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In this paper, we prove a sum set estimate and the exact sum set theorem for unimodular Kac algebras. Combining the characterization of minimizers of the Donoso–Stark uncertainty principle and the Hirschman–Beckner uncertainty principle, we characterize the extremal pairs of Young's inequality and extremal operators of the Hausdorff–Young inequality for unimodular Kac algebras.

1. Introduction

Young's inequality for the real line $\mathbb{R}$ was first studied by Young [1912]. Beckner [1975] characterized the extremal pairs of Young's inequality for $\mathbb{R}$ with the sharp constant and consequently characterized the extremal functions of the Hausdorff–Young inequality. For general cases, Fournier [1977] characterized the extremal pairs of Young's inequality and the extremal functions of the Hausdorff–Young inequality for unimodular locally compact groups. (Note that Russo [1974] characterized the extremal functions of the Hausdorff–Young inequality directly.) For a long time, Young's inequality was showed for commutative algebras. Recently, S. Wang and the authors [Liu et al. 2017] proved Young's inequality for locally compact quantum groups. Bobkov, Madiman and Wang [Bobkov et al. 2011] conjectured that a fractional generalization of Young's inequality for $\mathbb{R}$ is true.

Kac algebras were introduced independently by L. I. Vainerman and G. I. Kac [Vaĭnerman 1974; Vaĭnerman and Kac 1973] and by Enock and Nest [Enock and Schwartz 1973; 1974; 1975]. These algebras generalized locally compact groups and their duals. Locally compact quantum groups introduced by J. Kustermans and S. Vaes [Kustermans and Vaes 2000; 2003] generalized Kac algebras. It is natural to ask what extremal pairs of Young's inequality for locally compact quantum groups are. Unfortunately, the methods to characterize extremal pairs of Young's inequality for locally compact groups [Fournier 1977] can not be applied to locally compact quantum groups. We plan to characterize the extremal pairs of Young's inequality for locally compact quantum groups. Our first aim in this direction is

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to characterize extremal pairs of Young's inequality for unimodular Kac algebras. Our proof for noncommutative algebras is quite different from the classical proof for commutative algebras.

In this paper, we will characterize extremal pairs of Young's inequality and extremal operators of the Hausdorff–Young inequality for unimodular Kac algebras. We show that extremal pairs and extremal operators are exactly bishifts of biprojections introduced in [Liu and Wu 2017] and we will use the notations therein. Prior to the characterization, we prove a sum set theorem for unimodular Kac algebras.

Main Theorem (sum set theorem[1], Theorem 3.1, Theorem 3.9). *Suppose $\mathbb{G}$ is a unimodular Kac algebra with a Haar tracial weight φ. Let p, q be projections in $L^\infty(\mathbb{G})$. Then*

$$\max\{\varphi(p), \varphi(q)\} \leq \mathcal{S}(p * q),$$

where $\mathcal{S}(x) = \varphi(\mathcal{R}(x))$ and $\mathcal{R}(x)$ is the range projection of x, $x \in L^\infty(\mathbb{G})$. Moreover the following are equivalent:

(1) $\mathcal{S}(p * q) = \varphi(p) < \infty$;

(2) $\varphi(q)^{-1} p * q$ *is a projection in* $L^1(\mathbb{G})$;

(3) $\mathcal{S}(p * (q * R(q)^{*(m)}) * q^{*(j)}) = \varphi(p)$ *for some* $m \geq 0$, $j \in \{0, 1\}$, $m + j > 0$, $q^{*(0)}$ *means q vanishes*;

(4) *there exists a biprojection B such that q is a right subshift of B and* $p = \mathcal{R}(x * B)$ *for some* $x > 0$.

Combining the results above and the characterization of minimizers of the Hirschman–Beckner and the Donoho–Stark uncertainty principles for unimodular Kac algebras, we are able to characterize the extremal pairs of Young's inequality.

Theorem 4.8. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let x, y be φ-measurable. Then the following are equivalent*:

(1) $\|x * y\|_r = \|x\|_t \|y\|_s$ *for some* $1 < r, t, s < \infty$ *such that* $1/r + 1 = 1/t + 1/s$;

(2) $\|x * y\|_r = \|x\|_t \|y\|_s$ *for any* $1 \leq r, t, s \leq \infty$ *such that* $1/r + 1 = 1/t + 1/s$;

(3) *there exists a biprojection B such that* $x = ({}_hBa_x) * \mathcal{F}(\tilde{B}_g)$ *and* $y = \mathcal{F}(\tilde{B}_g) * (B_f a_y)$, *where $\tilde{B}$ is the range projection of $\mathcal{F}(B)$; B_g, B_f are right shifts of B; ${}_hB$ is left shift of B, and a_x, a_y are elements such that x, y are nonzero.*

Furthermore, we characterize the extremal operators of the Hausdorff–Young inequality for unimodular Kac algebra.

Theorem 5.2. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let x be φ-measurable. Then the following are equivalent*:

[1] We refer the reader to [Tao and Vu 2006] for a classical sum set theorem

(1) $\|\mathcal{F}(x)\|_{t/(t-1)} = \|x\|_t$ *for some* $1 < t < 2$;

(2) $\|\mathcal{F}(x)\|_{t/(t-1)} = \|x\|_t$ *for any* $1 \le t \le 2$;

(3) x *is a bishift of a biprojection.*

This paper is organized as follows. In Section 2, we recall some basic notations and properties of unimodular Kac algebras. In Section 3, we prove the sum set estimate and the exact inverse sum set theorem for unimodular Kac algebras. In Section 4, we characterize extremal pairs of Young's inequality for unimodular Kac algebras. In Section 5, we characterize extremal operators of the Hausdorff–Young inequality for unimodular Kac algebras.

2. Preliminaries

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a normal semifinite faithful tracial weight φ.

A closed densely defined operator x affiliated with $\mathcal{M}$ is called φ-measurable if for all $\epsilon > 0$ there exists a projection $p \in \mathcal{M}$ such that $p\mathcal{H} \subset \mathfrak{D}(x)$, and $\varphi(1-p) \le \epsilon$, where $\mathfrak{D}(x)$ is the domain of x. Denote by $\widetilde{\mathcal{M}}$ the set of φ-measurable closed densely defined operators. Then $\widetilde{\mathcal{M}}$ is $*$-algebra with respect to strong sum, strong product, and adjoint operation. If x is a positive self-adjoint φ-measurable operator, then $x^\alpha \log x$ is φ-measurable for any $\alpha \in \mathbb{C}$ with $\Re\alpha > 0$, where $\Re\alpha$ is the real part of α.

For any positive self-adjoint operator x affiliated with $\mathcal{M}$, we put

$$\varphi(x) = \sup_{n\in\mathbb{N}} \varphi\left(\int_0^n t \, \mathrm{d}e_t\right),$$

where $x = \int_0^\infty t \, \mathrm{d}e_t$ is the spectral decomposition of x. Then for $t \in [1, \infty)$, the noncommutative L^t space $L^t(\mathcal{M})$ with respect to φ is given by

$$L^t(\mathcal{M}) = \{x \text{ densely defined, closed, affiliated with } \mathcal{M} \mid \varphi(|x|^t) < \infty\}.$$

The t-norm $\|x\|_t$ of x in $L^t(\mathcal{M})$ is given by $\|x\|_t = \varphi(|x|^t)^{1/t}$. We have that $L^p(\mathcal{M}) \subseteq \widetilde{\mathcal{M}}$. For more details on noncommutative L^p space we refer to [Terp 1981; 1982].

Now let us recall the definition of locally compact quantum groups in [Kustermans and Vaes 2000].

Let $\mathcal{M}$ be a von Neumann algebra with a normal semifinite faithful weight φ. Then $\mathfrak{N}_\varphi = \{x \in \mathcal{M} \mid \varphi(x^*x) < \infty\}$, $\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^*\mathfrak{N}_\varphi$, $\mathfrak{M}_\varphi^+ = \{x \ge 0 \mid x \in \mathfrak{M}_\varphi\}$. Denote by $\mathcal{H}_\varphi$ the Hilbert space by taking the closure of $\mathfrak{N}_\varphi$. The map $\Lambda_\varphi : \mathfrak{N}_\varphi \mapsto \mathcal{H}_\varphi$ is the inclusion map.

A locally compact quantum group $\mathbb{G} = (\mathcal{M}, \Delta, \varphi, \psi)$ consists of

(1) a von Neumann algebra $\mathcal{M}$,

(2) a normal, unital, *-homomorphism $\Delta : \mathcal{M} \to \mathcal{M} \overline{\otimes} \mathcal{M}$ such that $(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta$,

(3) a normal, semifinite, faithful weight φ such that $(\iota \otimes \varphi)\Delta(x) = \varphi(x)1$, $\forall x \in \mathfrak{M}_\varphi^+$; a normal, semifinite, faithful weight ψ such that $(\psi \otimes \iota)\Delta(x) = \psi(x)1$, $\forall x \in \mathfrak{M}_\psi^+$,

where $\overline{\otimes}$ denotes the von Neumann algebra tensor product and ι denotes the identity map. The normal, unital, *-homomorphism Δ is a comultiplication of $\mathcal{M}$, φ is the left Haar weight, and ψ is the right Haar weight.

We assume that $\mathcal{M}$ acts on $\mathcal{H}_\varphi$. There exists a unique unitary operator $W \in \mathcal{B}(\mathcal{H}_\varphi \otimes \mathcal{H}_\varphi)$ which is known as the multiplicative unitary defined by

$$W^*(\Lambda_\varphi(a) \otimes \Lambda_\varphi(b)) = (\Lambda_\varphi \otimes \Lambda_\varphi)(\Delta(b)(a \otimes 1)), \quad a, b \in \mathfrak{N}_\varphi.$$

Moreover for any $x \in \mathcal{M}$, $\Delta(x) = W^*(1 \otimes x)W$.

For the locally compact quantum group $\mathbb{G}$, there exists an antipode S, a scaling automorphism group τ, and a unitary antipode R and there also exists a dual locally compact quantum group $\hat{\mathbb{G}} = (\hat{\mathcal{M}}, \hat{\Delta}, \hat{\varphi}, \hat{\psi})$ of $\mathbb{G}$. The antipode, the scaling group, and the unitary antipode of $\hat{\mathbb{G}}$ are denoted by $\hat{S}$, $\hat{\tau}$, and $\hat{R}$ respectively. We refer to [Kustermans and Vaes 2000; 2003] for more details.

For any $\omega \in \mathcal{M}_*$, $\lambda(\omega) = (\omega \otimes \iota)(W)$ is the Fourier representation of ω, where $\mathcal{M}_*$ is the Banach space of all bounded normal functionals on $\mathcal{M}$. For any ω, θ in $\mathcal{M}_*$, the convolution $\omega * \theta$ is given by

$$\omega * \theta = (\omega \otimes \theta)\Delta.$$

S. Wang and the authors [Liu et al. 2017] defined the convolution of $x \in L^t(\mathbb{G})$ and $y \in L^s(\mathbb{G})$ for $1 \le t, s \le 2$. If the left Haar weights φ, $\hat{\varphi}$ of $\mathbb{G}$ and $\hat{\mathbb{G}}$ respectively are tracial weights, we have that the convolution is well-defined for $1 \le t, s \le \infty$ by the results in [Liu et al. 2017].

For any locally compact quantum group $\mathbb{G}$, the Fourier transforms $\mathcal{F}_t : L^t(\mathbb{G}) \to L^s(\hat{\mathbb{G}})$ are well-defined, where $1/t + 1/s = 1$, $1 \le t \le 2$. (See [Cooney 2010; Caspers 2013] for the definition of Fourier transforms and [Van Daele 2007] for the definition of the Fourier transform for algebraic quantum groups.) For any x in $L^1(\mathbb{G})$, we denote by $x\varphi$ the bounded linear functional on $L^\infty(\mathbb{G})$ given by $(x\varphi)(y) = \varphi(yx)$ for any y in $L^\infty(\mathbb{G})$. Recall that a projection p in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ is a biprojection if $\mathcal{F}_1(p\varphi)$ is a multiple of a projection in $L^\infty(\hat{\mathbb{G}})$. A projection x in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is called a left shift of a biprojection B if $\varphi(x) = \varphi(B)$ and $x * B = \varphi(B)x$. A projection x in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is called a right shift of a biprojection B if $\varphi(x) = \varphi(B)$ and $B * x = \varphi(B)x$. Denote by $\tilde{B}$ the range projection of $\mathcal{F}(B)$. A nonzero element x in $L^\infty(\mathbb{G})$ is said to be a bishift of a biprojection B if there exists a right shift B_g of the biprojection B and a right shift $\tilde{B}_h$ of the biprojection

$\tilde{B}$ and an element y in $L^\infty(\mathbb{G})$ such that

$$x = \widehat{\mathcal{F}}(\tilde{B}_h) * (B_g y).$$

(We refer to [Jiang et al. 2017; Liu and Wu 2017; Liu et al. 2017] for more properties of biprojections and bishifts of biprojections.)

Throughout this paper, we focus on a unimodular Kac algebra $\mathbb{G}$, which is a locally compact quantum group subject to the condition that $\varphi = \psi$ is tracial. (See [Enock and Schwartz 1992] for more details.) For a unimodular Kac algebra $\mathbb{G}$, we denote by $\mathcal{F}$ the Fourier transform for simplicity.

Proposition 2.1. *Suppose $\mathbb{G}$ is a unimodular Kac algebra and $a \in L^t(\mathbb{G})$, $b \in L^s(\mathbb{G})$, $c \in L^r(\mathbb{G})$ such that $1 \le r, t, s \le \infty$ and $1/r + 1/t + 1/s = 2$. Then*

$$\varphi((a * b)c) = \varphi\big((R(c) * a)R(b)\big) = \varphi\big((b * R(c))R(a)\big).$$

Proof. Suppose that $a, b, c \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. Then

$$\begin{aligned}
\varphi((a * b)c) &= (a\varphi \otimes b\varphi)(\Delta(c)) \\
&= (a\varphi R)\big((\iota \otimes \varphi)(\Delta(c)(1 \otimes b))\big) \\
&= (a\varphi R \otimes \varphi c)(\Delta(b)) \quad \text{strong left invariance} \\
&= (a\varphi R \otimes R(c)\varphi R)(\Delta(b)) \\
&= (R(c)\varphi \otimes a\varphi)(\Delta(R(b))) \\
&= \varphi\big((R(c) * a)R(b)\big) \\
&= \varphi\big((b * R(c))R(a)\big).
\end{aligned}$$

By Young's inequality [Liu et al. 2017], we see that the proposition is true for $a \in L^t(\mathbb{G})$, $b \in L^s(\mathbb{G})$, $c \in L^r(\mathbb{G})$ such that $1 \le r, t, s \le \infty$ and $1/r + 1/t + 1/s = 2$. □

Proposition 2.2. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let $x \in L^t(\mathbb{G})$, $y \in L^s(\mathbb{G})$ be positive such that $1 \le t, s \le \infty$, $1/t + 1/s = 1$ Then*

$$\mathcal{R}(x * y) = \sup\{\mathcal{R}(p * q) \mid p \le \mathcal{R}(x), q \le \mathcal{R}(y),\ p, q \text{ are projections in } L^1(\mathbb{G})\}.$$

Proof. Let e_n be the spectral projection of x corresponding to $[1/n, n]$ and f_m the spectral projection of y corresponding to $[1/m, m]$ for $n, m \in \mathbb{N}$. Then we have that $e_n, f_m \in L^1(\mathbb{G})$ and

$$\frac{1}{nm} e_n * f_m \le e_n x e_n * f_m y f_m \le mn e_n * f_m.$$

Hence $\mathcal{R}(e_n * f_m) \le \mathcal{R}(x * y)$. Let $Q = \sup\{\mathcal{R}(p * q) \mid p \le \mathcal{R}(x), q \le \mathcal{R}(y), p, q$ are projections in $L^1(\mathbb{G})\}$. Then we have that $Q = \sup_{n,m} \mathcal{R}(e_n * f_m)$. Therefore $Q \le \mathcal{R}(x * y)$. Assume that there is a nonzero vector $\xi \in \mathcal{H}_\varphi$ such that $Q\xi = 0$ and $\mathcal{R}(x * y)\xi = \xi$. We have that $(e_n * f_m)\xi = 0$ for any $n, m \in \mathbb{N}$ and then

$(e_n x e_n * f_m y f_m)\xi = 0$. Therefore $(x * y)\xi = 0$, which leads a contradiction and $Q = \mathcal{R}(x * y)$. □

Definition 2.3. Suppose $\mathbb{G}$ is a unimodular Kac algebra and $x \in L^t(\mathbb{G})$, $y \in L^s(\mathbb{G})$ are positive for $1 \le t, s \le \infty$. We define the symbol $\mathcal{R}(x * y)$ in terms of x, y as

$$\mathcal{R}(x * y) = \sup\{\mathcal{R}(p * q) \mid p \le \mathcal{R}(x), q \le \mathcal{R}(y),\ p, q \text{ are projections in } L^1(\mathbb{G})\},$$

and

$$\mathcal{S}(x * y) = \varphi(\mathcal{R}(x * y)).$$

Remark 2.4. In Definition 2.3, $\mathcal{R}(x * y)$ and $\mathcal{S}(x * y)$ are symbols. Proposition 2.2 shows that the symbol $\mathcal{R}(x * y)$ is the usual one when $x * y$ is well-defined.

3. The exact inverse sum set theorem

The sum set estimate is a theory of counting the cardinalities of additive sets in additive combinatorics [Tao and Vu 2006]. The sum set estimate for Kac algebras has a different behavior, because of the different types of topology. In this section, we prove a sum set estimate and the exact inverse sum set theorem for unimodular Kac algebras.

Theorem 3.1 (sum set estimate). *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let p, q be projections in $L^\infty(\mathbb{G})$. Then*

$$\max\{\varphi(p), \varphi(q)\} \le \mathcal{S}(p * q).$$

Moreover, $\mathcal{S}(p * q) = \varphi(p) < \infty$ if and only if $\varphi(q)^{-1} p * q$ is a projection in $L^1(\mathbb{G})$.

Proof. First, we assume that p, q are projections in $L^1(\mathbb{G})$. If $\mathcal{S}(p * q) = \infty$, then the inequality is true. We assume that $\mathcal{S}(p * q) < \infty$. By Hölder's inequality,

$$\|p * q\|_1 \le \|\mathcal{R}(p * q)\|_2 \|p * q\|_2.$$

Note that

$$\|p * q\|_1 = \varphi(p * q) = \varphi(p)\varphi(q),$$

and

$$\|\mathcal{R}(p * q)\|_2^2 = \mathcal{S}(p * q).$$

By Young's inequality [Liu et al. 2017], we have

$$\|p * q\|_2 \le \|p\|_1 \|q\|_2 = \varphi(p)\varphi(q)^{1/2},$$

and

$$\|p * q\|_2 \le \|q\|_1 \|p\|_2 = \varphi(q)\varphi(p)^{1/2}.$$

Now we obtain that

$$\varphi(p)\varphi(q) \leq \mathcal{S}(p*q)^{1/2}\varphi(p)\varphi(q)^{1/2},$$

i.e., $\mathcal{S}(p*q) \geq \varphi(q)$. Similarly, we have $\mathcal{S}(p*q) \geq \varphi(p)$. Hence

$$\max\{\varphi(p), \varphi(q)\} \leq \mathcal{S}(p*q).$$

For arbitrary projections p, q in $L^\infty(\mathbb{G})$, by Definition 2.3, we have that

$$\max\{\varphi(p), \varphi(q)\} \leq \mathcal{S}(p*q).$$

If $\mathcal{S}(p*q) = \varphi(p) < \infty$, the inequalities above are equalities. Thus $p*q = \lambda\mathcal{R}(p*q)$ for some $\lambda > 0$ (by the equality of Hölder's inequality) and $\|p*q\|_2 = \|p\|_1\|q\|_2$. Now we see that $p*q = \varphi(q)^{-1}\mathcal{R}(p*q)$.

If $\varphi(q)^{-1}p*q$ is a projection, we have $\mathcal{S}(p*q) = \varphi(q)^{-1}\varphi(p*q) = \varphi(p)$. □

Remark 3.2. By the results in [Jiang et al. 2016], there is an upper bound for the finite dimensional case. But this is not always true for unimodular Kac algebras. For the real line $\mathbb{R}$, we let p be the characteristic function on the interval [0, 1] and q the characteristic function on the set $\cup_{k\in\mathbb{Z}}[k, k+1/(k^2+1)]$. Then $\mathcal{R}(p*q) = 1$.

Corollary 3.3. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let v, w be partial isometries in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ such that $\|v*w\|_1 = \|v\|_1\|w\|_1$. Then*

$$\max\{\varphi(|v|), \varphi(|w|)\} \leq \mathcal{S}(v*w).$$

Moreover $\mathcal{S}(v*w) = \varphi(|w|) < \infty$ if and only if $1/\varphi(|v|)\, v*w$ is a partial isometry in $L^1(\mathbb{G})$.

Proof. The proof is similar to the one of Theorem 3.1. □

Proposition 3.4. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let p, q be projections in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. Then the following are equivalent*:

(1) $\|p*q\|_t = \|p\|_t\|q\|_1$ *for some* $1 < t < \infty$;

(2) $\|p*q\|_t = \|p\|_t\|q\|_1$ *for any* $1 \leq t \leq \infty$;

(3) $\mathcal{S}(p*q) = \varphi(p)$.

Proof. (1) ⇒ (3): Suppose that $\|p*q\|_t = \|p\|_t\|q\|_1$ for some $1 < t < \infty$. Note that $\|p*q\|_\infty \leq \|q\|_1$. By the spectral decomposition, we have

$$\frac{1}{\varphi(q)}p*q = \int_0^1 \lambda\, \mathrm{d}E_\lambda,$$

where $\{E_\lambda\}_\lambda$ is a resolution of the identity for $p*q$. By the assumption, we obtain

$$\int_0^1 \lambda^t\varphi(\mathrm{d}E_\lambda) = \varphi(p).$$

Note that $\|p*q\|_1 = \|p\|_1\|q\|_1$, i.e.,

$$\int_0^1 \lambda\varphi(\mathrm{d}E_\lambda) = \varphi(p).$$

Combining the two equations above, we see that $E(\{1\}) = 1/\varphi(q)\, p*q$ and $\varphi(E(\{1\})) = \varphi(p)$, i.e., $\mathcal{S}(p*q) = \varphi(p)$.

(3) $\Rightarrow$ (2): Suppose that $\mathcal{S}(p*q) = \varphi(p)$. By Theorem 3.1, we have that $1/\varphi(q)\, p*q$ is a projection. Hence for any $1 \le t \le \infty$,

$$\frac{1}{\varphi(q)}\|p*q\|_t = \left\|\frac{1}{\varphi(q)}p*q\right\|_t = \|\mathcal{R}(p*q)\|_t = \mathcal{S}(p*q)^{1/t} = \varphi(p)^{1/t} = \|p\|_t,$$

i.e., $\|p*q\|_t = \|p\|_t\|q\|_1$.

(2) $\Rightarrow$ (1): It is obvious. $\square$

Proposition 3.5. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let $x, y \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ be nonzero positive elements. Then the following are equivalent*:

1. $\|x*y\|_2 = \|x\|_1\|y\|_2$;
2. *there exists a biprojection B such that $\mathcal{R}(R(x)*x) \le B$, $(y*R(y))B = \|y*R(y)\|_\infty B$ and $\|y*R(y)\|_\infty = \|y\|_2^2$.*

Proof. (1) $\Rightarrow$ (2): Note that

$$\begin{aligned}\|x*y\|_2^2 &= \varphi((x*y)(x*y))\\ &= \varphi\big(R(y)((R(y)*R(x))*x)\big)\\ &= \varphi\big((R(x)*x)(y*R(y))\big)\\ &\le \|R(x)*x\|_1\|y*R(y)\|_\infty\\ &\le \|x\|_1^2\|y\|_2^2.\end{aligned}$$

If $\|x*y\|_2 = \|x\|_1\|y\|_2$, then $\varphi((R(x)*x)(y*R(y))) = \|R(x)*x\|_1\|y*R(y)\|_\infty$ and $\|y*R(y)\|_\infty = \|y\|_2^2$. Let B be the spectral projection of $y*R(y)$ corresponding to $\|y*R(y)\|_\infty$. By [Liu and Wu 2017, Proposition 3.14 and Corollary 3.16], we have that B is a biprojection.

(2) $\Rightarrow$ (1): It follows by the argument above. $\square$

Proposition 3.6. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. If there exists a nonzero positive element $x \in L^1(\mathbb{G}) \cap L^t(\mathbb{G})$ for some $t > 1$ such that $x*x = x$, then $1/\hat{\varphi}(\mathcal{F}(x))\, x$ is a biprojection.*

Proof. By assumption, we obtain that $\|x\|_1 = 1$ and $\mathcal{F}(x)^2 = \mathcal{F}(x)$. By the Hausdorff–Young inequality [Cooney 2010], we have that $\|\mathcal{F}(x)\|_\infty \le \|x\|_1 = 1$. Hence $\mathcal{F}(x)$ is a contractive idempotent, i.e., $\mathcal{F}(x)$ is a projection. We see that $\mathcal{F}(x) = \mathcal{F}(x)^*$ and $x = R(x)$.

If $t \geq 2$, we have that $x \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$. If $1 < t < 2$, we let $\epsilon = t - 1$, then $\|x\|_{1+\epsilon} < \infty$. We show that $x \in L^2(\mathbb{G})$. Let $K(s) = 2s/(1+s)$. Then $K(s) < s$ when $s > 1$ and $K^n(s) \to 1$ as $n \to \infty$ for any $s \geq 1$. By Young's inequality [Liu et al. 2017], we have that

$$\|x\|_2 = \|x * x\|_2 \leq \|x\|_{K(2)}^2 = \|x * x\|_{K(2)}^2 \leq \cdots \leq \|x\|_{K^n(2)}^{2^n} \leq \|x\|_{1+\epsilon}^{2^n} < \infty$$

for some n large enough. Hence $x \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$. Note that $\|x * R(x)\|_2 = \|x\|_2 = \|x\|_2 \|x\|_1$. By Proposition 3.5, we see that there exists a biprojection B such that $\mathcal{R}(R(x) * x) \leq B$ and $(R(x) * x)B = \|R(x) * x\|_\infty B$. Hence

$$x = R(x) * x = \|R(x) * x\|_\infty B = \|x\|_2^2 B = \hat{\varphi}(\mathcal{F}(x))B. \qquad \square$$

Definition 3.7. Suppose $\mathbb{G}$ is a unimodular Kac algebra and there exists a biprojection B in $L^1(\mathbb{G})$. A projection q in $L^1(\mathbb{G})$ is said to be a right (left) subshift of the biprojection B if there exists a right (left) shift B_g of B such that $q \leq B_g$.

Proposition 3.8. *Suppose $\mathbb{G}$ is a unimodular Kac algebra and B is a biprojection in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. Let q be a projection in $L^\infty(\mathbb{G})$. Then*

$$\mathcal{R}(q * R(q)) \leq B \textit{ if and only if } q \textit{ is a right subshift of } B,$$

and

$$\mathcal{R}(R(q) * q) \leq B \textit{ if and only if } q \textit{ is a left subshift of } B.$$

Proof. Suppose that $\mathcal{R}(q * R(q)) \leq B$. Let $p_1 = \mathcal{R}(B * q)$. We shall show that p_1 is a projection in $L^1(\mathbb{G})$. Since

$$\mathcal{R}(B * q * R(q) * B) \leq \mathcal{R}(B * B * B) = B,$$

by Theorem 3.1, we see that $p_1,\ \mathcal{R}(p_1 * R(p_1)) \in L^1(\mathbb{G})$, and

$$\varphi(p_1) \leq \mathcal{S}(p_1 * R(p_1)) \leq \varphi(B) < \infty.$$

On the other hand, $\varphi(p_1) \geq \varphi(B)$ by Theorem 3.1. Then we obtain that $\varphi(p_1) = \varphi(B)$. By Theorem 3.1, we have that $1/\varphi(q)\, B * q$ is a projection and $p_1 = 1/\varphi(q)\, B * q$.

Suppose q is a right subshift of B. Let p_1 be the right subshift of B such that $q \leq p_1$. Then $q * R(q) \leq p_1 * R(p_1) = \varphi(B)B$. $\square$

Theorem 3.9 (exact inverse sum set theorem). *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let p, q be projections in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. Then the following are equivalent*:

(1) $\mathcal{S}(p * q) = \varphi(p)$;

(2) $\mathcal{S}(p * (q * R(q))^{*(m)}) * q^{*(j)} = \varphi(p)$ *for some* $m \geq 0$, $j \in \{0, 1\}$, $m + j > 0$, $q^{*(0)}$ *means* q *vanishes*;

(3) *there exists a biprojection B such that q is a right subshift of B and $p = \mathcal{R}(x * B)$ for some $x > 0$.*

Proof. (1) ⇒ (3): By Proposition 3.4, we have that $\|p * q\|_2 = \|p\|_2 \|q\|_1$. By Proposition 3.5, we see that there is a biprojection B such that $\mathcal{R}(q * R(q)) \leq B$ and $(R(p) * p)B = \|p\|_2^2 B$. Since

$$\varphi((p * B)p) = \varphi(B(R(p) * p)) = \varphi(p)\varphi(B) = \varphi(p * B),$$

we obtain that $\mathcal{R}(p * B) \leq p$. By Theorem 3.1, we have that $\mathcal{R}(p * B) = p$.

(3) ⇒ (2): Let $p = \mathcal{R}(x * B)$. Then $\mathcal{R}(p * B) = p$ and hence $p * B = \varphi(B)p$ by Theorem 3.1. Note that

$$\mathcal{R}((q * R(q))^{*(m+j)}) \leq \mathcal{R}(B^{*(m+j)}) = B.$$

By Theorem 3.1, we have

$$\begin{aligned}\varphi(p) &\leq \mathcal{S}(p * (q * R(q))^{*(m)} * q^{*(j)}) \\ &\leq \mathcal{S}\big(\mathcal{R}(p * (q * R(q))^{*(m)} * q^{*(j)}) * R(q)^{*(j)}\big) \\ &\leq \mathcal{S}(p * B) = \varphi(p),\end{aligned}$$

i.e.,

$$\mathcal{S}(p * (q * R(q))^{*(m)} * q^{*(j)}) = \varphi(p).$$

(2) ⇒ (1): By Theorem 3.1, we have that

$$\varphi(p) = \mathcal{S}(p * (q * R(q))^{*(m)} * q^{*(j)}) \geq \mathcal{S}(p * q) \geq \varphi(p).$$

Hence $\mathcal{S}(p * q) = \varphi(p)$. □

4. Extremal pairs of Young's inequality

In this section, we characterize extremal pairs of Young's inequality for unimodular Kac algebras.

Proposition 4.1. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let v, w be partial isometries in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. Then the following are equivalent:*

(1) $\|v * w\|_t = \|v\|_t \|w\|_1$ *for some* $1 < t < \infty$;

(2) $\|v * w\|_t = \|v\|_t \|w\|_1$ *for any* $1 \leq t \leq \infty$;

(3) $1/(\varphi(|w|))\,|v * w|$ *is a projection and* $\|v * w\|_1 = \|v\|_1 \|w\|_1$.

Proof. By Corollary 3.3 and a similar argument of Proposition 3.4, we have the proposition proved. □

Proposition 4.2. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let v, w be partial isometries. Then the following are equivalent:*

(1) $\|v * w\|_r = \|v\|_t \|w\|_s$ *for some* $1 < t, r, s < \infty$ *such that* $1/r + 1 = 1/t + 1/s$;

(2) $\|v * w\|_r = \|v\|_t \|w\|_s$ *for any* $1 \leq r, t, s \leq \infty$ *such that* $1/r + 1 = 1/t + 1/s$.

(3) *there exists a biprojection* B *such that* $v = ({}_h B y_v) * \hat{\mathscr{F}}(\tilde{B}_g)$ *and* $w = \hat{\mathscr{F}}(\tilde{B}_g) * (B_f y_w)$, *where* B_g, B_f *are right shifts of* B, ${}_h B$ *is a left shift of* B *and* y_v, y_w *are elements such that* v, w *are nonzero partial isometries.*

Proof. (1) $\Rightarrow$ (2): Suppose that $\|v * w\|_r = \|v\|_t \|w\|_s$ for some $1 < r, t, s < \infty$ such that $1/r + 1 = 1/t + 1/s$. By Young's inequality in [Liu et al. 2017], we have

$$\|v * w\|_r \leq \|v\|_r \|w\|_1, \quad \|v * w\|_r \leq \|v\|_1 \|w\|_r,$$

and hence $\varphi(|v|) = \varphi(|w|)$. By Proposition 4.1, we see that $\|v * w\|_{\tilde{r}} = \varphi(|v|)^{1+1/\tilde{r}}$ for any $1 \leq \tilde{r} \leq \infty$. Therefore $\|v * w\|_{\tilde{r}} = \|v\|_{\tilde{t}} \|w\|_{\tilde{s}}$ for any $1 \leq \tilde{r}, \tilde{t}, \tilde{s} \leq \infty$ with $1/\tilde{r} + 1 = 1/\tilde{t} + 1/\tilde{s}$.

(2) $\Rightarrow$ (3): Let $r = 2$. Then $1 \leq t, s \leq 2$. By Hölder's inequality and the Hausdorff–Young inequality [Cooney 2010], we obtain that

$$\begin{aligned} \|v\|_t \|w\|_s &= \|v * w\|_2 \\ &= \|\mathscr{F}(v)\mathscr{F}(w)\|_2 \leq \|\mathscr{F}(v)\|_{t/(t-1)} \|\mathscr{F}(w)\|_{s/(s-1)} \leq \|v\|_t \|w\|_s. \end{aligned}$$

Hence for any $1 \leq t, s \leq 2$,

$$\|\mathscr{F}(v)\|_{t/(t-1)} = \|v\|_t, \quad \|\mathscr{F}(w)\|_{s/(s-1)} = \|w\|_s, \tag{1}$$

and

$$\|\mathscr{F}(v)\mathscr{F}(w)\|_2 = \|\mathscr{F}(v)\|_{t/(t-1)} \|\mathscr{F}(w)\|_{s/(s-1)}. \tag{2}$$

For (1), by [Liu and Wu 2017, Proposition 3.6], we have that v, w are minimizers of the Hirschman–Beckner uncertainty principle for unimodular Kac algebras. By [Liu and Wu 2017, Theorem 3.15], we see that v, w are bishifts of biprojections. By Lemma 4.4, we have that

$$\|v\|_t \|w\|_s = \|v * w\|_r \leq \||v| * |w|\|_r^{1/2} \||v^*| * |w^*|\|_r^{1/2} \leq \|v\|_t \|w\|_s.$$

For (2), we have that

$$|\mathscr{F}(v)| = |\mathscr{F}(w)^*|,$$

i.e., $\mathscr{R}((\mathscr{F}(v))^*) = \mathscr{R}(\mathscr{F}(w))$.

By Theorem 3.9, we have that there exists a biprojection B such that $|v|$ is a left shift ${}_h B$ of B and $|w|$ is a right shift B_f of B. By the definition of a bishift of a biprojection, we have that $w = \hat{\mathscr{F}}(\tilde{B}_g) * (B_f y_w)$ for some right shift of $\tilde{B}$. By [Liu and Wu 2017, Proposition 3.11], $\mathscr{R}(\mathscr{F}(w)) = \tilde{B}_g$. Hence $\mathscr{R}((\mathscr{F}(v))^*) = \tilde{B}_g$. By [Liu and Wu 2017, Theorem 3.18], we have that $v = ({}_h B y_v) * \hat{\mathscr{F}}(\tilde{B}_g)$.

(3) $\Rightarrow$ (2): Since v, w are bishifts of the biprojection B, we have that system of equations (1) are true. By [Liu and Wu 2017, Proposition 3.11], we have

$$\mathcal{R}(\mathcal{F}(w)) = \mathcal{R}((\mathcal{F}(v))^*) = \tilde{B}_g.$$

Note that $\varphi(|v|) = \varphi({}_hB) = \varphi(B_f) = \varphi(|w|)$ and $\mathcal{F}(v), \mathcal{F}(w)$ are multiples of partial isometries. We see $|\mathcal{F}(v)| = |\mathcal{F}(w)^*|$ and that

$$\|v * w\|_2 = \|v\|_t \|w\|_s$$

for any $1 \le t, s \le 2$ from the argument for "(2) $\Rightarrow$ (3)". By Proposition 4.1, we have $\|v * w\|_r = \|v\|_r \|w\|_1$ for any $1 \le r \le \infty$. Therefore (2) is true.

(2) $\Rightarrow$ (1): It is obvious. □

Lemma 4.3. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. For any a, b in $L^2(\mathbb{G}) \cap L^\infty(\mathbb{G})$, we define $v_{a,b} : L^2(\mathbb{G}) \otimes L^2(\mathbb{G}) \to L^2(\mathbb{G})$ given by*

$$v_{a,b}(\Lambda_\varphi(x_1) \otimes \Lambda_\varphi(x_2)) = \Lambda_\varphi(ax_1 * bx_2)$$

for any x_1, x_2 in $L^2(\mathbb{G}) \cap L^\infty(\mathbb{G})$. Then $v_{a,b}$ is bounded and $v_{a,b} v_{a,b}^* = aa^* * bb^*$. Moreover,

$$\|v_{a,b}\|_\infty = \|aa^* * bb^*\|_\infty^{1/2}.$$

Proof. For any $y \in L^2(\mathbb{G}) \cap L^\infty(\mathbb{G})$, we have that

$$\begin{aligned}
\langle v_{a,b}(\Lambda_\varphi(x_1) \otimes \Lambda_\varphi(x_2)), \Lambda_\varphi(y)\rangle &= \langle \Lambda_\varphi(ax_1 * bx_2), \Lambda_\varphi(y)\rangle \\
&= \varphi(y^*(ax_1 * bx_2)) \\
&= (\varphi \otimes \varphi)((1 \otimes y^*)(R \otimes \iota)(\Delta(bx_2))(ax_1 \otimes 1)) \\
&= (\varphi \otimes \varphi)((R \otimes \iota)(\Delta(bx_2))(ax_1 \otimes y^*)) \\
&= \varphi(R((\iota \otimes \varphi)(\Delta(bx_2)(1 \otimes y^*))ax_1) \\
&= (\varphi \otimes \varphi)((ax_1 \otimes bx_2)\Delta(y^*)) \\
&= (\varphi \otimes \varphi)((x_1 \otimes x_2)\Delta(y^*)(a \otimes b)).
\end{aligned}$$

Note that

$$\begin{aligned}
(\varphi \otimes \varphi)(\Delta(y^*)(aa^* \otimes bb^*)\Delta(y)) &= (\varphi \otimes \varphi)((aa^* \otimes bb^*)\Delta(yy^*)) \\
&= \varphi((yy^*)(aa^* * bb^*)) \le \|aa^* * bb^*\|_\infty \varphi(yy^*) \\
&\le \|y\|_2^2 \|a\|_2^2 \|b\|_\infty^2;
\end{aligned}$$

the last inequality follows from Young's inequality in [Liu et al. 2017]. Then we have that

$$\begin{aligned}
&\big\langle v_{a,b}(\Lambda_\varphi(x_1) \otimes \Lambda_\varphi(x_2)), \Lambda_\varphi(y)\big\rangle \\
&\qquad = \big\langle \Lambda_\varphi(x_1) \otimes \Lambda_\varphi(x_2), (\Lambda_\varphi \otimes \Lambda_\varphi)((a^* \otimes b^*)\Delta(y))\big\rangle.
\end{aligned}$$

Therefore

$$\|v_{a,b}\| \le \|a\|_2 \|b\|_\infty \quad \text{and} \quad v_{a,b}^* \Lambda_\varphi(y) = (\Lambda_\varphi \otimes \Lambda_\varphi)((a^* \otimes b^*)\Delta(y))$$

whenever $y \in L^2(\mathbb{G}) \cap L^\infty(\mathbb{G})$.

Now we have to check that $vv^* = aa^* * bb^*$. First, write $\Delta(y) = \sum_\beta y_{\beta 1} \otimes y_{\beta 2}$. Then

$$\begin{aligned}
v_{a,b} v_{a,b}^* \Lambda_\varphi(y) &= v_{a,b}((\Lambda_\varphi \otimes \Lambda_\varphi)((a^* \otimes b^*)\Delta(y))) \\
&= \sum_\beta v_{a,b}(\Lambda_\varphi(a^* y_{\beta 1}) \otimes \Lambda_\varphi(b^* y_{\beta 2})) \\
&= \sum_\beta \Lambda_\varphi((aa^* y_{\beta 1}) * (bb^* y_{\beta 2})) \\
&= \sum_\beta \hat{\Lambda}(\lambda((aa^* y_{\beta 1}\varphi) * (bb^* y_{\beta 2}\varphi))) \\
&= \sum_\beta \hat{\Lambda}(\lambda((aa^* y_{\beta 1}\varphi) \otimes (bb^* y_{\beta 2}\varphi)\Delta)) \\
&= \hat{\Lambda}(\lambda((aa^* * bb^*) y\varphi)) \\
&= (aa^* * bb^*)\Lambda_\varphi(y),
\end{aligned}$$

i.e., $v_{a,b} v_{a,b}^* = aa^* * bb^*$. □

For any element x in a von Neumann algebra, $x = w_x |x|$ is the polar decomposition of x.

Lemma 4.4. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let $x \in L^t(\mathbb{G})$ and $y \in L^s(\mathbb{G})$ such that $1 + 1/r = 1/t + 1/s$. Then*

$$\|x * y\|_r \le \||x| * |y|\|_r^{1/2} \||x^*| * |y^*|\|_r^{1/2}.$$

Proof. We assume that x, y are in $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$. By Lemma 4.3, we define $v_{w_x|x|^{1/2}, w_y|y|^{1/2}}$ and $v_{|x|^{1/2},|y|^{1/2}}$. Then by Lemma 4.3 again,

$$v_{w_x|x|^{1/2}, w_y|y|^{1/2}} v_{|x|^{1/2},|y|^{1/2}}^* = x * y.$$

Let $\tilde{x} = v_{w_x|x|^{1/2}, w_y|y|^{1/2}}$ and $\tilde{y} = v_{|x|^{1/2},|y|^{1/2}}$. Then by the polar decomposition, we obtain that

$$\tilde{x} = |\tilde{x}^*| w_{\tilde{x}}, \quad \tilde{y} = |\tilde{y}^*| w_{\tilde{y}}.$$

By Lemma 4.3, we have

$$|\tilde{x}^*|^2 = |x^*| * |y^*|, \quad |\tilde{y}^*|^2 = |x| * |y|.$$

By Hölder's inequality, we have

$$\begin{aligned}\|x*y\|_r &= \|\tilde{x}\tilde{y}^*\|_r = \||\tilde{x}^*|w_{\tilde{x}}w_{\tilde{y}}^*|\tilde{y}^*|\|_r\\ &\leq \|(|x|*|y|)^{1/2}\|_{2r}\|(|x^*|*|y^*|)^{1/2}\|_{2r} = \||x|*|y|\|_r^{1/2}\||x^*|*|y^*|\|_r^{1/2}.\end{aligned}$$

For any x in $L^t(\mathbb{G})$ and y in $L^s(\mathbb{G})$ such that $1+1/r=1/t+1/s$, there exists nets $\{x_\alpha\}_\alpha$ and $\{y_\beta\}_\beta$ such that $x_\alpha, y_\beta \in L^1(\mathbb{G})\cap L^\infty(\mathbb{G})$ are positive and $\lim_\alpha x_\alpha = |x|$, $\lim_\beta y_\beta = |y|$ in $L^t(\mathbb{G})$ and $L^s(\mathbb{G})$ respectively. Therefore we have that

$$\|x * y\|_r \leq \||x| * |y|\|_r^{1/2}\||x^*| * |y^*|\|_r^{1/2}$$

is true for any $x \in L^t(\mathbb{G})$, $y \in L^s(\mathbb{G})$, $1 + 1/r = 1/t + 1/s$. □

Proposition 4.5. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let $x \in L^t(\mathbb{G})$, $y \in L^1(\mathbb{G})$ for some $1 < t < \infty$. If $\|x * y\|_t = \|x\|_t\|y\|_1$ for some $1 < t < 2$, then for any $0 \leq \Re z \leq 1$,*

$$\|w_x|x|^{t(1+z)/2} * y\|_{2/(1+\Re z)} = \|w_x|x|^{t(1+z)/2}\|_{2/(1+\Re z)}\|y\|_1.$$

If $\|x * y\|_t = \|x\|_t\|y\|_1$ for some $2 < t < \infty$, then for any $0 \leq \Re z \leq 1$,

$$\|w_x|x|^{t(1-z)/2} * y\|_{2/(1-\Re z)} = \|w_x|x|^{t(1-z)/2}\|_{2/(1-\Re z)}\|y\|_1.$$

Proof. Suppose that $\|x\|_t = 1$ and $\|y\|_1 = 1$. When $1 < t < 2$, we define a complex function $F_1(z)$ given by

$$\begin{aligned}F_1(z) &= \varphi((w_x|x|^{t(1+z)/2} * y)|x * y|^{t(1-z)/2}w_{x*y}^*),\\ |F_1(z)| &\leq \|w_x|x|^{t(1+z)/2} * y\|_{2/(1+\Re z)}\||x * y|^{t(1-z)/2}w_{x*y}^*\|_{2/(1-\Re z)}\\ &\leq \||x|^{t(1+z)/2}\|_{2/(1+\Re z)}\|y\|_1\varphi(|x * y|^t)^{(1-\Re z)/2} = 1.\end{aligned}$$

Hence $F_1(z)$ is a bounded analytic function on $0 < \Re z < 1$. Note that

$$F_1\left(\frac{2}{t} - 1\right) = \varphi((x * y)|x * y|^{t-1}w_{x*y}^*) = 1.$$

Therefore $F_1(z) \equiv 1$ on $0 \leq \Re z \leq 1$ by the maximum modulus theorem.

When $2 < t < \infty$, we consider the function $F_2(z)$ given by

$$F_2(z) = \varphi\big((w_x|x|^{t(1-z)/2} * y)|x * y|^{t(1+z)/2}w_{x*y}^*\big).$$

Similarly, we have the proposition proved. □

Proposition 4.6. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let $x \in L^t(\mathbb{G})$, $y \in L^s(\mathbb{G})$ be such that $\|x*y\|_r = \|x\|_t\|y\|_s$ for some $1 < r, t, s < \infty$, where $1/r+1 = 1/t + 1/s$. Then for any $-r + 1 \leq \Re z \leq r - 1$,*

$$\|w_x|x|^{t\frac{r+1-z}{2r}} * w_y|y|^{s\frac{r+1+z}{2r}}\|_r = \|w_x|x|^{t\frac{r+1-z}{2r}}\|_{\frac{2r}{r+1-\Re z}}\|w_y|y|^{s\frac{r+1+z}{2r}}\|_{\frac{2r}{r+1+\Re z}}.$$

Proof. Suppose that $\|x\|_t = \|y\|_s = 1$. We define a function $F(z)$ on $-r+1 \leq \Re z \leq r-1$ given by

$$F(z) = \varphi\big((w_x|x|^{t\frac{r+1+z}{2r}} * w_y|y|^{s\frac{r+1-z}{2r}})|x*y|^{r-1}w_{x*y}^*\big).$$

$$\begin{aligned}|F(z)| &\leq \|w_x|x|^{t\frac{r+1+z}{2r}} * w_y|y|^{s\frac{r+1-z}{2r}}\|_r \||x*y|^{r-1}\|_{\frac{r}{r-1}} \\ &\leq \|w_x|x|^{t\frac{r+1+z}{2r}}\|_{\frac{2r}{r+1+\Re z}} \|w_y|y|^{s\frac{r+1-z}{2r}}\|_{\frac{2r}{r+1-\Re z}} \varphi(|x*y|^r)^{\frac{r-1}{r}} = 1.\end{aligned}$$

Hence $F(z)$ is a bounded analytic function on $-r+1 \leq \Re z \leq r-1$. Since

$$F\Big(\frac{2r}{t} - r - 1\Big) = \varphi((x*y)|x*y|^{r-1}w_{x*y}^*) = \varphi(|x*y|^r) = 1,$$

we have that $F(z) \equiv 1$ on $-r+1 \leq \Re z \leq r-1$ by the maximum modulus theorem. Therefore we have the proposition proved. □

Proposition 4.7. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. If there exist positive elements $x \in L^t(\mathbb{G})$, $y \in L^s(\mathbb{G})$ such that $\|x*y\|_r = \|x\|_t\|y\|_s$ for some $1 < r, t, s < \infty$ and $1/r + 1 = 1/t + 1/s$, then there exists a biprojection B such that x is a multiple of left shift of B and y is a multiple of a right shift of B.*

Proof. By Proposition 4.6, we have

$$\|x^{t/r} * y^s\|_r = \|x^{t/r}\|_r\|y^s\|_1, \quad \|x^t * y^{s/r}\|_r = \|x^t\|_1\|y^{s/r}\|_r.$$

By Proposition 4.5, we have that

$$\|x^{t/2} * y^s\|_2 = \|x^{t/2}\|_2\|y^s\|_1, \quad \|x^t * y^{s/2}\|_2 = \|y^{s/2}\|_2\|x^t\|_1.$$

By Proposition 3.5, we have that there exist projections B_1, B_2 such that

$$\mathscr{R}(y^s * R(y^s)) \leq B_1, \quad (R(x^{t/2}) * x^{t/2})B_1 = \|R(x^{t/2}) * x^{t/2}\|_\infty B_1,$$

and

$$\mathscr{R}(x^t * R(x^t)) \leq B_2, \quad (y^{s/2} * R(y^{s/2})B_2 = \|y^{s/2}) * R(y^{s/2})\|_\infty B_2,$$

and

$$\|R(x^{t/2}) * x^{t/2}\|_\infty = \|x^{t/2}\|_2^2, \quad \|y^{s/2} * R(y^{s/2})\|_\infty = \|y^{s/2}\|_2^2.$$

Then we see that $B_1 = B_2 (= B)$. In Proposition 3.5, to obtain that B is a biprojection, it requires that $x^{t/2} \in L^1(\mathbb{G})$, but we only have $x^{t/2} \in L^2(\mathbb{G})$ here. To see that B is a biprojection, we focus on

$$R(x^{t/2}) * x^{t/2} = \|R(x^{t/2}) * x^{t/2}\|_\infty B. \tag{3}$$

Note that $B = R(B)$.

Let $q \leq B$ be a projection in $L^1(\mathbb{G})$. Then

$$\begin{aligned}\|x^{t/2}\|_2^2\varphi(q) &= \varphi\big((R(x^{t/2}) * x^{t/2})R(q)\big)\\ &= \varphi(x^{t/2}(x^{t/2} * q)) \quad \text{(by Proposition 2.1)}\\ &\leq \|x^{t/2}\|_2\|x^{t/2} * q\|_2\\ &\leq \|x^{t/2}\|_2^2\varphi(q).\end{aligned}$$

Thus $x^{t/2} * q = \varphi(q)x^{t/2}$. Then we have that $\mathcal{F}(x^{t/2})\mathcal{F}(q) = \varphi(q)\mathcal{F}(x^{t/2})$. So

$$\mathcal{R}(\mathcal{F}(x^{t/2})) \leq E, \tag{4}$$

where E is the spectral projection of $\mathcal{F}(q)$ corresponding to $\varphi(q)$.

Recall that q is a projection in $L^1(\mathbb{G})$, so q is in $L^2(\mathbb{G})$. Thus

$$\frac{1}{\varphi(q)} = \left\|\frac{\mathcal{F}(q)}{\varphi(q)}\right\|_2^2 \geq \|\mathcal{R}(\mathcal{F}(x^{t/2}))\|_2^2 = \mathcal{S}(\mathcal{F}(x^{t/2})).$$

Then we have that

$$\varphi(B) = \sup_{q\leq B}\{\varphi(q)\} \leq \frac{1}{\mathcal{S}(\mathcal{F}(x^{t/2})}. \tag{5}$$

So $x^{t/2}$ is in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$. By [Liu and Wu 2017, Proposition 3.14], B is a biprojection.

Note that since $\mathcal{S}(B)\mathcal{S}(\mathcal{F}(B)) = 1$, we obtain that $\mathcal{S}(\mathcal{F}(x^{t/2})) \leq \mathcal{S}(\mathcal{F}(B))$. Applying Theorem 3.1 to (3), we see that $\mathcal{S}(x^{t/2}) \leq \mathcal{S}(B)$. Thus

$$\mathcal{S}(F(x^{t/2}))\mathcal{S}(x^{t/2}) \leq \mathcal{S}(\mathcal{F}(B))\mathcal{S}(B) = 1.$$

By [Liu and Wu 2017, Proposition 3.6 and Theorem 3.15], $x^{t/2}$ is a multiple of left shift of B. So is x. Similarly, y is a multiple of right shift of B. □

Theorem 4.8. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let x, y be φ-measurable and nonzero. Then the following are equivalent*:

(1) $\|x * y\|_r = \|x\|_t\|y\|_s$ *for some* $1 < r, t, s < \infty$ *such that* $1/r + 1 = 1/t + 1/s$;

(2) $\|x * y\|_r = \|x\|_t\|y\|_s$ *for any* $1 \leq r, t, s \leq \infty$ *such that* $1/r + 1 = 1/t + 1/s$;

(3) *there exists a biprojection B such that $x = ({}_hBa_x) * \hat{\mathcal{F}}(\tilde{B}_g)$ and $y = \hat{\mathcal{F}}(\tilde{B}_g) * (B_f a_y)$, where B_g, B_f are right shifts of B, ${}_hB$ is a left shift of B and a_x, a_y are elements such that x, y are nonzero.*

Proof. (1) ⇒ (3): By Lemma 4.4, we have that $\||x| * |y|\|_r = \|x\|_t\|y\|_s$. By Proposition 4.7, we have that $|x|, |y|$ are multiples of projections. By Proposition 4.2, we see (3) is true.

(3) ⇒ (2): It is true from Proposition 4.2.

(2) $\Rightarrow$ (1): It is obvious. □

5. Extremal operators of the Hausdorff–Young inequality

In this section, we will characterize the extremal operators of the Hausdorff–Young inequality for unimodular Kac algebras.

Proposition 5.1. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let $x \in L^t(\mathbb{G})$ for some $1 < t < 2$. If $\|\mathcal{F}(x)\|_{t/(t-1)} = \|x\|_t$, then for any complex number z, we have*

$$\|\mathcal{F}(w_x|x|^{t(1+z)/2})\|_{2/(1-\Re z)} = \|w_x|x|^{t(1+z)/2}\|_{2/(1+\Re z)}.$$

Proof. We assume that $\|x\|_t = 1$, $t' = t/(t-1)$, and consider the function $F(z)$ given by

$$F(z) = \varphi\big(\mathcal{F}(w_x|x|^{t(1+z)/2})|\mathcal{F}(x)|^{t'(1+z)/2} w^*_{\mathcal{F}(x)}\big).$$

Since

$$\begin{aligned}|F(z)| &\leq \|\mathcal{F}(w_x|x|^{t(1+z)/2})\|_{2/(1-\Re z)} \||\mathcal{F}(x)|^{t'(1+z)/2} w^*_{\mathcal{F}(x)}\|_{2/(1+\Re z)} \\ &\leq \|w_x|x|^{t(1+z)/2}\|_{2/(1+\Re z)} \||\mathcal{F}(x)|^{t'(1+z)/2}\|_{2/(1+\Re z)} = 1,\end{aligned}$$

we see that $F(z)$ is a bounded analytic function on $0 \leq \Re z \leq 1$. Note that

$$F\Big(\frac{2}{t} - 1\Big) = \varphi(\mathcal{F}(x)|\mathcal{F}(x)|^{1/(t-1)} w^*_{\mathcal{F}(x)}) = \|\mathcal{F}(x)\|^{t'}_{t'} = 1.$$

By the maximum modulus theorem, we have that $F(z) \equiv 1$ on $0 \leq \Re z \leq 1$ and the proposition is proved. □

Theorem 5.2. *Suppose $\mathbb{G}$ is a unimodular Kac algebra. Let x be measurable. Then the following are equivalent*:

(1) $\|\mathcal{F}(x)\|_{t/(t-1)} = \|x\|_t$ *for some* $1 < t < 2$;

(2) $\|\mathcal{F}(x)\|_{t/(t-1)} = \|x\|_t$ *for any* $1 \leq t \leq 2$;

(3) *x is a bishift of a biprojection.*

Proof. (1) $\Rightarrow$ (3): By Proposition 5.1, we have that

$$\|\mathcal{F}(w_x|x|^{3t/4})\|_4 = \|w_x|x|^{3t/4}\|_{4/3}.$$

Let $y = w_x|x|^{3t/4}$. Then

$$\|y^* * R(y)\|_2 = \||\mathcal{F}(y)|^2\|_2 = \|\mathcal{F}(y)\|_4^2 = \|y\|_{4/3}^2.$$

By Theorem 4.8, we have that y is a bishift of a biprojection and so is x.

(3) $\Rightarrow$ (2): It can be checked directly.

(2) $\Rightarrow$ (1): It is obvious. □

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