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A CASSELMAN–SHALIKA FORMULA FOR THE GENERALIZED SHALIKA MODEL OF SO_{4n}

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We compute the explicit formula (sometimes called the Casselman–Shalika formula) of the generalized Shalika model for unramified principal series of p-adic SO_{4n}. The method mainly used is the Casselman–Shalika method, modified by Y. Hironaka and applied by Y. Sakellaridis to the case of the Shalika model of GL_{2n}.

1. Introduction

Let $G = \mathrm{SO}_{4n}(F)$, the F-split $4n$-dimensional special orthogonal group, where F is a nonarchimedean local field of characteristic 0.

By P, we denote the Siegel parabolic subgroup of G and by N, the unipotent radical of P. Once we identify G with a subgroup of the isotropy group of the quadratic form defined by

$$\xi = \begin{pmatrix} & \mathbb{1}_{2n} \\ \mathbb{1}_{2n} & \end{pmatrix},$$

N is identified with the subgroup consisting of matrices of the form

$$\begin{pmatrix} \mathbb{1}_{2n} & X \\ & \mathbb{1}_{2n} \end{pmatrix} \quad \text{with } X + {}^tX = 0_{2n}.$$

Let M be the Levi component of P consisting of matrices of the form

$$\begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix} \quad \text{with } A \in \mathrm{GL}_{2n}(F).$$

Jiang and Qin [2007] introduced the notion of a generalized Shalika model for representations of G as follows. Take any nontrivial additive character ψ of F with conductor 0. The expression

$$\psi\left(\tfrac{1}{2}\,\mathrm{tr}(JX)\right)$$

defines a character Ψ on N, where

$$J = \begin{pmatrix} & \mathbb{1}_n \\ -\mathbb{1}_n & \end{pmatrix}.$$

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The stabilizer of this character in M is naturally isomorphic to $\mathrm{Sp}_{2n}(F)$, the symplectic group with respect to J,

$$\mathrm{Sp}_{2n}(F) = \{x \in \mathrm{GL}_{2n}(F) \mid {}^t x J x = J\}.$$

Define the subgroup (called the "generalized Shalika subgroup") H of P by

$$H := \mathrm{Stab}_M(\Psi)N \cong \mathrm{Sp}_{2n}(F) \ltimes N$$

and extend Ψ to a character of H, which will be again denoted by Ψ.

An admissible representation π of G is said to have a generalized Shalika model if there is a nonzero G-morphism from π to $\mathrm{Ind}_H^G(\Psi)$. Because of Frobenius reciprocity, this is equivalent to saying that there is a nonzero H-morphism from π to Ψ.

In this article, we will treat the case of unramified principal series $I(\chi)$ of G and determine a necessary and sufficient condition for $I(\chi)$ to have a generalized Shalika model. Moreover, we will give an explicit formula (a Casselman–Shalika formula) for the spherical vector in the generalized Shalika model of $I(\chi)$.

We will explain our results more precisely. Take any nonzero H-morphism Λ from $I(\chi)$ to Ψ. Let $K = \mathrm{SO}_{4n}(\mathfrak{o})$, the standard maximal compact subgroup of G, where $\mathfrak{o}$ is the ring of integers of F. There is a unique K-invariant vector ϕ_K in $I(\chi)$ which satisfies $\phi_K(1) = 1$. Let $\Omega(g) = \Lambda(R_g \phi_K)$. Our goal is to give an explicit formula for this function Ω.

The Weyl group of G is denoted by W. The main result involves the subgroup Γ of W. Let $\Sigma = \{e_i \pm e_j,\ 1 \le i,\ j \le 2n,\ i \ne j\}$ be the root system of G and $E_i = e_{2i-1} + e_{2i}$. Then, $\Phi = \{E_i - E_j,\ \pm E_k,\ 1 \le i,\ j,\ k \le n,\ i \ne j\}$ is a root system of type B_n and Γ is the Weyl group of Φ realized by the subgroup of W. For each root $\alpha \in \Sigma$, Casselman defined a certain constant $c_\alpha(\chi)$ (see [Casselman 1980, Section 3]). If $\beta \in \Phi$ is a short root, then β is in Σ and a_β is already defined. In this case, let $d_\beta(\chi) = \chi(a_\beta)$. If $\beta = E_i - E_j$ is a long root of Φ, define $a_\beta = a_{e_{2i-1} - e_{2j-1}}$. In this case, let

$$d_\beta(\chi) = \chi(a_\beta) \frac{1 - q^{-2}\chi(a_{-\beta})}{1 - q^{-2}\chi(a_\beta)}.$$

Our main result is as follows.

Theorem 1.1. *For every* $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ *with* $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$,

$$\Omega(g_\lambda) = \prod_{\alpha > 0} c_\alpha(\chi) \sum_{w \in \Gamma} (-1)^{l_\Gamma(w)} (w\chi)^{-1} \delta^{1/2}(h_\lambda) \prod_{\beta > 0,\ w\beta < 0} d_\beta(\chi),$$

where $g_\lambda = \mathrm{diag}(\varpi^{\lambda_1}, \dots, \varpi^{\lambda_n}, 1, \dots, 1)$, $h_\lambda = \mathrm{diag}(\varpi^{\lambda_1}, 1, \varpi^{\lambda_2}, 1, \dots, \varpi^{\lambda_n}, 1) \in M$ *and* l_Γ *is the length function of* Γ.

Note that Ω satisfies $\Omega(hgk) = \Psi(h)\Omega(g)$ for every $h \in H,\ k \in K,\ g \in G$ and hence we only need to compute the value of Ω for representatives $\{g_\lambda\}$ of $H \backslash G / K$.

The method we will use is based on works of Casselman and Shalika (see [Casselman 1980; 1980]) and the outline of this paper is essentially the same as that of [Sakellaridis 2006], where an explicit formula for the Shalika model is given.

2. Preliminaries

Notation. Let F be a nonarchimedean local field of characteristic 0. Let ϖ be a uniformizer, q the order of the residue field, $\mathfrak{o}$ the ring of integers, and $\mathfrak{p}$ the maximal ideal of F.

Let $G = \mathrm{SO}_{4n}(F)$, the F-split $4n$-dimensional special orthogonal group. The group G is identified with the subgroup of $\mathrm{SL}_{4n}(F)$ consisting of matrices satisfying

$$ {}^t g \xi g = \xi, \quad \xi = \begin{pmatrix} & \mathbb{1}_{2n} \\ \mathbb{1}_{2n} & \end{pmatrix}. $$

Denote by $\mathrm{Mat}_{2n}(F)$ the set of matrices of degree $2n$.

By P, we denote the Siegel parabolic subgroup of G, consisting of matrices of the form

$$ \begin{pmatrix} x & \\ & {}^t x^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1}_{2n} & X \\ & \mathbb{1}_{2n} \end{pmatrix} \tag{2-1} $$

with $x \in \mathrm{GL}_{2n}(F)$ and $X \in \mathrm{Mat}_{2n}(F),\ X + {}^t X = 0$. Let N be the unipotent radical of P and M the Levi component with Levi decomposition $P = MN$ as (2-1). We will frequently identify M with $\mathrm{GL}_{2n}(F)$ without notice.

The Bruhat–Tits building of G is denoted by $\mathscr{B}(G)$. Each maximal F-split torus defines an apartment of $\mathscr{B}(G)$. We denote the split maximal torus consisting of diagonal matrices by T and corresponding apartment by $\mathscr{A}(T)$. Fix a special point $o \in \mathscr{A}(T)$ and identify $\mathscr{A}(T)$ with $2n$-dimensional Euclid space with origin o.

Let Σ be the set of roots of G with respect to T. By taking differentials, we identify elements of Σ with linear functions on $\mathfrak{t}$, the Lie algebra of T. We will naturally identify $\mathfrak{t}$ with an F-linear space of diagonal matrices:

$$ \mathfrak{t} = \{\mathrm{diag}(t_1, \dots, t_{2n}, -t_1, \dots, -t_{2n}) \mid t_1, \dots, t_{2n} \in F\}. $$

For $1 \le i \le 2n$, the element e_i of the dual space of $\mathfrak{t}$ is defined by

$$ e_i : \mathrm{diag}(t_1, \dots, t_{2n}, -t_1, \dots, -t_{2n}) \mapsto t_i. $$

Then, under identifications mentioned above, $\Sigma = \{e_i \pm e_j \mid 1 \le i \ne j \le 2n\}$. Let $\Pi = \{\alpha_i := e_i - e_{i+1} \mid 1 \le i \le 2n-1,\ \alpha_{2n} := e_{2n-1} + e_{2n}\}$; this is a basis of the root system Σ. Elements of Σ are regarded as linear functions on $\mathscr{A}(T)$ and the set Σ_{aff} of affine roots of G as a subset of affine functions on $\mathscr{A}(T)$:

$$ \Sigma_{\mathrm{aff}} = \{\alpha + m \mid \alpha \in \Sigma, m \in \mathbb{Z}\}. $$

Let $C = \{x \in \mathscr{A}(T) \mid 0 < \alpha(x) < 1 \text{ for all } \alpha \in \Pi\}$ be an alcove of $\mathscr{B}(G)$. Let B be the Iwahori subgroup of G stabilizing C.

We denote the Weyl group of G by W. By $s_i \in W$, we denote the simple reflection attached to the simple root α_i.

Generalized Shalika model. Following [Jiang and Qin 2007], we define the generalized Shalika model for representations of G as follows. Let $\mathcal{A}$ be the set of nonsingular skew-symmetric matrices of degree $2n$. Take any nontrivial additive character ψ of F with conductor 0 and a skew-symmetric matrix $b \in \mathcal{A}$. The expression

$$\psi\left(\tfrac{1}{2}\operatorname{tr}(bX)\right)$$

defines a character Ψ^b on N. The stabilizer of this character in M is naturally isomorphic to $\mathrm{Sp}_{2n}^b(F)$, the symplectic group with respect to b,

$$\mathrm{Sp}_{2n}^b(F) = \{x \in \mathrm{GL}_{2n}(F) \mid {}^t x b x = b\}.$$

Form a group

$$H^b := \mathrm{Sp}_{2n}^b(F) \ltimes N$$

and extend Ψ^b to a character of H^b, which is again denoted by Ψ^b.

Let $J = \left(\begin{smallmatrix} & \mathbb{1}_n \\ -\mathbb{1}_n & \end{smallmatrix}\right) \in \mathcal{A}$. We will simply denote Ψ^b, H^b and $\mathrm{Sp}_{2n}^b(F)$ by Ψ (or sometimes by Ψ_H), H and $\mathrm{Sp}_{2n}(F)$ when $b = J$.

Definition. Let (π, V) be an irreducible admissible representation of G. We say that π has a generalized Shalika model if $\mathrm{Hom}_{H^b}(\pi, \Psi^b)$ is nonzero for some $b \in \mathcal{A}$.

Nien proved the uniqueness of generalized Shalika models:

Theorem 2.1 [Nien 2010]. *For any irreducible admissible representation π of $\mathrm{SO}_{4n}(F)$ and $b \in \mathcal{A}$,*

$$\dim \mathrm{Hom}_{H^b}(\pi, \Psi^b) \le 1.$$

We will consider the generalized Shalika model for unramified principal series of G. The Borel subgroup of G consisting of matrices in the form of (2-1) with upper triangular $x \in \mathrm{GL}_{2n}(F)$ will be denoted by P_ϕ. Let $\chi = (|\cdot|^{z_1}, |\cdot|^{z_2}|, \dots, |\cdot|^{z_{2n}})$ be an unramified character of P_ϕ (i.e., $\chi : \mathrm{diag}(t_1, \dots, t_{2n}, t_1^{-1}, \dots, t_{2n}^{-1}) \mapsto |t_1|^{z_1} \cdots |t_{2n}|^{z_{2n}}$) and $I(\chi)$ the smooth unramified principal series of G. The representation space of $I(\chi)$ is realized by the space of locally constant functions on G which satisfy

$$f(pg) = \chi\delta^{1/2}(p) f(g)$$

for every $p \in P_\phi,\ g \in G$, where $\delta = (|\cdot|^{4n-2}, |\cdot|^{4n-4}, \dots, |\cdot|^2, |\cdot|^0)$ is the modular character of P_ϕ. Then G acts on this space by right translations R. There is a surjective map $\mathscr{P}_\chi$ to this space from $C_c^\infty(G)$ defined by

$$\mathscr{P}_\chi(f)(g) = \int_{P_\phi} \chi^{-1}\delta^{1/2}(p) f(pg)\, dp$$

for $f \in C_c^\infty(G)$ and $g \in G$. We will always assume that $P_\phi(\mathfrak{o})$ has total measure 1. Let $K = \mathrm{SO}_{4n}(\mathfrak{o})$ be the standard maximal compact subgroup of G and $\phi_K = \phi_{K,\chi}$ be the unique K-invariant element of $I(\chi)$ satisfying $\phi_K(1) = 1$. It is easy to see that ϕ_K is the image under $\mathcal{P}_\chi$ of the characteristic function of K.

Definition. Take a nontrivial element $\Lambda (= \Lambda_H = \Lambda_{H,\chi}) \in \mathrm{Hom}_H(I(\chi), \Psi))$. We define a generalized Shalika function

$$\Omega(g) (= \Omega_H(g) = \Omega_{H,\chi}(g)) = \Lambda(R_g \phi_K).$$

The aim of this paper is to give an explicit formula of this function.

The main results. We will briefly explain the statement of the main results in this subsection. At first, we need to introduce some more notation.

Since the function Ω satisfies

$$\Omega(hgk) = \Psi(h)\Omega(g)$$

for every $h \in H$, $g \in G$ and $k \in K$, it suffices to compute it for a set of double coset representatives in $H \backslash G/K$. By Iwasawa decomposition,

$$H \backslash G/K = H \backslash PK/K \cong \mathrm{Sp}_{2n}(F) \backslash \mathrm{GL}_{2n}(F)/\mathrm{GL}_{2n}(\mathfrak{o}).$$

Considering transitive right action of $\mathrm{GL}_{2n}(F)$ on $\mathcal{A}$ defined by $X * g := {}^t g X g$, we can naturally identify these double cosets with orbits in $\mathcal{A}$ under the action of $\mathrm{GL}_{2n}(\mathfrak{o})$.

Proposition 2.2. *We have the following double coset decomposition*:

$$\mathrm{GL}_{2n}(F) = \bigsqcup_\lambda \mathrm{Sp}_{2n}(F) g_\lambda \mathrm{GL}_{2n}(\mathfrak{o}),$$

where $g_\lambda := \mathrm{diag}(\varpi^{\lambda_1}, \varpi^{\lambda_2}, \dots \varpi^{\lambda_n}, 1, \dots, 1)$ *with* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

Proof. The elementary divisor theorem shows that representatives of orbits in $\mathcal{A}$ under the action of $\mathrm{GL}_{2n}(\mathfrak{o})$ can be taken as follows:

$$X_\lambda = \left(\begin{array}{cccc|cccc} & & & & \varpi^{\lambda_1} & & & \\ & & & & & \varpi^{\lambda_2} & & \\ & & & & & & \ddots & \\ & & & & & & & \varpi^{\lambda_n} \\ \hline -\varpi^{\lambda_1} & & & & & & & \\ & -\varpi^{\lambda_2} & & & & & & \\ & & \ddots & & & & & \\ & & & -\varpi^{\lambda_n} & & & & \end{array}\right).$$

Since $X_\lambda = J * g_\lambda$, we obtain the double coset decomposition. □

By an abuse of notation, we will write g_λ as $\mathrm{diag}(g_\lambda^{-1}, g_\lambda) \in G$. Then we only have to compute $\Omega(g_\lambda)$ for each λ.

Lemma 2.3. *If some λ_i is negative, then $\Omega(g_\lambda) = 0$.*

Proof. Assume that $\lambda_n < 0$ and let $X \in \mathrm{Mat}_{2n}(\mathfrak{o})$ be a matrix whose only nonzero entries are $X_{n,2n} = u$ and $X_{2n,n} = -u$, where $u \in \mathfrak{o}^\times$. Then

$$a = \begin{pmatrix} \mathbb{1}_{2n} & X \\ & \mathbb{1}_{2n} \end{pmatrix}$$

is an element of K and we have $\Omega(g_\lambda) = \Omega(g_\lambda a) = \psi^{-1}(u\varpi^{\lambda_n})\Omega(g_\lambda)$. Since the conductor of ψ is 0, we can choose u so that $\psi(u\varpi^{\lambda_n}) \neq 1$. □

Consequently, we only have to treat the case where λ is a dominant partition of some positive integer. Hereafter, we assume that λ denotes these partitions.

For each $w \in W$, there is an intertwining operator $T_w : I(\chi) \to I(w\chi)$ which satisfies the following relations (see [Casselman 1980]):

$$T_w(\phi_{K,\chi}) = c_w(\chi)\phi_{K,w\chi},$$

where

$$c_w(\chi) = \prod_{\alpha>0,\, w\alpha<0} c_\alpha(\chi), \qquad c_\alpha(\chi) = \frac{1 - q^{-1}\chi(a_\alpha)}{1 - \chi(a_\alpha)}.$$

Here α is a root of G and a_α is a diagonal matrix attached to α. For details, see [Casselman 1980]. Taking the adjoint, we get a G-morphism $T_w^* : I(w\chi)^* \to I(\chi)^*$, where * denotes the dual space of a complex linear space.

Denote the space of distributions on G by $\mathfrak{D}(G)$. By $\mathcal{P}_\chi : C_c^\infty(G) \to I(\chi)$, we obtain the adjoint G-morphism $\mathcal{P}_\chi^* : I(\chi)^* \to \mathfrak{D}(G)$. Let $\Delta(= \Delta_H = \Delta_{H,\chi}) := \mathcal{P}_\chi^*(\Lambda) \in \mathfrak{D}(G)$. Based on the work of Sakellaridis [2006] (also see [Casselman 1980] and [Hironaka 1999]), we get

$$\Omega(g) = Q^{-1} \sum_w \Biggl(\prod_{\alpha>0,\, w\alpha>0} c_\alpha(\chi) \Biggr) T_{w^{-1}}^* \Delta(R_g \,\mathrm{ch}_B), \tag{2-2}$$

where ch_B denotes the characteristic function of B, Q the volume of Bw_lB and w_l is the longest element of W. Hence the problem is reduced to computing $T_{w^{-1}}^* \Delta(R_g \,\mathrm{ch}_B)$ for $w \in W$ and $g = g_\lambda$.

The statement of our formula involves the subgroup Γ of W, which is isomorphic to the Weyl group of type B_n, and its root system. Therefore, let us fix some notation.

Let $E_i = e_{2i-1} + e_{2i}$, $\beta_i = E_i - E_{i+1}$ $(1 \le i < n)$ and $\beta_n = E_n$. Then, $\Phi := \{E_i - E_j, \pm E_k \mid 1 \le i, j, k \le n, i \neq j\}$ is a root system of type B_n and $\{\beta_i \mid 1 \le i \le n\}$ is a basis of Φ.

The subgroup Γ is generated by

$$\begin{array}{c} \\ w_i := \end{array} \begin{array}{c} \begin{array}{ccccc} & & i & & \\ & & \vee & & \end{array} \\ \begin{pmatrix} \mathbb{1}_2 & & \vdots & & & \\ & \ddots & \vdots & & & \\ & & 0_2 & \mathbb{1}_2 & & \\ & & \mathbb{1}_2 & 0_2 & & \\ & & & & \ddots & \\ & & & & & \mathbb{1}_2 \end{pmatrix} \end{array} \in M, \quad (1 \le i \le n-1)$$

and

$$w_n := \left(\begin{array}{cc|cc} \mathbb{1}_{2(n-1)} & & & \\ & 0_2 & & \varepsilon \\ \hline & & \mathbb{1}_{2(n-1)} & \\ & \varepsilon & & 0_2 \end{array}\right) \in G, \quad \text{where } \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that Γ is naturally identified with the Weyl group of the root system Φ and under this identification, w_i is the simple reflection corresponding to β_i.

Definition. For each long root $\beta = E_i - E_j \in \Phi$, let $a_\beta = a_{e_{2i-1}-e_{2j-1}}$. For a short root $\beta \in \Phi$, a_β is already defined since $\beta \in \Sigma$.

We define $d_\beta(\chi)$ for each $\beta \in \Phi$ as follows: if β is a short root,

$$d_\beta(\chi) = \chi(a_\beta)$$

and if β is a long root,

$$d_\beta(\chi) = \chi(a_\beta)\frac{1-q^{-2}\chi(a_{-\beta})}{1-q^{-2}\chi(a_\beta)}.$$

Our main theorem is as follows.

Theorem 2.4. *Let $\chi = (|\cdot|^{z_1}, |\cdot|^{z_2}, \ldots, |\cdot|^{z_{2n}})$ be an unramified character on P_ϕ and assume that this character satisfies $z_{2i-1} = 1 + z_{2i}$ for all $a \le i \le n$.*

(i) *If χ is not of the form as above (or its W-translate), then $I(\chi)$ does not have a generalized Shalika model.*

(ii) *For every $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$,*

$$\Omega(g_\lambda) = Q^{-1} \prod_{\alpha>0} c_\alpha(\chi) \sum_{w\in\Gamma} (-1)^{l_\Gamma(w)} (w\chi)^{-1} \delta^{1/2}(h_\lambda) \prod_{\substack{\beta>0 \\ w\beta<0}} d_\beta(\chi),$$

where $h_\lambda = \mathrm{diag}(\varpi^{\lambda_1}, 1, \varpi^{\lambda_2}, 1, \ldots, \varpi^{\lambda_n}, 1) \in M$ and l_Γ is the length function of Γ.

3. The open coset

In this section, we will determine which double cosets in $P_\phi \backslash G/H$ are open (if they exist). We don't analyze this quotient space directly but consider $P_\phi \backslash G/P$, which is easily described by using Weyl groups. Since the unique open coset in $P_\phi \backslash G/P$ is $P_\phi \xi P$, the open cosets in $P_\phi \backslash G/H$ are in this coset (if they exist). So we will treat the following quotient space: $P_\phi \backslash P_\phi \xi P/H \cong (\xi^{-1}P_\phi\xi \cap P)\backslash P/H \cong P_0 \backslash G_0/H_0$, where $G_0 = \mathrm{GL}_{2n}(F)$, $H_0 = \mathrm{Sp}_{2n}(F)$ and P_0 is the Borel subgroup of G_0 consisting of lower triangular matrices.

The transitive left action of G_0 on $\mathcal{A}$ is defined by $g * X := gX{}^tg$. Then there is a natural surjective map θ from G_0 to $\mathcal{A}$ defined by $\theta(g) = g * J$. For $X \in G_0$ and each $1 \le i \le n$, X_i denotes the top left $2i \times 2i$-block and $d_i(X)$ its Pfaffian. Let $\mathcal{A}' = \{X \in \mathcal{A} \mid d_i(X) \neq 0\,(1 \le i \le n)\}$ be an open set in $\mathcal{A}$. We will show that the inverse image of this set under the map θ is a double coset in $P_0 \backslash G_0/H_0$. Identifying W_0, the Weyl group of G_0, with the symmetric group of degree $2n$, define the element w_0 of W_0 as a permutation such that

$$w_0(i) = \begin{cases} 2i-1 & (1 \le i \le n) \\ 2i-2n & (n+1 \le i \le 2n). \end{cases}$$

Let $\varepsilon = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. Then

$$\theta(w_0) = w_0 * J = \begin{pmatrix} \varepsilon & & & \\ & \varepsilon & & \\ & & \ddots & \\ & & & \varepsilon \end{pmatrix} \in \mathcal{A}'.$$

Lemma 3.1. $\mathcal{A}' = \theta(P_0 w_0 H_0)$. *In particular*, $P_0 w_0 H_0 = \theta^{-1}(\mathcal{A}')$ *is open.*

Proof. Since

$$p * X = \begin{pmatrix} p_i & \\ * & * \end{pmatrix}\begin{pmatrix} X_i & * \\ * & * \end{pmatrix}\begin{pmatrix} {}^tp_i & * \\ & * \end{pmatrix} = \begin{pmatrix} p_i * X_i & * \\ * & * \end{pmatrix},$$

we have $d_i(p * X) = \det(p_i)^2 d_i(X) \neq 0$ and this shows that $\mathcal{A}'$ is P_0-stable. Hence $\theta(P_0 w_0 H_0) \subset \mathcal{A}'$. By induction on i, we have to show that if X_i is of the form

$$\begin{pmatrix} \varepsilon & & \\ & \ddots & \\ & & \varepsilon \end{pmatrix},$$

there is a lower triangular matrix $p \in \mathrm{GL}_{2(i+1)}(F)$ such that

$$p * X_{i+1} = \begin{pmatrix} \varepsilon & & \\ & \ddots & \\ & & \varepsilon \end{pmatrix}.$$

Let A_j be 2×2-matrices and assume that X_{i+1} is expressed as

$$\begin{pmatrix} \varepsilon & & & A_1 \\ & \ddots & & \vdots \\ & & \varepsilon & A_i \\ -{}^tA_1 & \cdots & -{}^tA_i & A_{i+1} \end{pmatrix}.$$

Let $p_1 \in \mathrm{GL}_{2(i+1)}(F)$ denote a lower triangular matrix of the form

$$\begin{pmatrix} \mathbb{1}_2 & & & \\ & \ddots & & \\ & & \mathbb{1}_2 & \\ B_1 & \cdots & B_i & \mathbb{1}_2 \end{pmatrix},$$

where $B_j := {}^tA_j\varepsilon^{-1}$. Then there is a skew-symmetric matrix $C \in \mathrm{GL}_2(F)$ satisfying

$$p_1 * X_{i+1} = \begin{pmatrix} \varepsilon & & & \\ & \ddots & & \\ & & \varepsilon & \\ & & & C \end{pmatrix}.$$

It is clear that there is a diagonal matrix p_2 of degree $2(i+1)$ so that $(p_2p_1)*X_{i+1}$ becomes the desired form. $\square$

Remark. From the proof of Lemma 3.1, we easily obtain the following slight refinement. For any $X \in \mathcal{A}'$, there is a $p \in P_0$ with diagonal component $(c_1, \ldots, c_{2n})$ which sends X to $\theta(w_0)$ and satisfies

$$c_{2i} = 1, \quad c_{2i+1} = \frac{d_{i+1}(X)}{d_i(X)}.$$

Let

$$B_0 = \begin{pmatrix} \mathfrak{o}^\times & & \mathfrak{p} \\ & \ddots & \\ \mathfrak{o} & & \mathfrak{o}^\times \end{pmatrix}$$

be the standard Iwahori subgroup corresponding to P_0 and

$$Y_\lambda = \begin{pmatrix} \varpi^{\lambda_1}\varepsilon & & 0 \\ & \ddots & \\ 0 & & \varpi^{\lambda_n}\varepsilon \end{pmatrix}$$

be an element of $\mathcal{A}'$.

Lemma 3.2. *For all $b \in B_0$ and λ, $1 \le i \le n$,*

$$|d_i(b * Y_{-\lambda})| = |d_i(Y_{-\lambda})|.$$

In particular, $\mathcal{A}' = P_0B_0 * Y_{-\lambda} = \theta(P_0B_0w_0H_0)$.

Proof. By Lemma 3.1, $\mathcal{A}' = P_0 * Y_{-\lambda} \subset P_0 B_0 * Y_{-\lambda} = \theta(P_0 B_0 w_0 H_0)$. The other inclusion follows once we prove the first equation. This is clear for elements in $P_0 \cap B_0$. Thus, by Iwahori decomposition, it suffices to prove this equation for elements in

$$N_0 := \begin{pmatrix} 1 & & \mathfrak{p} \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

This will be proved by induction on the size of matrices. Let $n \in N_0$ and $X = n * Y_{-\lambda}$. Then for $1 \le i \le n-1$,

$$X_i \in n_i * (Y_{-\lambda})_i + \varpi^{-\lambda_{i+1}+2} \mathrm{M}_{2i}(\mathfrak{o}).$$

Since $-\lambda_i \le -\lambda_{i+1}$, any component of $n_i * (Y_{-\lambda})_i$ does not lie in $\varpi^{-\lambda_{i+1}+2}\mathfrak{o} = \mathfrak{p}^{-\lambda_{i+1}+2}$. Hence we see by induction hypothesis,

$$|d_i(X)| = |\det X_i|^{1/2} = |\det(n_i * (Y_{-\lambda})_i)|^{1/2} = |d_i(Y_{-\lambda})|. \qquad \square$$

From the two lemmas above, we have $\theta(P_0 w_0 H_0) = \mathcal{A}' = P_0 B_0 * Y_{-\lambda}$. Let $h_\lambda = \mathrm{diag}(\varpi^{\lambda_1}, 1, \varpi^{\lambda_2}, 1, \dots, \varpi^{\lambda_n}, 1)$. Then $\theta(h_\lambda w_0) = Y_\lambda$ and $P_0 w_0 H_0 = P_0 B_0 h_{-\lambda} w_0 H_0$. In other words, $B_0 w_0 g_{-\lambda} \subset P_0 w_0 H_0$ since $g_{-\lambda} = w_0^{-1} h_{-\lambda} w_0$.

Lemma 3.3. *For all* λ, $B \xi w_0 g_{-\lambda} \subset P_\phi \xi w_0 H$.

Proof. Identifying G_0 with M by

$$g \mapsto \begin{pmatrix} g & \\ & {}^t g^{-1} \end{pmatrix},$$

we see that $\xi P_0 \xi^{-1} \subset P_\phi$, $H_0 \subset H$ and $\xi B_0 \xi^{-1} \subset B$. From the previous argument, $P_\phi \xi B_0 w_0 g_{-\lambda} H \subset P_\phi \xi P_0 w_0 H_0 H$.

Since ξ and w_0 are commutative, we obtain

$$P_\phi \xi P_0 w_0 H_0 H = P_\phi (\xi P_0 \xi^{-1}) \xi w_0 H = P_\phi \xi w_0 H.$$

On the other hand, $P_\phi \xi B_0 w_0 g_{-\lambda} H = P_\phi (\xi B_0 \xi^{-1}) \xi w_0 g_{-\lambda} H$. By Iwahori decomposition, $B = (B \cap P_\phi)(\xi B_0 \xi^{-1})(B \cap \xi N \xi^{-1})$ and since $w_0 g_{-\lambda} \in G_0$, $\xi N \xi^{-1} = (\xi w_0 g_{-\lambda}) N (\xi w_0 g_{-\lambda})^{-1}$. Therefore, $P_\phi (\xi B_0 \xi^{-1}) \xi w_0 g_{-\lambda} H = P_\phi B \xi w_0 g_{-\lambda} H$ and the desired inclusion follows. $\square$

Let $\eta = \xi w_0$, $S = \eta H \eta^{-1}$. Hereafter, we will treat S instead of H and so we need to translate all things defined above as follows:

$$\Psi_S(s) = \Psi_H(\eta^{-1} s \eta), \qquad \Lambda_S = \Lambda_H \circ R_{\eta^{-1}} \in \mathrm{Hom}_S(I(\chi), \Psi_S),$$
$$\Delta_S = \mathcal{P}_\chi^*(\Lambda_S) \in \mathfrak{D}(G), \quad \Omega_S(g) = \Omega_H(\eta^{-1} g) = \Omega_H(\eta^{-1} g \eta).$$

We have to compute $\Omega_H(g_\lambda) = \Omega_S((\xi w_0)g_\lambda(\xi w_0)^{-1}) = \Omega_S(h_{-\lambda})$. Since by Lemma 3.3, we have $\mathrm{supp}(R_{h_{-\lambda}}\,\mathrm{ch}_B) = Bh_\lambda \subset P_\phi S$, and taking (2-2) into consideration, we obtain the following result:

Proposition 3.4. *Let $\chi = (|\cdot|^{z_1}, |\cdot|^{z_2}, \dots, |\cdot|^{z_{2n}})$ be an unramified character on P_ϕ and assume that this character satisfies $z_{2i-1} = 1 + z_{2i}$ for all $1 \le i \le n$.*

(i) *If χ is not of the form above (or its W-translate), then $I(\chi)$ does not have a generalized Shalika model.*

(ii) *For $w \notin \Gamma$, we have $T^*_{w^{-1}}\Delta_S(R_{h_{-\lambda}}\,\mathrm{ch}_B) = 0$ for every λ, where Γ is the subgroup of W generated by*

$$w_i := \begin{array}{c} \begin{matrix} & & \overset{i}{\vee} & & & \end{matrix} \\ \begin{pmatrix} \mathbb{1}_2 & & \vdots & & & \\ & \ddots & \vdots & & & \\ & & 0_2 & \mathbb{1}_2 & & \\ & & \mathbb{1}_2 & 0_2 & & \\ & & & & \ddots & \\ & & & & & \mathbb{1}_2 \end{pmatrix} \end{array} \in G_0,\ (1 \le i \le n-1) \textit{ and}$$

$$w_n := \left(\begin{array}{cc|cc} \mathbb{1}_{2(n-1)} & & & \\ & 0_2 & & \varepsilon \\ \hline & & \mathbb{1}_{2(n-1)} & \\ & \varepsilon & & 0_2 \end{array}\right) \in G.$$

Proof. (essentially the same as [Sakellaridis 2006, Proposition 5.2])

(i) Let $I_S(w\chi)$ be the subspace of $I(w\chi)$ consisting of elements supported in $P_\phi S$. Then the restriction map induces an isomorphism $I_S(w\chi) \to \text{c-ind}^S_{P_\phi\cap S}(w\chi)\delta^{1/2}$. On the other hand, there is a surjective map $\mathcal{P}_r : C_c^\infty(S) \to \text{c-ind}^S_{P_\phi\cap S}(w\chi)\delta^{1/2}$ defined by

$$\mathcal{P}_r(f)(s) = \int_{P_\phi\cap S} (w\chi)\delta^{1/2}(p)^{-1} f(ps)\, d_r p$$

where $d_r p$ is a right Haar measure on $P_\phi \cap S$. Composed with these maps, $T^*_{w^{-1}}\Lambda_S$ can be taken as a distribution on S. Then $\Psi_S \cdot T^*_{w^{-1}}\Lambda_S$ is a right S-invariant distribution, which must be a Haar measure on S:

$$T^*_{w^{-1}}\Lambda_S = \Psi_S^{-1}\, ds.$$

For $x \in P_\phi \cap S$, $f \in C_c^\infty(S)$,

$$\begin{aligned}(w\chi)\delta^{1/2}\delta_{P_\phi\cap S}^{-1}(x)\int_S f(s)T^*_{w^{-1}}\Lambda_S(s)\,ds &= \int_S f(xs)T^*_{w^{-1}}\Lambda_S(s)\,ds\\ &= \int_S f(xs)\Psi_S^{-1}(s)\,ds\\ &= \Psi_S(x)\int_S f(s)\Psi_S^{-1}(s)\,ds\\ &= \Psi_S(x)\int_S f(s)T^*_{w^{-1}}\Lambda_S(s)\,ds,\end{aligned}$$

where $\delta_{P_\phi\cap S}$ is the modular character of $P_\phi\cap S$. Since $P_\phi\cap S$ consists of matrices of the form

$$p = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n\end{pmatrix}, \quad A_i = \begin{pmatrix} a_i & b_i \\ & a_i^{-1}\end{pmatrix} \in \mathrm{SL}_2(F)$$

and is contained in $G_0 \cong M$, Ψ_S is trivial on $P_\phi\cap S$. So we have

$$\delta_{P_\phi\cap S}(x) = (w\chi)\delta^{1/2}(x) \tag{3-1}$$

for all $x \in P_\phi \cap S$.

An easy calculation shows that $\delta_{P_\phi\cap S}(p) = \delta(p) = \prod_i |a_i|^2$ and hence we get $(w\chi)(p) = \prod_i |a_i|$. If we put $w\chi = (|\cdot|^{z_1},\dots,|\cdot|^{z_{2n}})$, then $(w\chi)(p) = \prod_i |a_i|^{z_{2i-1}-z_{2i}}$ and so it is necessary for the existence of a generalized Shalika model that $z_{2i-1}-z_{2i}=1$ for every $1\le i\le n$.

(ii) Note that Γ is isomorphic to the Weyl group of type B_n, in particular, it is a Coxeter group. It is easy to see that Γ consists of elements which preserves the condition (3-1) and the claim follows immediately. □

This proposition proves the first half of the main theorem. Throughout this paper, assume that χ satisfies the conditions stated in Proposition 3.4.

Since $\chi\delta^{1/2}\delta_{P_\phi\cap S}^{-1} = 1$ on $P_\phi\cap S$ and S is unimodular, there exists a nonzero right S-invariant linear functional $I : \text{c-ind}_{P_\phi\cap S}^S \chi\delta^{1/2} \to \mathbb{C}$ (where the action of S on $\mathbb{C}$ is trivial). We habitually use an integral expression

$$I(\varphi) = \int_{P_\phi\cap S\backslash S} \varphi(s)\,d\dot{s}$$

for $\varphi \in \text{c-ind}_{P_\phi\cap S}^S \chi\delta^{1/2}$. Note that this is not an integral in the usual sense since "integrands" are twisted by characters. This functional is uniquely determined by right S-invariance up to a positive constant factor (see [Bushnell and Henniart 2006, Proposition 3.4]). For an element φ of $\text{c-ind}_{P_\phi\cap S}^S \chi\delta^{1/2}$, $\varphi\cdot\Psi_S^{-1}$ is also an element of $\text{c-ind}_{P_\phi\cap S}^{S}\,\chi\delta^{1/2}$ and it follows that

$$\int_{P_\phi\cap S\backslash S}\varphi(s)\Psi_S^{-1}(s)\,d\dot{s}$$

is well defined. On the other hand, $\mathcal{P}_r^* I$ is a right S-invariant distribution on S, which is a Haar measure on S. Therefore, by the argument in the proof of Proposition 3.4,

$$\mathcal{P}_r^*\Lambda_S=\Psi_S^{-1}\mathcal{P}_r^* I=\mathcal{P}_r^*(\Psi_S^{-1}I).$$

In other words, the restriction of Λ_S to $I_S(\chi)$ has an integral expression:

Lemma 3.5. *For $\varphi\in I_S(\chi)$,*

$$\Lambda_S(\varphi)=\int_{P_\phi\cap S\backslash S}\varphi(s)\Psi_S^{-1}(s)\,d\dot{s}. \tag{3-2}$$

In a similar way, using uniqueness of invariant distributions and the linear functional $C_c^\infty(P_\phi\times S)\to C_c^\infty(P_\phi S)$ defined by

$$P_\phi S\ni ps\mapsto\int_{p_\phi\cap S}f(px^{-1},xs)\,d_rx,\qquad f\in C_c^\infty(P_\phi\times S),$$

we obtain the following result:

Lemma 3.6. *The map $\Theta_\chi:P_\phi S\to\mathbb{C}$ defined by $\Theta_\chi(ps)=\chi^{-1}\delta^{1/2}(p)\Psi_S^{-1}(s)$ for $ps\in P_\phi S$ is well defined and for every $f\in C_c^\infty(P_\phi S)$ and*

$$\Delta_S(f)=\int_{P_\phi S}\Theta_\chi(x)f(x)\,dx, \tag{3-3}$$

where dx is a suitably normalized Haar measure on G.

Proposition 3.7. *Assume that* $\operatorname{Re} z_i>0$ *for all i. Then* (3-2) *converges absolutely for every $\varphi\in I(\chi)$.*

Proof. (essentially the same as [Sakellaridis 2006, Proposition 7.1])

We will treat Λ_H in place of Λ_S. The equation (3-2) is equivalent to saying that for every $\varphi\in I(\chi)$ with support contained in $P_\phi\eta H$,

$$\Lambda_H(\varphi)=\int_{{}_HP_\phi\backslash H}\varphi(\eta h)\Psi_H^{-1}(h)\,d\dot{h}. \tag{3-4}$$

Here, ${}_HP_\phi=\eta^{-1}P_\phi\eta\cap H$. Hence we need to prove that (3-4) converges absolutely for every $\varphi\in I(\chi)$.

Since every element of $I(\chi)$ is dominated by some multiple of ϕ_K, it suffices to treat the case $\varphi=\phi_K$. By Iwasawa decomposition and K-invariance of ϕ_K, (3-4) is reduced to

$$\int \phi_K\left(\eta \begin{pmatrix} \mathbb{1}_{2n} & X \\ & \mathbb{1}_{2n} \end{pmatrix} \begin{pmatrix} m & \\ & {}^t m^{-1} \end{pmatrix}\right) \psi\left(\tfrac{1}{2}\operatorname{tr}(JX)\right) dX\, dm,$$

where X is a skew-symmetric matrix and

$$m = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1}_{2n} & Y \\ & \mathbb{1}_{2n} \end{pmatrix} \in \mathrm{Sp}_{2n}(F)$$

with an upper triangular unipotent matrix $a \in \mathrm{GL}_{2n}(F)$ and a symmetric matrix $Y \in \mathrm{Mat}_n(F)$. The integral over a is taken modulo matrices of the form

$$\begin{pmatrix} \mathbb{1}_n & b \\ & \mathbb{1}_n \end{pmatrix} \in \mathrm{GL}_{2n}(F),$$

where $b \in \mathrm{Mat}_n(F)$ is a diagonal matrix. Then

$$\begin{aligned} &\int \phi_K\left(\eta \begin{pmatrix} \mathbb{1}_{2n} & X \\ & \mathbb{1}_{2n} \end{pmatrix} \begin{pmatrix} m & \\ & {}^t m^{-1} \end{pmatrix}\right) \psi\left(\tfrac{1}{2}\operatorname{tr}(JX)\right) dX\, d\dot{m} \\ &\qquad = \int \phi_K\left(\eta \begin{pmatrix} m & \\ & {}^t m^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1}_{2n} & m^{-1}X\,{}^t m^{-1} \\ & \mathbb{1}_{2n} \end{pmatrix}\right) \psi\left(\tfrac{1}{2}\operatorname{tr}(JX)\right) dX\, d\dot{m}. \end{aligned}$$

Then $m^{-1}X\,{}^t m^{-1}$ can be replaced by X since H is unimodular and $m \in \mathrm{Sp}_{2n}(F)$.

Since $\phi_K \in I(\chi)$, m on the left factor can be assumed to be of the form

$$\begin{pmatrix} c & d \\ & \mathbb{1}_n \end{pmatrix},$$

with an upper triangular unipotent matrix $c \in \mathrm{GL}_n(F)$ and an upper triangular nilpotent matrix $d \in \mathrm{Mat}_n(F)$ (here, the integral is taken in the usual sense, not in that of Lemma 3.5). Therefore, the integral above is dominated absolutely by the integral representing the intertwining operator T_η (see [Casselman 1995, Lemma 6.4.2]), which converges absolutely when $\operatorname{Re} z_i > 0$ for all i by [Casselman 1980, Lemma 3.2]. □

Thanks to Proposition 3.7, exactly the same argument given in [Sakellaridis 2006, Section 7] suggests that for any $f \in C_c^\infty(G)$, $\Delta_{S,\chi}(f)$ is a rational function of χ.

4. End of calculations

Normalize the Haar measure on G so that $\mathrm{vol}(B) = 1$.

Lemma 4.1. *For any* λ, $\Delta_S(\mathrm{ch}_{Bh_\lambda}) = \chi^{-1}\delta^{1/2}(h_\lambda)$.

Proof. Since $Bh_\lambda \subset P_\phi S$ and (3-3), $\Delta_S(\mathrm{ch}_{Bh_\lambda}) = \int_{Bh_\lambda} \Theta_\chi(x)\, dx$. Using Iwahori decomposition of B and B_0, every $b \in B$ can be expressed in the form $b = pqr$, where

$$p = \begin{pmatrix} \mathbb{1}_{2n} & * \\ & \mathbb{1}_{2n} \end{pmatrix} \in B \cap N \subset P_\phi, \quad q = \begin{pmatrix} * & 0_{2n} \\ 0_{2n} & * \end{pmatrix} \in \xi B_0 \xi^{-1},$$
$$r = \begin{pmatrix} \mathbb{1}_{2n} & \\ * & \mathbb{1}_{2n} \end{pmatrix} \in B \cap \xi N \xi^{-1}.$$

Then

$$bh_\lambda = pqr\eta g_{-\lambda}\eta^{-1} = p\xi \cdot \underbrace{\xi^{-1} q \xi}_{\in B_0} \cdot w_0 g_{-\lambda} \cdot \underbrace{(\eta g_{-\lambda})^{-1} r (\eta g_{-\lambda})}_{\in N} \cdot \eta^{-1}.$$

Since $B_0 w_0 g_{-\lambda} \subset P_0 w_0 H_0$, there are $p_0 \in P_0$ and $h_0 \in H_0$ satisfying $b_0 w_0 g_{-\lambda} = p_0 w_0 h_0$, where $b_0 := \xi^{-1} q \xi$. In other words,

$$b_0 Y_{-\lambda} {}^t b_0 = \theta(b_0 w_0 g_{-\lambda}) = \theta(p_0 w_0 h_0) = p_0 Y_0 {}^t p_0 =: X \in \mathcal{A}.$$

By this and Lemma 3.2,

$$|d_i(X)| = |d_i(b_0 * Y_{-\lambda})| = |d_i(Y_{-\lambda})| = |d_i(p_0 * Y_0)| = |\det(p_0)_i|.$$

Denote the diagonal component of p_0 by $(c_1, \ldots, c_{2n})$. Then we have $|d_i(X)| = \prod_{j=1}^{2i} |c_j| = q^{\lambda_1 + \cdots + \lambda_i}$ and therefore the remark on page 481 shows that p_0 can be chosen so that $c_{2i} = 1,\ \ |c_{2i-1}| = q^{\lambda_i}$ for each i.

Let $n_0 = (\eta g_{-\lambda})^{-1} r (\eta g_{-\lambda})$. Then

$$bh_\lambda = p\xi p_0 w_0 h_0 n_0 \eta^{-1} = \underbrace{p \cdot \xi p_0 \xi^{-1}}_{\in P_\phi} \cdot \underbrace{\eta h_0 n_0 \eta^{-1}}_{\in S}$$

Hence,

$$\Theta_\chi(bh_\lambda) = \chi^{-1}\delta^{1/2}(p\xi p_0 \xi^{-1}) \Psi_H(h_0 n_0) = \prod_{i=1}^{n} q^{-(2n-2i+1-z_{2i-1})\lambda_i} \Psi_H(n_0)$$

Express r in the form

$$\begin{pmatrix} \mathbb{1}_{2n} & \\ X & \mathbb{1}_{2n} \end{pmatrix},$$

where X is an element of $\mathrm{Mat}_{2n}(\mathfrak{p})$. Since

$$n_0 = (w_0 g_{-\lambda})^{-1} \begin{pmatrix} \mathbb{1}_{2n} & X \\ & \mathbb{1}_{2n} \end{pmatrix} (w_0 g_{-\lambda}) = \begin{pmatrix} \mathbb{1}_{2n} & g_\lambda {}^t w_0 X w_0 g_\lambda \\ & \mathbb{1}_{2n} \end{pmatrix}$$

and the conductor of ψ is assumed to be 0,

$$\Psi_H(n_0) = \psi\big(\tfrac{1}{2}\,\mathrm{tr}(J g_\lambda {}^t w_0 X w_0 g_\lambda)\big) = \psi\big(\tfrac{1}{2}\,\mathrm{tr}(X \cdot Y_\lambda)\big) = 1.$$

Some additional simple computations show that $\Delta_S(\mathrm{ch}_{Bh_\lambda}) = \chi^{-1}\delta^{1/2}(h_\lambda)$. $\square$

Proposition 4.2.

$$\Omega_S(h_{-\lambda}) = Q^{-1} \sum_{w\in\Gamma} \Bigl(\prod_{\substack{\alpha>0 \\ w\alpha>0}} c_\alpha(\chi) \Bigr) (w\chi)^{-1} \delta^{1/2}(h_\lambda) T^*_{w^{-1}} \Delta_{S,\chi}(\mathrm{ch}_B).$$

Proof. By the uniqueness of the generalized Shalika model, $T^*_{w^{-1}}\Lambda_{S,\chi}$ is a scalar multiple of $\Lambda_{S,w\chi}$. Hence,

$$\begin{aligned}\frac{T^*_{w^{-1}}\Delta_{S,\chi}(R_{h_{-\lambda}}\,\mathrm{ch}_B)}{T^*_{w^{-1}}\Delta_{S,\chi}(\mathrm{ch}_B)} = \frac{T^*_{w^{-1}}\Lambda_{S,\chi}(R_{h_{-\lambda}}\phi_B)}{T^*_{w^{-1}}\Lambda_{S,\chi}(\phi_B)} &= \frac{\Lambda_{S,w\chi}(R_{h_{-\lambda}}\phi_B)}{\Lambda_{S,w\chi}(\phi_B)} \\ &= \frac{\Delta_{S,w\chi}(R_{h_{-\lambda}}\,\mathrm{ch}_B)}{\Delta_{S,w\chi}(\mathrm{ch}_B)} \\ &= (w\chi)^{-1}\delta^{1/2}(h_\lambda).\end{aligned}$$

Applying this to (2-2), the desired result follows. □

We denote the length function of W by l and that of Γ by l_Γ. The following lemma suggests that we only have to treat the case $w = w_i$ in the notation of Proposition 3.4.

Lemma 4.3. *For $w, w' \in \Gamma$, $l_\Gamma(ww') = l_\Gamma(w) + l_\Gamma(w')$ implies that $l(ww') = l(w) + l(w')$.*

Notice that a reduced expression of each w_i is given as follows:

$$w_i = s_{2i}s_{2i-1}s_{2i+1}s_{2i},\ (1 \le i \le n-1), \quad w_n = s_{2n}.$$

Following [Casselman 1980], we denote $\mathcal{P}_\chi(\mathrm{ch}_{BwB})$ by $\phi_{w,\chi}$ for each $w \in W$.

Let N_ϕ be the unipotent radical of P_ϕ and N_ϕ^- be that of the opposite of P_ϕ. For $\alpha \in \Sigma$, N_ϕ^α (resp. $N_\phi^{-,\alpha}$) will denote the image of standard embedding $F \to N_\phi$ (resp. $F \to N_\phi^-$) corresponding to α. We will use $N_\phi^{\hat\alpha}$ (resp. $N_\phi^{-,\widehat{-\alpha}}$) to denote the product (in any order) of all N_ϕ^β (resp. $N_\phi^{-,-\beta}$), $(0 < \beta \neq \alpha)$. Similarly, for a subset $\Sigma' \subset \Sigma$, we define $N_\phi^{\Sigma'}$, $N_\phi^{\widehat{\Sigma'}}$, etc. Let $P_\phi^\alpha = T \cdot N_\phi^\alpha$ and so on.

We use the following fundamental equation of intertwining operators T_w and functions ϕ (see [Casselman 1980, Theorem 3.4]): for each simple reflection s_k and $w \in W$ with $l(s_k w) = l(w) + 1$, we have

$$T_{s_k}(\phi_{w,s_k\chi}) = (c_{\alpha_k}(s_k\chi) - 1)\phi_{w,\chi} + q^{-1}\phi_{s_kw,\chi}, \tag{4-1}$$

$$T_{s_k}(\phi_{w,s_k\chi}) = \phi_{w,\chi} + (c_{\alpha_k}(s_k\chi) - q^{-1})\phi_{s_kw,\chi}. \tag{4-2}$$

Lemma 4.4. *Let $w = w_n$ and $\beta = \beta_n$. Then, $T^*_{w^{-1}}\Delta_{S,\chi}(\mathrm{ch}_B) = -c_\beta(\chi)\chi(a_\beta)$.*

Proof. Since $w = s_{2n}$ is a simple reflection, we can apply (4-1) and obtain $T_{w^{-1}}(\phi_{B,w\chi}) = (c_\beta(w\chi) - 1)\phi_{1,\chi} + q^{-1}\phi_{w,\chi}$. Using the integral expression (3-3), it follows that $\Lambda_S(\phi_{1,\chi}) = 1$ (with Haar measure normalized so that the volume of B is 1). Therefore, it remains to compute $\Lambda_{S,\chi}(\phi_{w,\chi})$.

Assume $\operatorname{Re} z_i > 0$ for all i so that Δ_S is given by (3-3). In order to use the integral expression (3-3) again, we need to express elements of BwB in the form $P_\phi S$. Note that BwB need not be contained in $P_\phi S$, but almost all (i.e., except elements in certain set of measure 0) elements must be contained.

We use the following measure-preserving decomposition where all compact groups which appear are assumed to be total measure 1:

$$BwB = P_\phi(\mathfrak{o}) w N^\beta(\mathfrak{o}) N^{-,\widehat{-\beta}}(\mathfrak{p}).$$

An easy calculation shows that $\mathrm{Lie}(N^{-,\widehat{-\beta}})(\mathfrak{p}) \subset \mathrm{Lie}(P_\phi^{\hat{\beta}})(\mathfrak{p}) + \mathrm{Lie}(S)(\mathfrak{p})$, and by an argument similar to the proof of [Sakellaridis 2006, Lemma 5.1], we have $N^{-,\widehat{-\beta}}(\mathfrak{p}) \subset P_\phi^{\hat{\beta}}(\mathfrak{o}) S(\mathfrak{o})$. Consequently,

$$\Lambda_S(\phi_{w,\chi}) = \Delta_S(\mathrm{ch}_{BwB}) = \int_{BwB} \Theta_\chi(x)\,dx = q \int_{wN_\phi^\beta(\mathfrak{o})} \Theta_\chi(x)\,dx.$$

The domain of the integral $wN_\phi^\beta(\mathfrak{o})$ consists of elements of the form

$$\left(\begin{array}{cc|cc} \mathbb{1}_{2(n-1)} & & & \\ & 0_2 & & \varepsilon \\ \hline & & \mathbb{1}_{2(n-1)} & \\ & \varepsilon & & -x \cdot \mathbb{1}_2 \end{array}\right) =: A(x),$$

with $x \in \mathfrak{o}$. If $x \neq 0$,

$$A(x) = \left(\begin{array}{cc|cc} \mathbb{1}_{2(n-1)} & & & \\ & x^{-1} \cdot \mathbb{1}_2 & & -\varepsilon \\ \hline & & \mathbb{1}_{2(n-1)} & \\ & & & x \cdot \mathbb{1}_2 \end{array}\right) \left(\begin{array}{cc|cc} \mathbb{1}_{2(n-1)} & & & \\ & -\mathbb{1}_2 & & \\ \hline & & \mathbb{1}_{2(n-1)} & \\ & x^{-1}\varepsilon & & -\mathbb{1}_2 \end{array}\right) \in P_\phi S.$$

Therefore,

$$\Theta_\chi(A(x)) = |x|^{z_{2n-1}+z_{2n}-1} \psi\big(\tfrac{1}{2}\,\mathrm{tr}(x^{-1}\varepsilon^2)\big) = |x|^{z_{2n-1}+z_{2n}-1} \psi^{-1}(x^{-1})$$

and

$$\Lambda_S(\phi_w, \chi) = q \int_{\mathfrak{o}} |x|^{z_{2n-1}+z_{2n}-1} \psi^{-1}(x^{-1})\,dx = q \sum_{i=0}^{\infty} (q\chi(a_\alpha))^i \int_{\mathfrak{p}^i - \mathfrak{p}^{i+1}} \psi^{-1}(x^{-1})\,dx.$$

Substituting

$$\int_{\mathfrak{p}^i - \mathfrak{p}^{i+1}} \psi^{-1}(x^{-1})\,dx = \begin{cases} 1 - q^{-1} & (i = 0) \\ -q^{-2} & (i = 1) \\ 0 & (i \geq 2) \end{cases}$$

for the above equation, it follows that

$$\Lambda_S(\phi_{w,\chi}) = q(1 - q^{-1} - q^{-1}\chi(a_\beta)).$$

By putting all this together and after some simple algebraic manipulation, the desired equation follows. By rationality, we can drop the assumption of $\operatorname{Re} z_i > 0$ and the result follows for all χ. □

Lemma 4.5. *Let $w = w_i$ for fixed $1 \le i \le n-1$. Then*

$$T^*_{w^{-1}}\Delta_S(\mathrm{ch}_B) = \frac{q^{-1}(q^{-2}x-1)(x-q^{-1})}{(q^{-1}x-1)(x-1)},$$

where $x = \chi(a_{\alpha_{2i}})$.

Proof. Applying (4-1), we obtain $T_{s_j^{-1}}\phi_{1,s_j\chi} = (c_{\alpha_j}(s_j\chi)-1)\phi_{1,\chi} + q^{-1}\phi_{s_j\chi}$ for each $2i-1 \le j \le 2i+1$. Since $s_j \notin \Gamma$, $T^*_{s_j^{-1}}\Lambda_{S,\chi}(\phi_{1,s_j\chi}) = 0$ and we get

$$q^{-1}\Lambda_{S,\chi}(\phi_{s_j,\chi}) = -(c_{\alpha_j}(s_j\chi)-1). \tag{4-3}$$

Repeating the same argument gives us the following equations: for every distinct $j,k,l \in \{2i-1, 2i, 2i+1\}$,

$$q^{-2}\Lambda_{S,\chi}(\phi_{s_k s_j,\chi}) = (c_{\alpha_k}(s_k\chi)-1)(c_{\alpha_j}(s_j\chi)-1), \tag{4-4}$$

$$q^{-3}\Lambda_{S,\chi}(\phi_{s_l s_k s_j,\chi}) = -(c_{\alpha_l}(s_l\chi)-1)(c_{\alpha_k}(s_k\chi)-1)(c_{\alpha_j}(s_j\chi)-1). \tag{4-5}$$

For $j \in \{2i-1, 2i+1\}$, we also obtain

$$\begin{aligned} &q^{-3}\Lambda_{S,\chi}(\phi_{s_{2i}s_j s_{2i}},\chi) \\ &= (c_{\alpha_{2i+1}}(s_{2i+1}s_{2i}\chi)-1)(c_{\alpha_{2i}}(s_{2i}\chi)-q^{-1})(c_{\alpha_{2i}}(s_{2i}\chi)-1) \\ &\qquad - q^{-1}(c_{\alpha_{2i+1}}(s_{2i+1}s_2\chi)-1) - (c_{\alpha_{2i}}(s_{2i}\chi)-1)^2(c_{\alpha_{2i+1}}(s_{2i+1}\chi)-1). \end{aligned} \tag{4-6}$$

Using (4-1) and (4-2) repeatedly, we can express $T_{w^{-1}}(\phi_{1,w\chi})$ as a linear combination of functions ϕ. Substituting (4-3), (4-4) and (4-5), we obtain

$$\begin{aligned} \Lambda_S(T_{w^{-1}}(\phi_{1,w\chi})) &= (c_{\alpha_{2i}}(s_{2i}\chi)-1)^2(c_{\alpha_{2i-1}}(s_{2i-1}\chi)-1)(c_{\alpha_{2i+1}}(s_{2i+1}s_{2i}\chi)-1) \\ &\quad + (c_{\alpha_{2i-1}}(s_{2i-1}s_{2i}\chi)-1)(c_{\alpha_{2i}}(s_{2i}\chi)-q^{-1}) \\ &\qquad\qquad (c_{\alpha_{2i}}(s_{2i}\chi)-1)(c_{\alpha_{2i+1}}(s_{2i+1}s_{2i}\chi)-1) \\ &\quad - q^{-1}(c_{\alpha_{2i-1}}(s_{2i-1}s_{2i}\chi)-1)(c_{\alpha_{2i+1}}(s_{2i+1}s_{2i}\chi)-1) \\ &\quad - (c_{\alpha_{2i}}(s_{2i}\chi)-1)^2(c_{\alpha_{2i-1}}(s_{2i-1}\chi)-1)(c_{\alpha_{2i+1}}(s_{2i+1}s_{2i}\chi)-1) \\ &\quad - (c_{\alpha_{2i}}(s_{2i}\chi)-1)^2(c_{\alpha_{2i-1}}(s_{2i-1}\chi)-1)(c_{\alpha_{2i+1}}(s_{2i+1}\chi)-1) \\ &\quad + q^{-4}\Lambda_S(\phi_{w,\chi}). \end{aligned}$$

Simple computations using

$$\begin{aligned}
c_{\alpha_{2i+1}}(s_{2i+1}\chi) = c_{\alpha_{2i-1}}(s_{2i-1}\chi) &= 0,\\
c_{\alpha_{2i}}(s_{2i}\chi) - 1 &= \frac{1-q^{-1}}{\chi(a_{\alpha_{2i}})-1},\\
c_{\alpha_{2i+1}}(s_{2i+1}s_{2i}\chi) - 1 = c_{\alpha_{2i-1}}(s_{2i-1}s_{2i+1}s_{2i}\chi) - 1 &= \frac{1-q^{-1}}{q^{-1}\chi(a_{\alpha_{2i}})-1},\\
c_{\alpha_{2i}}(s_{2i}\chi) - q^{-1} &= \frac{1-q^{-1}}{\chi(a_{\alpha_{2i}})-1}\chi(a_{\alpha_{2i}})
\end{aligned}$$

show that

$$\begin{aligned}
\Lambda_S(T_{w^{-1}}(\phi_{1,w\chi})) &= \frac{(1-q^{-1})^4}{(q^{-1}x-1)^2(x-1)^2}x - q^{-1}\frac{(1-q^{-1})^2}{(q^{-1}x-1)^2}\\
&\qquad - \frac{(1-q^{-1})^2}{(x-1)^2} + q^{-4}\Lambda_S(\phi_{w,\chi})\\
&= -\frac{(1+q^{-1})(1-q^{-1})^2}{(q^{-1}x-1)(x-1)} + q^{-4}\Lambda_S(\phi_{w,\chi}),
\end{aligned}$$

where $x = \chi(a_{\alpha_{2i}})$.

It remains to compute $\Lambda_S(\phi_{w,\chi})$. This can be done by essentially the same method as the proof of Lemma 4.4.

Assume $\operatorname{Re} z_1 > \operatorname{Re} z_2 > \cdots > \operatorname{Re} z_{2n} > 0$ so that Δ_S is given by (3-3). For later use, we make a stronger assumption. Let

$$\Sigma_i = \{\alpha \in \Sigma \mid \alpha > 0,\ w\alpha < 0\} = \{e_{2i-1}-e_{2i+1},\, e_{2i-1}-e_{2i+2},\, e_{2i}-e_{2i+1},\, e_{2i}-e_{2i+2}\}.$$

Then $BwB = P_\phi(\mathfrak{o})wN_\phi^{\Sigma_i}(\mathfrak{o})N_\phi^{-,\widehat{-\Sigma_i}}(\mathfrak{p})$. An easy calculation shows that

$$\mathrm{Lie}(N_\phi^{-,\widehat{-\Sigma_i}})(\mathfrak{p}) \subset \mathrm{Lie}(S)(\mathfrak{p}) + \mathrm{Lie}(P_\phi^{\widehat{\Sigma_i}})(\mathfrak{p}),$$

and by an argument similar to the proof of Lemma 5.1 of [Sakellaridis 2006], we have $N_\phi^{-,\widehat{-\Sigma_i}}(\mathfrak{p}) \subset P_\phi^{\widehat{\Sigma_i}}(\mathfrak{o})S(\mathfrak{o})$. Therefore,

$$\Lambda_S(\phi_{w,\chi}) = \Delta_S(\mathrm{ch}_{BwB}) = \int_{BwB} \Theta_\chi(x)\,dx = q^4 \int_{wN_\phi^{\Sigma_i}(\mathfrak{o})} \Theta_\chi(x)\,dx$$

Then $wN_\phi^{\Sigma_i}(\mathfrak{o})$ consists of elements of the form

$$B(a) = \begin{pmatrix} \mathbb{1}_{2(i-1)} & & & \\ & 0_2 & \mathbb{1}_2 & \\ & \mathbb{1}_2 & a & \\ & & & \mathbb{1}_{2(n-i-1)} \end{pmatrix} \in G_0$$

with $a \in \mathrm{Mat}_2(\mathfrak{o})$. If $\det a \neq -1$, let

$$b = \begin{pmatrix} 1+\det a & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} \mathbb{1}_{2(i-1)} & & & \\ & b & \varepsilon b^{-1} {}^t a \varepsilon & \\ & & b^{-1} & \\ & & & \mathbb{1}_{2(n-i-1)} \end{pmatrix} B(a) \in S \cap G_0.$$

Thus,

$$\Theta_\chi(B(a)) = |1+\det a|^{z_{2i-1}-z_{2i+1}-2}$$

and

$$\begin{aligned} \Lambda_S(\phi_{w,\chi}) &= q^4 \int_{\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathrm{Mat}_2(o)} |1+xw-yz|^{z_{2i-1}-z_{2i+1}-2}\, dx\, dy\, dz\, dw \\ &= q^4 \cdot \mathrm{vol}(\mathrm{Mat}_2(\mathfrak{o}) - \mathrm{GL}_2(\mathfrak{o})) \\ &\qquad + q^4 \int_{\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathfrak{o})} |1+xw-yz|^{z_{2i-1}-z_{2i+1}-2}\, dx\, dy\, dz\, dw. \end{aligned}$$

The first term can be computed as follows. Since the restriction of a Haar measure on $\mathrm{Mat}_2(\mathfrak{o})$ to $\mathrm{GL}_2(\mathfrak{o})$ is equal to the restriction of a Haar measure on $\mathrm{GL}_2(F)$,

$$\mathrm{vol}(\mathrm{GL}_2(\mathfrak{o})) = (q+1)\cdot \mathrm{vol}\begin{pmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o}^\times \end{pmatrix} = q^{-3}(q-1)^2(q+1),$$

and hence $\mathrm{vol}(\mathrm{Mat}_2(\mathfrak{o}) - \mathrm{GL}_2(\mathfrak{o})) = 1 - q^{-3}(q-1)^2(q+1)$.

Next, we need to compute the second term. There is a diffeomorphism f between $\mathrm{GL}_2(\mathfrak{o})$ and $\mathfrak{o}^\times \times \mathrm{SL}_2(\mathfrak{o})$ given by

$$\mathfrak{o}^\times \times \mathrm{SL}_2(\mathfrak{o}) \ni \left(t, \begin{pmatrix} x & y \\ z & w \end{pmatrix}\right) \mapsto \begin{pmatrix} tx & ty \\ z & w \end{pmatrix} \in \mathrm{GL}_2(\mathfrak{o}).$$

The Jacobian of f on the region $\mathfrak{o}^\times \times \{w \neq 0\} \subset \mathfrak{o}^\times \times \mathrm{SL}_2(\mathfrak{o})$ is $Jf = t/w$. Since the complement of this region is a set of measure 0, we can transform the second term into an integral on $\mathfrak{o}^\times \times \mathrm{SL}_2(\mathfrak{o})$:

$$\begin{aligned} &\int_{\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathfrak{o})} |1+xw-yz|^{z_{2i-1}-z_{2i+1}-2}\, dx\, dy\, dz\, dw \\ &\qquad = \int_{\mathfrak{o}^\times} \int_{\{w \neq 0\}} |1+t|^{z_{2i-1}-z_{2i+1}-2} |tw^{-1}|\, d^\times t\, dy\, dz\, dw \\ &\qquad = \int_{\mathfrak{o}^\times} |1+t|^{z_{2i-1}-z_{2i+1}-2}\, dt \cdot \int_{\{w \neq 0\}} |w|^{-1}\, dy\, dz\, dw. \end{aligned}$$

First, we consider the integral $\int_{\mathfrak{o}^\times} |1+t|^{z_{2i-1}-z_{2i+1}-2}\,dt$. Split the integral into $1+t \in \mathfrak{o}^\times$ and $1+t \in \mathfrak{p}$. The former contributes $1-2q^{-1}$ and the latter meshing

$$\begin{aligned}\sum_{j=1}^{\infty} |\varpi^j|^{z_{2i-1}-z_{2i+1}-2} \cdot (1-q^{-1})q^{-j} &= (1-q^{-1}) \sum_{j=1}^{\infty} \chi(a_{\alpha_{2i}})^j \\ &= (1-q^{-1})\chi(a_{\alpha_{2i}})(1-\chi(a_{\alpha_{2i}})^{-1}.\end{aligned}$$

Here, we used the assumption $\operatorname{Re} z_1 > \operatorname{Re} z_2 > \cdots > \operatorname{Re} z_{2n} > 0$. This implies $|\chi(a_{\alpha_{2i}})| < 1$, which is necessary for convergence of the above power series.

Therefore, we have

$$\int_{\mathfrak{o}^\times} |1+t|^{z_{2i-1}-z_{2i+1}-2}\,dt = -q^{-1} + (1-q^{-1})(1-\chi(a_{\alpha_{2i}}))^{-1}.$$

Second, we compute the integral $\int_{\{w\neq 0\}} |w|^{-1}\,dy\,dz\,dw$. Splitting the integral into $w \in \mathfrak{o}^\times$ and $w \in \varpi^j\mathfrak{o}^\times$, we get

$$\begin{aligned}&\int_{\{w\neq 0\}} |w|^{-1}\,dy\,dz\,dw \\ &\qquad = \int_{w\in\mathfrak{o}^\times}\int_{y,z\in\mathfrak{o}} dy\,dz\,dw + \sum_{j=1}^{\infty}\int_{w\in\varpi^j\mathfrak{o}^\times}\int_{yz\in -1+\mathfrak{p}^j} |\varpi^j|^{-1}\,dy\,dz\,dw \\ &\qquad = 1-q^{-1} + \sum_{j=1}^{\infty}(1-q^{-1})^2 q^{-j} \\ &\qquad = 1-q^{-2}.\end{aligned}$$

Consequently, we obtain

$$\Lambda_S(\phi_{w,\chi}) = q^2 - q(q-1)^2(q+1)(\chi(a_{\alpha_{2i}})-1)^{-1}.$$

Putting all this together, the desired equation follows. By rationality, we can drop the assumption of $\operatorname{Re} z_1 > \operatorname{Re} z_2 > \cdots > \operatorname{Re} z_{2n} > 0$, and the result follows for all χ. □

Some more computation enables us to rewrite these results.

Corollary. *For $w = w_n$ we have*:

$$T^*_{w^{-1}}\Lambda_{S,\chi} = -\chi(a_\beta)c_\beta(\chi)\Lambda_{S,w\chi},$$

where $\beta = \beta_n$.

For $w = w_i$ $(1 \le i < n)$ we have:

$$T^*_{w^{-1}}\Lambda_{S,\chi} = -\chi(a_\beta)\frac{1-q^{-2}\chi(a_{-\beta})}{1-q^{-2}\chi(a_\beta)}\prod_{\alpha\in\Sigma_i} c_\alpha(\chi)\Lambda_{S,w\chi},$$

where $\beta = \beta_i$.

More compactly, for every $w \in \Gamma$,

$$T^*_{w^{-1}} \Lambda_{S,\chi} = (-1)^{l_\Gamma(w)} \prod_{\substack{\alpha>0 \\ w\alpha<0}} c_\alpha(\chi) \prod_{\substack{\beta>0 \\ w\beta<0}} d_\beta(\chi) \Lambda_{S,w\chi},$$

where $\alpha \in \Sigma$, $\beta \in \Phi$.

These complete the proof of Theorem 2.4.

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