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GOOD REDUCTION AND SHAFAREVICH-TYPE THEOREMS FOR DYNAMICAL SYSTEMS WITH PORTRAIT LEVEL STRUCTURES

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Let K be a number field, let S be a finite set of places of K, and let R_S be the ring of S-integers of K. A K-morphism $f:\mathbb{P}^1_K\to\mathbb{P}^1_K$ has simple good reduction outside S if it extends to an R_S-morphism $\mathbb{P}^1_{R_S}\to\mathbb{P}^1_{R_S}$. A finite Galois invariant subset $X\subset\mathbb{P}^1_K(\bar{K})$ has good reduction outside S if its closure in $\mathbb{P}^1_{R_S}$ is étale over R_S. We study triples (f,Y,X) with $X=Y\cup f(Y)$. We prove that for a fixed K, S, and d, there are only finitely many $\mathrm{PGL}_2(R_S)$-equivalence classes of triples with $\deg(f)=d$ and $\sum_{P\in Y}e_f(P)\ge 2d+1$ and X having good reduction outside S. We consider refined questions in which the weighted directed graph structure on $f:Y\to X$ is specified, and we give an exhaustive analysis for degree 2 maps on $\mathbb{P}^1$ when $Y=X$.

1. Introduction

Let K be a number field, let S be a finite set of places of K including all archimedean places, and let R_S be the ring of S-integers of K. We recall that an abelian variety A/K is said to have *good reduction outside S* if there exists a proper R_S-group scheme $\mathcal{A}/R_S$ whose generic fiber is K-isomorphic to A/K. Then we have

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the following famous conjecture of Shafarevich, which was proven by Shafarevich in dimension 1 and by Faltings in general.

Theorem 1 [Faltings 1983]. *There are only finitely many K-isomorphism classes of abelian varieties A/K having good reduction outside S.*

Our goal in this paper is to study an analogue of Shafarevich's conjecture for dynamical systems on projective space. The first requirement is a definition of good reduction for self-maps of $\mathbb{P}^N$, such as the following.

Definition [Morton and Silverman 1995]. Let $f : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be a nonconstant K-morphism. Then f has (*simple*) *good reduction outside S* if there exists an R_S-morphism $\mathbb{P}^N_{R_S} \to \mathbb{P}^N_{R_S}$ whose generic fiber is $\mathrm{PGL}_{N+1}(K)$-conjugate to f.

If f has simple good reduction outside S, and if $\varphi \in \mathrm{PGL}_{N+1}(R_S)$, then it is clear that the conjugate map

$$f^{\varphi} := \varphi^{-1} \circ f \circ \varphi : \mathbb{P}^N_K \to \mathbb{P}^N_K$$

also has simple good reduction. But even modulo this equivalence, it is easy to see that a dynamical analogue of Shafarevich's conjecture using simple good reduction is false. For example, every map $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ of the form

$$f(X, Y) = [X^d + a_1 X^{d-1} Y + \cdots + a_d Y^d, Y^d] \quad \text{with } a_i \in R_S$$

has simple good reduction outside S, and these maps represent infinitely many $\mathrm{PGL}_2(R_S)$-conjugacy classes. And as noted in [Morton and Silverman 1995, Example 4.1], there are also infinite nonpolynomial families such as

$$[aX^2 + bXY + Y^2, X^2] \quad \text{with } a, b \in R_S.$$

It is thus of interest to formulate alternative definitions of good reduction for which a Shafarevich conjecture might hold in the dynamical setting. The literature contains several papers [Petsche 2012; Petsche and Stout 2015; Stout 2014; Szpiro and Tucker 2008] along these lines. We refer the reader to Section 2 for a description of these earlier results and a comparison with the present paper.

Our approach is to study pairs consisting of a map f and a set of points $Y \in \mathbb{P}^N$ such that the map $f : Y \to f(Y)$ "does not collapse" when it is reduced modulo $\mathfrak{p}$ for primes not in S; see Remark 5 for a discussion of why this is a natural analogue of the classical Shafarevich–Faltings result. To make this precise, we need to define good reduction for sets of points.

Definition. Let $X \subset \mathbb{P}^N(\bar{K})$ be a finite $\mathrm{Gal}(\bar{K}/K)$-invariant subset, say $X = \{P_1, \ldots, P_n\}$. Then X has *good reduction outside S* if for every prime $\mathfrak{p} \notin S$, and every prime $\mathfrak{P}$ of $K(P_1, \dots, P_n)$ lying over $\mathfrak{p}$, the reduction map[1]

$$X \to \tilde{X} \bmod \mathfrak{P} \quad \text{is injective.}$$

We observe that good reduction is preserved by the natural action of $\mathrm{PGL}_{N+1}(R_S)$ on $\mathbb{P}^N(\bar{K})$.

Our dynamical analogue of the Shafarevich–Faltings theorem is a statement about triples (f, Y, X) consisting of a morphism f and sets of points that have good reduction. We restrict attention to $\mathbb{P}^1$, since this is the setting for which we are currently able to prove a strong Shafarevich-type theorem; but see Section 8 for a brief discussion of possible extensions to $\mathbb{P}^N$ and why the naive generalization fails.

Definition. We define $\mathcal{GR}_d^1[n](K, S)$ to be the set of triples (f, Y, X), where $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ is a degree d morphism defined over K and $Y \subseteq X \subset \mathbb{P}^1(\bar{K})$ are finite sets, satisfying the following conditions:

- $X = Y \cup f(Y)$,
- X is $\mathrm{Gal}(\bar{K}/K)$-invariant,
- $\sum_{P \in Y} e_f(P) = n$, where $e_f(P)$ is the ramification index of f at P,
- f and X have good reduction outside S.

We also define a potentially larger set $\widetilde{\mathcal{GR}}_d^1[n](K, S)$ by dropping the requirement that f has good reduction. We observe that if $Y = X$, then the points in X have finite f-orbits, in which case we say that (f, X, X) is a *preperiodic triple*.

There is a natural action of $\mathrm{PGL}_2(R_S)$ on $\mathcal{GR}_d^1[n](K, S)$ and on $\widetilde{\mathcal{GR}}_d^1[n](K, S)$ given by

$$\varphi \cdot (f, Y, X) := \bigl(f^\varphi, \varphi^{-1}(Y), \varphi^{-1}(X)\bigr).$$

Our dynamical Shafarevich-type theorem for $\mathbb{P}^1$ says that if n is sufficiently large, then $\widetilde{\mathcal{GR}}_d^1[n](K, S)$ has only finitely many $\mathrm{PGL}_2(R_S)$-orbits.

Theorem 2 (dynamical Shafarevich theorem for $\mathbb{P}^1$). *Let $d \geq 2$.*

(a) *Let $K/\mathbb{Q}$ be a number field, and let S be a finite set of places of K. Then for all $n \geq 2d + 1$, the set*

$$\widetilde{\mathcal{GR}}_d^1[n](K, S)/\mathrm{PGL}_2(R_S) \text{ is finite.}$$

[1]In scheme-theoretic terms, the set X is a reduced 0-dimensional K-subscheme of $\mathbb{P}^N_K$. Let $\mathcal{X} \subset \mathbb{P}^N_{R_S}$ be the scheme-theoretic closure of X. Then X has good reduction outside S if $\mathcal{X}$ is étale over R_S.

(b) *Let S be the set of rational primes less than $2d-2$. Then*

$$\mathcal{GR}_d^1[2d](\mathbb{Q}, S)/\operatorname{PGL}_2(\mathbb{Z}_S) \textit{ is infinite.}$$

Indeed, there are infinitely many $\operatorname{PGL}_2(\mathbb{Z}_S)$-equivalence classes of preperiodic triples (f, X, X) in $\mathcal{GR}_d^1[2d](\mathbb{Q}, S)$.

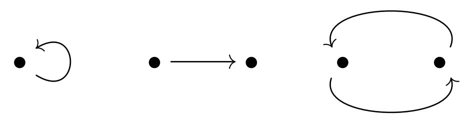
Proof. See Section 3 for the proof of (a), and Section 4, specifically Proposition 11, for the proof of (b). $\square$

In some sense, Theorem 2 is the end of the story for $\mathbb{P}^1$, since it says:

"The dynamical Shafarevich conjecture is true for sets of weight at least $2d+1$, but it is not true for sets of smaller weight."

However, rather than merely specifying the total weight, we might consider the weighted graph structure that $f : Y \to X$ imposes on X, where each point $P \in Y$ is assigned an outgoing arrow $P \to f(P)$ of weight $e_f(P)$. In dynamical parlance, we want to classify triples (f, Y, X) according to their *portrait structure*.[2] The following example of an (unweighted) portrait illustrates the general idea:

$\mathcal{P}$:



A model for this portrait $\mathcal{P}$ is a triple (f, Y, X) with $Y = \{P_1, P_2, P_4, P_5\}$ and $X = \{P_1, P_2, P_3, P_4, P_5\}$ satisfying

- P_1 is a fixed point of f,
- $f(P_2) = P_3$,
- P_4 and P_5 form a periodic 2-cycle for f.

If each point $P \in \mathcal{P}$ is assigned a weight $\epsilon(P)$, then we might further require that $e_f(P) = \epsilon(P)$, although there are other natural possibilities. Indeed, we consider three ways to define good reduction for dynamical systems and weighted portraits. We start with the largest set and work our way down:

Definition. Let $\mathcal{P}$ be a weighted portrait. We define $\mathcal{GR}_d^1[\mathcal{P}](K, S)$ to be the set of triples (f, Y, X), where $f : \mathbb{P}_K^1 \to \mathbb{P}_K^1$ is a degree d morphism defined over K and $Y \subseteq X \subset \mathbb{P}^1(\bar{K})$ are finite sets, satisfying the conditions

- $X = Y \cup f(Y)$ and $f : Y \to X$ looks like $\mathcal{P}$ (ignoring the weights),
- X is $\operatorname{Gal}(\bar{K}/K)$-invariant,
- f and X have good reduction outside S.

[2]Portrait structures, especially on critical point orbits, are important tools in the study of complex dynamics on $\mathbb{P}^1(\mathbb{C})$; see for example [Arfeux 2016].

We then define three subsets of $\mathcal{GR}_d^1[\mathcal{P}](K,S)$ by imposing the following additional conditions on the triple (f, Y, X) that reflect the weights assigned by $\mathcal{P}$:[3]

$$\begin{aligned}
&\mathcal{GR}_d^1[\mathcal{P}]^\bullet(K,S)\colon\ e_f(P) \ge \epsilon(P) \text{ for all } P \in Y,\\
&\mathcal{GR}_d^1[\mathcal{P}]^\circ(K,S)\colon\ e_f(P) = \epsilon(P) \text{ for all } P \in Y,\\
&\mathcal{GR}_d^1[\mathcal{P}]^\star(K,S)\colon\ e_{\tilde f}(\tilde P \bmod \mathfrak{p}) = \epsilon(P) \text{ for all } P \in Y \text{ and all } \mathfrak{p} \notin S.
\end{aligned}$$

We refer the reader to Section 6 for rigorous definitions of portraits, both weighted and unweighted, and their models. See also the companion paper [Doyle and Silverman $\ge$ 2018], in which we construct parameter spaces and moduli spaces for dynamical systems with portraits via geometric invariant theory and study some of their geometric and arithmetic properties.

This leads to fundamental questions:

Question 3. For a given $d \ge 2$, classify the portraits $\mathcal{P}$ having the property that for all number fields K and all finite sets of places S, the set

$$\mathcal{GR}_d^N[\mathcal{P}]^x(K,S)/\operatorname{PGL}_2(R_S) \text{ is finite,} \quad \text{where } x \in \{\bullet, \circ, \star\}.$$

If $\mathcal{P}$ has this property, then we say that $\mathcal{P}$ is an (x,d)*-Shafarevich portrait*, or that (x,d)*-Shafarevich finiteness holds for* $\mathcal{P}$.

For example, Theorem 2(a) says that if the total weight of the points in $\mathcal{P}$ is at least $2d+1$, then $(\bullet, d)$-Shafarevich finiteness holds for $\mathcal{P}$. This is quite satisfactory. But the converse result, which is Theorem 2(b), says only that there exists at least one portrait of total weight $2d$ such that $(\bullet, d)$-Shafarevich finiteness fails for $\mathcal{P}$. It says nothing about the full set of such portraits. And indeed, we will prove that among the many portraits of total weight 4, $(\bullet, 2)$-Shafarevich finiteness holds for some and not for others! Thus the answer to Question 3 appears to be fairly subtle for portraits of weight at most $2d$.

In those cases that $\mathcal{GR}_d^1[\mathcal{P}]^x(K,S)$ is infinite, we might ask for a more refined measure of its size. This is provided by looking at its image in the moduli space $\mathcal{M}_d^1$, where $\mathcal{M}_d^1 := \operatorname{End}_d^1 /\!/ \operatorname{SL}_2$ is the moduli space of dynamical systems of degree d morphisms on $\mathbb{P}^1$. (See [Milnor 1993; Silverman 1998] for the construction of $\mathcal{M}_d^1$, and [Levy 2011; Petsche et al. 2009] for an analogous construction for $\mathbb{P}^N$.) This prompts the following definition.

Definition. Let $d \ge 2$, let $x \in \{\bullet, \circ, \star\}$, and let $\mathcal{P}$ be a portrait. The (x,d)-*Shafarevich dimension of* $\mathcal{P}$ is the quantity

$$\operatorname{ShafDim}_d^1[\mathcal{P}]^x = \sup_{\substack{K \text{ a number field}\\ S \text{ a finite set of places}}} \dim \overline{\operatorname{Image}\big(\mathcal{GR}_d^1[\mathcal{P}]^x(K,S) \to \mathcal{M}_d^1\big)},$$

where the overline denotes the Zariski closure.

[3] We note that $\star$-good reduction was first defined and studied by Petsche and Stout [2015], specifically for $d = 2$ and $\mathcal{P}$ consisting of two fixed points or one 2-cycle.

$\mathcal{P}$	(1, 2)-Shafarevich finiteness true?
$\bullet \longrightarrow \bullet\circlearrowleft \quad \bullet\circlearrowleft \quad \bullet\circlearrowleft$	yes
$\bullet \longrightarrow \bullet\circlearrowleft \quad \bullet \longrightarrow \bullet\circlearrowleft$	no
$\bullet \longrightarrow \bullet \longrightarrow \bullet\circlearrowleft \quad \bullet\circlearrowleft$	yes
$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet\circlearrowleft$	yes
$\bullet\circlearrowleft \quad \bullet\circlearrowleft \quad \bullet \rightleftarrows \bullet$	yes
$\bullet \longrightarrow \bullet\circlearrowleft \quad \bullet \rightleftarrows \bullet$	no
$\bullet\circlearrowleft \quad \bullet \longrightarrow \bullet \rightleftarrows \bullet$	yes
$\bullet \longrightarrow \bullet \longrightarrow \bullet \rightleftarrows \bullet$	yes

Table 1. Some weight 4 portraits for degree 2 maps.

By definition, we have

$$\mathcal{P} \text{ is a } (x, d)\text{-Shafarevich portrait} \quad \Longrightarrow \quad \operatorname{ShafDim}_d^1[\mathcal{P}]^x = 0.$$

A natural generalization of Question 3 is to ask for a formula (or algorithm, or …) for $\operatorname{ShafDim}_d^1[\mathcal{P}]^x$ as a function of $\mathcal{P}$.

In this paper we start to answer this refined question by performing an exhaustive computation of $\operatorname{ShafDim}_2^1[\mathcal{P}]^x$ for preperiodic portraits of weight up to 4, since Theorem 2(a) says that the dimension is 0 for portraits whose weight is strictly greater than 4.

To partially illustrate the complete results that are given in Section 7, we refer the reader to Table 1. This table lists eight preperiodic portraits of weight 4 that arise for degree 2 maps of $\mathbb{P}^1$. For six of them, the $(\bullet, 2)$-Shafarevich finiteness property holds, while for two of them it does not. It is not clear (to this author) how to distinguish this dichotomy directly from the geometry of the portraits, other than by performing a detailed analysis. It turns out that there are 34 possible portraits of weight at most 4 for degree 2 maps of $\mathbb{P}^1$. See Section 7 for an analysis of all 34 portraits and a computation of their various Shafarevich dimensions.

We can also turn the question around by fixing $\mathcal{P}$ and letting $d \to \infty$. We note that the Shafarevich dimension is never more than $\dim \mathcal{M}_d^1 = 2d - 2$.

Question 4. For a given unweighted portrait $\mathcal{P}$, what is the limiting behavior of the *Shafarevich discrepancy*[4]

$$2d - 2 - \operatorname{ShafDim}_d^1[\mathcal{P}] \quad \text{as } d \to \infty?$$

We note that Question 4 is quite interesting even for $\mathcal{P} = \varnothing$. We will show in Proposition 12 that

$$d \le \operatorname{ShafDim}_d^1[\varnothing] \le 2d - 2.$$

This gives the exact value for $d = 2$, a result that is also proven in [Petsche and Stout 2015] using a slightly different argument.

Remark 5. Returning to the case of abelian varieties for motivation and inspiration, we note that an abelian variety is really a pair $(A, \mathcal{O})$ consisting of a variety and a marked point. As noted by Petsche and Stout [2015], if we discard the marked point, then Shafarevich finiteness is no longer true. For example, there may be infinitely many K-isomorphism classes of curves of genus 1 having good reduction outside S. Hence in order to prove Shafarevich finiteness for a collection of geometric object (varieties, maps, etc.), it is very natural to add level structure in the form of one or more points. We also remark that if we add further level structure to an abelian variety, for example specifying an n-torsion point Q, then an ostensibly stronger form of good reduction would require that the points Q and $\mathcal{O}$ remain distinct modulo the primes not in S. But if we enlarge S so that $n \in R_S^*$, then the two forms of good reduction are actually identical due to the standard result on injectivity of torsion under reduction; cf. [Hindry and Silverman 2000, Theorem C.1.4] or [Mumford 1970, Appendix II, Corollary 1]. To make the dynamical analogy complete, we note that torsion points are exactly the points of A that are preperiodic for the doubling map.

2. Earlier results

It has long been realized that dynamical Shafarevich finiteness does not hold for morphisms $f : \mathbb{P}^1 \to \mathbb{P}^1$ if the definition of good reduction is simple good reduction; cf. [Morton and Silverman 1995, Example 4.1]. This has led a number of authors to impose additional good reduction conditions on f and to prove a variety of finiteness theorems. We briefly mention a few of these results.

[4] If $\mathcal{P}$ has weights ϵ, it is more natural to consider the quantity $2d - 2 - \sum_{P \in Y} \bigl(\epsilon(P) - 1\bigr) - \operatorname{ShafDim}_d^1[\mathcal{P}]^x$ for $x \in \{\bullet, \circ, \star\}$.

Closest in spirit to the present paper is work of Petsche and Stout [2015] in which they study good reduction of degree 2 maps of $\mathbb{P}^1$. They define (with similar notation) the sets that we've denoted by $\mathcal{GR}_d^1(K,S)[\varnothing]$ and they pose the question of whether the maps in this set are Zariski dense in the moduli space $\mathcal{M}_d^1$. They prove that this is true for $d=2$, which is a special case of our Proposition 12. They also study maps with $\star$-good reduction relative to various portraits, i.e., the sets $\mathcal{GR}_d^1[\mathcal{P}]^\star$ defined earlier. For example, they prove that $\operatorname{ShafDim}_2^1[\mathcal{P}]^\star = 1$ when $\mathcal{P}$ is a portrait consisting of two unramified fixed points, and similarly when $\mathcal{P}$ is a portrait consisting of a single unramified 2-cycle. (These are the portraits labeled $\mathcal{P}_{2,3}$ and $\mathcal{P}_{2,4}$ in Table 2.) We will show later that $\operatorname{ShafDim}_2^1[\mathcal{P}]^\circ = 1$ and $\operatorname{ShafDim}_2^1[\mathcal{P}]^\bullet = 2$ for these two portraits. More generally, in Section 7 we compute the three Shafarevich dimensions for the 34 preperiodic portraits of weight at most 4 for degree 2 maps of $\mathbb{P}^1$.

Other approaches to a dynamical Shafarevich conjecture also consider pairs (f,X) or triples (f,Y,X) of maps and points, but impose different function-theoretic constraints. Thus in [Szpiro and Tucker 2008; Szpiro et al. 2017; Szpiro and West 2017], maps are classified according to what Szpiro characterizes as "differential good reduction". For a given map $f:\mathbb{P}^1\to\mathbb{P}^1$, let $\mathcal{R}(f)$ denote the set of ramified points of f and let $\mathcal{B}(f)=f(\mathcal{R}(f))$ denote the set of branch points.[5]

Definition. The map f has *critical good reduction* outside S if each of the sets $\mathcal{R}(f)$ and $\mathcal{B}(f)$ has good reduction outside S. The map f has *critical excellent reduction* outside S if the union $\mathcal{R}(f)\cup\mathcal{B}(f)$ has good reduction outside S.

Canci, Peruginelli, and Tossici [Canci et al. 2013] prove that f has critical good reduction if and only if f has simple good reduction and the branch locus $\mathcal{B}(f)$ has good reduction.

Theorem 6 [Szpiro et al. 2017; Szpiro and West 2017]. *Fix a number field K, a finite set of places S, and an integer $d\ge 2$. Then up to* $\mathrm{PGL}_2(K)$*-conjugacy, there are only finitely many degree d maps $f:\mathbb{P}_K^1\to\mathbb{P}_K^1$ that are ramified at* 3 *or more points and have critically good reduction outside S.*

Theorem 6 of Szpiro, Tucker, and West fits into the framework of our Theorem 2, since their maps f correspond to triples

$$\bigl(f,\mathcal{R}(f),\mathcal{R}(f)\cup\mathcal{B}(f)\bigr)\in\widetilde{\mathcal{GR}}_d^1[n](K,S),$$

where

$$n=\sum_{P\in\mathcal{R}(f)} e_f(P)=\sum_{P\in\mathcal{R}(f)}\bigl(e_f(P)-1\bigr)+\#\mathcal{R}(f)=2d-2+\#\mathcal{R}(f).$$

[5]In dynamical terminology, $\mathcal{R}(f)$ is the set of *critical points* and $\mathcal{B}(f)$ is the set of *critical values.*

If we assume that $\#\mathcal{R}(f) \geq 3$ as in Theorem 6, then $n \geq 2d+1$, so we see that Theorem 6 follows from Theorem 2(a).

The proof of Theorem 6 in [Szpiro et al. 2017; Szpiro and West 2017] uses a finiteness result for sets of points in $\mathbb{P}^1(K)$ having good reduction outside S, similar to our Lemmas 7 and 8, which in turn rely on classical results of Hermite and Minkowski together with the finiteness of solutions to the S-unit equation. The other ingredient used by Szpiro, Tucker, and West in their proof of Theorem 6 is a special case of a theorem of Grothendieck that computes the tangent space of the parameter scheme of morphisms. We remark that [Szpiro et al. 2017; Szpiro and West 2017; Szpiro and Tucker 2008] also deal with the case of function fields, which can present additional complications.

The earlier paper of Szpiro and Tucker [2008] proved a result similar to Theorem 6, but with a two-sided conjugation equivalence relation, i.e., f_1 and f_2 are considered equivalent if there are maps $\varphi, \psi \in \mathrm{PGL}_2$ such that $f_2 = \psi \circ f_1 \circ \varphi$. This equivalence relation, while interesting, is not well suited for studying dynamics.

There is an article of Stout [2014] in which he proves that for a fixed rational map f, there are only finitely many $\bar{K}/K$ twists of f having simple good reduction outside of S. And a paper of Petsche [2012] proves a Shafarevich finiteness theorem for certain families of critically separable maps, which he defines to be maps f of degree $d \geq 2$ such that for every prime not in S, the reduced map has $2d-2$ distinct critical points. In other words, $\#\mathcal{R}(f) = 2d-2$ and $\mathcal{R}(f)$ has good reduction outside S. This is not enough to obtain finiteness, so Petsche restricts to certain codimension 3 families in Rat_d^1 that are modeled after Lattès maps, and he proves that the dynamical Shafarevich conjecture holds for these families.

A number of authors have studied the resultant equation $\mathrm{Res}(F, G) = c$, where the coefficients of F and G are viewed as indeterminates [Evertse and Győry 1993; Győry 1990; 1993]. Taking c to be an S-unit, this is clearly related to the question of simple good reduction of the map $f = [F, G] \in \mathrm{End}_d^1$. Rephrasing the results in our notation,[6] Evertse and Győry [1993, Corollary 1] prove that up to $\mathrm{PGL}_2(R_S)$-equivalence, there are only finitely many $f = [F, G] \in \mathrm{End}_d^1$ having the property that FG is square-free and splits completely over K. Alternatively, their conditions may be phrased in terms of f as requiring that 0 and ∞ are not critical values of f and that the points in $f^{-1}(0) \cup f^{-1}(\infty)$ are in $\mathbb{P}^1(K)$, and their conclusion is that Shafarevich finiteness is true for this collection of maps. We note that the condition that $f^{-1}(0) \cup f^{-1}(\infty) \subset \mathbb{P}^1(K)$ means, more or less, that the maps in question correspond to S-integral points on a $2d$-to-1 finite cover of an open subset of End_d^1.

Finally, we mention two topics that seem at least tangentially related. There are a number of papers that fix a map f and a wandering point P and ask which portraits

[6] We have restricted to the case that $\deg(F) = \deg(G)$, although the cited papers do not require this.

arise when one reduces the orbit of P modulo various primes; see for example [Faber and Granville 2011; Ghioca et al. 2015]. And there are two articles of Doyle [2016; 2018] in which he classifies periodic point portraits that are permitted for unicritical polynomials, i.e., polynomials of the form $ax^d + b$. These results could be useful in studying the geometry and arithmetic of our portrait moduli spaces studied in [Doyle and Silverman $\ge$ 2018].

3. Dynamical Shafarevich finiteness holds on $\mathbb{P}^1$ for weight $\ge 2d+1$

In this section we prove Theorem 2(a); namely we prove that the dynamical Shafarevich finiteness holds for maps f of $\mathbb{P}^1$ and f-invariant sets X of weight at least $2d+1$. The first step is to show that there are only finitely many choices for the set X.

Definition. Let K be a number field, let S be a finite set of places including all archimedean places, and let $n \ge 1$ be an integer. We define $\mathcal{X}[n](K, S)$ to be the collection of subsets $X \subset \mathbb{P}^1(\bar{K})$ satisfying

- $\#X = n$,
- X is $\operatorname{Gal}(\bar{K}/K)$-invariant,
- X has good reduction outside S.

We note that if $\varphi \in \mathrm{PGL}_2(R_S)$ and $X = \{P_1, \dots, P_n\} \in \mathcal{X}[n](K, S)$, then

$$\varphi(X) := \{\varphi(P_1), \dots, \varphi(P_n)\} \in \mathcal{X}[n](K, S), \tag{1}$$

so there is a natural action of $\mathrm{PGL}_2(R_S)$ on $\mathcal{X}[n](K, S)$. More generally, we use (1) to define an action of $\mathrm{PGL}_2(\bar{K})$ on n-tuples of points in $\mathbb{P}^1(\bar{K})$.

The following lemma is well known, but for lack of a suitable reference and as a convenience to the reader, we include the proof.

Lemma 7. *Fix a number field K, a finite set of places S including all archimedean places, and an integer $n \ge 3$. Then*

$$\mathcal{X}[n](K, S)/\,\mathrm{PGL}_2(R_S)$$

is finite.

We start with a sublemma that will allow us to restrict attention to set of points defined over a single field K.

Sublemma 8. *Let K be a number field, let S be a finite set of places including all archimedean places, and let $n \ge 3$ be an integer. Then there is a constant $C(K, S, n)$ such the map*

$$\mathcal{X}[n](K, S)/\,\mathrm{PGL}_2(R_S) \to \{X \subset \mathbb{P}^1(\bar{K}) : \#X = n\}/\,\mathrm{PGL}_2(\bar{K})$$

is at most $C(K, S, n)$-to-1.

Proof. Let $X = \{P_1, \dots, P_n\} \in \mathcal{X}[n](K, S)$. The fact that X is Galois invariant implies that the field

$$K_X := K(P_1, \dots, P_n)$$

is a Galois extension of degree dividing $n!$. Further, the good reduction assumption on X implies that K_X/K is unramified outside S. It follows from the Hermite–Minkowski theorem [Neukirch 1999, Section III.2] that there are only finitely many possibilities for the field K_X.[7] It follows that the field

$$K' := \prod_{X \in \mathcal{X}[n](K,S)} K_X \tag{2}$$

is a finite Galois extension of K that depends only on K, S, and n.

We now fix an n-tuple $X_0 \in \mathcal{X}[n](K, S)$, say $X_0 = \{Q_1, \dots, Q_n\}$, and consider the set of n-tuples in $\mathcal{X}[n](K, S)$ that are $\mathrm{PGL}_2(\bar{K})$-equivalent to X_0. Our goal is to prove that the set

$$\mathrm{PGL}_2(K, S, X_0) := \{\varphi \in \mathrm{PGL}_2(\bar{K}) : \varphi(X_0) \in \mathcal{X}[n](K, S)\}$$

has the property that $\mathrm{PGL}_2(K, S, X_0)/\mathrm{PGL}_2(R_S)$ is finite and has order bounded solely in terms of K, S, and n.

Our first observation is that if $\varphi \in \mathrm{PGL}_2(K, S, X_0)$, then in particular we have $Q_i \in \mathbb{P}^1(K')$ and $\varphi(Q_i) \in \mathbb{P}^1(K')$ for all $1 \le i \le n$, where K' is the field (2). A fractional linear transformation is determined by its values at three points, so our assumption that $n \ge 3$ tells us that $\varphi \in \mathrm{PGL}_2(K')$, i.e., every $\varphi \in \mathrm{PGL}_2(K, S, X_0)$ is defined over the finite extension K' of K, where K' does not depend on X_0.

Next let S' be the places of K' lying over S. The good reduction assumption on X_0 and $\varphi(X_0)$ implies that $Q_1, \dots, Q_n$ remain distinct modulo all primes $\mathfrak{P}$ of L with $\mathfrak{P} \notin S'$, and similarly for $\varphi(Q_1), \dots, \varphi(Q_n)$. Since $n \ge 3$, we can apply the following elementary result to conclude that φ has simple good reduction at $\mathfrak{P}$, and since this holds for all $\mathfrak{P} \notin S'$, we see that $\varphi \in \mathrm{PGL}_2(R_{S'})$.

Sublemma 9. *Let R be a discrete valuation ring with maximal ideal $\mathfrak{P}$ and fraction field K. Let $P_1, P_2, P_3 \in \mathbb{P}^1(K)$ be points whose reductions modulo $\mathfrak{P}$ are distinct, and let $Q_1, Q_2, Q_3 \in \mathbb{P}^1(K)$ also be points with distinct mod $\mathfrak{P}$ reductions. Let $\varphi \in \mathrm{PGL}_2(K)$ be the unique linear fractional transformation satisfying $\varphi(P_i) = Q_i$ for $1 \le i \le 3$. Then $\varphi \in \mathrm{PGL}_2(R)$, i.e., φ has good reduction modulo $\mathfrak{P}$.*

Proof. The fact that the reductions of P_1, P_2, P_3 are distinct means that we can find a linear fractional transformation $\psi \in \mathrm{PGL}_2(R)$ satisfying $\psi(P_1) = 0$, $\psi(P_2) = 1$, $\psi(P_3) = \infty$. Similarly, we can find a $\lambda \in \mathrm{PGL}_2(R)$ satisfying $\lambda(Q_1) = 0$, $\lambda(Q_2) = 1$,

[7] More precisely, our assumptions imply that for $\mathfrak{p} \notin S$, we have $\mathrm{ord}_{\mathfrak{p}} \mathfrak{D}_{L/K} = 0$, while for all primes $\mathfrak{p}$ one has the standard estimate $\mathrm{ord}_{\mathfrak{p}} \mathfrak{D}_{L/K} \le [L : K] - 1$. This proves that $\mathsf{N}_{L/K}\mathfrak{D}_{L/K}$ is bounded, and then for a fixed K, Hermite–Minkowski says that there are only finitely many L.

$\lambda(Q_3) = \infty$. Then $\lambda \circ \varphi \circ \psi^{-1}$ fixes 0, 1, and ∞, so it is the identity map. Hence $\varphi = \lambda^{-1} \circ \psi \in \mathrm{PGL}_2(R)$. □

We next observe that if $\varphi \in \mathrm{PGL}_2(K, S, X_0)$, then by definition and from what we proved earlier, both of the sets X_0 and $\varphi(X_0)$ are composed of points in $\mathbb{P}^1(K')$ and both are $\mathrm{Gal}(K'/K)$-invariant. Hence for any $\sigma \in \mathrm{Gal}(K'/K)$, we find that

$$\varphi(X_0) = (\varphi(X_0))^\sigma = \varphi^\sigma(X_0^\sigma) = \varphi^\sigma(X_0).$$

Thus $\varphi^{-1} \circ \varphi^\sigma : X_0 \to X_0$, i.e., the map $\varphi^{-1} \circ \varphi^\sigma$ is a permutation of the set X_0. We thus obtain a map

$$\mathrm{PGL}_2(K, S, X_0) \to \mathrm{Map}_{\mathsf{Set}}(\mathrm{Gal}(K'/K), \mathcal{S}_{X_0}),$$
$$\varphi \mapsto (\sigma \mapsto \varphi^{-1} \circ \varphi^\sigma),$$

where $\mathcal{S}_{X_0}$ denotes the group of permutations of the set X_0. (The map $\sigma \mapsto \varphi^{-1} \circ \varphi^\sigma$ is actually some sort of cocycle, but that is irrelevant for our purposes.) Since $\mathrm{Gal}(K'/K)$ and $\mathcal{S}_{X_0}$ are both finite and have order bounded in terms of K, S, and n, it suffices to fix some $\varphi_0 \in \mathrm{PGL}_2(K, S, X_0)$ and to bound the number of $\mathrm{PGL}_2(R_S)$ equivalence classes of maps $\varphi \in \mathrm{PGL}_2(K, S, X_0)$ that have the same image in $\mathrm{Map}_{\mathsf{Set}}(\mathrm{Gal}(K'/K), \mathcal{S}_{X_0})$. This means that for all $\sigma \in \mathrm{Gal}(K'/K)$, the maps $\varphi^{-1} \circ \varphi^\sigma = \varphi_0^{-1} \circ \varphi_0^\sigma$ have the same effect on X_0; and since $\#X_0 = n \geq 3$ and linear fractional transformations are determined by their values on three points, it follows that $\varphi^{-1} \circ \varphi^\sigma = \varphi_0^{-1} \circ \varphi_0^\sigma$ as elements of $\mathrm{PGL}_2(K')$. Thus

$$\varphi \circ \varphi_0^{-1} = \varphi^\sigma \circ (\varphi_0^\sigma)^{-1} = (\varphi \circ \varphi_0^{-1})^\sigma \quad \text{for all } \sigma \in \mathrm{Gal}(K'/K).$$

Hence $\varphi \circ \varphi_0^{-1} \in \mathrm{PGL}_2(K)$. But we also know that φ_0 and φ are in $\mathrm{PGL}_2(R_{S'})$, so

$$\varphi \circ \varphi_0^{-1} \in \mathrm{PGL}_2(K) \cap \mathrm{PGL}_2(R_{S'}).$$

It remains to show that

$$\mathrm{PGL}_2(K) \cap \mathrm{PGL}_2(R_{S'}) = \mathrm{PGL}_2(R_S), \tag{3}$$

since that will show that up to composition with elements of $\mathrm{PGL}_2(R_S)$, there are only finitely many choices for φ. In order to prove (3), we start with some $\psi \in \mathrm{PGL}_2(K) \cap \mathrm{PGL}_2(R_{S'})$. Then for each prime $\mathfrak{p} \notin S$, we need to show that ψ has good reduction at $\mathfrak{p}$. We write ψ in normalized form as

$$\psi(X, Y) = [aX + bY, cX + dY] \quad \text{with } a, b, c, d \in K \text{ and} \tag{4}$$
$$\min\{\mathrm{ord}_\mathfrak{p}(a), \mathrm{ord}_\mathfrak{p}(b), \mathrm{ord}_\mathfrak{p}(c), \mathrm{ord}_\mathfrak{p}(d)\} = 0,$$

i.e., a, b, c, d are all $\mathfrak{p}$-integral, and at least one of them is a $\mathfrak{p}$-unit. Now let $\mathfrak{P}$ be a prime of K' lying above $\mathfrak{p}$. We are given that ψ has good reduction at $\mathfrak{P}$, which means that if we choose a $\mathfrak{P}$-normalized equation for ψ, its reduction modulo $\mathfrak{P}$

has good reduction. But (4) is already normalized for $\mathfrak{P}$, since $\operatorname{ord}_{\mathfrak{P}} = e(\mathfrak{P}/\mathfrak{p}) \operatorname{ord}_{\mathfrak{p}}$. Hence

$$ad - bc \text{ is a } \mathfrak{P}\text{-adic unit.}$$

But $ad - bc \in K$, so $ad - bc$ is a $\mathfrak{p}$-adic unit, and hence ψ has good reduction at $\mathfrak{p}$. This holds for all $\mathfrak{p} \notin S$, which completes the proof that $\psi \in \mathrm{PGL}_2(R_S)$, and thus completes the proof of Sublemma 8. □

Proof of Lemma 7. Let L/K be a finite Galois extension, and let T be a finite of places of L whose restriction to K contains S. Then we get a natural map

$$\mathcal{X}[n](K, S)/\mathrm{PGL}_2(R_S) \to \mathcal{X}[n](L, T)/\mathrm{PGL}_2(R_T), \tag{5}$$

since if $X \subset \mathbb{P}^1(\bar{K})$ is $\mathrm{Gal}(\bar{K}/K)$ invariant and has good reduction outside S, it is clear that X is also $\mathrm{Gal}(\bar{L}/L)$ invariant and has good reduction outside T. However, what is not clear *a priori* is that the map (5) is finite-to-one, since $\mathrm{PGL}_2(R_T)$ may be larger than $\mathrm{PGL}_2(R_S)$.

However Sublemma 8 says not only that the map

$$\mathcal{X}[n](K, S)/\mathrm{PGL}_2(R_S) \to \{X \subset \mathbb{P}^1(\bar{K}) : \#X = n\}/\mathrm{PGL}_2(\bar{K})$$

is finite-to-one, but it also says that the number of elements in each $\mathrm{PGL}_2(\bar{K})$-equivalence class of $\mathcal{X}[n](K, S)/\mathrm{PGL}_2(R_S)$ is bounded solely in terms of K, S, and n. Hence using (5), it suffices to prove Lemma 7 for any such L and T.

As shown in the proof of Sublemma 8, there is a finite extension K'/K such that every $X \in \mathcal{X}[n](K, S)$ is an n-tuple of points in $\mathbb{P}^1(K')$. We then let S' be a finite set of places of K' such that S' restricted to K contains S and such that $R_{S'}$ is a PID. Replacing K and S with K' and S', we are reduced to studying the $\mathrm{PGL}_2(R_S)$-equivalence classes of the set of $X \in \mathcal{X}[n](K, S)$ such that

$$X = \{P_1, \dots, P_n\} \quad \text{with } P_1, \dots, P_n \in \mathbb{P}^1(K),$$

with the further condition that R_S is a PID. This allows us to choose normalized coordinates for the points in X, say

$$P_i = [a_i, b_i] \quad \text{with } a_i, b_i \in R_S \text{ and } \gcd_{R_S}(a_i, b_i) = 1.$$

The good reduction assumption says that $P_1, \dots, P_n$ are distinct modulo all primes not in S, which given our normalization of the coordinates of the P_i, is equivalent to the statement that

$$a_i b_j - a_j b_i \in R_S^* \quad \text{for all } 1 \le i < j \le n.$$

This means that we can find a linear fractional transformation $\varphi \in \mathrm{PGL}_2(R_S)$ that moves the first three points in our list to the points

$$\varphi(P_1) = [1, 0], \quad \varphi(P_2) = [0, 1], \quad \varphi(P_3) = [1, 1].$$

Replacing X by $\varphi(X)$, the remaining points in X are S-integral points of the scheme

$$\mathbb{P}^1_{R_S} \smallsetminus \{[1,0],[0,1],[1,1]\}, \tag{6}$$

and it is well known that there are only finitely many such points. More precisely, a normalized point $P=[a,b]$ is an S-integral point of the scheme (6) if and only if a, b, and $a-b$ are S-units. But this implies that $\left(\frac{a}{a-b},\frac{b}{b-a}\right)$ is a solution to the S-unit equation $U+V=1$, and hence that there are only finitely many values for each of $\frac{a}{a-b}$ and $\frac{b}{b-a}$ [Silverman 2009, IX.4.1]. Further, each S-unit solution (u,v) to $u+v=1$ gives one point $P=[a,b]=[u,-v]$. This concludes the proof that there are only finitely many $\mathrm{PGL}_2(R_S)$-equivalence classes of sets X having n elements and good reduction outside S. □

The following geometric result is also undoubtedly well known, but for lack of a suitable reference and the convenience of the reader, we include the short proof.[8]

Lemma 10. *Let K be a field, and let $f,g:\mathbb{P}^1_K\to\mathbb{P}^1_K$ be rational maps of degree $d\ge 1$. Suppose that*

$$\sum_{\substack{P\in\mathbb{P}^1(K)\\ f(P)=g(P)}} \min\{e_f(P),e_g(P)\}\ge 2d+1.$$

Then $f=g$.

Proof. We may assume that K is algebraically closed. We fix a basepoint $P_0\in\mathbb{P}^1(K)$, and we take

$$H_1=\{P_0\}\times\mathbb{P}^1 \quad\text{and}\quad H_2=\mathbb{P}^1\times\{P_0\}$$

as generators for $\mathrm{Pic}(\mathbb{P}^1\times\mathbb{P}^1)\cong\mathbb{Z}\oplus\mathbb{Z}$. We consider the divisors

$$\begin{aligned}\Delta&=\{(P,P):P\in\mathbb{P}^1(K)\}\in\mathrm{Div}(\mathbb{P}^1\times\mathbb{P}^1),\\ \Gamma_{f,g}&=(f\times g)_*\Delta\in\mathrm{Div}(\mathbb{P}^1\times\mathbb{P}^1).\end{aligned}$$

We write $|\Delta|$ and $|\Gamma_{f,g}|$ for the supports of Δ and $\Gamma_{f,g}$, respectively, and we note that these supports are irreducible, since they are the images of $\mathbb{P}^1$ under, respectively, the diagonal map and the map $f\times g$.

We use the push-pull formula to compute the global intersection

$$\Gamma_{f,g}\cdot H_1=(f\times g)_*(\Delta)\cdot H_1=\Delta\cdot(f\times g)^*(H_1)=\Delta\cdot(f^*(P_0)\times\mathbb{P}^1)=d.$$

Similarly, we have $\Gamma_{f,g}\cdot H_2=d$. Hence

$$\Gamma_{f,g}=dH_1+dH_2\quad\text{in }\mathrm{Pic}(\mathbb{P}^1\times\mathbb{P}^1).$$

[8]Mike Zieve has pointed out that this lemma may also be proven by writing f and g as quotients of polynomials $f=f_1/f_2$ and $g=g_1/g_2$, and then analyzing the factorization of $f_1g_2-f_2g_1$.

This allows us to compute

$$\Gamma_{f,g} \cdot \Delta = dH_1 \cdot \Delta + dH_2 \cdot \Delta = 2d. \tag{7}$$

Choose some $P \in \mathbb{P}^1(K)$ satisfying $f(P) = g(P)$, and let z be a local uniformizer at P. We may assume that $z(f(P)) \neq \infty$, since otherwise we can replace z by $z/(z-1)$. By assumption we have $c := f(P) = g(P)$, so locally near P the functions f and g satisfy

$$f(z) \in c + az^{e_f(P)} + z^{e_f(P)+1} K[\![z]\!], \quad g(z) \in c + bz^{e_g(P)} + z^{e_g(P)+1} K[\![z]\!]$$

for some nonzero a and b. This allows us to estimate the following local intersection index:

$$\begin{aligned}\big((f \times g)_* \Delta \cdot \Delta\big)_{(f(P),g(P))} &= \dim_K \frac{K[\![x,y,z]\!]}{\big(x - f(z),\, y - g(z),\, x - y\big)} \\ &= \dim_K \frac{K[\![z]\!]}{\big(f(z) - g(z)\big)} \\ &\geq \min\{e_f(P), e_g(P)\}.\end{aligned} \tag{8}$$

Suppose that $|\Gamma_{f,g}| \cap |\Delta|$ is finite. Then we can calculate $\Gamma_{f,g} \cdot \Delta$ as a sum of local intersections. Combined with (8), this yields

$$\begin{aligned}2d &= \Gamma_{f,g} \cdot \Delta \quad \text{from (7)}, \\ &= \sum_{Q \in \mathbb{P}^1(K)} \big((f \times g)_* \Delta \cdot \Delta\big)_{(Q,Q)} \quad \text{since } |\Gamma_{f,g}| \cap |\Delta| \text{ is finite}, \\ &= \sum_{\substack{Q \in \mathbb{P}^1(K) \text{ such that} \\ \exists P \in \mathbb{P}^1(K) \text{ with } f(P) = g(P) = Q}} \big((f \times g)_* \Delta \cdot \Delta\big)_{(Q,Q)} \\ &\geq \sum_{\substack{P \in \mathbb{P}^1(K) \\ f(P) = g(P)}} \min\{e_f(P), e_g(P)\} \quad \text{from (8)}, \\ &\geq 2d + 1 \quad \text{by assumption}.\end{aligned}$$

Thus the assumption that $|\Gamma_{f,g}| \cap |\Delta|$ is finite leads to a contradiction. It follows that $|\Delta|$ and $|\Gamma_{f,g}|$ have a common positive dimensional component. But as noted earlier, both $|\Delta|$ and $|\Gamma_{f,g}|$ are irreducible curves, and hence $|\Delta| = |\Gamma_{f,g}|$. Thus f and g take on the same value at every point of $\mathbb{P}^1(K)$, and therefore $f = g$, which completes the proof of Lemma 10. □

We now have the tools needed to prove dynamical Shafarevich finiteness for $\mathbb{P}^1$.

Proof of Theorem 2(a). Our goal is to prove that

$$\widetilde{\mathcal{GR}}^1_d[n](K, S)/\operatorname{PGL}_2(R_S) \text{ is finite.}$$

Let $(f, Y, X) \in \widetilde{\mathcal{GR}}_d^1[n](K, S)$, and let $\ell = \#X$. We note that

$$2d + 1 \le n = \sum_{P \in Y} e_f(P) \le d \cdot \#Y \le d \cdot \#X = d\ell,$$

so $\ell \ge 3$. Further, the set X is $\mathrm{Gal}(\bar{K}/K)$-invariant and has good reduction outside of S. Lemma 7 tells us that up to $\mathrm{PGL}_2(R_S)$-equivalence, there are only finitely many possibilities for X. So without loss of generality, we may assume that $X = \{P_1, \ldots, P_\ell\}$ is fixed.

The set Y is subset of X, so there are only finitely many choices for Y. Relabeling the points in X, we may thus also assume that $Y = \{P_1, \ldots, P_m\}$ is fixed.

By definition, the map f satisfies $X = f(Y) \cup Y$, so in particular, $f(Y) \subset X$. Thus f induces a map

$$\nu_f : \{1, \ldots, m\} \to \{1, \ldots, \ell\} \quad \text{characterized by} \quad f(P_i) = P_{\nu_f(i)}.$$

There are only m^ℓ maps ν from the set $\{1, \ldots, m\}$ to the set $\{1, \ldots, \ell\}$, so again without loss of generality, we may fix one map ν and restrict attention to maps f satisfying $\nu_f = \nu$. This means that the value of f is specified at each of the points $P_1, \ldots, P_m$ in Y.

We define the map

$$\widetilde{\mathcal{GR}}_d^1[n](K, S) \to \mathbb{Z}^m, \quad (f, X) \mapsto \bigl(e_f(P_1), \ldots, e_f(P_m)\bigr).$$

Since $e_f(P)$ is an integer between 1 and d, there are only finitely many possibilities for the image. We may thus restrict attention to triples (f, Y, X) such that the ramification indices of f at the points in Y are fixed.

But now any two triples (f, Y, X) and (g, Y, X) have the same values and the same ramification indices at the points in Y, and by assumption the sum of those ramification indices is at least $2d + 1$, so Lemma 10 tells us that $f = g$. This completes the proof that $\widetilde{\mathcal{GR}}_d^1[n](K, S)$ contains only finitely many $\mathrm{PGL}_2(R_S)$-equivalence classes of triples (f, Y, X). □

4. Dynamical Shafarevich finiteness fails on $\mathbb{P}^1$ for weight $\le 2d$

In this section we prove Theorem 2(b). More precisely, we prove that the dynamical Shafarevich finiteness is false for maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ and f-invariant sets X containing $2d$ points. We do this by analyzing a particular family of maps.

Proposition 11. *Let $d \ge 2$, let $K/\mathbb{Q}$ be a number field, and let S be the set of primes of K dividing $(2d-2)!$. For each $a \in \bar{K}^*$, let $f_a(x)$ be the map*

$$f_a(x) = \frac{ax(x-1)(x-2)\cdots(x-d+1)}{(x+1)(x+2)\cdots(x+d-1)} \in \mathrm{End}_d^1,$$

and let $X \subset \mathbb{P}^1$ *be the set*

$$X = \{0, 1, 2, \dots, d-1\} \cup \{-1, -2, \dots, -(d-1)\} \cup \{\infty\}.$$

(a) *For all* $a \in R_S^*$, *we have*

$$(f_a, X, X) \in \mathcal{GR}_d^1[2d](K, S).$$

(b) *For a given* $a \in \bar{K}^*$, *there are only finitely many* $b \in \bar{K}^*$ *such that* f_b *is* $\mathrm{PGL}_2(\bar{K})$-*conjugate to* f_a.

(c) $\#\mathcal{GR}_d^1[2d](K, S)/\mathrm{PGL}_2(R_S) = \infty$.

Proof. (a) The resultant of f_a is

$$\mathrm{Res}(f_a) = a^d \prod_{i=0}^{d-1} \prod_{j=1}^{d-1} (i + j).$$

In particular, if $a \in R_S^*$, then our choice of S implies that $\mathrm{Res}(f_a) \in R_S^*$, so the map f_a has simple good reduction outside S. We also observe that our choice of S implies that the set X has good reduction outside S, and from the formula for f_a we see that $f_a(X) = \{0, \infty\} \subset X$. For example, the case $d = 4$ looks like

$$\begin{tikzcd}
1 \arrow[dr] & & -1 \arrow[dr] & & \\
2 \arrow[r] & 0 \arrow[loop right] & -2 \arrow[r] & \infty \arrow[loop right] & \text{with } S = \{2, 3, 5\}. \\
3 \arrow[ur] & & -3 \arrow[ur] & &
\end{tikzcd}$$

Since $\#X = 2d$, this completes the proof that $(f_a, X, X) \in \mathcal{GR}_d^1[2d](K, S)$.

(b) We consider the $\bar{K}$-valued points of the morphism

$$\bar{K}^* \to \mathcal{M}_d^1(\bar{K}) = \mathrm{End}_d^1(\bar{K})/\mathrm{PGL}_2(\bar{K}), \quad a \mapsto [f_a]. \tag{9}$$

We claim that the map (9) is nonconstant. To see this, we note that 0 is a fixed point of f_a, and that the multiplier of f_a at 0 is

$$\lambda(f_a, 0) := f_a'(0) = (-1)^{d-1} a.$$

But for any rational map $f \in \mathrm{End}_d^1$, the set of fixed point multipliers $\{\lambda(f, P) : P \in \mathrm{Fix}(f)\}$ is a PGL_2-conjugation invariant [Silverman 2007, Proposition 1.9]. So if (9) were constant, there would be a single map $g \in \mathrm{End}_d^1(\bar{K})$ with the property that for every $a \in \bar{K}^*$, the map f_a is $\mathrm{PGL}_2(\bar{K})$-conjugate to g. In particular, for every $a \in \bar{K}^*$, the multiplier $(-1)^{d-1} a = \lambda(f_a, 0)$ would be one of the finitely many fixed-point multipliers of g. This contradiction completes the proof of (b).

(c) It follows from (a) and (b) that $\{(f_a, X, X) : a \in R_S^*\}$ is contained in $\mathcal{GR}_d^1[2d](K, S)$ and that it contains infinitely many distinct $\mathrm{PGL}_2(R_S)$-conjugacy classes. $\square$

5. How large is the set of maps having simple good reduction?

As noted in the Introduction, it would be very interesting to know the behavior of the "Shafarevich discrepancy",

$$2d-2-\mathrm{ShafDim}_d^N[\mathcal{P}] \quad \text{as } d \to \infty,$$

even for the case $\mathcal{P} = \varnothing$. It has long been noted that monic polynomial maps on $\mathbb{P}^1$ have everywhere simple good reduction. This gives a set of such maps in $\mathcal{M}_d^1$ whose Zariski closure has dimension $d-1$. With a little work, we can increase this dimension by 1 for $d = 2$ and by 2 for $d \geq 3$.

Proposition 12. *We have*

$$\mathrm{ShafDim}_2^1[\varnothing] = \dim \mathcal{M}_2^1 = 2,$$

and for all $d \geq 3$ we have

$$\mathrm{ShafDim}_d^1[\varnothing] \geq d+1.$$

Proof. We fix a number field K and a set of places S so that R_S^* is infinite. For $\boldsymbol{a} = (a_0, a_2, \ldots, a_{d-1}, a_d)$ we define a rational map

$$f_{\boldsymbol{a}}(x) := \frac{a_0x^d + a_1x^{d-1} + \cdots + a_{d-2}x^2 + a_{d-1}x + a_d}{x(x-1)}.$$

We have

$$\mathrm{Res}(f_{\boldsymbol{a}}) = a_0^{d-2}a_d(a_0 + a_1 + \cdots + a_{d-1} + a_d).$$

Hence $f_{\boldsymbol{a}}$ will have simple good reduction if we take $a_0, a_d \in R_S^*$, $a_2, \ldots, a_{d-1} \in R_S$, and set $a_1 = w - a_0 - a_2 - \cdots - a_d$ for some $w \in R_S^*$. In other words, the image of the map

$$\begin{aligned} (R_S^*)^3 \times R_S^{d-2} &\to \mathbb{A}^{d+1}(K), \\ \big((u, v, w), (a_2, \ldots, a_{d-1})\big) &\mapsto (u, w-u-a_2-\cdots-a_{d-1}-v, a_2, \ldots, a_{d-1}, v), \end{aligned}$$

gives values of $\boldsymbol{a}$ for which $f_{\boldsymbol{a}}$ has simple good reduction. The image of this map is Zariski dense in $\mathbb{A}^{d+1}$, so it remains to show that the map $\mathbb{A}^{d+1} \to \mathcal{M}_d^1$ given by $\boldsymbol{a} \mapsto \langle f_{\boldsymbol{a}} \rangle$ is generically finite-to-one.

Suppose that $\varphi \in \mathrm{PGL}_2(\bar{K})$ has the property that $f_{\boldsymbol{a}}^{\varphi} = f_{\boldsymbol{b}}$. We start with the case $d \geq 4$. Then $f_{\boldsymbol{a}}$ is ramified at the fixed point ∞, since $e_{f_{\boldsymbol{a}}}(\infty) = d-2$, and similarly for $f_{\boldsymbol{b}}$. Generically, ∞ will be the only ramified fixed point of $f_{\boldsymbol{a}}$ and $f_{\boldsymbol{b}}$, so $\varphi(\infty) = \infty$. Next we use the fact that

$$f_{\boldsymbol{a}}^{-1}(\infty) = f_{\boldsymbol{b}}^{-1}(\infty) = \{\infty, 1, 0\}$$

to conclude that $\varphi(\{0,1\}) = \{0,1\}$. Thus φ fixes ∞ and either fixes or swaps 0 and 1, so the only possibilities are $\varphi(x) = x$ or $\varphi(x) = 1-x$. Thus $f_{\boldsymbol{a}}$ is PGL_2-conjugate to only one other map of the same form. This concludes the proof for $d \ge 4$. For $d = 3$, the point ∞ is fixed by $f_{\boldsymbol{a}}$ and $f_{\boldsymbol{b}}$, but ∞ is not a critical point, so we cannot conclude that φ fixes ∞. However, we can argue as follows. A generic map of the form $f_{\boldsymbol{a}}$ has 3 fixed points, say $\{\infty, \gamma_1, \gamma_2\}$, and each fixed point has 3 points in its inverse image, one of which is itself, say

$$f_{\boldsymbol{a}}^{-1}(\infty) = \{\infty, 0, 1\}, \quad f_{\boldsymbol{a}}^{-1}(\gamma_1) = \{\gamma_1, \alpha_1, \beta_1\}, \quad f_{\boldsymbol{a}}^{-1}(\gamma_2) = \{\gamma_2, \alpha_2, \beta_2\}.$$

It follows that $\varphi(\infty) \in \{\infty, \gamma_1, \gamma_2\}$, and that

$$\varphi(\{0,1\}) = \begin{cases} \{0,1\} & \text{if } \varphi(\infty) = \infty, \\ \{\alpha_1, \beta_1\} & \text{if } \varphi(\infty) = \gamma_1, \\ \{\alpha_2, \beta_2\} & \text{if } \varphi(\infty) = \gamma_2. \end{cases}$$

Since φ is determined by its values at on $\{0, 1, \infty\}$, we see that there are (at most) 6 maps φ for which $f_{\boldsymbol{a}}^{\varphi}$.

Finally, for $d = 2$, we first note that the above proof fails because ∞ is no longer a fixed point. And it is good that the proof fails, since otherwise we would conclude that $\mathrm{ShafDim}_2^1[\varnothing] \ge 3$, which would contradict $\dim \mathcal{M}_2^1 = 2$. So for $d = 2$ we instead use the family of maps

$$g_{a,b}(x) := \frac{ax^2 + x + b}{x}.$$

These satisfy $\mathrm{Res}(g_{a,b}) = ab$, so they have good reduction for all $a, b \in R_S^*$. We could argue as above that there are only finitely many φ preserving this form, and thus the image in $\mathcal{M}_2^1$ is 2-dimensional. But to illustrate an alternative method of proof, we instead use the Milnor isomorphism $s : \mathcal{M}_2^1 \xrightarrow{\sim} \mathbb{A}^2$; see [Silverman 2007, Theorem 4.5.6]. The map $g_{a,b}$ has Milnor coordinates

$$s(g_{a,b}) = \left(\frac{4a^2b - 2ab - a + b}{ab}, \frac{4a^3b - 4a^2b - a^2 + 5ab - 2b - 1}{ab} \right).$$

We used Magma [Bosma et al. 1997] to verify that these two rational functions are algebraically independent in $K(a, b)$. Hence under our assumption that $\#R_S^* = \infty$, we see that $\{s(a,b) : a, b \in R_S^*\}$ is Zariski dense in $\mathbb{A}^2$. □

6. Abstract portraits and models for portraits

In this section we briefly construct a category of portraits and use it to describe dynamical systems that model a given portrait. See [Doyle and Silverman $\ge$ 2018] for further development and the construction of parameter and moduli spaces for dynamical systems with portraits.

Definition. An (*abstract*) *weighted portrait* is a 4-tuple $\mathcal{P} = (\mathcal{W}, \mathcal{V}, \Phi, \epsilon)$, where

- $\mathcal{W} \subseteq \mathcal{V}$ are finite sets (of vertices),
- $\Phi : \mathcal{W} \to \mathcal{V}$ is a map (which specifies directed edges),
- $\mathcal{V} = \mathcal{W} \cup \Phi(\mathcal{W})$,
- $\epsilon : \mathcal{W} \to \mathbb{N}$ is a map (assigning weights to vertices).

The *weight of* $\mathcal{P}$ is the total weight

$$\mathrm{wt}(\mathcal{P}) := \sum_{w \in \mathcal{W}} \epsilon(w).$$

We say that the portrait is *unweighted* if $\epsilon(w) = 1$ for every $w \in \mathcal{W}$, or equivalently if $\mathrm{wt}(\mathcal{P}) = \#\mathcal{W}$, in which case we sometimes write $\mathcal{P} = (\mathcal{W}, \mathcal{V}, \Phi)$. We say that the portrait is *preperiodic* if $\mathcal{W} = \mathcal{V}$.

We now explain how a self-map of $\mathbb{P}^1$ can be used to model a portrait.

Definition. Let $\mathcal{P} = (\mathcal{W}, \mathcal{V}, \Phi, \epsilon)$ be a portrait. A *model for* $\mathcal{P}$ is a triple (f, Y, X) consisting of a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ and subsets $Y \subset X \subset \mathbb{P}^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\ i\ } & \mathbb{P}^1 \\ {\scriptstyle\Phi}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ \mathcal{V} & \xrightarrow{\ i\ } & \mathbb{P}^1 \end{array} \tag{10}$$

We say that (f, Y, X) is a $\bullet$-*model* if in addition

$$e_f(i(w)) \ge \epsilon(w) \quad \text{for all } w \in \mathcal{W};$$

and similarly we say that (f, Y, X) is a $\circ$-*model* if

$$e_f(i(w)) = \epsilon(w) \quad \text{for all } w \in \mathcal{W}.$$

With this formalism, we can now define our three Shafarevich-type sets.

Definition. Let $\mathcal{P} = (\mathcal{W}, \mathcal{V}, \Phi, \epsilon)$ be a portrait and let $n = \mathrm{wt}(\mathcal{P})$. Then

$$\mathcal{GR}_d^1[\mathcal{P}]^\bullet(K, S) = \{(f, Y, X) \in \mathcal{GR}_d^1[n](K, S) : (f, Y, X) \text{ is a } \bullet\text{-model for } \mathcal{P}\},$$

$$\mathcal{GR}_d^1[\mathcal{P}]^\circ(K, S) = \{(f, Y, X) \in \mathcal{GR}_d^1[n](K, S) : (f, Y, X) \text{ is a } \circ\text{-model for } \mathcal{P}\},$$

$$\begin{aligned}\mathcal{GR}_d^1[\mathcal{P}]^\star(K, S) = \{(f, Y, X) \in \mathcal{GR}_d^1[n](K, S) : e_{\tilde f_\mathfrak{p}}(\widetilde{i(w) \bmod \mathfrak{p}}) = \epsilon(w) \\ \text{for all } w \in \mathcal{W} \text{ and all } \mathfrak{p} \notin S\}.\end{aligned}$$

It may happen that a portrait has no models using maps of a given degree. For example, if the portrait $\mathcal{P}$ contains 4 fixed points, then it cannot be modeled by a map of degree 2, and similarly if $\mathcal{P}$ contains a pair of 2-cycles. In order to describe more generally the constraints on a model, we set an ad hoc piece of notation. (A better definition of $\mathcal{M}_d^1[\mathcal{P}]^\bullet$ as a $\mathbb{Z}$-scheme is given in [Doyle and Silverman ≥ 2018].)

Definition. Let $\mathcal{P}$ be a portrait and let $d \ge 2$. We define

$$\mathcal{M}_d^1[\mathcal{P}]^\bullet := \{f \in \mathcal{M}_d^1(\bar{K}) : \text{there exist sets } Y \subseteq X \subseteq \mathbb{P}^1(\bar{K}) \text{ such that } (f, Y, X) \text{ is a } \bullet\text{-model for } \mathcal{P}\}.$$

Proposition 13. *Let $d \ge 2$, and let $\mathcal{P} = (\mathcal{W}, \mathcal{V}, \Phi, \epsilon)$ be a portrait such that $\mathcal{M}_d^1[\mathcal{P}] \ne \varnothing$. Then $\mathcal{P}$ satisfies the following conditions*:

$$\text{(I)} \quad \sup_{v\in\mathcal{V}} \sum_{w\in\Phi^{-1}(v)} \epsilon(w) \le d, \qquad \text{(II)} \quad \sum_{w\in\mathcal{W}} \big(\epsilon(w) - 1\big) \le 2d - 2.$$

For all $n \ge 1$,

$$(\mathrm{III}_n) \quad \#\{w \in \mathcal{W} : \Phi^n(w) = w \text{ and } \Phi^m(w) \ne w \text{ for all } m < n\} \le \sum_{m|n} \mu\Big(\frac{n}{m}\Big)(d^m + 1).$$

(*Here μ is the Möbius function.*)

Proof. Constraint I comes from the fact that f is a map of degree d, constraint II follows from the Riemann–Hurwitz formula $\sum\big(e_f(P) - 1\big) = 2d - 2$ [Silverman 2007, Theorem 1.1], and constraint III_n from the fact that a degree d map on $\mathbb{P}^1$ has at most the indicated number of points of exact period n [Silverman 2007, Remark 43]. □

If we fix a preperiodic portrait $\mathcal{P}$ and allow the degree d to grow, then we expect that $\mathcal{M}_d^1[\mathcal{P}]^\bullet$ has exactly the expected dimension, as in the following conjecture. This is in marked contrast to our uncertainty regarding the size of $\mathrm{ShafDim}_d^1[\mathcal{P}]^\bullet$ as $d \to \infty$; cf. Question 4.

Conjecture 14. *Let $\mathcal{P} = (\mathcal{W}, \mathcal{V}, \Phi, \epsilon)$ be a preperiodic portrait. There is a constant $d_0(\mathcal{P})$ such that for all $d \ge d_0(\mathcal{P})$ we have*

$$\begin{aligned} \dim \overline{\mathcal{M}_d^1[\mathcal{P}]^\bullet} &= \dim \mathcal{M}_d^1 - \sum_{w\in\mathcal{W}} \big(\epsilon(w) - 1\big) \\ &= 2d - 2 - \mathrm{wt}(\mathcal{P}) + \#\mathcal{W}. \end{aligned}$$

Remark 15. The local conditions used to define $\mathcal{GR}_d^1[\mathcal{P}]^\star(K, S)$ reflect the viewpoint adopted by Petsche and Stout [2015]. We note that since f and $i(\mathcal{V})$ are assumed to have good reduction outside S, there is a well-defined map $\tilde{f}_{\mathfrak{p}} : \mathbb{P}^1 \to \mathbb{P}^1$ defined over the residue field of $\mathfrak{p}$, and so it makes sense to look at the ramification indices of $\tilde{f}_{\mathfrak{p}}$ at the $\mathfrak{p}$-reductions of the points in $i(\mathcal{W})$.

Remark 16. Since the primary goal of this paper is the study of Shafarevich-type finiteness theorems, we have been content to define our sets of good reduction purely as sets. In a subsequent paper [Doyle and Silverman $\geq$ 2018] we will take up the more refined question of constructing moduli spaces for dynamical systems with portraits, after which the results of the present paper can be reinterpreted as characterizing the S-integral points on these spaces, with the caveat that there may be field-of-moduli versus field-of-definition issues.

Since our goal is to understand the size of the various sets of good reduction triples (f, Y, X), we are prompted to make the following definitions.

Definition. Let $x \in \{\bullet, \circ, \star\}$. The associated *Shafarevich dimension* is the quantity

$$\operatorname{ShafDim}_d^1[\mathcal{P}]^x = \sup_{\substack{K \text{ a number field} \\ S \text{ a finite set of places}}} \dim \overline{\operatorname{Image}\bigl(\mathcal{GR}_d^1[\mathcal{P}]^x(K, S) \to \mathcal{M}_d^1\bigr)}.$$

We record some elementary properties for future reference.

Proposition 17. *Let $d \geq 2$, and let $\mathcal{P} = (\mathcal{W}, \mathcal{V}, \Phi, \epsilon)$ be a portrait.*

(a) *Let $\epsilon' : \mathcal{V} \to \mathbb{N}$ be a weight function satisfying $\epsilon' \geq \epsilon$, let $\mathcal{P}' = (\mathcal{W}, \mathcal{V}, \Phi, \epsilon')$, and let $x \in \{\bullet, \circ, \star\}$. Then*

$$\mathcal{GR}_d^1[\mathcal{P}']^x(K, S) \subseteq \mathcal{GR}_d^1[\mathcal{P}]^x(K, S).$$

(b) *We have*

$$\mathcal{GR}_d^1[\mathcal{P}]^\star(K, S) \subseteq \mathcal{GR}_d^1[\mathcal{P}]^\circ(K, S) \subseteq \mathcal{GR}_d^1[\mathcal{P}]^\bullet(K, S).$$

(c) *We have*

$$\operatorname{ShafDim}_d^1[\mathcal{P}]^\star \leq \operatorname{ShafDim}_d^1[\mathcal{P}]^\circ \leq \operatorname{ShafDim}_d^1[\mathcal{P}]^\bullet \leq \dim \mathcal{M}_d^1 = 2d - 2.$$

Proof. (a) and (b) are clear from the definitions of the various sets of good reduction, and (c) follows (b) and the definition of Shafarevich dimension. We note that if a map f has good reduction at $\mathfrak{p}$, then its ramification index can only increase when f is reduced modulo $\mathfrak{p}$. $\square$

Example 18. Consider the following two preperiodic portraits:

$$\mathcal{P}_1 \quad \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[dl] \\ \bullet \arrow[u] & \end{tikzcd} \qquad\qquad \mathcal{P}_2 \quad \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[dl] \\ \bullet \arrow[u, "2"] & \end{tikzcd}$$

We note that the portrait $\mathcal{P}_2$ is strictly larger than the portrait $\mathcal{P}_1$ in the sense of Proposition 17(a), so that result tells us that $\mathcal{GR}_d^1[\mathcal{P}_2]^\circ(K, S) \subseteq \mathcal{GR}_d^1[\mathcal{P}_1]^\circ(K, S)$. However, we will see in Section 7 that if $\#R_S^* = \infty$, then

$$\#\,\mathcal{GR}_2^1[\mathcal{P}_1]^\circ(K, S) < \infty \quad \text{and} \quad \#\,\mathcal{GR}_2^1[\mathcal{P}_2]^\circ(K, S) = \infty.$$

In words, there are only finitely many degree 2 rational maps with good reduction outside S that have an *unramified* good reduction 3-cycle, but if we allow one of the points in the 3-cycle to be ramified, then there are infinitely many such maps. In terms of Shafarevich dimensions, we have $\mathrm{ShafDim}_d^1[\mathcal{P}_1]^\circ = 0$ and $\mathrm{ShafDim}_d^1[\mathcal{P}_2]^\circ = 1$. On the other hand, we will show that with the more restrictive Petsche–Stout good reduction criterion, we have $\mathrm{ShafDim}_d^1[\mathcal{P}_1]^\star = \mathrm{ShafDim}_d^1[\mathcal{P}_2]^\star = 0$. Another example of this phenomenon, where more ramification leads to more maps of good reduction, is given by portraits $\mathcal{P}_{3,3}$ and $\mathcal{P}_{4,7}$ in Tables 2 and 3, respectively.

7. Good reduction for preperiodic portraits of weight ≤ 4 for degree 2 maps of $\mathbb{P}^1$

We know from Theorem 2 with $N = 1$ and $d = 2$ that if a portrait $\mathcal{P}$ satisfies $\mathrm{wt}(\mathcal{P}) \ge 5$, then $\mathrm{ShafDim}_2^1[\mathcal{P}]^\bullet = 0$. In other words, dynamical Shafarevich finiteness holds for degree 2 maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ that model a portrait $\mathcal{P}$ of weight at least 5. In this section we give a complete analysis of preperiodic portraits of weights 1 to 4. For example, it turns out that there are 22 such portraits of weight 4, and dynamical Shafarevich finiteness holds for some of them, but not for others. For notational convenience, we label portraits as $\mathcal{P}_{w,m}$, where w is the weight and $m \in \{1, 2, 3, \dots\}$.

Theorem 19. *We consider moduli spaces of degree* 2 *maps* $\mathbb{P}^1 \to \mathbb{P}^1$ *with weighted preperiodic portraits.*

(a) *There is* 1 *portrait* $\mathcal{P}$ *of weight* 1 *such that* $\mathcal{M}_2^1$ *contains a map that can be used to model* $\mathcal{P}$.

(b) *There are* 4 *portraits* $\mathcal{P}$ *of weight* 2 *such that* $\mathcal{M}_2^1$ *contains a map that can be used to model* $\mathcal{P}$.

(c) *There are* 8 *portraits* $\mathcal{P}$ *of weight* 3 *such that* $\mathcal{M}_2^1$ *contains a map that can be used to model* $\mathcal{P}$.

(d) *There are* 22 *portraits* $\mathcal{P}$ *of weight* 4 *such that* $\mathcal{M}_2^1$ *contains a map that can be used to model* $\mathcal{P}$.

These portraits are as catalogued in Tables 2, 3 *and* 4, *which also give the values of the following quantities*:

$$\mathcal{MD} := \dim \overline{\mathcal{M}_2^1[\mathcal{P}]^\bullet}, \qquad \mathcal{SD}^\bullet := \mathrm{ShafDim}_2^1[\mathcal{P}]^\bullet,$$
$$\mathcal{SD}^\circ := \mathrm{ShafDim}_2^1[\mathcal{P}]^\circ, \quad \mathcal{SD}^\star := \mathrm{ShafDim}_2^1[\mathcal{P}]^\star.$$

Proof. Since we will be dealing entirely with preperiodic portraits, we write the triple (f, X, X) as a pair (f, X). For degree 2 maps, we see that $\mathcal{M}_2^1[\mathcal{P}]^\bullet = \varnothing$ unless the following four conditions are true; cf. Proposition 13.

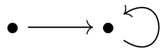
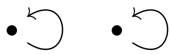
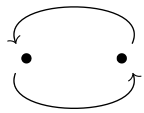
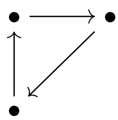
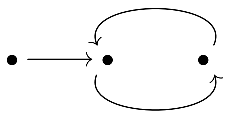
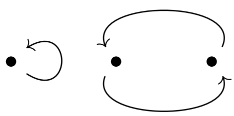
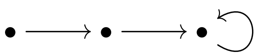
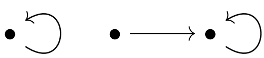
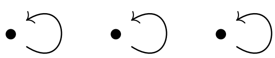
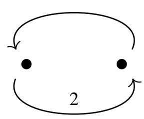
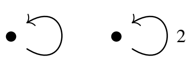
#	$\mathcal{P}$	wt($\mathcal{P}$)	$\mathcal{MD}$	$\mathcal{SD}^\bullet$	$\mathcal{SD}^\circ$	$\mathcal{SD}^\star$
$\mathcal{P}_{1,1}$		1	2	2	2	2
$\mathcal{P}_{2,1}$		2	1	1	1	1
$\mathcal{P}_{2,2}$		2	2	2	2	2
$\mathcal{P}_{2,3}$		2	2	2	2	1
$\mathcal{P}_{2,4}$		2	2	2	2	1
$\mathcal{P}_{3,1}$		3	2	1	0	0
$\mathcal{P}_{3,2}$		3	2	2	2	1
$\mathcal{P}_{3,3}$		3	2	1	0	0
$\mathcal{P}_{3,4}$		3	2	2	2	1
$\mathcal{P}_{3,5}$		3	2	2	2	1
$\mathcal{P}_{3,6}$		3	2	1	0	0
$\mathcal{P}_{3,7}$		3	1	1	1	1
$\mathcal{P}_{3,8}$		3	1	1	1	1

Table 2. Weight 1, 2, and 3 preperiodic portraits for degree 2 maps.

(I) Each point has at most weight 2 worth of incoming arrows.

(II) There are at most 2 critical points.

(III_1) There are at most 3 fixed points.

(III_2) There is at most one periodic cycle of length 2.

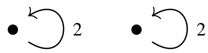
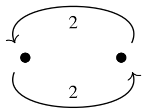
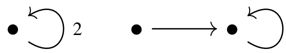
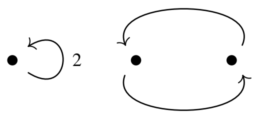
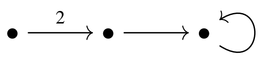
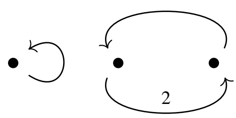
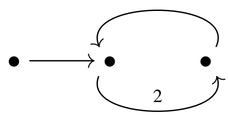
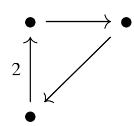
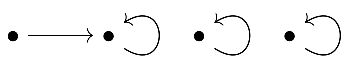
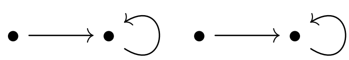
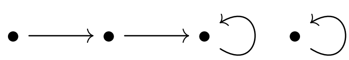
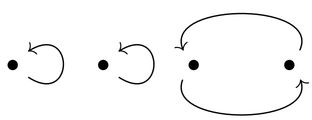
#	$\mathcal{P}$	wt($\mathcal{P}$)	$\mathcal{MD}$	$\mathcal{SD}^\bullet$	$\mathcal{SD}^\circ$	$\mathcal{SD}^\star$
$\mathcal{P}_{4,1}$		4	0	0	0	0
$\mathcal{P}_{4,2}$		4	0	0	0	0
$\mathcal{P}_{4,3}$		4	1	1	1	0
$\mathcal{P}_{4,4}$		4	1	1	1	1
$\mathcal{P}_{4,5}$		4	1	1	1	0
$\mathcal{P}_{4,6}$		4	1	1	1	1
$\mathcal{P}_{4,7}$		4	1	1	1	0
$\mathcal{P}_{4,8}$		4	1	1	1	1
$\mathcal{P}_{4,9}$		4	1	1	1	0
$\mathcal{P}_{4,10}$		4	2	0	0	0
$\mathcal{P}_{4,11}$		4	2	1	1	1
$\mathcal{P}_{4,12}$		4	2	0	0	0
$\mathcal{P}_{4,13}$		4	2	0	0	0

Table 3. Weight 4 preperiodic portraits for degree 2 maps (part 1).

Sublemma 8 says that in order to prove that $\mathcal{GR}_2^1[\mathcal{P}]^\circ(K, S)/\,\mathrm{PGL}_2(R_S)$ is finite for all K and S, it suffices to prove finiteness after extending K and enlarging S. And the definition of $\mathrm{ShafDim}_d^1[\mathcal{P}]^\bullet$ and its variants is a supremum over all K and all S. So we may assume throughout our discussion that in every model (f, X)

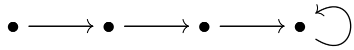
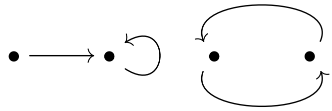
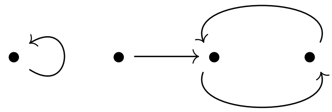
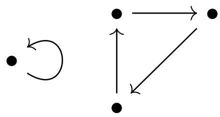
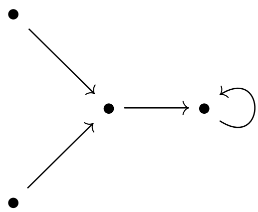
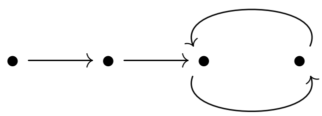
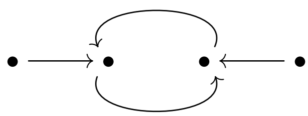
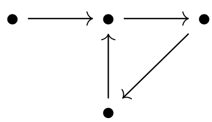
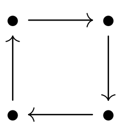
#	$\mathcal{P}$	wt($\mathcal{P}$)	$\mathcal{MD}$	$\mathcal{SD}^{\bullet}$	$\mathcal{SD}^{\circ}$	$\mathcal{SD}^{\star}$
$\mathcal{P}_{4,14}$		4	2	0	0	0
$\mathcal{P}_{4,15}$		4	2	0	0	0
$\mathcal{P}_{4,16}$		4	2	0	0	0
$\mathcal{P}_{4,17}$		4	2	0	0	0
$\mathcal{P}_{4,18}$		4	2	1	1	1
$\mathcal{P}_{4,19}$		4	2	0	0	0
$\mathcal{P}_{4,20}$		4	2	1	1	1
$\mathcal{P}_{4,21}$		4	2	0	0	0
$\mathcal{P}_{4,22}$		4	2	0	0	0

Table 4. Weight 4 preperiodic portraits for degree 2 maps (part 2).

for $\mathcal{P}$, the points in X are in $\mathbb{P}^1(K)$, and further that S is chosen so that

$$R_S \text{ is a PID}; \quad R_S^* \text{ is infinite}; \quad 2, 3 \in R_S^*.$$

Using the assumptions that the points in our portraits are in $\mathbb{P}^1(K)$ and that R_S is a PID, Sublemma 9 and the Chinese remainder theorem tell us that we can find an element of $\mathrm{PGL}_2(R_S)$ to move three of the points in X to the points 0, 1, and ∞. (Or just to 0 and ∞ if $\#X = 2$.)

As in the proof of Proposition 12, we will frequently use the Milnor isomorphism [Silverman 2007, Theorem 4.5.6]

$$s = (s_1, s_2) : \mathcal{M}_2^1 \xrightarrow{\sim} \mathbb{A}^2,$$

which we implemented in PARI [2016], to help distinguish the $\mathrm{PGL}_2(\bar{K})$-conjugacy classes of our maps, and we often use Magma [Bosma et al. 1997] to verify that the images of certain maps are Zariski dense in $\mathcal{M}_2^1$.

$\boldsymbol{\mathcal{P}_{1,1}}$: This case was done by Petsche and Stout [2015, Remark 3], but for completeness, we include a proof. Let $f(x) = (x^2 + ax)/(bx + 1)$ with $\mathrm{Res}(f) = 1 - ab$, so $(f, \{0\}) \in \mathcal{GR}_2^1[\mathcal{P}_{1,1}]^\bullet(K, S)$ for all $a, b \in R_S$ satisfying $1 - ab \in R_S^*$. Further, $f'(0) = a$, so if we take $a \in R_S^*$, then 0 is not critical modulo v for all $v \notin S$. This suggests that we change variables via $b = (1 - u)a^{-1}$. Then $f(x) = (ax^2 + a^2x)/((1 - u)x + a)$ with $\mathrm{Res}(f) = a^4u$ and $f'(0) = a$, so $(f, \{0\}) \in \mathcal{GR}_2^1[\mathcal{P}_{1,1}]^\star(K, S)$ for all $a, u \in R_S^*$. The Milnor image of this map in $\mathcal{M}_2 \cong \mathbb{A}^2$ is

$$s\left(\frac{ax^2 + a^2x}{(1-u)x + a}\right) = \left(\frac{a^2(u-1) + 2a - (u-1)^2}{au},\right.$$
$$\left.\frac{-a^4 + 2a^3 - a^2(u-1)(u-2) - 2a(u-1) - (u-1)^2}{a^2u}\right).$$

We used Magma to verify that the two rational functions $s_1(a, u)$ and $s_2(a, u)$ are algebraically independent in $K(a, u)$. Hence under our assumption that $\#R_S^* = \infty$, we see that $\{s(a, u) : a, u \in R_S^*\}$ is Zariski dense in $\mathbb{A}^2$. This completes the proof that $\mathrm{ShafDim}_2^1[\mathcal{P}_{1,1}]^\star = 2$, and the other Shafarevich dimensions are also 2 by the standard inequalities in Proposition 17(e).

$\boldsymbol{\mathcal{P}_{2,1}}$: Moving the totally ramified fixed point to ∞, the map f has the form $f(x) = ax^2 + bx + c$. It has good reduction if and only if $a \in R_S^*$. Then we can conjugate by a map of the form $x \mapsto a^{-1}x + e$ to put $f(x)$ in the form $f(x) = x^2 + c$. Since the ramification at ∞ can't increase when we reduce modulo primes not in S, we see that

$$(x^2 = c, \{\infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{2,1}]^\star(K, S) \quad \text{for all } c \in R_S^*.$$

The closure of the image in $\mathcal{M}_2^1$ is the line $s_1 = 2$ of polynomials.

$\boldsymbol{\mathcal{P}_{2,2}}$: Move the two points to $0, \infty$; then f has the form $f(x) = (ax^2 + bx + c)/dx$. This map has $\mathrm{Res}(f) = acd^2$, so we can dehomogenize $d = 1$. Thus $f(x) = ax + b + cx^{-1}$ with $ac \in R_S$. Conjugating by $x \to ux$ gives $u^{-1}f(ux) = ax + bu^{-1} + cu^{-2}x^{-1}$, so going to $K(\sqrt{c})$, which is unramified over S, we may assume that $c = 1$

and $f(x) = (ax^2 + bx + 1)/x$. We also observe that $f^{-1}(f(\infty)) = \{0, \infty\}$ and in $f^{-1}(f(0)) = \{0, \infty\}$, so 0 and ∞ are unramified modulo all primes. (Alternatively, one could compute derivatives, after moving ∞ to a more amenable point.) Hence

$$(f, \{0, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{2,2}]^\star(K, S) \quad \text{for all } a \in R_S^* \text{ and } b \in R_S.$$

The Milnor image is

$$s\left(\frac{ax^2+bx+1}{x}\right) = \left(\frac{4a^2 - ab^2 - 2a + 1}{a}, \frac{4a^3 - a^2b^2 - 4a^2 + 5a - b^2 - 2}{a}\right).$$

We used Magma to verify that the rational functions $s_1(a, u)$ and $s_2(a, u)$ are algebraically independent in $K(a, u)$. Hence under our assumption that $\#R_S^* = \infty$, we see that $\{s(a, u) : a, u \in R_S^*\}$ is Zariski dense in $\mathbb{A}^2$. This completes the proof that $\text{ShafDim}_2^1[\mathcal{P}_{2,2}]^\star = 2$, and the other Shafarevich dimensions are also 2 by the standard inequalities in Proposition 17(e).

$\boldsymbol{\mathcal{P}_{2,3}}$: Moving the two fixed points to 0 and ∞, the map f has the form $f(x) = (ax^2+bx)/(cx+d)$. The resultant is $\text{Res}(f) = ad(ad-bc)$. Good reduction implies in particular that $a, d \in R_S^*$, so we can dehomogenize $d = 1$ and replace f with $af(a^{-1}x) = (x^2 + bx)/(a^{-1}cx + 1)$. We can also replace $a^{-1}c$ with c, so $f(x) = (x^2 + bx)/(cx + 1)$ with $\text{Res}(f) = 1 - bc$. Hence

$$(f, \{0, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{2,3}]^\circ(K, S) \quad \text{for all } b, c \in R_S \text{ satisfying } 1 - bc \in R_S^*.$$

We note that this set of (b, c) is Zariski dense in $\mathbb{A}^2$, under our assumption that $\#R_S^* = \infty$. For example, if $u \in R_S^*$ has infinite order, then for every $n \geq 1$ we can take $b = 1 - u$ and $c = 1 + u + u^2 + \cdots + u^n$, and this set of points is Zariski dense. The Milnor image is

$$s\left(\frac{x^2+bx}{cx+1}\right) = \left(\frac{-b^2c - bc^2 + 2}{1 - bc}, \frac{-b^2c^2 - b^2 - bc + 2b - c^2 + 2c}{1 - bc}\right).$$

We used Magma to verify that the rational functions $s_1(a, u)$ and $s_2(a, u)$ are algebraically independent in $K(a, u)$. Hence under our assumption that $\#R_S^* = \infty$, we see that $\{s(a, u) : a, u \in R_S^*\}$ is Zariski dense in $\mathbb{A}^2$. This completes the proof that $\text{ShafDim}_2^1[\mathcal{P}_{2,3}]^\circ = 2$.

However, the set $\mathcal{GR}_2^1[\mathcal{P}_{2,3}]^\star(K, S)$ is more restrictive, since we need the fixed points to be unramified for all primes not in S. Thus $(f, \{0, \infty\})$ is in this set if and only if $f'(0) = b \in R_S^*$ and $f'(\infty) = c \in R_S^*$. We thus need $b, c, 1 - bc$ to be S-units. Then $(bc, 1 - bc)$ is a solution to the S-unit equation $u + v = 1$, so there are only finitely many possible values for bc. On the other hand, any fixed solution (u, v) gives a map $f(x) = (x^2 + bx)/(b^{-1}ux + 1)$ satisfying

$$(f, \{0, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{2,3}]^\star(K, S) \quad \text{for all } b \in R_S^*.$$

Each (u, v) value gives points lying on a curve in $\mathcal{M}_2^1$. And there is at least one such curve, since our assumption that $2 \in S$ says that we can take $(u, v) = (-1, 2)$, leading to the Milnor image

$$s\left(\frac{x^2 + bx}{-b^{-1}x + 1}\right) = \left(\frac{b^2 + 2b - 1}{2b}, \frac{-b^4 + 2b^3 - 2b - 1}{2b^2}\right).$$

Hence $\operatorname{ShafDim}[\mathcal{P}_{2,3}]^\star = 1$, a result that was first proven by Petsche and Stout [2015, Section 4].

$\boldsymbol{\mathcal{P}_{2,4}}$: We move the two points to 0 and ∞, so $f(x) = (ax + b)/(cx^2 + dx)$ with $\operatorname{Res}(f) = bc(ad - bc)$. Good reduction implies in particular that $b, c \in R_S^*$, so we can dehomogenize $b = 1$. Conjugating f gives $u^{-1} f(ux) = (aux + 1)/(cu^3x^2 + du^2x)$. Going to the field $K(\sqrt[3]{c})$, which is unramified outside S, we can take $u = c^{-1/3}$ and adjust a and d accordingly to put f in the form $f(x) = (ax + 1)/(x^2 + dx)$. Then

$$(f, \{0, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{2,4}]^\circ(K, S) \quad \text{for all } a, d \in R_S \text{ with } 1 - ad \in R_S^*.$$

The map f is unramified at 0 if and only if $d \neq 0$ and f is unramified at ∞ if and only if $a \neq 0$. The Milnor image is

$$s\left(\frac{ax + 1}{x^2 + dx}\right) = \left(\frac{a^3 + 4ad + d^3 - 6}{1 - ad}, \frac{-2a^3 - a^2d^2 - 7ad - 2d^3 + 12}{1 - ad}\right).$$

We used Magma to verify that the rational functions $s_1(a, u)$ and $s_2(a, u)$ are algebraically independent in $K(a, u)$. Hence under our assumption that $\#R_S^* = \infty$, we see that $\{s(a, u) : a, u \in R_S^*\}$ is Zariski dense in $\mathbb{A}^2$. This completes the proof that $\operatorname{ShafDim}_2^1[\mathcal{P}_{2,4}]^\circ = 2$.

The multiplier of the 2-cycle is $(f^2)'(0) = ad$, so the points 0 and ∞ are unramified modulo all primes not in S if and only if $a, d \in R_S^*$. So in this case $(ad, 1 - ad)$ is a solution to the S-unit equation $u + v = 1$, and each of the finitely many such solutions yields a family of maps $f(x) = (ax + 1)/(x^2 + ua^{-1}x)$ with

$$(f, \{0, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{2,4}]^\star(K, S) \quad \text{for all } a \in R_S^*.$$

The Zariski closure of these points form a nonempty finite collection of curves, since for example $(u, v) = (-1, 2)$ gives

$$s\left(\frac{ax + 1}{x^2 - a^{-1}x}\right) = \left(\frac{a^6 - 10a^3 - 1}{2a^3}, \frac{-a^6 + 9a^3 + 1}{a^3}\right).$$

Hence $\operatorname{ShafDim}[\mathcal{P}_{2,4}]^\star = 1$, a result that was first proven by Petsche and Stout [2015, Section 5].

$\boldsymbol{\mathcal{P}_{3,1}}$: We first note that almost all rational maps of degree 2 have a 3-cycle [Beardon 1991, Section 6.8]. Hence the image of $\mathcal{M}_2^1[\mathcal{P}_{3,1}]^\bullet$ omits only finitely many points, and thus $\dim \overline{\mathcal{M}_2^1[\mathcal{P}_{3,1}]^\bullet} = 2$. We next move the 3-cycle to $0, 1, \infty$, so f has the form $f(x) = (ax^2 - (a+c)x + c)/(ax^2 + ex)$ with $\operatorname{Res}(f) = ac(a+e)(c+e)$. We dehomogenize $a = 1$. Then

$$(f, \{0, 1, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{3,1}]^\bullet(K, S) \iff c(1+e)(c+e) \in R_S^*.$$

This leads to solutions to the 4-term S-unit equation

$$c + (1+e) - (c+e) - 1 = 0.$$

The multivariable S-unit sum theorem [Evertse 1984; van der Poorten and Schlickewei 1991] says that there are finitely many solutions with no subsum equal to 0. Ignoring those finitely many solutions, there are three subsum 0 cases:

(1) $c + (1+e) = 0$, which implies that $e_f(\infty) = 2$.

(2) $c - (c+e) = 0$, which implies that $e_f(0) = 2$.

(3) $c - 1 = 0$, which implies that $e_f(1) = 2$.

This gives three families of pairs (f, X) in $\mathcal{GR}_2^1[\mathcal{P}_{3,1}]^\bullet(K, S)$, but every f is ramified at one of the three points in X, so these pairs are not in $\mathcal{GR}_2^1[\mathcal{P}_{3,1}]^\circ(K, S)$. Instead, they are in $\mathcal{GR}_2^1[\mathcal{P}_{4,9}]^\circ(K, S)$. These three families are in fact $\mathrm{PGL}_2(R_S)$-conjugate via permutation of the points in $\{0, 1, \infty\}$. Taking, say, the $e = 0$ family, we have good reduction for all $c \in R_S^*$, and the Milnor image is

$$s\left(\frac{x^2 - (1+c)x + c}{x^2}\right) = \left(\frac{-c^3 - 5c^2 + c - 1}{c^2}, \frac{2c^3 + 7c^2 - 2c + 1}{c^2}\right).$$

This proves that $\operatorname{ShafDim}_2^1[\mathcal{P}_{3,1}]^\bullet = 1$ and $\operatorname{ShafDim}_2^1[\mathcal{P}_{3,1}]^\circ = 0$.

$\boldsymbol{\mathcal{P}_{3,2}}$: We move the three points to $1, 0, \infty$, and then f has the form $f(x) = a(x-1)/(bx^2+cx)$. This map has $\operatorname{Res}(f) = -a^2b(b+c)$, so we can dehomogenize $a = 1$ and replace c with $c-b$. This gives the map $f(x) = (x-1)/(bx^2 + (c-b)x)$ with $\operatorname{Res}(f) = bc$. Hence

$$(f, \{1, 0, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{3,2}]^\circ(K, S) \iff b, c \in R_S^*,$$

and it is in $\mathcal{GR}_2^1[\mathcal{P}_{3,2}]^\circ(K, S)$ if further f is not ramified at the points $\{0, 1, \infty\}$. The map f is never ramified at 1, while its multiplier at the 2-cycle is $(f^2)'(0) = (b-c)/c$. The Milnor image is

$$s\left(\frac{x-1}{bx^2 + (c-b)x}\right) = \left(\frac{b^3 - 3b^2c - 2b^2 + 3bc^2 - 4bc + b - c^3}{bc}, \right.$$
$$\left. \frac{-2b^3 + 6b^2c + 4b^2 - 6bc^2 + 9bc - 2b + 2c^3 - c^2}{bc}\right).$$

We used Magma to verify that the rational functions $s_1(b, c)$ and $s_2(b, c)$ are algebraically independent in $K(b, c)$. Hence under our assumption that $\#R_S^* = \infty$, we find that $\mathrm{ShafDim}_2^1[\mathcal{P}_{3,2}]^\circ = 2$.

However, if we also require the reduction of f to be unramified at $\{0, 1, \infty\}$ for all primes not in S, then we must also require that $b - c \in R_S^*$. Then $(c/b, 1 - c/b)$ is a solution to the S-unit equation $u + v = 1$, so there are only finitely many choices for the ratio c/b. For each such choice, say $c = ub$ with u fixed, the image in $\mathcal{M}_2^1$ lies on a curve. And taking, say, $u = -1$ gives the set of points

$$s\left(\frac{x-1}{bx^2-2bx}\right) = \left(\frac{-8b^2-2b-1}{b} \frac{16b^2+6b+2}{b}\right), \quad b \in R_S^*.$$

The Zariski closure of this set in $\mathcal{M}_2^1$ is a curve, more precisely, it is the line $2s_1 + s_2 = 2$. Hence $\mathrm{ShafDim}_2^1[\mathcal{P}_{3,2}]^\star = 1$.

$\boldsymbol{\mathcal{P}_{3,3}}$: We move the fixed point to ∞ and the 2-cycle to $\{0, 1\}$, which puts f into the form $f(x) = (x-1)(ax+b)/(cx-b)$. The resultant is $\mathrm{Res}(f) = ab(a+c)(b-c)$, so we may dehomogenize $b = 1$. This puts f in the form $f(x) = (x-1)(ax+1)/(cx-1)$ with resultant $\mathrm{Res}(f) = a(a+c)(1-c)$. Thus f has good reduction if and only if $a, a+c, 1-c \in R_S^*$, which gives a solution to the 4-term S-unit equation

$$a - (a+c) - (1-c) + 1 = 0.$$

The multivariable S-unit sum theorem [Evertse 1984; van der Poorten and Schlickewei 1991] says that there are finitely many solutions with no subsum equal to 0. Ignoring those finitely many solutions, there are three subsum 0 cases:

(1) $a - (a+c) = 0$, which implies that $e_f(\infty) = 2$.

(2) $a - (1-c) = 0$, which implies that $e_f(0) = 2$.

(3) $a + 1 = 0$, which implies that $e_f(1) = 2$.

This gives three families of pairs (f, X) in $\mathcal{GR}_2^1[\mathcal{P}_{3,3}]^\bullet(K, S)$, but every f is ramified at one of the three points in X, so these pairs are not in $\mathcal{GR}_2^1[\mathcal{P}_{3,3}]^\circ(K, S)$. Instead, they are $\mathcal{GR}_2^1[\mathcal{P}_{4,5}]^\circ(K, S)$ in case (1) and in $\mathcal{GR}_2^1[\mathcal{P}_{4,7}]^\circ(K, S)$ in cases (2) and (3). These give sets of points whose closures are curves:

$$\mathcal{P}_{4,5}: \qquad s\big(-ax^2 + (a-1)x + 1\big) = (2, -a^2 - 3), \quad a \in R_S^*,$$

$$\mathcal{P}_{4,7}: \qquad s\left(\frac{-x^2+2x-1}{cx-1}\right) = \left(\frac{-c^3+2}{(c-1)^2} \frac{2c^3-4}{(c-1)^2}\right), \quad c \in R_S^*.$$

More precisely, they give the curves $s_1 = 2$ and $2s_1 + s_2 = 0$. This completes the proof that $\mathrm{ShafDim}_2^1[\mathcal{P}_{3,3}]^\bullet = 1$ and $\mathrm{ShafDim}_2^1[\mathcal{P}_{3,3}]^\circ = 0$.

$\boldsymbol{\mathcal{P}_{3,4}}$: We move the three points to $1, 0, \infty$, and then f has the form $f(x) = (x-1)(ax+b)/cx$. This map has $\mathrm{Res}(f) = -abc^2$, so we can dehomogenize $c = 1$.

Then $f(x) = (x-1)(ax+b)/x$ has good reduction if and only if $a, b \in R_S^*$. The multiplier at the fixed point is $f'(\infty) = a^{-1}$, so f is not ramified at ∞, and similarly since $f^{-1}(f(0)) = \{0, \infty\}$, the map f is not ramified at 0. And these statements are true even modulo primes not in S. Finally we observe that $f'(1) = a+b$, so f is ramified at 1 if and only if $a+b = 0$. The Milnor image is

$$s\left(\frac{(x-1)(ax+b)}{x}\right) = \left(\frac{a^3+2a^2b+ab^2-2ab+b}{ab},\right.$$
$$\left.\frac{a^4+2a^3b+a^2b^2-4a^2b+a^2+3ab+b^2-2b}{ab}\right),$$

We used Magma to verify that the rational functions $s_1(b,c)$ and $s_2(b,c)$ are algebraically independent in $K(b,c)$. Hence under our assumption that $\#R_S^* = \infty$, we find that $\mathrm{ShafDim}_2^1[\mathcal{P}_{3,4}]^\circ = 2$.

However, if we also require the reduction of f to be unramified at $\{0, 1, \infty\}$ for all primes not in S, then we must also require that $a+b \in R_S^*$. Then $(-b/a, 1+b/a)$ is a solution to the S-unit equation $u+v=1$, so there are only finitely many choices for the ratio b/a. For each such choice, say $b = ua$ with u fixed, the image in $\mathcal{M}_2^1$ lies on a curve. And taking, say, $u=1$ gives the set of points

$$s\left(\frac{a(x^2-1)}{x}\right) = \left(\frac{4a^2-2a+1}{a}, \frac{4a^3-4a^2+5a-2}{a}\right), \quad a \in R_S^*.$$

The Zariski closure in $\mathcal{M}_2^1$ is a curve. Hence $\mathrm{ShafDim}_2^1[\mathcal{P}_{3,4}]^\star = 1$.

$\boldsymbol{\mathcal{P}_{3,5}}$: We move the three points to $0, 1, \infty$, which puts f in the form $f(x) = (ax^2+(b-a)x)/c(x-1)$ with $\mathrm{Res}(f) = abc^2$. We dehomogenize $c = 1$, so $f(x) = (ax^2+(b-a)x)/(x-1)$. We have $f'(\infty) = a^{-1}$ and $f^{-1}(f(1)) = \{1, \infty\}$, so $a \in R_S^*$ implies that f is unramified at ∞ and at 1, even modulo primes not in S. Further, $f'(0) = a-b$, so f is unramified at 0 if and only if $a \ne b$. The Milnor image is

$$s\left(\frac{ax^2+(b-a)x}{x-1}\right) = \left(\frac{-a^3+2a^2b+2a^2-ab^2-a+b}{ab},\right.$$
$$\left.\frac{-a^4+2a^3b+2a^3-a^2b^2-2a^2b-2a^2+3ab+2a-b^2-1}{ab}\right).$$

We used Magma to verify that the rational functions $s_1(a,b)$ and $s_2(a,b)$ are algebraically independent in $K(a,b)$. Hence under our assumption that $\#R_S^* = \infty$, we find that $\mathrm{ShafDim}_2^1[\mathcal{P}_{3,5}]^\circ = 2$.

However, if we also require the reduction of f to be unramified at $\{0, 1, \infty\}$ for all primes not in S, then we must also require that $a-b \in R_S^*$. Then $(b/a, 1-b/a)$ is a solution to the S-unit equation $u+v=1$, so there are only finitely many choices for the ratio b/a. For each such choice, say $b = ua$ with u fixed, the image in $\mathcal{M}_2^1$ lies on a curve. And taking, say, $u = -1$ gives the set of points

$$s\left(\frac{a(x^2-2x)}{x-1}\right) = \left(\frac{4a^2-2a+2}{a}, \frac{4a^4-4a^3+6a^2-2a+1}{a^2}\right), \quad a \in R_S^*.$$

The Zariski closure in $\mathcal{M}_2^1$ is a curve, so $\mathrm{ShafDim}_2^1[\mathcal{P}_{3,5}]^\star = 1$.

$\boldsymbol{\mathcal{P}_{3,6}}$: We move the three fixed points to 0, 1, ∞, so that f has the form $f(x) = (ax^2+bx)/((a-c)x+b+c)$ with $\mathrm{Res}(f) = ac(a+b)(b+c)$. We dehomogenize $a = 1$, so $f(x) = (x^2+bx)/((1-c)x+b+c)$, and we compute the three multipliers: $f'(0) = b/(b+c)$, $f'(1) = (b+c+1)/(b+c)$, $f'(\infty) = 1-c$. We have

$$(f, \{0, 1, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{3,6}]^\bullet(K, S) \quad \Longleftrightarrow \quad c, 1+b, b+c \in R_S^*.$$

These maps give a solution to the 4-term S-unit equation

$$(b+c) - c - (1+b) + 1 = 0.$$

The multivariable S-unit sum theorem [Evertse 1984; van der Poorten and Schlickewei 1991] says that there are finitely many solutions with no subsum equal to 0. Ignoring those finitely many solutions, there are three subsum 0 cases:

$$\begin{aligned}
(b+c)-c=0 &\implies f(x) = \frac{x^2}{(1-c)x+c} &&\implies e_f(0)=2,\\
(b+c)-(1+b)=0 &\implies f(x) = \frac{x^2+bx}{b+1} &&\implies e_f(\infty)=2,\\
(b+c)+1=0 &\implies f(x) = \frac{x^2+bx}{(b+2)x-1} &&\implies e_f(1)=2.
\end{aligned}$$

This proves that $\mathrm{ShafDim}_2^1[\mathcal{P}_{3,6}]^\circ = 0$, since the subsum 0 cases have a ramified point, and hence are actually in $\mathcal{GR}_2^1[\mathcal{P}_{4,3}]^\circ(K, S)$. The closure of these maps in $\mathcal{M}_2^1$ is a finite set of curves, since for example the family with $c = 1$ gives the family of polynomials $f(x) = (x^2+bx)/(b+1)$ whose closure in $\mathcal{M}_2$ for $b+1 \in R_S^*$ is the line $s_1 = 2$. This proves that $\mathrm{ShafDim}_2^1[\mathcal{P}_{3,6}]^\bullet = 1$, and also (for future reference) that $\mathrm{ShafDim}_2^1[\mathcal{P}_{4,3}]^\circ = 1$.

$\boldsymbol{\mathcal{P}_{3,7}}$: Moving the two points to 0 and ∞ with 0 critical, the map f has the form $f(x) = (ax+b)/cx^2$ with $\mathrm{Res}(f) = b^2c^2$. Dehomogenizing $c = 1$ gives the map $f(x) = (ax+b)/x^2$, which has good reduction if and only if $b \in R_S^*$. We conjugate $u^{-1}f(ux)$ with $u = \sqrt[3]{b}$, which is okay since $K(\sqrt[3]{b})$ is unramified outside S. This puts f into the form $f(x) = (ax+1)/x^2$ with $\mathrm{Res}(f) = 1$. We also note that f is ramified at ∞ if and only if $a = 0$, so taking $a \in R_S^*$ gives maps such that ∞ is unramified modulo all primes not in S. This map has Milnor coordinates

$$s\left(\frac{ax+1}{x^2}\right) = (a^3-6, -2a^3+12),$$

so taking the Zariski closure for $a \in R_S^*$ gives the line $2s_1 + s_2 = 0$. Hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{3,7}]^\bullet = \operatorname{ShafDim}_2^1[\mathcal{P}_{3,7}]^\star = 1$.

$\boldsymbol{\mathcal{P}_{3,8}}$: Moving the totally ramified fixed point to ∞ and the other fixed point to 0, we have $f(x) = ax^2 + bx$ with $\operatorname{Res}(f) = a^2$. Conjugating by $x \mapsto a^{-1}x$ puts f into the form $f(x) = x^2 + bx$, and then $(f, \{0, \infty\})$ is in $\mathcal{GR}_2^1[\mathcal{P}_{3,8}]^\circ(K, S)$ for all $b \in R_S$ with $b \neq 0$, and $\mathcal{GR}_2^1[\mathcal{P}_{3,8}]^\star(K, S)$ for all $b \in R_S^*$. The Zariski closure of the Milnor image of these maps in $\mathcal{M}_2^1 \cong \mathbb{A}^2$ is the line $s_1 = 2$. Hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{3,8}]^\bullet = \operatorname{ShafDim}_2^1[\mathcal{P}_{3,8}]^\star = 1$.

This completes our analysis of the 13 portraits of weights 1, 2, and 3 in Table 2. We move on to analyzing the 22 portraits of weight 4 in Tables 3 and 4.

$\boldsymbol{\mathcal{P}_{4,1}}$: Moving the two totally ramified fixed points to 0 and ∞, the map has the form $f(x) = ax^2$. Good reduction forces $a \in R_S^*$, and then conjugation $af(a^{-1}x)$ yields $f(x) = x^2$. Hence $\mathcal{GR}_2^1[\mathcal{P}_{4,1}]^\bullet(K, S)/\operatorname{PGL}_2(R_S)$ consists of a single element.

$\boldsymbol{\mathcal{P}_{4,2}}$: Moving the two totally period 2 points to 0 and ∞, the map has the form $f(x) = ax^{-2}$. Good reduction forces $a \in R_S^*$, and then conjugation $a^{-1}f(ax)$ yields $f(x) = x^{-2}$. Hence $\mathcal{GR}_2^1[\mathcal{P}_{4,2}]^\bullet(K, S)/\operatorname{PGL}_2(R_S)$ consists of a single element.

$\boldsymbol{\mathcal{P}_{4,3}}$: Moving the fixed points to $0, 1, \infty$ with ∞ ramified, the map f has the form $f(x) = ax^2 + (1-a)x$ with $\operatorname{Res}(f) = a^2$. Conjugating gives $af(a^{-1}x) = x^2 + (1-a)x$. The multipliers at 0 and 1 are $f'(0) = 1-a$ and $f'(1) = 3-a$. The Milnor image is $s\bigl(x^2 + (1-a)x\bigr) = (2, 1-a^2)$, so $a \in R_S^*$ gives a Zariski dense set of points in the line $s_1 = 2$, and the same is true if we disallow $a = 1$ and $a = 3$. This proves that $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,3}]^\circ = 1$; cf. the analysis of $\mathcal{P}_{3,6}$. However, if we also insist that 0 and 1 are unramified modulo all primes outside S, then we need $1-a \in R_S^*$ and $3-a \in R_S^*$. In particular, $(a, 1-a)$ is a solution to the S-unit equation $u + v = 1$, so there are only finitely many values of a. This proves that $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,3}]^\star = 0$.

$\boldsymbol{\mathcal{P}_{4,4}}$: Moving the ramified fixed point to ∞, the unramified fixed point to 0, and the other point to 1, we find that f has the form $f(x) = ax^2 - ax$ with $\operatorname{Res}(f) = a^2$. Since $f'(0) = -a$ and $f'(1) = a$, we see that f is unramified at 0 and 1 modulo all primes not in S, and hence $(f, \{0, 1, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{4,4}]^\star(K, S)$ for all $a \in R_S^*$. The Milnor image is $s(ax^2 - ax) = (2, -a^2 - 2a)$, so $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,4}]^\star = 1$.

$\boldsymbol{\mathcal{P}_{4,5}}$: We move the ramified fixed point to ∞ and the other two points to 0 and 1. Then f has the form $f(x) = ax^2 - (a+1)x + 1$ with $\operatorname{Res}(f) = a^2$ and Milnor image $s(ax^2 - (a+1)x + 1) = (2, -a^2 - 3)$. The multiplier for the 2-cycle is

$(f^2)'(0) = 1 - a^2$. Hence

$$(f, \{0, 1, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{4,5}]^\circ(K, S) \iff a \in R_S^* \text{ and } a \neq \pm 1.$$

In particular, we see that $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,5}]^\circ = 1$; cf. the analysis of $\mathcal{P}_{3,3}$. However, if we also require that the 2-cycle be unramified modulo all primes not in S, then we need $1 - a^2 \in R_S^*$. This gives solutions $(a, 1 - a)$ to the S-unit equation $u + v = 1$, so there are only finitely many maps, and hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,5}]^\star = 0$.

$\boldsymbol{\mathcal{P}_{4,6}}$: We move the points to $0, 1, \infty$ so that $1 \to 0 \to \infty \to \infty$. Before imposing the condition that f is ramified at 1, this put f in the form $f(x) = (ax^2 + bx + c)/ex$ with $a + b + c = 0$ and $\operatorname{Res}(f) = ace^2$. We dehomogenize $e = 1$, and then setting $f'(1) = 0$, we find that f has the form $f(x) = a(x - 1)^2/x$. Then $f'(\infty) = a^{-1}$ and $f^{-1}(f(0)) = \{0, \infty\}$, so f is unramified at 0 and ∞ modulo all primes not in S. This gives

$$(f, \{0, 1, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{4,6}]^\star(K, S) \iff a \in R_S^*.$$

The Milnor image is

$$s\left(\frac{a(x-1)^2}{x}\right) = \left(\frac{-2a+1}{a}, \frac{-4a^2+a-2}{a}\right),$$

so the Zariski closure is a curve, and hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,6}]^\star = 1$.

$\boldsymbol{\mathcal{P}_{4,7}}$: We move 0 to the fixed point and ∞ and 1 to the 2-cycle with ∞ ramified. Ignoring the ramification at ∞ for the moment, we find that f has the form $(ax^2+bx)/(x - 1)(ax + c)$. Then we see that f is ramified at ∞ if and only if $c = a + b$, so $f(x) = (ax^2 + bx)/(x - 1)(ax + a + b)$. We compute $\operatorname{Res}(f) = a^2(a + b)^2$, so we can dehomogenize $a = 1$, and for convenience replace b with $b - 1$, to get $f(x) = (x^2 + (b-1)x)/(x-1)(x+b)$ with $\operatorname{Res}(f) = b^2$. Further, we see that f is unramified at 0 if and only if $b \neq 1$ and f is unramified at 1 if and only if $b \neq -1$. Hence

$$(f, \{0, 1, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{4,7}]^\circ(K, S) \iff b \in R_S^* \text{ and } b \neq \pm 1.$$

The Milnor image of f is

$$s\left(\frac{x^2+(b-1)x}{(x-1)(x+b)}\right) = \left(\frac{b^3+3b^2-3b+1}{b}, \frac{-2b^3-6b^2+6b-2}{b}\right),$$

which proves that $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,7}]^\circ = 1$. Indeed, we have again landed on the line $2s_1 + s_2 = 0$; cf. the analysis of $\mathcal{P}_{3,3}$. However, if we want f to be unramified at 0 and 1 modulo all primes not in S, then we need $1 \pm b \in R_S^*$. In particular, $(b, 1 - b)$ is one of the finitely many solutions of the S-unit equation $u + v = 1$, so $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,7}]^\star = 0$.

$\boldsymbol{\mathcal{P}_{4,8}}$: We move the 2-cycle to 0 and ∞ with 0 ramified and the other point to 1. Then f has the form $f(x) = a(x-1)/bx^2$ with $\operatorname{Res}(f) = a^2b^2$, so we can dehomogenize $a = 1$ to get $f(x) = (x-1)/bx^2$. Assuming that $b \in R_S^*$, we observe that f is unramified at 1 and ∞, even modulo primes not in S. Hence

$$(f, \{0, 1, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{4,8}]^\star(K, S) \quad \text{for all } b \in R_S^*.$$

The Milnor image is

$$s\left(\frac{x-1}{bx^2}\right) = \left(\frac{-6b+1}{b}, \frac{12b-2}{b}\right),$$

so the Zariski closure in $\mathcal{M}_2^1$ of $\mathcal{GR}_2^1[\mathcal{P}_{4,8}]^\star(K, S)$ is the line $2s_1 + s_2 = 0$.

$\boldsymbol{\mathcal{P}_{4,9}}$: We move the 3-cycle to $1 \to 0 \to \infty \to 1$ with 1 a ramification point. This puts f in the form $f(x) = a(x-1)^2/(ax^2+ex)$ with $\operatorname{Res}(f) = a^2(a+e)^2$. We dehomogenize $a = 1$ and replace e with $e-1$ to get $f(x) = (x-1)^2/(x^2+(e-1)x)$ with $\operatorname{Res}(f) = e^2$. The fact that 1 is a ramification point in a 3-cycle tells us that $(f^3)'(1) = 0$, and one of the other points in the 3-cycle will also be ramified if and only if $(f^3)''(1) = 2(1-e^2)/e = 0$. Hence

$$(f, \{0, 1, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{4,9}]^\circ(K, S) \quad \Longleftrightarrow \quad e \in R_S^* \text{ and } e \neq \pm 1.$$

The Milnor image is

$$s\left(\frac{(x-1)^2}{x^2+(e-1)x}\right) = \left(\frac{e^3-5e^2-e-1}{e^2}, \frac{-2e^3+7e^2+2e+1}{e^2}\right),$$

so the closure of $\mathcal{GR}_2^1[\mathcal{P}_{4,9}]^\circ(K, S)$ is a curve and $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,9}]^\circ = 1$. However, if we want the 3-cycle to contain only one ramification point modulo primes not in S, then we need $e^2 - 1 \in R_S^*$. This yields solutions $(e, 1-e)$ to the S-unit equation $u + v = 1$, so there are only finitely many such maps and $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,9}]^\star = 0$.

$\boldsymbol{\mathcal{P}_{4,10}}$: We move the three fixed points to 0, 1, and ∞, and let the fourth point be α with $f(\alpha) = 0$. Then f has the form $f(x) = (ax^2+bx)/(ex+a+b-e)$ with $\alpha = -b/a$ and

$$\operatorname{Res}(f) = a(a+b)(a-e)(a+b-e).$$

We dehomogenize $a = 1$, so $f(x) = (x^2+bx)/(ex+1+b-e)$ and $\alpha = -b$. Then

$$\begin{aligned} \{0, 1, \infty, -b\} \text{ has good reduction} &\iff b, 1+b \in R_S^*, \\ f \text{ has good reduction} &\iff 1+b, 1-e, 1+b-e \in R_S^*, \\ \big(f(x), \{0, 1, \infty, -b\}\big) \in \mathcal{GR}_2^1[\mathcal{P}_{4,10}]^\bullet(K, S) &\iff b, 1+b, 1-e, 1+b-e \in R_S^*. \end{aligned}$$

But this means that $(-b, 1+b)$ is a solution to the S-unit equation $u+v = 1$, so there are only finitely many values for b; and then the fact that $\big(b^{-1}(e-1), b^{-1}(1+b-e)\big)$

is also a solution to the S-unit equation proves that there are only finitely many values for e. This completes the proof that $\mathcal{GR}_2^1[\mathcal{P}_{4,10}]^\bullet(K, S)/\operatorname{PGL}_2(R_S)$ is finite.

$\boldsymbol{\mathcal{P}_{4,11}}$: We move the points so that 0 and ∞ are fixed by f and $f(1) = 0$. This puts f in the form $f(x) = ax(x-1)/(bx-c)$, with $\operatorname{Res}(f) = a^2c(c-b)$. We dehomogenize $c = 1$, so $f(x) = ax(x-1)/(bx-1)$. Then $f^{-1}(\infty) = \{\infty, b^{-1}\}$, and our assumption that we have a good reduction model for $\mathcal{P}_{4,11}$ requires that b^{-1} be distinct from $\{0, 1, \infty\}$ for all primes not in S. Thus $b^{-1} \in R_S^*$ and $b^{-1} - 1 \in R_S^*$. The S-unit equation $u - v = 1$ has only finitely many solutions, so there are finitely many values for b. We observe that for these b values, the map f is unramified modulo all primes not in S, since $f^{-1}(f(0)) = f^{-1}(f(1)) = \{0, 1\}$ and $f^{-1}(f(\infty)) = f^{-1}(f(b^{-1})) = \{\infty, b^{-1}\}$. We also note that we can take $b = 2$, since $2 \in R_S^*$ by assumption. Thus for every $a \in R_S^*$, we see that $\bigl(ax(x-1)/(2x-1), \{0, 1, 2^{-1}, \infty\}\bigr)$ is in $\mathcal{GR}_2^1[\mathcal{P}_{4,11}]^\star(K, S)$. The Milnor image is

$$s\left(\frac{ax(x-1)}{2x-1}\right) = \left(\frac{2a^2 - 2a + 4}{a}, \frac{a^4 - 2a^3 + 6a^2 - 4a + 4}{a^2}\right),$$

and hence the Zariski closure of $\mathcal{GR}_2^1[\mathcal{P}_{4,11}]^\circ(K, S)$ in $\mathcal{M}_2^1$ is a nonempty finite union of curves. (We remark that the pairs (f, X) studied in Section 4, when restricted to the case $d = 2$, have portrait $\mathcal{P}_{4,11}$.)

$\boldsymbol{\mathcal{P}_{4,12}}$: We move the points so that 0 and ∞ are fixed by f and $f(1) = 0$. This puts f in the form $f(x) = ax(x-1)/(bx-c)$, with $\operatorname{Res}(f) = a^2c(c-b)$. We dehomogenize $a = 1$, so $f(x) = x(x-1)/(bx-c)$. The portrait $\mathcal{P}_{4,12}$ includes a point in $f^{-1}(1) = \{x^2 - (1+b)x + c = 0\}$, and this point is in K, since the portrait is assumed to be $\operatorname{Gal}(\bar{K}/K)$-invariant. Thus $(1+b)^2 - 4c = t^2$ for some $t \in K$. Then $(1+b+t)(1+b-t) = 4c \in R_S^*$, so if we have a good reduction portrait for f, then $c, c-b, 1+b \pm t \in R_S^*$. This gives us a 5-term S-unit sum

$$(1+b+t) + (1+b-t) + 2(c-b) - 2c - 2 = 0.$$

There are only finitely many solutions with no subsum equal to 0 [Evertse 1984; van der Poorten and Schlickewei 1991], so it remains to analyze the 10 cases where some subsum vanishes.

$\underline{(1+b+t) + (1+b-t) = 0.}$ So $b = -1$ and $f(x) = -x(x-1)/(x+c)$. Then c and $c+1$ are in R_S^*, so there are only finitely many choices for c.

$\underline{(1+b \pm t) + 2(c-b) = 0.}$ So $a - b + 2c \pm t = 0$. Substituting into $(1+b)^2 - 4c = t^2$ to eliminate t yields $b = c(c+2)/(c+1)$, and from that we find that $c/(b-c) = 1+c$. We know that $c, b-c \in R_S^*$, so this shows that $1+c \in R_S^*$. But then $(1+c, -c)$ is a solution to the S-unit equation $u+v = 1$, so there are only finitely many possibilities for c.

$\underline{(1+b\pm t)-2c=0.}$ So $1+b\pm t=2c$. Substituting into $(1+b)^2-4c=t^2$ to eliminate t yields $c^2-bc=0$, so either $c=0$ or $c-b=0$. This contradicts the fact that c and $c-b$ are S-units.

$\underline{(1+b\pm t)-2=0.}$ So $1+b\pm t=2$. Substituting into $(1+b)^2-4c=t^2$ to eliminate t yields $c-b=0$, contradicting the fact that $c-b\in R_S^*$.

$\underline{2(c-b)-2c=0.}$ So $b=0$ and $f(x)=x(x-1)/c$. We have $c\in R_S^*$ and $1-4c=t^2$. We write $a=\gamma u^3$ with $u\in R_S^*$ and γ chosen from a finite set of representatives for $R_S^*/(R_S^*)^3$. Then (u,t) is an R_S-integral point on the genus 1 curve $y^2=1-4\gamma x^3$. Siegel's theorem [Hindry and Silverman 2000, D.9.1] says that there are only finitely many such points.

$\underline{2(c-b)-2=0.}$ So $c=b+1$ and $f(x)=x(x-1)/(bx-b-1)$. We have $c\in R_S^*$ and $c^2-4c=t^2$. We write $c=\gamma u^3$ with $u\in R_S^*$ and γ chosen from a finite set of representatives for $R_S^*/(R_S^*)^2$. Then $(u,t/u)$ is an R_S-integral point on the genus 1 curve $y^2=\gamma^2x^4-4\gamma x$. Siegel's theorem [Hindry and Silverman 2000, D.9.1] says that there are only finitely many such points.

$\underline{-2c-2=0.}$ So $c=-1$ and $f(x)=x(x-1)/(bx+1)$. We have $1+b\in R_S^*$ and $(1+b)^2+4=t^2$. We write $1+b=\gamma u^2$ with $u\in R_S^*$ and γ chosen from a finite set of representatives for $R_S^*/(R_S^*)^2$. Then (u,t) is an R_S-integral point on the genus 1 curve $y^2=\gamma^2u^4+4$. Siegel's theorem [Hindry and Silverman 2000, D.9.1] says that there are only finitely many such points.

$\boldsymbol{\mathcal{P}_{4,13}}$: We move the 2-cycle to $0,\infty$, so $f(x)=(ax+b)/(cx^2+dx)$. The resultant is $-bc(ad-bc)$, so we can dehomogenize $c=1$. Moving a fixed point to 1, we have $a+b=d+1$, so $f(x)=(ax+b)/(x^2+(a+b-1)x)$ with $\operatorname{Res}(f)=-b(a-1)(a+b)$. The good reduction assumption for f tells us that $b, a-1, a+b\in R_S^*$, so we obtain a 4-term S-unit equation

$$(a+b)-(a-1)-b-1=0.$$

The multivariable S-unit sum theorem [Evertse 1984; van der Poorten and Schlickewei 1991] says that there are finitely many solutions with no subsum equal to 0. Ignoring those finitely many solutions, there are three subsum 0 cases:

(1) $(a+b)-(a-1)=0$, so $b=-1$.

(2) $(a+b)-b=0$, so $a=0$.

(3) $(a+b)-1=0$, so $a=1-b$.

The portrait $\mathcal{P}_{4,13}$ has a second fixed point. The fixed points of f are the roots of

$$(x-1)(x^2+(a+b)x+b)=0.$$

We have assumed that the points in $\mathcal{P}_{4,13}$ are defined over K, so the quadratic has a root in K. Thus there is a $t \in K$ such that

$$(a+b)^2 - 4b = t^2.$$

And since $a, b \in R_S$, we have $t \in R_S$. From earlier we know that $a+b$ and b are in R_S^*, so we can write $a+b = \gamma u^2$ and $b = \delta v^4$, with $u, v \in R_S^*$ and γ, δ chosen from a finite set of representatives for $R_S^*/(R_S^*)^4$. Then (uv^{-1}, tv^{-2}) is a R_S-integral point on the genus 1 curve $y^2 = \gamma^2 x^4 - 4\delta$. Siegel's theorem [Hindry and Silverman 2000, D.9.1] says that there are only finitely many such points. Hence there are only finitely many possibilities for the ratio u/v, and thus only finitely many possibilities for $\gamma^2\delta^{-1}(u/v)^4 = (a+b)^2/b$. But we know from the three cases described earlier that either $b = -1$ or $a = 0$ or $a = 1-b$. Substituting these into $(a+b)^2/b$, we find that there are finitely many values for, respectively, $-(a-1)^2$, b, and $1/b$.

$\boldsymbol{\mathcal{P}_{4,14}}$: We move the points to $\alpha \to 1 \to 0 \to \infty$ with ∞ fixed. Ignoring α for the moment, this means that f has the form $f(x) = (ax^2 - (a+c)x + c)/ex$. We have $\operatorname{Res}(f) = -ace^2$, so good reduction forces $a, c, e \in R_S^*$. We dehomogenize by setting $e = 1$. At this stage the pair $(f, \{0, 1, \infty\})$ has good reduction. However, we need to adjoin the point α to the set X. The point α is a root of the numerator of $f(x) - 1$, so α is a root of the polynomial

$$ax^2 - (a+c+1)x + c = 0. \tag{11}$$

Since we are assuming that $\alpha \in K$, the discriminant of this quadratic polynomial is a square in K, say

$$t^2 = (a+c+1)^2 - 4ac \quad \text{with } t \in R_S.$$

Then

$$(a+c+1+t)(a+c+1-t) = 4ac \in R_S^*,$$

so $a+c+1 \pm t \in R_S^*$. So we now know four S-units,

$$a,\ c,\ a+c+1+t,\ a+c+1-t \in R_S^*,$$

which yields a 5-term S-unit sum

$$(a+c+1+t) + (a+c+1-t) - 2a - 2c - 2 = 0.$$

There are only finitely many solutions with no subsum equal to 0 [Evertse 1984; van der Poorten and Schlickewei 1991], so it remains to analyze the 10 cases where some subsum vanishes.

<u>$(a+c+1+t)+(a+c+1-t) = 0.$</u> Substituting $c = -a-1$, we find that $t^2 = -4a(a-1)$. Since $a \in R_S^*$, we may write $a = \gamma u^3$ with $u \in R_S^*$ and γ chosen from a finite set of representatives for $R_S^*/(R_S^*)^3$. Then (u, tu^{-1}) is an S-integral point on

the genus 1 curve $y^2 = -4\gamma^2 x^4 + 4\gamma^2 x$. Siegel's theorem [Hindry and Silverman 2000, D.9.1] says that there are only finitely many such points.

$\underline{(a+c+1\pm t)-2a=0.}$ Then $0=(a+c+1)^2-4ac-t^2=4a$, contradicting $a\in R_S^*$.

$\underline{(a+c+1\pm t)-2c=0.}$ Then $0=(a+c+1)^2-4ac-t^2=4c$, contradicting $c\in R_S^*$.

$\underline{(a+c+1\pm t)-2=0.}$ Then

$$0=(a+c+1)^2-4ac-t^2=4(a+c-ac).$$

Hence $1=(a-11)(c-1)$, so $a-1$ and $c-1$ are S-units. Thus $(1-a,a)$ and $(1-c,c)$ are each solutions to the S-unit equation $u+v=1$, which has finitely many solutions.

$\underline{-2a-2c=0.}$ Substituting $a=-c$, we find that $t^2=1+4a^2$. Since $a\in R_S^*$, we may write $a=\gamma u^2$ with $u\in R_S^*$ and γ chosen from a finite set of representatives for $R_S^*/(R_S^*)^2$. Then (u,t) is an R_S-integral point on the genus 1 curve $y^2=1+4\gamma^2x^4$. Siegel's theorem [Hindry and Silverman 2000, D.9.1] says that there are only finitely many such points.

$\underline{-2a-2=0.}$ Substituting $a=-1$, we find that $t^2=c^2+4c$. Since $c\in R_S^*$, we may write $c=\gamma u^3$ with $u\in R_S^*$ and γ chosen from a finite set of representatives for $R_S^*/(R_S^*)^3$. Then (u,tu^{-1}) is an S-integral point on the genus 1 curve $y^2=\gamma^2x^4+4\gamma x$. Siegel's theorem [Hindry and Silverman 2000, D.9.1] says that there are only finitely many such points.

$\underline{-2c-2=0.}$ Substituting $c=-1$, we find that $t^2=a^2+4a$. The analysis is then identical to the previous case with $-2a-2=0$.

$\boldsymbol{\mathcal{P}_{4,15}}$: The portrait $\mathcal{P}_{4,15}$ contains the portrait $\mathcal{P}_{3,3}$ as a subportrait, and we already proved that $\operatorname{ShafDim}_2^1[\mathcal{P}_{3,3}]^\circ=0$, so the same is true for $\mathcal{P}_{4,15}$. On the other hand, if we allow any of the points in $\mathcal{P}_{4,15}$ to have weight greater than 1, then the total weight would be at least 5, in which case Theorem 2(a) gives us finiteness. Hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,15}]^\bullet=0$.

$\boldsymbol{\mathcal{P}_{4,16}}$: The portrait $\mathcal{P}_{4,16}$ contains the portrait $\mathcal{P}_{3,3}$ as a subportrait, and we already proved that $\operatorname{ShafDim}_2^1[\mathcal{P}_{3,3}]^\circ=0$, so the same is true for $\mathcal{P}_{4,16}$. On the other hand, if we allow any of the points in $\mathcal{P}_{4,16}$ to have weight greater than 1, then the total weight would be at least 5, in which case Theorem 2(a) gives us finiteness. Hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,16}]^\bullet=0$.

$\boldsymbol{\mathcal{P}_{4,17}}$: The portrait $\mathcal{P}_{4,17}$ contains the portrait $\mathcal{P}_{3,1}$ as a subportrait, and we already proved that $\operatorname{ShafDim}_2^1[\mathcal{P}_{3,1}]^\circ=0$, so the same is true for $\mathcal{P}_{4,17}$. On the other hand, if we allow any of the points in $\mathcal{P}_{4,17}$ to have weight greater than 1, then the total weight would be at least 5, in which case Theorem 2(a) gives us finiteness. Hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,17}]^\bullet=0$.

$\boldsymbol{\mathcal{P}_{4,18}}$: Moving the four points to $b, 1, 0, \infty$, we see that $f(x) = a(x-1)(x-b)/ex$ with $\operatorname{Res}(f) = a^2be^2$, so we can dehomogenize $a = 1$. Then

$$\bigl(f(x), \{b, 1, 0, \infty\}\bigr) \in \mathcal{GR}_2^1[\mathcal{P}_{4,18}]^\bullet(K, S) \iff b, 1-b, e \in R_S^*.$$

(Note that $b, 1-b \in R_S^*$ is the condition for $\{b, 0, 1, \infty\}$ to have good reduction outside S.) Then $(b, 1-b)$ is a solution to the S-unit equation $u + v = 1$, so there are finitely many values for b. Each value of b, for example $b = 2$, yields a curve in $\mathcal{M}_2^1$, for example, the Milnor image with $b = 2$ is

$$s\left(\frac{(x-1)(x-2)}{ex}\right) = \left(\frac{2e^2 - 4e + 17}{2e}, \frac{-4e^3 + 19e^2 - 8e + 17}{2e^2}\right).$$

Hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,18}]^\circ = 1$. However, since $f^{-1}(f(1)) = f^{-1}(f(b)) = \{1, b\}$ and $f^{-1}(f(0)) = f^{-1}(f(\infty)) = \{0, \infty\}$, we see that f modulo primes not in S is unramified at the points in $\{b, 1, 0, \infty\}$, so the above maps with $b = 2$ and $e \in R_S^*$ are in $\mathcal{GR}_2^1[\mathcal{P}_{4,18}]^\star(K, S)$, and hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,18}]^\circ = 1$.

$\boldsymbol{\mathcal{P}_{4,19}}$: Moving $0, \infty$ to the 2-cycle and 1 to the incoming point, we see that $f(x) = a(x-1)/(bx^2 + cx)$. This has $\operatorname{Res}(f) = a^2b(b+c)$. We dehomogenize $b = 1$, so $f(x) = a(x-1)/x(x+c)$ with $a, 1+c \in R_S^*$. The fourth point of the portrait is in $f^{-1}(1)$, so it is a root of $x^2 + (c-a)x + a$. Since that point is in K by assumption, we see that the discriminant $(c-a)^2 - 4a$ must be a square in K, say equal to t^2. Then

$$(c - a + t)(c - a - t) = 4a \in R_S^*,$$

so $c - a \pm t \in R_S^*$. This gives us a 5-term S-unit sum

$$(c - a + t) + (c - a - t) - 2(1 + c) + 2a + 2 = 0.$$

There are only finitely many solutions with no subsum equal to 0 [Evertse 1984; van der Poorten and Schlickewei 1991], so it remains to analyze the 10 cases where some subsum vanishes.

$\underline{-2(1+c) + 2a = 0.}$ Then $a = c + 1$ and $f(x) = a(x-1)/x(x+a-1)$. We have $1 - 4a = t^2$. We write $a = \gamma u^4$ with $u \in R_S^*$ and γ chosen from a finite set of representatives for $R_S^*/(R_S^*)^4$. Then $1 - 4\gamma u^4 = t^2$, so (u, t) is an R_S-integral point on the genus 1 curve $Y^2 = 1 - 4\gamma X^4$. Siegel's theorem [Hindry and Silverman 2000, D.9.1] tells us that there are only finitely many solutions.

$\underline{-2(1+c) + 2 = 0.}$ Then $c = 0$ and $f(x) = a(x-1)/x^2$. This map has $e_f(0) = 2$, so we do not get the portrait $\mathcal{P}_{4,19}$ in which every point has multiplicity 1.

$\underline{2a + 2 = 0.}$ Then $a = -1$ and $f(x) = (-x+1)/x(x+c)$. We have $(c+1)^2 + 4 = t^2$. We write $c + 1 = \gamma u^2$ with $u \in R_S^*$ and γ chosen from a finite set of representatives for $R_S^*/(R_S^*)^2$. Then $\gamma^2u^4 + 4 = t^2$, so (u, t) is an R_S-integral point on the genus 1 curve $Y^2 = \gamma^2 X^4 + 4$. Siegel's theorem [Hindry and Silverman 2000, D.9.1] tells us that there are only finitely many solutions.

$\underline{(c-a+t)+(c-a-t)=0.}$ Then $a=c$ and $f(x)=a(x-1)/x(x+a)$. We have $a,\,1+a \in R_S^*$, so $(-a,\,1+a)$ is a solution to the S-unit equation $u+v=1$. Here there are only finitely many choices for a.

$\underline{(c-a\pm t)-2(1+c)=0.}$ Then $\pm t = a+c+2$, and the equation $(c-a)^2-4a=t^2$ becomes $ac+2a+c+1=0$. We rewrite this as $a(c+1)+(c+1)+1=0$. Thus $\bigl(a(c+1),\,c+1\bigr)$ is a solution to the S-unit equation $u+v+1=0$, so has only finitely many solutions.

$\underline{(c-a\pm t)+2a=0.}$ Then $\pm t = a+c$, and the equation $(c-a)^2-4a=t^2$ becomes $4a(c+1)=0$. This contradicts the fact that a and c are in R_S^*.

$\underline{(c-a\pm t)+2=0.}$ Then $\pm t = c-a+2$, and the equation $(c-a)^2-4a=t^2$ becomes $4(c+1)=0$, contradicting $c+1 \in R_S^*$.

This completes the proof that $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,19}]^\circ = 0$. But if we assign a weight greater than 1 to any of the points in $\mathcal{P}_{4,19}$, then the resulting portrait will have total weight at least 5, so Theorem 2(a) gives us finiteness. Hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,19}]^\bullet = 0$.

$\boldsymbol{\mathcal{P}_{4,20}}$: Moving $0,\infty$ to the 2-cycle and 1 to an incoming point, we see that $f(x)=a(x-1)/(bx^2+cx)$. This has $\operatorname{Res}(f)=a^2b(b+c)$. In particular, $a,b\in R_S^*$, so we can dehomogenize $b=1$ and $f(x)=a(x-1)/x(x+c)$ with $a,\,1+c\in R_S^*$. The fourth point of the portrait in $f^{-1}(\infty)$, so it is the point $-c$. Then $\{0,1,\infty,-c\}$ has good reduction if and only if $c,\,1+c\in R_S^*$, so $(-c,\,1+c)$ is a solution to the S-unit equation $u+v=1$. There are thus only finitely many choices for c. For example, since $2\in R_S^*$, we may could take $c=1$. Then $a(x-1)/x(x+1)\in \mathcal{GR}_2^1[\mathcal{P}_{4,20}]^\circ(K,S)$ for all $a\in R_S^*$. The Milnor image is

$$s\left(\frac{a(x-1)}{x(x+1)}\right)=\left(\frac{a^2-10a-1}{2a},\,\frac{-a^2+9a+1}{a}\right),$$

which shows that the Zariski closure of $\mathcal{GR}_2^1[\mathcal{P}_{4,20}]^\circ(K,S)$ in $\mathcal{M}_2^1$ is a nonempty finite union of curves. Further, since

$$f^{-1}(f(1)) = f^{-1}(f(\infty)) = \{1,\infty\},$$
$$f^{-1}(f(0)) = f^{-1}(f(-1)) = \{0,-1\},$$

we see that f modulo primes not in S is unramified at the points in $\{-1,1,0,\infty\}$, so the above maps with $c=1$ and $a\in R_S^*$ are in $\mathcal{GR}_2^1[\mathcal{P}_{4,20}]^\star(K,S)$, and hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,20}]^\star=1$. Finally, we note that $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,20}]^\bullet=\operatorname{ShafDim}_2^1[\mathcal{P}_{4,20}]^\circ$, since if we assign a weight greater than 1 to any of the points in $\mathcal{P}_{4,20}$, then the resulting portrait will have total weight at least 5, so Theorem 2(a) gives us finiteness. Hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,20}]^\bullet = 1$.

$\boldsymbol{\mathcal{P}_{4,21}}$: The portrait $\mathcal{P}_{4,21}$ contains the portrait $\mathcal{P}_{3,1}$ as a subportrait, and we already proved that $\operatorname{ShafDim}_2^1[\mathcal{P}_{3,1}]^\circ = 0$, so the same is true for $\mathcal{P}_{4,21}$. On the other hand, if we allow any of the points in $\mathcal{P}_{4,21}$ to have weight greater than 1, then the total weight would be at least 5, in which case Theorem 2(a) gives us finiteness. Hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,21}]^\bullet = 0$.

$\boldsymbol{\mathcal{P}_{4,22}}$: We move three of the points in the 4-cycle to 0, 1, and ∞, and we denote the fourth point by c. The map f then has the form

$$f(x) = \frac{c(x-1)(x+a)}{x\bigl(x-c+(c-1)(c+a)\bigr)},$$
$$\operatorname{Res}(f) = ac^2(1-c)(2-c)(a+c)(1-a-c).$$

The set $\{c, 0, 1, \infty\}$ has good reduction outside S if and only if $c, 1-c \in R_S^*$. Hence $(f, \{c, 0, 1, \infty\}) \in \mathcal{GR}_2^1[\mathcal{P}_{4,22}]^\bullet$ if and only if

$$a, c, 1-c, 2-c, a+c, 1-a-c \in R_S^*.$$

Then $(c, 1-c)$ is a solution to the S-unit equation $u+v=1$, so there are only finitely many values of c. Then the fact that $(a+c, 1-a-c)$ is also a solution to the S-unit equation shows that there are only finitely many values of a. Hence $\operatorname{ShafDim}_2^1[\mathcal{P}_{4,22}]^\bullet = 0$.

This completes our analysis of the 22 weight 4 portraits in Tables 3 and 4, and with it, the proof of Theorem 19. □

8. Possible generalizations

It is natural to attempt to generalize Theorem 2(a) to self-maps of $\mathbb{P}^N$ with $N \geq 2$. The naive generalization fails. Indeed, suppose that we define $\mathcal{GR}_d^N[n](K, S)$ to be the set of triples (f, Y, X) such that $f : \mathbb{P}_K^N \to \mathbb{P}_K^N$ is a degree d morphism defined over K and $Y \subseteq X \subset \mathbb{P}^1(\bar{K})$ are finite sets satisfying the following conditions:[9]

- $X = Y \cup f(Y)$,
- X is $\operatorname{Gal}(\bar{K}/K)$-invariant,
- $\#Y = n$,
- f and X have good reduction outside S.

Then it is easy to see that for any fixed d and N, the set $\mathcal{GR}_d^N[n](K, S)$ can be infinite for arbitrarily large n. We illustrate with $d = N = 2$, since the general case is then clear.

Consider the family of maps $f_{a,b} : \mathbb{P}^2 \to \mathbb{P}^2$ defined by

$$f_{a,b}(X, Y, Z) = [aXZ + X^2, bYZ + Y^2, Z^2] \quad \text{with } a, b \in R_S. \tag{12}$$

[9] This definition is not entirely consistent with our definition of $\mathcal{GR}_d^1[n](K, S)$, since we've replaced the earlier ramification condition on Y with the simpler condition that Y contain n points.

Then $f_{a,b}$ has good reduction at all primes $\mathfrak{p} \notin S$. And it is not an isotrivial family, since for example the characteristic polynomial of $f_{a,b}$ acting on the tangent space at the fixed point $[0, 0, 1]$ is easily computed to be $(T - a)(T - b)$. For a given n, we take $K = \mathbb{Q}$ and we take S to be the set of primes dividing $2\prod_{i=1}^{n}(2^i - 1)$, and we let

$$X_n := \{[1, 2^i, 0] \in \mathbb{P}^N(\mathbb{Q}) : 0 \le i \le n\}.$$

Then X_n has good reduction at all $p \notin S$, and, since $f([1, y, 0]) = [1, y^2, 0]$, we see that $f_{a,b}(X_{n-1}) \subset X_n$. Hence

$$(f_{a,b}, X_{n-1}, X_n) \in \mathcal{GR}_2^2[n](\mathbb{Q}, S)/\operatorname{PGL}_3(\mathbb{Z}_S)$$

gives infinitely many inequivalent triples as a and b range over $\mathbb{Z}_S$.

One key step in the proof of Theorem 2(a) that goes wrong when we try to generalize to $\mathbb{P}^N$ is Lemma 10, which says that if two maps $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ agree at enough points, then $f = g$. This is false in higher dimension, and indeed, the maps $f_{a,b}$ defined by (12) take identical values at all points on the line $Z = 0$.

This suggests two ways to rescue the theorem.

First, we might simply say that two maps are "the same" if they take the same values on a nontrivial subvariety of $\mathbb{P}^N$. This is a somewhat drastic solution, but the following partial generalization of Lemma 10, whose proof we leave to the reader, makes it a reasonable solution.

Lemma 20. *Let K be a field, and let $f, g : \mathbb{P}_K^N \to \mathbb{P}_K^N$ be morphisms of degrees d and e, respectively. Suppose that*

$$\#\{P \in \mathbb{P}^N(K) : f(P) = g(P)\} \ge (d + e)^N + 1.$$

Then there is a curve $C \subset \mathbb{P}_K^N$ such that $f(P) = g(P)$ for all $P \in C$.

Second, we might insist that the marked points in the set X are in sufficiently general position to ensure that $f|_X = g|_X$ forces $f = g$. Thus writing End_d^N for the space of degree d self-morphisms of $\mathbb{P}^N$, we might say that a set $Y \subset \mathbb{P}^N$ is in *d-general position for $\mathbb{P}^N$* if the map

$$\operatorname{End}_d^N \to (\mathbb{P}^N)^{\#Y}, \quad f \mapsto (f(P))_{P \in Y}$$

is injective. Then a version of Theorem 2(a) might be true if we restrict to triples $(f, Y, X) \in \mathcal{GR}_d^N[n](K, S)$ for which Y is in d-general position for $\mathbb{P}^N$.

In this paper, we will not further pursue these, or other potential, generalizations of Theorem 2(a) to $\mathbb{P}^N$.

A second possible generalization of our results would be to extend them to other types of fields, for example taking $K = k(C)$ to be the function field of a curve over an algebraically closed field k. If k has characteristic 0, then much of the argument in this paper should carry over, although there may be issues with isotrivial maps;

while if k has characteristic $p > 0$, then issues of wild ramification arise, as does the fact that the theorem on S-unit equations is more restrictive in requiring more than the simple "no vanishing subsum" condition. Again, we have chosen not to pursue such function field generalizations in the present paper.

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