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We prove that every homomorphism from the fundamental group of a planar Peano continuum to the fundamental group of a planar or one-dimensional Peano continuum is induced by a continuous map up to conjugation. This is used to provide an uncountable family of planar Peano continua with pairwise nonisomorphic fundamental groups each of which is not homotopy equivalent to a one-dimensional space.

1. Introduction

Every continuous map between topological spaces induces a homomorphism between their respective homotopy and homology groups. This provides a method to translate questions about continuous functions of topological spaces into questions about homomorphisms of abstract groups. The converse statement is not true even for relatively nice spaces. For example, $\mathbb{R}P^\infty \times S^2$ and $\mathbb{R}P^2$ have isomorphic homotopy groups but there does not exist any continuous map which induces an isomorphism on all homotopy groups; see [Hatcher 2002, p. 345]. When only considering the first homotopy group, it is a classical result that any homomorphism from the fundamental group of a connected CW complex into the fundamental group of a $K(G, 1)$ space is induced by a continuous map; see [Hatcher 2002, Proposition 1B.9].

However, for spaces with local topological complications, the converse could fail even when only considering homomorphisms of the fundamental group. For example, an inner automorphism of the fundamental group of a one-dimensional continuum which is not locally simply connected at the chosen basepoint cannot be induced by a continuous map; see [Conner and Kent 2017, Proposition 3.12].

In the literature, the phrase *induced by a continuous map* has been used to mean both strictly induced by a continuous map and induced by a continuous map up to conjugation. To avoid confusion, we will say a homomorphism φ between fundamental groups is *induced by a continuous map* if $\varphi = f_*$ for some continuous map f. We will say that φ is *conjugate to a homomorphism induced by a continuous*

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map if there exists a path α such that $\hat{\alpha} \circ \varphi = f_*$ for some continuous map f where $\hat{\alpha}$ is the change of basepoint isomorphism induced by the path α.

Katsuya Eda [1998] was the first to prove that arbitrary homomorphisms between fundamental groups of certain spaces which are not locally simply connected are induced by continuous maps up to conjugation by showing that any endomorphism of the fundamental group of the Hawaiian earring is conjugate to one induced by a continuous map. Later, Eda proved the following generalization.

Theorem A [Eda 2010]. *Every homomorphism between fundamental groups of one-dimensional Peano continua is conjugate to a homomorphism induced by a continuous map.*

Eda actually proves a stronger statement [2010, Theorem 1.2] by allowing the range to be the fundamental group of any one-dimensional metric space. Understanding the extent to which homomorphisms of fundamental groups are induced by continuous maps of the underlying topological spaces provides an additional tool to understand the homotopy type of locally complicated spaces using their fundamental groups, see [Cannon and Conner 2006; Eda 2002; Conner and Kent 2011]. Knowing when homomorphisms are induced by continuous maps allowed Eda to prove that the fundamental group is a perfect invariant of homotopy type for one-dimensional Peano continua [Eda 2010] and is the key tool to prove that the set of points at which a space is not semilocally simply connected is constructible from the fundamental group for one-dimensional and planar Peano continua [Conner and Eda 2005; Conner and Kent 2011].

In [Conner and Kent 2011], Greg Conner and the author show that many of the known results about fundamental groups of one-dimensional spaces extend to planar spaces. Specifically, it is proved that any homomorphism from the fundamental group of a one-dimensional Peano continuum to the fundamental group of a planar Peano continuum is induced by a continuous map after composing with a change of basepoint isomorphism (Theorem A when the range is a planar Peano continuum). Here we will prove the following theorem.

Theorem 2.7. *Let $\varphi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ be a homomorphism from the fundamental group of a planar Peano continuum X into the fundamental group of a one-dimensional or planar Peano continuum Y. Then there exists a continuous function $f : X \to Y$ and a path $\alpha : (I, 0, 1) \to (Y, y_0, y)$, with the property that $f_* = \hat{\alpha} \circ \varphi$.*

In one-dimensional spaces every path class contains a unique (up to reparametrization) minimal representative and every other representative can be homotoped to the unique minimal one by removing backtracking; see [Curtis and Fort 1959, Lemma 3.1] or [Cannon and Conner 2006, Theorem 3.9] for the existence and uniqueness of reduced representatives. We will say that a loop in a one-dimensional space is *reduced* if it is the unique minimal representative in its path class. Every one-dimensional Peano continuum deformation retracts to a one-dimensional Peano continuum in which every point is contained in some reduced loop [Conner and Meilstrup 2012, Theorems 4.3 and 3.1]. With these tools in hand, to prove that homomorphisms from the fundamental group of one-dimensional Peano continua are continuous up to conjugation, one starts with a one-dimensional Peano continuum such that each point is contained in a reduced loop and then uses the homomorphism to understand where to send each reduced loop.

Two of the difficulties of the planar case are the lack of a canonical deformation retract and the lack of representatives for path classes which are analogous to reduced paths in one-dimensional spaces. To prove Theorem 2.7, we will find a one-dimensional *core* of a planar Peano continuum to which we can apply Theorem A. We will show how to continuously extend this map to all of the planar continuum.

The property that homomorphisms are induced by continuous maps up to conjugation does not hold for more general spaces. For example there exists uncountable many homomorphisms from the fundamental group of the Hawaiian earring into the fundamental group of the projective plane which are not induced by a continuous function [Conner and Spencer 2005].

Homotopy dimension. The *homotopy dimension* of a space X is the smallest covering dimension of a space homotopy equivalent to X. A space is *homotopically at most k-dimensional* if its homotopy dimension is at most k.

Cannon and Conner [2007] asked the following question:

Question. If X is a planar Peano continuum whose fundamental group is isomorphic to the fundamental group of some one-dimensional Peano continuum, is it true that X is homotopy equivalent to a one-dimensional Peano continuum?

Let $\boldsymbol{S}$ be the Sierpinski curve in $\mathbb{R}^2$ obtained by the standard Cantor construction performed on the unit square in the plane. Let $\boldsymbol{S}_i$ be the planar Peano continuum obtained from $\boldsymbol{S}$ by filling in i of the removed discs, i.e.,

$$\boldsymbol{S}_i = \boldsymbol{S} \cup \Big(\bigcup_{n=1}^{i} D_n \Big),$$

where D_n are distinct bounded components of $\mathbb{R}^2 \setminus \boldsymbol{S}$. Cannon, Conner and Zastrow showed that $\boldsymbol{S}_1$ is not homotopy equivalent to any one-dimensional space [Cannon et al. 2002]. Their example, $\boldsymbol{S}_1$, illustrates that there exists some rigidity in planar sets and at least provides some motivation as to why the previous question is interesting. Karimov, Repovš, Rosicki, and Zastrow [Karimov et al. 2005] give additional examples of planar sets spaces which are not homotopically one-dimensional.

By applying Theorem 2.7, we will show that $\boldsymbol{S}_i$ cannot have the same fundamental group as any one-dimensional Peano continua and that the $\boldsymbol{S}_i$, $\boldsymbol{S}_j$ do not have isomorphic fundamental groups for $i \neq j$.

As an application of Theorem 2.7, we prove the following result.

Theorem 2.18. *There exists an uncountable family of planar Peano continua whose fundamental groups are pairwise nonisomorphic and also not isomorphic to the fundamental group of any one-dimensional Peano continuum.*

Our family of examples is constructed by filling infinitely many of the removed squares of $\boldsymbol{S}$ in a discrete fashion and then studying the limit set of the filled squares.

2. Planar to one-dimensional or planar

We will use $\mathbb{D}$ to denote the unit disc in the Euclidean plane $\mathbb{R}^2$ and I to denote the interval $[0, 1]$. For a metric space X, let $B_r^X(x) = \{y \in X \mid d(x, y) < r\}$ and $S_r^X(x) = \{y \in X \mid d(x, y) = r\}$. For planar sets X, $B_r^X(x) = B_r^{\mathbb{R}^2}(x) \cap X$ and $S_r^X(x) = S_r^{\mathbb{R}^2}(x) \cap X$. For A a subset of a metric space X, we let $\mathcal{N}_\epsilon(A) = \{x \in X \mid d(x, A) < \epsilon\}$, the open ϵ-neighborhood of A.

For a path $f : I \to X$, let $\overline{f}(t)$ denote the path $\overline{f}(t) = f(1-t)$. For a path $\alpha : (I, 0, 1) \to (X, x_0, x_1)$, let $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by the standard change of base point isomorphism, i.e., $\hat{\alpha}([g]) = [\bar{\alpha} * g * \alpha]$. This isomorphism has inverse $\hat{\bar{\alpha}}$.

We will use $\operatorname{int}(X)$ to denote the interior of X as a subset of the plane, $\operatorname{cl}(X)$ for the closure of X in the plane and ∂X for $\operatorname{cl} X \setminus \operatorname{int}(X)$.

Theorem 2.1 [Eda 2010; Conner and Kent 2011]. *Let $\varphi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ be a homomorphism from the fundamental group of a one-dimensional Peano continuum X into the fundamental group of a one-dimensional or planar Peano continuum Y. Then there exists a continuous function $f : X \to Y$ and a path $\alpha : (I, 0, 1) \to (Y, y_0, y)$, with the property that $f_* = \hat{\alpha} \circ \varphi$.*

Lemma 2.2. *Suppose that $f : \partial\mathbb{D} \to X$ is a nullhomotopic loop into a planar or one-dimensional set. Then f is nullhomotopic in the $B_r^X(f(0))$ for every $r > 2\operatorname{diam}(\operatorname{im} f)$.*

Cannon and Conner [2007, Section 6: proof of Theorem 1.4] prove that every nullhomotopic loop in a planar Peano continuum bounds a disc contained in the convex hull of its image (in which case the multiplicative constant is unnecessary). However, they do not explicitly state this corollary of their proof. Another proof using the Riemann mapping theorem can be found in [Fischer and Zastrow 2005, Lemma 13]. Here we will prove a slightly weaker bound, which will be sufficient for our needs, using the Phragmén–Brouwer properties.

Proof. The lemma is trivial when X is one-dimensional since every nullhomotopic loop factors through a dendrite [Cannon and Conner 2006, Theorem 3.7] which implies that it is nullhomotopic inside of its image.

Suppose that $f : \partial\mathbb{D} \to X$ is a nullhomotopic loop into a planar set X. We will denote the smallest convex set containing a set A by $\mathrm{Hull}(A)$.

Claim. *Suppose that l is a line in the plane which is disjoint from* $\mathrm{im}\, f$ *and A is the component of $\mathbb{R}^2 \setminus l$ containing* $\mathrm{im}\, f$*. For every $\epsilon > 0$ and any extension $h : \mathbb{D} \to X$ of f, there exists an extension $\tilde{h} : \mathbb{D} \to X$ of f such that*

$$\tilde{h}(\{x \in \mathbb{D} \mid h(x) \neq \tilde{h}(x)\}) \subset X \cap \mathrm{Hull}(l \cap h(\mathbb{D})) \quad \textit{and} \quad \mathrm{im}\, \tilde{h} \subset X \cap \mathrm{cl}(\mathcal{N}_\epsilon(A)).$$

Proof of claim. Suppose that $h : \mathbb{D} \to X$ is a nullhomotopy of f. Let $\mathcal{C}$ be the components of $h^{-1}(\mathbb{R}^2 \setminus A)$ which intersect $\mathbb{R}^2 \setminus \mathcal{N}_\epsilon(A)$. Since $\mathbb{D}$ is compact, $\mathcal{C}$ is finite. For each $C \in \mathcal{C}$, let $\partial_M C$ be the boundary of the unbounded component of $\mathbb{R}^2 \setminus C$. Then $\partial_M C$ is a closed connected subset of $\mathbb{D}$ such that the closure of the bounded components of $\mathbb{R}^2 \setminus \partial_M C$ contains C. (This is the second of the Phragmén–Brouwer properties in [Wilder 1949, p. 47] applied to the unbounded component of $\mathbb{R}^2 \setminus C$.) We will denote the closure of the bounded components of $\mathbb{R}^2 \setminus \partial_M C$ by $\mathrm{wHull}(C)$.

By passing to a subset of $\mathcal{C}$, we may assume that for any two distinct elements $C, C' \in \mathcal{C}$ we have that C' is contained in the unbounded component of $\mathbb{R}^2 \setminus C$ while still maintaining the property that $h^{-1}\big(\mathbb{R}^2 \setminus \mathcal{N}_\epsilon(A)\big) \subset \bigcup_{c \in \mathcal{C}} \mathrm{wHull}(C)$.

Since $\partial_M C$ is connected, $h(\partial_M C)$ is contained in a connected component of $l \cap X$.

By the Tietze extension theorem, there exists $h_C : \mathrm{wHull}(C) \to l \cap X$ such that $h_C(x) = h(x)$ for all $x \in \partial_M C$. Since $h(\partial_M C)$ is contained in a connected component of $l \cap X$, we have that $X \cap \mathrm{Hull}(h(\partial_M C)) \subset l \cap X$ and h_C can be chosen to have image contained in $X \cap \mathrm{Hull}(h(\partial_M C))$.

By the pasting lemma for continuous functions, the function $\tilde{h} : \mathbb{D} \to X$ defined by $\tilde{h}(x) = h_C(x)$ if $x \in \mathrm{wHull}(C)$ for some $C \in \mathcal{C}$ and $\tilde{h}(x) = h(x)$ otherwise is a continuous function which extends f. By our choice of $\mathcal{C}$, $\mathrm{im}\, \tilde{h}$ is contained in $X \cap \mathrm{cl}(\mathcal{N}_\epsilon(A))$. □

Fix $\epsilon > 0$ such that $2\,\mathrm{diam}(\mathrm{im}\, f) > \sqrt{2}\,\mathrm{diam}(\mathrm{im}\, f) + (1 + \sqrt{2})\epsilon$. Let l_1, l_2 be the two distinct vertical lines and l_3, l_4 the two distinct horizontal lines such that $d(f(0), l_i) = \mathrm{diam}(\mathrm{im}\, f) + \epsilon$ for $i \in \{1, \dots, 4\}$. Notice this implies that $\mathrm{im}\, f$ is contained in the unique bounded component of $\mathbb{R}^2 \setminus \{l_1, \dots, l_4\}$. By applying the previous claim to each l_i in turn, we obtain a nullhomotopy of f which is contained in the closure of an ϵ-neighborhood of the bounded component of $\mathbb{R}^2 \setminus \{l_1, \dots, l_4\}$.

By our choice of ϵ, this is contained in the ball of radius r for any $r > 2\,\mathrm{diam}(\mathrm{im}\, f)$ which completes the proof of the lemma. □

Lemma 2.3. *Every bounded open set U of $\mathbb{R}^2$ is the union of a sequence of dyadic squares with disjoint interiors whose diameters form a null sequence. In addition,* *the squares can be chosen such that if A_i is the union of squares with side length at least $1/2^i$, then $U \setminus A_i \subset \mathcal{N}_{1/2^{i-1}}(\partial U)$.*

This is standard and well known. We present a proof to introduce notation that we will use later.

Proof. Set $\chi_i = \{(x, y) \mid 0 \le x \le 1/2^i,\ 0 \le y \le 1/2^i\}$ and let

$$Q_i = \{(n, m) + \chi_i \mid n, m \in (1/2^i)\mathbb{Z}\}$$

be the set of closed squares in the standard tiling of the plane by squares with side length $1/2^i$.

Let D_0 be the maximal subset of Q_0 such that $A_0 \subset U$ where $A_0 = \bigcup_{s \in D_0} s$. Then $U \setminus A_0 \subset \mathcal{N}_{1/2^{-1}}(\partial U)$.

We will inductively define D_i and A_i as follows. Let D_i be the maximal subset of Q_i such that $\bigcup_{s\in D_i} s \subset U \setminus \operatorname{int}(A_{i-1})$. Let $A_i = \left(\bigcup_{s\in D_i} s\right) \cup A_{i-1}$. Suppose $x \in U \setminus A_i$, then there exists some $s \in Q_i$ such that $x \in s$. Since the tilings are nested, if $s \cap \operatorname{int}(A_{i-1}) \neq \varnothing$, then $s \subset A_{i-1}$. Thus $s \cap \operatorname{int}(A_{i-1}) = \varnothing$. Since s is not in D_i and is disjoint from $\operatorname{int}(A_{i-1})$, we have $s \not\subset U$ and $d(x, \partial U) \le \operatorname{diam}(s) = \sqrt{2}/2^i < 1/2^{i-1}$. Thus $U \setminus A_i \subset \mathcal{N}_{1/2^{i-1}}(\partial U)$ and $\bigcup_{i=1}^{\infty} A_i = U$. □

Lemma 2.4. *Let $f : I \to X$ be a continuous function into a metric space X and $\mathcal{V}$ be a covering of I by closed, possibly degenerate, intervals with disjoint interiors. Suppose that $g : I \to X$ is a mapping such that, for every $V \in \mathcal{V}$, the maps g and f agree on the endpoints of V and $g|_V$ is continuous. If there exists an L such that, for every $V \in \mathcal{V}$, $\operatorname{diam}(g(V)) \le L \operatorname{diam}(f(V))$ then g is continuous.*

In addition; if there exists a K such that $g|_V$ is homotopic to $f|_V$ rel endpoints, for every $V \in \mathcal{V}$, by a homotopy of diameter at most $K \operatorname{diam}(f(V))$, then g is homotopic rel endpoints to f.

Proof. Let f, g, $\mathcal{V}$, and L be defined as in the lemma. Fix $\epsilon > 0$. Since the elements of $\mathcal{V}$ have disjoint interiors and f is uniformly continuous, there exists a cofinite subset $\mathcal{V}_0 \subset \mathcal{V}$ such that the $\operatorname{diam}(f(V)) < \epsilon/(3L)$ for all $V \in \mathcal{V}_0$. Thus $\operatorname{diam}(g(V)) \le \epsilon/3$ for all $V \in \mathcal{V}_0$.

Fix $\delta > 0$ satisfying these conditions:

(i) $d(f(x), f(y)) < \epsilon/3$ for all $x, y \in I$ such that $|x - y| < \delta$.

(ii) $d(g(x), g(y)) < \epsilon/3$ for all $x, y \in V$ for some $V \in \mathcal{V} \setminus \mathcal{V}_0$ such that $|x - y| < \delta$.

Take $x, y \in I$ such that $|x - y| < \delta$. If $x, y \in V \in \mathcal{V}$, then $d(g(x), g(y)) < \epsilon/3$ by our choice of δ and $\mathcal{V}_0$. We may assume x, y are in distinct elements of $\mathcal{V}$ and without loss of generality $x < y$. There exist points x', y' such that $x \le x' \le y' \le y$ where x', y' are endpoints of the intervals of $\mathcal{V}$ containing x, y respectively. Then $|x - x'|, |y - y'|, |x' - y'| < \delta$. Thus

$$d(g(x), g(y)) \le d(g(x), g(x')) + d(f(x'), f(y')) + d(g(y'), g(y)) < \epsilon.$$

Therefore g is uniformly continuous.

Suppose $g|_V$ is homotopic to $f|_V$ rel endpoints, for each $V \in \mathcal{V}$, by a homotopy of diameter at most K diam$(f(V))$. For each $V \in \mathcal{V}$, let $h_V : V \times I$ be a homotopy rel endpoints of $f|_V$ to $g|_V$ such that diam$(h_V(V \times I)) \leq K$ diam$(f(V))$.

Define $h : I \times I \to X$ by $h(x,t) = h_V(x,t)$ for any $V \in \mathcal{V}$ such that $x \in V$. Since $h_V(x,t) = f(x)$ for all t if x is an endpoint of V, h is well defined. Notice that $h(x,0) = f(x)$ and $h(x,1) = g(x)$.

As before, there exists a cofinite subset $\mathcal{V}_1$ of $\mathcal{V}$ such that diam(im $h_V) < \epsilon/3$ for all $V \in \mathcal{V}_1$. Fix $\delta > 0$ satisfying the following:

(i) $d(f(x), f(y)) < \epsilon/3$ for all $x, y \in I$ such that $|x - y| < \delta$.

(ii) $d(h(x,t), h(y,s)) < \epsilon/3$ for all $x, y \in V$ for some $V \in \mathcal{V} \setminus \mathcal{V}_1$ such that $|x - y| + |s - t| < \delta$.

Suppose that $(x,s), (y,t) \in I \times I$ such that $|x - y| + |s - t| < \delta$. If $x, y \in V$ for some $V \in \mathcal{V}$, then $d(h(x,t), h(y,s)) < \epsilon/3$ by our choice of δ and $\mathcal{V}_1$. Thus we may assume x, y are in distinct elements of $\mathcal{V}$ and without loss of generality $x < y$. There exist points x', y' such that $x \leq x' \leq y' \leq y$ where x', y' are endpoints of the intervals of $\mathcal{V}$ containing x and y, respectively. Then $|x - x'|, |y - y'|, |x' - y'| < \delta$. Thus

$$\begin{aligned} d(h(x,t), h(y,s)) &\leq d(h(x,t), h(x',t)) + d(h(x',t), h(y',s)) + d(h(y',s), h(y,s)) \\ &< \epsilon/3 + d(f(x'), f(y')) + \epsilon/3 < \epsilon. \end{aligned}$$ □

Remark 2.5. For a planar Peano continuum X considered as a subset of $\mathbb{R}^2$, int(X) is on open bounded subset of the plane. By Lemma 2.3, int(X) can be tiled by a null sequence of dyadic squares with disjoint interiors. If A_i is the union of squares from the tiling of int(X) with side length at least $1/2^i$, then A_i has a natural CW structure given by the tiling and we will denote the one-skeleton of A_i by $A_i^{(1)}$. Then $X^{(1)} = \partial X \cup \left(\bigcup_i A_i^{(1)}\right)$ can be considered as a type of one-skeleton for X.

The following lemma is immediate from the construction of A_i and the diameter condition of the squares composing A_i. Alternatively, given a surjective map $f : I \to X$, it is a straightforward exercise to show how to modify it to construct a surjective map from I to $X^{(1)}$.

Lemma 2.6. *Let X be a planar Peano continuum and $X^{(1)} = \partial X \cup \left(\bigcup_i A_i^{(1)}\right)$, where A_i is as in Lemma 2.3 for the bounded open set* int(X). *Then $X^{(1)}$ is a one-dimensional Peano continuum.*

Theorem 2.7. *Let $\varphi : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ be a homomorphism from the fundamental group of a planar Peano continuum X into the fundamental group of a one-dimensional or planar Peano continuum Y. Then there exists a continuous* *function* $f : X \to Y$ *and a path* $\alpha : (I, 0, 1) \to (Y, y_0, y)$, *with the property that* $f_* = \hat{\alpha} \circ \varphi$.

Proof. Let $X^{(1)} = \partial X \cup \big(\bigcup_i A_i^{(1)}\big)$, where A_i is as in Lemma 2.3 for the bounded open set $\text{int}(X)$, and let $i : X^{(1)} \to X$ be the inclusion map. Since we are only concerned about the homomorphism up to conjugation, we may assume that $x_0 \in X^{(1)}$.

Let $B = \text{int}(X) \setminus X^{(1)}$. Then B is the disjoint union of open square discs whose diameters form a null sequence.

Fix a loop $\beta : I \to X$ in X. Notice that $\beta^{-1}(B)$ is the disjoint union of open intervals in I. Let $\mathcal{V}$ be the covering of I by disjoint intervals consisting of two types: (1) the closure of a component of $\beta^{-1}(B)$ and (2) a point not contained in the closure of any interval of $\beta^{-1}(B)$. Then $\mathcal{V}$ is a cover of I by intervals with disjoint interiors.

For every nondegenerate $V \in \mathcal{V}$ there exists s_V a closed square from the tiling of $\text{int}(X)$ such that $\beta(V) \subset s_V$. For every degenerate $V \in \mathcal{V}$, let $s_V = V$. Define $\beta' : I \to X$ by letting $\beta'|_V$ be a shortest path from $\beta(a)$ to $\beta(b)$ contained in ∂s_V where $V = [a, b]$. It is an elementary computation to show that $\text{diam}(\beta'(V)) \leq 2d(\beta(a), \beta(b)) \leq 2\,\text{diam}(\beta(V))$. Since s_V is convex and contained in X, the map $h : I \times V \to X$ given by $h(t, v) = t\beta(v) + (1-t)\beta'(v)$ is a homotopy rel endpoints from $\beta|_V$ to $\beta'|_V$ with $\text{diam}\,(\text{im}\, h_V) \leq 4\,\text{diam}\,(f(V))$. Lemma 2.4 implies that β' is continuous and homotopic to β. Hence i_* is surjective.

By Theorem 2.1, $\varphi \circ i_* : \pi_1(X^{(1)}, x_0) \to \pi_1(Y, y_0)$ is conjugate to being induced by a continuous map, i.e., $\varphi \circ i_* = \hat{\hat{\alpha}} \circ f_*$ where $f : X^{(1)} \to Y$ is a continuous map and $\alpha : I \to Y$ is a continuous path.

Let s be a square for our tiling of $\text{int}(X)$. Then $f|_{\partial s}$ is a nullhomotopic loop in Y. Thus we can extend f to all of s such that $\text{diam}\,(f(s)) \leq 2\,\text{diam}\,(f(\partial s))$. Doing this for all the components of B defines an extension $\overline{f}$ of f to all of X. The diameter condition guarantees the continuity of $\overline{f}$ (the details are analogous to those of Lemma 2.4).

Let β be a loop in X. Then there exists a loop β' in $X^{(1)}$ homotopic (in X) to β. Then

$$\begin{aligned}\varphi([\beta]) &= \varphi \circ i_*([\beta']) = \hat{\hat{\alpha}} \circ f_*([\beta']) \\ &= \hat{\hat{\alpha}}([f \circ \beta']) = \hat{\hat{\alpha}}([\overline{f} \circ \beta']) \\ &= \hat{\hat{\alpha}}([\overline{f} \circ \beta]) = \hat{\hat{\alpha}} \circ \overline{f}_*([\beta])\end{aligned}$$

as desired. □

Applications. The Sierpinski curve in $\mathbb{R}^2$, which we will denote by $\boldsymbol{S}$, is constructed by iterating the process of subdividing $[0, 1] \times [0, 1]$ into 9 squares, removing the center one and repeating on each of the remaining 8 squares.

To be explicit, let $C_0 = ([0, 1] \times [0, 1])$ and define C_n inductively as follows.

$$C_n = C_{n-1} \setminus \Big\{ \bigcup_{0 \le i, j < 3^{n-1}} \Big(\frac{1+3i}{3^n}, \frac{2+3i}{3^n}\Big) \times \Big(\frac{1+3j}{3^n}, \frac{2+3j}{3^n}\Big) \Big\}.$$

Then $S = \bigcap_n C_n$. Notice that $\mathbb{R}^2 \setminus S$ is the union of countably many open squares with disjoint closures and a single unbounded component. Let $\{D_n\}$ be an enumeration of the bounded components of the complement of S.

For $A \subset \mathbb{N}$, let $S_A = S \cup \big(\bigcup_{n \in A} D_n\big)$; i.e., S_A is the space obtained from S by filling in the squares with indices in A. For $i \in \mathbb{N}$, let $S_i = S \cup \big(\bigcup_{n=1}^{i} D_n\big)$.

We will say that a sequence of subsets A_n of X converges to a set $A \subset X$, if for every $\epsilon > 0$ there exists an N such that $A_n \subset \mathcal{N}_\epsilon(A)$ and $A \subset \mathcal{N}_\epsilon(A_n)$ for all $n > N$.

Lemma 2.8. *For every $x \in S$, there exists a subsequence of natural numbers (i_n) such that D_{i_n} converges to $\{x\}$. Thus S is one-dimensional and $\bigcup_{n=1}^{\infty} \partial D_n$ is dense in S.*

Proof. Notice that C_n is contained in the closed $\sqrt{2}/3^n$-neighborhood of the boundaries of the open squares removed from C_{n-1} to obtained C_n. Thus every point in S is at most $\sqrt{2}/3^n$ from the boundary of an open square contained in $\mathbb{R}^2 \setminus S$ with side length $1/3^n$. For every n, we can choose an i_n such that D_{i_n} is a square with side length $1/3^n$ which is at most $\sqrt{2}/3^n$ from x. Then ∂D_{i_n} converges to x. Thus S is one-dimensional and $\bigcup_{n=1}^{\infty} \partial D_n$ is dense in S. □

Zastrow's example in [Cannon et al. 2002] and Example (2) in [Karimov et al. 2005] appear to suggest the following lemma.

Lemma 2.9. *Suppose that $h : X \to X$ is a continuous map of a planar Peano continuum such that every loop is freely homotopic to its image under h. Then h fixes the set of points at which X is not semilocally simply connected.*

Proof. Suppose that X is not semilocally simply connected at x and $h(x) \neq x$. Then we would be able to find an $\epsilon > 0$ such that the balls $B_\epsilon^{\mathbb{R}^2}(x)$ and $B_\epsilon^{\mathbb{R}^2}(h(x))$ are disjoint and $S_\epsilon^X(x) \subsetneq S_\epsilon^{\mathbb{R}^2}(x)$. This implies that $S_\epsilon^X(x)$ is the disjoint union of closed intervals.

Since any loop is freely homotopic to its image under h, any sufficiently small loop in $B_\epsilon^X(x)$ can be homotoped into $B_\epsilon^X(h(x))$. However, any map of an annulus which takes one boundary component into $B_\epsilon^X(x)$ and the other into $B_\epsilon^X(h(x))$ can be cut along $S_\epsilon^X(x)$ to construct a nullhomotopy of the boundary loops. The details for the cutting procedure are analogous to the proof of the claim on page 47. Thus any sufficiently small loop in $B_\epsilon(x)$ must be nullhomotopic. However this contradicts the assumption that X is not semilocally simply connected at x. □

Cannon, Conner, and Zastrow showed that S_1 is not homotopy equivalent to any one-dimensional Peano continuum. We can now use Theorem 2.7 to show even more, that the fundamental group of $\boldsymbol{S}_1$ is not *one-dimensional* in the following sense.

Theorem 2.10. *For any $s_0 \in \boldsymbol{S}$, the fundamental group $\pi_1(\boldsymbol{S}_i, s_0)$ is not isomorphic to the fundamental group of any one-dimensional Peano continuum.*

Proof. Suppose that there exists X a one-dimensional Peano continuum such that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(\boldsymbol{S}_i, s_0)$. By Theorem 2.7, there exists a continuous map $f : \boldsymbol{S}_i \to X$ which induces an isomorphism f_* of fundamental groups.

By applying Theorem 2.1 to the homomorphism f_*^{-1}, we can find a map $g : X \to \boldsymbol{S}_i$ such that $g \circ f \circ \beta$ is freely homotopic to β for every loop β based at x_0. (Note that β might not be homotopic to $g \circ f \circ \beta$ relative to endpoints.)

Since $\boldsymbol{S}_i$ is obtained by only adding finitely many discs, every neighborhood of every point in $\boldsymbol{S}$ contains a loop which is essential in $\boldsymbol{S}_i$. Thus $g \circ f$ must fix $\boldsymbol{S}$ by Lemma 2.9.

Let D_k be a square which was filled in the construction of $\boldsymbol{S}_i$. Since f maps ∂D_k to a nullhomotopic loop in a one-dimensional space, the map f must identify two distinct points x, y on the boundary of D_k. However this is a contradiction since $\partial D_k \subset \boldsymbol{S}$. □

Corollary 2.11. *For any $s_0 \in \boldsymbol{S}$ and any pair of distinct natural numbers i and j, the groups $\pi_1(\boldsymbol{S}_i, s_0)$ and $\pi_1(\boldsymbol{S}_j, s_0)$ are not isomorphic.*

Proof. We will assume that $i > j$ and proceed by way of contradiction. As in the proof of Theorem 2.10, we may assume that there are maps $f : \boldsymbol{S}_i \to \boldsymbol{S}_j$ and $g : \boldsymbol{S}_j \to \boldsymbol{S}_i$ such that $g \circ f \circ \beta$ is freely homotopic to β for any loop β based at s_0. As before, $g \circ f$ must fix $\boldsymbol{S}$.

Let D_k be a square which was filled in the construction of $\boldsymbol{S}_i$. Notice that ∂D_k must map to a simple closed curve which is nullhomotopic in $\boldsymbol{S}_j$ ($f|_{\boldsymbol{S}}$ must be injective). Hence it must map to the boundary of a square which was filled in the construction of $\boldsymbol{S}_j$. (A simple closed curve α in the plane is nullhomotopic if and only if the bounded component of $\mathbb{R}^2 \setminus \operatorname{im} \alpha$ is simply connected.) Since $i > j$, f must map two boundary circles to the same boundary circle which contradicts that fact that f restricted to $\boldsymbol{S}$ must be injective. □

We will now show how to extend Corollary 2.11 to certain nice fillings of $\boldsymbol{S}$.

Definition 2.12. Let $A \subset \mathbb{N}$. We will use $B(\boldsymbol{S}_A)$ to denote the set of points at which $\boldsymbol{S}_A$ is not semilocally simply connected. Let $K(\boldsymbol{S}_A)$ be the set of accumulation points of $\{D_n \mid n \in A\}$, i.e.,

$$K(\boldsymbol{S}_A) = \big\{x \in \boldsymbol{S} \mid \{n \in A \mid D_n \subset B_r(x)\} \text{ is infinite for every } r > 0\big\}.$$

We will say that $\boldsymbol{S}_A$ is a *discrete* filling of $\boldsymbol{S}$ if $\operatorname{cl}(D_n) \cap K(\boldsymbol{S}_A) = \varnothing$ for all $n \in A$. We will say that Y is *sparse* in $\boldsymbol{S}$ if $Y \subset \mathcal{N}_\delta(\boldsymbol{S} \setminus Y)$ for every $\delta > 0$.

Lemma 2.13. *If $\boldsymbol{S}_A$ is a discrete filling then $\partial D_n \subset B(\boldsymbol{S}_A)$ for all $n \in A$ and $B(\boldsymbol{S}_A) = \boldsymbol{S}$.*

Proof. It is clear that $B(\boldsymbol{S}_A) \subset \boldsymbol{S}$. By construction, $\mathrm{cl}(D_n) \cap \mathrm{cl}(D_m) = \varnothing$ for all $n \neq m$. For $n \in A$, let ϵ_n be the distance from $\mathrm{cl}(D_n)$ to $K(\boldsymbol{S}_A) \cup \left(\bigcup_{i \in A \setminus \{n\}} D_i\right)$. Since $\mathrm{cl}(D_n) \cap K(\boldsymbol{S}_A) = \varnothing$, ϵ_n is strictly positive. This implies that $\mathcal{N}_{\epsilon_n}(D_n)$ is not simply connected. Even more, $B(\boldsymbol{S}_A) \cap \mathcal{N}_{\epsilon_n}(D_n) = \mathcal{N}_{\epsilon_n}(D_n) \setminus D_n$. Thus the only points of $\boldsymbol{S}$ which might possibly have simply connected neighborhoods in $\boldsymbol{S}_A$ are those in $K(\boldsymbol{S}_A)$.

Suppose that $x \in \boldsymbol{S} \cap K(\boldsymbol{S}_A)$ and let U be a neighborhood of x. We must show that U is not simply connected. Since $x \in K(\boldsymbol{S}_A)$, we can find $n \in A$ such that $\mathrm{cl}(D_n) \subset U$. Therefore $\mathcal{N}_\epsilon(D_n) \subset U$ for some choice of $n \in A$ and $0 < \epsilon \leq \epsilon_n$ which implies that U is not simply connected since $\partial D_n \subset B(\boldsymbol{S}_A)$. □

The proof of the following lemma is similar to the proof of Theorem 2.10.

Lemma 2.14. *If $\boldsymbol{S}_A$ is a discrete filling then $\pi_1(\boldsymbol{S}_A, s)$ is not isomorphic to the fundamental group of a one-dimensional Peano continuum.*

Lemma 2.15. *Every simply connected subset of $\boldsymbol{S}$ is a sparse subset of $\boldsymbol{S}$.*

Proof. Let Y be a simply connected (not necessarily connected) subset of $\boldsymbol{S}$. Since $\boldsymbol{S}$ is one-dimensional, this implies that Y can contain no simply closed curves. Fix $y \in Y$. Then there exists a sequence of natural numbers i_n such that ∂D_{i_n} converges to y. Since ∂D_{i_n} cannot be entirely contained in Y, there exists an $x_n \in \partial D_{i_n}$ such that $x_n \in \boldsymbol{S} \setminus Y$. The diameter of ∂D_{i_n} must converge to 0, thus x_n converges to y and $Y \subset \mathcal{N}_\delta(\boldsymbol{S} \setminus Y)$ for every $\delta > 0$. □

Lemma 2.16. *Let Y be a sparse closed subset of $\boldsymbol{S}$. Then there exists a subset $A \subset \mathbb{N}$ such that $\boldsymbol{S}_A$ is a discrete filling of $\boldsymbol{S}$ and $K(\boldsymbol{S}_A) = Y$.*

Proof. A subset B of a metric space X is *δ-separated* if $d(x, y) \geq \delta$ for all $x, y \in B$. A δ-separated subset B of a space X is *maximal* if $X \subset \mathcal{N}_\delta(B)$. It is an exercise to show that any δ-separated subset of X can be extended to a maximal δ-separated subset.

Since Y is compact, any δ-separated subset of Y is finite. Let Y_1 be a maximal 1-separated subset of Y. Define Y_n to be a maximal $\frac{1}{n}$-separated subset of Y which extends Y_{n-1}.

For every $y \in Y_n$ there exists $s_{y,n} \in \boldsymbol{S} \setminus Y$ such that $d(s_{y,n}, y) \leq 1/n$. Fix $\delta_n > 0$ such that $\delta_n < d(s_{y,n}, Y)$ for all $y \in Y_n$. Then we may choose $i(y, n) \in \mathbb{N}$ such that $d(D_{i(y,n)}, s_{y,n}) \leq \delta_n/3$ and $D_{i(y,n)}$ has side length less than $\delta_n/3$. This implies $\mathrm{cl}(D_{i(y,n)}) \cap Y = \varnothing$.

Let $A = \{i(y, n) \mid n \in \mathbb{N} \text{ and } y \in Y_n\}$. By constructions $K(\boldsymbol{S}_A) = Y$. Thus $\boldsymbol{S}_A$ is a discrete filling. □

Proposition 2.17. *Suppose that $\boldsymbol{S}_A$, $\boldsymbol{S}_B$ are discrete fillings of $\boldsymbol{S}$. If $\pi_1(\boldsymbol{S}_A, s_0)$ is isomorphic to $\pi_1(\boldsymbol{S}_B, s_1)$, then $K(\boldsymbol{S}_A)$ is homeomorphic to $K(\boldsymbol{S}_B)$.*

Proof. Suppose that $\boldsymbol{S}_A$, $\boldsymbol{S}_B$ are discrete fillings of $\boldsymbol{S}$ and $\pi_1(\boldsymbol{S}_A, s_0)$ is isomorphic to $\pi_1(\boldsymbol{S}_B, s_1)$. Since the fundamental group is basepoint invariant, we may assume that $s_0 = s_1 \in \boldsymbol{S}$.

Using Theorem 2.7 and Lemma 2.9, we can find maps $f : \boldsymbol{S}_A \to \boldsymbol{S}_B$ and $g : \boldsymbol{S}_B \to \boldsymbol{S}_A$ such that both $g \circ f$ and $f \circ g$ are the identity on $\boldsymbol{S}$. (For discrete fillings $B(\boldsymbol{S}_A) = \boldsymbol{S}$ by Lemma 2.13.) For $n \in A$, the loop ∂D_n is a nullhomotopic simple closed curve in $\boldsymbol{S} \subset \boldsymbol{S}_A$ which implies that $f(\partial D_n)$ is a nullhomotopic simple closed curve in $\boldsymbol{S}_B$. (A simple closed curve α in the plane is nullhomotopic if and only if the bounded component of $\mathbb{R}^2 \setminus \operatorname{im}\alpha$ is simply connected.) Thus $f(\partial D_n) = D_m$ for some $m \in B$.

Thus $f(K(\boldsymbol{S}_A)) \subset K(\boldsymbol{S}_B)$. We can similarly show $g(K(\boldsymbol{S}_B)) \subset K(\boldsymbol{S}_A)$. Since $K(\boldsymbol{S}_A), K(\boldsymbol{S}_B) \subset \boldsymbol{S}$ and $g \circ f$ is the identity on $\boldsymbol{S}$, it follows $K(\boldsymbol{S}_A)$ is homeomorphic to $K(\boldsymbol{S}_B)$. □

Theorem 2.18. *There exists an uncountable family of planar Peano continua whose fundamental groups are pairwise nonisomorphic and also not isomorphic to the fundamental group of any one-dimensional Peano continuum.*

Proof. Let $\{U_1, U_2, \ldots\}$ be a countable set of disjoint open subsets of $(0, 1) \times (0, 1)$ such that U_i converges to a point. In each U_i we can find a subset X_i such that $X_i \subset \boldsymbol{S}$ and X_i is homeomorphic to the wedge of i closed intervals. Note that Sierpinski [1916] showed that any one-dimensional planar continuum embeds into $\boldsymbol{S}$. Since every open set of $\boldsymbol{S}$ contains a scaled copy of $\boldsymbol{S}$, it is always possible to find X_i in $U_i \cap \boldsymbol{S}$.

For every $A \subset \mathbb{N}$, let $X_A = \operatorname{cl}\left(\bigcup_{i \in A} X_i\right)$ which is simply connected. It is a trivial exercise to show that X_A is homeomorphic to X_B if and only if $A = B$.

Notice that for any $A \subset \mathbb{N}$, X_A is sparse. Thus for $A \subset \mathbb{N}$, we may choose $\widetilde{A} \subset \mathbb{N}$ such that $K(\boldsymbol{S}_{\widetilde{A}}) = X_A$. The corollary then follows from Proposition 2.17. □

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