces $T^1\mathbb{H}^3$, $T^1\Sigma$ respectively. We denote by π the projection from $T^1\mathbb{H}^3$ to $\mathbb{H}^3$. The restriction of π to $T^1\Sigma$ will still be denoted by π. There are two distances we can consider on Σ. The intrinsic one, defined as the infimum of the length of curves staying on Σ and the extrinsic one, where we take the distance in $\mathbb{H}^3$. We will denote by d_Σ and d these two distances.

First let us remark that there is no riemanniann metric on Σ which induces d. If such a metric existed, our Theorem 1.3 would be a particular case of [Knieper 1995].

Proposition 3.1. *If Σ is not totally geodesic, there is no riemannian metric on Σ which induces d.*

Proof. Assume there is such a riemannian metric, named g'. Let $\varepsilon > 0$ be such that the exponential map for g' is an embedding at every point. Let $c_{g'} : [0, \varepsilon] \to \Sigma$ be a minimizing geodesic for g' on Σ, then for all $t \in [0, \varepsilon]$,

$$d_{g'}(c_{g'}(0), c_{g'}(t)) + d_{g'}(c_{g'}(t), c_{g'}(\varepsilon)) = d_{g'}(c_{g'}(0), c_{g'}(\varepsilon)).$$

But since we suppose that g' induces d we have the same equality for d,

$$d(c_{g'}(0), c_{g'}(t)) + d(c_{g'}(t), c_{g'}(\varepsilon)) = d(c_{g'}(0), c_{g'}(\varepsilon)),$$

and this implies that $c_{g'}$ is a geodesic for $\mathbb{H}^3$. Hence every point of Σ is included in a totally geodesic disc, therefore Σ is totally geodesic. □

Consider the function a defined by

$$T^1\Sigma \times \mathbb{R} \to \mathbb{R}, \qquad (v, t) \mapsto d(\pi\phi_t^\Sigma(v), \pi(v)).$$

Letting $t_1, t_2 \in \mathbb{R}$ and $v \in T^1\Sigma$, we have by the triangle inequality,

$$\begin{aligned} a(v, t_1 + t_2) &= d(\pi\phi^\Sigma_{t_1+t_2}(v), \pi(v)) \\ &\le d(\pi\phi^\Sigma_{t_1+t_2}(v), \pi\phi^\Sigma_{t_1}(v)) + d(\pi\phi^\Sigma_{t_1}(v), \pi(v)) \\ &\le d(\pi\phi^\Sigma_{t_2}(\phi_{t_1}v), \pi\phi^\Sigma_{t_1}(v)) + d(\pi\phi^\Sigma_{t_1}(v), \pi(v)) \\ &\le a(\phi^\Sigma_{t_1}v, t_2) + a(v, t_1). \end{aligned}$$

Hence a is a subadditive cocycle for the geodesic flow ϕ_t^Σ. Since a is Γ-invariant it defines a subadditive cocycle on $T^1\Sigma$, still denoted by a.

The following is a consequence of Kingman's subadditive ergodic theorem [Kingman 1973].

Theorem 3.2. *Les μ be a ϕ_t^Σ invariant probability measure on $T^1\Sigma$. Then*

$$I_\mu(\Sigma, M, v) := \lim_{t\to\infty} \frac{a(v, t)}{t}$$

exists for μ-almost $v \in T^1\Sigma$ and defines a μ-integrable function on $T^1\Sigma$, invariant under the geodesic flow and we have

$$\int_{T^1\Sigma} I_\mu(\Sigma, M, v)\, d\mu = \lim_{t\to\infty} \int_{T^1\Sigma} \frac{a(v, t)}{t}\, d\mu.$$

Moreover if μ is ergodic, $I_\mu(\Sigma, M, v)$ is constant μ-almost everywhere. In this case, we write $I_\mu(\Sigma, M)$.

3B. ***Patterson–Sullivan measures.*** We call Λ the limit set of Γ acting on $\mathbb{H}^3$. Since Γ acts cocompactly on Σ, and on the convex core $C(\Lambda)$, the three geometric spaces Γ (seen as its Cayley graph), Σ and $C(\Lambda)$ are quasi-isometric. We assume from now on that (Σ, g) has negative curvature, hence there is a unique geodesic in each homotopy class of curves, and for every pair of points in Σ there is a unique geodesic which joins them. Let c_Σ be a geodesic on Σ, and denote by $c_\Sigma(\pm\infty)$ its limit points on Λ. There is a unique $\mathbb{H}^3$-geodesic $c_{\mathbb{H}^3}$ whose endpoints are $c_\Sigma(\pm\infty)$. Since Σ is quasi-isometric to $C(\Lambda)$, the two geodesics $c_{\mathbb{H}^3}$ and c_Σ are at bounded distance.

Let $p \in \Sigma$ and call pr_p^Σ the projection from Σ to Λ defined as follows. For any point $x \in \Sigma$ call $c_{p,x}^\Sigma$ the geodesic on Σ which joins p to x, then

$$pr_p^\Sigma(x) = c_{p,x}^\Sigma(+\infty).$$

We will denote the equivalent projection in $\mathbb{H}^3$ by $pr_p^{\mathbb{H}^3}$. There are two small distinctions to notice between $pr_p^{\mathbb{H}^3}$ and pr_p^Σ. First, $pr_p^{\mathbb{H}^3}$ is defined for every point in $\mathbb{H}^3$, whereas pr_p^Σ is only defined for points in Σ. Second is that the codomain of pr_p^Σ is exactly Λ whereas the codomain of $pr_p^{\mathbb{H}^3}$ is all S^2.

As we have just stated, for all $\xi \in \Lambda$ the geodesics, $c_{p,\xi}^\Sigma$ and $c_{p,\xi}^{\mathbb{H}^3}$ are at bounded distance, and this bound depends only on the quasi-isometry between Σ and $C(\Lambda)$. There exists C_1 such that for all $\xi \in \Lambda$ the Hausdorff distance between geodesics $c_{p,\xi}^\Sigma$ and $c_{p,\xi}^{\mathbb{H}^3}$ is less than C_1.

Let $x \in \Sigma,\ R > 0$ and consider the ball $B_{\mathbb{H}^3}(x, R)$ in $\mathbb{H}^3$ of center x and radius R. Now take $\xi \in pr_p^{\mathbb{H}^3}(B(x, R - C_1)) \cap \Lambda$; this means that the $\mathbb{H}^3$-geodesic from p to ξ crosses the ball $B_{\mathbb{H}^3}(x, R - C_1)$. This $\mathbb{H}^3$-geodesic is at bounded distance C_1 of the Σ-geodesic joining p to ξ. Hence,

$$c_{p,\xi}^\Sigma \cap (B_{\mathbb{H}^3}(x, R) \cap \Sigma) \neq \varnothing,$$

which proves that

$$\xi \in pr_p^\Sigma(B_{\mathbb{H}^3}(x, R) \cap \Sigma).$$

The same argument shows that

$$pr_p^\Sigma(B_{\mathbb{H}^3}(x, R) \cap \Sigma) \subset pr_p^{\mathbb{H}^3}(B_{\mathbb{H}^3}(x, R + C_1)) \cap \Lambda \subset pr_p^{\mathbb{H}^3}(B_{\mathbb{H}^3}(x, R + C_1)).$$

The distances on Σ and on $\mathbb{H}^3$ are locally equivalent: for every $R > 0$ there exists C_2 such that all balls satisfy

$$B_\Sigma(x, R - C_2) \subset B_{\mathbb{H}^3}(x, R) \cap \Sigma \subset B_\Sigma(x, R + C_2).$$

Set $C = \max(C_1, C_2)$, which leads to the following theorem:

Theorem 3.3.

$$\begin{array}{c} pr_p^\Sigma(B_\Sigma(x, R - C)) \\ \cap \\ pr_p^{\mathbb{H}^3}(B_{\mathbb{H}^3}(x, R - C)) \cap \Lambda \subset pr_p^\Sigma(B_{\mathbb{H}^3}(x, R) \cap \Sigma) \subset pr_p^{\mathbb{H}^3}(B_{\mathbb{H}^3}(x, R + C)) \\ \cap \\ pr_p^\Sigma(B_\Sigma(x, R + C)). \end{array}$$

Before proving Theorem 1.3, we will recall some basic facts about Patterson–Sullivan measure. Some classical references for this are [Patterson 1976] and [Sullivan 1979], the lecture of J-F. Quint [2006] and the monograph of T. Roblin [2003]. Let (X, g) be a simply connected manifold with negative curvature and $X(\infty)$ its geometric boundary. If Γ is a discrete group acting on (X, g) we can associate to it a family of measures $\{\mu_p^g\}_{p\in X}$ on $X(\infty)$ constructed as follows. Let x, y be two points of X and consider the Poincaré series

$$P(s) := \sum_{\gamma\in\Gamma} e^{-sd(\gamma x, y)}.$$

The convergence of $P(s)$ is independent of x and y by the triangle inequality. It converges for $s > \delta(\Gamma)$ and diverges for $s < \delta(\Gamma)$. If the action is cocompact, $\delta(\Gamma) = h(g)$ and the series diverges at $h(g)$. Then we define the probability measure

$$\mu^g_{p,x}(s) := \frac{\sum_{\gamma\in\Gamma} e^{-sd(\gamma x,p)}\delta_{\gamma x}}{\sum_{\gamma\in\Gamma} e^{-sd(\gamma p,p)}}.$$

By compactness of the set of probability measures on $X(\infty)$, we obtain a measure on $X(\infty)$ by taking a weak limit of a sequence $\mu^g_{p,x}(s_n)$[3],

$$\mu_p^g := \lim_{s_n\to h(g)} \mu_p^g(s_n).$$

It is supported on the accumulation points of G, that is to say the limit set.

These measures, called *Patterson–Sullivan measures*, have the following properties. They are quasiconformal, i.e., for all $p \in X$ and all $\xi, \eta \in \Lambda$, we have

$$\frac{d\mu_p^g}{d\mu_q^g}(\xi) = e^{-h(g)\beta_\xi(p,q)},$$

where $\beta_\xi(p, q) = \lim_{z\to\xi} d_g(p, z) - d_g(q, z)$.

They are also Γ-equivariant, i.e., for all $\gamma \in \Gamma$ and all $p \in X$, we have

$$\mu_p^g \circ \gamma = \mu^g_{\gamma^{-1}p}.$$

Moreover we know these measures behave locally like $h(g)-$Hausdorff measures. See [Quint 2006, Lemma 4.10], for example.

Lemma 3.4 (shadowing). *For $R > 0$ sufficiently large, there exists $c > 1$ such that for all $x \in X$,*

$$\frac{1}{c} e^{-h(g)d_g(x,p)} \le \mu_p^g(pr_p^g(B_g(x, R))) \le ce^{-h(g)d_g(x,p)}.$$

Suppose that X/Γ is compact; from the Patterson–Sullivan measure, we can construct an invariant measure on T^1X/Γ. Let $\Lambda^{(2)} := \{(x, y) \in \Lambda^2 \mid x \neq y\}$. There is a natural identification of $\Lambda^{(2)} \times \mathbb{R}$ and T^1X; a vector $v \in T^1X$ is identified with $(c_v(+\infty), c_v(-\infty), \beta_{c_v(+\infty)}(p, \pi v))$. The Bowen–Margulis measure is defined by

$$d\mu_{BM}(\xi, \eta, t) = e^{2h(g)\langle\xi|\eta\rangle_p} d\mu_p^g(\xi) d\mu_p^g(\eta)\, dt,$$

[3]It is a classical result of Sullivan that there is in fact a unique limit, up to normalization. It is equivalent to the ergodicity of Bowen–Margulis measure [Roblin 2003, Chapter 1]

where $\langle \xi \mid \eta \rangle_p$ is the Gromov product:

$$\langle \xi \mid \eta \rangle_p = \tfrac{1}{2}\big(\beta_\xi(z,p) + \beta_\eta(z,p)\big),$$

where z is any point on the geodesic (ξ, η).

Let us recall the classical fact that the measure μ_{BM} is Γ-invariant and define therefore a measure on T^1X/Γ. Letting $z \in (\xi, \eta)$,

$$\begin{aligned}
\langle \gamma\xi \mid \gamma\eta \rangle_p &= \tfrac{1}{2}\big(\beta_{\gamma\xi}(\gamma z, p) + \beta_{\gamma\eta}(\gamma z, p)\big) \\
&= \tfrac{1}{2}\big(\beta_{\gamma\xi}(\gamma z, \gamma p) + \beta_{\gamma\xi}(\gamma p, p) + \beta_{\gamma\eta}(\gamma z, \gamma p) + \beta_{\gamma\eta}(\gamma p, p)\big) \\
&= \tfrac{1}{2}\big(\beta_{\xi}(z, p) + \beta_{\eta}(z, p) + \beta_{\gamma\xi}(\gamma p, p) + \beta_{\gamma\eta}(\gamma p, p)\big) \\
&= \langle \xi \mid \eta \rangle_p + \tfrac{1}{2}\big(\beta_{\gamma\xi}(\gamma p, p) + \beta_{\gamma\eta}(\gamma p, p)\big).
\end{aligned}$$

By the quasiconformal behavior of μ_p^g, we have

$$\begin{aligned}
&e^{2h(g)\langle \gamma\xi \mid \gamma\eta \rangle_p} d\mu_p^g(\gamma\xi)\, d\mu_p^g(\gamma\eta) \\
&\qquad = e^{2h(g)\langle \xi \mid \eta \rangle_p} e^{h(g)\beta_{\gamma\xi}(\gamma p, p))} d\mu_p^g(\gamma\xi)\, e^{h(g)\beta_{\gamma\eta}(\gamma p, p))} d\mu_p^g(\gamma\eta) \\
&\qquad = e^{2h(g)\langle \xi \mid \eta \rangle_p} d\mu_p^g(\xi)\, d\mu_p^g(\eta).
\end{aligned}$$

The invariance by the geodesic flow is clear by definition and it is shown in [Nicholls 1989] that μ_{BM} is ergodic.

Finally we will need the following theorem, which is classical for compact manifolds endowed with two different negatively curved metrics. Since we treat a slightly different case, we give a proof.

Theorem 3.5. *If μ_p^Σ and $\mu_p^{\mathbb{H}^3}$ are equivalent, then the marked length spectrum of Σ is proportional to the marked length spectrum of M.*

Note that in the Fuchsian case, any surface equidistant to the totally geodesic one has a metric proportional to $\mathbb{H}^2$ and therefore satisfies the hypothesis of the theorem. It seems likely that it is the only case where the length spectrum is proportional to the one of the ambient manifold, however this is still uncertain.

Definition 3.6. For all $\xi, \eta \in \partial X^{(2)}$, we define the function D_X by

$$D_X(\xi, \eta) = \exp(-\langle \xi \mid \eta \rangle_p).$$

It is shown in [Ghys and de la Harpe 1990] that D_X^a for $a > 0$ small enough is a distance, called Gromov distance. However, we do not need such renormalization here.

The proof of Theorem 3.5 is in two steps. In the first, we prove that if the Patterson Sullivan measures are equivalent then the functions D_Σ and $D_{\mathbb{H}^3}$ are Hölder equivalent. In the second, we prove that this last condition implies the proportionality of the length spectrum.

Lemma 3.7. *If μ_p^{Σ} and $\mu_p^{\mathbb{H}^3}$ are equivalent, then the functions $D_{\mathbb{H}^3}$ and D_Σ are Hölder equivalent.*

Proof. Let us consider on $\Lambda^{(2)}$ the Bowen–Margulis currents defined by

$$\nu_\Sigma(\xi,\eta) = \frac{d\mu_\Sigma^p(\xi)d\mu_\Sigma^p(\eta)}{D_\Sigma(\xi,\eta)^{2\delta(\Sigma)}},$$

$$\nu_{\mathbb{H}^3}(\xi,\eta) = \frac{d\mu_{\mathbb{H}^3}^p(\xi)d\mu_{\mathbb{H}^3}^p(\eta)}{D_{\mathbb{H}^3}(\xi,\eta)^{2\delta(\mathbb{H}^3)}}.$$

These two measures are Γ-invariant by the previous computations for the Bowen–Margulis measures.

By assumption, μ_Σ^p and $\mu_p^{\mathbb{H}^3}$ are equivalent, therefore ν_Σ and $\nu_{\mathbb{H}^3}$ are also equivalent. The ergodicity and the Γ-invariance imply the existence of $c > 0$ such that

$$\nu_\Sigma = c\nu_{\mathbb{H}^3}.$$

Since μ_p^Σ and $\mu_p^{\mathbb{H}^3}$ are equivalent, there exists a function $f : \Lambda \to \mathbb{R}^+$ such that $\mu_p^\Sigma(\xi) = f(\xi)\mu_p^{\mathbb{H}^3}$. We have

$$f(\xi)f(\eta)D_{\mathbb{H}^3}^{\delta(\mathbb{H}^3)}(\xi,\eta) = cD_\Sigma^{\delta(\Sigma)}(\xi,\eta).$$

We see that f is equal almost everywhere to a continuous function. We can therefore suppose that f is continuous on Λ and hence strictly positive. By compacity, there exists $C > 1$ such that $\frac{1}{C} \le f(\xi) \le C$. Finally we get what we stated:

$$\frac{c}{C^2}D_\Sigma^{\delta(\Sigma)}(\xi,\eta) \le D_{\mathbb{H}^3}^{\delta(\mathbb{H}^3)}(\xi,\eta) \le C^2 c D_\Sigma^{\delta(\Sigma)}(\xi,\eta). \qquad \square$$

We now show the second part.

Lemma 3.8. *If D_Σ and $D_{\mathbb{H}^3}$ are Hölder equivalent the marked length spectra of Σ and $M = \mathbb{H}^3/\Gamma$ are proportional.*

Proof. In [Paulin et al. 2015, Section 3.5], the authors show that in a very general setting we have

$$\lim_{n\to\infty}\frac{1}{n}\log[g^-, g+, g^n(\xi), \xi] = \ell(g),$$

where $\ell(g)$ is the displacement of g and

$$[g^-, g^+, g^n(\xi), \xi] = \frac{D(g^-, g^n(\xi))D(g^+,\xi)}{D(g^-,\xi)D(g^+, g^n(\xi))}.$$

In particular, we can apply this result to Σ and $\mathbb{H}^3$ to get

$$\lim_{n\to\infty}\frac{1}{n}\log[g^-, g^+, g^n(\xi), \xi]_\Sigma = \ell_\Sigma(g)$$

and

$$\lim_{n\to\infty}\frac{1}{n}\log[g^-, g^+, g^n(\xi), \xi]_{\mathbb{H}^3} = \ell_{\mathbb{H}^3}(g).$$

By assumption on the distances D_Σ, $D_{\mathbb{H}^3}$, there exists $C > 1$ such that

$$\frac{1}{C}[g^-, g^+, g^n(\xi), \xi]^r_{\mathbb{H}^3} \leq [g^-, g^+, g^n(\xi), \xi]_\Sigma \leq C[g^-, g^+, g^n(\xi), \xi]^r_{\mathbb{H}^3}.$$

Hence,

$$\ell_\Sigma(g) = r\ell_{\mathbb{H}^3}(g). \qquad \square$$

Theorem 3.5 follows directly from Lemmas 3.7 and 3.8.

We will show at the very end of this article that if Σ has the same length spectrum as $M = \mathbb{H}^3/\Gamma$ then Γ is Fuchsian, to prove Corollary 1.4. It might be also true even when we only suppose that they are proportional, however this does not follow from our proof.

3C. *Entropy comparison.* We finally get to the proof of Theorem 1.3. First we prove the inequality using the behavior of Patterson–Sullivan measures and a volume comparison of a subset of Σ; the proof follows the same lines as [Knieper 1995, Theorem 3.4]. Then we prove the equality case using Theorem 3.5.

Theorem 3.9. *Let $\Sigma \subset \mathbb{H}^3$ be a Γ-invariant embedded disk, whose induced metric g has negative curvature, then*

$$h(g) \leq I_{\mu_{BM}}(\Sigma, M)\delta(\Gamma).$$

Moreover, the equality occurs if and only if the marked length spectrum of Σ is proportional to the marked length spectrum of M. In this case, the proportionality factor is given by $\ell_\Sigma(g)I(\Sigma, M) = \ell_M(g)$.

Proof. The geodesic flow is ergodic with respect to the Bowen–Margulis measure μ_{BM}, hence for μ_{BM}-almost all $v \in T^1\Sigma$ we have

$$\lim_{t\to\infty} \frac{a(v,t)}{t} = I_\mu(\Sigma, M).$$

Let v and v' be two unit vectors on the same weak stable manifold. Then

$$d(c_{v'}(t), c_{v'}(0)) \leq d(c_{v'}(t), (c_v(t)) + d(c_v(t), (c_v(0)) + d(c_v(0), (c_{v'}(0)),$$

and the same inequality holds interchanging the role of v and v'. Moreover $d(c_{v'}(t), (c_v(t))$ decreases exponentially since v and v' are on the same weak stable manifold. Hence $\lim_{t\to\infty} \frac{1}{t}a(v, t)$ exists if and only if $\lim_{t\to\infty} \frac{1}{t}a(v', t)$ does.

Let $v_p(\xi)$ denote the unitary vector in $T^1_p\Sigma$ such that $c_{v_p(\xi)}(\infty) = \xi$. The previous fact and the product structure of $d\mu_{BM}$ ensure that for μ^g_p-almost all $\xi \in \partial\Sigma$,

$$\lim_{t\to\infty} \frac{a(v_p(\xi), t)}{t} = I_\mu(\Sigma, M).$$

For all $\varepsilon > 0$ and $T > 0$, we define the set

$$A_p^{T,\varepsilon} = \left\{ \xi \in \partial\Sigma \;\middle|\; \left| \frac{a(v_p(\xi), t)}{t} - I_\mu(\Sigma, M) \right| \le \varepsilon,\ t \ge T \right\}.$$

For all $d \in]0, 1[$ and all $\varepsilon > 0$, there exists $T > 0$ such that $\mu_p^\Sigma(A_p^{T,\varepsilon}) \ge d$. For $t > T$, consider the subset $\{c_{p,\xi}(t) \mid \xi \in A_p^{T,\varepsilon}\} \subset S_g(p, t)$ of the geodesic sphere of radius t and center p on Σ.

Choose $\{B_\Sigma(x_i, R) \mid i \in I\}$ a covering of this subset of fixed radius $R > 0$ such that $x_i \in S_\Sigma(p, t)$ and $B_\Sigma(x_i, R/4)$ are pairwise disjoint. Then, by the local behavior of μ_p^Σ, there exists a constant $c > 1$, independent of t, such that

$$\frac{1}{c} e^{-h(g)t} \le \mu_p^\Sigma(pr_p^\Sigma(B_\Sigma(x_i, R))) \le c e^{-h(g)t}.$$

It is clear that $A_p^{T,\varepsilon} \subset \bigcup_{i\in I} pr_p^\Sigma(B_\Sigma(x_i, R))$ and therefore,

$$d \le \mu_p^\Sigma\Big(\bigcup_{i\in I} pr_p^\Sigma(B_\Sigma(x_i, R))\Big) \le \sum_{i\in I} \mu_p^\Sigma(pr_p^\Sigma(B_\Sigma(x_i, R))) \le c\,\mathrm{Card}(I) e^{-h(g)t}.$$

Since $\mathbb{H}^3/\Gamma$ is convex cocompact, $C_Q(\Lambda)/\Gamma$ is compact, where $C_Q(\Lambda)$ is the Q neighborhood of the convex core of Λ. Hence for any $Q > 0$,

$$\delta(\Gamma) = \lim_{R\to\infty} \mathrm{Vol}(B_{\mathbb{H}^3}(o, R) \cap C_Q(\Lambda)).$$

Now take Q sufficiently large such that Σ is inside $C_Q(\Lambda)$. There exists K such that $B_\Sigma(x_i, R/4) \subset B_{\mathbb{H}^3}(x_i, R + K) \cap C_Q(\Lambda)$.

From the definition of the set $A_p^{T,\varepsilon}$, we then have that the disjoint union

$$\bigcup_{i\in I} B_\Sigma(x_i, R/4) \subset B_{\mathbb{H}^3}(p, t(I_{\mu_{BM}}(\Sigma, \mathbb{H}^3) + \varepsilon) + R + K) \cap C_Q(\Lambda).$$

It follows that

$$\begin{aligned} e^{h(g)t} &\le \frac{c}{d}\,\mathrm{Card}(I) \le \frac{c}{dV} \sum_{i\in I} \mathrm{Vol}_{\mathbb{H}^3}(B_{\mathbb{H}^3}(x_i, R/4)) \cap C_Q(\Lambda)) \\ &\le \frac{c}{dV}\,\mathrm{Vol}_{\mathbb{H}^3}\big(B_{\mathbb{H}^3}(p, t(I_{\mu_{BM}}(\Sigma, \mathbb{H}^3) + \varepsilon) + R + K) \cap C_Q(\Lambda)\big). \end{aligned}$$

Hence,

$$h(g) \le \frac{1}{t}\Big(\log \frac{c}{dV} + \log \mathrm{Vol}_{\mathbb{H}^3}(B_{\mathbb{H}^3}(p, t(I_{\mu_{BM}}(\Sigma, \mathbb{H}^3) + \varepsilon) + R + K) \cap C_Q(\Lambda))\Big).$$

Taking the limit $t \to \infty$, we get

$$h(g) \le (I_{\mu_{BM}}(\Sigma, \mathbb{H}^3) + \varepsilon)\delta(\Gamma),$$

which concludes the proof since ε is arbitrary. $\square$

For the proof of the equality case in Theorem 1.3 we will use the result equivalent to [Knieper 1995, Corollary 3.6] in our context, that is:

Lemma 3.10 [Knieper 1995]. *Letting $p \in \Sigma$ and μ_p^g be the Patterson–Sullivan measure with respect to p and g, there exists a constant L such that for μ_p^g-almost all $\xi \in \partial\Sigma$ there is a sequence $t_n \to \infty$ such that*

$$|d(p, \pi\phi_{t_n}^{\Sigma} v_p(\xi)) - I_{\mu_{BM}}(\Sigma, \mathbb{H}^3)t_n| \leq L.$$

Proof. It follows from Lemma 3.5 of [Knieper 1995], that our lemma is true provided there exists a constant $C > 0$ such that, for all $t_1, t_2 > 0$ and all $v \in T^1\Sigma$,

$$a(v, t_1) + a(\phi_{t_1}^{\Sigma} v, t_2) \leq C + a(v, t_1 + t_2).$$

Let $v \in T^1\Sigma$ and c_v^{Σ} be the geodesic on Σ directed by v. Recall that there exists C_1 such that the $\mathbb{H}^3$-geodesic from $\pi(v)$ to $c_v^{\Sigma}(t_1 + t_2)$ is at bounded distance C_1 of $c_v^{\Sigma}(t_1 + t_2)$, independent of t_1 and t_2. The $\mathbb{H}^3$-geodesic from p to $c_v^{\Sigma}(t_1)$ and the one from $c_v^{\Sigma}(t_1)$ to $c_v^{\Sigma}(t_1 + t_2)$ are also at bounded distance C_1 of c_v^{Σ}. This implies the desired property with $C = 2C_1$. □

Proof of the equality case in 1.3. Suppose that $h(g) = I_{\mu_{BM}}(\Sigma, \mathbb{H}^3)\delta(\Gamma)$. Choose a point $p \in \Sigma$ and $\xi \in \Lambda$, set $y_n := \pi\phi_{t_n}^{\Sigma} v_p(\xi)$. From the above lemma, for μ_p^{Σ}-almost all ξ we have

$$|d(p, y_n) - I_{\mu_{BM}}(\Sigma, \mathbb{H}^3)t_n| \leq L.$$

Setting a fixed constant, $R > 0$, by the local property of the Patterson–Sullivan measure on $\mathbb{H}^3$, there is c_1 such that

$$\frac{1}{c_1} e^{-\delta(\Gamma)d(p,y_n)} \leq \mu_p^{\mathbb{H}^3}(pr_{\mathbb{H}^3} B_{\mathbb{H}^3}(y_n, R)) \leq c_1 e^{-\delta(\Gamma)d(p,y_n)},$$

and by Theorem 3.3,

$$pr_{\mathbb{H}^3}(B_{\mathbb{H}^3}(x, R - C)) \cap \Lambda \subset pr_{\Sigma}(B_{\mathbb{H}^3}(x, R) \cap \Sigma) \subset pr_{\mathbb{H}^3}(B_{\mathbb{H}^3}(x, R + C)).$$

Hence there is a constant c_2 such that

$$\frac{1}{c_2} e^{-\delta(\Gamma)d(p,y_n)} \leq \mu_p^{\mathbb{H}^3}(pr_{\Sigma} B_{\mathbb{H}^3}(y_n, R) \cap \Sigma) \leq c_1 e^{-\delta(\Gamma)d(p,y_n)}.$$

By the local property of the Patterson–Sullivan measure on Σ, there is c_3 such that

$$\frac{1}{c_3} e^{-h(\Sigma)d_{\Sigma}(p,y_n)} \leq \mu_p^{\Sigma}(pr_{\Sigma} B_{\Sigma}(y_n, R)) \leq c_3 e^{-h(\Sigma)d_{\Sigma}(p,y_n)},$$

and by Theorem 3.3,

$$pr_{\Sigma}(B_{\Sigma}(x, R - C)) \subset pr_{\Sigma}(B_{\mathbb{H}^3}(x, R) \cap \Sigma) \subset pr_{\Sigma}(B_{\Sigma}(x, R + C)).$$

Hence there is c_4 such that

$$\frac{1}{c_4} e^{-h(\Sigma)d_{\Sigma}(p,y_n)} \leq \mu_p^{\Sigma}(pr_{\Sigma} B_{\mathbb{H}^3}(y_n, R) \cap \Sigma) \leq c_4 e^{-h(\Sigma)d_{\Sigma}(p,y_n)}.$$

From the choice of y_n and since $h(\Sigma) = I_{\mu_{BM}}(\Sigma, \mathbb{H}^3)\delta(\Gamma)$,

$$e^{-L}e^{-\delta(\Gamma)d(p,y_n)} \leq e^{-h(g)d_\Sigma(p,y_n)} \leq e^L e^{-\delta(\Gamma)d(p,y_n)}.$$

Hence there is $c_5 > 0$ such that

$$\frac{1}{c_5}e^{-\delta(\Gamma)d(p,y_n)} \leq \mu_p^\Sigma(pr_\Sigma B_{\mathbb{H}^3}(y_n, R) \cap \Sigma) \leq c_5 e^{-\delta(\Gamma)d(p,y_n)}.$$

Finally we have a constant c_6 such that

$$c_6 \leq \frac{\mu_p^\Sigma(pr_\Sigma B_{\mathbb{H}^3}(y_n, R) \cap \Sigma)}{\mu_p^{\mathbb{H}^3}(pr_\Sigma B_{\mathbb{H}^3}(y_n, R) \cap \Sigma)} \leq c_6.$$

Since $pr_\Sigma(B_{\mathbb{H}^3}(y_n, R) \cap \Sigma) \to \xi$, the measures μ_p^Σ and $\mu_p^{\mathbb{H}^3}$ are equivalent. Theorem 3.5 concludes the proof. □

We finish this article with the proof of Corollary 1.4:

Corollary 1.4. *Under the assumptions of Theorem 1.3 we have*

$$h(\Sigma) \leq \delta(\Gamma, \mathbb{H}^3),$$

with equality if and only if Γ *is fuchsian and* Σ *is the totally geodesic hyperbolic plane, preserved by* Γ.

Proof. The inequality is obvious. Suppose the equality occurs. Then by Theorem 1.3, we have that the length spectrum is proportional to the one of $\mathbb{H}^3/\Gamma$ and moreover that $I(\Sigma, M) = 1$. In other words the two length spectra are equal.

Since Σ is embedded in $\mathbb{H}^3$, we can prove that the equality between the spectra implies that Σ is totally geodesic by the following argument:

Let $\gamma \in \Gamma$, and consider A its axis in Σ. Then for all $p \in A$, we have

$$\ell_\Sigma(\gamma) = d_\Sigma(\gamma p, p) \geq d_{\mathbb{H}^3}(\gamma p, p) \geq \ell_{\mathbb{H}^3}(\gamma).$$

Since the two spectra are equal, these inequalities are equalities. In particular, it implies that p lies in the axis of γ in $\mathbb{H}^3$. Therefore A is a geodesic of $\mathbb{H}^3$.

Let c be the closed geodesic on Σ represented by g and consider c' any geodesic that intersects c. Let g' be a representative of this closed geodesic such that the axis A' of g' on Σ intersects A. By similar computations as before, we see that A' is a geodesic of $\mathbb{H}^3$.

Since the two geodesics intersect, the endpoints of A and A' are cocyclic on the boundary of $\mathbb{H}^3$, and in particular bound a copy of $\mathbb{H}^2$ inside $\mathbb{H}^3$. By similar arguments for any element $g \in \Gamma$ such that its axis A_g intersects A and A' we see that A_g is a geodesic of $\mathbb{H}^3$ and therefore that A_g is included in the copy of $\mathbb{H}^2$. This last fact implies that Σ is included, therefore equal, to this copy of $\mathbb{H}^2$ and finishes the proof of the corollary. □

3C1. *A remark on length spectrum rigidity.* As we said in the introduction, the proof of the last corollary raises the following question: If a quasi-Fuchsian has the same length spectrum as a negatively curved surface, is it Fuchsian? Or more generally, if the two length spectra are proportional does it imply that it is Fuchsian? The latter question seems to be unanswered even if we suppose that the surface has constant negative curvature equal -1, and the problem in general seems to be quite hard.

We answer the case of constant negative curvature:

Theorem 1.5. *Let M be a quasi-Fuchsian manifold and Σ a hyperbolic* (*in the sense that it has constant curvature* -1) *surface. Suppose that M and Σ have proportional length spectrum* (*i.e., there exists $k \in \mathbb{R}^+$ such that for all $\gamma \in \Gamma$, $\ell_M(\gamma) = k\ell_\Sigma(\gamma)$*), *then M is Fuchsian, $k = 1$ and Σ is isometric to the totally geodesic surface in M.*

In this case we cannot use the entropy argument that is used when we suppose the equality of the two spectra. Our proof is inspired by the work of F. Dal'bo and I. Kim [2000] and based on the following theorem of Benoist:

Theorem 3.11 [Benoist 1997]. *Let G be a semisimple linear connected Lie group. Let $\Gamma < G$ be a Zariski dense subgroup. Then the limit cone is convex with nonempty interior.*

The limit cone is the smallest closed cone of a Cartan subspace of $\mathfrak{g}$ containing $\log(\lambda(\Gamma))$ where $\lambda(\gamma)$ is the Jordan projection.

Proof of Theorem 1.5. Consider Γ a surface group and ρ_{QF} a quasi-Fuchsian representation into $\mathrm{PSL}_2(\mathbb{C})$ and ρ_0 a Teichmüller representation in $\mathrm{PSL}_2(\mathbb{R})$. Consider the diagonal representation,

$$\rho = (\rho_{QF}, \rho_0) : \Gamma \to \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{R}).$$

The group $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{R})$ is a semisimple linear connected Lie group of rank 2. The Jordan projection of an element (γ_1, γ_2) is given by $(\ell_{\mathbb{H}^3}(\gamma_1), \ell_{\mathbb{H}^2}(\gamma_2))$ where ℓ_X is the translation length in X.

Therefore if the two representations have proportional length spectra, then the limit cone of $\rho(\Gamma)$ is a line, in particular it has empty interior. Using Benoist's theorem we conclude that $\rho(\Gamma)$ is not Zariski dense, which implies that M is Fuchsian. Therefore the length spectrum of Σ is k times the length spectrum of the hyperbolic surface $\Sigma_0 = \mathbb{H}^2/\rho(\Gamma)$. By Otal's theorem [1990] we get

$$(\Sigma, g) = (\Sigma_0, k^2 g_{\mathbb{H}}),$$

hence since Σ is hyperbolic, we have $k = 1$ and $\Sigma = \Sigma_0$. □

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References

[Benoist 1997] Y. Benoist, "Propriétés asymptotiques des groupes linéaires", *Geom. Funct. Anal.* **7**:1 (1997), 1–47. MR Zbl

[Besson et al. 1995] G. Besson, G. Courtois, and S. Gallot, "Entropies et rigidités des espaces localement symétriques de courbure strictement négative", *Geom. Funct. Anal.* **5**:5 (1995), 731–799. MR Zbl

[Bowen 1979] R. Bowen, "Hausdorff dimension of quasicircles", *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 11–25. MR Zbl

[Bridgeman and Taylor 2000] M. Bridgeman and E. C. Taylor, "Length distortion and the Hausdorff dimension of limit sets", *Amer. J. Math.* **122**:3 (2000), 465–482. MR Zbl

[Dal'Bo and Kim 2000] F. Dal'Bo and I. Kim, "A criterion of conjugacy for Zariski dense subgroups", *C. R. Acad. Sci. Paris Sér. I Math.* **330**:8 (2000), 647–650. MR Zbl

[Ghys and de la Harpe 1990] E. Ghys and P. de la Harpe, "Espaces métriques hyperboliques", pp. 27–45 in *Sur les groupes hyperboliques d'après Mikhael Gromov* ((Bern, 1988)), Progr. Math. **83**, Birkhäuser, Boston, 1990. MR

[Hamenstädt 2002] U. Hamenstädt, "Ergodic properties of function groups", *Geom. Dedicata* **93** (2002), 163–176. MR Zbl

[Kifer 1990] Y. Kifer, "Large deviations in dynamical systems and stochastic processes", *Trans. Amer. Math. Soc.* **321**:2 (1990), 505–524. MR Zbl

[Kingman 1973] J. F. C. Kingman, "Subadditive ergodic theory", *Ann. Probability* **1** (1973), 883–909. MR Zbl

[Knieper 1995] G. Knieper, "Volume growth, entropy and the geodesic stretch", *Math. Res. Lett.* **2**:1 (1995), 39–58. MR Zbl

[Nicholls 1989] P. J. Nicholls, *The ergodic theory of discrete groups*, London Mathematical Society Lecture Note Series **143**, Cambridge University Press, 1989. MR Zbl

[Otal 1990] J.-P. Otal, "Le spectre marqué des longueurs des surfaces à courbure négative", *Ann. of Math.* (2) **131**:1 (1990), 151–162. MR Zbl

[Patterson 1976] S. J. Patterson, "The limit set of a Fuchsian group", *Acta Math.* **136**:3-4 (1976), 241–273. MR Zbl

[Paulin et al. 2015] F. Paulin, M. Pollicott, and B. Schapira, *Equilibrium states in negative curvature*, Astérisque **373**, Société Mathématique de France, Paris, 2015. MR Zbl

[Quint 2006] J. F. Quint, "An overview of Patterson–Sullivan theory", workshop on the barycenter method, Forschungsinstitut für Mathematik , Zurich, 2006, available at https://tinyurl.com/quint-pdf.

[Roblin 2003] T. Roblin, *Ergodicité et équidistribution en courbure négative*, Mém. Soc. Math. Fr. **95**, Société Mathématique de France, Paris, 2003. MR Zbl

[Sullivan 1979] D. Sullivan, "The density at infinity of a discrete group of hyperbolic motions", *Inst. Hautes Études Sci. Publ. Math.* 50 (1979), 171–202. MR Zbl

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