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## A VARIANT OF A THEOREM BY AILON-RUDNICK FOR ELLIPTIC CURVES

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Given a smooth projective curve $C$ defined over $\overline{\mathbb{Q}}$ and given two elliptic surfaces $\mathcal{E}_{1} \rightarrow C$ and $\mathcal{E}_{2} \rightarrow C$ along with sections $\sigma_{P_{i}}, \sigma_{Q_{i}}$ (corresponding to points $P_{i}, Q_{i}$ of the generic fibers) of $\mathcal{E}_{i}($ for $i=1,2)$, we prove that if there exist infinitely many $t \in C(\overline{\mathbb{Q}})$ such that for some integers $m_{1, t}, m_{2, t}$, we have $\left[m_{i, t}\right]\left(\sigma_{P_{i}}(t)\right)=\sigma_{Q_{i}}(t)$ on $\mathcal{E}_{i}($ for $i=1,2)$, then at least one of the following conclusions must hold:
i. There exist isogenies $\varphi: E_{1} \rightarrow E_{2}$ and $\psi: E_{2} \rightarrow E_{2}$ such that $\varphi\left(P_{1}\right)=\psi\left(P_{2}\right)$. ii. $Q_{i}$ is a multiple of $P_{i}$ for some $i=1,2$.

A special case of our result answers a conjecture made by Silverman.

## 1. Introduction

Ailon and Rudnick [2004] showed that for two multiplicatively independent nonconstant polynomials $a, b \in \mathbb{C}[x]$ there is a nonzero polynomial $h \in \mathbb{C}[x]$, depending on $a$ and $b$ such that $\operatorname{gcd}\left(a^{n}-1, b^{n}-1\right) \mid h$ for all positive integer $n$. In this paper, we prove a similar result for elliptic curves; instead of working with the multiplicative group $\mathbb{G}_{m}$, we work with the group law on an elliptic curve defined over a function field. The result of Ailon-Rudnick relies crucially on the Serre-Ihara-Tate theorem (see [Lang 1965]), while our result relies crucially on recent Bogomolov-type results for elliptic surfaces due to DeMarco and Mavraki [2017].

Throughout our article, we work with elliptic surfaces over $\overline{\mathbb{Q}}$; more precisely, given a projective, smooth curve $C$ defined over $\overline{\mathbb{Q}}$, an elliptic surface $\mathcal{E} / C$ is given by a morphism $\pi: \mathcal{E} \rightarrow C$ over $\overline{\mathbb{Q}}$ where the generic fiber of $\pi$ is an elliptic curve $E$ defined over $K=\overline{\mathbb{Q}}(C)$, while for all but finitely many $t \in C(\overline{\mathbb{Q}})$, the fiber $\mathcal{E}_{t}:=\pi^{-1}(\{t\})$ is an elliptic curve defined over $\overline{\mathbb{Q}}$. Recall that a section $\sigma$ of $\pi$ (i.e., a map $\sigma: C \rightarrow \mathcal{E}$ such that $\pi \circ \sigma=\left.\mathrm{id}\right|_{C}$ ) gives rise to a $K$-rational point of $E$. Conversely, a point $P \in E(K)$ corresponds to a section of $\pi$; if we need to indicate the dependence on $P$, we will denote it by $\sigma_{P}$. So, for all but finitely many $t \in C(\overline{\mathbb{Q}})$, the intersection of the image of $\sigma_{P}$ in $\mathcal{E}$ with the fiber above $t$

[^0]is a point $P_{t}:=\sigma_{P}(t)$ on the elliptic curve $\mathcal{E}_{t}:=\pi^{-1}(\{t\})$. For any integer $k$, the multiplication-by- $k$ map [ $k$ ] on $E$ extends to a morphism on $\mathcal{E}$; if there is no risk of confusion, we still denote the extension by $[k]$.

We prove the following result:
Theorem 1-1. Let $\pi_{i}: \mathcal{E}_{i} \rightarrow C$ be elliptic surfaces over a curve $C$ defined over $\overline{\mathbb{Q}}$ with generic fibers $E_{i}$, and let $\sigma_{P_{i}}, \sigma_{Q_{i}}$ be sections of $\pi_{i}$ (for $\left.i=1,2\right)$ corresponding to points $P_{i}, Q_{i} \in E_{i}(\overline{\mathbb{Q}}(C))$. If there exist infinitely many $t \in C(\overline{\mathbb{Q}})$ for which there exist some $m_{1, t}, m_{2, t} \in \mathbb{Z}$ such that $\left[m_{i, t}\right] \sigma_{P_{i}}(t)=\sigma_{Q_{i}}(t)$ for $i=1,2$, then at least one of the following properties must hold:
(i) There exist isogenies $\varphi: E_{1} \rightarrow E_{2}$ and $\psi: E_{2} \rightarrow E_{2}$ such that $\varphi\left(P_{1}\right)=\psi\left(P_{2}\right)$.
(ii) For some $i \in\{1,2\}$, there exists $k_{i} \in \mathbb{Z}$ such that $\left[k_{i}\right] P_{i}=Q_{i}$ on $E_{i}$.

We note here that, in contrast to similar results such as [Ailon and Rudnick 2004], the ambient algebraic group ( $\mathcal{E}_{1} \times \mathcal{E}_{2}$ in our case, as opposed to $\mathbb{G}_{m}$ for Ailon and Rudnick) need not be defined over the field of constants in $k(C)$.

A special case of our result (when both $Q_{1}$ and $Q_{2}$ are the zero elements) answers in the affirmative [Silverman 2004b, Conjecture 7]; this is carried out in a more general setting (over the complex numbers and also, giving a more precise connection to the original GCD problem of Ailon-Rudnick) in our Proposition 4-3 from Section 4. We also note that the special case of Theorem 1-1 when $Q_{1}=Q_{2}=0$ was solved by Masser and Zannier [2014] when both elliptic surfaces are defined over $\mathbb{C}$.

Silverman's question [2004b, Conjecture 7] was motivated by work of Ailon and Rudnick [2004], who showed that the greatest common divisor of $a^{n}-1$ and of $b^{n}-1$ for multiplicatively independent polynomials $a, b \in \mathbb{C}[T]$ has bounded degree (see also the generalization in [Corvaja and Zannier 2013b] along with the related results from [Corvaja and Zannier 2008; 2011; 2013a]). In turn, the result of Ailon and Rudnick was motivated by the work of Bugeaud-Corvaja-Zannier [Bugeaud et al. 2003] who obtained an upper bound for $\operatorname{gcd}\left(a^{k}-1, b^{k}-1\right)$ (as $k$ varies in $\mathbb{N}$ ) for given $a, b \in \overline{\mathbb{Q}}$. On the other hand, Silverman [2004a] showed that the degree of $\operatorname{gcd}\left(a^{m}-1, b^{n}-1\right)$ could be quite large when $a, b \in \overline{\mathbb{F}}_{p}[T]$; see also the authors' previous paper [Ghioca et al. 2017], where (using as technical ingredient [Ghioca 2014] in place of [DeMarco and Mavraki 2017]) we study the $\operatorname{gcd}\left(a^{m}-1, b^{n}-1\right)$ when $a$ and $b$ are polynomials over arbitrary fields of positive characteristic, along with other generalizations on the same theme. Finally, we mention the work of Denis [2011] who studied the same problem of the greatest common divisor in the context of Drinfeld modules.

As hinted in [Silverman 2004b], this greatest common divisor (GCD) problem may be studied in much higher generality; for example, if one knew the result of [DeMarco and Mavraki 2017] (see Theorem 2-3 below) in the context of abelian varieties, then our method would extend to a similar conclusion for arbitrary abelian
schemes over a base curve. DeMarco-Mavraki's theorem can be interpreted as an extension of Masser-Zannier's theorem (see [Masser and Zannier 2012]) in the same spirit as the Bogomolov conjecture is an extension of the classical Manin-Mumford conjecture. So, even though the extension to arbitrary abelian varieties of the results from [DeMarco and Mavraki 2017] is expected to be quite challenging, we mention that there is some progress in this direction due to Cinkir [2011], Gubler [2007], and Yamaki [2017], who proved various cases of the Bogomolov conjecture for abelian varieties defined over function fields.

Our Theorem 1-1 is related also to [Barroero and Capuano 2016, Theorem 1.1] (see also the extension from [Barroero and Capuano 2017]) where it is shown that given $n$ linearly independent sections $P_{i}$ on the Legendre elliptic family $y^{2}=$ $x(x-1)(x-t)$, there are at most finitely many parameters $t$ such that the points $\left(P_{i}\right)_{t}$ satisfy two independent linear relations on the corresponding elliptic curve. Therefore, a special case of the result by Barroero and Capuano is that given sections $P_{1}, P_{2}, Q_{1}, Q_{2}$ on the Legendre elliptic surface, if these four sections are linearly independent, then there are at most finitely many $t$ such that for some $m_{t}, n_{t} \in \mathbb{Z}$ we have $\left[m_{t}\right]\left(P_{1}\right)_{t}=\left(Q_{1}\right)_{t}$ and $\left[n_{t}\right]\left(P_{2}\right)_{t}=\left(Q_{2}\right)_{t}$. However, in our Theorem 1-1 we obtain the same conclusion under the weaker hypothesis that $Q_{i}$ is not a multiple of $P_{i}$ for $i=1,2$ and also that $P_{1}$ and $P_{2}$ are linearly independent. We also note that the constant case of Barroero and Capuano's theorem is covered by the results of Habegger and Pila [2016].

A special case of our Theorem 1-1 bears a resemblance to the classical MordellLang problem proven by Faltings [1994] (see also [Hrushovski 1996] for the case of semiabelian varieties defined over function fields). Indeed, with the notation as in Theorem 1-1, assume there exist infinitely many $t \in C(\overline{\mathbb{Q}})$ such that for some $m_{t} \in \mathbb{Z}$ we have

$$
\begin{equation*}
\left[m_{t}\right]\left(P_{i}\right)_{t}=\left(Q_{i}\right)_{t} \quad \text { for } i=1,2 \tag{1-2}
\end{equation*}
$$

Also assume there is no $m \in \mathbb{Z}$ such that $[m] P_{i}=Q_{i}$ for $i=1,2$. Then the conclusion of Theorem 1-1 yields the existence of isogenies $\varphi: E_{1} \rightarrow E_{2}$ and $\psi: E_{2} \rightarrow E_{2}$ such that $\varphi\left(P_{1}\right)=\psi\left(P_{2}\right)$. Thus, using that (1-2) holds for infinitely many $t \in C(\overline{\mathbb{Q}})$ we see that

$$
\begin{equation*}
\varphi\left(Q_{1}\right)=\psi\left(Q_{2}\right) \tag{1-3}
\end{equation*}
$$

Therefore, if we let $X \subset \mathcal{A}:=\mathcal{E}_{1} \times \mathcal{E}_{2}$ be the 1-dimensional subscheme corresponding to the section $\left(Q_{1}, Q_{2}\right)$, and we let $\Gamma \subset \mathcal{A}$ be the subgroup spanned by $\left(0, P_{2}\right)$ and $\left(P_{1}, 0\right)$, then the existence of infinitely many $\gamma \in \Gamma$ such that for some $t \in C(\overline{\mathbb{Q}})$ we have $\gamma_{t} \in X$ implies that $X$ is contained in a proper algebraic subgroup of $\mathcal{A}$ (as given by (1-3)). Such a statement can be viewed as a relative version of the classical Mordell-Lang problem; note that if $\mathcal{E}_{1}$ and $\mathcal{E}_{2}$ are constant elliptic
surfaces with generic fibers $E_{i}^{0}$ defined over $\overline{\mathbb{Q}}$, while $\Gamma \subset\left(E_{1}^{0} \times E_{2}^{0}\right)(\overline{\mathbb{Q}})$, then this question is a special case of Faltings's theorem [1994] (formerly known as the Mordell-Lang conjecture). It is natural to ask whether the above relative version of the Mordell-Lang problem holds more generally when $\mathcal{A} \rightarrow C$ is an arbitrary semiabelian scheme, $X \subset \mathcal{A}$ is a 1-dimensional scheme and $\Gamma \subset \mathcal{A}$ is an arbitrary finitely generated group. This more general question is also related to the bounded height problems studied in [Amoroso et al. 2017] in the context of pencils of finitely generated subgroups of $\mathbb{G}_{m}^{n}$.

In the next section of this paper, we review some preliminary material. Following that, in Section 3, we prove Theorem 1-1. The proof in the case of nonconstant sections is quite similar to the proofs of the main results of [Ailon and Rudnick 2004] and [Hsia and Tucker 2017], while the case of constant sections requires a different argument. In Section 4, we give a positive answer to Silverman's conjecture [2004b, Conjecture 7].

## 2. Preliminaries

From now on, we fix an elliptic surface $\pi: \mathcal{E} \rightarrow C$, where $C$ is a projective, smooth curve defined over $\overline{\mathbb{Q}}$. We denote by $E$ the generic fiber of $\mathcal{E}$; this is an elliptic curve defined over $\overline{\mathbb{Q}}(C)$. For all but finitely many $t \in C(\overline{\mathbb{Q}})$, we have $\mathcal{E}_{t}:=\pi^{-1}(\{t\})$ is an elliptic curve defined over $\overline{\mathbb{Q}}$.
2.1. Isotriviality. We say that $\mathcal{E}$ is isotrivial if the $j$-invariant of the generic fiber is a constant function (on $C$ ); for isotrivial elliptic surfaces $\mathcal{E}$, all smooth fibers of $\pi$ are isomorphic (to the generic fiber $E$ ). If $\mathcal{E}$ is isotrivial, then there exists a finite cover $C^{\prime} \rightarrow C$ such that $\mathcal{E}^{\prime}:=\mathcal{E} \times{ }_{C} C^{\prime}$ is a constant (elliptic) surface over $C^{\prime}$, i.e., there exists an elliptic curve $E^{0}$ defined over $\overline{\mathbb{Q}}$ such that $\mathcal{E}^{\prime}=E^{0} \times_{\operatorname{Spec}(\overline{\mathbb{Q}})} C^{\prime}$. Furthermore, for a constant elliptic surface $E^{0} \times_{\operatorname{Spec}(\overline{\mathbb{Q}})} C^{\prime}$, we say that $\sigma_{P}$ is a constant section if $P \in E^{0}(\overline{\mathbb{Q}})$.
2.2. Canonical height on an elliptic surface. For each $t \in C(\overline{\mathbb{Q}})$ such that $\mathcal{E}_{t}$ is an elliptic curve, we let $\hat{h}_{\mathcal{E}_{t}}$ be the Néron-Tate canonical height for the points in $\mathcal{E}_{t}(\overline{\mathbb{Q}})$ (for more details, see [Silverman 1986]). There are two important properties of the canonical height which we will use:
(i) $\hat{h}_{\mathcal{E}_{t}}\left(P_{t}\right)=0$ if and only if $P_{t}$ is a torsion point of $\mathcal{E}_{t}$, i.e., there exists a positive integer $k$ such that $[k] P_{t}=0$; and
(ii) for each $k \in \mathbb{Z}$ we have $\hat{h}_{\mathcal{E}_{t}}\left([k] P_{t}\right)=k^{2} \cdot \hat{h}_{\mathcal{E}_{t}}\left(P_{t}\right)$.

Also, we let $\hat{h}_{E}$ be the Néron-Tate canonical height on the generic fiber $E$ constructed with respect to the Weil height on the function field $\overline{\mathbb{Q}}(C)$; for more details, see [Silverman 1994a]. Property (ii) above holds also on the generic fiber,
i.e., $\hat{h}_{E}([k] P)=k^{2} \cdot \hat{h}_{E}(P)$. On the other hand, property (i) above holds only if $\mathcal{E}$ is nonisotrivial. Furthermore, if $\mathcal{E}=E \times{ }_{C} C$ is a constant family (where $E$ is an elliptic curve defined over $\overline{\mathbb{Q}})$, then for any $P \in E(\overline{\mathbb{Q}}(C))$, we have that $\hat{h}_{E}(P)=0$ if and only if $P \in E(\overline{\mathbb{Q}})$.
2.3. Variation of the canonical height. We let $h_{C}$ be a given Weil height for points in $C(\overline{\mathbb{Q}})$ corresponding to a divisor of degree 1 on $C$.

Let $\sigma_{P}$ be a section of the elliptic surface $\mathcal{E} \rightarrow C$ corresponding to a point $P$ on the generic fiber $E$. Then, for all but finitely many $t \in C(\overline{\mathbb{Q}})$, the intersection of the image of $\sigma_{P}$ in $\mathcal{E}$ with the fiber above $t$ is a point $P_{t}$, on the elliptic curve $\mathcal{E}_{t}$. The following important fact will be used in our proof (see [Tate 1983; Silverman 1983]):

$$
\begin{equation*}
\lim _{h_{C}(t) \rightarrow \infty} \frac{\hat{h}_{\mathcal{E}_{t}}\left(P_{t}\right)}{h_{C}(t)}=\hat{h}_{E}(P) \tag{2-1}
\end{equation*}
$$

Furthermore, the following more precise result holds, as proven by Silverman [1994b]:

$$
\begin{equation*}
\hat{h}_{\mathcal{E}_{t}}\left(P_{t}\right)=h_{C, \eta(P)}(t)+O_{P}(1), \tag{2-2}
\end{equation*}
$$

where $\eta(P)$ is a divisor on $C$ of degree equal to $\hat{h}_{E}(P)$ and $h_{C, \eta(P)}$ is a given Weil height for the points in $C(\overline{\mathbb{Q}})$ corresponding to the divisor $\eta(P)$, while the implicit constant from the term $O_{P}(1)$ is only dependent on the section $\sigma_{P}$ (and implicitly on the divisor $\eta(P))$, but not on $t \in C(\overline{\mathbb{Q}})$.
2.4. Points of small height on sections. We will use [DeMarco and Mavraki 2017, Theorem 1.4], which extends [DeMarco et al. 2016] (and in turn, uses the extensive analysis from [Silverman 1994b] regarding the variation of the canonical height in an elliptic fibration). We also note that the case of isotrivial elliptic curves from Theorem 2-3 was previously proven by Zhang [1998], as part of Zhang's famous proof of the classical Bogomolov conjecture.

Theorem 2-3 [DeMarco and Mavraki 2017, Theorem 1.4]. Let $\mathcal{E}_{1}, \mathcal{E}_{2}$ be elliptic fibrations over the same $\overline{\mathbb{Q}}$-curve $C$. Let $P_{i}$ be a section of $\mathcal{E}_{i}($ for $i=1,2)$ with the property that there exists an infinite sequence $\left\{t_{n}\right\} \subset C(\overline{\mathbb{Q}})$ such that

$$
\lim _{n \rightarrow \infty} \hat{h}_{\left(\mathcal{E}_{i}\right)_{t_{n}}}\left(\left(P_{i}\right)_{t_{n}}\right)=0 \quad \text { for } i=1,2
$$

Then there exist group homomorphisms $\phi: \mathcal{E}_{1} \rightarrow \mathcal{E}_{2}$ and $\psi: \mathcal{E}_{2} \rightarrow \mathcal{E}_{2}$, not both trivial, such that $\phi\left(P_{1}\right)=\psi\left(P_{2}\right)$.

## 3. Proof of our main result

Propositions 3-1 and 3-9 are key to our proof.

Proposition 3-1. Let $C$ be a projective, smooth curve defined over $\overline{\mathbb{Q}}$, and let $h_{C}(\cdot)$ be a Weil height for the algebraic points of $C$ corresponding to a divisor of degree 1. Let $P$ and $Q$ be sections of an elliptic surface $\pi: \mathcal{E} \rightarrow C$ with generic fiber $E$, and assume there exists no $k \in \mathbb{Z}$ such that $[k] P=Q$. In addition, assume $\hat{h}_{E}(P)>0$. If there exists an infinite sequence $\left\{t_{i}\right\} \subset C(\overline{\mathbb{Q}})$ such that for each $i \in \mathbb{N}$ there exists some $m_{i} \in \mathbb{Z}$ such that $\left[m_{i}\right] P_{t_{i}}=Q_{t_{i}}$, then $h_{C}\left(t_{i}\right)$ is uniformly bounded and $\lim _{i \rightarrow \infty} \hat{h}_{\mathcal{E}_{t_{i}}}\left(P_{t_{i}}\right)=0$.

We note that the special case of Proposition 3-1 when $\pi: \mathcal{E} \rightarrow C$ is a constant elliptic surface follows from [Silverman 1983].

Proof. Since $\left[m_{i}\right] P_{t_{i}}=Q_{t_{i}}$, we have

$$
\begin{equation*}
m_{i}^{2} \cdot \hat{h}_{\mathcal{E}_{t_{i}}}\left(P_{t_{i}}\right)=\hat{h}_{\mathcal{E}_{t_{i}}}\left(Q_{t_{i}}\right) \tag{3-2}
\end{equation*}
$$

Since $[k] P \neq Q$ for any $k \in \mathbb{Z}$ and the sequence $\left\{t_{i}\right\}$ is infinite, then

$$
\begin{equation*}
\lim _{i \rightarrow \infty}\left|m_{i}\right|=\infty \tag{3-3}
\end{equation*}
$$

We claim first that $h_{C}\left(t_{i}\right)$ is uniformly bounded. Indeed, assuming (at the expense, perhaps, of replacing $\left\{t_{i}\right\}$ by an infinite subsequence) that $\lim _{i \rightarrow \infty} h_{C}\left(t_{i}\right)=\infty,(2-1)$ coupled with (3-2) and (3-3) yields a contradiction. To see this, we divide both sides of (3-2) by $h_{C}\left(t_{i}\right)$ and then take limits. Because $\hat{h}_{E}(P)>0$, (3-3) implies that the left-hand side equals

$$
\begin{equation*}
\lim _{i \rightarrow \infty} m_{i}^{2} \cdot \frac{\hat{h}_{\mathcal{E}_{t_{i}}}\left(P_{t_{i}}\right)}{h_{C}\left(t_{i}\right)}=\infty \tag{3-4}
\end{equation*}
$$

while the right-hand side equals

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \frac{\hat{h}_{\mathcal{E}_{i}}\left(Q_{t_{i}}\right)}{h_{C}\left(t_{i}\right)}=\hat{h}_{E}(Q)<\infty \tag{3-5}
\end{equation*}
$$

which is a contradiction. So, indeed, $h_{C}\left(t_{i}\right)$ must be uniformly bounded.
Next we prove that also $\hat{h}_{\mathcal{E}_{i}}\left(Q_{t_{i}}\right)$ is uniformly bounded. Using (2-2) (see [Silverman 1994b]) we know that there exists a divisor $\eta(Q)$ of $C$ of degree equal to $\hat{h}_{E}(Q)$ such that

$$
\begin{equation*}
\hat{h}_{\mathcal{E}_{t}}\left(Q_{t}\right)=h_{C, \eta(Q)}(t)+O(1) \tag{3-6}
\end{equation*}
$$

where $h_{C, \eta(Q)}$ is a Weil height on $C(\overline{\mathbb{Q}})$ corresponding to the divisor $\eta(Q)$. Since $h_{C}$ is a Weil height associated to a divisor $D$ on $C$ of degree 1 , then for any positive integer $m>\operatorname{deg}(\eta(Q))$, the divisor $D_{1}:=m D-\eta(Q)$ has positive degree and therefore, is ample. Then [Hindry and Silverman 2000, Proposition B.3.2] implies
that any Weil height $h_{C, D_{1}}$ associated to the divisor $D_{1}$ satisfies $h_{C, D_{1}}(t) \geq O(1)$ for all $t \in C(\overline{\mathbb{Q}})$. So,

$$
\begin{equation*}
m h_{C}(t)+O(1) \geq h_{C, \eta(Q)}(t) \quad \text { for } t \in C(\overline{\mathbb{Q}}) \tag{3-7}
\end{equation*}
$$

Therefore $h_{C, \eta(Q)}\left(t_{i}\right)$ is uniformly bounded (since $h_{C}\left(t_{i}\right)$ is uniformly bounded). Then (3-6) provides the desired claim that

$$
\begin{equation*}
\hat{h}_{\mathcal{E}_{t_{i}}}\left(Q_{t_{i}}\right) \text { is bounded as } i \rightarrow \infty \tag{3-8}
\end{equation*}
$$

Finally, the fact that $\lim _{i \rightarrow \infty} \hat{h}_{\mathcal{E}_{i}}\left(P_{i}\right)=0$ follows easily from combining equations (3-2), (3-3), and (3-8).

Proposition 3-9. Let $P$ and $Q$ be sections of a constant elliptic fibration $\pi: \mathcal{E} \rightarrow C$, and assume there exists no $k \in \mathbb{Z}$ such that $[k] P=Q$. In addition, assume $P$ is a nontorsion, constant section. If there exists an infinite sequence $\left\{t_{i}\right\} \subset C(\overline{\mathbb{Q}})$ such that for each $i \in \mathbb{N}$ there exists some $m_{i} \in \mathbb{Z}$ such that $\left[m_{i}\right] P_{t_{i}}=Q_{t_{i}}$, then $\lim _{i \rightarrow \infty} h_{C}\left(t_{i}\right)=\infty$.
Proof. Each fiber $\mathcal{E}_{t_{i}}$ is isomorphic to the generic fiber $E^{0}$, and so, because $P$ is a constant section,

$$
\begin{equation*}
\hat{h}_{\mathcal{E}_{t_{i}}}\left(P_{t_{i}}\right)=\hat{h}_{E^{0}}\left(P^{0}\right), \tag{3-10}
\end{equation*}
$$

where $P^{0}$ is the intersection of $P$ with the generic fiber and $\hat{h}_{E^{0}}(\cdot)$ is the Néron-Tate canonical height of the elliptic curve $E^{0}$ defined over $\overline{\mathbb{Q}}$ (i.e., it is not the canonical height on the generic fiber of $\mathcal{E}$ seen as an elliptic curve defined over the function field $\overline{\mathbb{Q}}(C)$ ).

Furthermore, since $P^{0}$ is not a torsion point of $E^{0}$, then $\hat{h}_{E^{0}}\left(P^{0}\right)>0$. Thus, from the equality $\left[m_{i}\right] P_{t_{i}}=Q_{t_{i}}$, along with (3-10) coupled with the fact that $\left|m_{i}\right| \rightarrow \infty$ (because $[k] P \neq Q$ for all integers $k$ ), we must have

$$
\begin{equation*}
\hat{h}_{\mathcal{E}_{i}}\left(Q_{t_{i}}\right)=m_{i}^{2} \hat{h}_{E^{0}}\left(P^{0}\right) \rightarrow \infty \tag{3-11}
\end{equation*}
$$

Then, using (2-2), we have

$$
\begin{equation*}
\hat{h}_{\mathcal{E}_{t_{i}}}\left(Q_{t_{i}}\right)=h_{C, \eta(Q)}\left(t_{i}\right)+O(1) \tag{3-12}
\end{equation*}
$$

where $h_{C, \eta(Q)}$ is a Weil height on $C$ corresponding to a certain divisor $\eta(Q)$. So, (3-11) and (3-12) yield $h_{C, \eta(Q)}\left(t_{i}\right) \rightarrow \infty$ and thus, $h_{C}\left(t_{i}\right) \rightarrow \infty$ (see [Hindry and Silverman 2000, Proposition B.3.5], along with our similar argument from the proof of Proposition 3-1).

Now we can prove our main result.
Proof of Theorem 1-1. First we note that if $P_{i}$ is a torsion section (for some $i \in\{1,2\}$ ), then conclusion (ii) holds trivially since then we would obtain that there
exist infinitely many $t \in C(\overline{\mathbb{Q}})$ such that $\left(Q_{i}\right)_{t}=[k]\left(P_{i}\right)_{t}$ for the same integer $k$. So, from now on, we assume that both $P_{1}$ and $P_{2}$ are nontorsion sections on $\mathcal{E}_{1}, \mathcal{E}_{2}$, respectively. In particular, this means that if $\hat{h}_{E_{i}}\left(P_{i}\right)=0$, then $\mathcal{E}_{i}$ must be an isotrivial elliptic surface.

We assume there exists an infinite sequence $\left\{t_{i}\right\} \subset C(\overline{\mathbb{Q}})$ such that for each $i \in \mathbb{N}$ there exist $m_{i, 1}, m_{i, 2} \in \mathbb{Z}$ with the property that $\left[m_{i, 1}\right]\left(P_{1}\right)_{t_{i}}=\left(Q_{1}\right)_{t_{i}}$ and also $\left[m_{i, 2}\right]\left(P_{2}\right)_{t_{i}}=\left(Q_{2}\right)_{t_{i}}$. In addition, we assume conclusion (ii) does not hold, i.e., there is no $m \in \mathbb{Z}$ such that $[m] P_{i}=Q_{i}$ for some $i \in\{1,2\}$. We split our analysis into two cases.
Case 1. $\hat{h}_{E_{i}}\left(P_{i}\right)>0$ for each $i=1,2$.
Applying then Proposition 3-1 to the sections $P_{i}$ and $Q_{i}$, we obtain

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \hat{h}_{\left(\mathcal{E}_{1}\right)_{t_{i}}}\left(\left(P_{1}\right)_{t_{i}}\right)=\lim _{i \rightarrow \infty} \hat{h}_{\left(\mathcal{E}_{2}\right)_{t_{i}}}\left(\left(P_{2}\right)_{t_{i}}\right)=0 \tag{3-13}
\end{equation*}
$$

Equation (3-13) along with Theorem 2-3 implies that conclusion (i) must hold in Theorem 1-1. Note that we obtain in this case that the morphisms $\varphi: E_{1} \rightarrow E_{2}$ and $\psi: E_{2} \rightarrow E_{2}$ from the conclusion of Theorem 2-3 are both isogenies since $P_{1}$ and $P_{2}$ are nontorsion sections.
Case 2. Either $\hat{h}_{E_{1}}\left(P_{1}\right)=0$ or $\hat{h}_{E_{2}}\left(P_{2}\right)=0$.
Without loss of generality, we assume $\hat{h}_{E_{1}}\left(P_{1}\right)=0$. Therefore (since $P_{1}$ is not torsion) $\mathcal{E}_{1}$ is an isotrivial elliptic surface, and furthermore, at the expense of replacing $C$ by a finite cover (and also performing a base extension for $\mathcal{E}_{1}$ and $\mathcal{E}_{2}$ ), we may assume that $\mathcal{E}_{1}$ is a constant family. Thus, $\mathcal{E}_{1}=E_{1}^{0} \times_{C} C$ for some elliptic curve $E_{1}^{0}$ defined over $\overline{\mathbb{Q}}$. Then also $P_{1}$ is a constant (nontorsion) section, because $\hat{h}_{\mathcal{E}_{1}}\left(P_{1}\right)=0$. Finally, we let $h_{C}(\cdot)$ be a Weil height for the algebraic points of $C$ with respect to a divisor of degree 1.

If $\hat{h}_{E_{2}}\left(P_{2}\right)>0$, then Proposition 3-1 applied to $P_{2}$ and $Q_{2}$ implies that $h_{C}\left(t_{i}\right)$ is uniformly bounded, which contradicts the conclusion of Proposition 3-9 applied to $P_{1}$ and $Q_{1}$. Therefore, we must have $\hat{h}_{E_{2}}\left(P_{2}\right)=0$, and also $\mathcal{E}_{2}$ is an isotrivial elliptic surface. At the expense of (yet another) base extension, we may assume that also $\mathcal{E}_{2}=E_{2}^{0} \times C$ is a constant fibration. Then $P_{2}$ is a constant, nontorsion section on $\mathcal{E}_{2}$. We let $P_{i}^{0}$ be the intersection of $P_{i}$ (for $i=1,2$ ) with the generic fiber of $\mathcal{E}_{i}$.

Now, if either $Q_{1}$ or $Q_{2}$ is also a constant section, then we get a contradiction since we assumed conclusion (ii) does not hold. Indeed, if for some $i=1,2$ both $P_{i}$ and $Q_{i}$ are constant sections on the constant elliptic surface $\mathcal{E}_{i}$, then the existence of a point $t \in C(\overline{\mathbb{Q}})$ such that for some $k \in \mathbb{Z}$ we have $[k]\left(P_{i}\right)_{t}=\left(Q_{i}\right)_{t}$ implies that actually $[k] P_{i}=Q_{i}$ on $\mathcal{E}_{i}$. So, we may assume that $Q_{1}$ and $Q_{2}$ are both nonconstant sections on $\mathcal{E}_{1}$, respectively $\mathcal{E}_{2}$. Then, there is a (neither vertical, nor horizontal) curve $X \subset E_{1}^{0} \times E_{2}^{0}$ containing all points $\left(\left(Q_{1}\right)_{t},\left(Q_{2}\right)_{t}\right)$ for $t \in C(\overline{\mathbb{Q}})$. Furthermore, our hypothesis means that this curve $X$ intersects the subgroup $\Gamma \subset E_{1}^{0} \times E_{2}^{0}$ spanned
by the points $\left(P_{1}^{0}, 0\right)$ and $\left(0, P_{2}^{0}\right)$ in an infinite set. The classical Mordell-Lang conjecture (proven by Faltings [1994]) implies that $X$ itself is a coset of an algebraic subgroup of $E_{1}^{0} \times E_{2}^{0}$. Hence, because $X$ projects dominantly onto each coordinate, there exists a nontrivial isogeny $\tau: E_{1}^{0} \rightarrow E_{2}^{0}$, and also there exist endomorphisms $\phi_{i}$ of $E_{i}^{0}$, not both trivial, such that

$$
\begin{equation*}
\tau\left(\phi_{1}\left(Q_{1}\right)\right)=\phi_{2}\left(Q_{2}\right) \tag{3-14}
\end{equation*}
$$

Then, using (for any $i$ such that $m_{i, 1}$ and $m_{i, 2}$ are nonzero) that

$$
\left[m_{i, 1}\right] P_{1}^{0}=\left(Q_{1}\right)_{t_{i}} \quad \text { and } \quad\left[m_{i, 2}\right] P_{2}^{0}=\left(Q_{2}\right)_{t_{i}}
$$

along with the fact that $\tau\left(\phi_{1}\left(\left(Q_{1}\right)_{t_{i}}\right)\right)=\phi_{2}\left(\left(Q_{2}\right)_{t_{i}}\right)$, we obtain the conclusion in Theorem 1-1 with $\varphi:=\tau \circ\left[m_{i, 1}\right] \circ \phi_{1}$ and $\psi:=\left[m_{i, 2}\right] \circ \phi_{2}$. Finally, note that since $P_{1}$ and $P_{2}$ are nontorsion, then also $\varphi$ and $\psi$ are dominant morphisms. Indeed, if $\varphi$ were trivial, then using that $\tau$ is an isogeny and that $m_{i, 1} \neq 0$, we would obtain that $\phi_{1}$ must be trivial. But then $\phi_{2}\left(Q_{2}\right)=0$ (using (3-14)), which implies that $\phi_{2}=0$ because we assumed that $Q_{2}$ is a nontorsion section. So, if $\varphi$ were trivial (and a completely similar argument works assuming $\psi$ were trivial), we would get that both $\phi_{1}$ and $\phi_{2}$ are trivial, a contradiction.

This concludes the proof of Theorem 1-1.

## 4. Common divisors of elliptic sequences

In this section, we apply Theorem 1-1 to prove Silverman's conjecture [2004b, Conjecture 7] concerning common divisors of elliptic sequences; actually, our Proposition 4-3 provides a slightly more general statement than the original conjecture. We first recall the terminology and notation from [Silverman 2004b] that we will use in this section.

Let $k$ be an algebraically closed field of characteristic 0 . Let $C$ be a smooth projective curve defined over $k$ and let $K=k(C)$ be the function field of $C$. For any point $\gamma \in C(k)$, we let $\operatorname{ord}_{\gamma}(D)$ denote the coefficient of $\gamma$ in $D \in \operatorname{Div}(C)$. The greatest common divisor for any two effective divisors $D_{1}, D_{2} \in \operatorname{Div}(C)$ is defined as

$$
\operatorname{GCD}\left(D_{1}, D_{2}\right)=\sum_{\gamma \in C} \min \left\{\operatorname{ord}_{\gamma}\left(D_{1}\right), \operatorname{ord}_{\gamma}\left(D_{2}\right)\right\} \cdot(\gamma) \in \operatorname{Div}(C)
$$

For an elliptic curve $E$ defined over $K$, let $\pi: \mathcal{E} \rightarrow C$ be an elliptic surface whose generic fiber is $E$ and let $P \in E(K)$. Recall that the section corresponding to $P$ is denoted by $\sigma_{P}: C \rightarrow \mathcal{E}$. We denote the image of $\sigma_{P}$ by $\bar{P}:=\sigma_{P}(C) \subset \mathcal{E}$.

Let $E_{1}$ and $E_{2}$ be elliptic curves defined over $K$, let $\mathcal{E}_{i} / C$ be elliptic surfaces with generic fibers $E_{i}$, and let $P_{i} \in E_{i}(K)$ for $i=1,2$. The greatest common divisor
of $P_{1}$ and $P_{2}$ is given by

$$
\operatorname{GCD}\left(P_{1}, P_{2}\right)=\operatorname{GCD}\left(\sigma_{P_{1}}^{*}\left(\bar{O}_{\mathcal{E}_{1}}\right), \sigma_{P_{2}}^{*}\left(\bar{O}_{\mathcal{E}_{2}}\right)\right),
$$

where $\bar{O}_{\mathcal{E}_{i}}:=\sigma_{O_{i}}(C)$ is the zero section on $\mathcal{E}_{i}$ corresponding to the identity $O_{i}$ of $E_{i}$ and $\sigma_{P_{i}}^{*}\left(\bar{O}_{\mathcal{E}_{i}}\right)$ is the pull-back under $\sigma_{i}: C \rightarrow \mathcal{E}_{i}$ of $\bar{O}_{\mathcal{E}_{i}}$ as a divisor of $\mathcal{E}_{i}$ for $i=1,2$. Thus, for any given $Q_{i} \in E_{i}(K), \operatorname{GCD}\left(P_{1}-Q_{1}, P_{2}-Q_{2}\right)$ is the greatest common divisor of the two points $P_{i}-Q_{i} \in E_{i}$ for $i=1,2$. In the following, points $P_{1}$ and $P_{2}$ are called ( $K-$ ) dependent if there are morphisms $\varphi: E_{1} \rightarrow E_{2}$ and $\psi: E_{2} \rightarrow E_{2}$ not both trivial such that $\varphi\left(P_{1}\right)=\psi\left(P_{2}\right)$; otherwise they are called independent. Note that if one of $P_{1}$ and $P_{2}$ is a torsion point, then they are automatically dependent.

Motivated by the result of [Ailon and Rudnick 2004], Silverman conjectured that an elliptic analogue also exists. For the convenience of the reader, we recall his conjecture.
Conjecture 4-1 Silverman [1994b, Conjecture 7]. Let $K=k(C)$ be the function field of a smooth projective curve $C$ over an algebraically closed field $k$ of characteristic 0 , let $E_{1} / K$ and $E_{2} / K$ be elliptic curves, and let $P_{1} \in E_{1}(K)$ and $P_{2} \in E_{2}(K)$ be $K$-independent points.
(i) There is a constant $c=c\left(K, E_{1}, E_{2}, P_{1}, P_{2}\right)$ such that

$$
\operatorname{deg} \operatorname{GCD}\left(\left[n_{1}\right] P_{1},\left[n_{2}\right] P_{2}\right) \leq c \quad \text { for all } n_{1}, n_{2} \geq 1
$$

(ii) Further, there is an equality

$$
\operatorname{GCD}\left([n] P_{1},[n] P_{2}\right)=\operatorname{GCD}\left(P_{1}, P_{2}\right) \quad \text { for infinitely many } n \geq 1 .
$$

Remark 4-2. Silverman [1994b, Theorem 8] showed that Conjecture 4-1 is true provided that both $E_{1}$ and $E_{2}$ have constant $j$-invariant as a consequence of Raynaud's theorem [1983].

As an application of Theorem 1-1, we prove that Conjecture 4-1 holds (even in a slightly stronger form); we strengthen further the conclusion from Conjecture 4-1 when $k=\overline{\mathbb{Q}}$.

Proposition 4-3. Let $k$ be an algebraically closed field of characteristic 0. Let C be a smooth projective curve defined over $k$, let $K=k(C)$ and let $E_{i} / K, i=1,2$, be elliptic curves defined over $K$. Let $P_{i}, Q_{i} \in E_{i}(K)$ for $i=1,2$ and furthermore, assume that $P_{1}$ and $P_{2}$ are $K$-independent.
(i) If $k=\overline{\mathbb{Q}}$, then there exists an effective divisor $D \in \operatorname{Div}(C)$ such that

$$
\operatorname{GCD}\left(\left[n_{1}\right] P_{1}-Q_{1},\left[n_{2}\right] P_{2}-Q_{2}\right) \leq D
$$

for all integers $n_{i}$ such that $\left[n_{i}\right] P_{i} \neq Q_{i}, i=1,2$.
(ii) For an arbitrary algebraically closed field $k$ of characteristic 0 , there exists an effective divisor $D_{0} \in \operatorname{Div}(C)$ such that

$$
\operatorname{GCD}\left(\left[n_{1}\right] P_{1},\left[n_{2}\right] P_{2}\right) \leq D_{0}
$$

for all nonzero integers $n_{i}$.
(iii) The set

$$
\left\{n \geq 1: \operatorname{GCD}\left([n] P_{1},[n] P_{2}\right)=\operatorname{GCD}\left(P_{1}, P_{2}\right)\right\}
$$

has positive density in $\mathbb{N}$.
(iv) For all but finitely many primes $q$, we have

$$
\operatorname{GCD}\left([q] P_{1},[q] P_{2}\right)=\operatorname{GCD}\left(P_{1}, P_{2}\right)
$$

Remark 4-4. The conclusion of Proposition 4-3 (i) for an arbitrary algebraically closed field $k$ of characteristic 0 would follow from our method once the validity of DeMarco-Mavraki's result [DeMarco and Mavraki 2017] (see Theorem 2-3) is extended over function fields. In turn, their result is contingent on establishing the smooth variation of the canonical height in fibers of an elliptic surface defined over a function field (over $\overline{\mathbb{Q}}$ ).

The proof of Proposition 4-3 relies on Theorem 1-1 and the following lemma which is a variant of [Silverman 2004b, Lemma 4] bounding $\operatorname{ord}_{\gamma}\left(\sigma_{[n] P}^{*}\left(\bar{O}_{\mathcal{E}}\right)\right)$ for $\gamma \in C$ and all integers $n \neq 0$.
Lemma 4-5. Let $k$ be an algebraically closed field of characteristic 0 . Let $E$ be an elliptic curve defined over $k(C)$ and let $\mathcal{E} \rightarrow C$ be an elliptic surface whose generic fiber is $E$. Let $\gamma \in C(k)$ and let $P, Q \in E(k(C))$ be given. There exists a constant $m=m(\gamma, E, P, Q)$ such that $\operatorname{ord}_{\gamma}\left(\sigma_{[n] P}^{*}(\bar{Q})\right) \leq m$ for all integers $n$ such that $[n] P \neq Q$.
Proof. Observe that $\operatorname{ord}_{\gamma}\left(\sigma_{[n] P}^{*}(\bar{Q})\right) \geq 1$ if and only if $\sigma_{[n] P}(\gamma)=\sigma_{Q}(\gamma)$. Moreover, $\sigma_{Q}(\gamma)$ is a torsion point of $\mathcal{E}_{\gamma}$ if and only if there are more than one $n$ such that $\operatorname{ord}_{\gamma}\left(\sigma_{[n] P}^{*}(\bar{Q})\right) \geq 1$.

It suffices to prove the assertion when $\operatorname{ord}_{\gamma}\left(\sigma_{[n] P}^{*}(\bar{Q})\right) \geq 1$ for more than one integer $n$. Thus, we assume that $\sigma_{Q}(\gamma)$ is a torsion point of $\mathcal{E}_{\gamma}$. Let $\ell$ be the order of $\sigma_{Q}(\gamma)$ and assume that $\operatorname{ord}_{\underline{\gamma}}\left(\sigma_{[n] P}^{*} P(\bar{Q})\right) \geq 1$ for some integer $n$ such that $[n] P \neq Q$. It follows that $\operatorname{ord}_{\gamma}\left(\sigma_{[n] P}^{*}(\bar{Q})\right)$ is finite and

$$
\begin{equation*}
\sigma_{[\ell n] P}(\gamma)=[\ell] \sigma_{[n] P}(\gamma)=[\ell] \sigma_{Q}(\gamma)=O_{\mathcal{E}_{\gamma}} \tag{4-6}
\end{equation*}
$$

which is the zero element for the elliptic curve $\mathcal{E}_{\gamma}$.
If $Q$ is the zero element of $E$, then it follows from [Silverman 2004b, Lemma 4] that the value of $\operatorname{ord}_{\gamma}\left(\sigma_{[n] P}^{*}\left(\bar{O}_{\mathcal{E}}\right)\right)$ is bounded independently of $n \neq 0$ and we are done in this case.

Assume that $Q \neq O$. Then (4-6) yields the inequality

$$
\operatorname{ord}_{\gamma}\left(\sigma_{[n] P}^{*}(\bar{Q})\right) \leq \operatorname{ord}_{\gamma}\left(\sigma_{[\ell n] P}^{*}\left(\bar{O}_{\mathcal{E}}\right)\right)
$$

Note that the right-hand side of the above inequality involves only $\operatorname{ord}_{\gamma}\left(\sigma_{[m] P}^{*}\left(\bar{O}_{\mathcal{E}}\right)\right)$, which is bounded independently of the integer $m$ in question as remarked above. Hence, we conclude that $\operatorname{ord}_{\gamma}\left(\sigma_{[n] P}^{*}(\bar{Q})\right.$ ) is bounded independently of $n \neq 0$ (and $n$ such that $[n] P \neq Q)$. As $Q \neq O$, we also have that $\operatorname{ord}_{\gamma}\left(\sigma_{[n] P}^{*}(\bar{Q})\right)$ is finite if $n=0$. Thus we obtain that $\operatorname{ord}_{\gamma}\left(\sigma_{[n] P}^{*}(\bar{Q})\right)$ is bounded independently of $n$ such that $[n] P \neq Q$, which concludes our proof.

Proof of Proposition 4-3. We first prove part (i) in Proposition 4-3. So, for each $\gamma \in C(\overline{\mathbb{Q}})$, let $m_{i, \gamma}$ be an upper bound for $\operatorname{ord}_{\gamma}\left(\sigma_{[n] P_{i}}^{*}\left(\bar{Q}_{i}\right)\right)$ as in Lemma 4-5. Set $m_{\gamma}=\min \left\{m_{1, \gamma}, m_{2, \gamma}\right\}$. Since $P_{1}$ and $P_{2}$ are independent, by Theorem 1-1 we may take $m_{\gamma}=0$ for all but finitely many points $\gamma \in C(\overline{\mathbb{Q}})$; let $S$ be the finite set of points $\gamma \in C(\overline{\mathbb{Q}})$ for which $m_{\gamma}>0$. Let

$$
D:=\sum_{\gamma \in S} m_{\gamma}(\gamma)
$$

Then, $D$ is an effective divisor of $C$. Now it follows directly from Lemma 4-5 that $\operatorname{GCD}\left(\left[n_{1}\right] P_{1}-Q_{1},\left[n_{2}\right] P_{2}-Q_{2}\right) \leq D$ for all $n_{i}$ such that $\left[n_{i}\right] P \neq Q_{i}$ for both $i=1,2$. Indeed,

$$
\begin{aligned}
\operatorname{GCD}\left(\left[n_{1}\right] P_{1}-Q_{1},\left[n_{2}\right] P_{2}-\right. & \left.Q_{2}\right) \\
& =\operatorname{GCD}\left(\sigma_{\left[n_{1}\right] P_{1}-Q_{1}}^{*}\left(\bar{O}_{\mathcal{E}_{1}}\right), \sigma_{\left[n_{2}\right] P_{2}-Q_{2}}^{*}\left(\bar{O}_{\mathcal{E}_{2}}\right)\right) \\
& =\operatorname{GCD}\left(\sigma_{\left[n_{1}\right] P}^{*}\left(\overline{Q_{1}}\right), \sigma_{\left[n_{2}\right] P_{2}}^{*}\left(\overline{Q_{2}}\right)\right) \\
= & \sum_{\gamma \in C(\overline{\mathbb{Q}})} \min \left\{\operatorname{ord}_{\gamma}\left(\sigma_{\left[n_{1}\right] P_{1}}^{*}\left(\overline{Q_{1}}\right)\right), \operatorname{ord}_{\gamma}\left(\sigma_{\left[n_{2}\right] P_{2}}^{*}\left(\overline{Q_{2}}\right)\right)\right\} \\
\leq & \sum_{\gamma \in C(\overline{\mathbb{Q}})} \min \left\{m_{1, \gamma}, m_{2, \gamma}\right\} \cdot(\gamma) \leq \sum_{\gamma \in S} m_{\gamma}(\gamma)
\end{aligned}
$$

For the proof of part (ii) in Proposition 4-3, we let $Q_{i}=O_{i}$ be the zero element of $E_{i}$ for $i=1$, 2. If $k=\overline{\mathbb{Q}}$, then the result follows immediately from part (i). Now, for the general case, we note that it suffices to prove the existence of at most finitely many $t \in C(k)$ such that both $\left(P_{1}\right)_{t}$ and $\left(P_{2}\right)_{t}$ are torsion points on the elliptic fiber $\mathcal{E}_{1, t}$ and $\mathcal{E}_{2, t}$ respectively; indeed, the fact that the multiplicity of each such $t$ appearing in a divisor $\operatorname{GCD}\left(\left[n_{1}\right] P_{1},\left[n_{2}\right] P_{2}\right)$ is bounded follows exactly as in the proof of part (i), using Lemma 4-5. On the other hand, if there exist infinitely many $t \in C(k)$ such that both $\left(P_{1}\right)_{t}$ and $\left(P_{2}\right)_{t}$ are torsion, then (according to [Masser and Zannier 2014, Theorem, p. 117]) $P_{1}$ and $P_{2}$ are related, which yields a contradiction.

The conclusion of part (iii) in Proposition 4-3 was proven by Silverman [2004b, Theorem 8 (b)] in the case where both $E_{1}, E_{2}$ have constant $j$-invariants. We generalize his argument as follows. For each of the finitely many $\gamma \in C(k)$ which does not appear in the support of $\operatorname{GCD}\left(P_{1}, P_{2}\right)$, but for which there exists some positive integer $n$ such that $\gamma$ is contained in the support of the divisor $\operatorname{GCD}\left([n] P_{1},[n] P_{2}\right)$, or equivalently,

$$
\begin{equation*}
\text { the divisor } \operatorname{GCD}\left([n] P_{1},[n] P_{2}\right)-(\gamma) \text { is effective, } \tag{4-7}
\end{equation*}
$$

we let $n_{\gamma}$ be the smallest such positive integer $n$ for which (4-7) holds. Then, it is easy to see that $\gamma$ is contained in the support of $\operatorname{GCD}\left([n] P_{1},[n] P_{2}\right)$ if and only if $n_{\gamma} \mid n$. Also, for each of these points $\gamma$ which are not in the support of $\operatorname{GCD}\left(P_{1}, P_{2}\right)$, we have $n_{\gamma}>1$. This implies that for any positive integer $n$ which is not divisible by any of the finitely many integers $n_{\gamma}$, we have

$$
\operatorname{GCD}\left([n] P_{1},[n] P_{2}\right)=\operatorname{GCD}\left(P_{1}, P_{2}\right)
$$

The conclusion in part (iv) in Proposition 4-3 follows from the proof of part (iii) since $\operatorname{GCD}\left([q] P_{1},[q] P_{2}\right)=\operatorname{GCD}\left(P_{1}, P_{2}\right)$ for all primes $q$ which do not divide any of the finitely many numbers $n_{\gamma}>1$.

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