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In a previous paper, we defined the *growth rate of the tunnel number of knots*, an invariant that measures the asymptotic behavior of the tunnel number under connected sum. In this paper we calculate the growth rate of the tunnel number of m-small knots in terms of their bridge indices.

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Part I. Introduction and background material

1. Introduction

Let M be a compact connected orientable 3-manifold and $K \subset M$ a knot. K is called *admissible* if $g(E(K)) > g(M)$ and *inadmissible* otherwise (throughout this paper $E(\cdot)$ denotes knot exterior and $g(\cdot)$ denotes the Heegaard genus; see Section 2 for these and other basic definitions). Let nK denote the connected sum of n copies of K. In [Kobayashi and Rieck 2006b] we defined the *growth rate* of the tunnel number of K to be:

$$\mathrm{gr}_t(K) = \limsup_{n\to\infty} \frac{g(E(nK)) - ng(E(K)) + n - 1}{n-1}.$$

The main result of [Kobayashi and Rieck 2006b] shows that if K is admissible then $\mathrm{gr}_t(K) < 1$, and $\mathrm{gr}_t(K) = 1$ otherwise. This concept was the key to constructing a counterexample to Morimoto's conjecture [Kobayashi and Rieck 2008; 2009]. Unless explicitly stated otherwise, all knots considered are assumed to be admissible (note that this is always the case for knots in the 3-sphere S^3).

In this paper we continue our investigation of the growth rate of the tunnel number. In Part II we give an upper bound on the growth rate of admissible knots (this is an improvement of the bound given in [Kobayashi and Rieck 2006b]), and in Part III we obtain a lower bound on the growth rate of admissible m-small knots (a knot is called *m-small* if its meridian is not a boundary slope of an essential surface). With this we obtain an exact calculation of the growth rate of m-small knots. Before stating this result we define the following notation that will be used extensively throughout the paper:

Notation 1.1. Let $K \subset M$ be an admissible knot. We denote $g(E(K)) - g(M)$ by g and for $i = 1, \dots, g$ we denote the bridge index of K with respect to Heegaard surfaces of genus $g(E(K)) - i$ by b_i^*. That is, b_i^* is the minimal integer so that K admits a b_i^* bridge position with respect to some Heegaard surface of M of genus $g(E(K)) - i$; we call such a decomposition a $(g(E(K)) - i, b_i^*)$ *decomposition*. Note that for a knot $K \subset S^3$ we have that $g = g(E(K))$, $b_g^*(K)$ is the bridge index of K, and $b_{g-1}^*(K)$ is the torus bridge index of K.

We note that, for any knot $K \subset M$, b_i^* forms an increasing sequence of positive integers: $0 < b_1^* < \cdots < b_g^*$. To see this, fix $i \geq 1$ and let Σ be a Heegaard surface that realizes the bridge index b_i^*, that is, Σ is a genus $g(E(K)) - i$ Heegaard surface for M with respect to which K has bridge index b_i^*. By tubing Σ once (see Definition 5.3) we obtain a Heegaard surface of genus $g(E(K)) - (i-1)$ that realizes a $(g(E(K)) - (i-1), b_i^* - 1)$ decomposition for K. This shows that $b_{i-1}^* \leq b_i^* - 1$.

We are now ready to state the following theorem:

Theorem 1.2. *Let M be a compact connected orientable 3-manifold and $K \subset M$ be an admissible knot. Then $gr_t(K) \leq \min_{i=1,\dots,g}\{1 - i/b_i^*\}$. If, in addition, K is m-small then equality holds*:

$$gr_t(K) = \min_{i=1,\dots,g}\left\{1 - \frac{i}{b_i^*}\right\}.$$

Moreover, for m-small knots the limit of $\frac{1}{n-1}(g(E(nK)) - ng(E(K)) + n - 1)$ *exists.*

Remark 1.3. Let X be a manifold whose boundary ∂X is a single torus. By Hatcher [1982], only finitely many slopes on ∂X are boundary slopes of an essential surface. Let M be a manifold obtained by filling any slope not in this finite set, and $K \subset M$ be the core of the attached solid torus. By construction, K is an m-small knot; this shows that m-small knots are very common indeed.

As noted in Notation 1.1, the indices b_i^* form an increasing series of positive integers. It follows that $b_i^* \geq i$; moreover, $b_i^* = i$ implies that $b_1^* = 1$. Applying this to an index i that realizes that the equality $\mathrm{gr}_t(K) = 1 - i/b_i^*$ we obtain the following simple and useful consequence of Theorem 1.2 that strengthens the main result of [Kobayashi and Rieck 2006b] in the case of m-small knots:

Corollary 1.4. *If $K \subset M$ is an admissible m-small knot, then*

$$0 \leq gr_t(K) < 1.$$

Moreover, $gr(K) = 0$ if and only if $b_1^* = 1$.

There are several results about the spectrum of the growth rate and we summarize them here. It is well known that there exist manifolds M that admit minimal genus Heegaard splittings Σ of genus at least 2 and of Hempel distance at least 3. We fix such M and Σ and for simplicity we assume that M is closed. Let C be a handlebody obtained by cutting M along Σ and K a core of C, that is, K is a core of a solid torus obtained by cutting C along appropriately chosen meridian disks. Then Σ is a Heegaard surface for $E(K)$; it follows that K is inadmissible. Clearly, the Hempel distance does not go down after drilling K. Hence the Hempel distance of $\Sigma \subset E(K)$ is at least 3. It is a well known consequence of the Thurston–Perelman geometrization theorem that manifolds that admit a Heegaard surface of genus at least 2 and Hempel distance at least 3 are hyperbolic. Thus $K \subset M$ is a hyperbolic knot in a hyperbolic manifold. As mentioned above, the growth rate of inadmissible knots is 1. This proves the existence of hyperbolic knots in hyperbolic manifolds with growth rate 1. It was shown in [Kobayashi and Rieck 2006b] that torus knots and 2-bridge knots have growth rate 0. Kobayashi and Saito [2010] constructed knots with growth rate $-\frac{1}{2}$. Theorem 1.2 enables us to calculate the growth rate of the knots constructed by Morimoto, Sakuma and Yokota in [Morimoto et al. 1996] (perhaps with finitely many exceptions), which we denote by K_{MSY}. We explain this here. The knots K_{MSY} enjoy the following properties:

(1) K_{MSY} are hyperbolic and m-small: this was announced by Morimoto [2008].

(2) $g(E(K_{\text{MSY}})) = 2$: this was proved in [Morimoto et al. 1996].

(3) $b_1^*(K_{\text{MSY}}) = 2$ (in other words, the torus bridge index of K_{MSY} is 2): it was shown in [Morimoto et al. 1996] that $b_1^* > 1$, and it is easy to observe that $b_1^* \leq 2$ (see, for example, [Kobayashi and Rieck 2006b]).

(4) $b_2^*(K_{\text{MSY}}) \geq 4$ (in other words, the bridge index of K_{MSY} is at least 4): since $b_2^*(K_{\text{MSY}}) > b_1^*(K_{\text{MSY}})$, we only need to exclude the possibility $b_2^*(K_{\text{MSY}}) = 3$. Assume for a contradiction that $b_2^*(K_{\text{MSY}}) = 3$. Then K_{MSY} is a 3-bridge knot of tunnel number 1. Kim [2005] proved that every 3-bridge knot of tunnel number 1 has torus bridge index 1, contradicting the previous point. We note that R. Bowman, S. Taylor and A. Zupan [Bowman et al. 2015] showed that $b_2^*(K_{\text{MSY}}) = 7$ for all but finitely many of the knots K_{MSY} (see Remark 1.7).

Using these facts, Theorem 1.2 implies that $\text{gr}_t(K_{\text{MSY}}) = \frac{1}{2}$. This is the first example of knots with growth rate in the open interval (0, 1) and provides a partial answer to questions posed in [Kobayashi and Rieck 2006b]. In summary we have the following; we emphasize that only (4) is new:

Corollary 1.5.

(1) *There exist hyperbolic knots in hyperbolic manifolds with growth rate* 1.

(2) *There exist hyperbolic knots in S^3 with growth rate* 0.

(3) *There exist knots in S^3 with growth rate* $-\frac{1}{2}$.

(4) *There exist hyperbolic knots in S^3 with growth rate* $\frac{1}{2}$.

Remark 1.6. In joint work with K. Baker [Baker et al. 2016], we use Theorem 1.2 to show that for any $\epsilon > 0$ there exists a hyperbolic knot $K \subset S^3$ with $1-\epsilon < \text{gr}_t(K) < 1$. This implies, in particular, that the spectrum of the growth rate is infinite.

Remark 1.7. We take this opportunity to mention a few recent results about b_i^* that appeared since we first started writing this paper; for precise statements see references.

(1) Given positive integers $g_M < i \leq g_K$ and n, K. Ichihara and T. Saito [2013] constructed manifolds M and knots $K \subset M$ so that $g(M) = g_M$, $g(E(K)) = g_K$, and $b_i^*(K) - b_{i-1}^*(K) \geq 2$ (see [Ichihara and Saito 2013, Corollary 2]; the notation there is different from ours); their arguments can easily be applied to construct knots such that $b_i^*(K) - b_{i-1}^*(K) \geq n$ (informally, we may phrase this as an *arbitrarily large gap*).

(2) Zupan [2014] studied the bridge indices of iterated torus knots showing, in particular, that there exist iterated torus knots realizing arbitrarily large gaps between b_{i-1}^* and b_i^* for any i in the range where both indices are defined.

An easy argument shows that iterated torus knots are m-small; every knot K considered by Zupan fulfills $b_1^*(K) = 1$, and so has $\mathrm{gr}(K) = 0$ by Corollary 1.4.

(3) Bowman, Taylor, and Zupan [2015] calculated the bridge indices of generic iterated torus knots (see [Bowman et al. 2015] for definitions). They gave conditions on the parameters that imply that $b_g^* = p$, where here the knot considered is obtained by twisting the torus knot $T_{p,q}$, $p < q$. (We note that for the twisted torus knot $g = 2$). Applying this to K_{MSY} we see that all but finitely many of these knots have $b_2^* = 7$, improving on our estimate $b_2^* \geq 4$. We remark that in [Bowman et al. 2015] a linear lower bound on b_1^* was also obtained, showing that many twisted torus knots have a gap between b_1^* and b_2^*; since b_2^* can be made arbitrarily large, this can be seen as a second gap.

Before describing the structure and contents of this paper in more detail we introduce some necessary concepts. Let Σ be a Heegaard surface of a compact 3-manifold M, and A an essential annulus properly embedded in M. The annulus A is called a *Haken annulus for* Σ (Definition 4.1) if it intersects Σ in a single simple closed curve that is essential in A. For an integer $c \geq 0$, the manifold obtained by drilling c curves simultaneously parallel to meridians of K out of $E(K)$ is denoted by $E(K)^{(c)}$ (note that $E(K)^{(0)} = E(K)$). The c tori $\partial E(K)^{(c)} \setminus \partial E(K)$ are denoted by $T_1, \ldots, T_c$. There are c annuli properly embedded disjointly in $E(K)^{(c)}$, denoted by $A_1, \ldots, A_c$, so that one component of ∂A_i is a meridian on $\partial E(K)$ and the other is a longitude of T_i $(i = 1, \ldots, c)$. (We note that in general these annuli are not uniquely determined up to isotopy.) Annuli with these properties are called *a complete system of Hopf annuli* (Definition 5.1). Let Σ be a Heegaard surface for $E(K)^{(c)}$. The Hopf annuli $A_1, \ldots, A_c$ are called *a complete system of Hopf–Haken Annuli for* Σ (Definition 5.2) if $\Sigma \cap A_i$ is a single simple closed curve that is essential in A_i $(i = 1, \ldots, c)$.

Part II starts with Section 4 where we describe basic behavior of Haken annuli under amalgamation. In Section 5 we consider (g', b) decomposition of K (that is, b-bridge decomposition of K with respect to a genus g' Heegaard surface) and relate it to existence of Hopf–Haken Annuli. Specifically, we prove that K admits a $(g(E(K)) - c, c)$ decomposition if and only if $E(K)^{(c)}$ admits a complete system of Hopf–Haken Annuli for some Heegaard surface of genus $g(E(K))$ (Theorem 5.4).

In Section 6 we prove that given knots $K_1, \ldots, K_n$ and integers $c_1, \ldots, c_n \geq 0$ with $\sum_{i=1}^n c_i = n - 1$, $E(K_1 \# \cdots \# K_n)$ admits a system of $n - 1$ essential tori $\mathcal{T}$ (called *swallow follow tori*) so that the components of $E(K_1 \# \cdots \# K_n)$ cut open along $\mathcal{T}$ are homeomorphic to $E(K_1)^{(c_1)}, \ldots, E(K_n)^{(c_n)}$. By amalgamating Heegaard surfaces of $E(K_1)^{(c_1)}, \ldots, E(K_n)^{(c_n)}$ along the tori of $\mathcal{T}$ we obtain a Heegaard surface for $E(K_1 \# \cdots \# K_n)$; this implies this special case of Corollary 6.4:

$$g(E(K_1 \# \cdots \# K_n)) \leq \sum_{i=1}^n g(E(K_i)^{(c_i)}) - (n - 1).$$

In the final section of Part II, Section 7, we combine these facts to prove that for each i we have:

$$\mathrm{gr}_t(K) \leq 1 - i/b_i^*.$$

Thus we obtain the upper bound stated in Theorem 1.2.

To some degree, Part III complements Part II. We begin with Section 8 that complements Sections 4 and 5. As mentioned above, in Sections 4 and 5 we prove that K admits a $(g(E(K)) - c, c)$ decomposition if and only if $E(K)^{(c)}$ admits a complete system of Hopf–Haken Annuli for some Heegaard surface of genus $g(E(K))$. We are now ready to state the strong Hopf–Haken annulus theorem, which generalizes the Hopf–Haken annulus theorem (Theorem 6.3 of [Kobayashi and Rieck 2006a]), and is one of the highlights of this work. The proof is given in Section 8. For the definition of a Heegaard splitting of $(N; F_1, F_2)$ (where N is a manifold and F_1, F_2 are partitions of some of the components of ∂N), see Section 2 .

Theorem 1.8 (Strong Hopf–Haken annulus theorem). *For $i = 1, \dots, n$, let M_i be a compact connected orientable 3-manifold and $K_i \subset M_i$ be a knot. Suppose that $E(K_i) \not\cong T^2 \times I$, that $E(K_i)$ is irreducible, and that $\partial N(K_i)$ is incompressible in $E(K_i)$. Let F_1, F_2 be a partition of some of the components of ∂M, where $M = \#_{i=1}^n M_i$. Let $c \geq 0$ be an integer. Then one of the following holds*:

(1) *There exists a minimal genus Heegaard surface for $(E(\#_{i=1}^n K_i)^{(c)}; F_1, F_2)$ admitting a complete system of Hopf–Haken annuli.*

(2) *For some $1 \leq i \leq n$, $E(K_i)$ admits an essential meridional surface S with $\chi(S) \geq 6 - 2g(E(\#_{i=1}^n K_i)^{(c)}; F_1, F_2)$.*

One curious consequence of Theorem 1.8 (which is proved in Section 8) is the following, where b_g^* is as in Notation 1.1:

Corollary 1.9. *Let $K \subset S^3$ be a connected sum of $n \geq 1$ m-small knots. Then for $c \geq b_g^*$,*

$$g(E(K)^{(c)}) = c.$$

Section 9 complements Section 6. Recall that in Section 6 we used swallow follow tori to show that given *any* collection of integers $c_1, \dots, c_n \geq 0$ whose sum is $n-1$ we have that $g(E(K_1\#\cdots\#K_n)) \leq \sum_{i=1}^n g(E(K_i)^{(c_i)}) - (n-1)$. In Section 9 we prove that if K_i is m-small for each i, then any Heegaard splitting for $E(K_1\#\cdots\#K_n)$ admits an iterated weak reduction to $n-1$ swallow follow tori. This implies that any minimal genus Heegaard splitting admits an iterated weak reduction to *some* $n-1$ swallow follow tori that decompose $E(K_1\#\cdots\#K_n)$ as $E(K_1)^{(c_1)}, \dots, E(K_n)^{(c_n)}$, giving *some* integers $c_1, \dots, c_n \geq 0$ whose sum is $n-1$. The integers $c_1, \dots, c_n$ are very special (see Example 9.3).

In Section 10, which complements Section 7, we combine these results to give a lower bound on the growth rate of the tunnel number of m-small knots. Given K, we "expect" that $g(E(K)^{(c)}) = g(E(K)) + c$; so we define the function f_K that measures to what extent $g(E(K)^{(c)})$ fails to behave "as expected":

$$f_K(c) = g(E(K)) + c - g(E(K)^{(c)}).$$

For any knot K and any integer $c \geq 0$, we show that f_K fulfills

$$f_K(0) = 0 \text{ and } f_K(c) \leq f_K(c+1) \leq f_K(c) + 1.$$

We study f_K for m-small knots, calculating it exactly in terms of the bridge indices of K (Proposition 10.4). In particular, for m-small knots, f_K is bounded. In fact, for large enough c, Proposition 10.4 implies

$$f_K(c) = g(E(K)) - g(M).$$

We do not know much about the behavior of f_K in general; for example, we do not know if there exists a knot for which f_K is unbounded (see Question 10.5).

We express the growth rate of tunnel number of m-small knots in terms of f_K by showing (Corollary 10.3) that

$$\frac{g(E(nK)) - ng(E(K)) + n - 1}{n-1} = 1 - \frac{\max\left\{\sum_{i=1}^{n} f_K(c_i)\right\}}{n-1},$$

where the maximum is taken over all collections of integers $c_1, \ldots, c_n \geq 0$ whose sum is $n - 1$. The growth rate is then the limit superior of this sequence. We combine this interpretation of the growth rate with the calculation of f_K to obtain the exact calculation of the growth rate of m-small knots stated in Theorem 1.2.

2. Preliminaries

By *manifold* we mean a smooth 3-dimensional manifold. All manifolds considered are assumed to be connected orientable and compact. We assume the reader is familiar with the basic terms of 3-manifold topology (see, for example, [Schultens 2014; Jaco 1980; Hempel 1976]). Thus we assume the reader is familiar with terms such as compression, boundary compression, boundary parallel, and essential surface.

We use the notation ∂, cl, and int to denote boundary, closure, and interior, respectively. For a submanifold H of a manifold M, $N(H, M)$ denotes a closed regular neighborhood of H in M. When M is understood from context we often abbreviate $N(H, M)$ to $N(H)$.

By a *knot* K in a 3-manifold M we mean a smooth embedding of S^1 into M, taken up to ambient isotopy. $E(K)$, the *exterior* of K, is $\mathrm{cl}(M \setminus N(K))$. The slope on the torus $\partial E(K) \setminus \partial M = \partial N(K)$ that bounds a disk in $N(K)$ is called the *meridian* of K. A knot K is called *m-small* if there is no essential meridional surface in $E(K)$, that is, there is no essential surface $S \subset E(K)$ with nonempty boundary so that ∂S consists of meridians of K.

We assume the reader is familiar with the basic terms regarding Heegaard splittings, such as handlebody, compression body, meridian disk, etc. Recall that a compression body C is a connected 3-manifold obtained from $F \times [0, 1]$ (where here F is a possibly empty disjoint union of closed surfaces) and a (possibly empty) collection of 3-balls by attaching 1-handles to $F \times \{1\}$ and the boundary of the balls. Following standard conventions, we refer to $F \times \{0\}$ as $\partial_- C$ and $\partial C \setminus \partial_- C$ as $\partial_+ C$. We use the notation $C_1 \cup_\Sigma C_2$ for the Heegaard splitting given by the compression bodies C_1 and C_2. The basic concepts of reductions of a Heegaard splitting are summarized here:

Definitions 2.1. (1) A Heegaard splitting $C_1 \cup_\Sigma C_2$ is called *stabilized* if there exist meridian disks $D_1 \subset C_1$ and $D_2 \subset C_2$ such that ∂D_1 intersects ∂D_2 transversely (as submanifolds of Σ) in one point. Otherwise, the Heegaard splitting is called *nonstabilized.*

(2) A Heegaard splitting $C_1 \cup_\Sigma C_2$ is called *reducible* if there exist meridian disks $D_1 \subset C_1$ and $D_2 \subset C_2$ such that $\partial D_1 = \partial D_2$. Otherwise, the Heegaard splitting is called *irreducible.*

(3) A Heegaard splitting $C_1 \cup_\Sigma C_2$ is called *weakly reducible* if there exist meridian disks $D_1 \subset C_1$ and $D_2 \subset C_2$ such that $\partial D_1 \cap \partial D_2 = \varnothing$. Otherwise the splitting is called *strongly irreducible.*

(4) A Heegaard splitting $C_1 \cup_\Sigma C_2$ is called *trivial* if C_1 or C_2 is a trivial compression body, that is, a compression body with no 1-handles. Otherwise the Heegaard splitting is called *nontrivial.*

Let $C_1 \cup_\Sigma C_2$ be a weakly reducible Heegaard splitting of a manifold M. Let $\Delta_i \subset C_i$ be a non empty set of disjoint meridian disks so that $\Delta_1 \cap \Delta_2 = \varnothing$. By *weak reduction* along $\Delta_1 \cup \Delta_2$ we mean the (possibly disconnected) surface obtained by first compressing Σ along $\Delta_1 \cup \Delta_2$, and then removing any component that is contained in C_1 or C_2. Casson and Gordon [1987] showed that if an irreducible Heegaard splitting is weakly reducible, then an appropriately chosen weak reduction yields a (possibly disconnected) essential surface, say, F.

With F as in the previous paragraph, let $M_1, \dots, M_k$ be the components of M cut open along F. It is well known that Σ induces a Heegaard surface on each M_i, say, Σ_i. We say that Σ is obtained by *amalgamating* $\Sigma_1, \dots, \Sigma_k$. This is a special case of amalgamation; the general definition will be given below as the converse of iterated weak reduction. The genus after amalgamation is given in the following lemma; see Remark 2.7 of [Schultens 1993] for the case $m = 1$ (we leave the proof of the general case to the reader):

Lemma 2.2. *Let $C_1 \cup_\Sigma C_2$ be a weakly reducible Heegaard splitting and suppose that after weak reduction we obtain F* (*as above*). *Suppose that M cut open along*

F consists of two components, and denote the induced Heegaard splittings by $C_1^{(1)} \cup_{\Sigma_1} C_2^{(1)}$ and $C_1^{(2)} \cup_{\Sigma_2} C_2^{(2)}$. Let $F_1, \dots, F_m$ be the components of F. Then,

$$g(\Sigma) = g(\Sigma_1) + g(\Sigma_2) - \sum_{i=1}^{m} g(F_i) + (m-1).$$

In particular, if F is connected then $g(\Sigma) = g(\Sigma_1) + g(\Sigma_2) - g(F)$.

It is distinctly possible that not all the Heegaard splittings induced by weak reduction are strongly irreducible. When that happens we may weakly reduce some (possibly all) of the induced Heegaard splittings, and repeat this process. We refer to this as *repeated* or *iterated* weak reduction. The converse is called amalgamation. Scharlemann and Thompson [1994] proved that any Heegaard splitting admits a repeated weak reduction so that the induced Heegaard splittings are all strongly irreducible; we refer to this as *untelescoping*.

Let N be a manifold and $\{F_1, F_2\}$ be a partition of some components of ∂N. Note that we do *not* require every component of ∂N to be in F_1 or F_2. We say that $C_1 \cup_\Sigma C_2$ is a Heegaard splitting of $(N; F_1, F_2)$ if $F_1 \subset \partial_- C_1$ and $F_2 \subset \partial_- C_2$. We extend the terminology of Heegaard splittings to this context, so, for example, $g(N; F_1, F_2)$ is the genus of a minimal genus Heegaard splitting of $(N; F_1, F_2)$.

The following proposition allows us, in some cases, to consider weak reduction instead of iterated weak reduction. The proof is simple and left to the reader.

Proposition 2.3. *Let F be a component of the surface obtained by repeated weak reduction of $C_1 \cup_{\Sigma_1} C_2$. If F is separating, then some weak reduction of $C_1 \cup_{\Sigma_1} C_2$ yields exactly F.*

3. Relative Heegaard surfaces

In this section we study relative Heegaard surfaces. The results of this section will be used in Section 8 and the reader may postpone reading it until that section. Let $b \geq 1$ be an integer and T be a torus. For $1 \leq i \leq 2b$, let $A_i \subset T$ be an annulus. We say that $\{A_1, \dots, A_{2b}\}$ gives a *decomposition of T into annuli* (or simply a *decomposition of T*) if the following two conditions hold:

(1) $\bigcup_{i=1}^{2b} A_i = T$, and

(2) (a) If $b > 1$, then $A_i \cap A_j = \varnothing$ whenever $i \neq j$ are nonconsecutive integers (modulo $2b$), and $A_i \cap A_{i+1} = \partial A_i \cap \partial A_{i+1}$ is a single simple closed curve.

(b) If $b = 1$, then $A_1 \cap A_2 = \partial A_1 = \partial A_2$.

We begin by defining a relative Heegaard surface; note that the definition can be made more general by considering an arbitrary collection of boundary components (below we only consider a single torus) and a decomposition into arbitrary subsurfaces (below we only consider annuli); however Definition 3.1 suffices for our purposes:

Definition 3.1 (relative Heegaard surface). Let M be a compact connected orientable 3-manifold and T a torus component of ∂M. Let $\{A_1, \ldots, A_{2b}\}$ be a decomposition of T into annuli. A compact surface $S \subset M$ is called a *Heegaard surface for M relative to* $\{A_1, \ldots, A_{2b}\}$ (or simply a *relative Heegaard surface,* when no confusion may arise) if the following conditions hold:

(1) $\partial S = \bigcup_{i=1}^{2b} \partial A_i$.

(2) M cut open along S consists of two components (say, C_1 and C_2).

(3) For $i = 1, 2$, C_i admits a set of compressing disks Δ_i with $\partial \Delta_i \subset S$, so that C_i compressed along Δ_i consists of

  (a) exactly b solid tori, each containing exactly one A_i as a longitudinal annulus;

  (b) a (possibly empty) collection of collar neighborhoods of components of $\partial M \setminus T$;

  (c) a (possibly empty) collection of balls.

The genus of a minimal genus relative Heegaard surface is called the *relative genus.*

For an integer $c \geq 1$, let $Q^{(c)}$ be (an annulus with c holes)$\times S^1$. (To avoid confusion we remark that $Q^{(c)}$ can be described as (a sphere with $c+2$ holes)$\times S^1$, but in the context of this paper an annulus is more natural.) Note that $Q^{(c)}$ admits a unique Seifert fibration. Our goal is to calculate the genus of $Q^{(c)}$ relative to a given decomposition of a component of $\partial Q^{(c)}$ into annuli. We say that slopes β and γ of a torus are *complementary* if they are represented by simple closed curves that intersect each other transversely once.

Proposition 3.2. *Let* $\{A_1, \ldots, A_{2b}\}$ *be a decomposition of a component of* $\partial Q^{(c)}$ (*say*, T) *into annuli, and denote the slope defined by these annuli by* β. *Denote the slope defined by the Seifert fibers on* T *by* γ. *Then,*

(1) *when β and γ are complementary slopes, the genus of $Q^{(c)}$ relative to $\{A_1,\ldots,A_{2b}\}$ is c;*

(2) *when β and γ are not complementary slopes, the genus of $Q^{(c)}$ relative to $\{A_1,\ldots,A_{2b}\}$ is $c+1$.*

We immediately obtain:

Corollary 3.3. *The surfaces in Figure 1 are minimal genus Heegaard splittings for $Q^{(c)}$ relative to* $\{A_1, \ldots, A_{2b}\}$; *Figure 1* (*left*) *is of complementary slopes and Figure 1* (*right*) *is of noncomplementary slopes.*

We postpone the proof of Proposition 3.2 to the end of this section, as it will be an application of the next proposition which is of independent interest. We fix the following notation: glue $Q^{(b)}$ to $Q^{(c)}$ along a single boundary component and

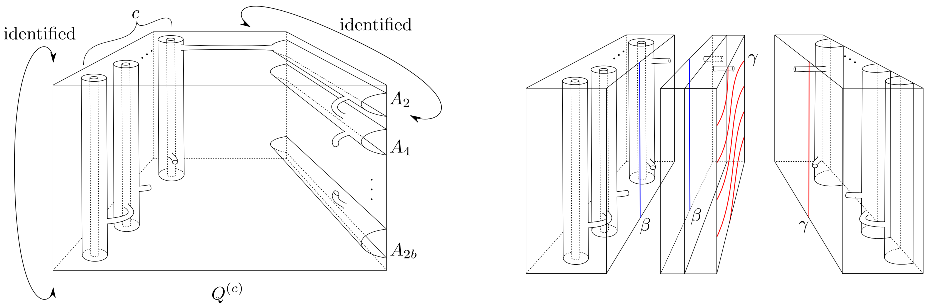

Figure 1. Relative Heegaard surfaces.

denote the slope of the Seifert fiber of $Q^{(b)}$ on the torus $Q^{(b)} \cap Q^{(c)}$ by β and the slope of the Seifert fiber of $Q^{(c)}$ by γ. The manifold obtained is denoted $Q^{(b,c)}_{\beta,\gamma}$.

Proposition 3.4. *The genus of $Q^{(b,c)}_{\beta,\gamma}$ satisfies the following*:

(1) *If β and γ are complementary slopes, then $g(Q^{(b,c)}_{\beta,\gamma}) = b + c$.*

(2) *If β and γ are not complementary slopes, then $g(Q^{(b,c)}_{\beta,\gamma}) = b + c + 1$.*

We immediately obtain:

Corollary 3.5. *The surfaces in Figure 2 (left) and in Figure 2 (right) are minimal genus Heegaard splittings for $Q^{(b,c)}_{\beta,\gamma}$ corresponding to (1) and (2) of Proposition 3.4, respectively.*

A surface in a Seifert fibered space is called *vertical* if it is everywhere tangent to the fibers and *horizontal* if it is everywhere transverse to the fibers. It is well known that given an essential surface in a Seifert fibered space we may assume it is vertical or horizontal; see, for example, [Jaco 1980].

Proof of Proposition 3.4. The surfaces in Figure 2 are Heegaard surfaces for $Q^{(b,c)}_{\beta,\gamma}$, showing the following, which we record here for future reference:

Remark 3.6. When β and γ are complementary, $g(Q^{(b,c)}_{\beta,\gamma}) \le b + c$. When β and γ are not complementary, $g(Q^{(b,c)}_{\beta,\gamma}) \le b + c + 1$.

Hence we only need to show that when β and γ are complementary, $g(Q^{(b,c)}_{\beta,\gamma}) \ge b + c$ and when β and γ are not complementary, $g(Q^{(b,c)}_{\beta,\gamma}) \ge b + c + 1$.

If $\beta = \gamma$ then $Q^{(b,c)}_{\beta,\gamma}$ is a $(b+c)$-times punctured annulus cross S^1 and the result was proved by Schultens [1993]. For the remainder of the proof we assume that $\beta \ne \gamma$. Then $Q^{(b,c)}_{\beta,\gamma}$ is a graph manifold whose underlying graph consists of two vertices connected by a single edge. We apply [Schultens 2004, Theorem 1.1] and refer the reader to that paper for notation and details. Following Schultens' notation, we decompose $Q^{(b,c)}_{\beta,\gamma}$ along two parallel copies of $Q^{(b)} \cap Q^{(c)}$ as $Q^{(b,c)}_{\beta,\gamma} = Q_b \cup M_e \cup Q_c$. Q_b and Q_c are called the *vertex manifolds* and M_e is the *edge manifold*. Note that $Q_b \cong Q^{(b)}$, $M_e \cong T^2 \times [0, 1]$, and $Q_c \cong Q^{(c)}$.

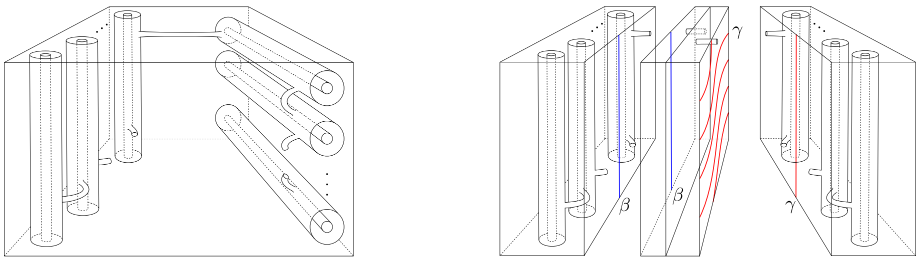

Figure 2. Heegaard surfaces for $Q^{(b,c)}_{\beta,\gamma}$.

Let S be a minimal genus Heegaard splitting for $Q^{(b,c)}_{\beta,\gamma}$. In the following claim we analyze completely what happens when $g(S) = 2$ or when S is strongly irreducible:

Claim 3.7. The following three conditions are equivalent:

(1) S is strongly irreducible.

(2) β and γ are complementary, $g(S) = 2$, and $b = c = 1$.

(3) $g(S) = 2$.

Proof of Claim 3.7. (1) implies (2). Suppose that S is strongly irreducible. By [Schultens 2004] we may assume that S is standard. In particular, $S \cap Q_b$ (respectively, $S \cap Q_c$) is either horizontal, pseudohorizontal, vertical, or pseudovertical. However, the first two cases are impossible as they require S to meet every boundary component of Q_b (respectively, Q_c). Hence $S \cap Q_b$ and $S \cap Q_c$ consist of vertical or pseudovertical components. In particular, the intersection of S with the torus $Q_b \cap M_e$ (respectively, $Q_c \cap M_e$) is a Seifert fiber of Q_b (respectively, Q_c).

Assume first that $S \cap M_e$ is as in [Schultens 2004, Theorem 1.1(1)], that is, $S \cap M_e$ is obtained from a collection of incompressible annuli, say, $\mathcal{A}$, by tubing along at most one boundary parallel arc (in [Schultens 2004], tubings are referred to as 1-*surgery*). Suppose that $\mathcal{A}$ consists of boundary parallel annuli. Since the tubing is performed, if at all, along a boundary parallel arc, we see that no component of $S \cap M_e$ connects the components of ∂M_e. This contradicts the fact that S is connected and must meet both Q_b and Q_c. Hence some component of $\mathcal{A}$ meets both components of ∂M_e, showing that $\beta = \gamma$, contradicting our assumption.

Hence [Schultens 2004, Theorem 1.1(2)] holds, and $S \cap M_e$ consists of a single component that is obtained by tubing together two boundary parallel annuli, one at each boundary component of M_e; moreover, [Schultens 2004, Theorem 1.1] shows that these annuli define complementary slopes. See Figure 3 (left). As argued above, the slopes defined by these annuli are β and γ. This gives the first condition of (2).

On the right side of Figure 3 we see two surfaces. One is $S \cap M_e$, and in its center we marked the boundary of the obvious compressing disk. It is easy to see that the other surface is isotopic to $S \cap M_e$. On it we marked the boundary of four

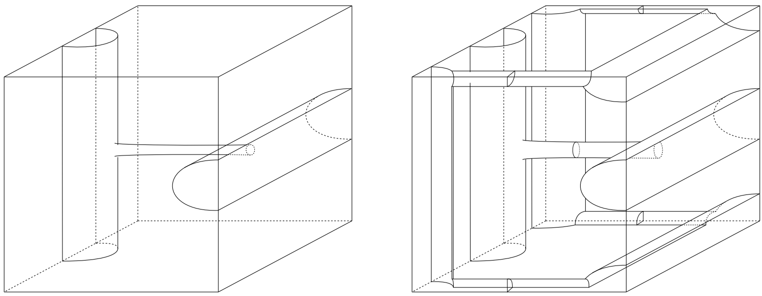

Figure 3. Heegaard surfaces in M_e.

disks, each shaped like a 90° sector. After gluing opposite sides of the cube to M_e, these sectors form a compressing disk on the opposite side of the obvious disk. This demonstrates that $S \cap M_e$ compresses into both sides. If $S \cap Q_b$ is pseudovertical then it compresses, and together with one of the compressing disks for $S \cap M_e$ we obtain a weak reduction, contradicting our assumption. Hence $S \cap Q_b$ consists of annuli; similarly, $S \cap Q_c$ consists of annuli. Hence,

$$\chi(S) = \chi(S \cap M_e) = -2.$$

The second condition of (2) follows.

Since $g(S) = 2$, $\partial Q^{(b,c)}_{\beta,\gamma}$ consists of at most four tori. On the other hand, $\partial Q^{(b,c)}_{\beta,\gamma}$ consists of $b+c+2$ tori, for $b, c \geq 1$. Hence $b = c = 1$, fulfilling the third and final condition of (2). This completes the proof that (1) implies (2).

It is trivial that (2) implies (3).

To see that (3) implies (1), assume that S weakly reduces. Since S is a minimal genus Heegaard surface and $g(S) = 2$, an appropriate weak reduction yields an essential sphere, which contradicts the fact that $Q^{(b,c)}_{\beta,\gamma}$ is irreducible.

This completes the proof of Claim 3.7. □

If S is strongly irreducible, Proposition 3.4 follows from Claim 3.7. For the reminder of the proof we assume, as we may, that S weakly reduces to a (possibly disconnected) essential surface, say, F. By the construction of $Q^{(b,c)}_{\beta,\gamma}$ we see that every component of F separates; hence by Proposition 2.3 we may assume that F is connected. Recall that we assumed that $\beta \neq \gamma$. This clearly implies that we may suppose that (after isotopy if necessary) F is disjoint from the torus $Q^{(b)} \cap Q^{(c)}$; without loss of generality we assume that $F \subset Q^{(b)}$.

We induct on $b+c$.

Base case: $b + c = 2$**.** Note that in the base case $b = c = 1$. It is easy to see that the only connected essential surface in $Q^{(1,1)}_{\beta,\gamma}$ is the torus $Q^{(b)} \cap Q^{(c)}$. Hence F is isotopic to this surface and the weak reduction induces Heegaard splittings Σ_b and Σ_c on $Q^{(b)}$ and $Q^{(c)}$, respectively; note that both $Q^{(b)}$ and $Q^{(c)}$ are homeomorphic to $Q^{(1)}$. By Schultens [1993], $g(Q^{(1)})=2$. By Lemma 2.2, amalgamation gives

$$g(Q^{(1,1)}_{\beta,\gamma})=g(S)=g(\Sigma_b)+g(\Sigma_c)-g(F)\geq g(Q^{(1)})+g(Q^{(1)})-g(F)=2+2-1=3.$$

By Remark 3.6, if β and γ are complementary slopes then $g(Q^{(1,1)}_{\beta,\gamma})\leq 2$; hence β and γ are not complementary slopes and together with Remark 3.6 the proposition follows in this case.

Inductive case: $b+c>2$**.** Assume, by induction, that the proposition holds for any integers b', $c'>0$, with $b'+c'<b+c$.

Case One: *F is isotopic to $Q^{(b)}\cap Q^{(c)}$.* Then weak reduction induces Heegaard splittings on $Q^{(b)}$ and $Q^{(c)}$. Similar to the argument above (using that $g(Q^{(b)})=b+1$ and $g(Q^{(c)})=c+1$ by [Schultens 1993]) we have

$$g(Q^{(b,c)}_{\beta,\gamma})\geq g(Q^{(b)})+g(Q^{(c)})-g(F)=b+c+1.$$

As in the base case it follows from Remark 3.6 that β and γ are not complementary slopes. Together with Remark 3.6, the proposition follows in this case.

Case Two: *F is not isotopic to $Q^{(b)}\cap Q^{(c)}$.* Then F is essential in $Q^{(b)}$ and is therefore isotopic to a vertical or horizontal surface. Since F is closed and $\partial Q^{(b)}\neq\varnothing$, we have that F cannot be horizontal. We conclude that F is a vertical torus and decomposes $Q^{(b)}$ as $Q^{(b')}$ (for some $b'<b$) and a disk with $b-b'+1$ holes cross S^1. By induction, the genus of $Q^{(b',c)}_{\beta,\gamma}$ fulfills the conclusion of Proposition 3.4; by [Schultens 1993], the genus of a disk with $b-b'+1$ holes cross S^1 is $b-b'+1$, and similar to the argument above we get

$$g(Q^{(b,c)}_{\beta,\gamma})\geq g(Q^{(b',c)}_{\beta,\gamma})+(b-b'+1)-1=g(Q^{(b',c)}_{\beta,\gamma})+b-b'.$$

Together with Remark 3.6, this completes the proof of Proposition 3.4. □

We are now ready to prove Proposition 3.2:

Proof of Proposition 3.2. The surfaces in Figure 1 are relative Heegaard surfaces realizing the values given in Proposition 3.2. To complete the proof we only need to show that these surfaces realize the minimal relative genus.

Let Σ be a minimal genus Heegaard surface for $Q^{(c)}$ relative to $\{A_1,\dots,A_{2b}\}$. By tubing $\partial\Sigma$ along the annuli A_{2i} and drilling a curve parallel to the core of A_{2i} ($i=1,\dots,b$; recall Figure 1) we obtain a Heegaard surface for $Q^{(b,c)}_{\beta,\gamma}$ of genus $g(S)+b$. Thus $g(\Sigma)\geq g(Q^{(b,c)}_{\beta,\gamma})-b$. By Proposition 3.4, when β and γ are complementary $g(Q^{(b,c)}_{\beta,\gamma})=b+c$ and when β and γ are not complementary $g(Q^{(b,c)}_{\beta,\gamma})=b+c+1$. Thus we see that $g(\Sigma)\geq c$ (when the β and γ are complementary) and $g(\Sigma)\geq c+1$ (otherwise).

This completes the proof of Proposition 3.2. □

Part II. An upper bound on the growth rate of the tunnel number of knots

4. Haken annuli

A primary tool in our study is the use of Haken annuli. Haken annuli were first defined in [Kobayashi and Rieck 2006a], where only a single annulus was considered. We generalize the definition to a collection of annuli below. Note the similarity between a Haken annulus and a Haken sphere or Haken disk (by a *Haken sphere* we mean a sphere that meets a Heegaard surface in a single simple closed curve that is essential in the Heegaard surface, see [Haken 1968] or [Jaco 1980, Chapter 2], and by a *Haken disk* we mean a disk that meets a Heegaard surface in a single simple closed curve that is essential in the Heegaard surface [Casson and Gordon 1987]).

Definition 4.1. Let $C_1 \cup_\Sigma C_2$ be a Heegaard splitting of a manifold M. A collection of essential annuli $\mathcal{A} \subset M$ are called *Haken annuli* for $C_1 \cup_\Sigma C_2$ (or simply *Haken annuli,* when no confusion may arise) if for every annulus $A \in \mathcal{A}$ we have that $A \cap \Sigma$ consists of a single simple closed curve that is essential in A.

Remark 4.2. For an integer $n \geq 2$, let $D(n)$ be (a disk with n holes)$\times S^1$ and denote the components of $\partial D(n)$ by $T_0, T_1, \ldots, T_n$. By the construction of minimal genus Heegaard splittings given in the proof of Proposition 2.14 of [Kobayashi and Rieck 2006a], we see that for each positive integer p with $1 \leq p \leq n$ there is a genus n Heegaard surface of $(D(n); \bigcup_{i=0}^{p-1} T_i, \bigcup_{i=p}^{n} T_i)$ which admits a collection $\{A_1, \ldots, A_p\}$ of Haken annuli connecting T_i to T_n $(i = 0, \ldots, p-1)$. By Schultens [1993], we see that this is a minimal genus Heegaard splitting of $D(n)$. See Figure 4.

In Propositions 3.5 and 3.6 of [Kobayashi and Rieck 2006a] we studied the behavior of Haken annuli under amalgamation. We generalize these propositions as Proposition 4.3 below. We first explain the construction that is used in Proposition 4.3. Let $C_1 \cup_\Sigma C_2$ be a Heegaard splitting for a manifold M that weakly reduces to a (possibly disconnected) essential surface F. Suppose that M cut open along F consists of two components, say, $M^{(i)}$ $(i = 1, 2)$. We

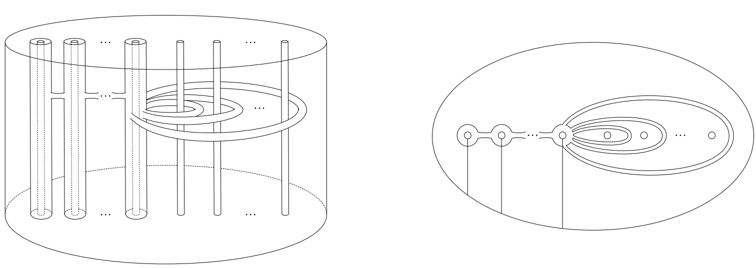

Figure 4. Heegaard surface in $D(n)$.

denote the image of F in $M^{(i)}$ by $F^{(i)}$ and the Heegaard splitting induced on $M^{(i)}$ by $C_1^{(i)} \cup_{\Sigma^{(i)}} C_2^{(i)}$. Suppose there are Haken annuli for $C_1^{(i)} \cup_{\Sigma^{(i)}} C_2^{(i)}$, say, $\mathscr{A}^{(i)}$, satisfying these conditions:

- There exists a unique component of $\mathscr{A}^{(1)}$, say, $A^{(1)}$, which intersects $F^{(1)}$ in a single simple closed curve, and other components are disjoint from $F^{(1)}$.
- Each component of $\mathscr{A}^{(2)}$ intersects $F^{(2)}$ in a single simple closed curve isotopic in F to $A^{(1)} \cap F^{(1)}$.

Then let $\tilde{\mathscr{A}}^{(1)}$ be a collection of mutually disjoint annuli obtained from $\mathscr{A}^{(1)}$ by substituting $A^{(1)}$ with $|\mathscr{A}^{(2)}|$ parallel copies of $A^{(1)}$ whose boundaries are identified with $\mathscr{A}^{(2)} \cap F^{(2)}$. Finally, let $\tilde{\mathscr{A}}$ equal $\tilde{\mathscr{A}}^{(1)} \cup \mathscr{A}^{(2)}$. Note that $\tilde{\mathscr{A}}$ is a system of mutually disjoint annuli properly embedded in M. It is easy to adopt the proofs of Propositions 3.5 and 3.6 of [Kobayashi and Rieck 2006a] and obtain:

Proposition 4.3. *Let M, $C_1 \cup_\Sigma C_2$, and $\tilde{\mathscr{A}}$ be as above. Then the components of $\tilde{\mathscr{A}}$ form Haken annuli for $C_1 \cup_\Sigma C_2$.*

5. Various decompositions of knot exteriors

In this section we compare two structures: Hopf–Haken annuli and (h, b) decompositions. After defining the two we prove (Theorem 5.4) that they are equivalent.

Let K be a knot in a 3-manifold M and $h \geq 0$, $b \geq 1$ be integers. We say that K admits a (h, b) *decomposition* (some authors use the term genus h, b-bridge position) if there exists a genus h Heegaard splitting $C_1 \cup_\Sigma C_2$ of M such that $K \cap C_i$ is a collection of b simultaneously boundary parallel arcs ($i = 1, 2$; note that in this paper we do not consider $(h, 0)$ decomposition).

Let K be a knot in a compact manifold M. Recall that $E(K)^{(c)}$ is obtained from $E(K)$ by removing c curves that are simultaneously isotopic to meridians of K. The trace of the isotopy forms c annuli which motivates the definition below (Definitions 5.1 and 5.2 generalize Definition 6.1 of [Kobayashi and Rieck 2006a]):

Definition 5.1 (a complete system of Hopf annuli). Let $K \subset M$ be a knot in a compact manifold and $c > 0$ be an integer. Let $A_1, \dots, A_c$ be annuli disjointly embedded in $E(K)^{(c)}$ so that for each i, one component of ∂A_i is a meridian of $\partial N(K)$ and the other is a longitude of T_i (recall $T_1, \dots, T_c$ denote the components of $\partial E(K)^{(c)} \setminus \partial E(K)$). Then $\{A_1, \dots, A_c\}$ is called a *complete system of Hopf annuli*. We emphasize that the complete system of Haken annuli for $E(K)^{(c)}$ is *not* unique up to isotopy.

Definition 5.2 (a complete system of Hopf–Haken annuli). Let $K \subset M$ be a knot in a compact manifold, $c > 0$ be an integer, Σ be a Heegaard surface for $E(K)^{(c)}$, and $\{A_1, \dots, A_c\}$ be a complete system of Hopf annuli. $\{A_1, \dots, A_c\}$ is called a

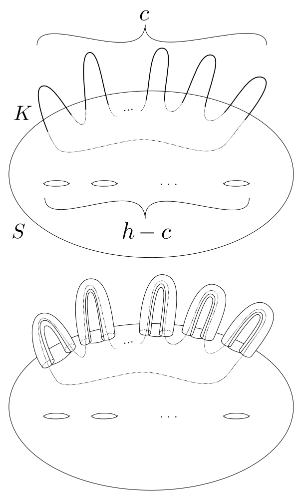

Figure 5. Tubing a $(h-c, c)$-decomposition.

complete system of Hopf–Haken annuli for Σ if for each i, $\Sigma \cap A_i$ is a single simple closed curve that is essential in A_i.

Definition 5.3 (tubing bridge decomposition). Let $K \subset M$ be a knot in a compact manifold, Σ a Heegaard surface for $E(K)$, and $c > 0$ an integer. Suppose that there exists a genus $h-c$ Heegaard surface for M (say, S) so that K is c-bridge with respect to S, and the surface obtained by tubing S along c arcs of K cut along S on one side of S is isotopic to Σ. Then we say that Σ is obtained by *tubing S to one side (along K)*. See Figure 5.

Theorem 5.4. *Let M be a compact manifold and $K \subset M$ be a knot and suppose the meridian of K does not bound a disk in $E(K)$. Let c and h be positive integers. The following two conditions are equivalent*:

(1) *K admits an $(h-c, c)$ decomposition.*

(2) *$E(K)^{(c)}$ admits a genus h Heegaard splitting that admits a complete system of Hopf–Haken annuli.*

Proof. (1) $\Longrightarrow$ (2): Let $S \subset M$ be a surface defining a $(h-c, c)$ decomposition. Then S separates M into two sides, say, "above" and "below". Pick one, say, above. Since the arcs of K above S form c boundary parallel arcs (say, $\alpha_1, \dots, \alpha_c$), there are c disjointly embedded disks above K (say, $D_1, \dots, D_c$) so that ∂D_i consists of two arcs, one α_i and the other along S (for this proof, see Figure 5). Tubing S c times along $\alpha_1, \dots, \alpha_c$, we obtain a Heegaard surface for $E(K)$ (say, Σ). We may assume that the tubes are small enough so that they intersect each D_i in a single spanning arc. Denote the compression bodies obtained by cutting $E(K)$

along Σ by C_1 and C_2 with $\partial N(K) \subset \partial_- C_1$. Then each $D_i \cap C_2$ is a meridional disk. Let $A_1, \dots, A_c$ be c meridional annuli properly embedded in C_1 near the maxima of K. Then $(\bigcup_i A_i) \cap \partial N(K)$ consists of c meridians, say, $\alpha'_1, \dots, \alpha'_c$. For each i, we isotope α'_i along the annulus A_i to the curve $A_i \cap \Sigma$ and then push it slightly into C_2, obtaining c curves, say, $\beta_1, \dots, \beta_c$, parallel to meridians. Drilling $\bigcup_i \beta_i$ out of $E(K)$ gives $E(K)^{(c)}$. Using the disks $D_i \cap C_2$ it is easy to see that Σ is a Heegaard surface for $E(K)^{(c)}$. Clearly, the trace of the isotopy from $\bigcup_{i=1}^n \alpha'_i$ to $\bigcup_{i=1}^n \beta_i$ forms a complete system of Hopf annuli, and by construction every one of these annuli intersects Σ in a single curve that is essential in the annulus. This completes the proof of (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (1): Assume that $E(K)^{(c)}$ admits a Heegaard surface of genus h, say, Σ, with a complete system of Hopf–Haken annuli, say, $\{A_1, \dots, A_c\}$. Let

$$E(K)' = \mathrm{cl}\big(E(K)^{(c)} \setminus \textstyle\bigcup_i N(A_i)\big).$$

Note that $E(K)'$ is homeomorphic to $E(K)$. Let S' be the meridional surface $\Sigma \cap E(K)'$. We may consider M as obtained from $E(K)'$ by meridional Dehn filling and K as the core of the attached solid torus. By capping off S' we obtain a closed surface $S \subset M$. The following claim completes the proof of (2) $\Longrightarrow$ (1):

Claim 5.5. S defines a $(h-c, c)$ decomposition for K.

Proof of claim. Recall that the components of $\partial E(K)^{(c)} \setminus \partial E(K)$ were denoted by $T_1, \dots, T_c$, as in Definition 5.2, so that $A_i \cap T_i \neq \varnothing$ and $A_i \cap T_j = \varnothing$ (for $i \neq j$). Let C_1, C_2 be the compression bodies obtained from $E(K)^{(c)}$ by cutting along Σ, where $\partial N(K) \subset \partial_- C_1$. Since $\Sigma \cap A_i$ is a single simple closed curve which is essential in A_i we have $T_i \subset \partial_- C_2$ ($i = 1, \dots, c$). Denote the annulus $A_j \cap C_i$ by $A_{i,j}$ ($i = 1, 2$, $j = 1, \dots, c$).

Let $C'_i = C_i \cap E(K)'$ ($i = 1, 2$). It is clear that S' cuts $E(K)'$ into C'_1 and C'_2. Since $A_i \cap \partial N(K)$ is a meridian of K, and by assumption the meridian of K does not bound a disk in $E(K)$, we have that $A_{i,j}$ is incompressible in C_i. Hence a standard innermost disk, outermost arc argument shows that there is a system of meridian disks $\mathfrak{D}_i$ of C_i which cuts C_i into $\partial_- C_i \times [0, 1]$ such that $\mathfrak{D}_i \cap \big(\bigcup A_{i,j}\big) = \varnothing$.

Now we consider C_2 cut along $\bigcup A_{2,j}$. Since $\mathfrak{D}_2 \cap \big(\bigcup A_{2,j}\big) = \varnothing$, there are components $T_1 \times [0, 1], \dots, T_c \times [0, 1]$ of C_2 cut along $\mathfrak{D}_2$, where $A_{2,j} \subset T_j \times [0, 1]$ ($j = 1, \dots, c$). Here we note that $T_j \times [0, 1]$ cut along $A_{2,j}$ is a solid torus in which the image of $T_j \times \{0\}$ is a longitudinal annulus (note that the image of $T_j \times \{0\}$ is exactly $T_j \cap C'_2$). This shows that $\{T_1 \cap C'_2, \dots, T_c \cap C'_2\}$ is a primitive system of annuli in C'_2, that is, there is a system of meridian disks $D_{2,1}, \dots, D_{2,c}$ in C'_2 such that $D_{2,j} \cap (T_j \cap C'_2)$ consists of a spanning arc of $T_j \cap C'_2$, and $D_{2,j} \cap (T_k \cap C'_2) = \varnothing$ ($j \neq k$). Let C''_2 be the manifold obtained from C'_2 by adding c 2-handles along $T_1 \cap C'_2, \dots, T_c \cap C'_2$. Since $\{T_1 \cap C'_2, \dots, T_c \cap C'_2\}$ is primitive, C''_2 is a genus

$(h-c)$ compression body, and the union of the co-cores of the attached 2-handles, which can be regarded as $K \cap C_2''$, are simultaneously isotopic (through the disks $\bigcup D_{2,j}$) into $\partial_+ C_2''$.

Analogously since $\mathcal{D}_1 \cap \left(\bigcup A_{1,j}\right) = \varnothing$, there are c components of C_1 cut by $\mathcal{D}_1 \cup \left(\bigcup A_{1,j}\right)$ which are solid tori such that $\partial N(K)$ intersects each solid torus in a longitudinal annulus. Then the arguments in the last paragraph show that $K \cap C_1''$ consists of c arcs which are simultaneously parallel to S.

These show that S gives a $(h-c, c)$ decomposition for K, completing the proof of the claim, and thus also of Theorem 5.4. □

Corollary 5.6. *Let K be a knot in a compact manifold M, and suppose that for some positive integers h and c, K admits a $(h-c, c)$ decomposition. Then,*

$$g(E(K)^{(c)}) \leq h.$$

Proof. This follows immediately from (1) $\Longrightarrow$ (2) of Theorem 5.4. □

6. Existence of swallow follow tori and bounding $g(E(K_1\#\cdots\#K_n)^{(c)})$ above

Definition 6.1 (swallow follow torus). Let $K \subset M$ be a knot and $c \geq 0$ an integer. An essential separating torus $T \subset E(K)^{(c)}$ is called a *swallow follow torus* if there exists an embedded annulus $A \subset E(K)^{(c)}$ with one component of ∂A a meridian of $E(K)^{(c)}$ and the other an essential curve of T, so that $\text{int}(A) \cap T = \varnothing$.

In this definition (and throughout this paper) we allow K to be the unknot in S^3, in which case $E(K)^{(c)}$ is homeomorphic to a disk with c holes cross S^1, and it admits swallow follow tori whenever $c \geq 3$.

Given a swallow follow torus T and an annulus A as above, we can surger T along A to obtain a separating meridional annulus. It is easy to see that since T is an essential torus, the annulus obtained is essential as well. Conversely, given an essential separating meridional annulus we can tube the annulus to itself along the boundary obtaining a swallow follow torus (this can be done in two distinct ways).

How does a swallow follow torus decompose a knot exterior? We first consider the case $c = 0$. Let $K = K_1\#K_2$ be a composite knot (here we are not assuming that K_1 or K_2 is prime). Let $\mathcal{A}$ be a decomposing annulus corresponding to the decomposition of K as $K_1\#K_2$. Thus $E(K) = E(K_1) \cup_{\mathcal{A}} E(K_2)$. Tubing $\mathcal{A}$ along the boundary (say, into $E(K_2)$) we obtain a swallow follow torus, say, T. Clearly, one component of $E(K)$ cut open along T is homeomorphic to $E(K_2)$. The other component is homeomorphic to $E(K_1)$ with two meridional annuli identified, and hence homeomorphic to $E(K_1)^{(1)}$. Thus we see that a swallow follow torus $T \subset E(K)$ decomposes $E(K)$ as $E(K_1)^{(1)} \cup_T E(K_2)$. More generally, given K, K_1, and K_2 as above and integers c, c_1, $c_2 \geq 0$ with $c_1 + c_2 = c$, let $\mathcal{A}$ be a decomposing annulus for $E(K)^{(c)}$, so that $E(K)^{(c)} = E(K_1)^{(c_1)} \cup_{\mathcal{A}} E(K_2)^{(c_2)}$. The swallow follow torus obtained by tubing $\mathcal{A}$ into $E(K_2)^{(c_2)}$ decomposes $E(K)^{(c)}$ as $E(K_1)^{(c_1+1)} \cup_T E(K_2)^{(c_2)}$. Since the components of $E(K)^{(c)}$ cut open along a swallow follow torus are themselves of the form $E(K_1)^{(c_1+1)}$ and $E(K_2)^{(c_2)}$, we may now extend Definition 6.1 inductively:

Definition 6.2 (swallow follow tori). Let K and c be as in the previous paragraph. Let $T_1, \dots, T_r$ (for some r) be disjointly embedded tori in $E(K)^{(c)}$. Then $T_1, \dots, T_r$ are called *swallow follow tori* if the following two conditions hold, perhaps after reordering the indices:

(1) T_1 is a swallow follow torus for $E(K)^{(c)}$.

(2) For each $i \geq 2$, T_i is a swallow follow torus for some component of $E(K)^{(c)}$ cut open along $\bigcup_{j=1}^{i-1} T_j$.

We are now ready to state and prove:

Proposition 6.3 (existence of swallow follow tori). *For $i = 1, \dots, n$, let K_i be a (not necessarily prime) knot in a compact manifold and let $c \geq 0$ be an integer. Suppose that $E(K_i) \not\cong T^2 \times [0, 1]$ and $\partial N(K_i)$ is incompressible in $E(K_i)$.*

Then given any integers $c_1, \dots, c_n \geq 0$ whose sum is $c + n - 1$, there exist $n - 1$ swallow follow tori, denoted $\mathcal{T}$, that decompose $E(\#_{i=1}^n K_i)^{(c)}$ as

$$E(\#_{i=1}^n K_i)^{(c)} = \bigcup_{\mathcal{T}} E(K_i)^{(c_i)}.$$

Proof. We use the notation as in the statement of the proposition and induct on n. If $n = 1$ there is nothing to prove. We assume, as we may, that $n > 1$. We first claim that for some i we have that $c_i \leq c$. Assume, for a contradiction, that $c_i > c$ for every $1 \leq i \leq n$. Since c_i and c are integers, $c_i \geq c + 1$. Then we have:

$$c + n - 1 = \sum_{i=1}^{n} c_i \geq n(c + 1) = nc + n.$$

Moving all term to the right we get that

$$0 \geq (n - 1)c + 1,$$

which is absurd, since $n \geq 1$ and $c \geq 0$. By reordering the indices if necessary we may assume that $c_n \leq c$.

Let A be an annulus in $E(\#_{i=1}^n K_i)$ so that the components of $E(\#_{i=1}^n K_i)$ cut open along A are identified with $E(K_1 \# \cdots \# K_{n-1})$ and $E(K_n)$. Since the tori $\partial N(K_i)$ are incompressible, A is essential in $E(\#_{i=1}^n K_i)$. Recall that $E(\#_{i=1}^n K_i)^{(c)}$ is obtained from $E(\#_{i=1}^n K_i)$ by drilling c curves that are parallel to the meridian; since $c_n \leq c$ we may choose the curves so that exactly c_n components are contained in $E(K_n)$. After drilling, the components of $E(\#_{i=1}^n K)^{(c)}$ cut open along A are identified with $E(K_1 \# \cdots \# K_{n-1})^{(c-c_n)}$ and $E(K_n)^{(c_n)}$. Let T be the torus obtained by tubing

A into $E(K)^{(c_n)}$; clearly the components of $E(\#_{i=1}^n K)^{(c)}$ cut open along T are identified with $E(K_1\#\cdots\#K_{n-1})^{(c-c_n+1)}$ and $E(K_n)^{(c_n)}$. Since A is essential and $E(K_i)\not\cong T^2\times[0,1]$, we have that T is essential in $E(\#_{i=1}^n K_i)^{(c)}$. By construction, there is an essential curve on T that cobounds an annulus with a meridian of $E(\#_{i=1}^n K_i)^{(c)}$ and we conclude that T is a swallow follow torus.

We induct on $K_1,\dots,K_n$. Let $c'=c-c_n+1$. Then we have

$$\begin{aligned}\sum_{i=1}^{n-1} c_i = \sum_{i=1}^{n} c_i - c_n &= c+n-1-c_n \\ &= (c-c_n+1)+n-2 = c'+(n-1)-1.\end{aligned}$$

By induction, $E(K_1\#\cdots\#K_{n-1})^{(c')}$ admits $n-2$ swallow follow tori, which we will denote by $\mathcal{T}'$, so that $\mathcal{T}'$ decomposes

$$E(K_1\#\cdots\#K_{n-1})^{(c')} = E(K_1\#\cdots\#K_{n-1})^{(c-c_n+1)},$$

as

$$\bigcup\nolimits_{\mathcal{T}'} E(K_i)^{(c_i)}.$$

It follows that $\mathcal{T}=T\cup\mathcal{T}'$ are swallow follow tori for $E(K)^{(c)}$, and the components of $E(K)^{(c)}$ cut open along $\mathcal{T}$ are homeomorphic to $E(K_1)^{(c_1)},\dots,E(K_n)^{(c_n)}$. □

By Proposition 6.3 and repeated use of Lemma 2.2 we obtain the following.

Corollary 6.4. *With notation as in Proposition 6.3 (and in particular for any integer $c\ge 0$ and any integers $c_1,\dots,c_n$ whose sum is $c+n-1$), we get:*

$$g(E(K)^{(c)}) \le \Sigma_{i=1}^n g(E(K_i)^{(c_i)}) - (n-1).$$

7. An upper bound for the growth rate

Using the results in the previous sections we can easily bound the growth rate:

Proposition 7.1. *Let K be an admissible knot in a closed manifold M. Let $g = g(E(K))-g(M)$ and the bridge indices $\{b_1^*,\dots,b_g^*\}$ be as in Notation 1.1. Then,*

$$gr_t(K) \le \min_{i=1,\dots,g}\left\{1-\frac{i}{b_i^*}\right\}.$$

Proof. Fix $1\le i\le g$ and a positive integer n. Let $k_i>0$ and $0\le r<b_i^*$ be the quotient and remainder when dividing $(n-1)$ by b_i^*; that is:

$$k_i b_i^* + r = n-1.$$

Consider the nonnegative integers $b_i^*,\dots,b_i^*,r,0,\dots,0$ (where b_i^* appears k_i times and the symbol 0 appears $n-(k_i+1)$ times). Applying Corollary 6.4 to $E(nK)^{(0)}$

we get (recalling that $E(nK)^{(0)} = E(nK)$):

$$g(E(nK)) \leq k_i g(E(K)^{(b_i^*)}) + g(E(K)^{(r)}) + (n-(k_i+1))g(E(K)) - (n-1).$$

By definition of b_i^*, K admits a $(g(E(K))-i, b_i^*)$ decomposition. Applying Corollary 5.6 with $h-c = g(E(K))-i$ and $c = b_i^*$ gives

$$g(E(K)^{(b_i^*)}) \leq g(E(K)) - i + b_i^*.$$

Thus we get:

$$\begin{aligned} g(E(nK)) &\leq k_i(g(E(K))-i+b_i^*)+g(E(K)^{(r)})+(n-(k_i+1))g(E(K))-(n-1) \\ &= (n-1)g(E(K))+g(E(K)^{(r)})-k_i i+(k_i b_i^*-(n-1)) \\ &= (n-1)g(E(K))+g(E(K)^{(r)})-k_i i-r. \end{aligned}$$

By denoting the n-th element of the sequence in the definition of the growth rate by S_n, we get:

$$\begin{aligned} S_n &= \frac{g(E(nK)) - ng(E(K)) + (n-1)}{n-1} \\ &\leq \frac{1}{n-1}\big[(n-1)g(E(K)) + g(E(K)^{(r)}) - k_i i - r - ng(E(K)) + (n-1)\big] \\ &= \frac{1}{n-1}\big[g(E(K)^{(r)}) - g(E(K)) - r - k_i i + (n-1)\big] \\ &= \frac{g(E(K)^{(r)}) - g(E(K)) - r}{n-1} + 1 - \frac{k_i i}{k_i b_i^* + r}. \end{aligned}$$

In the last equality we used $k_i b_i^* + r = n-1$. Recall that $E(K)^{(r)}$ is obtained by drilling r curves parallel to $\partial E(K)$ out of $E(K)$. Therefore, by [Rieck 2000],

$$g(E(K)^{(r)}) \leq g(E(K)) + r.$$

Hence the first summand above is nonpositive, and we may remove that term. Further, since $r < b_i^*$, $k_i b_i^* + r < (k_i+1)b_i^*$, which implies

$$S_n < 1 - \frac{i}{b_i^*}\frac{k_i}{k_i+1}. \tag{1}$$

Since $\lim_{n\to\infty} k_i = \infty$ we have:

$$\mathrm{gr}_t(K) = \limsup_{n\to\infty} S_n \leq \lim_{k_i\to\infty}\Big(1 - \frac{i}{b_i^*}\frac{k_i}{k_i+1}\Big) = 1 - \frac{i}{b_i^*}.$$

As i was arbitrary, we get

$$\mathrm{gr}_t(K) \leq \min_{i=1,\dots,g}\Big\{1 - \frac{i}{b_i^*}\Big\}.$$

This completes the proof of Proposition 7.1. □

Part III. The growth rate of m-small knots

This part is devoted to calculating the growth rate of m-small knots, completing the proof of Theorem 1.2. Section 8 contains the main technical result of this paper, the strong Hopf–Haken annulus theorem (Theorem 1.8). This result guarantees the existence of Hopf–Haken annuli, and complements Sections 4 and 5. In Section 9 we prove existence of "special" swallow follow tori; this section complements Section 6. Finally, in Section 10 we calculate the growth rate of m-small knots by finding a lower bound that equals exactly the upper bound found in Section 7.

8. The strong Hopf–Haken annulus theorem

Given a knot K in a compact manifold M and an integer $c > 0$, recall that the exterior of K is denoted by $E(K)$, the manifold obtained by drilling out c curves simultaneously parallel to the meridian of $E(K)$ is denoted by $E(K)^{(c)}$, and the components of $\partial E(K)^{(c)} \setminus \partial E(K)$ are denoted by $T_1, \dots, T_c$. Recall also the definitions of Haken annuli for a given Heegaard splitting (Definition 4.1), a complete system of Hopf annuli (Definition 5.1), and a complete system of Hopf–Haken annuli for a given Heegaard splitting (Definition 5.2).

In this section we prove the strong Hopf–Haken annulus theorem (Theorem 1.8), stated in the introduction. Before proving Theorem 1.8 we prove three of its main corollaries:

Corollary 8.1. *Suppose that the assumptions of Theorem 1.8 are satisfied with $F_1 = F_2 = \emptyset$ and in addition, for each i, $E(K_i)$ does not admit an essential meridional surface S with $\chi(S) \geq 6 - 2g(E(K)^{(c)})$. Let $h \geq 0$ be an integer. Then K admits an $(h-c, c)$ decomposition if and only if $g(E(K)^{(c)}) \leq h$.*

Proof of Corollary 8.1. Assume first that K admits an $(h-c, c)$ decomposition. Then by Corollary 5.6, we have $g(E(K)^{(c)}) \leq h$. Note that this direction holds in general and does not require the assumption about meridional surfaces.

Next assume that $g(E(K)^{(c)}) \leq h$ and let $\Sigma \subset E(K)^{(c)}$ be a genus h Heegaard surface. By the assumptions of the corollary, Theorem 1.8(2) does not hold. Hence by that theorem $E(K)^{(c)}$ admits a genus h Heegaard surface that admits a complete system of Hopf–Haken annuli. By (2) $\Rightarrow$ (1) of Theorem 5.4, K admits an $(h-c, c)$ decomposition. □

Corollary 8.2. *Suppose that the assumptions of Theorem 1.8 hold and in addition, that each K_i is m-small. Then for any c and any choice of F_1 and F_2, there is a minimal genus Heegaard splitting of $(E(\#_{i=1}^n K_i)^{(c)}; F_1, F_2)$ that admits a complete system of Hopf–Haken annuli.*

Proof of Corollary 8.2. This follows immediately from Theorem 1.8. □

Next we prove Corollary 1.9 which was stated in the introduction:

Proof of Corollary 1.9. We fix the notation in the statement of the corollary. First we show that for any knot K (not necessarily the connected sum of m-small knots), if $c \geq b_g^*$, then the inequality $g(E(K)^{(c)}) \leq c$ holds: by definition of b_g^*, K admits a $(0, b_g^*)$ decomposition (recall that $K \subset S^3$ and hence b_g^* is the bridge index of K with respect to S^2). Thus for $c \geq b_g^*$, K admits a $(0, c)$ decomposition. By viewing this as a $(c - c, c)$ decomposition, Corollary 5.6 implies that $g(E(K)^{(c)}) \leq c$.

Next we note that the inequality $g(E(K)^{(c)}) \geq c$ holds for K that is a connected sum of m-small knots, and any $c \geq 0$: by Corollary 8.2, $E(K)^{(c)}$ admits a minimal genus Heegaard surface (say, Σ) admitting a complete system of Hopf–Haken annuli. Hence the c tori, $T_1, \dots, T_c$, are on the same side of Σ, which implies $g(\Sigma) \geq c$; hence $g(E(K)^{(c)}) = g(\Sigma) \geq c$. □

Proof of Theorem 1.8. We first fix the notation that will be used in the proof (in addition to the notation in the statement of the theorem). Let K denote $\#_{i=1}^n K_i$. For $c > 0$, $E(K)^{(c)}$ admits an essential torus T that decomposes $E(K)^{(c)}$ as

$$E(K)^{(c)} = X \cup_T Q^{(c)},$$

where $X \cong E(K)$ and $Q^{(c)} \cong$ (an annulus with c holes) $\times S^1$. Note that $Q^{(c)}$ fibers over S^1 in a unique way, and the fibers in T are meridian curves in $X \cap Q^{(c)}$. Since $Q^{(c)}$ is Seifert fibered it is contained in a unique component J of the characteristic submanifold [Jaco 1980; Jaco and Shalen 1979; Johannson 1979]. Since $\partial N(K_i)$ is incompressible in $E(K_i)$, using Miyazaki's result [1989] it was shown in [Kobayashi and Rieck 2006a, Claim 1] that K admits a unique prime decomposition. Therefore the number of prime factors of K is well defined. We suppose, as we may, that each knot K_i is prime; consequently, the integer n appearing in the statement of the theorem is the number of prime factors of K.

The structure of the proof. The proof is an induction on (n, c) ordered lexicographically. We begin with two preliminary special cases. In Case One we consider strongly irreducible Heegaard splittings. In Case Two we consider weakly reducible Heegaard splittings so that no component of the essential surface obtained by untelescoping is contained in J. In both cases we prove the theorem directly and without reference to the complexity (n, c). We then proceed to the inductive step assuming the theorem for $(n', c') < (n, c)$ in the lexicographic order. By Cases One and Two we may assume that a minimal genus Heegaard surface for $E(K)^{(c)}$ is weakly reducible and some component of the essential surface obtained by untelescoping it is contained in J; this component allows us to induct.

Case One: $(E(K)^{(c)}; F_1, F_2)$ *admits a strongly irreducible minimal genus Heegaard splitting.* Let $C_1 \cup_\Sigma C_2$ be a minimal genus strongly irreducible Heegaard splitting of $(E(K)^{(c)}; F_1, F_2)$. The swallow follow torus theorem [Kobayashi and Rieck 2006a, Theorem 4.1] implies that if $n > 1$, either Σ weakly reduces to a swallow follow torus (contradicting the assumption of Case One) or Theorem 1.8(2) holds. We assume, as we may, that $n = 1$ in the remainder of the proof of Case One.

Recall the notation $E(K)^{(c)} = X \cup_T Q^{(c)}$. Since $T \subset E(K)^{(c)}$ is essential and $\Sigma \subset E(K)^{(c)}$ is strongly irreducible, we may isotope Σ so that $\Sigma \cap T$ is transverse and every curve of $\Sigma \cap T$ is essential in T. Minimize $|\Sigma \cap T|$ subject to this constraint. If $\Sigma \cap T = \varnothing$ then T is contained in a compression body C_1 or C_2, and hence T is parallel to a component of $\partial_- C_1$ or $\partial_- C_2$. But then T is parallel to a component of $\partial E(K)^{(c)}$, a contradiction. Thus $\Sigma \cap T \neq \varnothing$.

Let F be a component of Σ cut open along T. Minimality of $|\Sigma \cap T|$ implies that F is not boundary parallel. Then $\partial F \subset T$; since T is a torus, boundary compression of F implies compression into the same side; this will be used extensively below. A surface in a Seifert fibered manifold is called *vertical* if it is everywhere tangent to the fibers and *horizontal* if it is everywhere transverse to the fibers (see, for example, [Jaco 1980] for a discussion). We first reduce Theorem 1.8 as follows:

Assertion 1. One of the following holds:

(1) $\Sigma \cap X$ is connected and compresses into both sides, and $\Sigma \cap Q^{(c)}$ is a collection of essential vertical annuli.

(2) Theorem 1.8 holds.

Proof. A standard argument shows that one component of Σ cut open along T compresses into both sides (in X or $Q^{(c)}$) and all other components are essential (in X or $Q^{(c)}$); for the convenience of the reader we sketch it here: Let D_1 be a compressing disk for C_1. After minimizing $|D_1 \cap T|$ either $D_1 \cap T = \varnothing$ (and hence some component of Σ cut open along T compresses into C_1) or an outermost disk of D_1 provides a boundary compression for some component of Σ cut open along T; since boundary compression implies compression into the same side, we see that in this case too some component of Σ cut open along T compresses into C_1. Similarly, some component of Σ cut open along T compresses into C_2. Strong irreducibility of Σ implies that the same component compresses into both sides and all other components are incompressible and boundary incompressible. Minimality of $|\Sigma \cap T|$ implies that no component is boundary parallel, and hence the incompressible and boundary incompressible components are essential.

The proof of Assertion 1 breaks up into three subcases:

Subcase 1: *no component of $\Sigma \cap X$ is essential.* Then $\Sigma \cap X$ is connected and compresses into both sides, and therefore $\Sigma \cap Q^{(c)}$ consists of essential surfaces. Since $Q^{(c)}$ is Seifert fibered, every component of $\Sigma \cap Q^{(c)}$ is either horizontal or vertical (see, for example, [Jaco 1980, VI.34]). Any horizontal surface in $Q^{(c)}$ must meet every component of $\partial Q^{(c)}$; by construction $\Sigma \cap \partial N(K) = \varnothing$; thus every component of $\Sigma \cap Q^{(c)}$ is vertical (we will use this argument below without reference). This gives Assertion 1(1).

Subcase 2a: *some component of $\Sigma \cap X$ is essential and some component of $\Sigma \cap Q^{(c)}$ is essential.* Let F denote an essential component of $\Sigma \cap X$. Since T is incompressible and the components of $\Sigma \cap T$ are essential in T, no component of Σ cut open along T is a disk; hence $\chi(F) \geq \chi(\Sigma)$. Let S denote an essential component of $\Sigma \cap Q^{(c)}$. Then S is a vertical annulus. In particular, $S \cap T$ consists of fibers in the Seifert fibration of $Q^{(c)}$. By construction, the fibers on T are meridians of X. We see that F is meridional, giving Theorem 1.8(2).

Subcase 2b: *some component of $\Sigma \cap X$ is essential and no component of $\Sigma \cap Q^{(c)}$ is essential.* As above let F be an essential component of $\Sigma \cap X$. By assumption, no component of $\Sigma \cap Q^{(c)}$ is essential. Hence $\Sigma \cap Q^{(c)}$ is connected and compresses into both sides. Let Δ_1 be a maximal collection of compressing disks for $\Sigma \cap Q^{(c)}$ into $Q^{(c)} \cap C_1$ and S_1 the surface obtained by compressing S along Δ_1. Since $\Delta_1 \neq \varnothing$, maximality of Δ_1 and the no nesting lemma [Scharlemann 1998] imply that S_1 is incompressible. Suppose first that some nonclosed component of S_1, say, S_1', is not boundary parallel (this is similar to Subcase 2a). Then S_1' is an essential and hence vertical annulus and we see that F is meridional, giving Theorem 1.8(2) and the assertion follows. We assume from now on that S_1 consists of boundary parallel annuli and, perhaps, closed boundary parallel surfaces and ball-bounding spheres. Furthermore, we see that:

(1) No two closed components of S_1 are parallel to the same component of $\partial Q^{(c)}$: this follows from the connectivity of $\Sigma \cap Q^{(c)}$ and strong irreducibility of Σ.

(2) No two boundary parallel annuli of S_1 are nested: otherwise, it follows from the connectivity of $\Sigma \cap Q^{(c)}$ and strong irreducibility of Σ that Σ can be isotoped out of $Q^{(c)}$; for more details see [Kobayashi and Rieck 2004, page 249].

We assume, as we may, that the analogous conditions hold after compressing $\Sigma \cap Q^{(c)}$ into $Q^{(c)} \cap C_2$. Hence $\Sigma \cap Q^{(c)}$ is a Heegaard surface for $Q^{(c)}$ relative to the annuli $\{C_1 \cap T, C_2 \cap T\}$ (relative Heegaard surfaces were defined in Definition 3.1). We may replace $\Sigma \cap Q^{(c)}$ with the minimal genus relative Heegaard surface for $Q^{(c)}$ relative to $\{C_1 \cap T, C_2 \cap T\}$ given in Corollary 3.3. By pasting this surface to $\Sigma \cap X$ we obtain a closed surface, say, Σ', satisfying the four following conditions:

(1) Σ' is a Heegaard surface for $E(K)^{(c)}$: the components of X cut open along $\Sigma \cap X$ are the same as the components of C_1 and C_2 cut open along $\{C_1 \cap T, C_2 \cap T\}$ that are contained in X. Since T is essential, the annuli $C_i \cap T$ are incompressible in C_i. It is well known that cutting a compression body along incompressible surfaces yields compression bodies; we conclude that the components of X cut open along $\Sigma \cap X$ are compression bodies. By definition of the relative Heegaard surface, the annuli of $\{C_1 \cap T, C_2 \cap T\}$ are primitive in the compression bodies obtained by cutting $Q^{(c)}$ open along any relative Heegaard surface; it follows that $E(K)^{(c)}$ cut open along Σ' consists of two compression bodies.

(2) Σ' is a Heegaard surface for $(E(K)^{(c)}; F_1, F_2)$: in addition to (1) above, we must show Σ' respects the same partition of $\partial E(K)^{(c)} \setminus (\partial N(K), T_1, \dots, T_c)$ as Σ. This follows immediately from the facts that the changes we made are contained in $Q^{(c)}$, every component of F_1 is contained in $C_1 \cap X$, and every component of F_2 is contained in $C_2 \cap X$. Note that (1) and (2) hold for any relative Heegaard surface for $Q^{(c)}$ relative to $\{C_1 \cap T, C_2 \cap T\}$.

(3) $g(\Sigma') = g(\Sigma)$: minimality of the genus of the relative Heegaard splitting used implies that $g(\Sigma') \leq g(\Sigma)$, and since Σ is a minimal genus Heegaard surface for $(E(K)^{(c)}; F_1, F_2)$, equality holds: $g(\Sigma') = g(\Sigma)$. Note that (3) holds for any minimal genus relative Heegaard surface for $Q^{(c)}$ relative to $\{C_1 \cap T, C_2 \cap T\}$.

(4) Σ' admits a complete system of Hopf–Haken annuli: by Figure 1 we see directly that Σ' admits a complete system of Hopf–Haken annuli.

Remark 8.3. As noted, in the construction above, (1), (2), and (3) hold for any minimal genus relative Heegaard surface. This is quite different in (4), when considering Hopf–Haken annuli: it is not hard to construct relative Heegaard surfaces that result in a minimal genus Heegaard surface for $(E(K)^{(c)}; F_1, F_2)$ so that all the tori $T_1, \dots, T_c$ are in the compression body containing $\partial N(K)$, and hence cannot admit even one Hopf–Haken annulus. This shows that in the course of the proof of Theorem 1.8 the given Heegaard surface must be replaced.

The Heegaard surface Σ' fulfills the conditions of Theorem 1.8(1). This completes that proof of Assertion 1. □

Before proceeding, we fix the following notation and conventions: denote $\Sigma \cap X$ by Σ_X. By Assertion 1 we may assume that Σ_X is connected and compresses into both sides and every component of $\Sigma \cap Q^{(c)}$ is an essential vertical annulus. Note that X cut open along Σ_X consists of exactly two components, denoted by $C_{i,X}$, where $C_{i,X} = C_i \cap X$ $(i = 1, 2)$. Denote the collection of annuli $T \cap C_{i,X}$ by $\mathscr{A}_i$, and the annuli in $\mathscr{A}_i$ by $A_{i,1}, \dots, A_{i,b}$, where b denotes the number of annuli in $\mathscr{A}_i$. We assume from now on that Theorem 1.8(2) does not hold.

Assertion 2. The number b satisfies $c \leq b \leq g(\Sigma)$.

Proof. Assume for a contradiction that $b < c$. Since $\Sigma \cap Q^{(c)}$ consists of b annuli, $Q^{(c)}$ cut open along $\Sigma \cap Q^{(c)}$ consists of $b+1 < c+1$ components. Hence some component of $Q^{(c)}$ cut open along $\Sigma \cap Q^{(c)}$ contains two of the components of $\partial Q^{(c)} \setminus T$. Hence there is a vertical annulus connecting these components which is disjoint from Σ. Since this annulus is disjoint from Σ it is contained in a compression body C_i and connects two components of $\partial_- C_i$, which is impossible.

Since Σ_X is obtained by removing the b annuli $\Sigma \cap Q^{(c)}$ and is connected, $b \leq g(\Sigma)$. □

Assertion 3. The surface Σ_X defines a $(g(\Sigma) - b, b)$ decomposition of K.

Proof. For $i = 1, 2$, let Δ_i be a maximal collection of compressing disks for Σ_X into $C_{i,X}$; by assumption, $\Delta_i \neq \varnothing$. Let S_i be the surface obtained by compressing Σ_X along Δ_i. By maximality and the no nesting lemma [Scharlemann 1998] S_i is incompressible. Since the components of $\Sigma \cap Q^{(c)}$ are vertical annuli, the boundary components of S_i are meridians. Hence, if some nonclosed component of S_i is essential, we obtain Theorem 1.8(2), contradicting our assumption. Thus S_i consists of boundary parallel annuli and, perhaps, closed boundary parallel surfaces and ball-bounding spheres. As above, strong irreducibility of Σ and connectivity of Σ_X imply that these annuli are not nested. We see that $C_{i,X}$ is a compression body and $T \cap C_{i,X}$ consists of b mutually primitive annuli. In fact, we see that Σ_X is a Heegaard surface relative to $\{\mathscr{A}_1, \mathscr{A}_2\}$. By the argument of Claim 5.5, Σ_X gives a $(g(\Sigma) - b, b)$ decomposition. □

By Assertion 3 and Theorem 5.4, $E(K)^{(b)}$ admits a genus $g(\Sigma)$ Heegaard surface admitting a complete system of Hopf–Haken annuli, say, Σ'. By Assertion 2, $c \leq b$. Hence $E(K)^{(c)}$ is obtained from $E(K)^{(b)}$ by filling the tori $T_{c+1}, \dots, T_b$. Clearly, Σ' is a Heegaard surface for $E(K)^{(c)}$, admitting a complete system of Hopf–Haken annuli. This completes the proof of Theorem 1.8 in Case One.

Before proceeding to Case Two we introduce notation that will be used in that case. Recall that since $Q^{(c)}$ is Seifert fibered, it is contained in a component of the characteristic submanifold of $E(K)^{(c)}$ denoted by J. Since $X \cong E(K)$ and $K = \#_{i=1}^n K_i$, X admits $n-1$ decomposing annuli which we will denote by $A_1, \dots, A_{n-1}$ ($A_1, \dots, A_{n-1}$ are not uniquely defined). The components of X cut open along $\bigcup_{i=1}^{n-1} A_i$ are homeomorphic to $E(K_1), \dots, E(K_n)$. Let

$$V = Q^{(c)} \cup N(A_1) \cup \cdots \cup N(A_{n-1}).$$

Then V is Seifert fibered and contains $Q^{(c)}$, and hence after isotopy $V \subset J$. Note that $V \cap \mathrm{cl}(E(K)^{(c)} \setminus V)$ consists of n tori, say, $T_1', \dots, T_n'$. Finally note that $X^{(c)}$ cut open along $\bigcup_{i=1}^n T_i{}'$ consists of $n+1$ components, one is V, and the others are homeomorphic to $E(K_1), \dots, E(K_n)$. We denote the component that corresponds to $E(K_i)$ by X_i. After renumbering if necessary we may assume that T_i' is a component of ∂X_i. By construction T_i' corresponds to $\partial N(K_i)$.

The proof of Assertion 4 is a simple argument using essential arcs in base orbifolds, and we leave it to the reader.

Assertion 4. If V is not isotopic to J then some $E(K_i)$ contains a meridional essential annulus.

For future reference we remark:

Remark 8.4. By Assertion 4, either we have Theorem 1.8(2), or $J = V$. Hence, in the following, we may assume that $J = V$; we will use the notation J from here on. By construction, J is homeomorphic to (a $(c+n)$-times punctured disk)$\times S^1$ and hence admits no closed nonseparating surfaces.

Case Two: $(E(K)^{(c)}; F_1, F_2)$ *admits a weakly reducible minimal genus Heegaard surface* Σ*, and no component of the essential surface obtained by untelescoping* Σ *is isotopic into* J. Let F be the (not necessarily connected) essential surface obtained by untelescoping Σ. The assumptions of Theorem 1.8 imply that $E(K)^{(c)}$ does not admit a nonseparating sphere; hence the Euler characteristic of every component of F is bounded below by $\chi(\Sigma)+4$. After an isotopy that minimizes $|F \cap \partial J|$, every component of $F \cap J$ is essential in J and every component of $F \cap \mathrm{cl}(E(K)^{(c)} \setminus J)$ is essential in $\mathrm{cl}(E(K)^{(c)} \setminus J)$. By the assumption of Case Two, if some component F' of F meets J, then $F' \not\subset J$ and hence each component of $F' \cap J$ is a vertical annulus and each component of $F' \cap \mathrm{cl}(E(K)^{(c)} \setminus J)$, say, S, is a meridional essential surface with $\chi(S) \geq \chi(F' \cap E(K)^{(c)}) = \chi(F') \geq \chi(F) \geq 6-2g(\Sigma)$, giving Theorem 1.8(2). Thus we may assume $F \cap J = \varnothing$.

Let M_J be the component of $E(K)^{(c)}$ cut open along F containing J, and let Σ_J be the strongly irreducible Heegaard surface induced on M_J by untelescoping. Then Σ_J defines a partition of $\partial M_J \setminus (T_1 \cup \cdots \cup T_c \cup \partial N(K))$, say, $F_{J,1}$, $F_{J,2}$. Since Σ is minimal genus, Σ_J is a minimal genus splitting of $(M_J; F_{J,1}, F_{J,2})$.

For $i = 1, \ldots, n$, denote $X_i \cap M_J$ by X_i'. Note that $X_i' \cap J = T_i'$; the meridian of X_i defines a slope of T_i', denoted by μ_i'. By filling X_i' along μ_i' we obtain a manifold, say, M_i', and the core of the attached solid torus is a knot, say, $K_i' \subset M_i'$. Then M_J is naturally identified with $E(\#_{i=1}^n K_i')^{(c)}$, and Σ_J is a strongly irreducible Heegaard surface for $(E(\#_{i=1}^n K_i')^{(c)}; F_{J,1}, F_{J,2})$. It is easy to see that the knots K_i' fulfill the assumptions of Theorem 1.8; in particular, the assumptions of Case Two imply that $E(K_i') \not\cong T^2 \times I$. Therefore, by Case One, one of the following holds:

(1) Theorem 1.8(1): there exists a Heegaard surface Σ_J' for M_J so that the following three conditions hold:
   (a) $g(\Sigma_J') = g(\Sigma_J)$,
   (b) Σ_J' is a Heegaard splitting for $(E(\#_{i=1}^n K_i')^{(c)}; F_{J,1}, F_{J,2})$,
   (c) Σ_J' admits a complete system of Hopf–Haken annuli.

(2) Theorem 1.8(2): for some i, X_i' admits a meridional essential surface F_i' with $\chi(F_i') \geq 6 - 2g(\Sigma_J) \geq 6 - 2g(\Sigma)$.

Assume first that (1) holds. By condition (1b), Σ_J' induces the same partition on the components of $\partial M_j \setminus \{T_1, \ldots, T_c, \partial N(K)\}$ as Σ_J. Thus we may amalgamate the Heegaard surfaces induced on the components of $\mathrm{cl}(E(K)^{(c)} \setminus M_J)$ with Σ_J', obtaining a Heegaard surface for $(E(K)^{(c)}; F_1, F_2)$, say, Σ''. By Proposition 4.3, Σ'' admits a complete system of Hopf–Haken annuli. Since $g(\Sigma_J') = g(\Sigma_J)$, we have that $g(\Sigma'') = g(\Sigma)$; hence Σ'' is a minimal genus Heegaard surface for $(E(K)^{(c)}; F_1, F_2)$. This gives Theorem 1.8(1).

Assume next that (2) happens. Since X_i' is a component of X_i cut open along the (possibly empty) surface $F \cap X_i$, and every component of $F \cap X_i$ is incompressible,

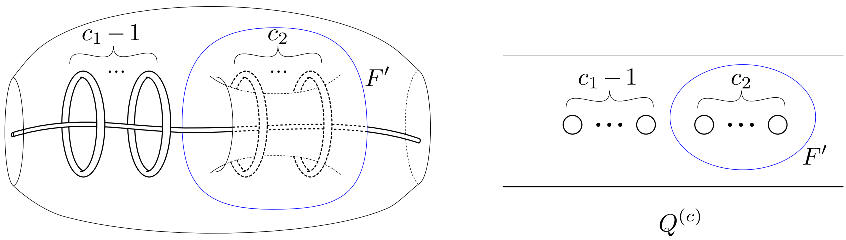

Figure 6. Subcase 1a.

we have that F_i' is essential in X_i. By construction, the meridians of X_i and X_i' are the same. Finally, $\chi(F_i') \geq 6 - 2g(\Sigma) = 6 - 2g(E(K)^{(c)}; F_1, F_2)$. This gives Theorem 1.8(2), completing the proof of Theorem 1.8 in Case Two.

With these two preliminary cases in hand we are now ready for the inductive step. For the remainder of the proof we assume that Theorem 1.8(2) does not hold. Fix $K_1, \dots, K_n$ and $c \geq 0$ and assume, by induction, that Theorem 1.8 holds for any example with complexity $(n', c') < (n, c)$ ordered lexicographically. Let Σ be a minimal genus Heegaard surface for $E(\#_{i=1}^n K_i)^{(c)}$. By Case One, we may assume that Σ is not strongly irreducible; hence Σ admits an untelescoping. By Case Two, we may assume that some component F' of the essential surface F obtained by untelescoping Σ is isotopic into J. By Remark 8.4, J is a Seifert fibered space over a punctured disk and the components of $E(\# K_i)^{(c)} \setminus J$ are identified with $E(K_1), \dots, E(K_n)$. After isotopy we may assume that F' is horizontal or vertical (see, for example, [Jaco 1980, VI.34]; recall that a surface in a Seifert fibered space is horizontal if it is everywhere transverse to the fibers and vertical if it is everywhere tangent to the fibers). However $\partial J \neq \varnothing$ and $\partial F' = \varnothing$, and therefore F' cannot be horizontal. We conclude that F' is a vertical torus that separates J and hence $E(\#_{i=1}^n K_i)^{(c)}$. Thus F' decomposes $E(\#_{i=1}^n K_i)^{(c)}$ as:

$$E(\#_{i=1}^n K_i)^{(c)} = E(\#_{i\in I} K_i)^{(c_1)} \cup_{F'} E(\#_{i\notin I} K_i)^{(c_2)},$$

where $c_1 + c_2 = c + 1$ and $I \subset \{1, \dots, n\}$. Since F' is connected and separating, by Proposition 2.3, Σ weakly reduces to F' and the weak reduction induces (not necessarily strongly irreducible) Heegaard splittings on $E(\#_{i\in I} K_i)^{(c_1)}$ and $E(\#_{i\notin I} K_i)^{(c_2)}$. We divide the proof into Cases 1 and 2 below:

Case 1: *$I = \varnothing$ or $I = \{1, \dots, n\}$.* By symmetry we may assume that $I = \{1, \dots, n\}$. Then F' decomposes $E(\#_{i=1}^n K_i)^{(c)}$ as $E(\#_{i=1}^n K_i)^{(c_1)} \cup_{F'} D(c_2)$ where $D(c_2)$ is a c_2 times punctured disk cross S^1. There are two possibilities: $\partial N(K) \subset E(\#_{i=1}^n K_i)^{(c_1)}$ (Subcase 1a) and $\partial N(K) \subset D(c_2)$ (Subcase 1b).

Subcase 1a: *$I = \{1, \dots, n\}$ and $\partial N(K) \subset E(\#_{i=1}^n K_i)^{(c_1)}$.* For this subcase, see Figure 6. Recall that $E(\#_{i=1}^n K_i)^{(c)} = E(\#_{i=1}^n K_i)^{(c_1)} \cup_{F'} D(c_2)$ with $c_1 + c_2 = c + 1$; reordering $T_1, \dots, T_c$ if necessary we may assume $T_1, \dots, T_{c_1-1} \subset \partial E(\#_{i=1}^n K_i)^{(c_1)}$

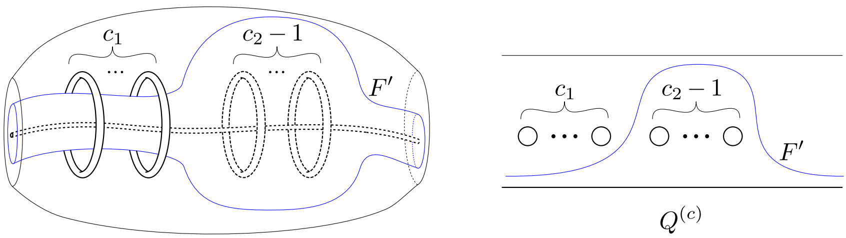

Figure 7. Subcase 1b.

and $T_{c_1}, \ldots, T_c \subset \partial D(c_2)$. Since F' is not boundary parallel, $c_2 \geq 2$; thus $c_1 < c$. Thus $(n, c_1) < (n, c)$ (in the lexicographic order) and hence we may apply induction to $E(\#_{i=1}^n K_i)^{(c_1)}$. Let Σ_1' be the Heegaard surface induced on $E(\#_{i=1}^n K_i)^{(c_1)}$ by the weak reduction of Σ. By assumption, Theorem 1.8(2) does not hold; it is easy to see that $E(\#_{i=1}^n K_i)^{(c_1)}$ satisfies the assumptions of Theorem 1.8, and since $g(\Sigma_1') < g(\Sigma)$, Theorem 1.8(2) does not hold for $E(\#_{i=1}^n K_i)^{(c_1)}$. Therefore the inductive hypothesis shows that $E(\#_{i=1}^n K_i)^{(c_1)}$ admits a Heegaard surface Σ_1 fulfilling the following three conditions:

(1) $g(\Sigma_1) = g(\Sigma_1')$.

(2) Σ_1 and Σ_1' induces the same partition of the components of $\partial E(\#_{i=1}^n K_i)^{(c_1)} \setminus \{T_1, \ldots, T_{c_1 - 1}, F', \partial N(K)\}$.

(3) Σ_1 admits a complete system of Hopf–Haken annuli.

Denote the union of the $c_1 - 1$ Hopf–Haken annuli connecting $\partial N(\#_{i=1}^n K_i)$ to $T_1, \ldots, T_{c_1-1}$ by $\mathscr{A}_1$ and the Hopf–Haken annulus connecting $\partial N(\#_{i=1}^n K_i)$ to F' by A (note that $c_1 - 1 = 0$ is possible; in that case $\mathscr{A}_1 = \emptyset$). There exists a minimal genus Heegaard surface Σ_2 for $D(c_2)$ that admits c_2 Haken annuli $A_{c_1}, \ldots, A_c$ so that one component of ∂A_i is a longitude of T_i and the other is on F' and parallel to $A \cap F'$ there (recall Remark 4.2). We denote $\bigcup_{i=c_1}^c A_i$ by $\mathscr{A}_2$. As shown in Proposition 4.3, the annuli obtained by attaching a parallel copy of A to each annulus of $\mathscr{A}_2$ union $\mathscr{A}_1$ are Haken annuli for the Heegaard surface obtained by amalgamating Σ_1 and Σ_2; we will denote this surface by $\hat{\Sigma}$. By construction, these annuli form a complete system of Hopf–Haken annuli for $\hat{\Sigma}$. Since $g(\hat{\Sigma}) = g(\Sigma_1) + g(\Sigma_2) - 1$ and $g(\Sigma) = g(\Sigma_1') + g(\Sigma_2) - 1$, by condition (1) above we have $g(\hat{\Sigma}) = g(\Sigma)$. By construction, Σ and $\hat{\Sigma}$ induce the same partition of the components of $\partial E(K)^{(c)} \setminus \{T_1, \ldots, T_c, \partial N(K)\}$. Theorem 1.8 holds in Subcase 1a.

Subcase 1b: $I = \{1, \ldots, n\}$ *and* $\partial N(K) \subset D(c_2)$. For this subcase, see Figure 7. Since Subcase 1b is similar to Subcase 1a we omit some of the easier details of the proof. As in Subcase 1a, F' decomposes $E(\#_{i=1}^n K_i)^{(c)}$ as $E(\#_{i=1}^n K_i)^{(c_1)} \cup_{F'} D(c_2)$ with $c_1 + c_2 = c + 1$; we reorder $T_1, \ldots, T_c$ so that $T_1, \ldots, T_{c_1} \subset \partial E(\#_{i=1}^n K_i)^{(c_1)}$ and $T_{c_1+1}, \ldots, T_c \subset \partial D(c_2)$. By induction there exists a minimal genus Heegaard

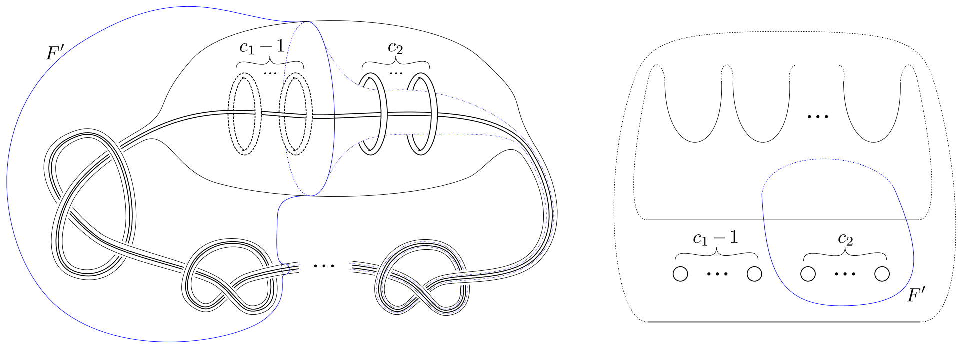

Figure 8. Case 2.

surface Σ_1 for $E(\#_{i=1}^n K_i)^{(c_1)}$ fulfilling conditions analogous to (1)–(3) listed in Subcase 1a. In particular, Σ_1 admits a complete system of c_1 Hopf– Haken annuli, say, $\mathcal{A}_1$, so that one boundary component of each annulus of $\mathcal{A}_1$ is a longitude of T_i ($i = 1, \dots, c_1$) and the other is a curve of F'. As in Subcase 1a, there exists a minimal genus Heegaard surface Σ_2 for $D(c_2)$ admitting a system of c_2 Haken annuli (recall Remark 4.2), denoted by $\mathcal{A}_2 \cup A$, so that $\mathcal{A}_2$ consists of $c_2 - 1$ annuli connecting meridians of $\partial N(\# K_i)$ to the longitudes of $T_{c_1+1}, \dots, T_c$, and A connects a meridian of $\partial N(\# K_i)$ to a curve of F'; by construction, this curve is parallel to the curves of $\mathcal{A}_1 \cap F'$. As shown in Proposition 4.3, the annuli obtained by attaching a parallel copy of A to each annulus of $\mathcal{A}_1$ union $\mathcal{A}_2$ are Haken annuli for the Heegaard surface obtained by amalgamating Σ_1 and Σ_2; we will denote this surface by $\hat{\Sigma}$. By construction, these annuli form a complete system of Hopf–Haken annuli for $\hat{\Sigma}$. As in Subcase 1a, $g(\hat{\Sigma}) = g(\Sigma)$ and $\hat{\Sigma}$ induces the same partition on the components of $\partial E(K)^{(c)} \setminus \{T_1, \dots, T_c, \partial N(K)\}$ as Σ. Theorem 1.8 holds in Subcase 1b.

Case 2: $\varnothing \neq I \neq \{1, \dots, n\}$. See Figure 8 for this case. Since Case 2 is similar to Subcase 1a we omit some of the easier details of the proof. By symmetry we may assume that $\partial N(K) \subset \partial E(\#_{i\in I} K_i)^{(c_1)}$. Let Σ_1' and Σ_2' be the Heegaard surfaces induced on $E(\#_{i\in I} K_i)^{(c_1)}$ and $E(\#_{i\notin I} K_i)^{(c_2)}$ (respectively) by Σ. Since both $|I|$ and $n - |I|$ are strictly less than n, we may apply induction to both $E(\#_{i\in I} K_i)^{(c_1)}$ and $E(\#_{i\notin I} K_i)^{(c_2)}$. By induction, there exist minimal genus Heegaard surfaces Σ_1 and Σ_2 for $E(\#_{i\in I} K_i)^{(c_1)}$ and $E(\#_{i\notin I} K_i)^{(c_2)}$ (respectively) fulfilling the following three conditions:

(1) $g(\Sigma_1) = g(\Sigma_1')$ and $g(\Sigma_2) = g(\Sigma_2')$.

(2) The partition of the components of $\partial E(\#_{i\in I} K_i)^{(c_1)} \setminus \{\partial N(K), T_1, \dots, T_{c_1-1}\}$ which Σ_1 induces is the same as that induced by Σ_1'. Similarly, Σ_2 induces the same partition of the components of $\partial E(\#_{i\notin I} K_i)^{(c_2)} \setminus \{T_{c_1}, \dots, T_{c_2}, F'\}$ as Σ_2'.

(3) Σ_1 admits a complete system of Hopf–Haken annuli, say, $A \cup \mathcal{A}_1$, where A connects $\partial N(K)$ to F' and the components of $\mathcal{A}_1$ connect $\partial N(K)$ to

$T_1, \dots, T_{c_1-1}$; similarly Σ_2 admits complete systems of Hopf–Haken annuli $\mathcal{A}_2$ whose components connect F' to $T_{c_1}, \dots, T_c$.

As shown in Proposition 4.3, the annuli obtained by attaching a parallel copy of A to each annulus of $\mathcal{A}_2$ union $\mathcal{A}_1$ are Haken annuli for the Heegaard surface obtained by amalgamating Σ_1 and Σ_2; we will denote this surface by $\hat{\Sigma}$. By construction, these annuli form a complete system of Hopf–Haken annuli for $\hat{\Sigma}$. As above $g(\hat{\Sigma}) = g(\Sigma)$ and $\hat{\Sigma}$ induces the same partition of the components of $\partial E(K)^{(c)} \setminus \{T_1, \dots, T_c, \partial N(K)\}$ as Σ. Theorem 1.8 holds in Case 2.

This completes the proof of Theorem 1.8. □

9. Weak reduction to swallow follow tori and calculating $g(E(K)^{(c)})$

Let $K_1 \subset M_1, \dots, K_n \subset M_n$ be knots in compact manifolds and $c > 0$ be an integer. When convenient, we will denote $\#_{i=1}^n K_i$ by K. Let $c_1, \dots, c_n \geq 0$ be integers such that $\sum_{i=1}^n c_i = c + n - 1$. By Proposition 6.3 there exist $n-1$ swallow follow tori $\mathcal{T} \subset E(K)^{(c)}$ that decompose it as $E(K)^{(c)} = \bigcup_{\mathcal{T}} E(K_i)^{(c_i)}$. By amalgamating minimal genus Heegaard surfaces for $E(K_i)^{(c_i)}$ we obtain a Heegaard surface for $E(K)^{(c)}$; however, it is distinctly possible that the surface obtained is not of minimal genus. This motivates the following definition:

Definition 9.1 (natural swallow follow tori). Let $K_1 \subset M_1, \dots, K_n \subset M_n$ be prime knots in compact manifolds and $c \geq 0$ an integer. Let $\mathcal{T} \subset E(\#_{i=1}^n K_i)^{(c)}$ be a collection of $n-1$ swallow follow tori giving the decomposition $E(\#_{i=1}^n K_i)^{(c)} = \bigcup_{\mathcal{T}} E(K_i)^{(c_i)}$, for some integers $c_i \geq 0$. We say that $\mathcal{T}$ is *natural* if it is obtained from a minimal genus Heegaard surface for $E(\#_{i=1}^n K_i)^{(c)}$ by iterated weak reduction; equivalently, $\mathcal{T}$ is called natural if

$$g(E(\#_{i=1}^n K_i)^{(c)}) = \sum_{i=1}^n g(E(K_i)^{(c_i)}) - (n-1).$$

Remark. As explained in Section 6, given *any* collection of $n-1$ swallow follow tori $\mathcal{T} \subset E(\#_{i=1}^n K_i)^{(c)}$ that give the decomposition $E(\#_{i=1}^n K_i)^{(c)} = \bigcup_{\mathcal{T}} E(K_i)^{(c_i)}$, the integers $c_1, \dots, c_n$ satisfy $\sum_{i=1}^n c_i = c+n-1$. We will often use this fact without reference; compare this to Proposition 6.3 where the converse was established.

Example 9.2 (knots with no natural swallow follow tori). In Theorem 9.4 below, we prove the existence of natural swallow follow tori under certain assumptions. The following example shows that a knot does not necessarily have swallow follow tori. We first analyze basic properties of knots that admit natural swallow follow tori: let $K_1, K_2 \subset S^3$ be prime knots and $T \subset E(K_1 \# K_2)$ be a natural swallow follow torus. By exchanging the subscripts if necessary we may assume that T decomposes $E(K_1 \# K_2)$ as $E(K_1)^{(1)} \cup_T E(K_2)$. By definition of naturality,

$$g(E(K_1 \# K_2)) = g(E(K_1)^{(1)}) + g(E(K_2)) - 1.$$

It is easy to see that $g(E(K_1)^{(1)}) \geq g(E(K_1))$. Combining these, we see that $g(E(K_1\#K_2)) \geq g(E(K_1))+g(E(K_2))-1$. Morimoto [1995] constructed examples of prime knots K_1, K_2 for which $g(E(K_1\#K_2)) = g(E(K_1)) + g(E(K_2)) - 2$. We conclude that for these knots, $E(K_1\#K_2)$ does not admit a natural swallow follow torus.

Example 9.3 (knots where only certain swallow follow tori are natural). This example is of a more subtle phenomenon. It shows that even when $E(K_1\#K_2)$ does admit a natural swallow follow torus, not every swallow follow torus is natural. In this sense, the weak reduction found in Theorem 9.4 is special as it finds natural swallow follow tori.

Let $K_{\mathrm{MSY}} \subset S^3$ be the knot constructed by Morimoto, Sakuma and Yokota [1996] and recall the notation $2K_{\mathrm{MSY}} = K_{\mathrm{MSY}}\#K_{\mathrm{MSY}}$. It was shown in [Morimoto et al. 1996] that $g(E(K_{\mathrm{MSY}})) = 2$ and $g(E(2K_{\mathrm{MSY}})) = 4$.

We claim that $g(E(K_{\mathrm{MSY}})^{(1)}) = 3$. By [Rieck 2000], $g(E(K_{\mathrm{MSY}})^{(1)}) = 2$ or 3. Assume for a contradiction that $g(E(K_{\mathrm{MSY}})^{(1)}) = 2$. By Corollary 6.4 (with $c = 0$, $c_1 = 1$, and $c_2 = 0$) we have

$$g(E(2K_{\mathrm{MSY}})) \leq g(E(K_{\mathrm{MSY}})^{(1)}) + g(E(K_{\mathrm{MSY}})) - 1 = 2+2-1 = 3,$$

a contradiction. Hence $g(E(K_{\mathrm{MSY}})^{(1)}) = 3$.

Let K be any nontrivial 2-bridge knot. It is well known that $g(E(K)) = 2$. We claim that $g(E(K_{\mathrm{MSY}}\#K)) = 3$. Since knots of tunnel number 1 are prime [Norwood 1982], $g(E(K_{\mathrm{MSY}}\#K)) \geq 3$. On the other hand, since K admits a $(1, 1)$ decomposition, by Theorem 5.4 we have that $g(E(K)^{(1)}) = 2$. As above, Corollary 6.4 gives

$$g(E(K_{\mathrm{MSY}}\#K)) \leq g(E(K_{\mathrm{MSY}})) + g(E(K)^{(1)}) - 1 = 2+2-1 = 3.$$

Hence $g(E(K_{\mathrm{MSY}}\#K)) = 3$.

$E(K_{\mathrm{MSY}}\#K)$ admits two swallow follow tori, say, T_1 and T_2, that decompose it as follows:

(1) $g(E(K_{\mathrm{MSY}}\#K)) = E(K_{\mathrm{MSY}})^{(1)} \cup_{T_1} E(K)$.

(2) $g(E(K_{\mathrm{MSY}}\#K)) = E(K_{\mathrm{MSY}}) \cup_{T_2} E(K)^{(1)}$.

In each case, amalgamating minimal genus Heegaard surfaces for the manifolds appearing on the right-hand side yields a Heegaard surface for $E(K_{\mathrm{MSY}}\#K)$ whose genus fulfills (Lemma 2.2):

(1) $g(E(K_{\mathrm{MSY}})^{(1)}) + g(E(K)) - g(T_1) = 3+2-1 = 4$.

(2) $g(E(K_{\mathrm{MSY}})) + g(E(K)^{(1)}) - g(T_2) = 2+2-1 = 3$.

We conclude that T_2 is a natural swallow follow torus but T_1 is not.

In this section we show that if K_i is m-small for all i, then any minimal genus Heegaard surface for $E(\#_{i=1}^n K_i)^{(c)}$ weakly reduces to a natural collection of swallow follow tori. The statement of Theorem 9.4 is more general and allows for nonminimal genus Heegaard surfaces.

Theorem 9.4. *Let $K_i \subset M_i$ be prime knots in compact manifolds so that $E(K_i)$ not homeomorphic to $T^2 \times I$, $E(K_i)$ is irreducible, and $\partial N(K_i)$ is incompressible in $E(K_i)$. Let Σ be a (not necessarily minimal genus) Heegaard surface for $E(\#_{i=1}^n K_i)^{(c)}$. Then one of the following holds:*

(1) *Σ admits iterated weak reductions that yield a collection of $n-1$ swallow follow tori, say, $\mathcal{T}$, giving the decomposition*

$$E(\#_{i=1}^n K_i)^{(c)} = \bigcup\nolimits_{\mathcal{T}} E(K_i)^{(c_i)},$$

where $c_1, \dots, c_n$ are integers such that $\sum_{i=1}^n c_i = c+n-1$.

(2) *For some i, K_i admits an essential meridional surface S with $\chi(S) \geq 6-2g(\Sigma)$.*

The main corollary of Theorem 9.4 allows us to calculate $g(E(\#_{i=1}^n K_i)^{(c)})$ in terms of $g(E(K_i)^{(c_i)})$.

Corollary 9.5. *In addition to the assumptions of Theorem 9.4, suppose that no K_i admits an essential meridional surface S with $\chi(S) \geq 6-2g(E(\#_{i=1}^n K_i)^{(c)})$. Then $E(\#_{i=1}^n K_i)^{(c)}$ admits a natural collection of $n-1$ swallow follow tori; equivalently, there exist integers $c_1, \dots, c_n \geq 0$ so that $\sum_{i=1}^n c_i = c+n-1$ and*

$$g(E(\#_{i=1}^n K_i)^{(c)}) = \sum_{i=1}^n g(E(K_i)^{(c_i)}) - (n-1).$$

Proof. Apply Theorem 9.4 to a minimal genus Heegaard splitting of $E(\#_{i=1}^n K_i)^{(c)}$ and apply Lemma 2.2. □

Corollary 9.6. *In addition to the assumptions of Theorem 9.4, suppose that no K_i admits an essential meridional surface S with $\chi(S) \geq 6-2g(E(\#_{i=1}^n K_i)^{(c)})$. Then*

$$g(E(\#_{i=1}^n K_i)^{(c)}) = \min\left\{\sum_{i=1}^n g(E(K_i)^{(c_i)}) - (n-1)\right\},$$

where the minimum is taken over all integers $c_1, \dots, c_n \geq 0$ with $\Sigma c_i = c+n-1$.

Proof. By Corollary 6.4, for any collection of integers $c_1, \dots, c_n$ such that $\sum_{i=1}^n c_i = c+n-1$ we have that

$$g(E(\#_{i=1}^n K_i)^{(c)}) \leq \sum_{i=1}^n g(E(K_i)^{(c_i)}) - (n-1)$$

and by Corollary 9.5, there exist integers $c_1, \dots, c_n$ for which equality holds. The corollary follows. □

Proof of Theorem 9.4. We induct on (n, c) ordered lexicographically. Recall that in the beginning of the proof of Theorem 1.8 we showed that (n, c) is well defined. If $n = 1$ there is nothing to prove; assume from now on $n > 1$.

Assume Theorem 9.4(2) does not hold, that is, for each i, $E(K_i)$ does not admit an essential meridional surface S with $\chi(S) \geq 6 - 2g(\Sigma)$. Then by the swallow follow torus theorem [Kobayashi and Rieck 2006a, Theorem 4.1] Σ weakly reduces to a swallow follow torus, say, T. T decomposes $E(\#_{i=1}^n K_i)^{(c)}$ as $E(K_I)^{(c_I)} \cup_T E(K_J)^{(c_J)}$, where $I \subseteq \{1, \dots, n\}$ (possibly empty), where c_I and c_J are nonnegative integers whose sum is $c+1$, $K_I = \#_{i \in I} K_i$, and $K_J = \#_{i \notin I} K_i$. Denote the Heegaard surfaces induced on $E(K_I)^{(c_I)}$ and $E(K_J)^{(c_J)}$ by Σ_I and Σ_J, respectively.

Case One: $\emptyset \neq I \neq \{1, \dots, n\}$. In this case both $E(K_I)^{(c_I)}$ and $E(K_J)^{(c_J)}$ are exteriors of knots with strictly less than n prime factors and hence we may apply induction to both. Since $g(\Sigma_I) < g(\Sigma)$, Theorem 9.4(2) does not hold for $E(K_I)^{(c_I)}$. Hence, by induction, Σ_I admits iterated weak reduction that yields a collection of $|I| - 1$ swallow follow tori (say, $\mathcal{T}_I \subset E(K_I)^{(c_I)}$) so that the following conditions hold:

(1) $\mathcal{T}_I$ decompose $E(K_I)^{(c_I)}$ as $\bigcup_{\mathcal{T}_I} E(K_i)^{(c_i)}$ (for $i \in I$).

(2) $\sum_{i \in I} c_i = c_I + |I| - 1$.

Similarly, Σ_J admits iterated weak reduction that yields a collection of $(n - |I|) - 1$ swallow follow tori (say, $\mathcal{T}_J \subset E(K_J)^{(c_J)}$) so that the following conditions hold:

(1) $\mathcal{T}_J$ decompose $E(K_J)^{(c_J)}$ as $\bigcup_{\mathcal{T}_J} E_i(K_i)^{(c_i)}$ (for $i \notin I$).

(2) $\sum_{i \notin I} c_i = c_J + (n - |I|) - 1$.

Thus, after iterated weak reduction of Σ we obtain $\mathcal{T} = T \cup \mathcal{T}_I \cup \mathcal{T}_J$. By the above, $\mathcal{T}$ decomposes $E(\#_{i=1}^n K_i)^{(c)}$ as $\bigcup_{\mathcal{T}} E(K_i)^{(c_i)}$, so that (recalling that $c_I + c_J = c + 1$)

$$\begin{aligned}\sum_{i=1}^n c_i &= \sum_{i \in I} c_i + \sum_{i \notin I} c_i \\ &= c_I + |I| - 1 + c_J + (n - |I|) - 1 = c + n - 1.\end{aligned}$$

This proves Theorem 9.4 in Case One.

Case Two: $I = \emptyset$ *or* $I = \{1, \dots, n\}$. By symmetry we may assume that $I = \{1, \dots, n\}$. In that case, $E(K_J)^{(c_J)} \cong D(c_J)$, (where $D(c_J)$ is a disk with c_J holes cross S^1), and T gives the decomposition:

$$E(\#_{i=1}^n K_i)^{(c)} = E(\#_{i=1}^n K_i)^{(c_I)} \cup_T D(c_J).$$

Since T is essential (and in particular, not boundary parallel), $c_J \geq 2$. Since $c_I + c_J = c + 1$, we have that $c_I < c$. Thus the complexity of $E(\#_{i=1}^n K_i)^{(c_I)}$ is $(n, c_I) < (n, c)$ and we may apply induction to $E(\#_{i=1}^n K_i)^{(c_I)}$. Let Σ_I be the

Heegaard surface for $E(\#_{i=1}^n K_i)^{(c_I)}$ induced by weak reduction. By induction, Σ_I admits a repeated weak reduction that yields a system of $n-1$ swallow follow tori, say, $\mathcal{T}_I$, that decomposes $E(\#_{i=1}^n K_i)^{(c_I)}$ as

$$E(\#_{i=1}^n K_i)^{(c_I)} = \bigcup_{\mathcal{T}_I} E(K_i)^{(c_i)}$$

with $\sum_{i=1}^n c_i = c_I + n - 1$. Let T' be a component of $\mathcal{T}_I$. Then T' decomposes $E(\#_{i=1}^n K_i)^{(c_I)}$ as

$$E(\#_{i=1}^n K_i)^{(c_I)} = E(\#_{i\in I'} K_i)^{(b_1)} \cup_{T'} E(\#_{i\notin I'} K_i)^{(b_2)},$$

for some $I' \subseteq \{1, \ldots, n\}$ and some integers $b_1, b_2 \geq 0$ with $b_1 + b_2 = c_I + 1$. Since $T' \subset \mathcal{T}_I$, we have that $\varnothing \neq I' \neq \{1, \ldots, n\}$. By Proposition 2.3, we see that Σ weakly reduces to T'. This reduces Case Two to Case One, completing the proof of Theorem 9.4. □

10. Calculating the growth rate of m-small knots

In this final section we complete the proof of Theorem 1.2. Let $K \subset M$ be an m-small admissible knot in a compact manifold. Recall the notation nK and $E(K)^{(c)}$.

The difference between $g(E(K)^{(c)})$ and $g(E(K)) + c$ is measured by a function denoted f_K that plays a key role our work:

Definition 10.1. Given a knot K, we define the function $f_K : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ to be

$$f_K(c) = g(E(K)) + c - g(E(K)^{(c)}).$$

We immediately see that f_K has the following properties, which we will often use without reference:

(1) $f_K(0) = 0$.

(2) For $c \geq 0$, $f_K(c) \leq f_K(c+1) \leq f_K(c) + 1$: this follows from the fact (proved in [Rieck 2000]) that for all $c \geq 0$,

$$g(E(K)^{(c)}) \leq g(E(K)^{(c+1)}) \leq g(E(K)^{(c)}) + 1.$$

(3) For $c \geq 0$, $0 \leq f_K(c) \leq c$ (this follows easily from (2)).

Before proceeding, we rephrase Corollaries 9.5 and 9.6 in terms of f_K:

Corollary 10.2. *Let $K \subset M$ be a knot in a compact manifold and let n be a positive integer. Suppose that $E(K)$ does not admit a meridional essential surface S with $\chi(S) \geq 6 - 2g(E(nK))$. Then there exist integers $c_1, \ldots, c_n \geq 0$ with $\Sigma c_i = n - 1$ so that*:

$$g(E(nK)) = ng(E(K)) - \sum_{i=1}^n f_K(c_i).$$

Proof. By Corollary 9.5 (with $c = 0$) there exist $c_1, \dots, c_n \geq 0$ with $\Sigma c_i = n - 1$, so that $g(E(nK)) = \sum_{i=1}^{n} g(E(K)^{(c_i)}) - (n-1)$. We get:

$$\begin{aligned} g(E(nK)) &= \Big[\sum_{i=1}^{n} g(E(K)^{(c_i)})\Big] - (n-1) \\ &= \Big[\sum_{i=1}^{n} g(E(K)) + c_i - f_K(c_i)\Big] - (n-1) \\ &= ng(E(K)) + \Big[\sum_{i=1}^{n} c_i\Big] - \Big[\sum_{i=1}^{n} f_K(c_i)\Big] - (n-1) \\ &= ng(E(K)) + (n-1) - \Big[\sum_{i=1}^{n} f_K(c_i)\Big] - (n-1) \\ &= ng(E(K)) - \sum_{i=1}^{n} f_K(c_i). \end{aligned}$$

□

A similar argument shows that Corollary 9.6 gives:

Corollary 10.3. *Let $K \subset M$ be a knot in a compact manifold and let n be a positive integer. Suppose that $E(K)$ does not admit a meridional essential surface S with $\chi(S) \geq 6 - 2g(E(nK))$. Then,*

$$\begin{aligned} g(E(nK)) &= \min\Big\{ng(E(K)) - \sum\nolimits_{i=1}^{n} f_K(c_i)\Big\} \\ &= ng(E(K)) - \max\Big\{\sum\nolimits_{i=1}^{n} f_K(c_i)\Big\}, \end{aligned}$$

where the minimum and maximum are taken over all integers $c_1, \dots, c_n \geq 0$ with $\sum_{i=1}^{n} c_i = n - 1$.

Recall (Notation 1.1) that we denote $g(E(K)) - g(M)$ by g and the bridge indices of K with respect to Heegaard surfaces of genus $g(E(K)) - i$ by b_i^* ($i = 1, \dots, g$), so that $0 < b_1^* < \cdots < b_i^* < \cdots < b_g^*$. We formally set $b_0^* = 0$ and $b_{g+1}^* = \infty$. Note that these properties imply that for every $c \geq 0$ there is a unique index i ($0 \leq i \leq g$), depending on c, so that $b_i^* \leq c < b_{i+1}^*$; we will use this fact below without reference.

In the following proposition we calculate $f_K(c)$ when $E(K)$ does not admit an essential meridional surface S with $\chi(S) \geq 6 - 2g(E(K)^{(c)})$.

Proposition 10.4. *Let K be a knot and $c \geq 0$ be an integer. Let $0 \leq i \leq g$ be the unique index for which $b_i^* \leq c < b_{i+1}^*$. Then $f_K(c) \geq i$. If, in addition, $E(K)$ does not admit an essential meridional surface S with $\chi(S) \geq 6 - 2g(E(K)^{(c)})$ then equality holds*:

$$f_K(c) = i.$$

Proof of Proposition 10.4. We first prove that $f_K(c) \geq i$ holds for any knot. Since f_K is a nonnegative function we may assume $i \geq 1$. By the definition of b_i^*, K admits a $(g(E(K)) - i,\, b_i^*)$ decomposition. Since $c \geq b_i^*$, K admits a $(g(E(K)) - i, c)$ decomposition. By Corollary 5.6 we have that $g(E(K)^{(c)}) \leq g(E(K)) - i + c$. Therefore,

$$f_K(c) = g(E(K)) + c - g(E(K)^{(c)}) \geq g(E(K)) + c - (g(E(K)) - i + c) = i.$$

Next we assume, in addition, that $E(K)$ does not admit an essential meridional surface S with $\chi(S) \geq 6 - 2g(E(K)^{(c)})$. We will complete the proof of the proposition by showing that $f_K(c) < i + 1$; suppose for a contradiction that $f_K(c) \geq i + 1$. Thus $g(E(K)^{(c)}) = g(E(K)) + c - f_K(c) \leq g(E(K)) + c - (i + 1)$.

Assume first that $i = g$. Then by Corollary 8.1 (with $g(E(K))+c-(g+1)$ corresponding to h) we see that k admits a $(g(E(K))+c-(g+1)-c,\, c)$ decomposition. In particular, M admits a Heegaard surface of genus $(g(E(K))) + c - (g + 1) - c$. Hence we see:

$$\begin{aligned} g(M) &\leq (g(E(K)) + c - (g + 1) - c \\ &= g(E(K)) - g - 1 \\ &= g(E(K)) - (g(E(K)) - g(M)) - 1 \\ &= g(M) - 1. \end{aligned}$$

This contradiction completes the proof when $i = g$.

Next assume $0 \leq i < g$. Applying Corollary 8.1 again (with $g(E(K))+c-(i+1)$ corresponding to h in Corollary 8.1) we see that K admits a $(g(E(K)) - (i + 1), c)$ decomposition. By definition, b_{i+1}^* is the smallest integer such that K admits a $(g(E(K)) - (i + 1), b_{i+1}^*)$ decomposition; hence $c \geq b_{i+1}^*$. This contradicts our choice of i in the statement of the proposition, showing that $f_K(c) < i + 1$. This completes the proof of Proposition 10.4. □

As an illustration of Proposition 10.4, let K be an m-small knot in S^3. Suppose that $g = 3$, $b_1^* = 5$, $b_2^* = 7$, and $b_3^* = 23$. (We do not know if a knot with these properties exists.) Then

$$f_K(c) = \begin{cases} 0 & \text{if } 0 \leq c \leq 4, \\ 1 & \text{if } 5 \leq c \leq 6, \\ 2 & \text{if } 7 \leq c \leq 22, \\ 3 & \text{if } 23 \leq c. \end{cases}$$

Not much is known about f_K for knots that are not m-small.

Question 10.5. Does there exist a knot K in a manifold M with unbounded f_K? Does there exist a knot K with $f_K(c) > g(E(K)) - g(M)$ (for sufficiently large c)? What can be said about the behavior of the function f_K?

With the preparation complete, we are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Fix the notation of Theorem 1.2. Since the upper bound was obtained in Proposition 7.1, we assume from now on that K is m-small. By Corollary 10.3, $g(E(nK)) = ng(E(K)) - \max\{\sum_{i=1}^{n} f_K(c_i)\}$, where the maximum is taken over all integers $c_1, \dots, c_n \geq 0$ with $\sum_{i=1}^{n} c_i = n-1$.

Fix n and let $c_1, \dots, c_n \geq 0$ be integers with $\sum_{i=1}^{n} c_i = n-1$ that maximize $\sum_{i=1}^{n} f_K(c_i)$.

Lemma 10.6. *We may assume that the sequence $c_1, \dots, c_n$ fulfills the following conditions for some $1 \leq l \leq n$:*

(1) $c_i \geq c_{i+1}$ $(i = 1, \dots, n-1)$.

(2) *For $i \leq l$, $c_i \in \{b_1^*, \dots, b_g^*\}$.*

(3) $c_{l+1} < b_1^*$.

(4) *For $i > l+1$, $c_i = 0$.*

Proof. By reordering the indices if necessary we may assume (1) holds.

Let l be the largest index for which $f_K(c_l) \neq 0$. For $i = 1, \dots, l$, let $0 \leq j(i) \leq g$ be the unique index for which $b_{j(i)}^* \leq c_i < b_{j(i)+1}^*$ (recall that we set $b_0^* = 0$ and $b_{g+1}^* = \infty$). Define $c'_1, \dots, c'_n$ as follows:

(1) For $i \leq l$, set $c'_i = b_{j(i)}^*$ (i.e., c'_i is the largest b_j^* that does not exceed c_i).

(2) Set $c'_{l+1} = n - 1 - \left(\sum_{i=1}^{l} c'_i\right)$.

(3) For $i > l+1$, set $c'_i = 0$.

By Proposition 10.4, for $i \leq l$, $f_K(c_i) = f_K(b_{j(i)}^*) = f_K(c'_i)$. We get:

$$\begin{aligned}\sum_{i=1}^{n} f_K(c'_i) &= \sum_{i=1}^{l} f_K(c'_i) + \sum_{i=l+1}^{n} f_K(c'_i) \\ &= \sum_{i=1}^{l} f_K(c_i) + \sum_{i=l+1}^{n} f_K(c'_i) \\ &\geq \sum_{i=1}^{l} f_K(c_i) = \sum_{i=1}^{n} f_K(c_i).\end{aligned}$$

(For the last equality, recall that $f_K(c_i) = 0$ for $i > l$.)

Since $c_1, \dots, c_n$ maximizes $\sum_{i=1}^{n} f_K(c_i)$, we conclude that

$$\sum_{i=1}^{n} f_K(c_i) = \sum_{i=1}^{n} f_K(c'_i)$$

and hence $f_K(c'_{l+1}) = 0$; so $c'_{l+1} < b_1^*$. Thus $c'_1, \dots, c'_n$ is a maximizing sequence; it is easy to see that it fulfills conditions (1)–(4). □

We will denote the n-th term of the defining sequence of the growth rate by S_n:

$$S_n = \frac{g(E(nK)) - ng(E(K)) + n - 1}{n-1}.$$

By Corollary 10.3

$$S_n = 1 - \frac{\max\left\{\sum_{i=1}^{n} f_K(c_i)\right\}}{n-1}. \tag{2}$$

In order to bound S_n below we need to understand the following optimization problem, where here we are assuming that the maximizing sequence fulfills the conditions listed in Lemma 10.6, and in particular, $f_K(c_i) = 0$ for $i > l$.

Problem 10.7. Find nonnegative integers l and $c_1,\ldots,c_l$ that maximize $\sum_{i=1}^{l} f_K(c_i)$ subject to the constraints

(1) $\sum_{i=1}^{l} c_i \leq n-1$,
(2) $c_i \in \{b_1^*, \ldots, b_g^*\}$ (for $1 \leq i \leq l$).

For $i = 1, \ldots, g$, let k_i be the number of times that b_i^* appears in $c_1, \ldots, c_l$. By Proposition 10.4, $f_K(b_i^*) = i$; thus Problem 10.7 can be rephrased as follows:

Problem 10.8. Maximize $\sum_{i=1}^{g} k_i i$ subject to the constraints

(1) $\sum_{i=1}^{g} k_i b_i^* \leq n-1$,
(2) k_i is a nonnegative integer.

We first solve this optimization problem over $\mathbb{R}$; we use the variables $x_1, \ldots, x_g$ instead of $k_1, \ldots, k_g$.

Problem 10.9. Given $n \in \mathbb{R}$, $n > 1$, maximize $\sum_{i=1}^{g} x_i i$ subject to the constraints

(1) $\sum_{i=1}^{g} x_i b_i^* \leq n-1$,
(2) $x_1 \geq 0, \ldots, x_g \geq 0$.

It is easy to see that for any sequence $x_1, \ldots, x_g$ that realizes the maximum we have that $\sum_{i=1}^{g} x_i b_i^* = n-1$, for otherwise we can increase the value of x_1, thus increasing $\sum_{i=1}^{g} x_i i$ and contradicting maximality. Problem 10.9 is an elementary linear programming problem (known as the standard maximum problem) and is solved using the simplex method which gives:

Lemma 10.10. *There is a (not necessarily unique) index i_0, which is independent of n, such that a solution of Problem 10.9 is given by*

$$x_{i_0} = \frac{n-1}{b_{i_0}^*}, \qquad x_i = 0\,(i \neq i_0).$$

Hence the maximum is

$$\frac{(n-1)i_0}{b_{i_0}^*}.$$

Proof of Lemma 10.10. The notation used in this proof was chosen to be consistent with notation often used in linear programming texts. Let $\vec{N}$, $\vec{F}$ and $\vec{x} \in \mathbb{R}^g$ denote the vectors

$$\vec{N} = (b_1^*, \dots, b_g^*), \quad \vec{F} = (1, \dots, g), \quad \text{and} \quad \vec{x} = (x_1, \dots, x_g).$$

For $n \in \mathbb{R}$, $n > 1$, let Δ_n be

$$\Delta_n = \left\{ \vec{x} \in \mathbb{R}^g \mid \vec{N} \cdot \vec{x} = n - 1, x_1 \geq 0, \dots, x_g \geq 0 \right\}.$$

Note that Δ_n is a simplex and its codimension k faces are obtained by setting k variables to zero. Problem 10.9 can be stated as:

$$\text{maximize } \vec{F} \cdot \vec{x}, \text{ subject to } \vec{x} \in \Delta_n.$$

Since the gradient of $\vec{F} \cdot \vec{x}$ is $\vec{F}$ and the normal to Δ_n is $\vec{N}$, the gradient of the restriction of $\vec{F} \cdot \vec{x}$ to Δ_n is the projection

$$\vec{P} = \vec{F} - \frac{\vec{F} \cdot \vec{N}}{|\vec{N}|^2} \vec{N}.$$

Note that $\vec{P}$ is independent of n. The maximum of $\vec{N} \cdot \vec{x}$ on Δ_n is found by moving along Δ_n in the direction of $\vec{P}$. This shows that the maximum is obtained along a face defined by setting some of the variables to zero, and the variables set to zero are independent of n. Lemma 10.10 follows by picking i_0 to be one of the variables not set to zero. □

Fix an index i_0 as in Lemma 10.10. If $b_{i_0}^* \mid n - 1$ then the maximum (over $\mathbb{R}$) found in Lemma 10.10 is in fact an integer and hence is also the maximum for Problem 10.7. This allows us to calculate S_n in this case:

Lemma 10.11. *If $b_{i_0}^* \mid n - 1$ then $S_n = 1 - i_0 / b_{i_0}^*$.*

Proof.
$$S_n = 1 - \frac{\max\left\{\sum_{i=1}^n f_K(c_i)\right\}}{n-1} = 1 - \frac{(n-1) i_0}{(n-1) b_{i_0}^*} = 1 - \frac{i_0}{b_{i_0}^*}.$$ □

We now turn our attention to the general case, where $b_{i_0}^*$ may not divide $n - 1$. We will only consider values of n for which $n > b_{i_0}^*$. As in Section 7, let k_{i_0} and r be the quotient and remainder when dividing $n - 1$ by $b_{i_0}^*$, so that

$$n - 1 = k_{i_0} b_{i_0}^* + r, \quad 0 \leq r < b_{i_0}^*. \tag{3}$$

Let $c_j \geq 0$ $(1 \leq j \leq n)$ be integers with $\sum_{j=1}^n c_j = n-1$ that maximize $\sum_{j=1}^n f_K(c_j)$. We denote $n - r$ by n'. Let $c_j' \geq 0$ $(1 \leq j \leq n')$ be integers with $\sum_{j=1}^{n'} c_j' = n' - 1$ that maximize $\sum_{j=1}^{n'} f_K(c_j')$.

Claim 10.12. $\sum_{j=1}^n f_K(c_j) \leq \sum_{j=1}^{n'} f_K(c_j') + r$.

Proof. Starting with the sequence $c_1, \ldots, c_n$, we obtain a new sequence by subtracting 1 from exactly one c_j (with $c_j > 0$). Let c'''_j be a sequence of nonnegative integers obtained by repeating this process r times. Then $\sum_{j=1}^{n} c'''_j = n-1-r = n'-1$. Let c''_j be the sequence obtained from c'''_j by removing r zeros (note that this is possible as there indeed are at least r zeros). We get

$$\begin{aligned}
\sum_{j=1}^{n'} f_K(c'_j) + r &\geq \sum_{j=1}^{n'} f_K(c''_j) + r \quad &&(\text{since } c'_j \text{ maximizes}) \\
&= \sum_{j=1}^{n} f_K(c'''_j) + r &&(\text{since } f_K(0) = 0) \\
&\geq \sum_{j=1}^{n} f_K(c_j) &&(\text{since } f_K(c) + 1 \geq f_K(c+1)). \qquad \square
\end{aligned}$$

Note that $b^*_{i_0} | n'-1$ and so we may apply Lemma 10.11 to calculate $S_{n'}$. Using Equation (2) from page 97 for the first line, we get:

$$\begin{aligned}
S_n &= 1 - \frac{\max\{\sum_{i=1}^{n} f(c_i)\}}{n-1} &&(\text{Equation (2) for } S_n) \\
&\geq 1 - \frac{\max\{\sum_{j=1}^{n'} f(c'_j) + r\}}{n-1} &&(\text{Claim 10.12}) \\
&= 1 - \frac{n'-1}{n-1} \frac{\max\{\sum_{j=1}^{n'} f(c'_j)\}}{n'-1} - \frac{r}{n-1} \\
&= \frac{n'-1}{n-1}\left(1 - \frac{\max\{\sum_{j=1}^{n'} f(c'_j)\}}{n'-1}\right) + \left(1 - \frac{n'-1}{n-1}\right) - \frac{r}{n-1} \\
&= \frac{n'-1}{n-1} S_{n'} + \left(1 - \frac{n'-1}{n-1}\right) - \frac{r}{n-1} &&(\text{Equation (2) for } S_{n'}) \\
&= \frac{n'-1}{n-1}\left(1 - \frac{i_0}{b^*_{i_0}}\right) + \left(1 - \frac{n'-1}{n-1}\right) - \frac{r}{n-1} &&(\text{Lemma 10.11}) \\
&= \frac{n'-1}{n-1}\left(1 - \frac{i_0}{b^*_{i_0}}\right) + \left(1 - \frac{n'+r-1}{n-1}\right) \\
&= \frac{n-r-1}{n-1}\left(1 - \frac{i_0}{b^*_{i_0}}\right) &&(\text{substituting } n' = n-r).
\end{aligned}$$

Recall that in the proof of Proposition 7.1 we proved Equation (1) (see page 78) which says (recalling that k_{i_0} was defined in Equation (3) above):

$$S_n < 1 - \frac{i_0}{b^*_{i_0}} \frac{k_{i_0}}{k_{i_0}+1}.$$

Combining these facts we obtain

$$\frac{n-r-1}{n-1}\left(1 - \frac{i_0}{b^*_{i_0}}\right) \le S_n < 1 - \frac{i_0}{b^*_{i_0}} \frac{k_{i_0}}{k_{i_0}+1}.$$

By Equation (3) above, $r < b^*_{i_0}$ and $\lim_{n\to\infty} k_{i_0} = \infty$. We conclude that as $n \to \infty$ both bounds limit on $1 - i_0/b^*_{i_0}$, and thus $\lim_{n\to\infty} S_n$ exists and equals $1 - i_0/b^*_{i_0}$.

This completes the proof of Theorem 1.2. □

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