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JASON VAN ZELM

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Let $[\overline{\mathcal{B}}_{2,0,20}]$ and $[\mathcal{B}_{2,0,20}]$ respectively be the classes of the loci of stable and of smooth bielliptic curves with 20 marked points where the bielliptic involution acts on the marked points as the permutation $(1\ 2)\cdots(19\ 20)$. Graber and Pandharipande proved that these classes are nontautological. In this note we show that their result can be extended to prove that $[\overline{\mathcal{B}}_g]$ is nontautological for $g \geq 12$ and that $[\mathcal{B}_{12}]$ is nontautological.

1. Introduction

The system of *tautological rings* $\{R^\bullet(\overline{\mathcal{M}}_{g,n})\}$ is defined (see [Faber and Pandharipande 2005]) to be the minimal system of $\mathbb{Q}$-subalgebras of the Chow rings $A^\bullet(\overline{\mathcal{M}}_{g,n})$ closed under pushforward along the natural gluing and forgetful morphisms

$$\begin{aligned} \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} &\to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}, \\ \overline{\mathcal{M}}_{g,n+2} &\to \overline{\mathcal{M}}_{g+1,n}, \\ \overline{\mathcal{M}}_{g,n+1} &\to \overline{\mathcal{M}}_{g,n}. \end{aligned}$$

The tautological ring $R^\bullet(\mathcal{M}_{g,n})$ of the moduli space of smooth curves is the image of $R^\bullet(\overline{\mathcal{M}}_{g,n})$ under the localization morphism $A^\bullet(\overline{\mathcal{M}}_{g,n}) \to A^\bullet(\mathcal{M}_{g,n})$. We denote the image of $R^\bullet(\overline{\mathcal{M}}_{g,n})$ under the cycle map $A^\bullet(\overline{\mathcal{M}}_{g,n}) \to H^{2\bullet}(\overline{\mathcal{M}}_{g,n})$ by $RH^{2\bullet}(\overline{\mathcal{M}}_{g,n})$ and define $RH^{2\bullet}(\mathcal{M}_{g,n})$ accordingly. We say a cohomology class is *tautological* if it lies in the tautological subring of its cohomology ring; otherwise we say it is *nontautological*. In this note we work over $\mathbb{C}$ and all Chow and cohomology rings are assumed to be taken with rational coefficients.

The tautological rings are relatively well understood. An additive set of generators for the vector spaces $R^\bullet(\overline{\mathcal{M}}_{g,n})$ is given by decorated boundary strata and there exists an algorithm for computing the intersection product; see [Graber and Pandharipande 2003]. The class of many "geometrically defined" loci can be shown to be tautological. For example, this is the case for the class of the locus $\overline{\mathcal{H}}_g$ of hyperelliptic curves in $\overline{\mathcal{M}}_g$; see [Faber and Pandharipande 2005, Theorem 1].

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Any odd cohomology class of $\overline{\mathcal{M}}_{g,n}$ is nontautological by definition. Deligne proved that $H^{11}(\overline{\mathcal{M}}_{1,11}) \neq 0$, thus providing a first example of the existence of nontautological classes. In fact, it is known that $H^{\bullet}(\overline{\mathcal{M}}_{0,n}) = RH^{\bullet}(\overline{\mathcal{M}}_{0,n})$ [Keel 1992] and that $H^{2\bullet}(\overline{\mathcal{M}}_{1,n}) = RH^{2\bullet}(\overline{\mathcal{M}}_{1,n})$ [Petersen 2014, Corollary 1.2].

Examples of geometrically defined loci which can be proven to be nontautological are still relatively scarce. In [Graber and Pandharipande 2003], Graber and Pandharipande hunt for algebraic classes in $H^{2\bullet}(\overline{\mathcal{M}}_{g,n})$ and in $H^{2\bullet}(\mathcal{M}_{g,n})$ which are nontautological. In particular, they show that the classes of the loci $\overline{\mathcal{B}}_{g,n,2m}$ and $\mathcal{B}_{g,n,2m}$ of, respectively, stable and smooth bielliptic curves of genus g, with n marked points fixed by the bielliptic involution and $2m$ marked points pairwise switched by the bielliptic involution, are nontautological when $g = 2$, $n = 0$ and $2m = 20$ (i.e., $[\overline{\mathcal{B}}_{2,0,20}] \notin RH^{\bullet}(\overline{\mathcal{M}}_{2,20})$ and $[\mathcal{B}_{2,0,20}] \notin RH^{\bullet}(\mathcal{M}_{2,20})$). They also show that for sufficiently high odd genus h, the class of the locus of stable curves of genus $2h$ admitting a map to a curve of genus h is nontautological in $\overline{\mathcal{M}}_{2h}$. Their result relies on the existence of odd cohomology in $H^{\bullet}(\overline{\mathcal{M}}_{h,1})$, which was proven in [Pikaart 1995] for all $h \geq 8069$. See [Faber and Pandharipande 2013] for a recent survey of different methods of detecting nontautological classes.

In [Petersen and Tommasi 2014; Petersen 2016], Petersen and Tommasi proved that $H^{2\bullet}(\overline{\mathcal{M}}_{2,n})$ is tautological for all $n < 20$ and that $H^{2\bullet}(\overline{\mathcal{M}}_{2,20})$ is additively generated by tautological classes, by the class $[\overline{\mathcal{B}}_{2,0,20}]$, and by its conjugates under the action of the symmetric group on 20 elements. In this sense the result of Graber and Pandharipande for the bielliptic locus is sharp.

In this note we prove the following two new results.

Theorem 1. *The cohomology class $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological for all $g+m \geq 12$, $0 \leq n \leq 2g-2$ and $g \geq 2$.*

Theorem 2. *The cohomology class $[\mathcal{B}_{g,0,2m}]$ is nontautological when $g+m = 12$ and $g \geq 2$.*

Theorem 1 reduces the genus for which algebraic nontautological classes on $\overline{\mathcal{M}}_g$ are known to exist from 16138 to 12. As far as the author is aware, Theorem 2 provides the first example of a nontautological algebraic class on $\mathcal{M}_g$.

2. Preliminaries

Let $\mathcal{B}_{g,n,2m} \subset \mathcal{M}_{g,n+2m}$ be the locus of smooth bielliptic curves for which the bielliptic involution acts on the last $2m$ markings as the involution $(1\ 2)\cdots(2m-1\ 2m)$ and fixes the remaining points, and let $\overline{\mathcal{B}}_{g,n,2m} \subset \overline{\mathcal{M}}_{g,n+2m}$ be its closure. A modular interpretation of $\overline{\mathcal{B}}_{g,n,2m}$ can be given by *admissible double covers* [Abramovich et al. 2003].

Definition 3. We define $\overline{\mathrm{Adm}}(g,h)_{2m}$ to be the stack parameterizing *admissible double covers* from curves of genus g to curves of genus h with $2m$ points switched by the involution. Specifically, $\overline{\mathrm{Adm}}(g, h)_{2m}$ parameterizes tuples

$$(C, D, f, y_1, \dots, y_{2m})$$

together with a *total ordering* of the smooth ramification points of f such that

- $f : C \to D$ is a double cover of connected nodal curves of arithmetic genus g and h, respectively,
- $y_1, \dots, y_{2m}$ are points in the smooth locus of C such that the covering involution swaps the points y_{2k-1} and y_{2k} pairwise,
- the image of each node of C under f is a node,
- the curve C, equipped with the markings given by the set of all ramification points and the points $y_1, \dots, y_{2m}$, is stable, and so is the curve D, equipped with the markings given by the ordered set of all smooth branch points and the images of the points $y_1, \dots, y_{2m}$.

There is a natural map $\phi_n : \overline{\mathrm{Adm}}(g, h)_{2m} \to \overline{\mathcal{M}}_{g,n+2m}$ which assigns to an admissible double cover $(C, D, f, y_1, \dots, y_{2m})$ the stabilization of the curve

$$(C, x_1, \dots, x_n, y_1, \dots, y_{2m}),$$

where the $(x_i)_{i=1}^n$ are the first n smooth ramification points of f. The space $\overline{\mathcal{B}}_{g,n,2m}$ equals the image of $\overline{\mathrm{Adm}}(g, 1)_{2m}$ under ϕ_n.

By using the Riemann–Hurwitz formula inductively on the number of nodes of D, we see that the map f must have $2g + 2 - 4h$ ramification points. The map $\overline{\mathrm{Adm}}(g, h)_{2m} \to \overline{\mathcal{M}}_{h,2g+2-4h+m}$, mapping each admissible cover to its target curve together with its marked points, is finite. In the bielliptic case it follows that the dimension of $\overline{\mathcal{B}}_{g,n,2m}$ is $2g - 2 + 2m$. The classes of these loci are denoted by

$$[\overline{\mathcal{B}}_{g,n,2m}] \in A^{g-1+n+2m}(\overline{\mathcal{M}}_{g,2m+n}) \quad \text{and} \quad [\mathcal{B}_{g,n,2m}] \in A^{g-1+n+2m}(\mathcal{M}_{g,2m+n}).$$

Similarly, we let $\mathcal{H}_{g,n,2m}$ be the locus of smooth hyperelliptic curves with n marked points fixed and $2m$ points pairwise permuted by the hyperelliptic involution. We denote its closure inside $\overline{\mathcal{M}}_{g,n+2m}$ by $\overline{\mathcal{H}}_{g,n,2m}$. This closure equals the image of $\overline{\mathrm{Adm}}(g, 0)_{2m}$ under ϕ_n.

Our proof of Theorem 1 relies on the following result for pullbacks along gluing morphisms.

Proposition 4 [Graber and Pandharipande 2003, Proposition 1]. *Let*

$$\xi : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$

be the gluing morphism and $\gamma \in RH^\bullet(\overline{\mathcal{M}}_{g_1+g_2,n_1+n_2})$. *Then*

$$\xi^*(\gamma) \in RH^\bullet(\overline{\mathcal{M}}_{g_1,n_1+1}) \otimes RH^\bullet(\overline{\mathcal{M}}_{g_2,n_2+1}).$$

We say that a cycle $\lambda \in H^\bullet(\overline{\mathcal{M}}_{g_1,n_1}) \otimes H^\bullet(\overline{\mathcal{M}}_{g_2,n_2})$ *admits a tautological Künneth decomposition* if $\lambda \in RH^\bullet(\overline{\mathcal{M}}_{g_1,n_1}) \otimes RH^\bullet(\overline{\mathcal{M}}_{g_2,n_2})$. Proposition 4 says that the pullback of a tautological class admits a tautological Künneth decomposition. It can be shown that the pullback of a tautological class under the gluing morphism $\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$ and the forgetful morphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is tautological. In this sense the tautological ring is also closed under pullbacks along the gluing and the forgetful morphisms.

3. Proof of Theorems 1 and 2

We are now ready to prove Theorem 1. We start by proving the following weaker result.

Proposition 5. *We have*

$$[\overline{\mathcal{B}}_{g,0,2m}] \notin RH^\bullet(\overline{\mathcal{M}}_{g,2m})$$

for $g+m=12$ and $g \geq 2$.

Proof. Let

$$i : \overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11} \to \overline{\mathcal{M}}_{g,2m} \tag{†}$$

be the gluing morphism that pairwise identifies the first $g-1$ points on the first curve with the first $g-1$ points on the second curve. In Lemma 6 we will prove that the restriction of $i^*[\overline{\mathcal{B}}_{g,0,2m}]$ to the interior $\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$ is a positive scalar multiple α of the class $[\Delta]$ of the diagonal. Let $\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$ denote the normalization of $(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}) \backslash (\mathcal{M}_{1,11} \times \mathcal{M}_{1,11})$. It follows from the localization sequence

$$A^{10}(\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})) \to A^{11}(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}) \to A^{11}(\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}) \to 0$$

that $i^*[\overline{\mathcal{B}}_{g,0,2m}] = \alpha \cdot \Delta + B$, with B supported on the image of $\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$.

The class B admits a tautological Künneth decomposition by Lemma 7(i). Given a homogeneous basis $\{e_i\}_{i\in I}$ for $H^\bullet(\overline{\mathcal{M}}_{1,11})$ with dual basis $\{\hat{e}_i\}_{i\in I}$, the cohomology class of the diagonal can be written as

$$[\Delta] = \sum_{i\in I} (-1)^{\deg e_i} e_i \otimes \hat{e}_i.$$

In particular, since $H^{11}(\overline{\mathcal{M}}_{1,11}) \neq 0$, the diagonal $[\Delta]$ does not admit a tautological Künneth decomposition. Since the pullback of a tautological class along a (composition of) gluing morphisms admits a tautological Künneth decomposition by repeated application of Proposition 4, this shows that $[\overline{\mathcal{B}}_{g,0,2m}]$ is nontautological. □

Lemma 6. *Let $g+m=12$ and $g \geq 2$. The pullback of $[\overline{\mathcal{B}}_{g,0,2m}]$ to $\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$, under the restriction j of the gluing map i defined in (†), is a scalar multiple α of the class of the diagonal Δ.*

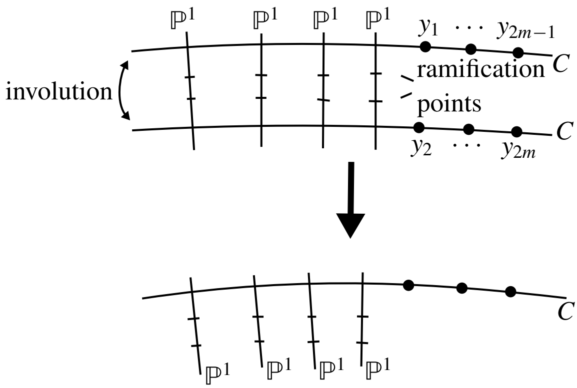

Figure 1. The image of C under η.

Proof. Let η be the map $\mathcal{M}_{1,11} \to \overline{\mathrm{Adm}}(g, 1)_{2m}$ which maps a curve $(C, x_1, \ldots, x_{11})$ to the admissible cover which has as a source curve two copies of C glued together by rational bridges attached to the first $g-1$ points of each copy of C, as covering involution the bielliptic involution which switches around the two copies of C and has two fixed points on each of the rational bridges, and as target curve a single copy of C with a rational component attached to the first $g-1$ points (see Figure 1). Let $\delta : \mathcal{M}_{1,11} \to \mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$ be the diagonal morphism. Consider the diagram

(‡)
$$\begin{tikzcd}
\mathcal{M}_{1,11} \arrow[rr, bend left, "\eta"] \arrow[r, dashed, "\zeta"] \arrow[dr, bend right, "\delta"'] & F \arrow[r] \arrow[d, "\tilde{\phi}_0"] \arrow[dr, phantom, "\square"] & \overline{\mathrm{Adm}}(g,1)_{2m} \arrow[d, "\phi_0"] \\
& \mathcal{M}_{1,11} \times \mathcal{M}_{1,11} \arrow[r, "j"'] & \overline{\mathcal{M}}_{g,2m}
\end{tikzcd}$$

By unwrapping definitions one verifies that $j \circ \delta = \phi_0 \circ \eta$. By the universal property of fiber products this defines a unique map $\zeta : \mathcal{M}_{1,11} \to F$, making diagram (‡) commute.

Claim: The morphism ζ is surjective on closed points.

Assuming the claim, it follows that $\tilde{\phi}_{0*}[F]$ is a positive scalar multiple of $\delta_*[\mathcal{M}_{1,11}] = [\Delta]$. Since

$$\operatorname{codim}_{\mathcal{M}_{1,11}\times\mathcal{M}_{1,11}} \Delta = 11 = \operatorname{codim}_{\overline{\mathcal{M}}_{g,2m}} \overline{\mathcal{B}}_{g,0,2m},$$

it follows that there is no excess of intersection between $\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}$ and $\overline{\mathcal{B}}_{g,0,2m} = \phi_0(\overline{\mathrm{Adm}}(g, 1)_{2m})$ in diagram (‡). We deduce that $j^*[\overline{\mathcal{B}}_{g,0,2m}] = \alpha[\Delta]$ for some $\alpha \in \mathbb{Q}_{>0}$.

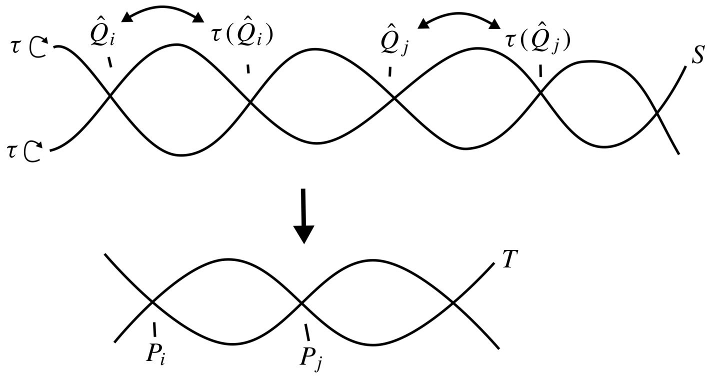

Figure 2. The admissible cover $S \to T$ when τ fixes C_1 and C_2.

Proof of the claim. By definition, an object of $F(\mathbb{C})$ consists of a curve $\widetilde{C} := (\tilde{C}_1, \widetilde{C}_2)$ in $\mathcal{M}_{1,11} \times \mathcal{M}_{1,11}(\mathbb{C})$, an object $(S \to T) \in \overline{\mathrm{Adm}}(g, 1)_{2m}(\mathbb{C})$ and an isomorphism $\gamma : j(\widetilde{C}) \xrightarrow{\sim} \phi_0(S \to T)$. To prove the claim, we show that $(\widetilde{C}, (S \to T), \gamma)$ is isomorphic to an object in the image of ζ. Let $f : \widetilde{C}_1 \cup \widetilde{C}_2 \to j(\widetilde{C})$ be the map of curves induced by j, set $C := j(\widetilde{C})$, $C_1 := f(\widetilde{C}_1)$ and $C_2 := f(\widetilde{C}_2)$, let τ be the involution on C induced by the bielliptic involution of $S \to T$ and let Q_i be the node of C corresponding to the i-th marking of $\widetilde{C}_1$ and $\widetilde{C}_2$ via the morphism f.

Since C_1 and C_2 are smooth, there are two possibilities for the action of τ on C: either it fixes C_1 and C_2 or it switches the whole of C_1 with the whole of C_2.

Suppose τ fixes C_1 and C_2. By construction the involution τ maps marked points lying on C_1 to marked points lying on C_2 so this is only possible if C has no marked points at all. In this case τ must fix the different branches of C at each Q_i. If the preimage of Q_i in S were to be a genus 0 curve R_i, contracted by the stabilization map, then R_i would have 2 marked ramification points which are not nodes. But this would imply that τ switches the nodes on R_i and it would therefore also switch the branches of C at Q_i. It follows that the preimage of each Q_i in S is a single node $\hat{Q}_i$. Since C_1 and C_2 are smooth, τ induces an involution on the set of nodes $\{\hat{Q}_1, \ldots, \hat{Q}_{11}\}$. We can thus find distinct $\hat{Q}_i$, $\hat{Q}_j \neq \tau(\hat{Q}_i)$ such that $S - \{\hat{Q}_i, \tau(\hat{Q}_i), \hat{Q}_j, \tau(\hat{Q}_j)\}$ is connected. If P_i and P_j are the images of Q_i and Q_j, respectively, under the admissible cover $S \to T$ then this means that $T - \{P_i, P_j\}$ is connected (see Figure 2). This implies that the arithmetic genus of T is at least 2, which is a contradiction.

We can therefore assume τ maps C_1 to C_2. Let us first suppose that τ does not fix all nodes, so there exist some distinct i, j such that $\tau(Q_i) = Q_j$ (see Figure 3). If the preimage of Q_i in S is a component of S contracted by the stabilization map, then this component must contain a ramification point. This would be a fixed point of the involution, contradicting the assumption that $\tau(Q_i) = Q_j$. So the preimage

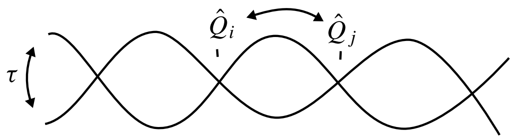

Figure 3. Nodes in S not fixed by τ.

of Q_i and Q_j in S are nodes $\hat{Q}_i$ and $\hat{Q}_j$. Let P be the image of $\{\hat{Q}_i, \hat{Q}_j\}$ under the bielliptic map. Arguing as at the end of the last paragraph, we see that $T\backslash\{P\}$ is connected. Therefore, since T has arithmetic genus 1, it has geometric genus 0. However, if S_1 is the irreducible component of S which surjects onto C_1 under the stabilization map, then S_1 is a smooth curve of geometric genus 1. This is a contradiction because $S_1 \to T_1$ is a birational map.

We have thus proven that τ switches the components C_1 and C_2 and fixes the nodes Q_i, which implies that $((\widetilde{C}_1, \widetilde{C}_2), (S \to T), \gamma)$ is isomorphic to an object in the image of $\mathcal{M}_{1,11}(\mathbb{C})$. This concludes the proof that the map $\mathcal{M}_{1,11} \to F$ is surjective on closed points. □

Lemma 7. (i) *Every algebraic class of codimension* 11 *in* $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ *supported on* $\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$ *admits a tautological Künneth decomposition.*

(ii) *Every algebraic class on* $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ *of codimension less than* 11 *admits a tautological Künneth decomposition.*

Proof. This is a slightly weaker version of [Graber and Pandharipande 2003, Lemma 3]; the proof given there requires that $RH^{2\bullet}(\overline{\mathcal{M}}_{1,n}) = H^{2\bullet}(\overline{\mathcal{M}}_{1,n})$ and $H^{\mathrm{odd}}(\overline{\mathcal{M}}_{1,n}) = 0$ for $n < 11$, for which there was no reference at the time of their paper. The first equation is [Petersen 2014, Corollary 1.2]. The second condition follows from the computations for $n < 11$ in [Getzler 1998]. □

We have now concluded the proof of Proposition 5. To prove Theorem 1 it remains to show that $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological for all g, n, m with $0 \le n \le 2g-2$ and $g+m > 12$.

Proof of Theorem 1. We will show in Lemma 8 and 9 that if $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological then so are $[\overline{\mathcal{B}}_{g,n+1,2m}]$ for $n \le 2g-3$, and $[\overline{\mathcal{B}}_{g,n,2m+2}]$. In Lemma 10 we will show that if $[\overline{\mathcal{B}}_{g,1,0}]$ is nontautological then so is $[\overline{\mathcal{B}}_{g+1}]$. Using these statements, and by induction with the statement of Proposition 5 as the base case, we conclude that $[\overline{\mathcal{B}}_{g,n,2m}]$ is nontautological for all $g+m \ge 12$. □

Lemma 8. *If* $[\overline{\mathcal{B}}_{g,n,2m}]$ *is nontautological and* $n \le 2g-3$, *then so is* $[\overline{\mathcal{B}}_{g,n+1,2m}]$.

Proof. Let $\pi : \overline{\mathcal{M}}_{g,n+1+2m} \to \overline{\mathcal{M}}_{g,n+2m}$ be the morphism that forgets the first point and stabilizes. By definition $\pi_*([\overline{\mathcal{B}}_{g,n+1,2m}])$ is a positive scalar multiple of $[\overline{\mathcal{B}}_{g,n,2m}]$. Because the pushforward of a tautological class by the forgetful morphism is tautological by definition, the result follows. □

Lemma 9. *If* $[\overline{\mathcal{B}}_{g,n,2m}]$ *is nontautological, then so is* $[\overline{\mathcal{B}}_{g,n,2m+2}]$.

Proof. If $n \le 2g-3$ then $[\overline{\mathcal{B}}_{g,n+1,2m}]$ is nontautological by Lemma 8. Consider the gluing morphism

$$\sigma : \overline{\mathcal{M}}_{g,n+2m+1} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{g,n+2m+2}$$

which glues the first points of both curves together; then $\sigma^{-1}(\overline{\mathcal{B}}_{g,n,2m+2}) = \overline{\mathcal{B}}_{g,n+1,2m}$. Since

$$\operatorname{codim}_{\overline{\mathcal{M}}_{g,n+2m+2}} \overline{\mathcal{B}}_{g,n,2m+2} = \operatorname{codim}_{\overline{\mathcal{M}}_{g,n+2m+1}} \overline{\mathcal{B}}_{g,n+1,2m},$$

it follows that $\sigma^*[\overline{\mathcal{B}}_{g,n,2m+2}] = \alpha[\overline{\mathcal{B}}_{g,n+1,2m}]$ for some $\alpha \in \mathbb{Q}_{>0}$. Since σ is a gluing morphism and the pullback of a tautological class along σ admits tautological Künneth decomposition, $[\overline{\mathcal{B}}_{g,n,2m+2}]$ is nontautological.

If $n = 2g-2$ we first prove that $[\overline{\mathcal{B}}_{g,n-1,2m+2}]$ is nontautological as above by pulling back along the map $\overline{\mathcal{M}}_{g,n+2m} \times \overline{\mathcal{M}}_{0,3} \to \overline{\mathcal{M}}_{g,n+2m+1}$ and then applying Lemma 8. □

Lemma 10. *If* $[\overline{\mathcal{B}}_{g,1,0}]$ *is nontautological, then so is* $[\overline{\mathcal{B}}_{g+1}]$.

Proof. Let $\epsilon : \overline{\mathcal{M}}_{g,1} \times \overline{\mathcal{M}}_{1,1} \to \overline{\mathcal{M}}_{g+1}$ be the gluing morphism. From the description of the boundary divisors of $\overline{\mathcal{B}}^{\mathrm{Adm}}_{g+1}$ [Pagani 2016, pp. 1275–1276], it follows that there exist $\alpha, \beta \in \mathbb{Q}_{>0}$ such that

$$\epsilon^*[\overline{\mathcal{B}}_{g+1}] = \alpha[\overline{\mathcal{B}}_{g,1,0} \times \overline{\mathcal{M}}_{1,1}] + \beta[(\overline{\mathcal{H}}_{g-1,0,2}, \overline{\mathcal{M}}_{1,1})] \in H^\bullet(\overline{\mathcal{M}}_{g,1} \times \overline{\mathcal{M}}_{1,1}),$$

where $(\overline{\mathcal{H}}_{g-1,0,2}, \overline{\mathcal{M}}_{1,1})$ denotes the locus of pairs $(C, E) \in \overline{\mathcal{M}}_{g,1} \times \overline{\mathcal{M}}_{1,1}$, where C consists of a genus $g-1$ hyperelliptic curve C' glued to an elliptic curve E' isomorphic to E, with the hyperelliptic involution switching the marked point of C' with the point of intersection with E'. The class $[(\overline{\mathcal{H}}_{g-1,0,2}, \overline{\mathcal{M}}_{1,1})]$ admits a tautological Künneth decomposition because the diagonal inside $\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1}$ does, the class of the hyperelliptic locus is tautological by [Faber and Pandharipande 2005, Theorem 1], and the pushforward of tautological classes under a gluing morphism is tautological by definition. The class $[\overline{\mathcal{B}}_{g,1} \times \overline{\mathcal{M}}_{1,1}]$ does not admit a tautological Künneth decomposition because $[\overline{\mathcal{B}}_{g,1}]$ is nontautological. It follows by Proposition 4 that $[\overline{\mathcal{B}}_{g+1}]$ is nontautological. □

We now complete the proof of Theorem 2.

Proof of Theorem 2. The case $g = 2$ is treated in [Graber and Pandharipande 2003, Section 3]. We use a similar argument to prove the remaining cases. The proof runs by contradiction.

Suppose $[\mathcal{B}_{g,0,2m}] \in RH^\bullet(\mathcal{M}_{g,2m})$; then there is a collection of cycles Z_k in $\overline{\mathcal{M}}_{g,2m}$, of codimension 11 and supported on $\partial\overline{\mathcal{M}}_{g,2m}$, such that $\sum[Z_k] + [\overline{\mathcal{B}}_{g,0,2m}]$ is tautological. Consider again the gluing morphism $i : \overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11} \to \overline{\mathcal{M}}_{g,2m}$ of (†). By assumption, the pullback of $\sum[Z_k] + [\overline{\mathcal{B}}_{g,0,2m}]$ to $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$

admits a tautological Künneth decomposition whereby the pullback of $\sum[Z_k]$ to $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$ must be nontautological (by Proposition 4 and since the pullback of $[\overline{\mathcal{B}}_{g,0,2m}]$ is nontautological, as we have shown in the proof of Theorem 1).

We denote by Δ_h the locus of curves in $\overline{\mathcal{M}}_{g,2m}$ consisting of two curves, one of which has genus h, glued together in a single node, and by Δ_{irr} the locus that generically parameterizes irreducible singular curves. So $\partial\overline{\mathcal{M}}_{g,2m} = \Delta_{\mathrm{irr}} \cup \bigcup_h \Delta_h$.

Suppose Z_k is supported on Δ_h for some h. Since $i(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$ does not have a separating node, we see that $i(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}) \not\subset \Delta_h$. The intersection

$$\Delta_h \cap (\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$$

therefore lies in the image of $\partial(\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11})$. It follows by Lemma 7(i) that $i^*[Z_k]$ admits a tautological Künneth decomposition.

Suppose now that Z_k is supported on Δ_{irr}. We decompose the map i as

$$\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11} \xrightarrow{i_1} \overline{\mathcal{M}}_{g-1,2m+2} \xrightarrow{i_2} \overline{\mathcal{M}}_{g,2m}.$$

Then there exist cycles Y_k in $\overline{\mathcal{M}}_{g-1,2m+2}$ such that $i_{2*}[Y_k] = [Z_k]$. Now

$$i^*[Z_k] = i_1^* i_2^*[Z_k] = i_1^*(c_1(N_{\overline{\mathcal{M}}_{g-1,2m+2}}\overline{\mathcal{M}}_{g,2m}) \cap [Y_k]).$$

We see that $i^*[Z_k]$ decomposes as a product of algebraic classes of codimension less than 11, all of which admit tautological Künneth decomposition by Lemma 7(ii).

We conclude that all the cycles $[Z_k]$ have a tautological Künneth decomposition when pulled back to $\overline{\mathcal{M}}_{1,11} \times \overline{\mathcal{M}}_{1,11}$, which is a contradiction. □

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JASON VAN ZELM
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF LIVERPOOL
LIVERPOOL
UNITED KINGDOM
jasonvanzelm@outlook.com

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