# Pacific Journal of Mathematics


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Volume 295 No. 1 July 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 295, No. 1, 2018
dx.doi.org/10.2140/pjm.2018.295.31

# STABILITY PROPERTIES OF POWERS OF IDEALS IN REGULAR LOCAL RINGS OF SMALL DIMENSION

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**Let $(R, \mathfrak{m})$ be a regular local ring or a polynomial ring over a field, and let $I$ be an ideal of $R$ which we assume to be graded if $R$ is a polynomial ring. Let $\operatorname{astab} I$, $\overline{\operatorname{astab}}\, I$ and $\operatorname{dstab} I$, respectively, be the smallest integers $n$ for which $\operatorname{Ass} I^n$, $\operatorname{Ass} \bar{I}^n$ and $\operatorname{depth} I^n$ stabilize. Here $\bar{I}^n$ denotes the integral closure of $I^n$.**

**We show that $\operatorname{astab} I = \overline{\operatorname{astab}}\, I = \operatorname{dstab} I$ if $\dim R \leq 2$, while already in dimension three, $\operatorname{astab} I$ and $\overline{\operatorname{astab}}\, I$ may differ by any amount. Moreover, we show that if $\dim R = 4$, there exist ideals $I$ and $J$ such that for any positive integer $c$ one has $\operatorname{astab} I - \operatorname{dstab} I \geq c$ and $\operatorname{dstab} J - \operatorname{astab} J \geq c$.**

## Introduction

Let $(R, \mathfrak{m})$ be a commutative Noetherian ring and $I$ be an ideal of $R$. Brodmann [1979a] proved that the set of associated prime ideals $\operatorname{Ass} I^k$ stabilizes. In other words, there exists an integer $k_0$ such that $\operatorname{Ass} I^k = \operatorname{Ass} I^{k_0}$ for all $k \geq k_0$. The smallest such integer $k_0$ is called the *index of Ass-stability* of $I$, and denoted by $\operatorname{astab} I$. Moreover, $\operatorname{Ass} I^{k_0}$ is called the *stable set of associated prime ideals* of $I$. It is denoted by $\operatorname{Ass}^\infty I$. For the integral closures $\overline{I^k}$ of the powers of $I$, McAdam and Eakin [1979] showed that $\operatorname{Ass} \overline{I^k}$ stabilizes as well. We denote the index of stability for the integral closures of the powers of $I$ by $\overline{\operatorname{astab}}\, I$, and denote its stable set of associated prime ideals by $\overline{\operatorname{Ass}}^\infty I$.

Brodmann [1979b] also showed that $\operatorname{depth} R/I^k$ stabilizes. The smallest power of $I$ for which depth stabilizes is denoted by $\operatorname{dstab} I$. This stable depth is called the *limit depth* of $I$, and is denoted by $\lim_{k\to\infty} \operatorname{depth} R/I^k$. These indices of stability have been studied and compared to some extent in [Herzog and Qureshi 2015; Herzog et al. 2013]. The purpose of this work is to compare once again these stability indices. The main result is that if $(R, \mathfrak{m})$ is a regular local ring with $\dim R \leq 2$, then all 3 stability indices are equal, but if $\dim R = 3$, then we still have $\operatorname{astab} I = \operatorname{dstab} I$, while $\operatorname{astab} I$ and $\overline{\operatorname{astab}}\, I$ may differ by any amount. On the other hand, if $\dim R \geq 4$, we will show by examples that in general a comparison

*MSC2010:* 13A15, 13A30, 13C15.
*Keywords:* associated primes, depth stability number.

between these stability indices is no longer possible. In other words, any inequality between these invariants may occur.

Quite often, but not always, depth $R/I^k$ is a nonincreasing function on $n$. In the last section we prove that if $(R, \mathfrak{m})$ is a 3-dimensional regular local ring and $I$ satisfies $I^{k+1} : I = I^k$ for all $k$, then depth $R/I^k$ is nonincreasing. For any unexplained notion or terminology, we refer the reader to [Bruns and Herzog 1993].

Several explicit examples were performed with help of the computer algebra systems [CoCoA] and [Macaulay2], as well as with the program in [Bayati et al. 2011] which allows one to compute $\operatorname{Ass}^\infty I$ of a monomial ideal $I$.

## 1. The case dim $R \leq 3$

In this section we study the behavior of the stability indices for regular rings of dimension $\leq 3$. In the proofs we will use the following elementary and well known fact: let $I \subset R$ be an ideal of height 1 in the regular local ring $R$. Then there exists $f \in R$ such that $I = fJ$ where either $J = R$ or otherwise height$(J) > 1$. Indeed, let $I = (f_1, \ldots, f_m)$. Since $R$ is factorial, the greatest common divisor of $f_1, \ldots, f_m$ exists. Let $f = \gcd(f_1, \ldots, f_m)$, and $g_i = f_i/f$ for $i = 1, \ldots, m$. Then $I = fJ$, where $J = (g_1, \ldots, g_m)$. Suppose height$(J) = 1$; then there exists a prime ideal $P$ of height 1 with $J \subset P$. Since $R$ is regular, $P$ is a principal ideal, say $P = (g)$. It follows then that $g$ divides all $g_i$, but $\gcd(g_1, \ldots, g_m) = 1$, a contradiction.

**Remark 1.1.** Let $(R, \mathfrak{m})$ be a regular local ring with $\dim R \leq 2$ and let $I$ be an ideal of $R$. Then

$$\operatorname{astab} I = \overline{\operatorname{astab}}\, I = \operatorname{dstab} I = 1.$$

*Proof.* If $\dim R \leq 1$, then either $R$ is a field or a principal ideal domain, and the statement is trivial. Now suppose $\dim R = 2$ that and $I \neq 0$. If height$(I) = 2$, then $\mathfrak{m}$ belongs to $\operatorname{Ass} I^k$ and $\operatorname{Ass} \overline{I^k}$ for all $k$, and the assertion is trivial. Hence, we may assume that height$(I) = 1$. Then $I = fJ$ with $J = R$ or height$(J) = 2$. In the first case $I$ is a principal ideal, and the assertion is trivial. In the second case, $I^k = f^k J^k$ for all $k$, and $J^k$ is $\mathfrak{m}$-primary. Thus there exists $g \notin J^k$ with $g\mathfrak{m} \in J^k$. Then $gf^k \notin f^k J^k$ and $gf^k\mathfrak{m} \in f^k J^k$. This shows that in the second case $\mathfrak{m} \in \operatorname{Ass} I^k$ for $k$, so that $\operatorname{astab} I = \operatorname{dstab} I = 1$.

Finally observe that in the second case, $\overline{I^k} = f^k \overline{J^k}$ for all $k$. This shows that $\mathfrak{m} \in \operatorname{Ass} \overline{I^k}$ for all $k$, so that also in this case $\overline{\operatorname{astab}}\, I = 1$. □

**Theorem 1.2.** *Let $(R, \mathfrak{m})$ be a regular local ring with* $\dim R \leq 3$ *and $I$ be an ideal of $R$. Then* $\operatorname{astab} I = \operatorname{dstab} I$.

*Proof.* By Remark 1.1, we may assume that $\dim R = 3$. If height$(I) \geq 2$, then $\operatorname{Ass} I^k \subseteq \operatorname{Min}(I) \cup \{\mathfrak{m}\}$ for all $k$. This implies at once that $\operatorname{astab} I = \operatorname{dstab} I$. Now suppose that height$(I) = 1$. If $I$ is a principal ideal, then the assertion is again trivial. Otherwise, $I = fJ$ with $\operatorname{height}(J) \geq 2$. Since $I^k$ is isomorphic to $J^k$ as an $R$-module, it follows that $\operatorname{proj\,dim} I^k = \operatorname{proj\,dim} J^k$ for all $k$. This implies that $\operatorname{proj\,dim} R/I^k = \operatorname{proj\,dim} R/J^k$ for all $k$, and consequently $\operatorname{depth} R/I^k = \operatorname{depth} R/J^k$, by the Auslander–Buchsbaum formula. Thus, $\operatorname{dstab} I = \operatorname{dstab} J$.

We claim that $\operatorname{astab} I = \operatorname{astab} J$. Since we have already seen that $\operatorname{astab} J = \operatorname{dstab} J$ if $\operatorname{height}(J) \geq 2$, the claim then implies that $\operatorname{astab} I = \operatorname{dstab} I$, as desired.

The claim follows once we have that shown

$$\operatorname{Ass} I^k = \operatorname{Ass} f^k J^k = \operatorname{Min}(f) \cup \operatorname{Ass} J^k.$$

For that we only need to prove the second equation. So let $P \in \operatorname{Spec} R$ with $f^k J^k \subset P$. Then $P \in \operatorname{Ass} f^k J^k$ if and only if $R_P/f^k J^k R_P$ has depth 0. If $J \not\subset P$, then $f^k J^k R_P = f^k R_P$, and hence $\operatorname{depth} R_P/f^k J^k R_P = 0$ if and only if $\operatorname{depth} R_P/f^k R_P = 0$, and this is the case if and only if $P \in \operatorname{Min}(f)$. If $J \subset P$, then the $R_P$-modules $f^k J^k R_P$ and $J^k R_P$ are isomorphic, so that with the arguments as above $\operatorname{depth} R_P/f^k J^k R_P = \operatorname{depth} R_P/J^k R_P$, which shows that in this case $P \in \operatorname{Ass} f^k J^k$ if and only if $P \in \operatorname{Ass} J^k$. This completes the proof. □

The statements shown so far and its proofs made for ideals in a regular local ring are valid as well for any graded ideal in a polynomial ring.

We now turn to some explicit examples. Hibi et al. [2016, Proposition 1.5] show that for any integer $t \geq 2$ the ideal $I = (x^t, xy^{t-2}z, y^{t-1}z) \subset K[x, y, z]$ satisfies $\operatorname{dstab} I = t$. Since by Theorem 1.2, $\operatorname{astab} I = \operatorname{dstab} I$, this example shows that in a 3-dimensional graded or local ring (we may pass to $K[|x, y, z|]$) both the index of depth stability as well as the index of Ass-stability may be any given number.

The following example shows that already for an ideal $I$ in a 3-dimensional polynomial ring the invariants $\operatorname{astab} I$ and $\overline{\operatorname{astab}}\, I$ may differ.

**Example 1.3.** Let $R = K[x, y, z]$ be a polynomial ring over a field $K$ and let $I = ((xy)^2, (xz)^2, (yz)^2) \subset R$. Then $\operatorname{astab} I = 2$ and $\overline{\operatorname{astab}}\, I = 1$.

*Proof.* We first claim that $I^n : (xy)^2 = I^{n-1} + z^{2n}(x^2, y^2)^{n-2}$. Indeed, let $J = ((xz)^2, (yz)^2)$. Then $I^n = J^n + (xy)^2 I^{n-1}$, and hence $I^n : (xy)^2 = J^n : (xy)^2 + I^{n-1}$. Since $J^n : (xy)^2 = z^{2n}(x^2, y^2)^n : (xy)^2 = z^{2n}(x^2, y^2)^{n-2}$, the assertion follows.

By symmetry, we also have $I^n : (xz)^2 = I^{n-1} + y^{2n}(x^2, z^2)^{n-2}$ and $I^n : (yz)^2 = I^{n-1} + x^{2n}(y^2, z^2)^{n-2}$. Thus, for all $n \geq 1$ we obtain

$$\begin{aligned} I^n : I &= (I^n : (xy)^2) \cap (I^n : (xz)^2) \cap (I^n : (yz)^2) \\ &= (I^{n-1} + z^{2n}(x^2, y^2)^{n-2}) \cap (I^{n-1} + y^{2n}(x^2, z^2)^{n-2}) \cap (I^{n-1} + x^{2n}(y^2, z^2)^{n-2}) \\ &= I^{n-1} + (z^{2n}(x^2, y^2)^{n-2}) \cap (y^{2n}(x^2, z^2)^{n-2}) \cap (x^{2n}(y^2, z^2)^{n-2}) = I^{n-1}. \end{aligned}$$

In other words, $I$ satisfies strong persistence in the sense of [Herzog and Qureshi 2015]. In particular, $\operatorname{Ass} I^n \subset \operatorname{Ass} I^{n+1}$ for all $n \geq 1$. Now since $\operatorname{Ass} I =$

$\{(x, y), (x, z), (y, z)\}$ and $\operatorname{Ass} I^2 = \{(x, y), (x, z), (y, z), (x, y, z)\}$, we deduce from this that $\operatorname{astab} I = \operatorname{dstab} I = 2$.

With Macaulay2 one checks that $\bar{I} = \big((xy)^2, (xz)^2, (yz)^2, xyz^2, xy^2z, x^2yz\big)$ and that $\operatorname{Ass} \bar{I} = \{(x, y), (x, z), (y, z), (x, y, z)\}$. By [McAdam 1983, Corollary 11.28], one has $\operatorname{Ass} \bar{I} \subset \operatorname{Ass} \overline{I^2} \subset \cdots \subset \overline{\operatorname{Ass}}^{\infty} I$. Since $\operatorname{Ass} \overline{I^n}$ is a subset of the monomial prime ideals containing $I$, and since this set is $\{(x, y), (x, z), (y, z), (x, y, z)\}$, we see that $\operatorname{Ass} \bar{I} = \operatorname{Ass} \overline{I^n}$ for all $n$. Hence, $\overline{\operatorname{astab}}\, I = 1$. □

The difference $\operatorname{astab} I - \overline{\operatorname{astab}}\, I$ may in fact be as big as we want:

**Theorem 1.4.** *Let $R = k[x, y, z]$ be the polynomial ring over a field $K$, $c$ be a positive integer and $I = (x^{2c+2}, xy^{2c}z, y^{2c+2}z)$. Then $\operatorname{astab} I = c+2$ and $\overline{\operatorname{astab}}\, I = 2$.*

*Proof.* Note that $I = (x^{2c+2}, z) \cap (x, y^{2c+2}) \cap (y^{2c}, x^{2c+2})$, from which it follows that $\dim R/I = \operatorname{depth} R/I = 1$.

In the next step we prove that $I^n : I = I^{n-1}$ for all $n$. Then [Herzog and Qureshi 2015, Theorem 1.3] implies that $\operatorname{Ass} I^n \subseteq \operatorname{Ass} I^{n+1}$ for all $n$. In particular, if $\operatorname{depth} R/I^k = 0$ for some $k$, then $\operatorname{depth} R/I^r = 0$ for all $r \geq 0$. Since $\operatorname{depth} R/I^k \leq 1$ for all $k$, it then follows that $\operatorname{depth} R/I^k \geq \operatorname{depth} R/I^{k+1}$ for all $k$.

In order to show that $I^n : I = I^{n-1}$, observe that

$$I^n : x^{2c+2} = I^{n-1} + ((y^{2c}z)^n (x, y^2)^n : x^{2c+2}) = I^{n-1} + (y^{2c}z)^n (x, y^2)^{n-2(c+1)},$$

and that

$$\begin{aligned} I^n : xy^{2c}z &= I^{n-1} + ((x^{2c+2}, y^{2c+2}z)^n : xy^{2c}z) \\ &\subseteq I^{n-1} + (((x^{2c+2}, y^{2c+2}z)^n : y^{2c+2}z) : x^{2c+2}) \\ &= I^{n-1} + (x^{2c+2}, y^{2c+2}z)^{n-2}. \end{aligned}$$

Similarly we have

$$\begin{aligned} I^n : y^{2c+2}z &= I^{n-1} + (x^n (x^{2c+1}, y^{2c}z)^n : y^{2c+2}z) \\ &\subseteq I^{n-1} + (x^n (x^{2c+1}, y^{2c}z)^n : y^{4c}z^2) \\ &= I^{n-1} + x^n (x^{2c+1}, y^{2c}z)^{n-2}. \end{aligned}$$

Now since

$$\begin{aligned} I^{n-1} &\subseteq (I^n : I) \\ &\subseteq I^{n-1} + (y^{2c}z)^n (x, y^2)^{n-2(c+1)} \cap (x^{2c+2}, y^{2c+2}z)^{n-2} \cap x^n (x^{2n+1}, y^{2c}z)^{n-2} \\ &\subseteq I^{n-1} + I^n = I^{n-1}, \end{aligned}$$

it follows that $I^n : I = I^{n-1}$ for all $n$, as desired.

Next we claim that $I^n : x^{2c+2} = I^{n-1}$ for all $n \leq c+1$.

If $n = 1$, there is nothing to prove. Let $1 < n \leq c+1$. By a calculation as before we see that

$$\begin{aligned} I^n : x^{2c+2} &= I^{n-1} + ((y^{2c}z)^n(x, y^2)^n : x^{2c+2}) = I^{n-1} + (y^{2c}z)^n \\ &= I^{n-1} + (y^{(2c+2)(n-1)+2c+2-2n}z^n) = I^{n-1} + (y^{2c+2}z)^{n-1}y^{2c+2-2n}z \\ &= I^{n-1}. \end{aligned}$$

We proceed by induction on $n$ to show that $\operatorname{depth} S/I^n = 1$ for $n \leq c+1$. We observed already that $\operatorname{depth} S/I = 1$, Now let $1 < n \leq c+1$. Then, since $I^n : x^{2c+2} = I^{n-1}$, we obtain the exact sequence

$$0 \to R/I^{n-1} \xrightarrow{x^{2c+2}} R/I^n \to R/(I^n, x^{2c+2}) \to 0.$$

Since by the induction hypothesis $\operatorname{depth} R/I^{n-1} = 1$, it follows that

$$\begin{aligned} \operatorname{depth} R/I^n &\geq \min\{\operatorname{depth} R/I^{n-1}, \operatorname{depth} R/(I^n, x^{2c+2})\} \\ &= \min\{1, \operatorname{depth} R/(I^n, x^{2c+2})\}. \end{aligned}$$

Note that $(I^n, x^{2c+2}) = ((y^{2c}z(x, y^2)^n, x^{2c+2})$, which implies that $R/(I^n, x^{2c+2}$ has depth 1. Thus we have $\operatorname{depth} R/I^n \geq 1$. On the other hand, we have seen before that $\operatorname{depth} R/I^n \leq \operatorname{depth} R/I = 1$, and so $\operatorname{depth} R/I^n = 1$ for all $n \leq c+1$.

In the next step we show that $\operatorname{depth} R/I^{c+2} = 0$, which then implies that $\operatorname{depth} R/I^n = 0$ for all $n \geq c+2$. In particular, it follows that $\operatorname{astab} I = c+2$.

In order to prove that $\operatorname{depth} R/I^{c+2} = 0$, we show that

$$x^{2c+2}y^{(c+1)(2c+2)-1}z^{c+1} \in (I^{c+2} : \mathfrak{m}) \setminus I^{c+2}.$$

Indeed, let $u = x^{2c+2}y^{(c+1)(2c+2)-1}z^{c+1}$. Then

$$\begin{aligned} ux &= x^{2c+2}(xy^{2c}z)(y^{2c+2}z)^c y^{2c}, \\ uy &= x^{2c+2}(y^{2c+2}z)^{c+1} \\ uz &= (xy^{2c}z)^{c+1}(xy^{2c}z)(yx^c). \end{aligned}$$

This shows that $u \in (I^{c+2} : \mathfrak{m})$.

Assume that $x^{2c+2}y^{(c+1)(2c+2)-1}z^{c+1} \in I^{c+2}$. Then

$$y^{(c+1)(2c+2)-1}z^{c+1} \in (I^{c+2} : x^{2c+2}) = I^{c+1} + (y^{2c}z)^{c+2},$$

and so $y^{(c+1)(2c+2)-1}z^{c+1} \in I^{c+1}$. Since $I^{c+1} = (x^{2c+2}, y^{2c}z(x, y^2))^{c+1}$, expansion of this power implies that

$$y^{(c+1)(2c+2)-1} \in \sum_{i=0}^{c+1}(x^{2c+2})^i(y^{2c}(x, y^2))^{c+1-i}.$$

It follows that $y^{(c+1)(2c+2)-1} \in (y^{2c}(x, y^2))^{c+1}$, which is a contradiction.

Now we compute $\overline{\operatorname{astab}} I$, and first prove that

$$\bar{I} = (I, (x^3y^{2c-1}z, x^4y^{2c-2}z, \ldots, x^{2c+1}yz)).$$

Let $J = (I, (x^3y^{2c-1}z, x^4y^{2c-2}z, \ldots, x^{2c+1}yz))$. For all $i \in \mathbb{Z}$ with $3 \le i \le 2c+1$, we have

$$\begin{aligned}(x^iy^{2c-i+2}z)^{2c} &= x^{2ic}y^{2c(2c-i+2)}z^{2c} = x^{2c(i-1)+i-2}x^{2c-i+2}y^{2c(2c-i+2)}z^{2c-i+2}z^{i-2}\\ &= x^{2c(i-1)+i-2}(xy^{2c}z)^{2c-i+2}z^{i-2}\\ &= (x^{2c+2})^{i-2}(xy^{2c}z)^{2c-i+2}z^{i-2}x^{2c+2-i} \in I^{2c}.\end{aligned}$$

Thus $J \subseteq \bar{I}$. We have $\operatorname{Ass} \bar{I}/J \subseteq \operatorname{Ass} J$. The primary decomposition of $J$ shows that $\operatorname{Ass} J = \{(x, z), (x, y)\}$. Let $P = (x, z)$. Then $(\bar{I})_P = \overline{I_P} = \overline{(x^{2c+2}, z)_P} = (x^{2c+2}, z)_P$. The last equality follows by [Huneke and Swanson 2006, Proposition 1.3.5], and so $(\bar{I}/J)_P = 0$. Hence $P \notin \operatorname{Ass} \bar{I}/J$. Now let $P = (x, y)$. Then

$$(\bar{I})_P = \overline{(x^{2c+2}, xy^{2c}, y^{2c+2})_P} \subset \overline{((x, y)^{2c+2}, xy^{2c})_P} = ((x, y)^{2c+2}, xy^{2c})_P = J_P.$$

The second equality follows by [Huneke and Swanson 2006, Exercise 1.19]. Thus we have $(\bar{I}/J)_P = 0$. This shows that $\operatorname{Ass} \bar{I}/J = \varnothing$, and hence $\bar{I} = J$, as desired. In particular, we see that

$$\operatorname{Ass} \bar{I} = \{(x, z), (x, y)\}.$$

Since $\operatorname{Ass} \bar{I} \subseteq \operatorname{Ass} \overline{I^k}$ for all $k$, it follows that $\{(x, z), (x, y)\} \subset \operatorname{Ass} I^k$ for all $k$. Suppose that $(y, z) \in \operatorname{Ass} \overline{I^k}$ for some $k$. Then $(y, z)$ is a minimal prime ideal of $I$. However, this is not the case, as can be seen from the primary decomposition of $I$.

Next we show that $\mathfrak{m} = (x, y, z)$ belongs to $\operatorname{Ass} \overline{I^2}$. Then it follows that

$$\operatorname{Ass} \overline{I^k} = \{(x, z), (x, y), (x, y, z)\} \quad \text{for all } k \ge 2,$$

thereby showing that $\overline{\operatorname{astab}} I = 2$.

In order to prove that $\mathfrak{m} \in \operatorname{Ass} \overline{I^2}$, we first show that the ideal $L$, which is equal to

$$\big(I^2, (x^4y^{4c-1}z^2, x^5y^{4c-2}z^2, \ldots, x^{2c+2}y^{2c+1}z^2), (x^{2c+5}y^{2c-1}z, x^{2c+6}y^{2c-2}z \ldots, x^{4c+3}yz)\big),$$

is contained in $\overline{I^2}$.

Since

$$I^2 = (x^{4c+4}, x^2y^{4c}z^2, y^{4c+4}z^2, x^{2c+3}y^{2c}z, x^{2c+2}y^{2c+2}z, xy^{4c+2}z^2),$$

it follows that for all integers $i$ with $4 \le i \le 2c+2$ the element

$$\begin{aligned}(x^iy^{4c-i+3}z^2)^{4c}x^{4ic}y^{4c(4c-i+3)}z^{8c} &= x^{2(4c-i+3)}y^{4c(4c-i+3)}z^{2(4c-i+3)}x^{4c(i-2)+2i-6}z^{2i-6}\\ &= (x^2y^{4c}z^2)^{4c-i+3}(x^{4c+4})^{i-3}x^{4c-2i+6}z^{2i-6}\end{aligned}$$

belongs to $(I^2)^{4c}$. Also, for all integers $i$ with $5 \le i \le 2c-2$, the element

$$\begin{aligned}(x^{2c+i}y^{2c+4-i}z)^{4c} &= x^{2(2c+4-i)}y^{4c(2c+4-i)}z^{2(2c+4-i)}x^{8c^2+4ic+2i-4c-8}z^{2i-8}\\ &= (x^2y^{4c}z^2)^{2c+4-i}x^{(4c+4)(2c+i-4)}x^{4c+8-2i}z^{2i-8}\\ &= (x^2y^{4c}z^2)^{2c+4-i}(x^{4c+4})^{2c+i-4}x^{4c+8-2i}z^{2i-8}\end{aligned}$$

belongs to $(I^2)^{4c}$. This shows $L \subseteq \overline{I^2}$.

By using primary decomposition for the ideal $L$, we see that

$$\operatorname{Ass} L = \{(x, z), (x, y), (x, y, z)\}.$$

On the other hand, by easy calculation, one verifies that $L : (x^{2c+2}y^{2c+1}z) = \mathfrak{m}$. Finally we show that $x^{2c+2}y^{2c+1}z \notin \overline{I^2}$, which then implies that $\mathfrak{m} \in \operatorname{Ass} \overline{I^2}$, as desired.

In order to prove this we show by induction on $n$ that $(x^{2c+2}y^{2c+1}z)^n \notin (I^2)^n$ for all $n$. For $n = 1$, if $x^{2c+2}y^{2c+1}z \in I^2$, then $y^{2c+1}z \in I^2 : x^{2c+2} = I + (y^{2c}z)^2 = I$, which is a contradiction.

Now let $n > 1$. Assume that $(x^{2c+2}y^{2c+1}z)^{n-1} \notin (I^2)^{n-1}$. using the induction hypothesis. If $(x^{2c+2}y^{2c+1}z)^n \in (I^2)^n$, then

$$x^{(2c+2)(n-1)}(y^{2c+1}z)^n \in (I^{2n} : x^{2c+2}) = I^{2n-1} + (y^{2c}z)^{2n}(x, y^2)^{2n-2(c+1)},$$

and so $x^{(2c+2)(n-1)}(y^{2c+1}z)^n \in I^{2n-1}$.

It follows that $x^{(2c+2)(n-1)}(y^{2c+1}z)^{n-1} \in (I^{2n-1} : y^{2c+1}z)$. Since

$$\begin{aligned}(I^{2n-1} : y^{2c+1}z) &= yI^{2n-2} + ((x^{2c+2}, xy^{2c}z)^{2n-1} : y^{2c+1}z)\\ &= yI^{2n-2} + (x^{2n-1}(x^{2c+1}, y^{2c}z)^{2n-2} : y)\\ &= yI^{2n-2} + x^{2n-1}(y^{2c-1}z(x^{2c+1}, y^{2c}z)^{2n-3} + (x^{2c+1})^{2n-2}),\end{aligned}$$

we see that $x^{(2c+2)(n-1)}(y^{2c+1}z)^{n-1} \in y(I^2)^{n-1}$, a contradiction.

Thus $(x^{2c+2}y^{2c+1}z)^n \notin (I^2)^n$ for all $n$, as desired. □

The theorem says that for any positive integer $c$ there exists a monomial ideal in $K[x, y, z]$ with $\operatorname{astab} I - \overline{\operatorname{astab}}\, I = c$. However we do not know whether for all ideals in $I \subset K[x, y, z]$ one has $\overline{\operatorname{astab}}\, I \le \operatorname{astab} I$.

## 2. The case dim $R > 3$

The purpose of this section is to show that for a polynomial ring $S$ in more than 3 variables, for a graded ideal $I \subset S$ the invariants $\operatorname{astab} I$ and $\operatorname{dstab} I$ may differ by any amount.

We begin with two examples.

**Example 2.1.** Let $R = k[x, y, z, u]$ be the polynomial ring over a field $k$ and consider the ideal $I = (xy, yz, zu)$ of $R$. Then $\operatorname{astab} I = 1$ and $\operatorname{dstab} I = 2$.

*Proof.* We have $\operatorname{Ass} I = \operatorname{Min}(I)$, and since $I$ may be viewed as the edge ideal of a bipartite graph it follows from [Herzog and Hibi 2011, Definition 1.4.5, Corollary 10.3.17] that $\operatorname{Ass} I = \operatorname{Ass} I^n$ for all $n \in \mathbb{N}$. Therefore $\operatorname{astab} I = 1$. By [Herzog and Hibi 2011, Corollary 10.3.18], $\lim_{k\to\infty} \operatorname{depth} R/I^k = 1$. Moreover, it can be seen that $\operatorname{depth} R/I = 2$ and $\operatorname{depth} R/I^2 = 1$. Since $I$ has a linear resolution, [Herzog and Hibi 2011, Theorem 10.2.6] implies that for all $k \geq 1$, $I^k$ has a linear resolution as well. Therefore, by [Herzog et al. 2013, Proposition 2.2] we have $\operatorname{depth} R/I^{k+1} \leq \operatorname{depth} R/I^k$ for all $k \in \mathbb{N}$. Hence $\operatorname{depth} R/I^k = 1$ for all $k \geq 2$, and so $\operatorname{dstab} I = 2$. □

**Example 2.2.** Let $R = K[x, y, z, u]$ be the polynomial ring in 4 variables over a field $K$, and let $I = (x^2z, uyz, u^3)$. Then $\operatorname{astab} I = 2$ and $\operatorname{dstab} I = 1$.

*Proof.* Set $J = (uyz, u^3)$. For all $n \in \mathbb{N}$, it follows that

$$I^n : x^2z = (J^n + x^2zI^{n-1}) : x^2z = I^{n-1} + (J^n : x^2z) = I^{n-1}.$$

Hence, $\operatorname{Ass} I^n \subseteq \operatorname{Ass} I^{n+1}$ for all $n \in \mathbb{N}$. By using Macaulay2 and the program in [Bayati et al. 2011], we see that $\operatorname{Ass}^\infty I = \operatorname{Ass} I^2 = \{(x, u), (z, u), (x, y, u), (x, z, u)\}$. Therefore $\operatorname{astab} I = 2$. As $\operatorname{Ass} I^n \subseteq \operatorname{Ass} I^{n+1}$ for all $n \in \mathbb{N}$, it follows that $\mathfrak{m} = (x, y, z, u) \notin \operatorname{Ass} I^n$ and so we have $\operatorname{depth} R/I^n \geq 1$. Moreover $y - z \in \mathfrak{m}$ is a nonzerodivisor on $R/I^n$ for all $n \in \mathbb{N}$. Set $\overline{R} = R/(y - z)$. Thus by [Bruns and Herzog 1993, Lemma 4.2.16] we have $\overline{R/I^n} = \overline{R}/\overline{I^n} \cong K[x, z, u]/(x^2z, uz^2, u^3)^n$. Since $xzu^{3n-1} \in (\overline{I^n}) : \overline{\mathfrak{m}} \setminus \overline{I^n}$, it follows $\operatorname{depth} \overline{R}/\overline{I^n} = 0$ and so $\operatorname{depth} R/I^n = 1$ for all $n \in \mathbb{N}$. Therefore $\operatorname{dstab} I = 1$. □

Now we come to the main result of this section.

**Theorem 2.3.** *Let $R = k[x, y, z, u]$ be the polynomial ring over a field $k$. Then for any nonnegative integer $c$, there exist two ideals $I$ and $J$ of $R$ such that the following statements hold*:

(i) $\operatorname{astab} I - \operatorname{dstab} I \geq c$.

(ii) $\operatorname{dstab} J - \operatorname{astab} J \geq c$.

*Proof.* We may assume that $c$ is a positive integer. Let $I = (x^{c+1}z^c, u^{2c-1}yz, u^{2c+1})$ and $J = (x^cy^{c-1}, y^{c-1}x^{c-1}z, z^cu^c)$. We claim that $\operatorname{astab} I = \operatorname{dstab} J = c + 1$ and $\operatorname{astab} J = \operatorname{dstab} I = 1$.

(i) In this case, by using Example 2.2, we can assume that $c \geq 2$. For all $n \in \mathbb{N}$, we have

$$\begin{aligned}(I^n : x^{c+1}z^c) &= (((u^{2c-1}yz, u^{2c+1})^n + x^{c+1}z^cI^{n-1}) : x^{c+1}z^c) \\ &= I^{n-1} + ((u^{2c-1}yz, u^{2c+1})^n : x^{c+1}z^c).\end{aligned}$$

Since $((u^{2c-1}yz, u^{2c+1})^n : x^{c+1}z^c) = ((u^{2c-1}yz, u^{2c+1})^n : z^c) \subseteq I^{n-1}$, it follows that $(I^n : x^{c+1}z^c) = I^{n-1}$ and so $\operatorname{Ass} I^n \subseteq \operatorname{Ass} I^{n+1}$. By using Macaulay2 and [Bayati et al. 2011], we have $\operatorname{Ass} I = \{(x, u), (z, u), (y, z, u), (x, y, u)\}$ and $\operatorname{Ass}^\infty I = \{(x, u), (z, u), (y, z, u), (x, z, u), (x, y, u)\}$. Set $\mathfrak{p} = (x, z, u)$. It is easily seen that $I^i : \mathfrak{p} = I^i$ for all $i \le c$ and $x^c y^{c+1} z^c u^{(2c+1)c} \in (I_{\mathfrak{p}}^{c+1} : \mathfrak{p}) \setminus I_{\mathfrak{p}}^{c+1}$. Hence $\operatorname{Ass} I = \operatorname{Ass} I^2 = \cdots = \operatorname{Ass} I^c$, $\operatorname{Ass} I^{c+1} = \operatorname{Ass}^\infty I$ and so $\operatorname{astab} I = c + 1$. By the same argument as in the proof of Example 2.2, we see that $\mathfrak{m} = (x, y, z, u) \notin \operatorname{Ass} I^n$ for all $n \in \mathbb{N}$ and so we have $\operatorname{depth} R/I^n \ge 1$ and $x - y - z \in \mathfrak{m}$ is a nonzerodivisor on $R/I^n$ for all $n \in \mathbb{N}$. Therefore $\overline{R/I^n} = \overline{R}/\overline{I^n} \cong K[y, z, u]/((y+z)^{c+1}z^c, u^{2c-1}yz, u^{2c+1})^n$, where $\overline{R} = R/(x-y-z)$. Since $z^{2c}u^{(2c+1)n-1} \in (\overline{I^n}) : \overline{\mathfrak{m}} \setminus \overline{I^n}$, it follows $\operatorname{depth} \overline{R}/\overline{I^n} = 0$ and so $\operatorname{depth} R/I^n = 1$ for all $n \in \mathbb{N}$. Therefore $\operatorname{dstab} I = 1$.

(ii) For all $n \in \mathbb{N}$, we have

$$\begin{aligned}(J^n : z^c u^c) &= (((x^c y^{c-1}, y^{c-1}x^{c-1}z)^n + z^c u^c J^{n-1}) : z^c u^c)\\ &= J^{n-1} + ((x^c y^{c-1}, y^{c-1}x^{c-1}z)^n : z^c u^c)\\ &= J^{n-1} + ((x^c y^{c-1}, y^{c-1}x^{c-1}z)^n : z^c).\end{aligned}$$

Since $((x^c y^{c-1}, y^{c-1}x^{c-1}z)^n : z^c) \subseteq J^{n-1}$, for all $n \in \mathbb{N}$ we have $(J^n : z^c u^c) = J^{n-1}$. Therefore, $\operatorname{Ass} J^n \subseteq \operatorname{Ass} J^{n+1}$ for all $n \in \mathbb{N}$. By using Macaulay2 and [Bayati et al. 2011] we have $\operatorname{Ass}^\infty J = \{(x, z), (x, u), (y, z), (y, u)\} = \operatorname{Min}(J)$ and so $\operatorname{astab} J = 1$. Since $\mathfrak{m} \notin \operatorname{Ass} J^n$ for all $n \in \mathbb{N}$, we have $2 = \dim R/J \ge \operatorname{depth} R/J^n \ge 1$ and $x - y \in \mathfrak{m}$ is a nonzerodivisor on $R/J^n$ for all $n \in \mathbb{N}$. Again by the above argument, $\overline{R/J^n} = \overline{R}/\overline{J^n} \cong K[x, z, u]/(x^{2c-1}, x^{2c-2}z, z^c u^c)^n$, where $\overline{R} = R/(x - y)$. Since $\overline{J^i} : \overline{\mathfrak{m}} = \overline{J^i}$ for all $i \le c$ and $x^{(2c-1)n}z^{n-1}u^{c-1} \in \overline{J^n} : \overline{\mathfrak{m}} \setminus \overline{J^n}$ for all $n \ge c + 1$, it follows that $\operatorname{depth} R/J = \operatorname{depth} R/J^2 = \cdots = \operatorname{depth} R/J^c = 2$ and $\operatorname{depth} R/J^n = 1$ for all $n \ge c + 1$. Hence $\operatorname{dstab} J = c + 1$. □

## 3. Nonincreasing depth functions

**Theorem 3.1.** *Let* $(R, \mathfrak{m})$ *be a regular local ring with* $\dim R = 3$ *and* $I$ *be an ideal of* $R$. *If* $I^{n+1} : I = I^n$ *for all* $n \in \mathbb{N}$, *then* $\operatorname{depth} R/I^n$ *is nonincreasing.*

*Proof.* Suppose $\operatorname{height}(I) \ge 2$. Since $I^{n+1} : I = I^n$ for all $n \in \mathbb{N}$, it follows that $\operatorname{depth} R/I^{n+1} \le \operatorname{depth} R/I^n$. Now, let $\operatorname{height}(I) = 1$. Then there exists an ideal $J$ of $R$ and an element $f \in R$ such that $I = fJ$ and $\operatorname{height}(J) \ge 2$. As in the proof of Theorem 1.2, $\operatorname{depth} R/I^n = \operatorname{depth} R/J^n$ for all $n \in \mathbb{N}$. Since $I^{n+1} : I = I^n$ for all $n \in \mathbb{N}$, we have $J^{n+1} : J = J^n$. Thus $\operatorname{depth} R/J^{n+1} \le \operatorname{depth} R/J^n$ and so $\operatorname{depth} R/I^{n+1} \le \operatorname{depth} R/I^n$. This completes the proof. □

**Corollary 3.2.** (i) *Let* $(R, \mathfrak{m})$ *be a regular local ring with* $\dim R = 3$. *Then* $\operatorname{depth} R/\overline{I^n}$ *is nonincreasing.*

(ii) *Let* $R = k[x, y, z]$ *be a polynomial ring in* 3 *indeterminates over a field* $k$. *If* $I$ *is an edge ideal of* $R$, *then* depth $R/I^n$ *is nonincreasing.*

**Example 3.3.** *Let* $R = k[x, y, z, u]$ *be a polynomial ring and consider the ideal* $I = \big(xy^2z, yz^2u, zu^2(x+y+z+u), xu(x+y+z+u)^2, x^2y(x+y+z+u)\big)$ *of* $R$. *Then* depth $R/I =$ depth $R/I^4 = 0$ *and* depth $R/I^2 =$ depth $R/I^3 = 1$. *Thus the depth function is neither nonincreasing nor nondecreasing.*

In view of Theorem 3.1 one may ask whether in a regular local ring (of any dimension), depth $R/I^n$ is a nonincreasing function of $n$, if $I^{n+1} : I = I^n$ for all $n$.

## Acknowledgements

The second author would like to thank Universität Duisburg–Essen, especially the Department of Mathematics for its hospitality during the preparation of this work.

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Received July 11, 2017. Revised November 15, 2017.

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Volume 295 No. 1 July 2018





0030-8730(201807)295:1;1-Q