Pacific Journal of Mathematics

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Volume 294 No. 2 June 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 294, No. 2, 2018
dx.doi.org/10.2140/pjm.2018.294.423

SOBOLEV INEQUALITIES ON A WEIGHTED RIEMANNIAN MANIFOLD OF POSITIVE BAKRY–ÉMERY CURVATURE AND CONVEX BOUNDARY

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To the memory of our friend A. El Soufi

In this paper, we study some nonlinear elliptic equations on a compact n-dimensional weighted Riemannian manifold of positive m-Bakry–Émery–Ricci curvature and convex boundary. Our main purpose is to find conditions which imply that such elliptic equations admit only constant solutions. As an application, we obtain weighted Sobolev inequalities with explicit constants that extend the inequalities obtained by Ilias [1983; 1996] in the Riemannian setting. In a last part of the article, as applications we derive a new Onofri inequality, a logarithmic Sobolev inequality and estimates for the eigenvalues of a weighted Laplacian and for the trace of the weighted heat kernel.

1. Introduction and main result

Sobolev inequalities with sharp constants play an important role in Riemannian and conformal geometries. For example, on the unit sphere $\mathbb{S}^n$ endowed with its standard metric, we have (see [Aubin 1982]), for all $f \in H_1^2(\mathbb{S}^n)$,

$$\|f\|^2_{L^{2n/(n-2)}(dv)} \le K(n,2)\|\nabla f\|^2_{L^2(dv)} + \operatorname{vol}(\mathbb{S}^n)^{-2/n}\|f\|^2_{L^2(dv)}, \tag{1-1}$$

where $K(n,2) := 4/(n(n-2))\operatorname{vol}(\mathbb{S}^n)^{-2/n}$, dv and $\operatorname{vol}(\mathbb{S}^n)$ are respectively the Riemannian measure and the Riemannian volume of $\mathbb{S}^n$. This inequality has been crucial in the study of the Yamabe problem on closed Riemannian manifolds. It corresponds to the limiting case in the Sobolev embedding

$$H_1^2 \hookrightarrow L^p \quad \Big(2 < p \le \hat{2} := \frac{2n}{n-2}\Big)$$

MSC2010: 35J60, 53C21, 58J32, 58J50, 58J60.
Keywords: Sobolev inequality, Onofri inequality, weighted Riemannian manifold, convex boundary, manifold with density, weighted Laplacian, drifting Laplacian, Bakry–Émery–Ricci curvature, Neumann boundary condition, eigenvalues.

and the constants appearing in it are the best possible constants (see [Aubin 1982; Lee and Parker 1987; Ilias 1983]). Note that, using a stereographic projection and the conformal nature of this inequality, one can show its equivalence with the Euclidean Sobolev inequality (see for instance [Lee and Parker 1987]),

$$\forall f \in H_1^2(\mathbb{R}^n), \quad \|f\|^2_{L^{2n/(n-2)}(dx)} \le K(n,2)\|\nabla f\|^2_{L^2(dx)}, \tag{1-2}$$

where dx is the Lebesgue measure and $K(n,2)$ is the best constant in this Euclidean Sobolev inequality [Aubin 1982]. The conformal nature of inequality (1-1) can also be used to deduce the following Sobolev inequality on the hyperbolic space:

$$\|f\|^2_{L^{2n/(n-2)}(dx)} \le K(n,2)\|\nabla f\|^2_{L^2(dx)} - \mathrm{vol}(\mathbb{S}^n)^{-2/n}\|f\|^2_{L^2(dx)} \tag{1-3}$$

for all $f \in H_1^2(\mathbb{H}^n)$ (see [Hebey 1996] for another proof of this inequality).

Beckner [1993] extended the spherical inequality (1-1) to all the Sobolev exponents, proving that for all $p \in (2, \hat{2}]$,

$$\|f\|^2_{L^p(dv)} \le \mathrm{vol}(\mathbb{S}^n)^{-\frac{p-2}{p}}\left(\frac{p-2}{n}\|\nabla f\|^2_{L^2(dv)} + \|f\|^2_{L^2(dv)}\right). \tag{1-4}$$

This inequality is attributed in the literature to Beckner but it was proved in 1991 independently by Bidaut-Véron and Véron [1991].

In 1983, the first author generalized the spherical inequality (1-1) to any closed Riemannian manifold with positive Ricci curvature, see [Ilias 1983]. In fact, if (M, g) is a compact n-dimensional Riemannian manifold of Ricci curvature bounded-below by a positive constant k, then every function $f \in H_1^2(M)$ satisfies

$$\|f\|^2_{L^{2n/(n-2)}(dv_g)} \le \mathrm{vol}_g(M)^{-2/n}\left(\frac{4(n-1)}{n(n-2)k}\|\nabla f\|^2_{L^2(dv_g)} + \|f\|^2_{L^2(dv_g)}\right), \tag{1-5}$$

where dv_g and $\mathrm{vol}_g(M)$ are respectively the Riemannian measure and the Riemannian volume of (M, g).

This last inequality is derived from (1-1) using the Levy–Gromov isoperimetric inequality and an adapted symmetrization. Moreover, we observe that if we use the inequality (1-4) instead of (1-1), the same arguments of symmetrization extend (1-5) for all the Sobolev exponents $p \in (2, \hat{2}]$.

Bidaut-Véron and Véron [1991] were able to give another proof of the inequality (1-5). Their proof is based on the Bochner formula and a uniqueness result for some nonlinear elliptic equations strongly related to that Sobolev inequality. In fact, they improve a technique developed by Gidas and Spruck [1981]. The technique developed by Gidas and Spruck seems mysterious, but it is in fact inspired by that of Obata in his study of the unicity of an Einstein metric in a conformal class [Yano and Obata 1970]. And as we mentioned above, Bidaut-Véron and Véron [1991] obtained simultaneously the Beckner inequality (1-4). Note that, Bakry and Ledoux

[1996] obtained another important and different proof of the inequality (1-5), and this "probabilistic" proof, generalizes in fact, that inequality to the so-called Markov generators. Another recent generalization of the inequality (1-5) to metric measured spaces is due to Profeta [2015].

A natural question that can be asked is

"Is there a similar Sobolev inequality when the manifold has a boundary?"

For the hemisphere $\mathbb{S}^n_+$ endowed with its standard metric, using the exact value of its relative Yamabe infimum (see for instance [Escobar 1988; 1992]), we immediately get,

$$\|f\|^2_{L^{2n/(n-2)}(dv)} \leq \operatorname{vol}(\mathbb{S}^n_+)^{-2/n}\left(\frac{4}{n(n-2)}\|\nabla f\|^2_{L^2(dv)} + \|f\|^2_{L^2(dv)}\right) \tag{1-6}$$

for all $f \in H^2_1(\mathbb{S}^n_+)$.

In 1996, after a generalization of the method used by Bidaut-Véron and Véron, the first author [Ilias 1996] gave an answer to the above question by obtaining the same inequality as (1-5) for manifolds with convex boundary. More precisely, he proved that, for a compact Riemannian manifold (M, g) with convex boundary and of Ricci curvature bounded below by a constant $k > 0$, we have for any $p \in (2, \hat{2}]$ and any $f \in H^2_1(M)$,

$$\|f\|^2_{L^p(dv_g)} \leq \operatorname{vol}_g(M)^{-\frac{p-2}{p}}\left(\frac{(n-1)(p-2)}{nk}\|\nabla f\|^2_{L^2(dv_g)} + \|f\|^2_{L^2(dv_g)}\right). \tag{1-7}$$

The purpose of the present paper is to adapt the technique that has been used in [Ilias 1996] to the setting of weighted Riemannian manifolds with positive Bakry–Émery–Ricci curvature. More precisely, for a compact n-dimensional weighted Riemannian manifold (M^n, g, σ) of convex boundary and of m-Bakry–Émery–Ricci curvature (for some $m \in [n, \infty)$) bounded below by a positive constant k, we prove the analogue of (1-7) for any $p \in (2, 2^* := 2m/(m-2)]$, where the constant of the gradient term is $(m-1)(p-2)/(mk)$. In fact, we prove a stronger inequality where the constant of the gradient term depends on m, p, k, and the first nonzero Neumann eigenvalue λ^h_1 of the weighted Laplacian (see Theorem 3.6 for more details). Concerning the limiting case $p = 2^*$, our result shows that for any $f \in H^2_1(d\sigma)$,

$$\|f\|^2_{L^{2m/(m-2)}(d\sigma)} \leq \operatorname{vol}_h(M)^{-2/m}\left(\frac{4(m-1)}{m(m-2)k}\|\nabla f\|^2_{L^2(d\sigma)} + \|f\|^2_{L^2(d\sigma)}\right), \tag{1-8}$$

where $d\sigma$ is the weighted measure and $\operatorname{vol}_h(M)$ is the volume of M with respect to $d\sigma$ (see Corollary 3.7). These inequalities are a consequence of two uniqueness results for some nonlinear elliptic equations involving the weighted Laplacian (see Propositions 3.2 and 3.4) which are respectively generalizations to weighted manifolds with convex boundary of the result obtained by the first author [Ilias 1996] and that obtained by Licois and Véron [1995] (independently by Fontenas [1997]) for closed manifolds.

Using inequality (1-8) we can extend many Riemannian results to weighted Riemannian manifolds. Without being exhaustive, we will treat only applications which seems to us the most important. More precisely, we can obtain an upper bound with explicit constants (depending on the Sobolev constants) for the trace of the weighted heat kernel (see Section 4) and deduce therefrom a lower bound for the eigenvalues of the weighted Laplacian.

We also derive from our Sobolev inequalities, the analogue of the Onofri inequality. In fact, we prove in Corollary 4.1 that for any surface M of convex boundary and of Gaussian curvature bounded below by a positive constant k, it holds that

$$\log\left(\frac{1}{\mathrm{vol}_g(M)}\int_M e^{\varphi}\,dv_g\right) \le \frac{1}{\mathrm{vol}_g(M)}\left(\frac{1}{4k}\int_M |\nabla\varphi|^2\,dv_g + \int_M \varphi\,dv_g\right)$$

for all $\varphi \in H_1^2(M)$. In the case of the unit sphere and the unit hemisphere, we recover the classical Onofri inequalities (see [Onofri 1982; Chang and Yang 1988; Osgood et al. 1988]). As another consequence of our Sobolev inequalities, we obtain a logarithmic Sobolev inequality (Corollary 4.3) for weighted Riemannian manifolds with boundary.

This paper is organized as follows. In Section 2, we establish two elementary lemmas (Lemmas 2.1 and 2.2). The uniqueness results (Propositions 3.2 and 3.4) are discussed in Section 3 as well as Theorem 3.6 and Corollary 3.7. Finally, Section 4 is dedicated to some applications.

2. Preliminaries

Throughout the paper, we consider (M^n, g) as a smooth compact n-dimensional Riemannian manifold of boundary ∂M, endowed with a measure $d\sigma := \sigma\, dv_g$, where $\sigma = e^{-h}$ is a positive density (h is a smooth real-valued function on M) and dv_g is the Riemannian measure associated to the metric g. We denote by $\mathrm{vol}_g(M)$ and $\mathrm{vol}_h(M)$ respectively the volume of M with respect to dv_g and that with respect to $d\sigma$. Such a triplet (M, g, σ) is known in the literature as a weighted Riemannian manifold, a manifold with density, a Bakry–Émery manifold, or a Riemannian measure space. The associated weighted Laplacian Δ_h (also called drifted Laplacian, h-Laplacian or Bakry–Émery Laplacian) is given by

$$\Delta_h u = \Delta u - \frac{1}{\sigma}\langle\nabla\sigma, \nabla u\rangle = \Delta u + \langle\nabla h, \nabla u\rangle, \tag{2-1}$$

where Δ and ∇ are respectively the nonnegative Laplacian and the gradient with respect to g. It is self-adjoint on the space of square integrable functions on M

with respect to the weighted measure $d\sigma$, henceforth $L^2(d\sigma)$. We will denote by $H_1^2(d\sigma)$, the Sobolev space of $L^2(d\sigma)$ functions, such that the norm of their gradient is also in $L^2(d\sigma)$. Note that, since the manifold is compact and h is smooth, this Sobolev space coincides with the Sobolev space $H_1^2(M)$ of the Riemannian manifold (M, g), and these two spaces differ only in their norms.

The m-dimensional Bakry–Émery–Ricci curvature tensor (where $m \in [n, \infty)$) is a modified Ricci tensor more suitable to control the geometry of weighted manifolds and is defined by

$$\operatorname{Ric}_h^m := \operatorname{Ric} + D^2 h - \frac{1}{m-n}\, dh \otimes dh \tag{2-2}$$

where D^2 is the Hessian operator on M and Ric is the usual Ricci curvature of (M, g). The equation $\operatorname{Ric}_h^m = \kappa g$ correspond to the so-called *quasi-Einstein metric*, which has been studied by many authors (see for instance [Case et al. 2011]). When $m = \infty$, (2-2) gives the tensor $\operatorname{Ric}_h = \operatorname{Ric} + D^2 h$ introduced by Lichnerowicz [1970; 1971/72] and independently by Bakry and Émery [1985]. For $m = n$, (2-2) makes sense only when the function h is constant and so Ric_h^m is the usual Ricci tensor of M and Δ_h in this case is nothing but the Laplace–Beltrami operator Δ of M.

Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame of M such that at $p \in \partial M$, the vectors $e_1, \dots, e_{n-1}$ are tangent to the boundary and the remaining vector $e_n := \nu$ is the outward unit normal vector to ∂M. The second fundamental form of ∂M at $p \in \partial M$ is defined as

$$\mathrm{II}(X, Y) := \langle \mathcal{A}X, Y \rangle = \langle \nabla_X \nu, Y \rangle$$

for any $X, Y \in T_p(\partial M)$, where $\mathcal{A}$ is the Weingarten endomorphism of $T_p M$. The mean curvature H of ∂M is defined as the trace of the second fundamental form II:

$$H = \sum_{i=1}^{n-1} \mathrm{II}(e_i, e_i).$$

In the sequel, we will need the following two lemmas. The first one is nothing but a little modification of the Bochner–Lichnerowicz–Weitzenböck formula for functions on weighted Riemannian manifolds which generalizes the Reilly identity ([Reilly 1977; Ma and Du 2010]). The version we present here is better suited to our purpose and its proof is a straightforward adaptation of that given by the first author [Ilias 1996] to the weighted setting.

Lemma 2.1 (generalized Reilly formula). *Let (M^n, g, σ) be a compact weighted Riemannian manifold with boundary ∂M. For any two smooth functions u and v on*

M, we have

$$\int_M v\big(|D^2u|^2-(\Delta_h u)^2\big)\,d\sigma = -\int_M \big\{(\Delta_h u)\langle\nabla u,\nabla v\rangle+\tfrac12|\nabla u|^2\Delta_h v+\mathrm{Ric}_h(\nabla u,\nabla u)v\big\}\,d\sigma + \int_{\partial M}\bigg\{-\tfrac12|\nabla u|^2\frac{\partial v}{\partial\nu}+\Big\langle\nabla^\partial\Big(\frac{\partial u}{\partial\nu}\Big),\nabla^\partial u\Big\rangle v+(\Delta^\partial u)\Big(\frac{\partial u}{\partial\nu}\Big)v - \mathrm{II}(\nabla^\partial u,\nabla^\partial u)v - Hv\Big(\frac{\partial u}{\partial\nu}\Big)^2+\langle\nabla u,\nabla h\rangle v\Big(\frac{\partial u}{\partial\nu}\Big)\bigg\}\,d\sigma,$$

where ∇^∂ and Δ^∂ denote the gradient and the Laplacian of ∂M and for the sake of simplicity, we still denote by $d\sigma$ the induced weighted measure on ∂M.

Proof. From the classical Riemannian Bochner formula applied to u, one can easily deduce the following weighted one (see for instance [Setti 1998; Bakry and Émery 1985]):

$$\langle\nabla(\Delta_h u),\nabla u\rangle=|D^2u|^2+\tfrac12\Delta_h(|\nabla u|^2)+\mathrm{Ric}_h(\nabla u,\nabla u). \tag{2-3}$$

Multiplying (2-3) by v and integrating over M with respect to $d\sigma$, we get:

$$\int_M v\langle\nabla(\Delta_h u),\nabla u\rangle\,d\sigma = \int_M v|D^2u|^2\,d\sigma+\tfrac12\int_M v\Delta_h(|\nabla u|^2)\,d\sigma+\int_M v\,\mathrm{Ric}_h(\nabla u,\nabla u)\,d\sigma.$$

Integration by parts in the left hand side and in the second term of the right hand side gives

$$\int_M v\big(|D^2u|^2-(\Delta_h u)^2\big)\,d\sigma = -\int_M \big\{(\Delta_h u)\langle\nabla u,\nabla v\rangle+\tfrac12|\nabla u|^2\Delta_h v+\mathrm{Ric}_h(\nabla u,\nabla u)v\big\}\,d\sigma + \int_{\partial M}\bigg\{-\tfrac12|\nabla u|^2\frac{\partial v}{\partial\nu}+(\Delta_h u)\frac{\partial u}{\partial\nu}v+\tfrac12\frac{\partial(|\nabla u|^2)}{\partial\nu}v\bigg\}\,d\sigma. \tag{2-4}$$

Now in the calculations that follow (at a point $x\in\partial M$), we will use an orthonormal local frame $\{e_1,\dots,e_n\}$ such that $e_1,\dots,e_{n-1}$ are tangent to the boundary and $e_n=\nu$ is the outward unit normal to ∂M. A direct calculation of the last two terms in (2-4) at a point $x\in\partial M$ yields

$$(\Delta_h u)\Big(\frac{\partial u}{\partial\nu}\Big)v+\tfrac12\frac{\partial(|\nabla u|^2)}{\partial\nu}v = \sum_{i=1}^{n-1}\big(D^2u(e_n,e_i)e_i(u)-D^2u(e_i,e_i)e_n(u)\big)v+\langle\nabla h,\nabla u\rangle\Big(\frac{\partial u}{\partial\nu}\Big)v \tag{2-5}$$

and

$$\begin{aligned}\sum_{i=1}^{n-1} D^2u(e_n, e_i)e_i(u) &= \Big\langle \nabla^\partial\Big(\frac{\partial u}{\partial \nu}\Big), \nabla^\partial u\Big\rangle - \mathrm{II}(\nabla^\partial u, \nabla^\partial u),\\ \sum_{i=1}^{n-1} D^2u(e_i, e_i)e_n(u) &= -(\Delta^\partial u)\Big(\frac{\partial u}{\partial \nu}\Big) + H\Big(\frac{\partial u}{\partial \nu}\Big)^2.\end{aligned} \tag{2-6}$$

After incorporating the two identities of (2-6) in (2-5), and the obtained result in (2-4), we conclude the proof of Lemma 2.1. □

The second lemma, which has an elementary proof, arises naturally when we need to estimate the Hessian D^2u in terms of $\Delta_h u$ (see for example [Li 2005] for a proof):

Lemma 2.2. *Let u be a smooth function on M. For every $m \geq n$, we have*

$$|D^2u|^2 + \mathrm{Ric}_h(\nabla u, \nabla u) \geq \frac{1}{m}(\Delta_h u)^2 + \mathrm{Ric}_h^m(\nabla u, \nabla u). \tag{2-7}$$

Moreover, the equality in (2-7) *holds if and only if*

$$D^2u = -\frac{1}{n}(\Delta u)g \quad \textit{and} \quad \Delta_h u = \frac{m}{m-n}\langle \nabla h, \nabla u\rangle.$$

3. Weighted Sobolev inequalities

Let (M, g, σ) be a compact weighted Riemannian manifold of dimension $n \geq 2$. In this section, we seek conditions that guarantee the uniqueness of the positive solution of a nonlinear elliptic PDE (see (3-17)). This is an important step towards the Sobolev inequality.

We first start by giving the keystone of both uniqueness results (Propositions 3.2 and 3.4):

Proposition 3.1. *Let $q > 1$, $\lambda > 0$ and φ be a positive solution of the following system*

$$\begin{cases} \Delta_h \varphi + \lambda \varphi = \varphi^q & \textit{in } M,\\ \dfrac{\partial \varphi}{\partial \nu} = 0 & \textit{on } \partial M.\end{cases} \tag{3-1}$$

Put

$$J := |D^2u|^2 - \frac{1}{m}(\Delta_h u)^2 + \mathrm{Ric}_h(\nabla u, \nabla u). \tag{3-2}$$

(i) *For any two nonzero real numbers α and β, we have*

$$\begin{aligned}&\int_M u^\beta J\, d\sigma + \int_{\partial M} u^\beta \mathrm{II}(\nabla^\partial u, \nabla^\partial u)\, d\sigma\\ &\quad = A_1 \int_M u^{\beta+(q-1)/\alpha}|\nabla u|^2\, d\sigma + B_1 \int_M u^\beta |\nabla u|^2\, d\sigma + C_1 \int_M u^{\beta-2}|\nabla u|^4\, d\sigma,\end{aligned} \tag{3-3}$$

where

$$A_1 = \left\{ \frac{m-1}{m}(\alpha\beta + q) - \tfrac{3}{2}\alpha\beta \right\},$$
$$B_1 = \left\{ -\lambda \frac{m-1}{m}(\alpha\beta + 1) + \tfrac{3}{2}\alpha\beta\lambda \right\},$$
$$C_1 = \left\{ \frac{m-1}{m}\Big(\frac{\alpha-1}{\alpha}\Big)^2 + \tfrac{3}{2}\beta\Big(\frac{\alpha-1}{\alpha}\Big) + \frac{\beta(\beta-1)}{2} \right\}.$$

(ii) *For any two nonzero real numbers* α *and* $\beta \neq -2$, *we have the following identity*:

$$\begin{aligned}(3\text{-}4)\quad & \int_M u^\beta J \, d\sigma + \int_{\partial M} u^\beta \mathrm{II}(\nabla^\partial u, \nabla^\partial u)\, d\sigma \\ &= A_2 \int_M \big(\Delta_h u^{(\beta+2)/2}\big)^2 \, d\sigma + B_2 \int_M u^\beta |\nabla u|^2 \, d\sigma + C_2 \int_M u^{\beta-2} |\nabla u|^4 \, d\sigma,\end{aligned}$$

where

$$A_2 = -\frac{2}{m(\beta+2)^2}\left\{ (m+2)\frac{\alpha\beta}{q} - 2(m-1) \right\},$$
$$B_2 = \frac{\alpha\beta}{q}\Big(\frac{m+2}{2m}\Big)\lambda(q-1),$$
$$C_2 = \frac{\alpha\beta}{q}\left[\frac{m+2}{2m}\left\{ \Big(\frac{\beta}{4} + \frac{\alpha-1}{\alpha}\Big)\Big(\beta + \frac{q}{\alpha}\Big) + \Big(\frac{\alpha-1}{\alpha}\Big)^2 \right\} + \frac{q(\beta-4)}{8\alpha} \right],$$

while if $\beta = -2$, *we have*

$$\begin{aligned}(3\text{-}5)\quad & \int_M u^{-2} J \, d\sigma + \int_{\partial M} u^{-2} \mathrm{II}(\nabla^\partial u, \nabla^\partial u)\, d\sigma \\ &= A_3 \int_M (\Delta_h \ln u)^2 \, d\sigma + B_3 \int_M |\nabla \ln u|^2 \, d\sigma + C_3 \int_M |\nabla \ln u|^4 \, d\sigma,\end{aligned}$$

where

$$A_3 = \frac{1}{m}\left\{ (m+2)\frac{\alpha}{q} + (m-1) \right\},$$
$$B_3 = \frac{-2\alpha}{q}\Big(\frac{m+2}{2m}\Big)\lambda(q-1),$$
$$C_3 = -\frac{2\alpha}{q}\left[\frac{m+2}{2m}\left\{ \Big(\frac{1}{2} - \frac{1}{\alpha}\Big)\Big(-2 + \frac{q}{\alpha}\Big) + \Big(\frac{\alpha-1}{\alpha}\Big)^2 \right\} - \frac{3q}{4\alpha} \right].$$

Proof. Let φ be a positive solution of (3-1). Let α and β be two nonzero real numbers to be determined later and take $u = \varphi^\alpha$ and $v = u^\beta$. Using (3-1), a direct calculation gives

$$\begin{cases} \Delta_h u = \alpha u^{1+(q-1)/\alpha} - \alpha\lambda u - \Big(\dfrac{\alpha-1}{\alpha}\Big)\dfrac{|\nabla u|^2}{u} & \text{in } M, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{3-6}$$

Applying Lemma 2.1 with u and v as above, and using (3-6), we obtain

$$\begin{aligned} \int_M u^\beta\Big(|D^2u|^2 - \frac{1}{m}(\Delta_h u)^2 + \mathrm{Ric}_h(\nabla u, \nabla u)\Big)\, d\sigma \\ = \frac{m-1}{m}\int_M u^\beta (\Delta_h u)^2\, d\sigma - \int_M (\Delta_h u)\langle \nabla u, \nabla u^\beta\rangle\, d\sigma \\ - \tfrac{1}{2}\int_M |\nabla u|^2 \Delta_h u^\beta\, d\sigma - \int_{\partial M} \mathrm{II}(\nabla^\partial u, \nabla^\partial u) u^\beta\, d\sigma, \end{aligned} \tag{3-7}$$

which is equivalent to

$$\begin{aligned} \int_M u^\beta J\, d\sigma + \int_{\partial M} \mathrm{II}(\nabla^\partial u, \nabla^\partial u) u^\beta\, d\sigma \\ = \frac{m-1}{m}\int_M u^\beta (\Delta_h u)^2\, d\sigma - \tfrac{3}{2}\beta\int_M u^{\beta-1}(\Delta_h u)|\nabla u|^2\, d\sigma \\ + \frac{\beta(\beta-1)}{2}\int_M u^{\beta-2}|\nabla u|^4\, d\sigma. \end{aligned} \tag{3-8}$$

Proof of (i). To prove identity (3-3), let us calculate the first two terms on the right hand side of (3-8). First:

$$\begin{aligned} I_1 :&= \frac{m-1}{m}\int_M u^\beta (\Delta_h u)^2\, d\sigma \\ &= \frac{m-1}{m}\int_M u^\beta\Big(\alpha u^{1+(q-1)/\alpha} - \alpha\lambda u - \Big(\frac{\alpha-1}{\alpha}\Big)\frac{|\nabla u|^2}{u}\Big)\Delta_h u\, d\sigma \\ &= \alpha\frac{m-1}{m}\Big(\beta+1+\frac{q-1}{\alpha}\Big)\int_M u^{\beta+(q-1)/\alpha}|\nabla u|^2\, d\sigma \\ &\qquad - \alpha\lambda\frac{m-1}{m}(\beta+1)\int_M u^\beta|\nabla u|^2\, d\sigma \\ &\quad - \frac{m-1}{m}\Big(\frac{\alpha-1}{\alpha}\Big)\int_M u^{\beta-1}|\nabla u|^2\Big(\alpha u^{1+(q-1)/\alpha} - \alpha\lambda u - \Big(\frac{\alpha-1}{\alpha}\Big)\frac{|\nabla u|^2}{u}\Big)\, d\sigma \\ &= \frac{m-1}{m}(\alpha\beta+q)\int_M u^{\beta+(q-1)/\alpha}|\nabla u|^2\, d\sigma - \lambda\frac{m-1}{m}(\alpha\beta+1)\int_M u^\beta|\nabla u|^2\, d\sigma \\ &\qquad + \frac{m-1}{m}\Big(\frac{\alpha-1}{\alpha}\Big)^2\int_M u^{\beta-2}|\nabla u|^4\, d\sigma, \end{aligned}$$

where we used integration by parts with respect to $d\sigma$. A similar type of calculation yields

$$
\begin{aligned}
I_2 :=& -\tfrac{3}{2}\beta\int_M u^{\beta-1}(\Delta_h u)|\nabla u|^2\,d\sigma \\
=& -\tfrac{3}{2}\beta\int_M u^{\beta-1}\Big(\alpha u^{1+(q-1)/\alpha}-\alpha\lambda u-\Big(\frac{\alpha-1}{\alpha}\Big)\frac{|\nabla u|^2}{u}\Big)|\nabla u|^2\,d\sigma \\
=& -\tfrac{3}{2}\alpha\beta\int_M u^{\beta+(q-1)/\alpha}|\nabla u|^2\,d\sigma+\tfrac{3}{2}\alpha\beta\lambda\int_M u^{\beta}|\nabla u|^2\,d\sigma \\
&+\tfrac{3}{2}\beta\Big(\frac{\alpha-1}{\alpha}\Big)\int_M u^{\beta-2}|\nabla u|^4\,d\sigma.
\end{aligned}
$$

Incorporating I_1 and I_2 in (3-8), we complete the proof of (3-3) by deducing that

$$
\begin{aligned}
&\int_M u^{\beta}J\,d\sigma+\int_{\partial M}\mathrm{II}(\nabla^{\partial}u,\nabla^{\partial}u)u^{\beta}\,d\sigma \\
&\quad=\Big\{\frac{m-1}{m}(\alpha\beta+q)-\tfrac{3}{2}\alpha\beta\Big\}\int_M u^{\beta+(q-1)/\alpha}|\nabla u|^2\,d\sigma \\
&\qquad+\Big\{-\lambda\frac{m-1}{m}(\alpha\beta+1)+\tfrac{3}{2}\alpha\beta\lambda\Big\}\int_M u^{\beta}|\nabla u|^2\,d\sigma \\
&\qquad+\Big\{\frac{m-1}{m}\Big(\frac{\alpha-1}{\alpha}\Big)^2+\tfrac{3}{2}\beta\Big(\frac{\alpha-1}{\alpha}\Big)+\frac{\beta(\beta-1)}{2}\Big\}\int_M u^{\beta-2}|\nabla u|^4\,d\sigma.
\end{aligned}\tag{3-9}
$$

Proof of (ii). Let us now give the proof of (3-4). The main idea here is to replace the second term on the right hand side of (3-8) $\big($i.e., $\int_M u^{\beta-1}(\Delta_h u)|\nabla u|^2\,d\sigma\big)$ by an expression in which the sign is controllable. For that, we multiply the first equation of (3-6) by $u^{\beta-1}|\nabla u|^2$, and we obtain after integrating with respect to $d\sigma$,

$$
\begin{aligned}
&\int_M u^{\beta-1}(\Delta_h u)|\nabla u|^2\,d\sigma \\
&\quad=\alpha\int_M (u^{\beta+(q-1)/\alpha}-\lambda u^{\beta})|\nabla u|^2\,d\sigma-\Big(\frac{\alpha-1}{\alpha}\Big)\int_M u^{\beta-2}|\nabla u|^4\,d\sigma.
\end{aligned}\tag{3-10}
$$

Similarly, multiplying the same equation of (3-6) by $u^{\beta}\Delta_h u$ and integrating by parts yields

$$
\begin{aligned}
&\int_M u^{\beta}(\Delta_h u)^2\,d\sigma \\
&=-\frac{\alpha-1}{\alpha}\int_M u^{\beta-1}(\Delta_h u)|\nabla u|^2\,d\sigma+\alpha\int_M\big(u^{\beta+1+(q-1)/\alpha}-\lambda u^{\beta+1}\big)\Delta_h u\,d\sigma \\
&=-\frac{\alpha-1}{\alpha}\int_M u^{\beta-1}(\Delta_h u)|\nabla u|^2\,d\sigma+\alpha\Big(\beta+1+\frac{q-1}{\alpha}\Big)\int_M u^{\beta+(q-1)/\alpha}|\nabla u|^2\,d\sigma \\
&\qquad-\alpha\lambda(\beta+1)\int_M u^{\beta}|\nabla u|^2\,d\sigma.
\end{aligned}\tag{3-11}
$$

In order to eliminate the term

$$\int_M u^{\beta+(q-1)/\alpha}|\nabla u|^2 d\sigma,$$

we multiply (3-10) by $(\beta+1+(q-1)/\alpha)$ and subtract it from (3-11) to get

$$\begin{aligned}(3\text{-}12)\quad \lambda(q-1)\int_M u^\beta|\nabla u|^2\,d\sigma &= -\Big(\beta+\frac{q}{\alpha}\Big)\int_M u^{\beta-1}(\Delta_h u)|\nabla u|^2\,d\sigma+\int_M u^\beta(\Delta_h u)^2\,d\sigma\\ &\quad-\Big(\frac{\alpha-1}{\alpha}\Big)\Big(\beta+1+\frac{q-1}{\alpha}\Big)\int_M u^{\beta-2}|\nabla u|^4\,d\sigma.\end{aligned}$$

On the other hand, by a straightforward calculation, we have for $\beta\neq -2$,

$$\begin{aligned}(3\text{-}13)\quad \int_M u^{\beta-1}(\Delta_h u)|\nabla u|^2\,d\sigma &= -\frac{4}{\beta(\beta+2)^2}\int_M (\Delta_h u^{(\beta+2)/2})^2\,d\sigma\\ &\quad+\frac{\beta}{4}\int_M u^{\beta-2}|\nabla u|^4\,d\sigma+\frac{1}{\beta}\int_M u^\beta(\Delta_h u)^2\,d\sigma.\end{aligned}$$

Now, we replace $\int_M u^{\beta-1}(\Delta_h u)|\nabla u|^2\,d\sigma$ by its expression given in (3-13) in the equations (3-8) and (3-12) respectively. So (3-8) gives

$$\begin{aligned}(3\text{-}14)\quad \int_M u^\beta J\,d\sigma+\int_{\partial M} u^\beta \mathrm{II}(\nabla^\partial u,\nabla^\partial u)\,d\sigma &= -\frac{m+2}{2m}\int_M u^\beta(\Delta_h u)^2\,d\sigma+\frac{6}{(\beta+2)^2}\int_M(\Delta_h u^{(\beta+2)/2})^2\,d\sigma\\ &\quad+\frac{\beta(\beta-4)}{8}\int_M u^{\beta-2}|\nabla u|^4\,d\sigma,\end{aligned}$$

and the Equation (3-12) gives

$$\begin{aligned}(3\text{-}15)\quad \lambda(q-1)\int_M u^\beta|\nabla u|^2\,d\sigma &= \frac{-q}{\alpha\beta}\int_M u^\beta(\Delta_h u)^2\,d\sigma+\frac{4(\beta+q/\alpha)}{\beta(\beta+2)^2}\int_M(\Delta_h u^{(\beta+2)/2})^2\,d\sigma\\ &\quad-\bigg(\frac{\beta}{4}\Big(\beta+\frac{q}{\alpha}\Big)+\Big(\frac{\alpha-1}{\alpha}\Big)\Big(\beta+1+\frac{q-1}{\alpha}\Big)\bigg)\int_M u^{\beta-2}|\nabla u|^4\,d\sigma.\end{aligned}$$

Thus in order to eliminate the term

$$\int_M u^\beta(\Delta_h u)^2\,d\sigma$$

from (3-14) and (3-15), we multiply (3-14) by $(q/\alpha\beta)$ and (3-15) by $(m+2)/2m$

and subtract them to obtain

$$A\int_M u^{\beta-2}|\nabla u|^4\,d\sigma = -\frac{q}{\alpha\beta}\left(\int_M u^\beta J\,d\sigma + \int_{\partial M} u^\beta \mathrm{II}(\nabla^\partial u,\nabla^\partial u)\,d\sigma\right) - B\int_M (\Delta_h u^{(\beta+2)/2})^2\,d\sigma + \left(\frac{m+2}{2m}\right)\lambda(q-1)\int_M u^\beta|\nabla u|^2\,d\sigma, \tag{3-16}$$

where the expressions of A and B are given by

$$A = -\frac{m+2}{2m}\left\{\left(\frac{\beta}{4}+\frac{\alpha-1}{\alpha}\right)\left(\beta+\frac{q}{\alpha}\right)+\left(\frac{\alpha-1}{\alpha}\right)^2\right\} - \frac{q(\beta-4)}{8\alpha},$$
$$B = \frac{2}{m(\beta+2)^2}\left\{(m+2)-2(m-1)\frac{q}{\alpha\beta}\right\}$$

This completes the proof of (3-4) in the case $\beta \neq -2$.

Concerning the case where $\beta=-2$, the second term on the right hand side of (3-16) can be written as

$$B\int_M (\Delta_h u^{(\beta+2)/2})^2\,d\sigma = \frac{1}{2m}\left\{(m+2)-2(m-1)\frac{q}{\alpha\beta}\right\}\int_M\left(\Delta_h\left(\frac{u^{(\beta+2)/2}-u^0}{(\beta+2)/2}\right)\right)^2 d\sigma,$$

and when β tends to -2 in (3-16), we obtain (3-5). Therefore the proof of Proposition 3.1 is completed. □

The first uniqueness result in this paper is the following:

Proposition 3.2. *Let (M^n, g, σ) be a compact weighted Riemannian manifold of dimension $n \geq 2$ with convex boundary ∂M. Assume that for some $m \in [n,\infty)$, the m-Bakry–Émery–Ricci curvature satisfies $\mathrm{Ric}_h^m \geq kg$, for a positive constant k. Let $q>1$, $\lambda>0$ and φ be a positive solution of the following system*:

$$\begin{cases} \Delta_h\varphi + \lambda\varphi = \varphi^q & \text{in } M,\\ \dfrac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial M.\end{cases} \tag{3-17}$$

Suppose that

$$q \leq \frac{m+2}{m-2}$$

with no restriction on q if $m=2$,

$$\lambda \leq \frac{mk}{(m-1)(q-1)}$$

and one of these two inequalities is strict, then φ is constant equal to $\lambda^{1/(q-1)}$.

In addition, if (M, g) *is of constant scalar curvature* R, *nonisometric to the hemisphere* $\mathbb{S}^n_+(\sqrt{n(n-1)/R})$, *then*

$$\lambda = \frac{mk}{(m-1)(q-1)} \quad \text{and} \quad q = \frac{m+2}{m-2}$$

ensure that φ *is constant equal to* $\lambda^{1/(q-1)}$.

Remark 3.3. • In the last Proposition, one can consider more generally an equation of the form

$$\Delta_h \varphi + \lambda\varphi = \mu\varphi^q,$$

where μ is a positive constant. In fact, if we take $\widetilde{\varphi} = \mu^{1/(q-1)}\varphi$, we can easily obtain $\Delta_h \widetilde{\varphi} + \lambda\widetilde{\varphi} = \widetilde{\varphi}^q$.

• *The solutions in the spherical case*: In the Riemannian case (i.e., the case $m = n$), the solutions are well known and they are related to the metrics conformal to the standard metric on the sphere, respectively the hemisphere (see for instance [Aubin 1982; Escobar 1990; 1992; Lee and Parker 1987]).

• *Unicity of an Einstein metric in a conformal class*: For (M^n, g) a compact Einstein manifold of totally geodesic boundary ∂M, Escobar [1990] proved that if g_1 is a metric conformally related to g with constant scalar curvature and for which ∂M is minimal, then g_1 is Einstein. Moreover, if (M^n, g) is not conformally equivalent to $\mathbb{S}^n_+$, then $g_1 = cg$ for some positive constant c. In fact, one can easily see that if $g_1 = u^{4/(n-2)}g$, then the scalar curvatures R_g and R_{g_1} satisfy

$$\begin{cases} \Delta u + \dfrac{n-2}{4(n-1)} R_g u = \dfrac{n-2}{4(n-1)} R_{g_1} u^{(n+2)/(n-2)} & \text{in } M, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases}$$

Now, when the scalar curvatures are positive (this is the difficult part in Escobar's result), using Proposition 3.2 one can easily deduce that u is constant. Note that Escobar's result is the generalization of the classical uniqueness result of Obata for Einstein compact manifolds without boundary (see for instance [Obata 1962]).

The situation is more complicated in the case of weighted Riemannian manifolds as one can see, for instance, in [Case et al. 2011; Case 2015; Chang et al. 2006; 2011]. Nevertheless, we expect that there is an analogue of Escobar's theorem for weighted Riemannian manifolds.

Proof of Proposition 3.2. One can easily infer from (3-3) the following:

$$\begin{aligned} \int_M u^\beta \big(J - \mathrm{Ric}_h^m(\nabla u, \nabla u)\big)\, d\sigma &+ \int_{\partial M} \mathrm{II}(\nabla^\partial u, \nabla^\partial u) u^\beta \, d\sigma \\ &= A_1 \int_M u^{\beta+(q-1)/\alpha} |\nabla u|^2 \, d\sigma + B_1 \int_M u^\beta |\nabla u|^2 \, d\sigma \\ &\qquad + C_1 \int_M u^{\beta-2} |\nabla u|^4 \, d\sigma - \int_M u^\beta \, \mathrm{Ric}_h^m(\nabla u, \nabla u). \end{aligned}$$

Using the hypothesis that the m-Bakry–Émery–Ricci curvature satisfies $\mathrm{Ric}_h^m \geq kg$, and the fact that ∂M is convex (i.e., $\mathrm{II} \geq 0$), we obtain:

$$\begin{aligned}(3\text{-}18)\quad &\int_M u^\beta\big(J - \mathrm{Ric}_h^m(\nabla u, \nabla u)\big)\, d\sigma \\ &\leq A_1 \int_M u^{\beta+(q-1)/\alpha} |\nabla u|^2\, d\sigma + (B_1 - k)\int_M u^\beta |\nabla u|^2\, d\sigma + C_1 \int_M u^{\beta-2}|\nabla u|^4\, d\sigma,\end{aligned}$$

where A_1, B_1 and C_1 are as given in Proposition 3.1. Since Lemma 2.2 asserts that J is bounded below by $\mathrm{Ric}_h^m(\nabla u, \nabla u)$, it suffices to show the existence of α and β such that $A_1 \leq 0$, $(B_1 - k) \leq 0$, $C_1 \leq 0$ and at least one of these three inequalities is strict, in order to conclude from Equation (3-18) that u (and hence φ) is a constant.

By arguing as in [Ilias 1996], we see that if

$$q \leq \frac{m+2}{m-2} \quad \text{and} \quad \lambda \leq \frac{mk}{(m-1)(q-1)}$$

then there exist α, β such that A_1, $(B_1 - k)$, $C_1 \leq 0$. Moreover, if one of the above two inequalities is strict, we may choose α, β such that A_1, $(B_1 - k)$, $C_1 \leq 0$ and at least one of these inequalities is strict.

Now suppose that (M, g) is of constant scalar curvature R, nonisometric to the hemisphere $\mathbb{S}_+^n(\sqrt{n(n-1)/R})$. If $q = (m+2)/(m-2)$ and $\lambda = mk/(m-1)(q-1)$, then one can choose α and β such that $A_1 = B_1 - k = C_1 = 0$. From (3-18) we conclude that $J - \mathrm{Ric}_h^m(\nabla u, \nabla u) = 0$, which is equivalent by Lemma 2.2, to

$$(3\text{-}19)\qquad D^2 u = -\frac{1}{n}(\Delta u) g \quad \text{and} \quad \Delta_h u = \frac{m}{m-n}\langle \nabla u, \nabla h\rangle.$$

Suppose that u is not constant and consider the vector field $Y = \nabla u$. First of all, Y is a conformal vector field because the first equality of (3-19) is nothing but $\mathcal{L}_Y g = (2/n)\rho g$, where $\rho = \operatorname{div} Y = -\Delta u$. Since Y is conformal and R is constant, we have (see equation (1.11) of [Yano and Obata 1970])

$$\mathcal{L}_Y(R) = Y(R) = \frac{2(n-1)}{n}\Delta\rho - \frac{2}{n}\rho R = 0,$$

and consequently

$$(3\text{-}20)\qquad \Delta u - \frac{R}{n-1} u = \text{constant}.$$

We immediately deduce from (3-20) that $\frac{\partial}{\partial \nu}(\Delta u)\,|_{\partial M} = 0$.

Differentiating (3-20) two times we get

$$(3\text{-}21)\qquad \begin{cases} D^2\rho + \dfrac{R}{n(n-1)}\rho g = 0 & \text{in } M, \\ \dfrac{\partial \rho}{\partial \nu} = 0 & \text{on } \partial M. \end{cases}$$

Since we have supposed that u is not constant, $\rho = -\Delta u$ is not identically zero on M and we can deduce from Escobar's theorem [1990, Theorem 4.2] that (M, g) is isometric to the upper hemisphere $\mathbb{S}^n_+(\sqrt{n(n-1)/R})$, which contradicts our hypothesis. Thus u (hence φ) is constant. □

Now, we shall prove another kind of uniqueness result (under different conditions on λ) for the same nonlinear elliptic PDE (3-17) which generalizes a result of Licois and Véron [1995]. This result involves the first nonzero eigenvalue λ_1^h of the weighted Laplacian Δ_h under the Neumann boundary condition.

Proposition 3.4. *Let (M^n, g, σ) be a compact weighted Riemannian manifold of dimension $n \geq 2$ with convex boundary ∂M. Assume that for some $m \in [n, \infty)$, the m-Bakry–Émery–Ricci curvature satisfies $\mathrm{Ric}_h^m \geq kg$, for a positive constant k. Let $q > 1$, $\lambda > 0$ and φ be a positive solution of the following system*

$$\begin{cases} \Delta_h \varphi + \lambda \varphi = \varphi^q & \text{in } M, \\ \dfrac{\partial \varphi}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{3-22}$$

Suppose that

$$q \leq \frac{m+2}{m-2} \tag{3-23}$$

with no restriction on q if $m = 2$,

$$(q-1)\lambda \leq \lambda_1^h + \frac{qm(m-1)}{q+m(m+2)}\left(k - \frac{m-1}{m}\lambda_1^h\right) \tag{3-24}$$

and one of these two inequalities is strict, then φ is constant equal to $\lambda^{1/(q-1)}$.

In addition, if (M, g) is of constant scalar curvature R nonisometric to the hemisphere $\mathbb{S}^n_+(\sqrt{n(n-1)/R})$, then

$$(q-1)\lambda = \lambda_1^h + \frac{qm(m-1)}{q+m(m+2)}\left(k - \frac{m-1}{m}\lambda_1^h\right) \quad \text{and} \quad q = \frac{m+2}{m-2}$$

ensure that φ is constant equal to $\lambda^{1/(q-1)}$.

Remark 3.5. We observe that the term $(k - ((m-1)/m)\lambda_1^h)$ in Equation (3-24) is always nonpositive and this is due to the Escobar–Lichnerowicz theorem (see [Escobar 1990, Theorem 4.3]) generalized to weighted Riemannian manifolds with convex boundary and satisfying $\mathrm{Ric}_h^m \geq kg > 0$ (see, for example, [Li and Wei 2015, Theorem 3] or [Ma and Du 2010, Theorem 2]). Moreover, (3-24) can be rewritten in the following form:

$$(q-1)\lambda \leq \left(1 - \frac{q(m-1)^2}{q+m(m+2)}\right)\lambda_1^h + \left(\frac{qm(m-1)}{q+m(m+2)}\right)k. \tag{3-25}$$

The coefficient of λ_1^h in (3-25) is positive if $q < (m+2)/(m-2)$ and equal to zero if $q = (m+2)/(m-2)$.

Proof of Proposition 3.4. From (3-4) of Proposition 3.1, we easily infer for $\beta \neq -2$,

$$(3\text{-}26)\quad A\int_M u^{\beta-2}|\nabla u|^4\,d\sigma = -\frac{q}{\alpha\beta}\left(\int_M u^\beta J\,d\sigma + \int_{\partial M} u^\beta \mathrm{II}(\nabla^\partial u, \nabla^\partial u)\,d\sigma\right) - B\int_M (\Delta_h u^{(\beta+2)/2})^2\,d\sigma + \left(\frac{m+2}{2m}\right)\lambda(q-1)\int_M u^\beta|\nabla u|^2\,d\sigma,$$

where the expressions of A and B are given by

$$A = -\frac{m+2}{2m}\left\{\left(\frac{\beta}{4} + \frac{\alpha-1}{\alpha}\right)\left(\beta + \frac{q}{\alpha}\right) + \left(\frac{\alpha-1}{\alpha}\right)^2\right\} - \frac{q(\beta-4)}{8\alpha},$$

$$B = \frac{2}{m(\beta+2)^2}\left\{(m+2) - 2(m-1)\frac{q}{\alpha\beta}\right\}.$$

The idea is to replace the integral associated to B by one of the form $\int_M u^\beta|\nabla u|^2\,d\sigma$. Since $\frac{\partial u}{\partial\nu} = 0$ over the boundary ∂M, the variational characterization of λ_1^h yields

$$(3\text{-}27)\quad \begin{aligned} &\int_M (\Delta_h u^{(\beta+2)/2})^2 d\sigma \geq \frac{(\beta+2)^2}{4}\lambda_1^h \int_M u^\beta|\nabla u|^2\,d\sigma \quad &&\text{for } \beta \neq -2,\\ &\int_M (\Delta_h \ln u)^2\,d\sigma \geq \lambda_1^h \int_M |\nabla \ln u|^2\,d\sigma &&\text{for } \beta = -2.\end{aligned}$$

Therefore if we can find a couple of nonzero real numbers (α, β) such that

$$(3\text{-}28)\quad \alpha\beta > 0, \quad A \geqq 0 \quad \text{and} \quad B \geqq 0,$$

then by using the relation (3-27), the hypothesis on the m-Bakry–Émery–Ricci curvature and the convexity of ∂M, we deduce from (3-26),

$$(3\text{-}29)\quad \begin{aligned} 0 &\leq A\int_M u^{\beta-2}|\nabla u|^4\,d\sigma\\ &\leq \frac{-q}{\alpha\beta}\int_M u^\beta\big(J - \mathrm{Ric}_h^m(\nabla u, \nabla u)\big)\,d\sigma + C\int_M u^\beta|\nabla u|^2\,d\sigma,\end{aligned}$$

where

$$\begin{aligned} C :&= \frac{-q}{\alpha\beta}k - \frac{(\beta+2)^2}{4}\lambda_1^h B + \left(\frac{m+2}{2m}\right)(q-1)\lambda\\ &= \frac{m+2}{2m}(\lambda(q-1) - \lambda_1^h) - \frac{q}{\alpha\beta}\left(k - \frac{m-1}{m}\lambda_1^h\right),\end{aligned}$$

and using the hypothesis (3-23) as well as (3-24) concerning λ, we can find a couple among the (α, β) satisfying the conditions (3-28) such that C is nonpositive.

Moreover, when the equality is not achieved in (3-23) or in (3-24) we will be able to conclude that u is constant.

For $\beta = -2$, an immediate modification where we use identity (3-5) instead of (3-4) permits us to conclude.

In the rest of the proof we will show how to find such a couple (α, β) when $\beta \neq -2$. First we simplify the expressions of A, B and C, by setting

$$X = \frac{-1}{\alpha\beta}, \quad \delta = \frac{1}{\beta} + \frac{1}{2}, \quad \text{and} \quad \widetilde{A} = \frac{2m}{(m+2)\beta^2} A$$

to obtain

$$\begin{aligned}
\widetilde{A} &= -\delta^2 + 2\frac{q-(m+2)}{m+2} X\delta + (q-1)X^2 + \frac{(m-1)Xq}{2(m+2)},\\
B &= \frac{2}{m(\beta+2)^2}\big((m+2) + 2(m-1)Xq\big),\\
C &= \frac{m+2}{2m}\big(\lambda(q-1) - \lambda_1^h\big) + Xq\Big(k - \frac{m-1}{m}\lambda_1^h\Big),
\end{aligned}$$

and then we maximize X in the interval

$$I := \Big[-\frac{m+2}{2(m-1)q}, 0\Big)$$

(which ensures that $B \geq 0$ and $\alpha\beta > 0$) such that

$$\widetilde{A} = -\delta^2 + 2\frac{q-(m+2)}{m+2} X\delta + (q-1)X^2 + \frac{(m-1)}{2(m+2)} Xq \geq 0.$$

The derivative of $\widetilde{A}$ with respect to δ is given by

$$\frac{d\widetilde{A}}{d\delta} = -2\Big(\delta - \frac{q-(m+2)}{m+2} X\Big).$$

Therefore the maximum of $\widetilde{A}$ with respect to δ is achieved for

$$\delta_0 := \frac{q-(m+2)}{m+2} X$$

and thus:

$$\begin{aligned}
\widetilde{A}(\delta_0, X) &= \delta_0^2 + (q-1)X^2 + \frac{(m-1)}{2(m+2)} Xq\\
&= \bigg(q - 1 + \Big(\frac{q-(m+2)}{m+2}\Big)^2\bigg)X^2 + \frac{(m-1)}{2(m+2)} Xq,
\end{aligned}$$

which admits a nontrivial negative solution

$$X_0 = -\frac{(m-1)(m+2)}{2\big(q + m(m+2)\big)}.$$

Using the hypothesis that $q \le (m+2)/(m-2)$, one has $X_0 \in I$ and therefore

$$\widetilde{A}(\delta_0, X) \ge 0 \quad \text{on } \Big[-\frac{m+2}{2(m-1)q}, X_0\Big] \subseteq I.$$

Moreover, a direct computation of C at the specific value $X = X_0$ gives

$$C(X_0) = \frac{m+2}{2m}\Big(\lambda(q-1) - \lambda_1^h - \frac{qm(m-1)}{q+m(m+2)}\Big(k - \frac{m-1}{m}\lambda_1^h\Big)\Big).$$

Thus, if we suppose that

$$q \le \frac{m+2}{m-2} \tag{3-30}$$

and

$$\lambda(q-1) \le \lambda_1^h + \frac{qm(m-1)}{q+m(m+2)}\Big(k - \frac{m-1}{m}\lambda_1^h\Big) \tag{3-31}$$

then we have two possibilities:

(1) The equality in (3-31) is not achieved (i.e., $C(X_0) < 0$). In this case, we obtain from (3-29) at $X = X_0$,

$$C(X_0)\int_M u^\beta |\nabla u|^2 \, d\sigma = 0,$$

and since $C(X_0) < 0$, we deduce that u is constant.

(2) The equality in (3-31) is achieved (i.e., $C(X_0) = 0$) and the inequality (3-30) is strict. In this case, one can deduce that all the inequalities used to obtain (3-29) are in fact equalities. In particular, one has

$$B(X_0)\int_M (\Delta_h u^{(\beta+2)/2})^2 d\sigma = B(X_0)\frac{(\beta+2)^2}{4}\lambda_1^h \int_M u^\beta |\nabla u|^2 \, d\sigma$$

and since (3-30) is strict, $B(X_0)$ is positive. Therefore

$$\int_M (\Delta_h u^{(\beta+2)/2})^2 \, d\sigma = \lambda_1^h \int_M |\nabla u^{(\beta+2)/2}|^2 \, d\sigma. \tag{3-32}$$

Thus, if u is not constant, then $u^{(\beta+2)/2}$ is an eigenfunction associated to λ_1^h and since $\frac{\partial u}{\partial \nu}|_{\partial M} = 0$, we have

$$\int_M u^{(\beta+2)/2} \, d\sigma = 0,$$

which contradicts the fact that u is positive. In conclusion u is constant.

To prove the last assertion in the proposition, suppose that (M, g) is of constant scalar curvature R, nonisometric to the hemisphere $\mathbb{S}^n_+(\sqrt{n(n-1)/R})$. If (3-30) and (3-31) are equalities, then we can conclude from (3-29) that $J - \mathrm{Ric}_h^m(\nabla u, \nabla u) = 0$.

Similar arguments as those used in the proof of Proposition 3.2 allow us to conclude that u is constant. □

Now we will deduce our Sobolev inequalities from the previous uniqueness results.

Theorem 3.6. *Let (M^n, g, σ) be a compact weighted Riemannian manifold of dimension $n \geq 2$ with convex boundary (i.e., $\mathrm{II} \geq 0$). Assume that for some $m \in [n, \infty)$, the m-Bakry–Émery–Ricci curvature satisfies $\mathrm{Ric}_h^m \geq kg$ for a positive constant k. Then every function $f \in H_1^2(d\sigma)$ satisfies*

$$\|f\|^2_{L^p(d\sigma)} \leq \mathrm{vol}_h(M)^{-(p-2)/p}\left(\frac{1}{\theta(m,p)}\|\nabla f\|^2_{L^2(d\sigma)} + \|f\|^2_{L^2(d\sigma)}\right), \tag{3-33}$$

for any $p \in (2, 2^*]$ with $2^* = 2m/(m-2)$ if $m > 2$ and for any $p \in (2, \infty)$ if $m = 2$ (i.e., $n = 2$ and h is constant) with $\theta(m, p) \in \{\theta_1(m, p), \theta_2(m, p)\}$ where

$$\theta_1(m,p) = \frac{mk}{(m-1)(p-2)},$$
$$\theta_2(m,p) = \frac{\lambda_1^h}{(p-2)} + \frac{m(m-1)(p-1)}{\big((p-1)+m(m+2)\big)(p-2)}\Big(k - \frac{m-1}{m}\lambda_1^h\Big).$$

Using the Escobar–Lichnerowicz theorem (see Section 4), we note that for $m > 2$

$$\theta_1(m,p) - \theta_2(m,p) = \frac{m(m+2) - m(m-2)(p-1)}{\big((p-1)+m(m+2)\big)(p-2)}\Big(\frac{mk}{m-1} - \lambda_1^h\Big) \leq 0$$

and for $m = 2$,

$$\theta_1(2,p) - \theta_2(2,p) = \frac{8}{((p-1)+8)(p-2)}(2k - \lambda_1^h) \leq 0$$

therefore, (3-33) is better with $\theta(m, p) = \theta_2(m, p)$ than with $\theta_1(m, p)$.

On the other hand, when $m > 2$ and p tends to the critical exponent $2^* = 2m/(m-2)$, we have

$$\theta_1(m, 2^*) = \theta_2(m, 2^*) = \frac{m(m-2)k}{4(m-1)}.$$

Therefore, one limiting case of Theorem 3.6 gives the following:

Corollary 3.7. *Let (M^n, g, σ) be a compact weighted Riemannian manifold of dimension $n \geq 2$ and of convex boundary (i.e., $\mathrm{II} \geq 0$). Assume that for some $m \in [n, \infty)$, the m-Bakry–Émery–Ricci curvature satisfies $\mathrm{Ric}_h^m \geq kg$, for a positive constant k. If $m > 2$, then every $f \in H_1^2(d\sigma)$ satisfies*

$$\|f\|^2_{L^{2m/(m-2)}(d\sigma)} \leq \mathrm{vol}_h(M)^{-2/m}\left(\frac{4(m-1)}{m(m-2)k}\|\nabla f\|^2_{L^2(d\sigma)} + \|f\|^2_{L^2(d\sigma)}\right). \tag{3-34}$$

Proof of Theorem 3.6. Suppose that $m > 2$, and consider the following family of functionals $\mathcal{J}_q$, defined by

$$\mathcal{J}_q(\varphi) = \int_M |\nabla\varphi|^2 \, d\sigma + \Theta(m, q)\int_M \varphi^2 \, d\sigma \quad \text{for } 1 < q < \frac{m+2}{m-2}$$

with $\Theta(m, q) \in \{\Theta_1(m, q), \Theta_2(m, q)\}$ where

$$\Theta_1(m, q) = \frac{mk}{(m-1)(q-1)}$$

$$\Theta_2(m, q) = \frac{\lambda_1^h}{(q-1)} + \frac{qm(m-1)}{(q-1)\big(q+m(m+2)\big)}\Big(k - \frac{m-1}{m}\lambda_1^h\Big)$$

and consider $\mu_q := \inf\{\mathcal{J}_q(\varphi),\ \varphi \in \mathcal{H}_q\}$, where

$$\mathcal{H}_q := \left\{\varphi \in H_1^2(d\sigma) : \int_M \varphi^{q+1} \, d\sigma = 1\right\}.$$

Another crucial key here is the fact that the real-valued function

$$g : x \longmapsto \frac{x+2}{x-2}$$

is decreasing. So

$$\frac{m+2}{m-2} = g(m) < g(n) = \frac{n+2}{n-2}$$

as $n < m$.

Using the compactness of the inclusions

$$H_1^2(d\sigma) \hookrightarrow L^2(d\sigma) \quad \text{and} \quad H_1^2(d\sigma) \hookrightarrow L^{q+1}(d\sigma)$$

for any $q + 1 < 2n/(n-2)$, we can prove that μ_q is achieved by a positive function $\psi_q \in \mathcal{H}_q$ and therefore, one can easily check that ψ_q verifies weakly the following system:

$$\text{(3-35)} \qquad \begin{cases} \Delta_h\psi_q + \Theta(m, q)\psi_q = \mu_q\psi_q^q & \text{in } M, \\ \dfrac{\partial\psi_q}{\partial\nu} = 0 & \text{on } \partial M. \end{cases}$$

Since h is smooth, the regularity result of Cherrier [1984, Theorem 1] shows that ψ_q is smooth, and hence by applying Proposition 3.2 if $\Theta(m, q) = \Theta_1(m, q)$ or Proposition 3.4 if $\Theta(m, q) = \Theta_2(m, q)$, we deduce that ψ_q is constant. Since $\psi_q \in \mathcal{H}_q$, we get

$$\psi_q = (\mathrm{vol}_h(M))^{-1/(q+1)} \quad \text{and} \quad \mu_q = \Theta(m, q)(\mathrm{vol}_h(M))^{(q-1)/(q+1)}.$$

Therefore, one can deduce from the definition of μ_q, that any $f \in H_1^2(d\sigma)$ satisfies

$$\left(\int_M |f|^{q+1}\,d\sigma\right)^{2/(q+1)} \le \mathrm{vol}_h(M)^{-(q-1)/(q+1)}\left(\frac{1}{\Theta(m,q)}\int_M |\nabla f|^2\,d\sigma + \int_M |f|^2\,d\sigma\right). \tag{3-36}$$

If we put $p = q+1$, then $\Theta(m,q) = \Theta(m,p-1) = \theta(m,p)$ and thus (3-36) completes the proof of Theorem 3.6 for $m > 2$. The case where $m = 2$, can be treated in a similar manner. □

4. Some applications

We can derive many interesting applications from the weighted Sobolev inequalities we obtain. The list is long but for brevity we will limit ourselves to a few and more significant examples.

(I) *The classical Escobar–Lichnerowicz lower bound*: As in [Bakry and Ledoux 1996], if we apply (3-34) to $f = 1 + t\phi$ $(t > 0)$, where $\int_M \phi\,d\sigma = 0$, and use the Taylor expansion $(1+x)^p \simeq_{x\to 0} 1 + px + \frac{1}{2}p(p-1)x^2$, then we obtain the analogue of the classical Escobar–Lichnerowicz theorem for measured spaces with convex boundary:

$$\lambda_1^h \ge \frac{mk}{m-1}. \tag{4-1}$$

We should also point out that many authors have obtained this estimate (see, for example, [Li and Wei 2015, Theorem 3; Ma and Du 2010, Theorem 2]).

Similarly, one can look for lower bounds depending on k for higher eigenvalues:

(II) *Lower bounds of higher eigenvalues with explicit constants*: Inspired by the work of Cheng and Li [1981] in the case of a Riemannian manifold and using Poincaré and Hölder inequalities, we can easily derive from (3-34) that any $f \in H_1^2(d\sigma)$ such that $\int_M f\,d\sigma = 0$ satisfies

$$\int_M |\nabla f|^2\,d\sigma \ge \mathcal{C}_1\left(\int_M f^2\,d\sigma\right)^{\frac{m+2}{m}}\left(\int_M |f|\,d\sigma\right)^{-\frac{4}{m}}, \tag{4-2}$$

where $\mathcal{C}_1 = \lambda_1^h/(\lambda_1^h\mathcal{C}_0+1)(\mathrm{vol}_h(M))^{2/m}$ with $\mathcal{C}_0 = 4(m-1)/(m(m-2)k)$. Applying this last inequality (4-2) to the weighted heat kernel $H_h(t,x,y)$ with Neumann condition on the boundary and after using its semigroup property (see for instance the book of Grigor'yan [2009] for the properties of such heat kernel), we are able to obtain an explicit upper bound for its trace. In fact, we obtain

$$\int_M \left(H_h(t,x,x) - \frac{1}{\mathrm{vol}_h(M)} \right) d\sigma = \sum_{i=1}^{\infty} e^{-\lambda_i^h t} \le 4\left(\frac{m}{2\mathcal{C}_1}\right)^{m/2} \mathrm{vol}_h(M) t^{-m/2}. \tag{4-3}$$

From this upper bound, taking $t = 1/\lambda_\ell^h$, where λ_ℓ^h is the ℓ-th Neumann eigenvalue of Δ_h on M, we can easily deduce a lower bound of λ_ℓ^h as follows:

$$\lambda_\ell^h \ge \left(\frac{1}{4e}\right)^{2/m} \left(\frac{2}{m}\right) \mathcal{C}_1 \left(\frac{\ell}{\mathrm{vol}_h(M)}\right)^{2/m}. \tag{4-4}$$

In the particular case, where the manifold is without boundary (respectively with convex boundary) and the density function is constant (hence $m = n$), λ_ℓ^h is nothing but λ_ℓ , the ℓ-th eigenvalue of the usual Laplacian (resp. the Neumann Laplacian), and thus we recover the lower bound obtained in [Cheng and Li 1981] but with explicit constants. It's worthwhile to point out that if we use the estimate (4-1) for λ_1^h obtained above, then the constant $\mathcal{C}_1$ in (4-2) can be replaced by the constant

$$\mathcal{C}_2 := \frac{m-2}{m+2} (\mathrm{vol}_h(M))^{2/m} \frac{mk}{m-1}$$

and thus (4-4) becomes

$$\lambda_\ell^h \ge 2\left(\frac{1}{4e}\right)^{2/m} \frac{(m-2)k}{(m+2)(m-1)} \ell^{2/m}.$$

In the same spirit, but inspired this time by the work of Li and Yau [1983], one can consider the Neumann heat kernel of the operator Δ_h/q, where q is a positive potential on M. In this case, using (4-2), one can deduce that

$$\lambda_\ell^h \left(\int_M q^{\frac{m}{2}} \, d\sigma \right)^{\frac{2}{m}} \ge \left(\frac{\mathcal{C}_1}{e}\right) \ell^{2/m}$$

and as above $\mathcal{C}_1$ can be replaced by $\mathcal{C}_2$. Using this last inequality, the same arguments as in Corollary 2 of [Li and Yau 1983] gives an explicit estimate of the number of eigenvalues for a weighted Schrödinger operator $\Delta_h + V$ which are less than or equal to a given value.

(III) *Lower bound for the Yamabe invariant*: • *The case of a compact Riemannian manifold without boundary*: Let (M, g) be a compact Riemannian manifold without boundary of dimension $n > 2$, and as before denote by dv_g its Riemannian measure and by R_g its scalar curvature. Let $[g]$ be the class of conformal metrics to g. The Yamabe invariant of the conformal class $[g]$ (see for instance [Aubin 1982; Hebey and Vaugon 1996; Lee and Parker 1987]) is given by

$$\mu(M,[g]) = \inf_{u \in C^1(M)\setminus\{0\}} \frac{\left(4(n-1)/(n-2) \int_M |\nabla u|^2 \, dv_g + \int_M R_g u^2 \, dv_g\right)}{\left(\int_M u^{2n/(n-2)} \, dv_g\right)^{(n-2)/n}} \tag{4-5}$$

and under the condition $\mathrm{Ric}(M, g) \geq kg$ (with $k > 0$), using the Sobolev inequality of Corollary 3.7 and the fact that $R_g \geq nk$, we obtain the following lower bound of (4-5):

$$\mu(M, [g]) \geq nk(\mathrm{vol}(M))^{2/n}. \tag{4-6}$$

Note that since

$$\mu(M, [g]) \leq \mu(\mathbb{S}^n, [\mathrm{can}]) = n\,(n-1)\,(\mathrm{vol}(\mathbb{S}^n))^{2/n}$$

(see [Aubin 1982]), we obtain the Bishop inequality:

$$\mathrm{vol}_g(M) \leq \Big(\frac{n-1}{k}\Big)^{n/2}\,\mathrm{vol}(\mathbb{S}^n).$$

We also observe that Petean [2005] deduced from (4-6) that if g_0 is the Fubini–Study metric on $\mathbb{C}P^2$ and g is any other metric on $\mathbb{C}P^2$ with $\mathrm{Ric}_g \geq \mathrm{Ric}_{g_0}$ then $\mathrm{vol}_g(\mathbb{C}P^2) \leq \mathrm{vol}_{g_0}(\mathbb{C}P^2)$.

• *The case of a compact Riemannian manifold with boundary*: This case is more complicated than the first one (see [Cherrier 1984; Escobar 1992; Akutagawa 2001]) even if the strategy is the same. In this case the boundary Yamabe invariant is given by

$$\mu(M, [g]) = \inf_{u \in C^1(M)\setminus\{0\}} \frac{\big(4(n-1)/(n-2)\int_M |\nabla u|^2\,dv_g + \int_M R_g u^2\,dv_g + 2\int_{\partial M} H u^2\,dv_g\big)}{\big(\int_M u^{2n/(n-2)}\,dv_g\big)^{(n-2)/n}}, \tag{4-7}$$

where H is the mean curvature of ∂M and dv_g denotes the induced Riemannian measure on ∂M. As before, under the conditions $\mathrm{Ric}(M, g) \geq kg$ (with $k > 0$) and ∂M convex, we deduce since, in this case, $R_g \geq n\,k$ and $H \geq 0$, the following lower bound for the boundary Yamabe invariant, similar to that for the Yamabe invariant:

$$\mu(M, [g]) \geq nk(\mathrm{vol}(M))^{2/n}. \tag{4-8}$$

Note that since

$$\mu(M, [g]) \leq \mu(\mathbb{S}^n_+, [\mathrm{can}]) = n(n-1)(\mathrm{vol}(\mathbb{S}^n_+))^{2/n}$$

(see [Escobar 1992]), we obtain the equivalent of the Bishop inequality when the boundary of the manifold is convex:

$$\mathrm{vol}_g(M) \leq \Big(\frac{n-1}{k}\Big)^{n/2}\,\tfrac{1}{2}\,\mathrm{vol}(\mathbb{S}^n).$$

• *The case of a measured Riemannian space*: Here we consider a weighted Riemannian manifold of dimension $n > 2$, and when the manifold is with boundary, H denotes the mean curvature of its boundary. As we observed in Remark 3.3, the situation is more complicated. We don't know if there's an equivalent of the Yamabe invariant related to our Sobolev inequality of Corollary 3.7. If we consider the following infimum:

$$\mu(M,m,g,\sigma) := \inf_{u\in C^1(M)\setminus\{0\}} \frac{\left(4(m-1)/(m-2)\int_M |\nabla u|^2\,d\sigma+\int_M R_h^m u^2\,d\sigma+2\int_{\partial M} Hu^2\,d\sigma\right)}{\left(\int_M u^{2m/(m-2)}\,d\sigma\right)^{(m-2)/m}}, \tag{4-9}$$

where $R_h^m = \frac{m}{n}\,\mathrm{trace}(\mathrm{Ric}_h^m)$ and as before, if we suppose that $\mathrm{Ric}_h^m \geq kg > 0$ and the boundary is convex in the case where $\partial M \neq \varnothing$, we obtain

$$\mu(M,m,g,\sigma) \geq mk(\mathrm{vol}_h(M))^{2/m}.$$

For an extension of the Yamabe invariant in the case of weighted manifolds, one can consult [Chang et al. 2011; Case 2015].

(IV) *Onofri and logarithmic Sobolev inequalities*: Another interesting application is the Onofri inequality (see for example [Onofri 1982; Beckner 1993]), which appears as an endpoint of various families of interpolation inequalities in dimension two, exactly like Sobolev inequality in higher dimensions. The following corollary gives the analogue of Onofri's inequality on any 2-dimensional compact Riemannian manifold (M^2, g) with positive curvature and convex boundary (see also [Ilias 1983] for nonsharp Onofri inequalities in all dimensions). Since in this case we take $m = 2$, we must have h constant, and finally the measure $d\sigma$ is just a multiple of the Riemannian measure. The inequality being invariant by homothety on the measure, we can restrict ourselves to the Riemannian case.

Corollary 4.1. *Let (M, g) be a compact Riemannian surface of convex boundary and such that $\mathrm{Ric}_g \geq kg$ for a positive constant k. We have for any $\varphi \in H_1^2$,*

$$\log\left(\frac{1}{\mathrm{vol}_g(M)}\int_M e^{\varphi}\,dv_g\right) \leq \frac{1}{\mathrm{vol}_g(M)}\left(\frac{1}{4k}\int_M |\nabla\varphi|^2\,dv_g + \int_M \varphi\,dv_g\right).$$

Proof. For any $p \in (2,\infty)$ and $f \in H_1^2(M)$, (3-33) yields

$$\left(\int_M |f|^p\,dv_g\right)^{\frac{2}{p}} \leq \mathrm{vol}_g(M)^{-\frac{p-2}{p}}\left(\frac{1}{\theta(2,p)}\int_M |\nabla f|^2\,dv_g + \int_M |f|^2\,dv_g\right), \tag{4-10}$$

where $\theta(2,p) \in \{\theta_1(2,p), \theta_2(2,p)\}$ as defined in Theorem 3.6. Proceeding as in [Beckner 1993], if we choose $f = 1+\varphi/p$, then (4-10) gives after applying the logarithm function to both sides

$$2\log\biggl(\int_M \Bigl|1+\frac{\varphi}{p}\Bigr|^p dv_g\biggr) \le 2\log(\mathrm{vol}_g(M)) + p\biggl\{\log\biggl(\frac{1}{\theta(2,p)}\frac{1}{p^2}\int_M |\nabla\varphi|^2\,dv_g + \mathrm{vol}_g(M) + \frac{2}{p}\int_M \varphi\,dv_g + \frac{1}{p^2}\int_M \varphi^2\,dv_g\biggr) - \log(\mathrm{vol}_g(M))\biggr\}.$$

Therefore when p tends to infinity, one can easily see that $1/\theta(2,p)p^2$ converges to zero, and thus the second term on the right hand side of the above equation converges to

$$\frac{1}{\mathrm{vol}_g(M)}\biggl(\frac{1}{2k}\int_M |\nabla\varphi|^2\,dv_g + 2\int_M \varphi\,dv_g\biggr). \qquad \square$$

Remark 4.2. • For any $\varphi \in H_1^2(\mathbb{S}^2)$ (respectively, $\varphi \in H_1^2(\mathbb{S}^2_+)$), Corollary 4.1 gives immediately

$$\log\biggl(\frac{1}{4\pi}\int_{\mathbb{S}^2} e^{\varphi}\,dv\biggr) \le \frac{1}{4\pi}\biggl(\frac{1}{4}\int_{\mathbb{S}^2} |\nabla\varphi|^2\,dv + \int_{\mathbb{S}^2} \varphi\,dv\biggr) \tag{4-11}$$

respectively,

$$\log\biggl(\frac{1}{2\pi}\int_{\mathbb{S}^2_+} e^{\varphi}\,dv\biggr) \le \frac{1}{2\pi}\biggl(\frac{1}{4}\int_{\mathbb{S}^2_+} |\nabla\varphi|^2\,dv + \int_{\mathbb{S}^2_+} \varphi\,dv\biggr), \tag{4-12}$$

where dv is the Riemannian measure of the unit 2-dimensional sphere $\mathbb{S}^2$ (respectively, hemisphere $\mathbb{S}^2_+$), see for instance [Chang and Yang 1988; Onofri 1982; Osgood et al. 1988] for different proofs. It is worth noting that our method (which is inspired by that of Beckner [1993] for the sphere) is the simplest among the existing ones concerning surfaces with boundary.

• We also observe that as in the proofs of [Ilias 1983] for surfaces without boundary, one can use the Levy–Gromov isoperimetric inequality and an adapted symmetrization to deduce the inequality of Corollary 4.1 from that of the 2-sphere.

In the last corollary, we give a logarithmic Sobolev inequality on a compact weighted Riemannian manifold (M, g, σ) of arbitrary dimension (for this kind of inequalities one can see for instance [Gross 1975]). In fact:

Corollary 4.3. *Let (M^n, g, σ) be a compact weighted Riemannian manifold of dimension $n \ge 2$ with convex boundary (i.e., $\mathrm{II} \ge 0$). Assume that for some $m \in [n, \infty)$, the m-Bakry–Émery–Ricci curvature satisfies $\mathrm{Ric}_h^m \ge kg$ for a positive constant k.*

Then for any $f \in H_1^2(d\sigma)$, *we have*:

$$\int_M |f|^2 \log |f|^2 \, d\sigma - \int_M |f|^2 \log\left(\frac{\|f\|^2_{L^2(d\sigma)}}{\mathrm{vol}_h(M)}\right) d\sigma \\ \leq \frac{p}{p-2} \|f\|^2_{L^2(d\sigma)} \log\left(\frac{1}{\theta(m,p)} \frac{\int_M |\nabla f|^2 \, d\sigma}{\|f\|^2_{L^2(d\sigma)}} + 1\right)$$

for any $p \in (2, 2^*]$ *with* $2^* = 2m/(m-2)$ *if* $m \geq 3$ *and* $p \in (2, \infty)$ *if* $m = 2$ (*i.e.*, $n = 2$ *and* h *is constant*) *with* $\theta(m, p)$ *as defined in Theorem 3.6.*

Proof. It is equivalent to prove that

$$\int_M |f|^2 \log |f|^2 \, d\sigma \leq \frac{p}{p-2} \log\left(\frac{1}{\theta(m,p)} \int_M |\nabla f|^2 \, d\sigma + 1\right) - \log(\mathrm{vol}_h(M)),$$

for any $f \in H_1^2(d\sigma)$ with $\|f\|^2_{L^2(d\sigma)} = 1$. From Theorem 3.6, we have

$$\frac{2}{p} \log\left(\int_M |f|^p \, d\sigma\right) \\ \leq -\frac{p-2}{p} \log(\mathrm{vol}_h(M)) + \log\left(\frac{1}{\theta(m,p)} \int_M |\nabla f|^2 \, d\sigma + 1\right). \tag{4-13}$$

Since the logarithmic function is concave, we use Jensen's inequality and the fact that $\int_M |f|^2 \, d\sigma = 1$, to obtain

$$\log\left(\int_M |f|^p \, d\sigma\right) = \log\left(\int_M |f|^{p-2} |f|^2 \, d\sigma\right) \geq \frac{p-2}{2} \int_M |f|^2 \log |f|^2 \, d\sigma.$$

Replacing the above equation in (4-13), we finish the proof of Corollary 4.3. □

Remark 4.4. If we suppose that the weighted measure $d\sigma$ is a probability measure on M, that is $\mathrm{vol}_h(M) = 1$, then one can reformulate Theorem 3.6 as follows:

$$\eta(m,p) \, \frac{F(p) - F(2)}{p-2} \leq \|\nabla f\|^2_{L^2(d\sigma)}, \tag{4-14}$$

where $F(p) = \|f\|^2_{L^p(d\sigma)} = \left(\int_M |f|^p \, d\sigma\right)^{2/p}$ and $\eta(m,p) \in \{\eta_1(m,p), \eta_2(m,p)\}$ with

$$\eta_1(m,p) = \frac{mk}{(m-1)},$$
$$\eta_2(m,p) = \lambda_1^h + \frac{m(m-1)(p-1)}{\big((p-1) + m(m+2)\big)} \left(k - \frac{m-1}{m} \lambda_1^h\right).$$

By taking the limit $p \to 2$ in (4-14), we obtain

$$\eta(m,2) \, F'(2) \leq \int_M |\nabla f|^2 \, d\sigma \tag{4-15}$$

but

$$F'(p) = \frac{2}{p}F(p)^{1-p/2}\int_M |f|^p \log|f|\,d\sigma - \frac{2}{p^2}F(p)\log\biggl(\int_M |f|^p\,d\sigma\biggr).$$

Substituting in (4-15), we obtain the following analogue for weighted Riemannian manifolds of the logarithmic Sobolev inequality:

$$\tfrac{1}{2}\eta(m,2)\biggl(\int_M |f|^2\log|f|^2\,d\sigma - \int_M |f|^2\log\biggl(\int_M |f|^2 d\sigma\biggr)d\sigma\biggr) \le \int_M |\nabla f|^2\,d\sigma.$$

In the case of a compact Riemannian manifold (without boundary), this last inequality was obtained by Fontenas [1997].

Acknowledgments

The authors are very grateful to the referee for his/her careful reading and valuable comments on the first version of this manuscript.

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Received April 17, 2017. Revised November 3, 2017.

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PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PUBLISHED BY

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PACIFIC JOURNAL OF MATHEMATICS

Volume 294 No. 2 June 2018

0030-8730(201806)294:2;1-R