Pacific Journal of Mathematics

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Volume 294 No. 2 June 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 294, No. 2, 2018
dx.doi.org/10.2140/pjm.2018.294.505

ADDENDUM: SINGULARITIES OF FLAT FRONTS IN HYPERBOLIC SPACE

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Volume **221**:2 (2005), 303–351

This is an addendum to the authors' previous paper in which criteria for cuspidal edges and swallowtails on surfaces are given by applying the so-called Zakalyukin's lemma. The original statement in Zakalyukin's paper assumed the properness of the mappings. However, the lemma in the appendix of our paper did not assume properness. Recently, we noticed that the proof given in the appendix was implicitly relying on properness. In this addendum, we prove that mappings satisfying the criteria of cuspidal edges and swallowtails have properness. Consequently, the criteria are clarified.

In [Kokubu et al. 2005], to which this note is an addendum, we found an omitted condition in the statement of Lemma 2.2, which was explained there as a lemma given by Zakalyukin. The original statement in [Zakalyukin 1976] assumed the properness of the mappings f_1 and f_2 in the lemma. We have discovered that the proof given in the appendix of [Kokubu et al. 2005] was implicitly using the properness of the mappings f_i, $(i = 1, 2)$.

In [Kokubu et al. 2005, Proposition 1.3], this lemma was applied to prove criteria for cuspidal edges and swallowtails. In this paper, we show that these criteria still remain valid. In fact, we prepare the following new lemma to replace Lemma 2.2 in [Kokubu et al. 2005].

Lemma. *Let $U(\subset \mathbb{R}^n)$ be a neighborhood of the origin, and let the mappings $f_i : (U, o) \to (\mathbb{R}^{n+1}, \mathbf{0})$, with $i = 1, 2$, be wave fronts, where o and $\mathbf{0}$ are the origins of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, respectively. Suppose that o is a singular point of f_i and the set of regular points of f_i is dense in U for each $i = 1, 2$. Moreover, suppose that $f_i^{-1}(\mathbf{0})$ is a finite set. Then the following two statements are equivalent*:

(1) *There exist neighborhoods $V_1, V_2(\subset \mathbb{R}^n)$ of the origin o and a local diffeomorphism on $\mathbb{R}^{n+1}$ which maps the image $f_1(V_1)$ to $f_2(V_2)$, namely the image of f_1 is locally diffeomorphic to that of f_2.*

MSC2010: primary 53C42, 57R45; secondary 53A05, 53A99.
Keywords: wave front, cuspidal edge, swallowtail.

(2) *There exists a local diffeomorphism h on $\mathbb{R}^{n+1}$ and a local contact diffeomorphism Φ on the unit cotangent bundle $T_1^*\mathbb{R}^{n+1}$ of $\mathbb{R}^{n+1}$ with respect to the Euclidean metric of $\mathbb{R}^{n+1}$ which sends fibers to fibers such that $\Phi \circ L_{f_1} = L_{f_2} \circ h$, namely the lift L_{f_1} is **Legendrian equivalent** to the lift L_{f_2}.*

Remark 1. Instead of the properness in the original Zakalyukin lemma, we assume the finiteness of the inverse images $f_i^{-1}(\mathbf{0})$, $i = 1, 2$, which was dropped in Lemma 2.2 in [Kokubu et al. 2005]. The condition that $f_i^{-1}(\mathbf{0})$ is a finite set relates to the $\mathcal{K}$-finiteness of the map f_i (cf. [Wall 1981]), which plays an important role in singularity theory.

We prepare the following assertion:

Proposition. *Let $U(\subset \mathbb{R}^n)$ be a neighborhood of a point $p \in \mathbb{R}^n$, and $B_{\mathbf{0}}(r)$ be an open ball of radius $r(>0)$ centered at the origin in $\mathbb{R}^N$, and let $f : (U, p) \to (\mathbb{R}^N, \mathbf{0})$ $(N \geq n)$ be a continuous map such that $f^{-1}(\mathbf{0})$ is a finite set. Then for sufficiently small $r > 0$, the connected component V of $f^{-1}(B_{\mathbf{0}}(r))$ containing p satisfies $\overline{V} \subset U$. Moreover, the restriction of the map f to V with image inside $B_{\mathbf{0}}(r)$ is a proper mapping.*

Proof. Take a ball $W := D_p(\epsilon)$ of radius ϵ centered at p such that $\overline{W}$ is contained in U. Since $f^{-1}(\mathbf{0})$ is a finite set, we may choose the radius ϵ so that

$$f^{-1}(\mathbf{0}) \cap W = \{p\} \tag{1}$$

holds. We denote by $V(r)$ the connected component of $f^{-1}(B_{\mathbf{0}}(r))$ containing p. It is sufficient to show that $\overline{V(1/k)} \subset W$ for any sufficiently large integers $k > 0$. If not, there exists a point $q_k \notin W$ lying in $\overline{V(1/k)}$. If $q_k \notin \partial W (:= \overline{W} \setminus W)$, then q_k is an exterior point of W. Then we can find a point $q_k' \in V(1/k)$ such that q_k' is also an exterior point of W. Since $V(1/k)$ is connected, there exists a continuous curve on $V(1/k)$ joining p and q_k'. By the intermediate value theorem, for each positive integer k, there exists a point p_k satisfying

$$p_k \in \overline{V(1/k)} \cap \partial W. \tag{2}$$

On the other hand, if $q_k \in \partial W$, then (2) trivially holds by setting $p_k := q_k$.

We then take a sequence $\{q_{j,k}\}_{j=1}^{\infty}$ lying in $V(1/k)$ converging to p_k. By definition, we have $f(q_{j,k}) \in B_{\mathbf{0}}(1/k)$. By the continuity of f,

$$f(p_k) \in \overline{B_{\mathbf{0}}(1/k)} \qquad (k = 1, 2, 3, \ldots) \tag{3}$$

holds, where $\overline{B_{\mathbf{0}}(1/k)}$ is the closure of the open ball $B_{\mathbf{0}}(1/k)$. Since ∂W is compact, we can take a subsequence $\{p_{k_m}\}_{m=1}^{\infty}$ of $\{p_k\}$ which converges to a point $p_\infty \in \partial W$. Letting $m \to \infty$, equation (3) yields that $f(p_\infty) = \mathbf{0}$, which contradicts (1).

We next prove the final assertion: Suppose that K is a compact subset of $B_{\mathbf{0}}(r)$ and $f^{-1}(K)$ is not compact. Then we can take a sequence $\{x_k\}$ in $f^{-1}(K)$ not accumulating to any point of V. Since $\overline{V}$ is compact, we may assume that $\{x_k\}$ converges to a point x_∞ on $\partial V := \overline{V} \setminus V$. Since f is a continuous map on U and $\overline{V} \subset U$, there exists a connected open neighborhood O of x_∞ such that $f(O) \subset B_{\mathbf{0}}(r)$. Then $V' := V \cup O$ is a connected open subset such that $f(V') \subset B_{\mathbf{0}}(r)$, which contradicts the definition of V, since $V \subsetneq V \cup O$. □

Proof of Lemma. (1) follows from (2) immediately, so it is sufficient to show (1) implies (2). By Fact A.3 in the appendix of [Kokubu et al. 2005], we may assume $f_1(V_1) = f_2(V_2)$. By the above proposition, we can take $r > 0$ such that $V(r) := f^{-1}(B_{\mathbf{0}}(r))$ satisfies $\overline{V(r)} \subset V_1 \cap V_2$. Then we have that

$$f_1(V(r)) = f_2(V(r)).$$

By [Kokubu et al. 2005, Fact A.1], we may assume that the associated Legendrian immersion $L_{f_i} : \tilde{U}_i \to T_1^*\mathbb{R}^{n+1}$ $(i = 1, 2)$ is an embedding. Since $\overline{V(r)}$ is compact, we have

$$f_1(\overline{V(r)}) = \overline{f_1(V(r))} = \overline{f_2(V(r))} = f_2(\overline{V(r)}).$$

Thus by [Kokubu et al. 2005, Proposition A.4], we have $L_{f_1}(\overline{V_1}) = L_{f_2}(\overline{V_2})$. In particular, we have

$$L_{f_1}(V_1) \subset L_{f_2}(U_2),$$

and by [Kokubu et al. 2005, Fact A.2], there exists a local diffeomorphism φ on $\mathbb{R}^n$ such that $L_{f_2} = L_{f_1} \circ \varphi$, which proves the assertion. □

We next show the following claim, that is, that wave fronts satisfying our criteria for cuspidal edges or swallowtails also satisfy the assumption of the above lemma. Consequently, the statement of [Kokubu et al. 2005, Proposition 1.3] is clarified.

Claim 1. *Let U be a domain in $\mathbb{R}^2$, and let $f : (U, p) \to (\mathbb{R}^3, \mathbf{0})$ be a wave front such that p is a nondegenerate singular point. Take a regular curve $\gamma(t)$ parametrizing the singular set such that $\gamma(0) = p$. If f satisfies one of the two conditions*

(1) *the null vector $\eta(0)$ is linearly independent of $\dot{\gamma}(0)$, or*

(2) *$\eta(0)$ is proportional to $\dot{\gamma}(0)$, and*

$$\left.\frac{d}{dt}\right|_{t=0} \det(\dot{\gamma}(t), \eta(t)) \neq 0,$$

then the inverse image $f^{-1}(\mathbf{0})$ is a finite set.

Proof. Let ν_0 be the unit normal vector of f at p, and T the plane passing through $\mathbf{0}$ perpendicular to ν_0. We denote by $\pi : \mathbb{R}^3 \to T$ the orthogonal projection. Then $\varphi := \pi \circ f : U \to \mathbb{R}^2$ is a smooth map having a singular point at p. Then the condition (1) (respectively, (2)) turns out to be a well-known criterion for a fold singularity (respectively, Whitney cusp singularity), see [Whitney 1955] or Theorem A1 in the appendix of [Saji et al. 2009] (A_1-Morin singularity means fold singularity, and A_2-Morin singularity means Whitney cusp singularity). So φ is right-left equivalent to the map germ $(u, v) \mapsto (u^2, v)$ (respectively, $(u, v) \mapsto (u^3 - 3uv, v)$). Thus, $f^{-1}(\mathbf{0})$ is a finite set. □

In [Saji et al. 2009], we gave a criterion for an A_{k+1}-singular point of a wave front for $k \geq 1$, as a generalization of the case of cuspidal edges and swallowtails. Then the same problem has arisen in that case as well, that is, to clarify the criterion, we must show that the map satisfies the condition that the inverse image of the singular point is a finite set. However, by the following claim, this is actually true:

Claim 2. *Let U be a domain in $\mathbb{R}^n$, and let $f : (U, p) \to (\mathbb{R}^{n+1}, \mathbf{0})$ be a wave front such that p is a nondegenerate singular point. If f satisfies the criterion given in [Saji et al. 2009, Theorem 2.4], then the inverse image $f^{-1}(\mathbf{0})$ consists of finitely many points.*

The proof is the same as for Claim 1: Taking the unit normal vector ν_0 of f at p, we define the orthogonal projection $\pi : \mathbb{R}^{n+1} \to H$, where H is the hyperplane passing through p orthogonal to the vector ν_0. Then $\varphi := \pi \circ f : U \to H (= \mathbb{R}^n)$ is a smooth map having a singular point at p. Then the criterion for an A_{k+1} singularity on the wave front f corresponds to the criterion for an A_k-Morin singularity of φ given in [Saji et al. 2009, Theorem A1]. Hence, the inverse image of the origin is a finite set. □

Remark 2. In [Izumiya and Saji 2010; Izumiya et al. 2010; Saji 2011], criteria for cuspidal lips, cuspidal beaks, cuspidal butterflies and D_4 singularities are given. In these cases as well, one can similarly show the finiteness of the inverse image of the singular point, assuming that the criteria given in those papers hold. However, the arguments are longer and will be given in a separate work by those authors.

Acknowledgement

The authors thank Professor Shyuichi Izumiya for valuable comments.

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Received November 16, 2016.

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PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 294 No. 2 June 2018