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We provide several ingredients towards a generalization of the Littlewood–Richardson rule from Chow groups to algebraic cobordism. In particular, we prove a simple product formula for multiplying classes of smooth Schubert varieties with any Bott–Samelson class in algebraic cobordism of Grassmannians. We also establish some results for generalized Schubert polynomials for hyperbolic formal group laws.

1. Introduction

Throughout the article, we fix an algebraically closed base field k with $\mathrm{char}(\mathsf{k}) = 0$. Recall that for G a reductive group over k and P a parabolic subgroup of G, there exists a Borel-type presentation of the algebraic cobordism ring $\Omega^*(G/P)$ for the homogeneous space G/P; see [Hornbostel and Kiritchenko 2011; Hudson and Matsumura 2016]. For a smooth projective variety X over k, we refer to [Levine and Morel 2007; Levine and Pandharipande 2009] for the foundations on $\Omega^*(X)$.

In this article, we adopt an alternative, more geometric point of view. Namely, it is known that an additive basis of any of these cobordism rings may be described via geometric generators, using resolutions of Schubert varieties; see below. Schubert calculus consists in multiplying these basis elements. One of the new features when passing from Chow groups to cobordism is the need to resolve the singularities of Schubert varieties. There are therefore many possible bases since a given basis element depends on the choice of a resolution of a Schubert variety. In this paper we shall mostly consider Bott–Samelson resolutions. Let us mention that some formulas for the multiplication with divisor classes are already available, see [Calmès et al. 2013; Hornbostel and Kiritchenko 2011], and that in the recent preprints of Hudson and Matsumura [2016; 2017], Giambelli-type formulas are obtained for special

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classes and for a group G of type A. There are several other recent preprints on related questions; see, e.g., [Lenart and Zainoulline 2017].

We also focus on groups of type A. In the first part we consider the classes of smooth Schubert varieties in Grassmannians and prove a formula for multiplying the class of a smooth Schubert variety with the class of any Bott–Samelson resolution. Several years ago, Buch [2002] achieved a beautiful generalization of the classical Littlewood–Richardson rule for K-theory instead of Chow groups, building on previous work of Lascoux and Schutzenberger, Fomin and Kirillov and others. In the language of formal group laws (FGL), Buch has generalized the Littlewood–Richardson rule from the additive FGL to the multiplicative FGL. In the second part, we analyse the work of Fomin and Kirillov [1996a; 1996b] used by Buch, and generalize parts of it to other formal group laws. One might hope that ultimately this will be part of a Littlewood–Richardson rule for the universal case, that is, a complete Schubert calculus for algebraic cobordism of Grassmannians.

Recall [Levine and Morel 2007] that algebraic cobordism is the universal oriented algebraic cohomology theory on smooth varieties over k. Its coefficient ring is the Lazard ring $\mathbb{L}$; see [Lazard 1955]. For any homogeneous space $X = G/P$ with G reductive and P a parabolic subgroup of G, we have a cellular decomposition of X given by the B-orbits ($B \subset P$ a Borel subgroup of G) called Schubert cells and denoted by $(\mathring{X}_w)_{w \in W^P}$, where W^P is a subset of the Weyl group W. Choosing resolutions $\widetilde{X}_w \to X_w$ of the closures X_w of $\mathring{X}_w$ defines an additive basis of $\Omega^*(X)$; see [Hornbostel and Kiritchenko 2011, Theorem 2.5]. Schubert calculus aims at understanding the product in terms of these basis elements.

Write $X = \mathrm{Gr}(k, n)$ for the Grassmannian variety of k-dimensional linear subspaces in k^n. This is a homogeneous space of the form G/P with $G = \mathrm{GL}_n(\mathsf{k})$ and P a maximal parabolic subgroup of G. In the first part of the article, we prove some simple product formulas in $\Omega^*(X)$. For Grassmannians, there is another indexing set for Schubert cells and their closures in terms of partitions, and we shall use this notation in the Grassmannian case. In the following statement, λ is a partition associated to a Schubert variety X_λ, that is, the closure of the Schubert cell $\mathring{X}_\lambda$ (see Section 2A). Recall also that for the Grassmannian X, all Bott–Samelson resolutions of the Schubert variety X_λ are isomorphic over X. We denote by $\widetilde{X}_\lambda$ this unique Bott–Samelson resolution. Finally, recall that any smooth Schubert variety in X is of the form X_{b^a} with b^a the partition with a parts of size b.

Before stating the main result of Section 2, recall the definition of the dual partition (see Section 2A for more details): for a partition λ contained in the $k \times (n-k)$ rectangle R, we denote by $\lambda^\vee$ the dual partition obtained by taking the complement of λ in R. For a partition μ in the $a \times b$ rectangle, we write $\mu^{\vee z}$ for its dual partition in the $a \times b$ rectangle.

Theorem 1.1 (Corollary 2.15). *Let $\lambda \in \mathcal{P}(k,n)$. Then in $\Omega^*(X)$, we have*

$$[X_{b^a}]\cdot[\widetilde{X}_\lambda] = \begin{cases} [\widetilde{X}_{(\lambda^\vee)^\vee z}] & \text{for } \lambda \geq (b^a)^\vee, \\ 0 & \text{for } \lambda \not\geq (b^a)^\vee. \end{cases}$$

Note that for Chow groups or for K-theory, the above results are well known and follow from the Pieri formulas (see for example [Manivel 2001] for the Chow group case and [Buch 2002] for K-theory, by which we always mean K_0).

Note also that there are other natural resolutions of Schubert varieties considered in the literature, such as Zelevinskiĭ's resolutions [1983]. We believe that for those resolutions (which contain as a special case the resolutions considered in the cobordism Giambelli formulas of [Hudson and Matsumura 2016]) similar formulas should exist for the multiplication with the class of a smooth Schubert variety.

In the second part (Sections 3 and 4), inspired by Buch's method for giving a Littlewood–Richardson rule for K-theory, we have a closer look at generalized Schubert polynomials for cobordism. Let us recall first that for the full flag variety $X = G/B$ with $G = \mathrm{GL}_n(\mathsf{k})$ and B a Borel subgroup, there is a Borel-type presentation of the cobordism ring; see [Hornbostel and Kiritchenko 2011, Theorem 1.1]:

Theorem 1.2. *There exists an isomorphism $\Omega^*(X) \simeq \mathbb{L}[x_1,\dots,x_n]/S$, where $\deg(x_i) = 1$ for all $i \in [1,n]$ and S is the ideal generated by homogeneous symmetric polynomials of positive degree.*

In particular, given a Schubert variety X_w and a Bott–Samelson resolution $\widetilde{X}_{\underline{w}} \to X_w$ (here $\underline{w}$ is a reduced expression of the permutation w), we may write the class $[\widetilde{X}_{\underline{w}}] \in \Omega^*(X)$ as a polynomial $\mathfrak{L}_{\underline{w}}$ in the $(x_i)_{i\in[1,n]}$. Fomin and Kirillov [1996a; 1996b] gave a very nice description of such polynomials for the K-theory case, and [Buch 2002] builds on these results. In Section 3, we compare the generalized Schubert polynomials for cobordism with those for K-theory (called *Grothendieck polynomials*); see Corollary 3.15. For this, we have to restrict to hyperbolic formal group laws, that is, to elliptic cohomology. Choosing a suitable generalization of the Hecke algebra, we are also able to generalize the main theorem of [Fomin and Kirillov 1996a] from K-theory to elliptic cohomology; see Theorem 3.13.

In the last section, we combine techniques and results from Sections 2 and 3 to compute some explicit generalized Schubert polynomials. In particular, we show that some of the smooth Schubert varieties satisfy a certain symmetry; see Corollary 4.3. For generalized Schubert polynomials associated to other cells, this is no longer true already when looking at Gr(2, 4); see Proposition 4.5.

We have tried to present the first two parts in a way that they can be read essentially independently of each other. However, we emphasize that they both are partial solutions to the quest of a Schubert calculus for arbitrary orientable cohomology theories. Both parts reflect that for general formal group laws with operators not satisfying the naive braid relation, Schubert cells will lead to different elements in the corresponding generalized cohomology theory. On the geometric side, we have different resolutions of a given Schubert variety, and on the combinatorial side we have different reduced words for a given permutation. We hope that forthcoming work will combine these two aspects, leading to a better understanding of general Schubert calculus.

2. Product with smooth Schubert varieties

2A. ***Notation.*** Let $X = \mathrm{Gr}(k, n)$ be the Grassmannian of k-dimensional subspaces in $E = \mathsf{k}^n$. Denote by $(e_i)_{i\in[1,n]}$ the canonical basis of k^n. Denote by B the subgroup of upper-triangular matrices in $\mathrm{GL}_n(\mathsf{k})$, by B^- the subgroup of lower-triangular matrices and by $T = B \cap B^-$ the subgroup of diagonal matrices. For any subset $I \subset [1, n]$ write E_I for the span $\langle e_i \mid i \in I\rangle$. Set $E_i = E_{[1,i]}$ and $E^i = E_{[n+1-i,n]}$ for $i \in [1, n]$.

Call any nonincreasing sequence $\lambda = (\lambda_i)_{i\geq 1}$ of nonnegative integers a partition. The length of a partition is $\ell(\lambda) = \max\{i \mid \lambda_i \neq 0\}$. For λ of length k, we identify λ with its first k parts, i.e., with $(\lambda_i)_{i\in[1,k]}$. The weight of λ is $|\lambda| = \sum_i \lambda_i$. We will also use the pictorial description via Young diagrams, which are left-aligned arrays of $|\lambda|$ boxes with λ_i boxes on the i-th line for all $i \geq 1$. A partition λ fits in the $k \times (n-k)$ rectangle if its Young diagram does or equivalently if $\ell(\lambda) \leq k$ and $\lambda_1 \leq n-k$. Denote by $\mathcal{P}(k, n)$ the set of partitions fitting in the $k \times (n-k)$ rectangle. For $\lambda \in \mathcal{P}(k, n)$ denote by $\lambda^\vee \in \mathcal{P}(k, n)$ its dual partition defined by $\lambda^\vee_i = n-k-\lambda_{k+1-i}$ for $i \in [1, k]$. We have $|\lambda^\vee| = k(n-k) - |\lambda|$. Define $\lambda \leq \mu$ if $\lambda_i \leq \mu_i$ for all i.

Recall the Bruhat decomposition: the B-orbits $(\mathring{X}_\lambda)_{\lambda\in\mathcal{P}(k,n)}$ form a cellular decomposition of X. The same result holds for the B^--orbits $(\mathring{X}^\lambda)_{\lambda\in\mathcal{P}(k,n)}$. Indeed these orbits are isomorphic to affine spaces: $\mathring{X}_w \simeq \mathbb{A}^{|\lambda|}_{\mathsf{k}}$ and $\mathring{X}^\lambda \simeq \mathbb{A}^{\dim X-|\lambda|}_{\mathsf{k}}$. This can easily be deduced from their explicit descriptions:

$$\mathring{X}_\lambda = \bigl\{V_k \in X \bigm| \dim(V_k \cap E_{i+\lambda_{k+1-i}}) = i \text{ for all } i \in [1, k]\bigr\},$$
$$\mathring{X}^\lambda = \bigl\{V_k \in X \bigm| \dim(V_k \cap E^{i+n-k-\lambda_i}) = i \text{ for all } i \in [1, k]\bigr\}.$$

Note that with this definition we have $\mathring{X}^{\lambda^\vee} = w_X \cdot \mathring{X}_\lambda$, where w_X is the matrix permutation associated to the permutation $i \mapsto n+1-i$ of $[1, n]$. Denote by X_λ the closure of $\mathring{X}_\lambda$ and by X^λ the closure of $\mathring{X}^\lambda$. We have

$$X_\lambda = \bigl\{V_k \in X \bigm| \dim(V_k \cap E_{i+\lambda_{k+1-i}}) \geq i \text{ for all } i \in [1, k]\bigr\},$$
$$X^\lambda = \bigl\{V_k \in X \bigm| \dim(V_k \cap E^{i+n-k-\lambda_i}) \geq i \text{ for all } i \in [1, k]\bigr\}.$$

Inclusion induces the order on partitions: $X_\lambda \subset X_\mu \iff \lambda \leq \mu$.

Remark 2.1. The bases $([X_\lambda])_{\lambda\in\mathcal{P}(k,n)}$ and $([X^\lambda])_{\lambda\in\mathcal{P}(k,n)}$ are dual bases in $CH^*(X)$; see [Manivel 2001, Proposition 3.2.7]. Since $X^{\lambda^\vee} = w_X \cdot X_\lambda$ we see that $([X_\lambda])_{\lambda\in\mathcal{P}(k,n)}$ and $([X_{\lambda^\vee}])_{\lambda\in\mathcal{P}(k,n)}$ are also dual bases. Note that this is no longer true in K-theory.

2B. ***Smooth Schubert varieties, Bott–Samelson resolution and cobordism.*** The smooth Schubert varieties in X are sub-Grassmannians; see for example [Lakshmibai and Brown 2015, Theorem 6.4.2] or [Gasharov and Reiner 2002, Theorem 1.1], and [Brion and Polo 1999] or [Perrin 2009] for more details on the singular locus and the type of singularities. The partitions corresponding to these smooth Schubert varieties are of the form $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_i = b$ for $i \in [1, a]$ and $\lambda_i = 0$ for $i > a$ for some integers $a \in [1, k]$ and $b \in [1, n-k]$. Denote this partition by $\lambda = b^a$. As a variety we have

$$\begin{aligned} X_{b^a} &= \{V_k \in X \mid E_{k-a} \subset V_k \subset E_{k+b}\}, \\ X^{b^{a\vee}} &= \{V_k \in X \mid E^{k-a} \subset V_k \subset E^{k+b}\}. \end{aligned}$$

Moreover we have $X_{b^a} \simeq \mathrm{Gr}(a, a+b) \simeq X^{b^{a\vee}}$.

As already mentioned, Schubert varieties are in general singular. There exist several resolutions of singularities. We recall here the Bott–Samelson resolutions of Schubert varieties, which were first introduced by Bott and Samelson [1958], as well as by Hansen [1973] and Demazure [1974] for full flag varieties. These constructions and their properties carry over easily to partial flags $G/P = \mathrm{Gr}(k, n)$. See, e.g., [Fulton 1998; Lakshmibai and Brown 2015] for more details. We give here an explicit description of these resolutions in the spirit of configuration spaces; see [Magyar 1998] or [Perrin 2007]. Note also that for Schubert varieties in X, these resolutions are *canonical* in the sense that they do not depend on the choice of a reduced expression.

For a partition λ and a pair of integers (i, j) write $(i, j) \in \lambda$ if $i \in [1, k]$ and $j \in [1, \lambda_i]$ and $(i, j) \notin \lambda$ else. Define $V_{(i,j)} = E_{k+j-i}$ for all $(i, j) \notin \lambda$, where E_i is the zero space for $i \leq 0$ and $E_i = \mathsf{k}^n = E_n$ for $i \geq n$. Define $Y_\lambda = \prod_{(i,j)\in\lambda} \mathrm{Gr}(k+j-i, n)$. Set

$$\widetilde{X}_\lambda = \left\{(V_{(i,j)})_{(i,j)\in\lambda} \in Y_\lambda \;\middle|\; V_{(i+1,j)} \subset V_{(i,j)} \subset V_{(i,j+1)} \text{ for all } (i, j) \in \lambda\right\}.$$

The projection $\pi_\lambda : \widetilde{X}_\lambda \to X$ defined by $\pi_\lambda((V_{(i,j)})_{(i,j)\in\lambda}) = V_{1,1}$ induces a birational morphism onto X_λ. Furthermore, one easily checks that $\widetilde{X}_\lambda$ has the structure of a tower of $\mathbb{P}^1$-bundles so that $\widetilde{X}_\lambda$ is smooth. The morphisms $\pi_\lambda : \widetilde{X}_\lambda \to X_\lambda$ are called the Bott–Samelson resolutions of X_λ.

These resolutions define classes $[\pi_\lambda : \widetilde{X}_\lambda \to X]$ in the cobordism $\Omega^*(X)$ of X. We write $[\widetilde{X}_\lambda]$ for these classes. The classes $([\widetilde{X}_\lambda])_{\lambda\in\mathcal{P}(k,n)}$ form a basis in any oriented cohomology theory and especially in cobordism:

$$\Omega^*(X) = \bigoplus_{\lambda \in \mathcal{P}(k,n)} \mathbb{L}[\widetilde{X}_\lambda],$$

where $\mathbb{L}$ is the Lazard ring; see [Hornbostel and Kiritchenko 2011].

2C. *Products in cobordism.* We want to understand the products with the classes $[X_{b^a}]$ in $\Omega^*(X)$. Note that the class $[X_{b^a}]$ is well defined without considering any resolution since $X^{b^a} \simeq \mathrm{Gr}(a, a+b)$ is smooth; hence its cobordism class is well defined.

2C1. *Sub-Grassmannians.* Let $Z = \mathrm{Gr}(a, a+b)$ be the Grassmannian of a-dimensional vector subspaces of k^{a+b}. Let $(f_i)_{i\in[1,a+b]}$ be the canonical basis of k^{a+b}. Define $F_i = \langle f_j \mid j \in [1, i]\rangle$ and $F^i = \langle f_j \mid j \in [a+b+1-i, a+b]\rangle$. For $\lambda \in \mathcal{P}(a, a+b)$ a partition contained in the $a \times b$ rectangle define the Schubert variety in Z (as above in X):

$$Z_\lambda = \{V_a \in Z \mid \dim(V_a \cap F_{i+\lambda_{a+1-i}}) \geq i \text{ for all } i \in [1, a]\},$$
$$Z^\lambda = \{V_a \in Z \mid \dim(V_a \cap F^{i+b-\lambda_i}) \geq i \text{ for all } i \in [1, a]\}.$$

If $w_Z : \mathsf{k}^{a+b} \to \mathsf{k}^{a+b}$ is the endomorphism defined by $f_i \mapsto f_{a+b+1-i}$, then $Z^\lambda = w_Z \cdot Z_{\lambda^{\vee_Z}}$ with $\mu = \lambda^{\vee_Z}$ defined by $\mu_i = b - \lambda_{a+1-i}$ for all $i \in [1, a]$.

Now define Bott–Samelson resolutions in Z. Define $W_{(i,j)} = F_{a+j-i}$ for all $(i, j) \notin \lambda$, where F_i is the zero space for $i \leq 0$ and $F_i = \mathsf{k}^{a+b} = F_{a+b}$ for $i \geq a+b$. Define $A_\lambda = \prod_{(i,j)\in\lambda} \mathrm{Gr}(a+j-i, a+b)$. Set

$$\widetilde{Z}_\lambda = \big\{(W_{(i,j)})_{(i,j)\in\lambda} \in A_\lambda \;\big|\; W_{(i+1,j)} \subset W_{(i,j)} \subset W_{(i,j+1)} \text{ for all } (i, j) \in \lambda\big\}.$$

The projection $\pi^Z_\lambda : \widetilde{Z}_\lambda \to Z$ defined by $\pi^Z_\lambda((W_{(i,j)})_{(i,j)\in\lambda}) = W_{1,1}$ induces a birational morphism onto Z_λ.

Embed Z in X with image $X_{(b^a)}$ as follows. Let $u : \mathsf{k}^{a+b} \to \mathsf{k}^n$ be the linear map defined by $u(f_i) = e_{k-a+i}$ for all $i \in [1, a+b]$. Note that $u(\mathsf{k}^{a+b}) = E_{[k-a+1,k+b]}$. Denote by $v : Z \to X$ the closed embedding defined by $W_a \mapsto E_{k-a} \oplus u(W_a)$.

Embed Z in X with image $X^{(b^a)^\vee}$ as follows. Let $u' : \mathsf{k}^{a+b} \to \mathsf{k}^n$ be the linear map defined by $u'(f_i) = e_{n-k-b+i}$ for all $i \in [1, a+b]$. Note that $u'(\mathsf{k}^{a+b}) = E_{[n-k-b+1,n-k+a]}$. Denote by $v' : Z \to X$ the closed embedding defined by $W_a \mapsto E^{k-a} \oplus u'(W_a)$.

2C2. *Intersection with Schubert varieties.* In this subsection we consider the classes of closed subvarieties $Y \subset X$ in Chow groups or in K-theory. To avoid introducing more notation we denote both theses classes by $[Y]$ and specify in which theory we are working. The product with the class $[X_{b^a}]$ in Chow groups or for K-theory is easy to compute.

Lemma 2.2. *Let $\lambda \in \mathcal{P}(k,n)$. We have*

$$v(Z) \cap X^\lambda = X_{b^a} \cap X^\lambda = \begin{cases} \varnothing & \text{for } \lambda \not\leq b^a, \\ v(Z^\lambda) & \text{for } \lambda \leq b^a. \end{cases}$$

Proof. Let $\mu = b^a$. As is well known, the intersection $X_\mu \cap X^\lambda$ is nonempty if and only if $\lambda \leq \mu$. Assume this holds. We also know that $X_\mu \cap X^\lambda$ is a Richardson variety thus reduced, irreducible of dimension $|\mu| - |\lambda|$. Since Z^λ has dimension $|\mu| - |\lambda|$ it is enough to prove the inclusion $v(Z^\lambda) \subset X_{b^a} \cap X^\lambda$. By construction, we have $v(Z) = X_{b^a}$ thus $v(Z^\lambda) \subset X_{b^a}$. We prove the inclusion $v(Z^\lambda) \subset X^\lambda$. Recall the definition

$$X^\lambda = \{V_k \in X \mid \dim(V_k \cap E^{i+n-k-\lambda_i}) \geq i \text{ for all } i \in [1,k]\}.$$

Since λ is contained in the $a \times b$ rectangle, we have $\ell(\lambda) \leq a$; thus the conditions $\dim(V_k \cap E^{i+n-k-\lambda_i}) \geq i$ for $i > a$ become $\dim(V_k \cap E^{i+n-k}) \geq i$ and are trivially satisfied. We need to check the conditions $\dim(V_k \cap E^{i+n-k-\lambda_i}) \geq i$ for $i \in [1,a]$ and $V_k = v(W_a)$ with $W_a \in Z^\lambda$. For all $i \in [1,a]$, we have $\dim(V_a \cap F^{i+b-\lambda_i}) \geq i$. Applying v we get the inequality

$$\dim\big(v(V_a) \cap v(F^{i+b-\lambda_i}) \cap E_{[k-a+1,k+b]}\big) \geq i.$$

But

$$v(F^{i+b-\lambda_i} \cap E_{[k-a+1,k+b]}) = E_{[k+1-\lambda_i-i,k+b]} \subset E_{[k+1-\lambda_i-i,n]} = E^{i+n-k-\lambda_i}.$$

In particular $\dim(v(V_a) \cap E^{i+n-k-\lambda_i}) \geq i$ for $i \in [1,a]$ proving the result. □

Remark that $v(w_Z(F^i)) = E_{k-a} \oplus u(F_i) = E_i$; thus for $\lambda \in \mathcal{P}(a, a+b)$, we have $v(Z_\lambda) = X_\lambda$. In particular, we have $v(w_Z \cdot Z^\lambda) = v(Z_{\lambda^{\vee_Z}}) = X_{\lambda^{\vee_Z}}$. Consider k^{a+b} as a subspace of k^n via the embedding u and let w^Z be the endomorphism of k^n obtained by extending w_Z with the identity on the complement $\langle e_i \mid i \notin [k-a, k+b]\rangle$. We have $w^Z \circ v = v \circ w_Z$.

Corollary 2.3. *Let $\lambda \in \mathcal{P}(a, a+b)$. We have*

$$v(Z) \cap X^\lambda = X_{b^a} \cap X^\lambda = \begin{cases} \varnothing & \text{for } \lambda \not\leq b^a, \\ w^Z \cdot X_{\lambda^{\vee_Z}} & \text{for } \lambda \leq b^a. \end{cases}$$

Corollary 2.4. *Let $\lambda \in \mathcal{P}(a, a+b)$. We have*

$$X_\lambda \cap v'(Z) = X_\lambda \cap X^{b^{a\vee}} = \begin{cases} \varnothing & \text{for } \lambda \not\geq (b^a)^\vee, \\ w_X w^Z \cdot X_{(\lambda^\vee)^{\vee_Z}} & \text{for } \lambda \geq (b^a)^\vee. \end{cases}$$

Proof. Set $\mu = \lambda^\vee$, apply Corollary 2.3 to μ and multiply with w_X. □

Corollary 2.5. *Let $\lambda \in \mathcal{P}(a, a+b)$. In $CH^*(X)$, we have*

$$[X_\lambda] \cup [X_{b^a}] = \begin{cases} [X_{(\lambda^\vee)^{\vee_Z}}] & \text{for } \lambda \geq (b^a)^\vee, \\ 0 & \text{for } \lambda \not\geq (b^a)^\vee. \end{cases}$$

Remark 2.6. The same result holds for K-theory; see [Buch 2002].

Our aim is to generalize the above results to Bott–Samelson resolutions and to cobordism. For this, the dual point of view of Corollary 2.4 is better suited.

2C3. *Fiber product.* Let μ be a partition in the $a \times b$ rectangle and let $\mu' = (\mu^{\vee z})^{\vee}$. We construct an embedding of $\widetilde{Z}_\mu \to \widetilde{X}_{\mu'}$. We denote by $v' : \mathrm{Gr}(i, a+b) \to \mathrm{Gr}(i+k-a, n)$ the embeddings induced by u' as follows: $v'(W_i) = u'(W_i) \oplus E^{k-a}$.

First remark that $\mu \leq \mu'$ and that we get μ' from μ by adding $k-a$ lines (with $n-k$ boxes) and $n-k-b$ columns (with k boxes). In other words, $\mu'_i = n-k$ for $i \in [1, k-a]$ and $\mu'_i = \mu_i + n - k - b$ for $i \in [k-a+1, k]$.

Let $(W_{(i,j)})_{(i,j)\in\mu} \in \widetilde{Z}_\mu$. We define $(V_{(i,j)})_{(i,j)\in\mu'}$ as follows:

- For $i \in [1, k-a]$ and $j \in [1, n-k-b]$, set
$$V_{(i,j)} = (v'(W_{(1,1)}) \oplus E_{j-1}) \cap E_{n+1-i}.$$
- For $i \in [k-a+1, k]$ and $j \in [1, n-k-b]$, set
$$V_{(i,j)} = (v'(W_{(i-(k-a),1)}) \oplus E_{j-1}) \cap E_{n+a-k}.$$
- For $i \in [1, k-a]$ and $j \in [n-k-b+1, n-k]$, set
$$V_{(i,j)} = (v'(W_{(1,j-(n-k-b))}) \oplus E_{n-k-b}) \cap E_{n+1-i}.$$
- For $i \in [1, k-a]$ and $j \in [1, n-k-b]$, set
$$V_{(i,j)} = (v'(W_{(i-(k-a),j-(n-k-b))}) \oplus E_{n-k-b}) \cap E_{n+a-k}.$$
- For $(i, j) \notin \mu'$, set
$$V_{(i,j)} = (v'(W_{(i-(k-a),j-(n-k-b))}) \oplus E_{n-k-b}) \cap E_{n+a-k}.$$

Lemma 2.7. *We have* $(V_{(i,j)})_{(i,j)\in\mu'} \in \widetilde{X}_{\mu'}$.

Proof. Recall that $u'(\mathsf{k}^{a+b}) = E_{n-k-b,n-k+a}$, that $E^{k-a} \subset v'(W)$ and that $v'(W) \subset E^{k+b}$ for any subspace $W \subset \mathsf{k}^{a+b}$. In particular, in the above definition all sums are direct and all intersections are transverse. This implies $\dim V_{(i,j)} = k + j - i$; thus $(V_{(i,j)})_{(i,j)\in\lambda'} \in Y_{\mu'}$. For $(i, j) \notin \mu'$ we have
$$V_{(i,j)} = \big(v'(W_{(i-(k-a),j-(n-k-b))}) \oplus E_{n-k-b}\big) \cap E_{n+a-k} = E_{k+j-i}.$$
One easily proves that $V_{(i+1,j)} \subset V_{(i,j)} \subset V_{(i,j+1)}$. The result follows. □

Lemma 2.8. *The map* $\varphi : \widetilde{Z}_\mu \to \widetilde{X}_{\mu'}$ *is a closed embedding.*

Proof. We have $u'(W_{(i,j)}) = V_{(i+k-a,j+n-k-b)} \cap E^{k+b}$. Since u is injective, the result follows. □

Lemma 2.9. *The map* $\psi : \widetilde{Z}_\mu \to X$ *defined by* $(W_{(i,j)})_{(i,j)\in\mu} \mapsto V_{(1,1)}$ *factors through* $v'(Z)$.

Proof. We have $V_{(1,1)} = v'(W_{(1,1)}) = u'(W_{(1,1)}) \oplus E^{k-a}$. In particular $E^{k-a} \subset V_{(1,1)} \subset E^{k+b}$. The result follows. □

Proposition 2.10. *Let $\mu \in \mathcal{P}(a, a+b)$ and consider $\widetilde{Z}_\mu$ as an X-scheme via ψ. We have $\widetilde{X}_{\mu'} \times_X v'(Z) = \widetilde{X}_{\mu'} \times_X X^{(b^a)^\vee} \simeq \widetilde{Z}_\mu$.*

Proof. We have morphisms $\varphi : \widetilde{Z}_\mu \to \widetilde{X}_{\mu'}$ and $\psi : \widetilde{Z}_\mu \to v'(Z)$ with φ a closed embedding. Furthermore the map $\pi_{\mu'} : \widetilde{X}_{\mu'} \to X$ is given by $(V_{(i,j)})_{(i,j)\in\mu'} \mapsto V_{(1,1)}$ so the composition $\pi_{\mu'} \circ \varphi$ is the map ψ. In particular we have a morphism $\varphi \times \psi : \widetilde{Z}_\mu \to \widetilde{X}_{\mu'} \times_X v'(Z)$. This is a closed embedding since φ is a closed embedding. To prove that this is an isomorphism, it is enough to prove that $\widetilde{X}_{\mu'} \times_X v'(Z)$ is irreducible and smooth of dimension $|\mu| = \dim \widetilde{Z}_\mu$. But $v'(Z) = X^{(b^a)^\vee}$ and $\widetilde{X}_{\mu'}$ are in general position. By Kleimann–Bertini [Kleiman 1974] any irreducible component is of dimension $|\mu| - \operatorname{codim}_X v'(Z) = |\mu|$. By Bertini again, the fiber product of $v'(Z)$ with the locus in $\widetilde{X}_{\mu'}$ where $\pi_{\mu'}$ is not an isomorphism, has dimension strictly less than $|\mu|$ and is therefore never an irreducible component. Now since $v'(Z) \cap X_{\mu'}$ is irreducible, the same holds for $\widetilde{X}_{\mu'} \times_X v'(Z)$. Furthermore by Bertini again this fiber product is smooth and therefore reduced. □

Corollary 2.11. *Let $\lambda \in \mathcal{P}(k, n)$. As X-schemes, we have*

$$\widetilde{X}_\lambda \times_X v'(Z) = \widetilde{X}_\lambda \times_X X^{b^a} \simeq \begin{cases} \varnothing & \text{for } \lambda \not\geq (b^a)^\vee, \\ \widetilde{Z}_\mu & \text{for } \lambda \geq (b^a)^\vee, \end{cases}$$

with $\mu = (\lambda^\vee)^{\vee_Z}$ for $\lambda \geq (b^a)^\vee$ and $\widetilde{Z}_\mu$ is considered as an X-scheme via ψ.

2C4. *Cobordism.* We construct another X-scheme isomorphism between $\widetilde{Z}_\mu$ and $w_X w^Z \cdot \widetilde{X}_\mu$. Here $\widetilde{Z}_\mu$ is an X-scheme via ψ, while $w_X w^Z \cdot \widetilde{X}_\mu$ is an X-scheme via $w_X w^Z \circ \pi_\mu$. The actions of w_X and w^Z on $\widetilde{X}_\mu$ being defined via the embedding of $\widetilde{X}_\mu$ in Y_μ and the actions on the later are given by the diagonal action on each factor (recall that Y_μ is a product of Grassmannians $\mathrm{Gr}(i, n)$ on which w_X and w^Z act).

Let $(W_{(i,j)})_{(i,j)\in\mu} \in \widetilde{Z}_\mu$. We define $(V_{(i,j)})_{(i,j)\in\mu}$ as follows. For $(i, j) \in \mu$, set $V_{(i,j)} = v'(W_{(i,j)})$. For $(i, j) \notin \mu$, set $V_{(i,j)} = w_X w^Z \cdot E_{k+j-i}$.

Lemma 2.12. *We have $(V_{(i,j)})_{(i,j)\in\mu} \in w_X w^Z \cdot \widetilde{X}_\mu$.*

Proof. For (i, j), $(i+1, j)$ and $(i, j+1)$ in μ, the conditions $V_{(i+1,j)} \subset V_{(i,j)} \subset V_{(i,j+1)}$ are clearly satisfied. We only need to check these conditions for $(i+1, j)$ or $(i, j+1)$ not in μ. But for $(i, j) \notin \mu$, we have $W_{(i,j)} = F_{a+j-i}$; thus

$$v'(W_{(i,j)}) = v'(F_{a+j-i}) = E^{k-a} \oplus E_{[n-k-b+1, n-k-b+a+j-i]} = w_X w^Z \cdot E_{k+j-i} = V_{(i,j)}$$

and the result follows. □

Proposition 2.13. *Let $\mu \in \mathcal{P}(a, a+b)$. The X-schemes $\widetilde{Z}_\mu$ (via ψ) and $w_X w^Z \cdot \widetilde{X}_\mu$ are isomorphic.*

Proof. The above morphism sending $(W_{(i,j)})_{(i,j)\in\mu} \in \widetilde{Z}_\mu$ to $(V_{(i,j)})_{(i,j)\in\mu} \in \widetilde{X}_\mu$ is a closed embedding. Since both schemes are smooth are irreducible of the same dimension, this map is an isomorphism. We need to check that the morphisms to X coincide. But the composition $\widetilde{Z}_\mu \to w_X w^Z \cdot \widetilde{X}_\mu \to X$ is given by $(W_{(i,j)})_{(i,j)\in\mu} \mapsto (V_{(i,j)})_{(i,j)\in\mu} \mapsto V_{(1,1)}$ and therefore maps $(W_{(i,j)})_{(i,j)\in\mu} \in \widetilde{Z}_\mu$ to $v'(W_{(1,1)}) = \psi(W_{(1,1)})$. It coincides with ψ. □

Corollary 2.14. *Let $\lambda \in \mathcal{P}(k,n)$. As X-schemes, we have*

$$\widetilde{X}_\lambda \times_X v'(Z) = \widetilde{X}_\lambda \times_X X^{b^a} \simeq \begin{cases} \varnothing & \text{for } \lambda \not\geq (b^a)^\vee, \\ w_X w^Z \cdot \widetilde{X}_{(\lambda^\vee)^{\vee_Z}} & \text{for } \lambda \geq (b^a)^\vee. \end{cases}$$

Corollary 2.15. *Let $\lambda \in \mathcal{P}(k,n)$. Then in $\Omega^*(X)$, we have*

$$[X_{b^a}] \cdot [\widetilde{X}_\lambda] = \begin{cases} [\widetilde{X}_{(\lambda^\vee)^{\vee_Z}}] & \text{for } \lambda \geq (b^a)^\vee, \\ 0 & \text{for } \lambda \not\geq (b^a)^\vee. \end{cases}$$

Proof. The product $[X_{b^a}] \cdot [\widetilde{X}_\lambda]$ is given by pulling back the exterior product $X_{b^a} \times \widetilde{X}_\lambda \to X \times X$ along the diagonal map $\Delta : X \to X \times X$; see [Levine and Morel 2007, Remark 4.1.14]. We thus have $[X_{b^a}] \cdot [\widetilde{X}_\lambda] = \Delta^*[X_{b^a} \times \widetilde{X}_\lambda \to X \times X]$. Applying Corollary 6.5.5.1 of the same book, we get $\Delta^*[X_{b^a} \times \widetilde{X}_\lambda \to X \times X] = [X_{b^a} \times_X \widetilde{X}_\lambda]$ in $\Omega^*(X)$. □

Remark 2.16. (1) These results were inspired by several similar results for other cohomology theories. In particular, the results explained in Corollary 2.5 are the classical part of Seidel symmetries [1997] in quantum cohomology. The results of Seidel are not explicit but were made explicit in [Chaput et al. 2007; 2009]. These results extend to quantum K-theory. This will be presented in a forthcoming work [Buch et al. ≥ 2018]. We expect the same results to be valid in quantum cobordism once the latter is defined.

(2) We expect more general results of the same type for other homogeneous spaces. These will be studied by the second author in forthcoming work.

3. Generalized Schubert polynomials and generalized Hecke algebras

Recall that classical Grothendieck polynomials are representatives of Schubert classes in Borel's presentation of K-theory. In this section, we discuss the difference between classical Grothendieck polynomials and the representatives in Borel's presentation of algebraic cobordism of Bott–Samelson resolutions of Schubert varieties. For K-theory (that is K_0), the computation of polynomial representatives for classes of Schubert varieties has been done in [Fomin and Kirillov 1996a; 1996b]. We establish a generalization of the main theorem of [Fomin and Kirillov 1996a]. Building on their work, Buch [2002] computed Littlewood–Richardson rules for K-theory.

3A. *Divided difference operators.* Recall that K-theory corresponds to the multiplicative formal group law. The methods of Buch and Fomin and Kirillov do not generalize to the universal formal group law, that is, to algebraic cobordism. However, we will show that they apply in a much weaker form to *hyperbolic formal group laws* (see Definition 3.6 below) since we need to impose one more relation in the Hecke algebra (see Definition 3.11 below). For $i \in [1, n-1]$, let s_i be the transposition of $[1, n]$ exchanging i and $i+1$.

Definition 3.1. Let F be a formal group law over R with inverse χ:

(1) For $i \in [1, n-1]$, define $\sigma_i \in \mathrm{End}(R[[x_1, \dots, x_n]])$ by

$$(\sigma_i f)(x_1, \dots, x_n) = f(x_{s_i(1)}, \dots, x_{s_i(n)}).$$

(2) For $i \in [1, n-1]$, define $C_i, \Delta_i \in \mathrm{End}(R[[x_1, \dots, x_n]])$ by

$$C_i = (\mathrm{Id} + \sigma_i)\frac{1}{F(x_i, \chi(x_{i+1}))} \quad \text{and} \quad \Delta_i = \frac{1}{F(x_{i+1}, \chi(x_i))}(\mathrm{Id} - \sigma_i).$$

Remark 3.2. Note that the above operators are well defined in $R[[x_1, \dots, x_n]]$ since $F(x, \chi(y))$ can be written $(x-y)g(x, y)$ with $g(x, y)$ invertible in $R[[x, y]]$.

This definition is taken from [Hornbostel and Kiritchenko 2011, p. 71] and [Calmès et al. 2013, Section 3]. When applying it to the additive formal group law, one recovers the usual definition as, e.g., in [Manivel 2001, Section 2.3.1] up to a sign (observe that $\sigma_i \circ F(x_{i+1}, \chi(x_i)) = F(x_i, \chi(x_{i+1}))$). For the multiplicative formal group law $F(x, y) = x + y + \beta xy$, the definition of C_i yields the β-DDO $\pi_i^{(\beta)}$ of [Fomin and Kirillov 1996a], which for $\beta = -1$ specializes to the isobaric DDO of [Buch 2002]. Moreover, still for the multiplicative formal group law $F(x, y) = x + y + \beta xy$, the operator Δ_i above, which equals the one of [Calmès et al. 2013, Section 3], coincides up to sign with the operator $\pi_i^{(\beta)} + \beta$ which appears in [Fomin and Kirillov 1996a, Lemma 2.5].

Recall [Bressler and Evens 1990] that the braid relations for the operators C_i only hold if the FGL is additive or multiplicative. We therefore need to keep track of reduced expressions to define generalized Schubert polynomials, which is not necessary in [Fomin and Kirillov 1996a, Definition 2.1].

3B. *Generalized Schubert polynomials.* The following definition generalizes both Schubert polynomials for Chow groups and Grothendieck polynomials for K-theory.

Definition 3.3. Let w be a permutation and $\underline{w}$ a reduced expression of w as product in the $(s_i)_{i \in [1, n-1]}$. Define the generalized Schubert polynomial $\mathfrak{L}_{\underline{w}}$ by induction:

(a) $\mathfrak{L}_1(x_1, \dots, x_n) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$.

(b) $\mathfrak{L}_{\underline{w}s_i} := C_i \mathfrak{L}_{\underline{w}}$ if $\underline{w}s_i$ is a reduced word.

Note that this notation is different from the one used in [Fomin and Kirillov 1996a] and elsewhere: our $\mathfrak{L}_1$ corresponds to their $\mathfrak{L}_{w_0}$ and our $\mathfrak{L}_{\underline{w}}$ to their $\mathfrak{L}_{w_0 w}$. We decided to adopt this notation since there is a unique class for the point as well as a unique reduced expression for 1, but there is a Bott–Samelson resolution and a polynomial $\mathfrak{L}_{\underline{w}_0}$ for each reduced expression $\underline{w}_0$ of the element w_0.

For any permutation w, the Bott–Samelson resolutions $\widetilde{X}_{\underline{w}} \to X_w$ of the Schubert variety X_w are indexed by the reduced words $\underline{w}$ of w. It was proved in [Hornbostel and Kiritchenko 2011, Theorem 3.2] that the polynomial $\mathfrak{L}_{\underline{w}}$ represents the class of the resolution $\widetilde{X}_{\underline{w}} \to X_w$ in $\Omega^*(G/B)$.

Let S be the ideal in $R[[x_1, \dots, x_n]]$ generated by symmetric polynomials of positive degree. The polynomial $\mathfrak{L}_1$ corresponds to the cobordism class of a point. Modulo S, the polynomial $n!\,\mathfrak{L}_1$ has several equivalent descriptions; compare to, e.g., [Hornbostel and Kiritchenko 2011, Remark 2.7], where it differs by a scalar from D_n below.

Lemma 3.4. *Let $A^*(-)$ be an oriented cohomology theory and F its FGL.*

(a) *We have*

$$D_n := \prod_{1 \le i < j \le n} (x_i - x_j) \equiv n!\, x_1^{n-1} x_2^{n-2} \cdots x_{n-1} = n!\,\mathfrak{L}_1 \mod S.$$

(b) *Setting $a -_F b = F(a, \chi(b))$, we have*

$$D_n \equiv D_n^F := \prod_{1 \le i < j \le n} (x_i -_F x_j) \mod S.$$

Proof. To show (a), one first verifies that modulo S we have $\prod_{1<i\le n}(x_1 - x_i) \equiv n x_1^{n-1}$, deriving the equality $\prod_{1\le i\le n}(x - x_i) \equiv x^n$ and setting $x = x_1$. Then one shows $x_1^{n-1} p(x_2, \dots, x_n) \equiv 0$ for any symmetric nonconstant polynomial $p(x_2, \dots, x_n)$, writing $p(x_2, \dots, x_n) \equiv x_1 q(x_1, \dots, x_n)$ and using that $x_1^n \equiv 0$ modulo S. Now proceed by induction on n. The claim holds for $n = 1$. Using the factorization

$$\prod_{1 \le i < j \le n} (x_i - x_j) = \prod_{1 < i < j \le n} (x_i - x_j) \prod_{1 < i \le n} (x_1 - x_i),$$

the claim for n follows using the induction hypothesis for $n - 1$ and the above two equalities modulo S.

For (b), note that $x_i -_F x_j = 0$ if $x_j = x_i$, which implies that $x_i -_F x_j$ is divisible by $x_i - x_j$. Hence $x_i -_F x_j = (x_i - x_j) a(x_i, x_j)$ with $a(x_i, x_j) = 1 + b(x_i, x_j)$ and $b \in (x_i, x_j)$. Thus $D_n^F = D_n + D_n q(x_1, \dots, x_n)$ with $q(0, \dots, 0) = 0$. Now using part (a) and the equality $x_1^n \equiv 0 \mod S$, we deduce that $D_n x_i \equiv 0 \mod S$ for $i = 1$ and thus (use a suitable permutation) for all i. Hence $D_n q(x_1, \dots, x_n) \equiv 0 \mod S$ as claimed. □

Remark 3.5. Some authors use $x_n^{n-1}x_{n-1}^{n-2}\cdots x_2$ in place of $x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$. Modulo S these two classes only differ by the sign $(-1)^{n(n-1)/2}$.

3C. ***Hyperbolic formal group laws.*** We now define hyperbolic formal group laws, which generalize the additive and multiplicative ones.

Definition 3.6. The hyperbolic formal group law F over $R=\mathbb{Z}[\mu_1,\mu_2]$ and its inverse χ are given by

$$F(x,y)=\frac{x+y-\mu_1xy}{1+\mu_2xy}\quad\text{and}\quad \chi(x)=-\frac{x}{1-\mu_1x}.$$

Recall that formal group laws are by definition power series in two variables, and all fractions here and below may be written as such. Note that any ring homomorphism $\mathbb{Z}[\mu_1,\mu_2]\to A$ induces a formal group law over A. Calling these induced formal group laws hyperbolic as well, we find that additive and multiplicative formal group laws are special cases of hyperbolic formal group laws. See, e.g., [Buchstaber and Bunkova 2010; Hoffnung et al. 2014, Example 2.2(d); Lenart and Zainoulline 2017, 2.2] for more on hyperbolic formal group laws. Combining their computations, we see that

$$F(x,y)=x+y-\mu_1xy+\mu_2(x^2y+xy^2)+\mu_2\mu_1x^2y^2+O(5).$$

In Section 4B below, we explain how these FGLs lead to certain elliptic cohomology theories $E^*(-)$. If $\mu_2=0$, these cohomology theories specialize to Chow groups (if $\mu_1=0$), K_0 (if μ_1 is invertible, thus sometimes called periodic K-theory), connective K_0 and (if $\mu_1=0$ but $\mu_2\neq 0$) theories associated with Lorentz FGLs.

Definition 3.7. Let F be a formal group law. Define

$$\kappa_i=\kappa_i^F=\frac{1}{F(x_i,\chi(x_{i+1}))}+\frac{1}{F(x_{i+1},\chi(x_i))}.$$

Remark 3.8. In the above definition, κ_i is a formal series. Indeed, writing

$$F(x,\chi(y))=(x-y)g(x,y)$$

with g a formal series with constant term equal to 1, we get

$$\kappa_i=\frac{g(y,x)-g(x,y)}{(x-y)g(x,y)g(y,x)}.$$

Since the numerator vanishes for $x=y$ there exists a formal series h such that $g(y,x)-g(x,y)=(x-y)h(x,y)$ and we get

$$\kappa_i=\frac{h(x,y)}{g(x,y)g(y,x)},$$

which can be written as a formal series.

Remark 3.9. An easy computation shows that $\Delta_i = \kappa_i - C_i$.

Example 3.10. The three formal group laws we have studied so far are F_a, F_m and F_e, namely the additive, the multiplicative and the elliptic (or hyperbolic) formal group laws:

$$F_a(x, y) = x + y, \quad F_m(x, y) = x + y - \mu_1 xy \quad \text{and} \quad F_e(x, y) = \frac{x + y - \mu_1 xy}{1 + \mu_2 xy}.$$

In these cases, we have $\kappa_i^{F_a} = 0$, $\kappa_i^{F_m} = \kappa_i^{F_e} = \mu_1$. So in all these examples, $\kappa := \kappa_i$ is independent of i.

We now define a variant of the Hecke algebra generalizing [Fomin and Kirillov 1996a, Definition 2.2] with respect to a fixed hyperbolic formal group law F. Setting $\mu_2 = 0$, we obtain the Hecke algebra of [Fomin and Kirillov 1996a], corresponding to (connective or periodic) K-theory.

Definition 3.11. For the hyperbolic formal group law F defined over $R = \mathbb{Z}[\mu_1, \mu_2]$ consider the commutative ring $\mathcal{R} := R[[x_1, \dots, x_n]]$. The *generalized Hecke algebra* $\mathcal{A}_n(\kappa)$ is the quotient of the associative $\mathcal{R}$-algebra $\mathcal{R}\langle u_1, \dots, u_{n-1}\rangle$ by the relations

- $u_i x_j = x_j u_i$ for all i, j,
- $u_i u_j = u_j u_i$ for $|i - j| > 1$,
- $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ for all i,
- $u_i^2 = -\mu_1 u_i$ for all i,
- $\mu_2 x_i x_{i+1} u_i = 0$ for all i.

Although this algebra generalizes the ones of [Fomin and Kirillov 1996a; Buch 2002] and others, note that it is different from the formal *Demazure* algebras studied in [Calmès et al. 2013; Hoffnung et al. 2014]. See Remark 3.18 below for more details on this.

Remark 3.12. Note that the elements u_i satisfy the braid relations. Hence for any permutation w, we can define the element u_w as $u_w = u_{i_1} \cdots u_{i_r}$, where $w = s_{i_1} \cdots s_{s_{i_r}}$ is any reduced expression of w.

We now generalize [Fomin and Kirillov 1996a, Theorem 2.3] from multiplicative to hyperbolic formal group laws. Define

$$\mathfrak{S}(x_1, \dots, x_{n-1}) = \prod_{j=1}^{n-1} \prod_{i=n-1}^{j} (1 + x_j u_i),$$

where the interchanged bounds for i mean that the corresponding factors are multiplied in descending order, starting with $i = n - 1$.

Theorem 3.13. *For any hyperbolic FGL, in the generalized Hecke algebra $\mathcal{A}_n(\kappa)$ of Definition 3.11, we have*

$$\mathfrak{S}(x_1, \dots, x_{n-1}) = \sum_{w \in \Sigma_n} \mathfrak{L}_{\underline{w}} u_{w_0 w},$$

where $\underline{w}$ is any reduced expression of w and $w_0(i) = n + 1 - i$ as usual.

Before proving this theorem, we compare the generalized Schubert polynomials $\mathfrak{L}_{\underline{w}}$ with the corresponding Grothendieck polynomials for K-theory.

Definition 3.14. Let w be a permutation and $w = s_{i_1} \cdots s_{s_{i_r}}$ any reduced expression:

(1) The support of w is the set $\mathrm{Supp}(w) = \{i_1, \dots, i_r\}$. This is independent of the chosen reduced expression since it is preserved by the braid relations.

(2) Define $I(w)$ as the ideal in $\mathcal{R}$ generated by the polynomials $\mu_2 x_i x_{i+1}$ for $i \in \mathrm{Supp}(w_0 w)$.

(3) Let $\mathfrak{L}_w^K$ be the K-theoretic Grothendieck polynomial representing X_w.

Corollary 3.15. *Let $\underline{w} = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_r}}$ be a reduced expression of w. Then for w a permutation and $\underline{w}$ any reduced expression for w, in $\mathcal{R}$ we have*

$$\mathfrak{L}_{\underline{w}} = \mathfrak{L}_w^K \mod I(w).$$

Some parts of the proof of [Fomin and Kirillov 1996a, Theorem 2.3] are formal and immediately generalize to arbitrary formal group laws. Lemma 2.5 of the same paper just rephrases Remark 3.9. Several other crucial parts of the proof do not generalize to arbitrary FGLs. However, they do generalize to hyperbolic FGLs when working with the generalized Hecke algebra $\mathcal{A}_n(\kappa)$. An important point in choosing hyperbolic FGL is the fact that the κ_i are independent of i, so we have an action of the symmetric group on $\mathcal{A}_n(\kappa)$ given by permutations on the variables x_i. From now on, we fix a hyperbolic formal group law F and a positive integer n.

Lemma 3.16. *Set $\alpha_i(x) = (1 + x u_{n-1}) \cdots (1 + x u_i)$. Then we have the following equalities in $\mathcal{A}_n(\kappa)$:*

(1) $\alpha_{i+1}(x_{i+1}) = \alpha_i(x_{i+1})(1 + \chi(x_{i+1}) u_i)$.

(2) $1 + \chi(x_i) u_i = (1 + F(x_{i+1}, \chi(x_i)) u_i)(1 + \chi(x_{i+1}) u_i)$.

(3) $\Delta_i(1 + \chi(x_{i+1}) u_i) = -(1 + \chi(x_{i+1}) u_i) u_i$.

Proof. (1) The equality $\alpha_{i+1}(x_{i+1})(1 + x_{i+1} u_i) = \alpha_i(x_{i+1})$ implies

$$\alpha_{i+1}(x_{i+1})(1 + x_{i+1} u_i)(1 + \chi(x_{i+1}) u_i) = \alpha_i(x_{i+1})(1 + \chi(x_{i+1}) u_i).$$

A straightforward computation shows that $(1 + x_{i+1} u_i)(1 + \chi(x_{i+1}) u_i) = 1$.

(2) To prove the claim, it suffices to prove that

$$\big(F(x_{i+1},\chi(x_i))+\chi(x_{i+1})-\chi(x_i)\big)u_i+\chi(x_{i+1})F(x_{i+1},\chi(x_i))u_i^2=0,$$

or equivalently that

$$\big(F(x_{i+1},\chi(x_i))+\chi(x_{i+1})-\chi(x_i)-\mu_1\chi(x_{i+1})F(x_{i+1},\chi(x_i))\big)u_i=0.$$

This holds by a computation using the explicit formulas for F and χ and the relation $\mu_2 x_i x_{i+1}(x_i-x_{i+1})u_i=0$. We use the stronger relation $\mu_2 x_i x_{i+1}u_i=0$ in the definition of our Hecke algebra since we need x_i-x_{i+1} to be a nonzero divisor for the next computation.

(3) We have

$$\begin{aligned}-\Delta_i(1+\chi(x_{i+1})u_i)&=\frac{(1+\chi(x_i)u_i)-(1+\chi(x_{i+1})u_i)}{F(x_{i+1},\chi(x_i))}\\&=\frac{1+F(x_{i+1},\chi(x_i))u_i-1}{F(x_{i+1},\chi(x_i))}(1+\chi(x_{i+1})u_i)\\&=(1+\chi(x_{i+1})u_i)u_i.\end{aligned}$$

The second equality follows from part (2). □

Proposition 3.17. *In the above notation, for all i we have the commutation*

$$\alpha_i(x_i)\alpha_i(x_{i+1})=\alpha_i(x_{i+1})\alpha_i(x_i).$$

Proof. Since we have the same relations for the u_i as in [Fomin and Kirillov 1996a], the proof of their Lemma 2.6 generalizes to our situation. More precisely, we may apply [Fomin and Kirillov 1996b, Corollary 5.4] as its assumptions (see Section 2 of that paper) are satisfied in our generalized Hecke algebra. □

Proof of Theorem 3.13. From $\mathfrak{S}(x_1,\dots,x_{n-1})=\alpha_1(x_1)\cdots\alpha_{n-1}(x_{n-1})$ we get

$$\mathfrak{S}(x_1,\dots,x_{n-1})=\alpha_1(x_1)\cdots\alpha_i(x_{i+1})(1+\chi(x_{i+1})u_i)\alpha_{i+2}(x_{i+2})\cdots\alpha_{n-1}(x_{n-1}).$$

Using Lemma 3.16(1), this implies $\Delta_i(\mathfrak{S}(x_1,\dots,x_{n-1}))$ is equal to the following formulas:

$$\begin{aligned}&\alpha_1(x_1)\cdots\alpha_{i-1}(x_{i-1})\Delta_i\alpha_i(x_i)\alpha_i(x_{i+1})(1+\chi(x_{i+1})u_i)\alpha_{i+2}(x_{i+2})\cdots\alpha_{n-1}(x_{n-1})\\&=\alpha_1(x_1)\cdots\alpha_{i-1}(x_{i-1})\alpha_i(x_i)\alpha_i(x_{i+1})\Delta_i(1+\chi(x_{i+1})u_i)\alpha_{i+2}(x_{i+2})\cdots\alpha_{n-1}(x_{n-1})\\&=-\alpha_1(x_1)\cdots\alpha_i(x_i)\alpha_i(x_{i+1})(1+\chi(x_{i+1})u_i)u_i\alpha_{i+2}(x_{i+2})\cdots\alpha_{n-1}(x_{n-1})\\&=-\alpha_1(x_1)\cdots\alpha_i(x_i)\alpha_i(x_{i+1})(1+\chi(x_{i+1})u_i)\alpha_{i+2}(x_{i+2})\cdots\alpha_{n-1}(x_{n-1})u_i.\end{aligned}$$

Here the first equality follows from Proposition 3.17 and the fact that, as an operator, Δ_i commutes with the operator given by multiplication with a polynomial which is symmetric in x_i and x_{i+1}. The second equality follows from Lemma 3.16(3). We thus have shown

$$-\Delta_i(\mathfrak{S}(x_1,\dots,x_{n-1})) = (\mathfrak{S}(x_1,\dots,x_{n-1}))u_i,$$

which corresponds precisely to the induction step in Definition 3.3, using that $\Delta_i = \kappa - C_i$ and $u_i^2 = -\kappa u_i$. More precisely, write $\mathfrak{S} = \sum \widehat{\mathfrak{L}}_w u_{w_0 w}$, where the sum is taken over all $w \in \Sigma_n$. We wish to show that $\widehat{\mathfrak{L}}_w u_{w_0 w} = \mathfrak{L}_{\underline{w}} u_{w_0 w}$ by an ascending induction on the length of w. For $w = 1$ the claim is obviously true. Now fix $w \neq 1$ and choose i such that ws_i is reduced. Consider the coefficient of $u_{w_0 w}$ in

$$(C_i - \kappa_i)\mathfrak{S} = -\Delta_i \mathfrak{S} = \mathfrak{S} u_i.$$

Using that $u_i^2 = -\kappa_i u_i$ and the fact that $w_0 w s_i < w_0 w$, we deduce that

$$(C_i - \kappa_i)\widehat{\mathfrak{L}}_w u_{w_0 w} = (\widehat{\mathfrak{L}}_{ws_i} - \kappa_i \widehat{\mathfrak{L}}_w) u_{w_0 w};$$

hence $C_i \widehat{\mathfrak{L}}_w u_{w_0 w} = \widehat{\mathfrak{L}}_{ws_i} u_{w_0 w}$ as required. □

Remark 3.18. Note that the computations from [Fomin and Kirillov 1996a] cannot be done in the formal Demazure algebra of [Hoffnung et al. 2014]. E.g., the equality

$$(1 + x_{i+1}u_i)(1 + \chi(x_{i+1})u_i) = 1,$$

which was used to prove Lemma 3.16 above, does not hold, even for the additive FGL. This is related to the failure of $\kappa_i \Delta_i = \Delta_i \kappa_i$.

As for hyperbolic formal group laws, κ_i is independent of i (see Example 3.10); several other parts in [Buch 2002] on the Littlewood–Richardson rule for K_0 easily generalize to hyperbolic formal group laws when working with the generalized Hecke algebra $\mathcal{A}_n(\kappa)$ of Definition 3.11. For example, similar to [Buch 2002, p. 41], it is possible to introduce a stable generalized Schubert polynomial colim $\mathfrak{L}_{1^m \times \underline{w}}$ of $\mathfrak{L}_{\underline{w}}$ and to try to analyze its behavior along the lines of [Fomin and Kirillov 1996b, Section 6]. Also, there is a well-defined analog $\mathfrak{L}_{\nu/\lambda}$ of the polynomial $G_{\nu/\lambda}$ which is crucial for [Buch 2002, Theorem 3.1], as the construction on pages 41–42 of the same paper provides a reduced word $\underline{w}$ rather than just a permutation w. However, for hyperbolic formal group laws the operators C_i no longer satisfy the classical braid relation but a twisted version of it, namely $C_i C_{i+1} C_i + \mu_2 C_i = C_{i+1} C_i C_{i+1} + \mu_2 C_{i+1}$ [Hoffnung et al. 2014]. This will lead to additional difficulties when arguing inductively using these C_i and the corresponding geometric operators as, e.g., in [Buch 2002, Section 8]. This is also related to the discussion in [Lenart and Zainoulline 2017, Section 6]. On the other hand, Proposition 3.17 is wrong already for small values of n and i when replacing the classical braid relation for the u_i by its twisted analog in the definition of $\mathcal{A}_n(\kappa)$. We hope to return to these questions in future work.

4. Some examples

4A. ***Polynomials representing some smooth Schubert varieties.*** We first compute generalized Schubert polynomials for some of the smooth Schubert varieties considered in Section 2. Let $X = \mathrm{Gr}(k, n)$ be a Grassmannian and let λ be a partition of the form b^a. Denote by $\mathfrak{G}_\lambda$ the polynomial in $\Omega^*(G/B) \simeq \mathbb{L}[x_1, \dots, x_n]/S$ representing the pull-back along the canonical quotient map $\pi : G/B \to X$ of the cobordism class $[X_\lambda \to X]$. Recall [Heller and Malagón-López 2013, Section 3.2.4] that the induced map $\pi^* : \Omega^*(\mathrm{Gr}(k, n)) \to \Omega^*(G/B)$ is a ring monomorphism which identifies $\Omega^*(\mathrm{Gr}(k, n))$ with an explicit subring of $\mathbb{L}[x_1, \dots, x_n]/S$. The results in the sequel may thus be stated in either of these rings. (Recall that there is a standard map, see, e.g., [Buch 2002, p. 42], from partitions to permutations that corresponds to π^* and geometric operators for K-theory and Chow groups.)

Lemma 4.1. *In $\Omega^*(X)$, we have $[X_{(n-k)^{k-1}}]^a = [X_{(n-k)^{k-a}}]$ and $[X_{(n-k-1)^k}]^b = [X_{(n-k-b)^k}]$.*

Proof. We need to prove the formula $[X_{(n-k)^{k-1}}] \cdot [X_{(n-k)^{k-a}}] = [X_{(n-k)^{k-a-1}}]$. But the first class is represented by the sub-Grassmannian $X_{n-k} = \{V_k \in X \mid E_1 \subset V_k\}$, while the second class is represented by $X^{(n-k)^{k-a\vee}} = X^{(n-k)^a} = \{V_k \in X \mid E^a \subset V_k\}$. The product is represented by the intersection of these varieties and since E_1 and E^a do not meet we get

$$X_{n-k} \cap X^{(n-k)^a} = \{V_k \in X \mid E_1 \oplus E^a \subset V_k\}.$$

This last variety is a $\mathrm{GL}_n(\mathrm{k})$-translate of $X_{(n-k)^{k-a-1}} = \{V_k \in X \mid E_{a+1} \subset V_k\}$, proving the first formula. The second one is obtained along the same lines or deduced from the first one using the isomorphism $\mathrm{Gr}(k, n) \simeq \mathrm{Gr}(n-k, n)$. □

Proposition 4.2. *In $\Omega^*(G/B)$, we have the formulas*

$$\mathfrak{G}_{(n-k)^a} = (x_{k+1} \cdots x_n)^{k-a} \quad \textit{and} \quad \mathfrak{G}_{b^k} = (x_1 \cdots x_k)^{n-k-b}.$$

Proof. By the previous lemma, we only need to compute the class $[X_{(n-k)^{k-1}}]$ in $\Omega^*(X)$. Since $X_{(n-k)^{k-1}}$ is the zero locus of a section of the tautological quotient bundle whose Chern roots are $x_{k+1}, \dots, x_n$, the first equality of the proposition follows; see for example the proof of [Levine and Morel 2007, Lemma 6.6.7]. For the second formula, we just need to remark that $X_{(k-1)^k}$ is the zero locus of a global section of the dual of the tautological subbundle and apply the same method (or use the isomorphism $\mathrm{Gr}(k, n) \simeq \mathrm{Gr}(n-k, n)$ again). □

Corollary 4.3. *The classes of $[X_{(n-k)^a} \to X]$ and $[X_{b^k} \to X]$ are represented by the same polynomial in any oriented cohomology theory.*

Proof. Indeed we have $[X_{(n-k)^a} \to X] = \mathfrak{G}_{(n-k)^a}$ and $[X_{b^k} \to X] = \mathfrak{G}_{b^k}$, so this is independent of the FGL. □

Remark 4.4. We will see in the next subsection that this is no longer the case for the other classes of smooth Schubert varieties. Indeed, in Proposition 4.5, we prove that the class of the line in the elliptic cohomology of Gr(2, 4) is given by $x_1x_2(x_1+x_2)-\mu_1x_1^2x_2^2$ and therefore depends on the FGL.

4B. *Elliptic cohomology of* Gr(2, 4). We now present explicit results concerning elliptic cohomology, i.e., for the hyperbolic FGL, of Gr(2, 4). We compute the polynomial representatives for all Bott–Samelson classes as well as their products.

Let $X = \mathrm{Gr}(2,4)$ and let λ be a partition. Denote by $\mathfrak{L}_\lambda$ the polynomial in $\Omega^*(G/B)\simeq\mathbb{L}[x_1,x_2,x_3,x_4]/S$ representing the pull-back along the map $G/B\to X$ of the cobordism class $[\widetilde{X}_\lambda\to X]$, where $\widetilde{X}_\lambda$ is the Bott–Samelson resolution of X_λ.

Recall the hyperbolic FGL of [Buchstaber and Bunkova 2010, Example 63] as in Section 3C above. By the universal property of the formal group law of Ω^* established in [Levine and Morel 2007], we have a unique morphism of formal group laws, which yields in particular a ring morphism $\mathbb{L}\to\mathbb{Z}[\mu_1,\mu_2]$. This map is called the "Krichever genus" and is studied in detail in [loc. cit.]. In particular, μ_i has cohomological degree $-i$ for $i=1,2$. Note that (unlike in the bigraded case, see, e.g., [Levine et al. 2013]) this always yields an oriented cohomology theory, as there is no Landweber exactness condition to check. As the theory $E^*(-)$ is oriented in the sense of [Levine and Morel 2007], the analogs of the above theorems also hold for $E^*(G/B)$ and $E^*(\mathrm{Gr}(2,4))$, and the natural transformation $\Omega^*(-)\to E^*(-)$ commutes in particular with the ring monomorphisms π^*. Below, we use the notations $\widetilde{X}_\lambda$ and $\mathfrak{L}_\lambda$ for elements in $E^*(-)$ as well.

Proposition 4.5. *In $E^*(\mathrm{Gr}(2,4))$, we have the following formulas*:

$$\begin{aligned}
\mathfrak{L}_{(00)} &= x_1^2x_2^2,\\
\mathfrak{L}_{(10)} &= x_1x_2(x_1+x_2)-\mu_1x_1^2x_2^2,\\
\mathfrak{L}_{(20)} &= x_1^2+x_1x_2+x_2^2-\mu_1x_1x_2(x_1+x_2)-\mu_2x_1^2x_2^2,\\
\mathfrak{L}_{(11)} &= x_1x_2-\mu_2x_1^2x_2^2,\\
\mathfrak{L}_{(21)} &= x_1+x_2-\mu_1x_1x_2-\mu_2x_1x_2(x_1+x_2)-\mu_1\mu_2x_1^2x_2^2,\\
\mathfrak{L}_{(22)} &= 1-\mu_2(x_1+x_2)^2+\mu_1^2\mu_2x_1^2x_2^2.
\end{aligned}$$

Proof. Since the fiber of the map $\pi : G/B\to\mathrm{Gr}(2,4)$ is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$, the pull-back $\pi^*[\widetilde{X}_\lambda]\in E^*(G/B)$ of a Bott–Samelson class in Gr(2, 4) is again a Bott–Samelson class $X_{\underline{w}}$. (Note that this is not true anymore in higher dimensions.) Moreover in this case, we can explicitly write down the reduced word $\underline{w}$ corresponding to λ under π^*. Now we wish to compute $\mathfrak{L}_\lambda\in E^*(\mathrm{Gr}(2,4))\subset E^*(G/B)$. The above, together with the results of [Hornbostel and Kiritchenko

2011], implies that both in $\Omega^*(G/B)$ and $E^*(G/B)$, we have

$$\pi^*[X_{(00)}] = C_1C_3(\mathfrak{L}_1), \qquad \pi^*[X_{(11)}] = C_1C_3C_2C_1(\mathfrak{L}_1),$$
$$\pi^*[X_{(10)}] = C_1C_3C_2(\mathfrak{L}_1), \qquad \pi^*[X_{(21)}] = C_1C_3C_2C_1C_3(\mathfrak{L}_1),$$
$$\pi^*[X_{(20)}] = C_1C_3C_2C_3(\mathfrak{L}_1), \quad \pi^*[X_{(22)}] = C_1C_3C_2C_1C_3C_2(\mathfrak{L}_1).$$

Now the results follow from $\mathfrak{L}_1 = x_1^3x_2^2x_3$ and explicit computations with the C_i done with the help of a computer. □

We computed everything in elliptic cohomology for sake of simplicity, but a similar computation can be done in $\Omega^*(X)$.

Remark 4.6. In elliptic cohomology, the multiplication formula for the square of the hyperplane class in the Bott–Samelson basis is the same as the one in K-theory, namely $\mathfrak{L}_{(21)}^2 = \mathfrak{L}_{(20)} + \mathfrak{L}_{(11)} - \mu_1\mathfrak{L}_{(10)}$.

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