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**BLOCKS IN FLAT FAMILIES OF  
FINITE-DIMENSIONAL ALGEBRAS**

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# BLOCKS IN FLAT FAMILIES OF FINITE-DIMENSIONAL ALGEBRAS

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We study the behavior of blocks in flat families of finite-dimensional algebras. In a general setting we construct a finite directed graph encoding a stratification of the base scheme according to the block structures of the fibers. This graph can be explicitly obtained when the central characters of simple modules of the generic fiber are known. We show that the block structure of an arbitrary fiber is completely determined by “atomic” block structures living on the components of a Weil divisor. As a byproduct, we deduce that the number of blocks of fibers defines a lower semicontinuous function on the base scheme. We furthermore discuss how to obtain information about the simple modules in the blocks by generalizing and establishing several properties of decomposition matrices by Geck and Rouquier.

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## 1. Introduction

It is a classical fact in ring theory that a nonzero noetherian ring  $A$  can be decomposed as a direct product  $A = \prod_{i=1}^n B_i$  of indecomposable rings  $B_i$ . Such a decomposition is unique up to permutation and isomorphism of the factors. Let us denote by  $\text{Bl}(A)$  the set of the  $B_i$ , called the *blocks* of  $A$ . The decomposition of

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$A$  into blocks induces the decomposition  $A\text{-Mod} = \bigoplus_{i=1}^n B_i\text{-Mod}$  of the category of (left)  $A$ -modules. In particular, a simple  $A$ -module is a simple  $B_i$ -module for a unique block  $B_i$  and so we get the induced decomposition  $\text{Irr } A = \bigsqcup_{i=1}^n \text{Irr } B_i$  of the set of simple modules. Let us denote by  $\text{Fam}(A)$  the set of the  $\text{Irr } B_i$ , called the *families* of  $A$ . The blocks and families of a ring are important invariants which help to organize and simplify its representation theory. The aim of this paper is to investigate how these invariants vary in a flat family of finite-dimensional algebras.

More precisely, we consider a *finite flat* algebra  $A$  over an integral domain  $R$ ; i.e.,  $A$  is finitely generated and flat as an  $R$ -module. This yields a family of algebras parametrized by  $\text{Spec}(R)$  consisting of the *specializations* (or *fibers*)

$$(1) \quad A(\mathfrak{p}) := k(\mathfrak{p}) \otimes_R A \simeq A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}A_{\mathfrak{p}},$$

where  $k(\mathfrak{p}) = \text{Frac}(R/\mathfrak{p})$  is the residue field of  $\mathfrak{p} \in \text{Spec}(R)$  in  $R$  and  $A_{\mathfrak{p}}$  is the localization of  $A$  in  $\mathfrak{p}$ . Note that the fiber  $A(\mathfrak{p})$  is a finite-dimensional  $k(\mathfrak{p})$ -algebra. Now, the primary goal would be to describe for any  $\mathfrak{p}$  the blocks of  $A(\mathfrak{p})$ , e.g., the number of blocks, and to describe the simple modules in each block, e.g., the number of such modules and their dimensions.

It is clear that there will be no general theory giving the precise solutions to these problems for arbitrary  $A$ . For example, we can take the group ring  $A = \mathbb{Z}S_n$  of the symmetric group. The fibers of  $A$  are precisely the group rings  $\mathbb{Q}S_n$  and  $\mathbb{F}_p S_n$  for all primes  $p$ , and the questions above are still unanswered. Nonetheless, and this is the point of this paper, there are some general phenomena, some patterns in the behavior of blocks and simple modules along the fibers, which are true quite generally.

**1A. The setting.** We assume that  $R$  is noetherian and normal, and that the generic fiber  $A^K$  is a *split*  $K$ -algebra, where  $K$  is the fraction field of  $R$ ; i.e., all simple modules of  $A^K$  remain simple under field extension. This setting includes many interesting examples in representation theory, like Brauer algebras, Hecke algebras, (restricted) rational Cherednik algebras, etc. We note that some results we mention below actually hold more generally and refer to the main body of the paper.

At the very end we also establish a semicontinuity property of blocks in the (important) case of a nonsplit generic fiber; see [Theorem 1.6](#). This then applies also to quantized enveloping algebras of semisimple Lie algebras at roots of unity, enveloping algebras of semisimple Lie algebras in positive characteristic, quantized function algebras of semisimple groups at roots of unity, etc. More generally, this applies to Hopf PI triples as introduced by Brown and Goodearl [2002]; see also [Brown and Gordon 2001; Gordon 2001].

**1B. Block stratification.** Under the assumptions described above, we prove the following theorem (see [Corollary 4.3](#)), which is the backbone of this paper:

**Theorem 1.1.** *For any  $\mathfrak{p} \in \text{Spec}(R)$  the natural map  $A_{\mathfrak{p}} \twoheadrightarrow A(\mathfrak{p})$  is **block bijective**; i.e., it induces a bijection between the block idempotents.*

This allows us to reduce the study of blocks of specializations to blocks of localizations, and this is much simpler from the general perspective. Since a block idempotent of a localization  $A_{\mathfrak{p}}$  splits into a sum of block idempotents of the generic fiber  $A^K$ , we can view the blocks of  $A_{\mathfrak{p}}$  as being a *partition* of the set of blocks of  $A^K$ ; see [Section 3](#) for details. This gives us a direct and natural way of comparing the block structures among the fibers — something which is in general, without the above theorem, not possible. Let

$$\text{Bl}_A : \text{Spec}(R) \rightarrow \text{Part}(\text{Bl}(A^K))$$

be the map just described. We equip the image  $\text{Bl}(A)$  of this map with the partial order  $\leq$  on partitions, where  $\mathcal{P}' \leq \mathcal{P}$  if the members of  $\mathcal{P}'$  are unions of members of  $\mathcal{P}$ . We let  $\text{Bl}_A^{-1}(\mathcal{P})$ , respectively  $\text{Bl}_A^{-1}(\leq \mathcal{P})$ , be the locus of all  $\mathfrak{p} \in \text{Spec}(R)$  such that the block structure of  $A_{\mathfrak{p}}$ , and thus of  $A(\mathfrak{p})$  under the above bijection, is equal to a given partition  $\mathcal{P}$ , respectively coarser than  $\mathcal{P}$ . We then obtain as [Theorem 3.3](#):

**Theorem 1.2.** *The sets  $\text{Bl}_A^{-1}(\leq \mathcal{P})$  are closed in  $\text{Spec}(R)$ , the sets  $\text{Bl}_A^{-1}(\mathcal{P})$  are locally closed in  $\text{Spec}(R)$ , and  $\text{Spec}(R) = \coprod_{\mathcal{P}} \text{Bl}_A^{-1}(\mathcal{P})$  is a stratification of  $\text{Spec}(R)$ .*

Denoting by  $\bullet$  the generic point of  $\text{Spec}(R)$ , so that  $A_{\bullet} = A(\bullet) = A^K$ , this implies in particular that the set

$$\begin{aligned} \text{BlGen}(A) &:= \text{Bl}_A^{-1}(\text{Bl}_A(\bullet)) \\ &= \{\mathfrak{p} \in \text{Spec}(R) \mid \text{Bl}_A(\mathfrak{p}) = \text{Bl}_A(\bullet)\} \end{aligned}$$

of primes  $\mathfrak{p}$  where the block structure of the fiber  $A(\mathfrak{p})$  is equal to the one of the generic fiber  $A^K$  is an open (dense) subset of  $\text{Spec}(R)$ . Hence, the set

$$\text{BlEx}(A) := \{\mathfrak{p} \in \text{Spec}(R) \mid \text{Bl}_A(\mathfrak{p}) < \text{Bl}_A(\bullet)\}$$

of primes where the block structure of the fiber is coarser than the one of the generic fiber is closed. This set has a nice property; see [Corollary 3.5](#):

**Theorem 1.3.** *If  $R$  is a Krull domain (e.g., if  $R$  is normal), then  $\text{BlEx}(A)$  is a reduced Weil divisor; i.e., it is either empty or pure of codimension 1 in  $\text{Spec}(R)$ .*

We thus call  $\text{BlEx}(A)$  the *block divisor* of  $A$ . This is an interesting new *discriminant* of  $A$ . Let  $\text{At}(A)$  be the set of irreducible components of  $\text{BlEx}(A)$ . On any  $Z \in \text{At}(A)$  there is a unique maximal block structure  $\text{Bl}_A(Z)$ , namely the one in the generic point. In [Section 3C](#) we show that these block structures have an *atomic character*:

**Theorem 1.4.** For  $\mathfrak{p} \in \text{Spec}(R)$  we have

$$\text{Bl}_A(\mathfrak{p}) = \bigwedge_{\substack{Z \in \text{At}(A) \\ \mathfrak{p} \in Z}} \text{Bl}_A(Z),$$

where  $\wedge$  is the meet of partitions; i.e., the members are the unions of all members with nonempty intersection.

Hence, once we know  $\text{At}(A)$  and the atomic block structures  $\text{Bl}_A(Z)$  for  $Z \in \text{At}(A)$ , we know the block structure for any  $\mathfrak{p} \in \text{Spec}(R)$ . By considering sets of the form

$$\bigcap_{Z \in \mathcal{Z}} Z \setminus \bigcup_{Z \notin \mathcal{Z}} Z$$

for subsets  $\mathcal{Z} \subseteq \text{At}(A)$ , we obtain a stratification of  $\text{Spec}(R)$  refining the one introduced above. We call this the *block stratification* of  $A$ .

**1C. Blocks via central characters.** In [Section 5](#) we discuss an approach to explicitly compute the block stratification and the block structures on the strata. This is based on the knowledge of central characters of simple  $A^K$ -modules. Since  $A^K$  splits, each simple  $A^K$ -module  $S$  has a central character  $\Omega'_S : Z(A) \rightarrow R$ , the image lying in  $R$  since  $R$  is normal. In [Theorem 5.9](#) we show:

**Theorem 1.5.** Two simple  $A^K$ -modules  $S$  and  $T$  lie in the same  $A_{\mathfrak{p}}$ -block if and only if  $\Omega'_S \equiv \Omega'_T \pmod{\mathfrak{p}}$ .

The key ingredient in the proof is a (rather nontrivial) result by B. Müller stating that the cliques of a noetherian ring, which is a finite module over its center, are fibered over the center. We address this in detail in [Section 5A](#).

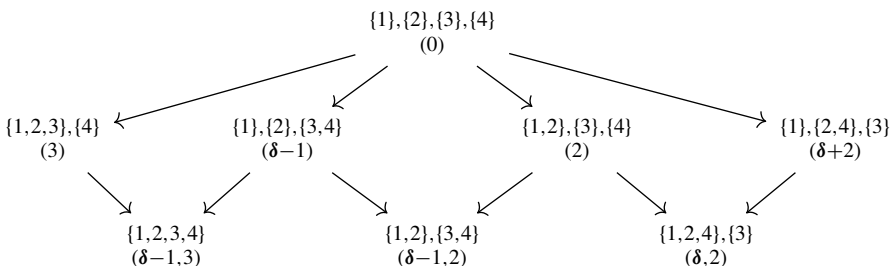
If  $z_1, \dots, z_n$  is an  $R$ -algebra generating system of  $Z(A)$ , then  $\Omega'_S \equiv \Omega'_T \pmod{\mathfrak{p}}$  if and only if  $\Omega'_S(z_i) \equiv \Omega'_T(z_i) \pmod{\mathfrak{p}}$  for all  $i$ . Hence, [Theorem 1.5](#) gives a computational tool to explicitly determine  $\text{Bl}_A(\mathfrak{p})$  once the central characters of the generic fiber are known. Moreover, it follows from [Theorem 1.5](#) that  $\text{At}(A)$  is the set of maximal irreducible components of the zero loci of the sets

$$(2) \quad \{\Omega'_S(z_i) - \Omega'_T(z_i) \mid i = 1, \dots, n\}$$

for  $\Omega'_S \neq \Omega'_T$ . The atomic block structures can then be determined by the vanishing of the differences  $\Omega'_S - \Omega'_T$  on the  $Z \in \text{At}(A)$ , and from these we obtain all block structures as described above.

**1D. An example.** Let us illustrate this with an explicit example. Let  $A$  be the generic Brauer algebra for  $n = 3$  over the polynomial ring  $R := \mathbb{Z}[\delta]$ ; see [\[Graham and Lehrer 1996\]](#). There are four simple  $A^K$ -modules, labeled by the partitions  $(0, (1, 1, 1))$ ,  $(0, (3))$ ,  $(0, (2, 1))$ , and  $(1, (1))$ . We will simply label these by

$1, \dots, 4$  from now on. Since  $A^K$  is semisimple, we can identify the blocks of  $A^K$  with the simple modules of  $A^K$ ; i.e., we can label the blocks by  $1, \dots, 4$ . We can thus view blocks of specializations of  $A$  as partitions of  $\{1, \dots, 4\}$  as described above. It is not too difficult to explicitly compute the central characters of the simple  $A^K$ -modules. From these we deduce that the block structure of the fibers of  $A$  over  $\mathbb{Z}[\delta]$  are distributed as in the following graph:



This graph encodes the block stratification of the two-dimensional base scheme  $\text{Spec}(\mathbb{Z}[\delta])$ , along with the block structures on the strata. We see that  $\text{BIE}x(A)$  has four components of codimension 1, the generic points of these components are  $3, \delta-1, 2,$  and  $\delta+2$ , respectively. The block structure on any other point  $p$  is uniquely determined as the meet of the block structures on the components of  $\text{BIE}x(A)$  containing  $p$ .

We want to point out that it is central for us to work with (affine) schemes. For example, we have one skeleton with generic point  $(2)$ ; i.e., we consider the Brauer algebra in characteristic 2. Now, we do not only have the case  $\delta \in \{0, 1\} = \mathbb{F}_2$ , which is described by the two strata below  $(2)$ , but we also have a generic characteristic-2 case, described by the generic point of  $\mathbb{F}_2[\delta]$ , and this is really different from the case of specialized  $\delta$ , as we can see from the block structures.

Note that the components of  $\text{BIE}x(A)$  are precisely the parameters where the Brauer algebra is not semisimple anymore (the precise parameters have been determined by Rui [2005] for all  $n \in \mathbb{N}$ ). We show in Lemma 6.7 that this is always the case for cellular algebras.

**1E. Blocks and decomposition matrices.** In Section 6 we address questions about the simple modules in a block. The main tool here is the decomposition matrices introduced by Geck and Rouquier. In Theorem 6.2 we show that they satisfy Brauer reciprocity in a rather general setting in which it was not known to hold before. In Section 6C we generalize the concept of Brauer graphs and show how these relate to blocks.

**1F. An open problem.** In Section 6B we contrast the preservation of simple modules with the preservation of blocks under specialization, and this leads to an interesting problem: In [Thiel 2016] we showed that decomposition matrices of  $A$

are trivial precisely on an open subset  $\text{DecGen}(A)$  of  $\text{Spec}(R)$ . In [Theorem 6.3](#) we show

$$\text{DecGen}(A) \subseteq \text{BGen}(A).$$

The obvious question is: are these two sets equal, and if not, when are they equal? We show in [Example 6.5](#) that in general we do not have equality. In [Lemma 6.7](#), on the other hand, we establish a context where we have equality (this includes Brauer algebras and explains why our Weil divisor is given by the nonsemisimple parameters). It is an open problem to understand the complement  $\text{BGen}(A) \setminus \text{DecGen}(A)$ .

**1G. *Semicontinuity of blocks in the case of a nonsplit generic fiber.*** In [Section 7](#) we consider the case of a nonsplit generic fiber. In this case we can no longer identify blocks of specializations with blocks of localizations, and so there is no natural way of comparing block structures among the fibers. However, it still makes sense to compare the *number* of blocks of the fibers, i.e., to consider the map  $\text{Spec}(R) \rightarrow \mathbb{N}$ ,  $\mathfrak{p} \mapsto \#\text{Bl}(A(\mathfrak{p}))$ . In the case  $R$  is normal and  $A^K$  splits, this map is lower semicontinuous by the results discussed above. Without the splitting of  $A^K$ , this is no longer true; see [Example 7.4](#). The problem is that we consider this map on all of  $\text{Spec}(R)$ . In [Corollary 7.1](#) we construct a setting in which the restriction of  $\mathfrak{p} \mapsto \#\text{Bl}(A(\mathfrak{p}))$  to certain subsets of  $\text{Spec}(R)$  is still lower semicontinuous without assuming that the generic fiber splits. From this we obtain a rather nice result; see [Corollary 7.3](#):

**Theorem 1.6.** *Suppose that  $R$  is a finite-type algebra over an algebraically closed field. Let  $X$  be the set of closed points of  $\text{Spec}(R)$ . Then the map  $X \rightarrow \mathbb{N}$ ,  $\mathfrak{m} \mapsto \#\text{Bl}(A(\mathfrak{m}))$ , is lower semicontinuous. In particular,  $X$  admits a stratification according to the number of blocks of fibers of  $A$  over  $X$ .*

**1H. *Remark.*** The behavior of blocks under specialization has been studied in several situations already. All of our results are well known in modular representation theory of finite groups due to the work of R. Brauer and C. Nesbitt [1941]. Our [Corollary 4.3](#) and [Theorem 6.9](#) generalize results by S. Donkin and R. Tange [2010] about algebras over Dedekind domains. Our results about lower semicontinuity of the number of blocks generalize a result by P. Gabriel [1975] to mixed characteristic and nonalgebraically closed settings; see also the corresponding result by I. Gordon [2001]. In general, K. Brown and I. Gordon [2001; 2002] used Müller's theorem [1976] to study blocks under specialization. [Theorem 5.8](#) has been treated in a more special setting by K. Brown and K. Goodearl [2002]. The codimension-1 property in [Corollary 4.3](#) and [Theorem 5.9](#) was proven by C. Bonnafé and R. Rouquier [2017] in a more special setting. Their work is without doubt one of the main motivations for this paper. Blocks and decomposition matrices of generically semisimple algebras

over discrete valuation rings have been studied by M. Geck and G. Pfeiffer [2000], and more generally by M. Chlouveraki [2009]. Brauer reciprocity has been studied more generally by M. Geck and R. Rouquier [1997], and by M. Neunhöffer [2003]. M. Neunhöffer and S. Scherotzke [2008] showed generic triviality of  $e_A^p$  over Dedekind domains.

## 2. Base change of blocks

The basic principle underlying the behavior of blocks in a family of algebras is *base change* of blocks. In this section, we introduce a few basic notions about this principle. Appendix A contains some further material which will later be used in some proofs.

Let us fix some basic notation for block theory. For us, a *ring* is always a ring with identity and a *module* is always a *left* module unless we explicitly say it is a *right* module. Let  $A$  be a ring and let  $Z$  be its center. If  $c$  is a central idempotent of  $A$ , then  $Ac = cA$  is a two-sided ideal of  $A$  and at the same time a ring with identity element equal to  $c$  (so, not a subring). This yields a bijection between the set of decompositions of  $1 \in A$  into a sum of pairwise orthogonal central idempotents and finite direct sum decompositions of the ring  $A$  into nonzero two-sided ideals of  $A$  up to permutation of the summands. Such decompositions are in turn in bijection with finite direct product decompositions of the ring  $A$  into nonzero rings up to permutation of the factors. Primitive idempotents of  $Z$  are also called *centrally primitive* idempotents of  $A$ . A central idempotent  $c$  is centrally primitive if and only if  $Ac$  is an indecomposable ring. It is a standard fact — and the starting point of block theory — that if there is a decomposition of  $1 = \sum_i c_i$  into pairwise orthogonal centrally primitive idempotents  $c_i$ , then this is unique and any central idempotent of  $A$  is a sum of a subset of the  $c_i$ . We then say that  $A$  has a *block decomposition*, call the centrally primitive idempotents of  $A$  also the *block idempotents*, and call the corresponding rings  $Ac$  the *blocks* of  $A$ . We denote by  $\text{Bl}(A)$  the set of centrally primitive idempotents of  $A$ . To avoid pathologies we set  $\text{Bl}(0) := \emptyset$  for the zero ring  $0$ . It is well known that noetherian rings have block decompositions (the block idempotents are the class sums with respect to the linkage relation of a decomposition of  $1 \in A$  into pairwise orthogonal primitive idempotents).

Let  $\mathcal{C} := \{c_i\}_{i \in I}$  be a finite set of pairwise orthogonal central idempotents whose sum is equal to  $1 \in A$ . Let  $B_i := Ac_i$ . If  $V$  is a nonzero  $A$ -module, then  $V = \bigoplus_{i \in I} c_i V$  as  $A$ -modules and each summand  $c_i V$  is a  $B_i$ -module. In this way we obtain a decomposition  $A\text{-Mod} = \bigoplus_{i \in I} B_i\text{-Mod}$  of module categories, which also restricts to a decomposition of the category of finitely generated modules. If a nonzero  $A$ -module  $V$  is under this decomposition obtained from a  $B_i$ -module, then  $V$  is said to *belong* to  $B_i$ . This is equivalent to  $c_i V = V$  and  $c_j V = 0$  for all  $j \neq i$ . An indecomposable  $A$ -module clearly belongs to a unique  $B_i$ , and so this



is true for any simple  $A$ -module. We thus get a decomposition  $\text{Irr } A = \coprod_{i \in I} \text{Irr } B_i$  of the set of (isomorphism classes of) simple modules. We call the sets  $\text{Irr } B_i$  the  $\mathcal{C}$ -families of  $A$  and denote the set of  $\mathcal{C}$ -families by  $\text{Fam}_{\mathcal{C}}(A)$ . Note that we have a natural bijection

$$(3) \quad \mathcal{C} \xrightarrow{\sim} \text{Fam}_{\mathcal{C}}(A)$$

given by  $c_i \mapsto \text{Irr } B_i$ . In the case  $\mathcal{C}$  is actually a block decomposition, we call the  $\mathcal{C}$ -families simply the *families* of  $A$  and set  $\text{Fam}(A) := \text{Fam}_{\mathcal{C}}(A)$ . Recall that any central idempotent of  $A$  is a sum of a subset of the block idempotents of  $A$ . Hence, for general  $\mathcal{C}$  as above the families are a finer partition of  $\text{Irr } A$  than the  $\mathcal{C}$ -families; i.e., any  $\mathcal{C}$ -family is a union of families.

Now, consider a morphism  $\phi : R \rightarrow S$  of commutative rings. If  $V$  is an  $R$ -module, we write

$$V^S := \phi^* V := S \otimes_R V$$

for the scalar extension of  $V$  to  $S$  and by  $\phi_V : V \rightarrow V^S$  we denote the canonical map  $v \mapsto 1 \otimes v$ . In most situations we consider, this map will be injective:

**Lemma 2.1.** *In each of the following cases the map  $\phi_V : V \rightarrow V^S$  is injective:*

- (a)  $\phi$  is injective and  $V$  is  $R$ -projective.
- (b)  $\phi$  is faithfully flat.
- (c)  $\phi$  is the localization morphism for a multiplicatively closed subset  $\Sigma \subseteq R$  and  $V$  is  $\Sigma$ -torsion-free.

*Proof.* The first case follows from [Bourbaki 1989, II, §5.1, Corollary to Proposition 4], the second follows from [Bourbaki 1972, I, §3.5, Proposition 8(i,iii)], and the last case follows from the fact that  $\phi$  is flat in conjunction with [Bourbaki 1972, I, §2.2, Proposition 4].  $\square$

If  $A$  is an  $R$ -algebra, then the  $S$ -module  $A^S$  is naturally an  $S$ -algebra and the map  $\phi_A : A \rightarrow A^S$  is a ring morphism. Moreover, if  $V$  is an  $A$ -module, then the underlying  $S$ -module of  $A^S \otimes_A V$  is simply  $V^S$ . Our aim is to study the behavior of blocks under the morphism  $\phi_A : A \rightarrow A^S$ . Clearly, if  $e \in A$  is an idempotent, also  $\phi_A(e) \in A^S$  is an idempotent, and if  $e$  is central, so is  $\phi_A(e)$  by the elementary fact

$$(4) \quad \phi_A(\mathbf{Z}(A)) \subseteq \mathbf{Z}(A^S).$$

**Definition 2.2.** We say that  $\phi_A$  is (central) *idempotent stable* if  $\phi_A(e) \neq 0$  for any nonzero (central) idempotent  $e$  of  $A$ . We say that  $\phi_A$  is *block bijective* if  $\phi_A$  induces a bijection between the centrally primitive idempotents of  $A$  and the centrally primitive idempotents of  $A^S$ .

Note that in the case  $\phi_A$  is idempotent stable, respectively central idempotent stable, it induces a map between the sets of decompositions of  $1 \in A$  and  $1 \in A^S$

into pairwise orthogonal idempotents, respectively into pairwise orthogonal central idempotents. The following lemma shows two situations in which  $\phi_A$  is idempotent stable (and thus central idempotent stable). We denote by  $\text{Rad}(A)$  the Jacobson radical of  $A$ .

**Lemma 2.3.** *If  $\text{Ker}(\phi_A) \subseteq \text{Rad}(A)$ , then  $\phi_A$  is idempotent stable. This holds in the following two cases:*

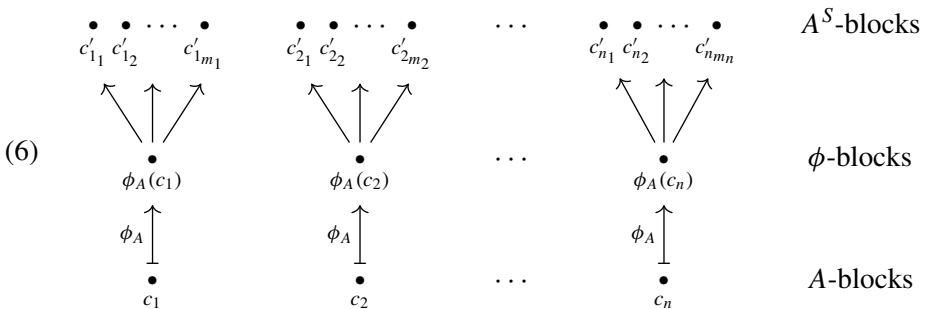
- (a)  $\phi_A$  is injective (see Lemma 2.1),
- (b)  $\phi$  is surjective,  $\text{Ker}(\phi) \subseteq \text{Rad}(R)$ , and  $A$  is finitely generated as an  $R$ -module.

*Proof.* If  $e \in A$  is an idempotent contained in  $\text{Rad}(A)$ , then by a well-known characterization of the Jacobson radical, see [Curtis and Reiner 1981, 5.10], we conclude that  $e^\dagger = 1 - e \in A^\times$  is a unit, and since  $e^\dagger$  is also an idempotent, we must have  $e^\dagger = 1$ , implying that  $e = 0$ . If  $\phi_A$  is injective, the condition clearly holds. In the second case we have  $\text{Ker}(\phi_A) = \text{Ker}(\phi)A \subseteq \text{Rad}(R)A \subseteq \text{Rad}(A)$ , where the last inclusion follows from [Lam 1991, Corollary 5.9].  $\square$

Suppose that  $\phi_A$  is idempotent stable and that both  $A$  and  $A^S$  have block decompositions. Let  $\{c_i\}_{i \in I}$  be the block idempotents of  $A$  and let  $\{c'_j\}_{j \in J}$  be the block idempotents of  $A^S$ . Since  $\phi_A$  is idempotent stable, the set  $\text{Bl}_\phi(A^S) := \phi_A(\{c_i\}_{i \in I})$  is a decomposition of  $1 \in A^S$  into pairwise orthogonal idempotents. We call the  $\phi_A(c_i)$  the  $\phi$ -blocks of  $A^S$  and call the corresponding families (see above) the  $\phi$ -families of  $A^S$ , denoted by  $\text{Fam}_\phi(A^S)$ . As explained above, each  $\phi$ -block  $\phi_A(c_i)$  is a sum of a subset of the block idempotents of  $A^S$  and the  $\phi$ -families are coarser than the families in the sense that each  $\phi$ -family is a union of  $A^S$ -families. In particular, we have

$$(5) \quad \# \text{Bl}(A) = \# \text{Bl}_\phi(A^S) \leq \# \text{Bl}(A^S).$$

The following picture illustrates this situation:



This paper is about this picture in the special case of specializations of an algebra in prime ideals. Before we begin investigating this, we record the following useful fact.

**Lemma 2.4.** *Suppose that  $\phi_A : A \rightarrow A^S$  is central idempotent stable. If  $A^S$  has a block decomposition, then  $A$  has a block decomposition.*

*Proof.* If  $A$  does not contain any nontrivial central idempotent, then  $A$  is indecomposable and thus has a block decomposition. So, assume that  $A$  is not indecomposable and let  $c$  be a nontrivial central idempotent. Then  $A = Ac \oplus Ac^\dagger$ . We can now continue this process to get finer and finer decompositions of  $A$  as a ring. Since  $\phi_A$  is central idempotent stable, we get decompositions of  $A^S$  of the same size. As  $A^S$  has a block decomposition, this process has to end after finitely many steps. We thus arrive at a ring decomposition of  $A$  with finitely many and indecomposable factors, hence, at a block decomposition of  $A$ .  $\square$

**Corollary 2.5.** *A nonzero finite flat algebra over an integral domain has a block decomposition.*

*Proof.* Let  $R$  be an integral domain with fraction field  $K$ , let  $\phi : R \hookrightarrow K$  be the embedding, and let  $A$  be a finite flat  $R$ -algebra. Since  $A$  is  $R$ -torsion-free, it follows from Lemma 2.1(c) that  $\phi_A$  is injective and so  $\phi_A$  is idempotent stable by Lemma 2.3(a). Since  $\phi_A^* A = A^K$  is a finite-dimensional algebra over a field, it has a block decomposition. Hence,  $A$  has a block decomposition by Lemma 2.4.  $\square$

The point of the corollary above is that we do *not* have to assume  $R$  to be noetherian — otherwise  $A$  is noetherian and we already know it has a block decomposition.

### 3. Blocks of localizations

Before we consider blocks of specializations, we first take a look at blocks of localizations as these are much easier to control and are still strongly related to blocks of specializations as we will see in the next paragraph.

*Throughout the next paragraph, we assume  $A$  is a finite flat algebra over an integral domain  $R$  with fraction field  $K$ .*

It follows from Corollary 2.5 that  $A$  and any localization  $A_{\mathfrak{p}}$  for  $\mathfrak{p} \in \text{Spec}(R)$  have a block decomposition, even if  $A$  is not necessarily noetherian. Since the canonical map  $\phi_{\mathfrak{p}} : A_{\mathfrak{p}} \rightarrow A^K$  is injective by Lemma 2.1, we have the notion of  $\phi_{\mathfrak{p}}$ -blocks and  $\phi_{\mathfrak{p}}$ -families of  $A^K$ , as defined in Section 2. To shorten notations, we call them the  $\mathfrak{p}$ -blocks and  $\mathfrak{p}$ -families, and write  $\text{Fam}_{\mathfrak{p}}(A^K)$  for the  $\mathfrak{p}$ -families. Recall that we have a natural bijection

$$(7) \quad \text{Bl}(A_{\mathfrak{p}}) \simeq \text{Fam}_{\mathfrak{p}}(A^K).$$

**3A. Block structure stratification.** There is the following more concrete point of view of  $\mathfrak{p}$ -blocks. Let  $(c_i)_{i \in I}$  be the block idempotents of  $A^K$ . If  $c \in A_{\mathfrak{p}}$  is any block idempotent, we know from Section 2 that there is  $I' \subseteq I$  with  $c = \sum_{i \in I'} c_i$  in  $A^K$ . Hence, to any block idempotent of  $A_{\mathfrak{p}}$  we can associate a subset of  $I$ , and if we take all block idempotents of  $A_{\mathfrak{p}}$  into account, we get a partition  $\text{Bl}_A(\mathfrak{p})$  of the

set  $I$ , from which we can recover the block idempotents of  $A_{\mathfrak{p}}$  by taking sums of the  $c_i$  over the members of  $\text{Bl}_A(\mathfrak{p})$ . In this way we get a map

$$(8) \quad \text{Bl}_A : \text{Spec}(R) \rightarrow \text{Part}(I)$$

to the set of partitions of the set  $I$ . We denote by

$$(9) \quad \text{Bl}(A) := \text{Im Bl}_A$$

the image of this map and call the partitions therein the *block structures* of  $A$ .

The set  $\text{Part}(I)$  is equipped with the partial order  $\leq$  defined by  $\mathcal{P} \leq \mathcal{Q}$  if  $\mathcal{P}$  is a coarser partition than  $\mathcal{Q}$ , i.e., the members of  $\mathcal{P}$  are unions of members of  $\mathcal{Q}$ . If  $\mathfrak{q} \subseteq \mathfrak{p}$ , then we have an embedding  $A_{\mathfrak{p}} \hookrightarrow A_{\mathfrak{q}}$  and by the same argument as above, the block idempotents of  $A_{\mathfrak{p}}$  are obtained by summing up block idempotents of  $A_{\mathfrak{q}}$ , so

$$(10) \quad \mathfrak{q} \subseteq \mathfrak{p} \implies \text{Bl}_A(\mathfrak{p}) \leq \text{Bl}_A(\mathfrak{q}).$$

Hence, the map  $\text{Bl}_A$  is actually a morphism of posets if we equip  $\text{Spec}(R)$  with the partial order  $\leq$  defined by  $\mathfrak{p} \leq \mathfrak{q}$  if  $\mathfrak{q} \subseteq \mathfrak{p}$  (i.e.,  $V(\mathfrak{p}) \subseteq V(\mathfrak{q})$ ).

For  $\mathcal{P} \in \text{Part}(I)$ , we call the fiber  $\text{Bl}_A^{-1}(\mathcal{P}) \subseteq \text{Spec}(R)$  the  $\mathcal{P}$ -*stratum* and

$$(11) \quad \text{Bl}_A^{-1}(\leq \mathcal{P}) := \bigcup_{\mathcal{P}' \leq \mathcal{P}} \text{Bl}_A^{-1}(\mathcal{P}') = \bigcup_{\substack{\mathcal{P}' \leq \mathcal{P} \\ \mathcal{P}' \in \text{Bl}(A)}} \text{Bl}_A^{-1}(\mathcal{P}')$$

the  $\mathcal{P}$ -*skeleton*. The  $\mathcal{P}$ -stratum ( $\mathcal{P}$ -skeleton) is simply the locus of all  $\mathfrak{p} \in \text{Spec}(R)$  where the block structure of  $A_{\mathfrak{p}}$  is equal to  $\mathcal{P}$  (respectively coarser than  $\mathcal{P}$ ). Since

$$(12) \quad \text{Bl}_A^{-1}(\mathcal{P}) = \text{Bl}_A^{-1}(\leq \mathcal{P}) \setminus \bigcup_{\mathcal{P}' < \mathcal{P}} \text{Bl}_A^{-1}(\leq \mathcal{P}'),$$

we can recover the strata from the skeleta. We get the finite decomposition

$$(13) \quad \text{Spec}(R) = \bigsqcup_{\mathcal{P}} \text{Bl}_A^{-1}(\mathcal{P})$$

and we call this the *block structure stratification*. Our aim is now to show that this is indeed a stratification, i.e., the strata are *locally closed* subsets of  $\text{Spec}(R)$  and the closure of a stratum is contained in its skeleton. The key ingredient in proving this is the following general proposition, which is essentially due to Bonnafé and Rouquier [2017, Proposition D.2.11] but is proven here in a more general form.

**Proposition 3.1.** *Let  $R$  be an integral domain with fraction field  $K$ , let  $A$  be a finite flat  $R$ -algebra, and let  $\mathcal{F} \subseteq A^K$  be a finite set. Then*

$$\text{Gen}_A(\mathcal{F}) := \{\mathfrak{p} \in \text{Spec}(R) \mid \mathcal{F} \subseteq A_{\mathfrak{p}}\}$$

is a **neighborhood** of the generic point of  $\text{Spec}(R)$ . If  $A$  is finitely presented flat, then  $\text{Gen}_A(\mathcal{F})$  is an **open** subset of  $\text{Spec}(R)$ , and if moreover  $R$  is a Krull domain, the complement  $\text{Ex}_A(\mathcal{F})$  of  $\text{Gen}_A(\mathcal{F})$  in  $\text{Spec}(R)$  is a **reduced Weil divisor**, i.e., it is either empty or pure of codimension 1 with finitely many irreducible components.

*Proof.* Let us first assume that  $A$  is actually  $R$ -free. For an element  $\alpha \in K$  we define  $I_\alpha := \{r \in R \mid r\alpha \in R\}$ . This is a nonzero ideal in  $R$ , and it has the property that  $\alpha \in R_{\mathfrak{p}}$  if and only if  $I_\alpha \not\subseteq \mathfrak{p}$ . To see this, suppose that  $\alpha \in R_{\mathfrak{p}}$ . Then we can write  $\alpha = r/x$  for some  $x \in R \setminus \mathfrak{p}$ . Hence,  $x\alpha = r \in R$  and therefore  $x \in I_\alpha$ . Since  $x \notin \mathfrak{p}$ , it follows that  $I_\alpha \not\subseteq \mathfrak{p}$ . Conversely, if  $I_\alpha \not\subseteq \mathfrak{p}$ , then there exists  $x \in I_\alpha$  with  $x \notin \mathfrak{p}$ . By the definition of  $I_\alpha$  we have  $x\alpha =: r \in R$  and since  $x \notin \mathfrak{p}$ , we can write  $\alpha = r/x \in R_{\mathfrak{p}}$ . Now, let  $(a_1, \dots, a_n)$  be an  $R$ -basis of  $A$ . Then we can write every element  $f \in \mathcal{F}$  as  $f = \sum_{i=1}^n \alpha_{f,i} a_i$  with  $\alpha_{f,i} \in K$ . Let  $I$  be the radical of the ideal

$$\prod_{f \in \mathcal{F}, i=1, \dots, n} I_{\alpha_{f,i}} \trianglelefteq R.$$

By the properties of the ideals  $I_\alpha$  we have the following logical equivalences:

$$\begin{aligned} \mathcal{F} \subseteq A_{\mathfrak{p}} &\iff \alpha_{f,i} \in R_{\mathfrak{p}} \quad \text{for all } f \in \mathcal{F}, i = 1, \dots, n \\ &\iff I_{\alpha_{f,i}} \not\subseteq \mathfrak{p} \quad \text{for all } f \in \mathcal{F}, i = 1, \dots, n \\ &\iff I \not\subseteq \mathfrak{p}, \end{aligned}$$

the last equivalence following from the fact that  $\mathfrak{p}$  is prime. Hence,

$$(14) \quad \text{Ex}_A(\mathcal{F}) = \text{Spec}(R) \setminus \text{Gen}_A(\mathcal{F}) = \text{V}(I) = \bigcup_{f \in \mathcal{F}, i=1, \dots, n} \text{V}(I_{\alpha_{f,i}}),$$

implying that  $\text{Gen}_A(\mathcal{F})$  is an open subset of  $\text{Spec}(R)$ .

Next, still assuming that  $A$  is  $R$ -free, suppose that  $R$  is a Krull domain. To show that  $\text{Ex}_A(\mathcal{F})$  is either empty or pure of codimension 1 in  $\text{Spec}(R)$  with finitely many irreducible components, it suffices to show this for the closed subsets  $\text{V}(I_\alpha) = \text{V}(\sqrt{I_\alpha})$ . If  $\alpha \in R$ , then  $I_\alpha = R$  and therefore  $\text{V}(I_\alpha) = \emptyset$ . So, let  $\alpha \notin R$ . Let  $\text{V}(I_\alpha) = \bigcup_{\lambda \in \Lambda} \text{V}(q_\lambda)$  be the decomposition into irreducible components. Note that this decomposition is unique and contains every irreducible component of  $\text{V}(I_\alpha)$ . The inclusion  $\text{V}(I_\alpha) \supseteq \text{V}(q_\lambda)$  is equivalent to  $I_\alpha \subseteq \sqrt{I_\alpha} \subseteq \sqrt{q_\lambda} = q_\lambda$ . Since an irreducible component is a maximal proper closed subset, we see that the  $q_\lambda$  are the minimal prime ideals of  $\text{Spec}(R)$  containing  $I_\alpha$ . Let  $q = q_\lambda$  for an arbitrary  $\lambda \in \Lambda$ . We will show that  $\text{ht}(q) = 1$ . Since  $I_\alpha \subseteq q$ , we have seen above that  $\alpha \notin R_q$ . As  $R$  is a Krull domain, also  $R_q$  is a Krull domain by [Matsumura 1986, Theorem 12.1]. By [Bourbaki 1972, VII, §1.6, Theorem 4] we have

$$R_q = \bigcap_{\substack{q' \in \text{Spec}(R_q) \\ \text{ht}(q')=1}} (R_q)_{q'} = \bigcap_{\substack{q' \in \text{Spec}(R) \\ q' \subseteq q \\ \text{ht}(q')=1}} R_{q'}.$$

Since  $\alpha \notin R_q$ , this shows that there exists  $q' \in \text{Spec}(R)$  with  $q' \subseteq q$ ,  $\text{ht}(q') = 1$  and  $\alpha \notin R_{q'}$ . The last property implies  $I_\alpha \subseteq q'$  and now the minimality in the choice of  $q$

implies that  $q' = q$ . Hence,  $\text{ht}(q) = 1$  and this shows  $V(I_\alpha)$  is pure of codimension 1. Since  $I_\alpha \neq 0$ , there is some  $0 \neq r \in I_\alpha$ . This element is contained in all the height-1 prime ideals  $q_\lambda$ . As  $R$  is a Krull domain, a nonzero element of  $R$  can only be contained in finitely many height-1 prime ideals, see [Huneke and Swanson 2006, 4.10.1], so  $\Lambda$  must be finite.

Now, assume that  $R$  is an arbitrary integral domain and that  $A$  is finite flat. Then Grothendieck's generic freeness lemma [1965, Lemme 6.9.2] shows that there exists a nonzero  $f \in R$  such that  $A_f$  is a free  $R_f$ -module. Note that  $\text{Spec}(R_f)$  can be identified with the distinguished open subset  $D(f)$  of  $\text{Spec}(R)$ . We obviously have

$$\text{Gen}_{A_f}(\mathcal{F}) = \text{Gen}_A(\mathcal{F}) \cap D(f).$$

By the arguments above,  $\text{Gen}_{A_f}(\mathcal{F})$  is an open subset of  $D(f)$ , and thus of  $\text{Spec}(R)$ . This shows that  $\text{Gen}_A(\mathcal{F})$  is a neighborhood in  $\text{Spec}(R)$ .

Next, let  $R$  be arbitrary and assume that  $A$  is finitely presented flat. It is a standard fact, see [Stacks 2005–, Tag 00NX], that the assumptions on  $A$  imply that  $A$  is already finite locally free; i.e., there exists a family  $(f_i)_{i \in I}$  of elements of  $R$  such that the standard open affines  $D(f_i)$  cover  $\text{Spec}(R)$  and  $A_{f_i}$  is a finitely generated free  $R_{f_i}$ -module for all  $i \in I$ . Since  $\text{Spec}(R)$  is quasicompact, see [Görtz and Wedhorn 2010, Proposition 2.5], we can assume that  $I$  is finite. Again note that  $\text{Spec}(R_{f_i})$  can be identified with  $D(f_i)$  and that

$$(15) \quad \text{Gen}_{A_{f_i}}(\mathcal{F}) = \text{Gen}_A(\mathcal{F}) \cap D(f_i).$$

By the above, the set  $\text{Gen}_{A_{f_i}}(\mathcal{F})$  is open and since the  $D(f_i)$  cover  $\text{Spec}(R)$ , it follows that  $\text{Gen}_A(\mathcal{F})$  is open. Now, suppose that  $R$  is a Krull domain. Much as in (15) we have

$$(16) \quad \text{Ex}_{A_{f_i}}(\mathcal{F}) = \text{Ex}_A(\mathcal{F}) \cap D(f_i).$$

Suppose that  $\text{Ex}_A(\mathcal{F})$  is not empty and let  $Z$  be an irreducible component of  $\text{Ex}_A(\mathcal{F})$ . There is an  $i \in I$  with  $Z \cap D(f_i) \neq \emptyset$ . The map  $T \mapsto \bar{T}$  defines a bijection between irreducible closed subsets of  $D(f_i)$  and irreducible closed subsets of  $\text{Spec}(R)$  which meet  $D(f_i)$ ; see [Görtz and Wedhorn 2010, §1.5]. This implies that  $Z \cap D(f_i)$  is an irreducible component of  $\text{Ex}_A(\mathcal{F}) \cap D(f_i) = \text{Ex}_{A_{f_i}}(\mathcal{F})$ . It follows from the above that  $Z \cap D(f_i)$  is of codimension 1 in  $D(f_i)$ . Hence,  $Z$  is of codimension 1 in  $\text{Spec}(R)$  by [Stacks 2005–, Tag 02I4]. All irreducible components of  $\text{Ex}_A(\mathcal{F})$  are thus of codimension 1 in  $\text{Spec}(R)$ . Since each set  $\text{Ex}_{A_{f_i}}(\mathcal{F})$  has only finitely many irreducible components and since  $I$  is finite, also  $\text{Ex}_A(\mathcal{F})$  has only finitely many irreducible components.  $\square$

**Remark 3.2.** We note that  $A$  is finitely presented flat if and only if it is finite projective; see [Lam 1999, Theorem 4.30; Stacks 2005–, Tag 058R]. Hence, we could have equally assumed that  $A$  is finite projective in Proposition 3.1 but we preferred the seemingly more general notion.

*From now on, we assume that  $A$  is finitely presented as an  $R$ -module.*

For  $\mathfrak{p} \in \text{Spec}(R)$  let us denote by  $\mathcal{B}_A(\mathfrak{p}) \subseteq A^K$  the set of block idempotents of  $A_{\mathfrak{p}}$ . Clearly,  $\mathcal{B}_A(\mathfrak{p})$  and  $\text{Bl}_A(\mathfrak{p})$  are in bijection by taking sums of the  $c_i$  over the subsets in  $\text{Bl}_A(\mathfrak{p})$ . Note that  $\mathcal{B}_A(\mathfrak{p})$  is constant on  $\text{Bl}_A^{-1}(\mathcal{P})$  for any  $\mathcal{P}$ . We can thus define  $\text{Gen}_A(\mathcal{P}) := \text{Gen}_A(\mathcal{B}_A(\mathfrak{p}))$  where  $\mathfrak{p} \in \text{Bl}_A^{-1}(\mathcal{P})$  is arbitrary.

**Theorem 3.3.** *Then  $\text{Bl}_A^{-1}(\mathcal{P})$  is a closed subset of  $\text{Spec}(R)$  for any partition  $\mathcal{P}$ . Thus, each stratum  $\text{Bl}_A^{-1}(\mathcal{P})$  is **open** in  $\text{Bl}_A^{-1}(\leq \mathcal{P})$ , and hence **locally closed** in  $\text{Spec}(R)$ , and*

$$(17) \quad \overline{\text{Bl}_A^{-1}(\mathcal{P})} \subseteq \text{Bl}_A^{-1}(\leq \mathcal{P}).$$

*In particular, the decomposition (13) is a **stratification** of  $\text{Spec}(R)$ .*

*Proof.* First, assume that  $\mathcal{P}$  is actually a block structure, i.e.,  $\mathcal{P} \in \text{Bl}(A)$ . Since  $\text{Spec}(R) = \coprod_{\mathcal{P}'} \text{Bl}_A^{-1}(\mathcal{P}')$ , we have

$$\text{Spec}(R) \setminus \text{Bl}_A^{-1}(\leq \mathcal{P}) = \bigcup_{\mathcal{P}' \not\leq \mathcal{P}} \text{Bl}_A^{-1}(\mathcal{P}').$$

Let  $\mathcal{P}' \not\leq \mathcal{P}$  and  $\mathfrak{p}' \in \text{Gen}_A(\mathcal{P}')$ . Then  $\mathcal{P}' \leq \text{Bl}_A(\mathfrak{p}')$ . But this implies  $\text{Bl}_A(\mathfrak{p}') \not\leq \mathcal{P}$  since otherwise  $\mathcal{P}' \leq \text{Bl}_A(\mathfrak{p}') \leq \mathcal{P}$ . Hence,  $\text{Gen}_A(\mathcal{P}') \subseteq \text{Spec}(R) \setminus \text{Bl}_A^{-1}(\leq \mathcal{P})$ . Conversely, we clearly have  $\text{Bl}_A^{-1}(\mathcal{P}') \subseteq \text{Gen}_A(\mathcal{P}')$ . This shows that

$$\text{Spec}(R) \setminus \text{Bl}_A^{-1}(\leq \mathcal{P}) = \bigcup_{\mathcal{P}' \not\leq \mathcal{P}} \text{Bl}_A^{-1}(\mathcal{P}') = \bigcup_{\mathcal{P}' \not\leq \mathcal{P}} \text{Gen}_A(\mathcal{P}').$$

This set is open by [Proposition 3.1](#), so

$$(18) \quad \text{Bl}_A^{-1}(\leq \mathcal{P}) = \bigcap_{\mathcal{P}' \not\leq \mathcal{P}} \text{Ex}_A(\mathcal{P}')$$

is closed. From (11) we see  $\text{Bl}_A^{-1}(\leq \mathcal{P})$  is also closed for an arbitrary partition  $\mathcal{P}$ . Using (12), we see  $\text{Bl}_A(\mathcal{P})$  is locally closed. Moreover, we have  $\text{Bl}_A^{-1}(\mathcal{P}) \subseteq \text{Bl}_A^{-1}(\leq \mathcal{P})$ , so

$$\overline{\text{Bl}_A^{-1}(\mathcal{P})} \subseteq \overline{\text{Bl}_A^{-1}(\leq \mathcal{P})} = \text{Bl}_A^{-1}(\leq \mathcal{P}). \quad \square$$

**Remark 3.4.** In general it is *not* true that we have  $\overline{\text{Bl}_A^{-1}(\mathcal{P})} = \text{Bl}_A^{-1}(\leq \mathcal{P})$ , so the stratification (13) is in general not a so-called *good* stratification: in the Brauer algebra example in the [Introduction](#) we have

$$\mathcal{P}' := \text{Bl}_A((3)) = \{\{1, 2, 3\}, \{4\}\} < \{\{1, 2\}, \{3\}, \{4\}\} = \text{Bl}_A((2)) =: \mathcal{P},$$

so  $(3) \in \text{Bl}_A^{-1}(\leq \mathcal{P})$ , but  $(3)$  is not contained in  $\overline{\text{Bl}_A^{-1}(\mathcal{P})} = \text{V}((2))$ . The problem here is that the skeleton  $\text{Bl}_A^{-1}(\leq \mathcal{P})$  has an irreducible component on which the maximal block structure is strictly smaller than the maximal one on the entire skeleton.

The poset  $\text{Bl}(A)$  has a unique maximal element, namely the block structure  $\text{Bl}_A(\bullet)$  of  $A$  in the generic point  $\bullet := (0)$  of  $\text{Spec}(R)$ ; i.e.,  $\text{Bl}_A(\bullet) = \{\{i\} \mid i \in I\}$  is the block structure of the generic fiber  $A^K = A_\bullet$ . The deviation of block structures from the generic one thus takes place on the closed subset

$$(19) \quad \text{BlEx}(A) := \{\mathfrak{p} \in \text{Spec}(R) \mid \text{Bl}_A(\mathfrak{p}) < \text{Bl}_A(\bullet)\} = \bigcup_{\mathcal{P} < \text{Bl}_A(\bullet)} \text{Bl}_A^{-1}(\leq \mathcal{P}).$$

We call this set the *block structure divisor* of  $A$ . In fact, since  $\text{BlEx}(A) = \text{Ex}_A(\mathcal{B}_A(\bullet))$ , [Proposition 3.1](#) implies:

**Corollary 3.5.** *Suppose that  $R$  is a Krull domain. Then  $\text{BlEx}(A)$  is a reduced Weil divisor.*

The generic block structure lives precisely on the open dense subset

$$(20) \quad \begin{aligned} \text{BlGen}(A) &:= \text{Spec}(R) \setminus \text{BlEx}(A) \\ &= \{\mathfrak{p} \in \text{Spec}(R) \mid \text{Bl}_A(\mathfrak{p}) = \text{Bl}_A(\bullet)\} = \text{Bl}_A^{-1}(\bullet). \end{aligned}$$

**3B. Block number stratification.** From the map  $\text{Bl}_A : \text{Spec}(R) \rightarrow \text{Part}(I)$  we obtain the numerical invariant

$$(21) \quad \#\text{Bl}_A : \text{Spec}(R) \rightarrow \mathbb{N}, \quad \mathfrak{p} \mapsto \#\text{Bl}_A(\mathfrak{p}) = \#\text{Bl}(A_{\mathfrak{p}}).$$

This map is again a morphism of posets, so

$$(22) \quad \mathfrak{q} \subseteq \mathfrak{p} \implies \#\text{Bl}_A(\mathfrak{p}) \leq \#\text{Bl}_A(\mathfrak{q}).$$

For  $n \in \mathbb{N}$  we have

$$(23) \quad \#\text{Bl}_A^{-1}(n) = \{\mathfrak{p} \in \text{Spec}(R) \mid \#\text{Bl}(A_{\mathfrak{p}}) = n\}$$

and we get the decomposition

$$(24) \quad \text{Spec}(R) = \bigsqcup_{n \in \mathbb{N}} \#\text{Bl}_A^{-1}(n).$$

We call this the *block number stratification*. This decomposition is of course coarser than the one defined by the fibers of  $\text{Bl}_A$ . We define

$$(25) \quad \#\text{Bl}_A^{-1}(\leq n) := \bigcup_{m \leq n} \#\text{Bl}_A^{-1}(m) = \{\mathfrak{p} \in \text{Spec}(R) \mid \#\text{Bl}(A_{\mathfrak{p}}) \leq n\}.$$

Since

$$(26) \quad \#\text{Bl}_A^{-1}(\leq n) = \bigcup_{\#\mathcal{P} \leq n} \text{Bl}_A^{-1}(\mathcal{P}),$$

this set is closed in  $\text{Spec}(R)$  by [Theorem 3.3](#). This means that the map  $\#\text{Bl}_A : \text{Spec}(R) \rightarrow \mathbb{N}$  is lower semicontinuous. Hence,  $\#\text{Bl}_A^{-1}(n)$  is open in  $\#\text{Bl}_A^{-1}(\leq n)$ , thus locally closed in  $\text{Spec}(R)$ , and

$$(27) \quad \overline{\#\text{Bl}_A^{-1}(n)} \subseteq \#\text{Bl}_A^{-1}(\leq n).$$



In particular, the partition (24) is a stratification of  $\text{Spec}(R)$ . Again, in general it will not be a *good* stratification. Note that

$$(28) \quad \text{BlEx}(A) = \{\mathfrak{p} \in \text{Spec}(R) \mid \#\text{Bl}_A(\mathfrak{p}) < \#\text{Bl}_A(\bullet)\} = \text{Bl}_A^{-1}(\leq \#\text{Bl}_A(\bullet) - 1).$$

**3C. Block stratification.** The poset  $\text{Part}(I)$  is actually a lattice; i.e., it has *meets*  $\wedge$  and *joins*  $\vee$ . The meet  $\mathcal{P} \wedge \mathcal{P}'$  of two partitions is the finest partition of  $I$  that is coarser than both  $\mathcal{P}$  and  $\mathcal{P}'$ , and this is obtained by joining members with nonempty intersection. The maximal elements in  $\text{Part}(I)$  not equal to the maximal element itself (the trivial partition) are the partitions  $\{i, j\} \cup (I \setminus \{i, j\})$  with  $i \neq j$ . We call these the *atoms* of  $\text{Part}(I)$  and we denote by  $\text{At}(I)$  the set of atoms. This terminology comes from the fact that an arbitrary partition  $\mathcal{P}$  is the meet of all atoms lying above it:

$$\mathcal{P} = \bigwedge_{\substack{T \in \text{At}(I) \\ \mathcal{P} \leq T}} T.$$

Because of this property, we say that  $\text{Part}(I)$  is *atomic*.

The poset of block structures of  $A$  has a similar atomic character. For  $i, j \in I$  with  $i \neq j$  let us write

$$(29) \quad \text{Gl}_A(\{i, j\}) := \text{Bl}_A^{-1}(\leq \{i, j\} \cup (I \setminus \{i, j\})).$$

This is the locus of all  $\mathfrak{p} \in \text{Spec}(R)$  such that the block idempotents  $c_i$  and  $c_j$  belong to the same block of  $A_{\mathfrak{p}}$ ; i.e., they are “glued” over  $\mathfrak{p}$ . We thus call this set a *gluing locus*. By [Theorem 3.3](#) it is a closed subset of  $\text{Spec}(R)$ . It is clear that

$$(30) \quad \text{BlEx}(A) = \bigcup_{i \neq j} \text{Gl}_A(\{i, j\}).$$

Let  $\text{At}(A)$  be the set of maximal elements of the set of irreducible components of the gluing loci, ordered by inclusion. Then we still have

$$(31) \quad \text{BlEx}(A) = \bigcup_{Z \in \text{At}(A)} Z.$$

**Lemma 3.6.** *The  $Z \in \text{At}(A)$  are precisely the irreducible components of  $\text{BlEx}(A)$ .*

*Proof.* Let  $Y$  be an irreducible component of  $\text{BlEx}(A)$  and let  $\xi$  be the generic point of  $Y$ . Since  $\text{Bl}_A(\xi)$  is not the trivial partition, there is  $i \neq j$  with  $\text{Bl}_A(\xi) \leq \{i, j\} \cup (I \setminus \{i, j\})$ . Hence,  $Y \subseteq \text{Gl}_A(\{i, j\})$ . Since  $Y$  is a maximal irreducible closed subset of  $\text{BlEx}(A)$  and  $\text{Gl}_A(\{i, j\}) \subseteq \text{BlEx}(A)$ , it is also a maximal irreducible closed subset of  $\text{Gl}_A(\{i, j\})$ , and thus equal to an irreducible component  $Z$  of  $\text{Gl}_A(\{i, j\})$ . It is clear that  $Z \in \text{At}(A)$ . Conversely, let  $Z \in \text{At}(A)$ . Since  $Z \subseteq \text{BlEx}(A)$ , there is an irreducible component  $Y$  of  $\text{BlEx}(A)$  containing  $Z$ . With the same argument as above, there is  $Z' \in \text{At}(A)$  with  $Y \subseteq Z'$ . Hence,  $Z \subseteq Y \subseteq Z'$ , and therefore  $Z = Y$  by maximality of the elements in  $\text{At}(A)$ .  $\square$

It now follows that for any  $\mathfrak{p} \in \text{Spec}(R)$  we have

$$(32) \quad \text{Bl}_A(\mathfrak{p}) = \bigwedge_{\substack{\mathcal{P} \in \text{At}(I) \\ \text{Bl}_A(\mathfrak{p}) \leq \mathcal{P}}} \mathcal{P} = \bigwedge_{\substack{Z \in \text{At}(A) \\ \mathfrak{p} \in Z}} \text{Bl}_A(Z),$$

where  $\text{Bl}_A(Z)$  denotes the block structure in the generic point of  $Z$ , i.e., the unique maximal block structure on  $Z$ . Hence, any block structure of  $A$  is a meet of *atomic block structures*  $\text{Bl}_A(Z)$  for  $Z \in \text{At}(A)$ . Recall from [Corollary 3.5](#) that if  $R$  is a Krull domain, the  $Z \in \text{At}(A)$  are all of codimension 1 in  $\text{Spec}(R)$ .

Following this observation, we introduce a refined stratification of  $\text{Spec}(R)$ . For a subset  $\mathcal{Z} \subseteq \text{At}(I)$  we define

$$(33) \quad \text{Bl}_A^{-1}(\leq \mathcal{Z}) := \bigcap_{Z \in \mathcal{Z}} Z,$$

$$(34) \quad \text{Bl}_A^{-1}(\mathcal{Z}) := \bigcap_{Z \in \mathcal{Z}} Z \setminus \bigcup_{Z \in \text{At}(A) \setminus \mathcal{Z}} Z.$$

It is clear that  $\text{Bl}_A^{-1}(\leq \mathcal{Z})$  is closed in  $\text{Spec}(R)$ , that  $\text{Bl}_A^{-1}(\mathcal{Z})$  is locally closed in  $\text{Spec}(R)$  and that the block structure on  $\text{Bl}_A^{-1}(\mathcal{Z})$  is in any point equal to  $\bigwedge_{Z \in \mathcal{Z}} \text{Bl}_A(Z)$ . Note that in this notation  $\text{Bl}_A^{-1}(\leq \emptyset) = \text{Spec}(R)$  and

$$(35) \quad \text{Bl}_A^{-1}(\emptyset) = \text{Spec}(R) \setminus \bigcup_{Z \in \text{At}(A)} Z = \text{BGen}(A).$$

Clearly,

$$(36) \quad \overline{\text{Bl}_A^{-1}(\mathcal{Z})} \subseteq \bigcap_{Z \in \mathcal{Z}} Z = \text{Bl}_A^{-1}(\leq \mathcal{Z}),$$

so we obtain the stratification

$$(37) \quad \text{Spec}(R) = \bigsqcup_{\mathcal{Z} \subseteq \text{At}(A)} \text{Bl}_A^{-1}(\mathcal{Z})$$

refining the block structure stratification [\(13\)](#). We call this the *block stratification* of  $A$ .

#### 4. Blocks of specializations

We now turn to our actual problem, namely blocks of specializations of  $A$ . Compared to blocks of localizations there is in general no possibility to compare the actual block structures of specializations. However, there is a rather general setting where blocks of specializations are naturally identified with blocks of localizations, namely when  $R$  is normal and  $A^K$  splits. In this case we can compare the actual block structures of specializations and all results from the preceding paragraph are actually also results about blocks of specializations. For the proof we need the following general result.

**Theorem 4.1.** *Let  $\phi : R \hookrightarrow S$  be a faithfully flat morphism of integral domains and let  $A$  be a finite flat  $R$ -algebra. Let  $K$  and  $L$  be the fraction fields of  $R$  and  $S$ , respectively. If  $\# \text{Bl}(A^K) = \# \text{Bl}(A^L)$ , then the morphism  $\phi_A : A \rightarrow A^S$  is block bijective.*

*Proof.* Recall from [Corollary 2.5](#) that both  $A$  and  $A^S$  have block decompositions. The map  $\phi_A : A \rightarrow A^S$  is injective by [Lemma 2.1\(b\)](#) since  $\phi$  is faithfully flat. Hence,  $\phi_A$  is idempotent stable by [Lemma 2.3\(a\)](#) and therefore  $\# \text{Bl}(A) \leq \# \text{Bl}(A^S)$  by [\(5\)](#). We thus have to show that  $\# \text{Bl}(A) \geq \# \text{Bl}(A^S)$ . We split the proof of this fact into several steps.

The case  $R = K$  and  $S = L$  holds by assumption. Assume that  $R = K$  and that  $S$  is general as in the theorem. Since  $A$  is  $R$ -flat, the extension  $A^S$  is  $S$ -flat and thus  $S$ -torsion-free. Hence, the map  $A^S \rightarrow A^L$  is injective by [Lemma 2.1\(c\)](#). In particular, it is idempotent stable by [Lemma 2.3\(a\)](#) and so  $\# \text{Bl}(A^S) \leq \# \text{Bl}(A^L)$  by [\(5\)](#). In total, we have

$$\# \text{Bl}(A) \leq \# \text{Bl}(A^S) \leq \# \text{Bl}(A^L) = \# \text{Bl}(A^K) = \# \text{Bl}(A).$$

Hence,  $\# \text{Bl}(A) = \# \text{Bl}(A^S)$ .

Finally, let both  $R$  and  $S$  be general as in the theorem. Let  $\Sigma := R \setminus \{0\}$  and  $\Omega := S \setminus \{0\}$ . Then  $K = \Sigma^{-1}R$  and  $L = \Omega^{-1}S$ . Set  $T := \Sigma^{-1}S$ . Since  $R$  and  $S$  are integral domains, we can naturally view all rings as subrings of  $L$  and so we get the two commutative diagrams

(38)

the right one being induced by the left one. All morphisms in the left diagram are clearly injective. We claim the same holds for the right diagram. We have noted at the beginning that the map  $A \rightarrow A^S$  is injective. Since  $A$  is  $R$ -flat, it is  $R$ -torsion-free and so the map  $A \rightarrow A^K$  is injective by [Lemma 2.1\(c\)](#). We have argued above already that the map  $A^S \rightarrow A^L$  is injective. Since  $S \hookrightarrow T$  is a localization map, the induced scalar extension functor is exact so that  $A^T$  is a flat  $T$ -module. In particular,  $A^T$  is  $T$ -torsion-free and so  $A^T \rightarrow A^L$  is injective by [Lemma 2.1\(c\)](#). The map  $A^K \rightarrow A^L$  is injective by [Lemma 2.1\(a\)](#). Due to the commutativity of the diagram, the remaining maps must be injective, too. We can thus view all scalar extensions of  $A$  naturally as subsets of  $A^L$ . We claim that

(39) 
$$A = A^K \cap A^S$$

as subsets of  $A^L$ . Because of the commutative diagram above, this intersection already takes place in  $A^T$ . Consider  $A^K$  as an  $R$ -module now. We have a natural identification

$$\begin{aligned}\phi^*(A^K) &= S \otimes_R A^K = S \otimes_R (\Sigma^{-1}A) \\ &= (\Sigma^{-1}S) \otimes_R A = T \otimes_R A = A^T\end{aligned}$$

as  $S$ -modules by [Bourbaki 1972, II, §2.7, Proposition 18]. Note that the map  $A^K \rightarrow A^T$  in the diagram above is the map  $\phi_{A^K}$ , when considering  $A^K$  as an  $R$ -module. The  $R$ -submodule  $A$  of  $A^K$  is now identified with  $\phi_{A^K}(A)$  and the  $S$ -submodule of  $A^T$  generated by  $A \subseteq A^T$  is identified with  $A^S$ . Since  $\phi$  is faithfully flat, it follows from [Bourbaki 1972, I, §3.5, Proposition 10(ii)] applied to the  $R$ -module  $A^K$  and the submodule  $A$  that

$$A = A^K \cap A^S$$

inside  $A^T$ . Let  $(c_i)_{i \in I}$  be the block idempotents of  $A^S$  and let  $(d_j)_{j \in J}$  be the block idempotents of  $A^K$ . By assumption the morphism  $A^K \rightarrow A^L$  is block bijective, which means that  $(d_j)_{j \in J}$  are the block idempotents of  $A^L$ . Since  $A^S \rightarrow A^L$  is idempotent stable, there exists by the arguments preceding (5) a partition  $(J_i)_{i \in I}$  of  $J$  such that the nonzero central idempotent  $c_i$  can in  $A^L$  be written as  $c_i = \sum_{j \in J_i} d_j$ . But this shows that  $c_i \in A^K \cap A^S$ , hence  $c_i \in A$  and so  $(c_i)_{i \in I}$  gives a decomposition of  $1 \in A$  into pairwise orthogonal centrally primitive idempotents of  $A$  by (39). Hence,  $\# \text{Bl}(A) = \# \text{Bl}(A^S)$ .  $\square$

To formulate the next proposition more generally, we use the property *block-split* introduced in Definition A.2 but note that the reader might just simply replace it by the more special property *split*. Moreover, we recall that a local integral domain  $R$  is called *unibranch* if its henselization  $R^h$  is again an *integral* (local) domain. This is equivalent to the normalization of  $R$  being again *local*; see [Raynaud 1970, IX, Corollaire 1]. This clearly holds if  $R$  is already *normal*. Examples of nonnormal unibranch rings are the local rings in ordinary cusp singularities of curves.

**Proposition 4.2.** *Let  $R$  be an integral domain and let  $A$  be a finite flat  $R$ -algebra with **block-split** generic fiber  $A^K$  (e.g., if  $A^K$  splits). Let  $\mathfrak{p} \in \text{Spec}(R)$  and suppose that  $R_{\mathfrak{p}}$  is unibranch (e.g., if  $R_{\mathfrak{p}}$  is normal). Then the quotient morphism  $A_{\mathfrak{p}} \twoheadrightarrow A(\mathfrak{p})$  is block bijective.*

*Proof.* By assumption,  $R_{\mathfrak{p}}$  and its henselization  $R_{\mathfrak{p}}^h$  are integral domains. Since  $A$  is  $R$ -flat, it follows that  $A_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R A$  is  $R_{\mathfrak{p}}$ -flat and that  $A_{\mathfrak{p}}^h := R_{\mathfrak{p}}^h \otimes_{R_{\mathfrak{p}}} A_{\mathfrak{p}}$  is  $R_{\mathfrak{p}}^h$ -flat. Hence, both  $A_{\mathfrak{p}}$  and  $A_{\mathfrak{p}}^h$  have block decompositions by Corollary 2.5. Let  $\mathfrak{p}_{\mathfrak{p}}^h$  be the maximal ideal of  $R_{\mathfrak{p}}^h$ . The henselization morphism  $R_{\mathfrak{p}} \rightarrow R_{\mathfrak{p}}^h$  is local and faithfully flat by [Grothendieck 1967, Théorème 18.6.6(iii)]. We now have the commutative

diagram

$$\begin{array}{ccc}
 A_{\mathfrak{p}} & \xrightarrow{\quad} & A_{\mathfrak{p}}^h \\
 \downarrow & & \downarrow \\
 A(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}A_{\mathfrak{p}} & \xrightarrow{\quad} & A_{\mathfrak{p}}^h/\mathfrak{p}_{\mathfrak{p}}^hA_{\mathfrak{p}}^h
 \end{array}$$

of idempotent stable morphisms. We know from Lemmas A.12(b) and A.10 that  $A_{\mathfrak{p}}^h \rightarrow A_{\mathfrak{p}}^h/\mathfrak{p}_{\mathfrak{p}}^hA_{\mathfrak{p}}^h$  is block bijective. Since  $A$  has block-split generic fiber and  $R_{\mathfrak{p}} \rightarrow R_{\mathfrak{p}}^h$  is a faithfully flat morphism of integral domains, we can use Theorem 4.1 to deduce that  $A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}^h$  is block bijective. In [Grothendieck 1967, Théorème 18.6.6(iii)] it is proven that  $R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \simeq R_{\mathfrak{p}}^h/\mathfrak{p}_{\mathfrak{p}}^h$ . Hence, the map  $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}^h/\mathfrak{p}_{\mathfrak{p}}^hA_{\mathfrak{p}}^h$  is an isomorphism and so in particular block bijective. We thus have

$$\# \text{BI}(A_{\mathfrak{p}}^h) = \# \text{BI}(A_{\mathfrak{p}}) \leq \# \text{BI}(A(\mathfrak{p})) = \# \text{BI}(A_{\mathfrak{p}}^h/\mathfrak{p}_{\mathfrak{p}}^hA_{\mathfrak{p}}^h) = \# \text{BI}(A_{\mathfrak{p}}^h)$$

by (5). Hence,  $\# \text{BI}(A_{\mathfrak{p}}) = \# \text{BI}(A(\mathfrak{p}))$ , so  $A_{\mathfrak{p}} \twoheadrightarrow A(\mathfrak{p})$  is block bijective. □

**Corollary 4.3.** *Suppose that  $R$  is normal and  $A^K$  splits. Then  $A_{\mathfrak{p}} \twoheadrightarrow A(\mathfrak{p})$  is block bijective for all  $\mathfrak{p} \in \text{Spec}(R)$ . Hence, all results from Section 3 apply also to blocks of specializations of  $A$ .*

### 5. Blocks via central characters

In this section we discuss an approach to explicitly compute the block structure of  $A$  in any point  $\mathfrak{p} \in \text{Spec}(R)$ , and so to compute the whole block stratification. This is based on the knowledge of the central characters of the generic fiber of  $A$ . Parts of the arguments presented here are due to Bonnafé and Rouquier [2017, Appendix D].

**5A. Müller’s theorem.** The central ingredient to establish a relationship between blocks and central characters is the general Lemma 5.6 below, which is usually referred to as *Müller’s theorem*. We were not able to find a proof of it in this generality in the literature, so we include a proof here but note that this is known. The main ingredient is an even more general result by B. Müller [1976] about the fibration of cliques of prime ideals in a noetherian ring over its center; see Lemma 5.5. We will recall only a few basic definitions from the excellent exposition in [Goodearl and Warfield 2004, §12] and refer to it for more details.

*Throughout the next paragraph, we assume that  $A$  is a noetherian ring.*

If  $\mathfrak{p}, \mathfrak{q}$  are prime ideals of  $A$ , we say that there is a *link* from  $\mathfrak{p}$  to  $\mathfrak{q}$ , written  $\mathfrak{p} \rightsquigarrow \mathfrak{q}$ , if there is an ideal  $\mathfrak{a}$  of  $A$  such that  $\mathfrak{p} \cap \mathfrak{q} \supseteq \mathfrak{a} \supseteq \mathfrak{p}\mathfrak{q}$  and  $(\mathfrak{p} \cap \mathfrak{q})/\mathfrak{a}$  is nonzero and torsion-free both as a left  $(A/\mathfrak{p})$ -module and as a right  $(A/\mathfrak{q})$ -module. The bimodule  $(\mathfrak{p} \cap \mathfrak{q})/\mathfrak{a}$  is then called a *linking bimodule* between  $\mathfrak{q}$  and  $\mathfrak{p}$ . The equivalence classes of the equivalence relation on  $\text{Spec}(A)$  generated by  $\rightsquigarrow$  are called the *cliques* of  $A$ .

We write  $\text{Clq}(A)$  for the set of cliques of  $A$  and  $\text{Clq}(\mathfrak{p})$  for the unique clique of  $A$  containing  $\mathfrak{p}$ . For the proof of [Lemma 5.6](#) we will need a few preparatory lemmas.

We call the supremum of lengths of chains of prime ideals in  $A$  the *classical Krull dimension* of  $A$ . The following lemma is standard.

**Lemma 5.1.** *Suppose that  $A$  is noetherian and of classical Krull dimension zero. Then there is a canonical bijection*

$$(40) \quad \begin{aligned} \text{Bl}(A) &\xrightarrow{\sim} \text{Clq}(A), \\ c &\longmapsto X_c := \{\mathfrak{m} \in \text{Max}(A) \mid c^\dagger \in \mathfrak{m}\}, \end{aligned}$$

where  $c^\dagger = 1 - c$ . If moreover  $A$  is **commutative**, then the cliques are singletons; i.e., there is a unique  $\mathfrak{m}_c \in \text{Max}(A)$  with  $c^\dagger \in \mathfrak{m}_c$ . Hence, in this case we have

$$\text{Bl}(A) \simeq \text{Max}(A) \simeq \text{Spec}(A).$$

*Proof.* The first assertion is proven in [[Goodearl and Warfield 2004](#), Corollary 12.13]. In a commutative noetherian ring the cliques are singletons, see [[loc. cit.](#), Exercise 12F], and this immediately implies the second assertion.  $\square$

**Lemma 5.2.** *Let  $\mathfrak{p}$  be a prime ideal of a noetherian ring  $A$  and let  $V$  be a nonzero  $A$ -module with  $\mathfrak{p} \subseteq \text{Ann}(V)$ . If  $V$  is torsion-free as an  $(A/\mathfrak{p})$ -module, then  $\mathfrak{p} = \text{Ann}(V)$ .*

*Proof.* Suppose that  $\mathfrak{p} \subsetneq \text{Ann}(V)$ . Then  $\text{Ann}(V)/\mathfrak{p}$  is a nonzero ideal of the noetherian prime ring  $A/\mathfrak{p}$  and thus contains a regular element  $\bar{x}$  by [[Jategaonkar 1986](#), Corollary 2.3.11]. But then  $\bar{x}V = 0$ , contradicting the assumption that  $V$  is a torsion-free  $(A/\mathfrak{p})$ -module.  $\square$

**Lemma 5.3.** *The following hold:*

- (a) *If  $\mathfrak{p}$  and  $\mathfrak{q}$  are prime ideals of  $A$  and if  $\mathfrak{b}$  is an ideal of  $A$  with  $\mathfrak{b} \subseteq \mathfrak{p} \cap \mathfrak{q}$  such that  $\mathfrak{p}/\mathfrak{b} \rightsquigarrow \mathfrak{q}/\mathfrak{b}$  in  $A/\mathfrak{b}$ , then  $\mathfrak{p} \rightsquigarrow \mathfrak{q}$  in  $A$ .*
- (b) *Let  $\mathfrak{p}$  and  $\mathfrak{q}$  be two prime ideals of  $A$  with  $\mathfrak{p} \rightsquigarrow \mathfrak{q}$  and let  $\mathfrak{b}$  be an ideal of  $A$ . If there exists a linking ideal  $\mathfrak{a}$  from  $\mathfrak{p}$  to  $\mathfrak{q}$  with  $\mathfrak{b} \subseteq \mathfrak{a}$ , then  $\mathfrak{p}/\mathfrak{b} \rightsquigarrow \mathfrak{q}/\mathfrak{b}$  in  $A/\mathfrak{b}$ .*

*Proof.* (a) We can write a linking ideal from  $\mathfrak{p}/\mathfrak{b}$  to  $\mathfrak{q}/\mathfrak{b}$  as  $\mathfrak{a}/\mathfrak{b}$  for an ideal  $\mathfrak{a}$  containing  $\mathfrak{b}$ . By definition, we have

$$(\mathfrak{p} \cap \mathfrak{q})/\mathfrak{b} = (\mathfrak{p}/\mathfrak{b}) \cap (\mathfrak{q}/\mathfrak{b}) \supseteq \mathfrak{a}/\mathfrak{b} \supseteq (\mathfrak{p}/\mathfrak{b}) \cdot (\mathfrak{q}/\mathfrak{b}) = (\mathfrak{p}\mathfrak{q})/\mathfrak{b},$$

implying that  $\mathfrak{p} \cap \mathfrak{q} \supseteq \mathfrak{a} \supseteq \mathfrak{p}\mathfrak{q}$ . Moreover, we have

$$((\mathfrak{p} \cap \mathfrak{q})/\mathfrak{b})/(\mathfrak{a}/\mathfrak{b}) \cong (\mathfrak{p} \cap \mathfrak{q})/\mathfrak{a}$$

as  $(A/\mathfrak{b})$ -bimodules. By definition,  $(\mathfrak{p} \cap \mathfrak{q})/\mathfrak{a}$  is torsion-free as a left module over the ring

$$(A/\mathfrak{b})/(\mathfrak{p}/\mathfrak{b}) \cong A/\mathfrak{p}.$$

Similarly, it follows that  $(\mathfrak{p} \cap \mathfrak{q})/\mathfrak{a}$  is torsion-free as a right module over the ring  $A/\mathfrak{q}$ . Hence,  $\mathfrak{a}$  is a linking ideal from  $\mathfrak{p}$  to  $\mathfrak{q}$ .

(b) We have

$$\mathfrak{p}/\mathfrak{b} \cap \mathfrak{q}/\mathfrak{b} = (\mathfrak{p} \cap \mathfrak{q})/\mathfrak{b} \supseteq \mathfrak{a}/\mathfrak{b} \supseteq (\mathfrak{p}\mathfrak{q} + \mathfrak{b})/\mathfrak{b} = (\mathfrak{p}/\mathfrak{b}) \cdot (\mathfrak{q}/\mathfrak{b}).$$

Since

$$((\mathfrak{p} \cap \mathfrak{q})/\mathfrak{b})/(\mathfrak{a}/\mathfrak{b}) \cong (\mathfrak{p} \cap \mathfrak{q})/\mathfrak{a}, \quad (A/\mathfrak{b})/(\mathfrak{p}/\mathfrak{b}) \cong A/\mathfrak{p}, \quad (A/\mathfrak{b})/(\mathfrak{q}/\mathfrak{b}) \cong A/\mathfrak{q},$$

it follows that  $\mathfrak{a}/\mathfrak{b}$  is a linking ideal from  $\mathfrak{p}/\mathfrak{b}$  to  $\mathfrak{q}/\mathfrak{b}$ .  $\square$

**Lemma 5.4.** *Let  $\mathfrak{p}$  and  $\mathfrak{q}$  be distinct prime ideals of a noetherian ring  $A$  with  $\mathfrak{p} \rightsquigarrow \mathfrak{q}$ . If  $\mathfrak{z}$  is a centrally generated ideal of  $A$  with  $\mathfrak{z} \subseteq \mathfrak{p}$  or  $\mathfrak{z} \subseteq \mathfrak{q}$ , then  $\mathfrak{z} \subseteq \mathfrak{p} \cap \mathfrak{q}$  and  $\mathfrak{p}/\mathfrak{z} \rightsquigarrow \mathfrak{q}/\mathfrak{z}$  in  $A/\mathfrak{z}$ .*

*Proof.* This is proven in [Müller 1985] but we also give a proof here for the sake of completeness. First note that since  $\mathfrak{z}$  is centrally generated and  $\mathfrak{p} \rightsquigarrow \mathfrak{q}$ , it follows from [Goodearl and Warfield 2004, Lemma 12.15] that already  $\mathfrak{z} \subseteq \mathfrak{p} \cap \mathfrak{q}$ . Let  $\mathfrak{a}$  be a linking ideal from  $\mathfrak{p}$  to  $\mathfrak{q}$ . We claim that  $\mathfrak{z}$  is contained in  $\mathfrak{a}$ . To show this, suppose that  $\mathfrak{z}$  is not contained in  $\mathfrak{a}$ . Then  $(\mathfrak{a} + \mathfrak{z})/\mathfrak{a}$  is a nonzero submodule of  $(\mathfrak{p} \cap \mathfrak{q})/\mathfrak{a}$  which is torsion-free as a left  $(A/\mathfrak{p})$ -module and as a right  $(A/\mathfrak{q})$ -module. In conjunction with the fact that  $\mathfrak{z}$  is centrally generated it now follows from Lemma 5.2 that

$$\mathfrak{p} = \text{Ann}_A((\mathfrak{a} + \mathfrak{z})/\mathfrak{a}) = \text{Ann}_A(((\mathfrak{a} + \mathfrak{z})/\mathfrak{a})_A) = \mathfrak{q},$$

contradicting the assumption  $\mathfrak{p} \neq \mathfrak{q}$ . Hence, we must have  $\mathfrak{z} \subseteq \mathfrak{a}$  and it thus follows from Lemma 5.3(b) that  $\mathfrak{p}/\mathfrak{z} \rightsquigarrow \mathfrak{q}/\mathfrak{z}$ .  $\square$

**Lemma 5.5.** *Let  $\mathfrak{z}$  be a centrally generated ideal of a noetherian ring  $A$ . Let  $\mathfrak{p}$  be a prime ideal of  $A$  with  $\mathfrak{z} \subseteq \mathfrak{p}$ . Then all prime ideals in  $\text{Clq}(\mathfrak{p})$  contain  $\mathfrak{z}$  and the map*

$$\begin{aligned} \text{Clq}(\mathfrak{p}) &\longrightarrow \text{Clq}(\mathfrak{p}/\mathfrak{z}), \\ \mathfrak{q} &\longmapsto \mathfrak{q}/\mathfrak{z}, \end{aligned}$$

*is a bijection between a clique of  $A$  and a clique of  $A/\mathfrak{z}$ .*

*Proof.* It follows immediately from [Goodearl and Warfield 2004, Lemma 12.15] that all prime ideals in  $\text{Clq}(\mathfrak{p})$  contain  $\mathfrak{z}$ . If  $\mathfrak{q} \in \text{Clq}(\mathfrak{p})$ , then there exists a chain  $\mathfrak{p} = \mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_{r-1}, \mathfrak{p}_r = \mathfrak{q}$  of prime ideals of  $A$  with  $\mathfrak{p}_i \rightsquigarrow \mathfrak{p}_{i+1}$  or  $\mathfrak{p}_{i+1} \rightsquigarrow \mathfrak{p}_i$  for all indices  $i$ . An inductive application of Lemma 5.4 shows now that  $\mathfrak{p}_i/\mathfrak{z} \rightsquigarrow \mathfrak{p}_{i+1}/\mathfrak{z}$  or  $\mathfrak{p}_{i+1}/\mathfrak{z} \rightsquigarrow \mathfrak{p}_i/\mathfrak{z}$  for all  $i$ . Hence,  $\mathfrak{p}/\mathfrak{z}$  and  $\mathfrak{q}/\mathfrak{z}$  lie in the same clique of  $A/\mathfrak{z}$  so that the map  $\text{Clq}(\mathfrak{p}) \rightarrow \text{Clq}(\mathfrak{p}/\mathfrak{z})$  is well-defined. On the other hand, similar arguments and Lemma 5.3(a) show that if  $\mathfrak{q}/\mathfrak{z} \in \text{Clq}(\mathfrak{p}/\mathfrak{z})$ , then also  $\mathfrak{q} \in \text{Clq}(\mathfrak{p})$ , so that we also have a well-defined map  $\text{Clq}(\mathfrak{p}/\mathfrak{z}) \rightarrow \text{Clq}(\mathfrak{p})$ . It is evident that both maps defined are pairwise inverse thus proving the first assertion. The second assertion is now obvious.  $\square$

**Lemma 5.6** (B. Müller). *Let  $A$  be a ring with center  $Z$  such that  $Z$  is noetherian and  $A$  is a finite  $Z$ -module. If  $\mathfrak{z}$  is a centrally generated ideal of  $A$  such that  $A/\mathfrak{z}A$  is of classical Krull dimension zero, then the inclusion  $(Z + \mathfrak{z})/\mathfrak{z} \hookrightarrow A/\mathfrak{z}A$  is block bijective. In other words, the block idempotents of  $A/\mathfrak{z}A$  are already contained in the central subalgebra  $(Z + \mathfrak{z})/\mathfrak{z}$ .*

*Proof.* Let  $\bar{A} := A/\mathfrak{z}$  and let  $\bar{Z} := (Z + \mathfrak{z})/\mathfrak{z}$ . Then  $\bar{A}$  is a finitely generated  $\bar{Z}$ -module since  $A$  is a finitely generated  $Z$ -module. Hence,  $\bar{Z} \subseteq \bar{A}$  is a finite centralizing extension and now it follows from going up in finite centralizing extensions [McConnell and Robson 2001, Theorem 10.2.9] that the classical Krull dimension of  $\bar{Z}$  is equal to that of  $\bar{A}$ , which is zero by assumption. Hence, by Lemma 5.1 we have  $\text{Bl}(\bar{Z}) \simeq \text{Clq}(\bar{Z})$  and  $\text{Bl}(\bar{A}) \simeq \text{Clq}(\bar{A})$ . Since  $\#\text{Bl}(\bar{Z}) \leq \#\text{Bl}(\bar{A})$ , the claim is thus equivalent to the claim that over each clique of  $\bar{Z}$ , there is just one clique of  $\bar{A}$ . So, let  $X, Y \in \text{Clq}(\bar{A})$  be two cliques. We pick  $\mathfrak{M}/\mathfrak{z} \in X$  and  $\mathfrak{N}/\mathfrak{z} \in Y$  with  $\mathfrak{M}, \mathfrak{N}$  maximal ideals of  $A$ . Assume that  $X$  and  $Y$  lie over the same clique of  $\bar{Z}$ . Since  $\bar{Z}$  is commutative, we know from Lemma 5.1 that all cliques are singletons and so the assumption implies that  $\mathfrak{M}/\mathfrak{z}$  and  $\mathfrak{N}/\mathfrak{z}$  lie over the same maximal ideal of  $\bar{Z}$ ; i.e.,

$$(\mathfrak{M}/\mathfrak{z}) \cap ((Z + \mathfrak{z})/\mathfrak{z}) = (\mathfrak{N}/\mathfrak{z}) \cap ((Z + \mathfrak{z})/\mathfrak{z}).$$

Hence

$$\mathfrak{M} \cap (Z + \mathfrak{z}) = \mathfrak{N} \cap (Z + \mathfrak{z}).$$

Since  $Z \subseteq Z + \mathfrak{z}$ , we thus get

$$\mathfrak{M} \cap Z = \mathfrak{M} \cap Z \cap (Z + \mathfrak{z}) = \mathfrak{N} \cap Z \cap (Z + \mathfrak{z}) = \mathfrak{N} \cap Z.$$

Now, Müller's theorem [Goodearl and Warfield 2004, Theorem 13.10] implies that  $\mathfrak{M}$  and  $\mathfrak{N}$  lie in the same clique of  $A$ . An application of Lemma 5.5 thus implies that  $\mathfrak{M}/\mathfrak{z}$  and  $\mathfrak{N}/\mathfrak{z}$  lie in the same clique of  $A/\mathfrak{z}$ , so  $X = Y$ .  $\square$

## 5B. Blocks as fibers of a morphism.

*We assume  $A$  is a finite flat algebra over a noetherian integral domain  $R$ .*

By Lemma B.2 the morphism

$$(41) \quad \Upsilon : \text{Spec}(Z) \rightarrow \text{Spec}(R),$$

induced by the canonical morphism from  $R$  to the center  $Z$  of  $A$  is finite, closed, and surjective. The center  $Z$  of  $A$  is naturally an  $R$ -algebra and so we can consider its fibers

$$(42) \quad Z(\mathfrak{p}) = k(\mathfrak{p}) \otimes_R Z/\mathfrak{p}Z = Z_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}Z_{\mathfrak{p}}$$

in prime ideals  $\mathfrak{p}$  of  $R$ . On the other hand, the image of  $Z_{\mathfrak{p}} = Z(A_{\mathfrak{p}})$  under the canonical (surjective) morphism  $A_{\mathfrak{p}} \rightarrow A(\mathfrak{p})$  yields a central subalgebra

$$(43) \quad Z_{\mathfrak{p}}(A) := (Z_{\mathfrak{p}} + \mathfrak{p}_{\mathfrak{p}}A_{\mathfrak{p}})/\mathfrak{p}_{\mathfrak{p}}A_{\mathfrak{p}}$$



of  $A(\mathfrak{p})$ . In general this subalgebra is *not* equal to the center of  $A(\mathfrak{p})$  itself. We have a surjective morphism

$$(44) \quad \phi_{\mathfrak{p}} : Z(\mathfrak{p}) \twoheadrightarrow Z_{\mathfrak{p}}(A)$$

of finite-dimensional  $k(\mathfrak{p})$ -algebras. This morphism is in general *not* injective — it is if and only if  $\mathfrak{p}_{\mathfrak{p}}A_{\mathfrak{p}} \cap Z_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}Z_{\mathfrak{p}}$ . Nonetheless, we have the following result.

**Lemma 5.7.** *The map  $\phi_{\mathfrak{p}} : Z(\mathfrak{p}) \rightarrow Z_{\mathfrak{p}}(A)$  in (44) is block bijective.*

*Proof.* Since  $\phi_{\mathfrak{p}}$  is surjective, the induced map  ${}^a\phi_{\mathfrak{p}} : \text{Spec}(Z_{\mathfrak{p}}(A)) \rightarrow \text{Spec}(Z(\mathfrak{p}))$  is injective, so  $\#\text{Bl}(Z_{\mathfrak{p}}(A)) \leq \#\text{Bl}(Z(\mathfrak{p}))$  by Lemma 5.1. Now we just need to show that  $\phi_{\mathfrak{p}}$  does not map any nontrivial idempotent to zero. Since  $R_{\mathfrak{p}}$  is noetherian, also  $A_{\mathfrak{p}}$  is noetherian. The Artin–Rees lemma [Matsumura 1986, Theorem 8.5] applied to the  $R_{\mathfrak{p}}$ -module  $A_{\mathfrak{p}}$ , the submodule  $Z_{\mathfrak{p}}$  of  $A_{\mathfrak{p}}$ , and the ideal  $\mathfrak{p}_{\mathfrak{p}}$  of  $R_{\mathfrak{p}}$  shows that there is an integer  $k \in \mathbb{N}_{>0}$  such that for any  $n > k$  we have

$$\mathfrak{p}_{\mathfrak{p}}^n A_{\mathfrak{p}} \cap Z_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}^{n-k} ((\mathfrak{p}_{\mathfrak{p}}^k A_{\mathfrak{p}}) \cap Z_{\mathfrak{p}}).$$

In particular, there is  $n \in \mathbb{N}_{>0}$  such that  $\mathfrak{p}_{\mathfrak{p}}^n A_{\mathfrak{p}} \cap Z_{\mathfrak{p}} \subseteq \mathfrak{p}_{\mathfrak{p}} Z_{\mathfrak{p}}$ . Now, let  $\bar{e} \in Z(\mathfrak{p}) = Z_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}Z_{\mathfrak{p}}$  be an idempotent with  $\phi_{\mathfrak{p}}(\bar{e}) = 0$ . By assumption,  $\bar{e} \in \text{Ker}(\phi_{\mathfrak{p}}) = (\mathfrak{p}_{\mathfrak{p}}A_{\mathfrak{p}} \cap Z_{\mathfrak{p}})/\mathfrak{p}_{\mathfrak{p}}Z_{\mathfrak{p}}$ . Hence, if  $e \in Z_{\mathfrak{p}}$  is a representative of  $\bar{e}$ , we have  $e \in \mathfrak{p}_{\mathfrak{p}}A_{\mathfrak{p}} \cap Z_{\mathfrak{p}}$ . We have  $e^n \in \mathfrak{p}_{\mathfrak{p}}^n A_{\mathfrak{p}} \cap Z_{\mathfrak{p}} \subseteq \mathfrak{p}_{\mathfrak{p}} Z_{\mathfrak{p}}$ , so already  $\bar{e} = 0$ .  $\square$

**Theorem 5.8.** *For any  $\mathfrak{p} \in \text{Spec}(R)$  there are canonical bijections*

$$(45) \quad \text{Bl}(A(\mathfrak{p})) \simeq \text{Bl}(Z_{\mathfrak{p}}(A)) \simeq \text{Bl}(Z(\mathfrak{p})) \simeq \Upsilon^{-1}(\mathfrak{p}).$$

*The first bijection  $\text{Bl}(A(\mathfrak{p})) \simeq \text{Bl}(Z_{\mathfrak{p}}(A))$  is induced by the embedding  $Z_{\mathfrak{p}}(A) \hookrightarrow A(\mathfrak{p})$ . In other words, all block idempotents of  $A(\mathfrak{p})$  are already contained in the central subalgebra  $Z_{\mathfrak{p}}(A)$  of  $A(\mathfrak{p})$ . The second bijection is the bijection from Lemma 5.7. The last bijection  $\text{Bl}(Z(\mathfrak{p})) \simeq \Upsilon^{-1}(\mathfrak{p})$  maps a block idempotent  $c$  of  $Z(\mathfrak{p})$  to the (by the theorem, unique) maximal ideal  $\mathfrak{m}_c$  of  $Z$  lying above  $\mathfrak{p}$  such that  $c^{\dagger} \in (\mathfrak{m}_c + \mathfrak{p}_{\mathfrak{p}}Z_{\mathfrak{p}})/\mathfrak{p}_{\mathfrak{p}}Z_{\mathfrak{p}}$ , where  $c^{\dagger} = 1 - c$ .*

*Proof.* The first bijection follows directly from Lemma 5.6 applied to  $A_{\mathfrak{p}}$  and the centrally generated ideal  $\mathfrak{z} := \mathfrak{p}_{\mathfrak{p}}A_{\mathfrak{p}}$ . Let  $\Upsilon_{\mathfrak{p}} : \text{Spec}(Z_{\mathfrak{p}}) \rightarrow \text{Spec}(R_{\mathfrak{p}})$  be the morphism induced by the canonical map  $R_{\mathfrak{p}} \rightarrow Z_{\mathfrak{p}}$ . Recall from Lemma B.2 that  $R_{\mathfrak{p}} \subseteq Z_{\mathfrak{p}}$  is a finite extension so that  $\Upsilon_{\mathfrak{p}}$  is surjective. We have

$$\begin{aligned} \Upsilon_{\mathfrak{p}}^{-1}(\mathfrak{p}_{\mathfrak{p}}) &= \{\mathfrak{Q} \in \text{Spec}(Z_{\mathfrak{p}}) \mid \mathfrak{Q} \cap R_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}\} \\ &= \{\mathfrak{Q} \in \text{Spec}(Z_{\mathfrak{p}}) \mid \mathfrak{p}_{\mathfrak{p}} \subseteq \mathfrak{Q}\} \\ &= \{\mathfrak{Q} \in \text{Spec}(Z_{\mathfrak{p}}) \mid \mathfrak{p}_{\mathfrak{p}}Z_{\mathfrak{p}} \subseteq \mathfrak{Q}\} \simeq \text{Spec}(Z(\mathfrak{p})). \end{aligned}$$

In the second equality we used the fact that  $R_{\mathfrak{p}} \rightarrow Z_{\mathfrak{p}}$  is a finite morphism and  $R_{\mathfrak{p}}$  is local with maximal ideal  $\mathfrak{p}_{\mathfrak{p}}$ . The identification with  $\text{Spec}(Z(\mathfrak{p}))$  is canonical

since  $Z(\mathfrak{p}) = Z_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}Z_{\mathfrak{p}}$ . The morphism  $\Theta_{\mathfrak{p}} : \text{Spec}(Z_{\mathfrak{p}}) \rightarrow \text{Spec}(Z)$  induced by the localization map  $Z \rightarrow Z_{\mathfrak{p}}$  is injective by [Eisenbud 1995, Proposition 2.2(b)]. We claim that this map induces  $\Upsilon_{\mathfrak{p}}^{-1}(\mathfrak{p}_{\mathfrak{p}}) \simeq \Upsilon^{-1}(\mathfrak{p})$ . If  $\Omega \in \Upsilon_{\mathfrak{p}}^{-1}(\mathfrak{p}_{\mathfrak{p}})$ , then clearly

$$(\Omega \cap Z) \cap R = \Omega \cap R \subseteq R \cap \mathfrak{p}_{\mathfrak{p}} = \mathfrak{p}$$

and therefore  $\Theta_{\mathfrak{p}}$  induces an injective map  $\Upsilon_{\mathfrak{p}}^{-1}(\mathfrak{p}_{\mathfrak{p}}) \rightarrow \Upsilon^{-1}(\mathfrak{p})$ . If  $\Omega \in \Upsilon^{-1}(\mathfrak{p})$ , then, since  $\Omega \cap R = \mathfrak{p}$ , we have  $\Omega \cap (R \setminus \mathfrak{p}) = \emptyset$  so that  $\Omega_{\mathfrak{p}} \in \text{Spec}(Z_{\mathfrak{p}})$  and clearly  $\mathfrak{p}_{\mathfrak{p}} \subseteq \Omega_{\mathfrak{p}}$ , implying that  $\Omega_{\mathfrak{p}} \in \Upsilon_{\mathfrak{p}}^{-1}(\mathfrak{p}_{\mathfrak{p}})$ . The map  $\Upsilon_{\mathfrak{p}}^{-1}(\mathfrak{p}_{\mathfrak{p}}) \rightarrow \Upsilon^{-1}(\mathfrak{p})$  is thus bijective. Hence, we have a canonical bijection  $\text{Spec}(Z(\mathfrak{p})) \simeq \Upsilon^{-1}(\mathfrak{p})$ . Now, recall from Lemma 5.1 that  $\text{Spec}(Z(\mathfrak{p})) \simeq \text{Bl}(Z(\mathfrak{p}))$ .  $\square$

### 5C. Blocks via central characters.

*We assume that  $R$  is noetherian and normal, and that  $A$  is a finite flat  $R$ -algebra with split generic fiber  $A^K$ .*

Recall from Corollary 4.3 that the quotient map  $A_{\mathfrak{p}} \twoheadrightarrow A(\mathfrak{p})$  induces  $\text{Bl}(A_{\mathfrak{p}}) \simeq \text{Bl}(A(\mathfrak{p}))$ , so together with Theorem 5.8 we have the canonical bijection

$$(46) \quad \text{Bl}(A_{\mathfrak{p}}) \simeq \Upsilon^{-1}(\mathfrak{p}).$$

Recall from Section 3 that  $\text{Fam}_{\mathfrak{p}}(A^K)$  is the partition of  $\text{Irr } A^K$  induced by the blocks of  $A_{\mathfrak{p}}$  and that we naturally have  $\text{Bl}(A_{\mathfrak{p}}) \simeq \text{Fam}_{\mathfrak{p}}(A^K)$ . Altogether, we now have canonical bijections

$$(47) \quad \text{Fam}_{\mathfrak{p}}(A) \simeq \text{Bl}(A_{\mathfrak{p}}) \simeq \Upsilon^{-1}(\mathfrak{p}) \simeq \text{Bl}(A(\mathfrak{p})).$$

Since  $A$  has split generic fiber  $A^K$ , we have a central character  $\Omega_S : Z(A^K) \rightarrow K$  for every simple  $A^K$ -module  $S$ . Recall that  $\Omega_S(z)$  is the scalar by which  $z \in Z(A^K)$  acts on  $S$ . Since  $R$  is normal, the image of the restriction of  $\Omega_S$  to  $Z(A) \subseteq Z(A^K)$  is contained in  $R \subseteq K$ . We thus get a well-defined  $R$ -algebra morphism

$$(48) \quad \Omega'_S : Z(A) \rightarrow R.$$

It is a classical fact that  $S, T \in \text{Irr } A^K$  lie in the same family if and only if  $\Omega'_S = \Omega'_T$ . We can thus label the central characters of  $A^K$  as  $\Omega_{\mathcal{F}}$  with  $\mathcal{F}$  a family (block) of  $A^K$ . Using Theorem 5.8 this description generalizes modulo  $\mathfrak{p}$  so that we get an explicit description of the  $\mathfrak{p}$ -families, and thus of the block stratification. For  $\mathfrak{p} \in \text{Spec}(R)$  let

$$(49) \quad \Omega_{\mathfrak{p}}^S : Z(A) \rightarrow R/\mathfrak{p}$$

be the composition of  $\Omega'_S$  with the quotient map  $R \twoheadrightarrow R/\mathfrak{p}$ .

**Theorem 5.9.** *Under the bijection  $\Upsilon^{-1}(\mathfrak{p}) \simeq \text{Fam}_{\mathfrak{p}}(A)$  from (47) the  $\mathfrak{p}$ -family of a simple  $A^K$ -module  $S$  corresponds to  $\text{Ker } \Omega_{\mathfrak{p}}^S$ . Hence, two simple  $A^K$ -modules  $S$  and  $T$  lie in the same  $\mathfrak{p}$ -family if and only if  $\Omega'_{\mathcal{F}}(z) \equiv \Omega'_{\mathcal{G}}(z) \pmod{\mathfrak{p}}$  for all  $z \in Z(A)$ .*

So, if  $z_1, \dots, z_n$  is an  $R$ -algebra generating system of  $Z(A)$  and  $\mathcal{F}, \mathcal{F}'$  are two distinct  $A^K$ -families, then the corresponding gluing locus is given by

$$(50) \quad \text{Gl}_A(\leq \{\mathcal{F}, \mathcal{F}'\}) = \text{V}(\{\Omega_{\mathcal{F}}(z_i) - \Omega_{\mathcal{F}'}(z_i) \mid i = 1, \dots, n\}).$$

*Proof.* Considering the explicit form of the bijection given in [Theorem 5.8](#) we see that the bijection (46) maps a block idempotent  $c$  of  $A_{\mathfrak{p}}$  to the (by the theorem, unique) maximal ideal  $\Omega_c$  of  $Z$  lying above  $\mathfrak{p}$  and satisfying  $c^\dagger \in (\Omega_c)_{\mathfrak{p}}$ . Let  $c_{\Omega}$  be the block idempotent of  $A_{\mathfrak{p}}$  corresponding to  $\Omega \in \Upsilon^{-1}(\mathfrak{p})$ .

For  $S \in \text{Irr } A^K$  let  $\Omega'_S : Z \rightarrow R/\mathfrak{p}$  be the composition of  $\Omega'_S$  and the quotient morphism  $R \rightarrow R/\mathfrak{p}$ . It is clear that  $\text{Ker}(\Omega'_S) \in \Upsilon^{-1}(\mathfrak{p})$ . Note that  $\Omega'_S(z) \equiv \Omega'_T(z) \pmod{\mathfrak{p}}$  for all  $z \in Z(A)$  if and only if  $\Omega'_S = \Omega'_T$ . We have an exact sequence

$$0 \longrightarrow \text{Ker}(\Omega'_S) \longrightarrow Z \xrightarrow{\Omega'_S} R \longrightarrow 0$$

of  $R$ -modules. Since  $\Omega'_S$  is an  $R$ -algebra morphism, the canonical map  $R \rightarrow Z$  is a section of  $\Omega'_S$  and therefore  $Z = R \oplus \text{Ker}(\Omega'_S)$  as  $R$ -modules. Similarly, we have  $Z = R \oplus \text{Ker}(\Omega'_T)$ . Since  $\text{Ker}(\Omega'_S) \subseteq \text{Ker}(\Omega'_T)$  and  $\text{Ker}(\Omega'_T) \subseteq \text{Ker}(\Omega'_S)$ , this implies that  $\Omega'_S = \Omega'_T$  if and only if  $\text{Ker}(\Omega'_S) = \text{Ker}(\Omega'_T)$ .

Now, suppose that  $\text{Ker}(\Omega'_S) = \text{Ker}(\Omega'_T)$ . Denote this common kernel by  $\Omega$ . Clearly,  $\Omega \in \Upsilon^{-1}(\mathfrak{p})$ . We know that the corresponding block idempotent  $c_{\Omega}$  of  $A_{\mathfrak{p}}$  has the property that  $c_{\Omega}^\dagger \in \Omega_{\mathfrak{p}}$ . Since  $\text{Ker}(\Omega'_S) \subseteq \text{Ker}(\Omega'_T) = \Omega = \text{Ker}(\Omega'_S) \supseteq \text{Ker}(\Omega'_T)$ , this certainly implies that  $c_{\Omega}^\dagger S = 0 = c_{\Omega}^\dagger T$ . Hence,  $S$  and  $T$  lie in the same  $\mathfrak{p}$ -family.

Conversely, suppose that  $S$  and  $T$  lie in the same  $\mathfrak{p}$ -family. We can write the corresponding block idempotent of  $A_{\mathfrak{p}}$  as  $c_{\Omega}$  for some  $\Omega \in \Upsilon^{-1}(\mathfrak{p})$ . By definition,  $c_{\Omega}^\dagger S = 0 = c_{\Omega}^\dagger T$ . We know that  $c_{\Omega}^\dagger \in \Omega_{\mathfrak{p}}$  and  $c_{\Omega} \notin \Omega_{\mathfrak{p}}$  and therefore  $\text{Ker}((\Omega'_S)_{\mathfrak{p}}) = \Omega_{\mathfrak{p}} = \text{Ker}((\Omega'_T)_{\mathfrak{p}})$ . Hence,  $\Omega \subseteq \text{Ker}(\Omega'_S) \subseteq \text{Ker}(\Omega'_T)$  and  $\Omega \subseteq \text{Ker}(\Omega'_T) \subseteq \text{Ker}(\Omega'_S)$ . Since  $\Omega, \text{Ker}(\Omega'_S), \text{Ker}(\Omega'_T) \in \Upsilon^{-1}(\mathfrak{p})$  and all prime ideals in  $\Upsilon^{-1}(\mathfrak{p})$  are incomparable, we thus conclude that  $\text{Ker}(\Omega'_S) = \text{Ker}(\Omega'_T)$ .

The equation for the gluing locus is now clear. □

### 6. Blocks and decomposition matrices

To obtain information about the actual members of the  $A(\mathfrak{p})$ -families we use decomposition maps as introduced by Geck and Rouquier [1997]; see also [Geck and Pfeiffer 2000; Thiel 2016]. For a ring  $A$  we denote by  $G_0(A) := K_0(A\text{-mod})$  the Grothendieck group and by  $K_0(A) := K_0(A\text{-proj})$  the projective class group. In the case  $A$  is semiperfect (e.g., artinian),  $K_0(A)$  is the free abelian group with basis the isomorphism classes of the projective indecomposable modules. In the case  $A$  is artinian,  $G_0(A)$  is the free abelian group with basis the isomorphism classes of simple modules and  $K_0(A) \simeq G_0(A)$  mapping  $P$  to  $\text{Hd}(P)$ .

For the theory of decomposition maps we need the following (standard) assumption:

*A is finite free with split generic fiber and for any nonzero  $\mathfrak{p} \in \text{Spec}(R)$  there is a discrete valuation ring  $\mathcal{O}$  with maximal ideal  $\mathfrak{m}$  in  $K$  dominating  $R_{\mathfrak{p}}$  such that the canonical map  $G_0(A(\mathfrak{p})) \rightarrow G_0(A^{\mathcal{O}}(\mathfrak{m}))$  of Grothendieck groups is an isomorphism.*

We call a ring  $\mathcal{O}$  as above a *perfect  $A$ -gate* in  $\mathfrak{p}$ . We refer to [Thiel 2016] for more details. The following lemma lists two standard situations in which the above assumptions hold. Part (a) is obvious and part (b) was proven in [Thiel 2016, Theorem 1.22].

**Lemma 6.1.** *A finite free  $R$ -algebra  $A$  with split generic fiber satisfies the above assumptions in the following two cases:*

- (a)  $R$  is a Dedekind domain.
- (b)  $R$  is noetherian and  $A$  has split fibers.

If  $\mathcal{O}$  is a perfect  $A$ -gate in  $\mathfrak{p}$ , then there is a group morphism

$$(51) \quad d_A^{\mathfrak{p}, \mathcal{O}} : G_0(A^K) \rightarrow G_0(A(\mathfrak{p}))$$

between Grothendieck groups generalizing reduction modulo  $\mathfrak{p}$ . In the case  $R$  is normal, it was proven by Geck and Rouquier [1997] that this map is independent of the choice of  $\mathcal{O}$  and in this case we just write  $d_A^{\mathfrak{p}}$ . We note that in the case  $R$  is noetherian and  $A$  has split fibers, any decomposition map in the sense of Geck and Rouquier can be realized by a perfect  $A$ -gate; see [Thiel 2016, Theorem 1.22].

**6A. Brauer reciprocity.** An important tool for relating decomposition maps and blocks is *Brauer reciprocity*, which we prove in Theorem 6.2 below in our general setup (this was known to hold before only in special settings). Recall that the *intertwining form* for a finite-dimensional algebra  $B$  over a field  $F$  is the  $\mathbb{Z}$ -linear pairing  $\langle \cdot, \cdot \rangle_B : K_0(B) \times G_0(B) \rightarrow \mathbb{Z}$  uniquely defined by

$$(52) \quad \langle [P], [V] \rangle := \dim_F \text{Hom}_B(P, V)$$

for a finite-dimensional projective  $B$ -module  $P$  and a finite-dimensional  $B$ -module  $V$ ; see [Geck and Rouquier 1997, §2]. Here,  $K_0(B)$  is the zeroth  $K$ -group of the category of finite-dimensional projective  $B$ -modules. The intertwining form is always nondegenerate; see Lemma A.6. Due to the nondegeneracy of  $\langle \cdot, \cdot \rangle_{A^K}$  there is at most one adjoint

$$(53) \quad e_A^{\mathfrak{p}, \mathcal{O}} : K_0(A(\mathfrak{p})) \rightarrow K_0(A^K)$$

of  $d_A^{\mathfrak{p}, \mathcal{O}} : G_0(A^K) \rightarrow G_0(A(\mathfrak{p}))$  with respect to  $\langle \cdot, \cdot \rangle_{A(\mathfrak{p})}$ , characterized by the relation

$$(54) \quad \langle e_A^{\mathfrak{p}, \mathcal{O}}([\bar{P}]), [V] \rangle_{A^K} = \langle [\bar{P}], d_A^{\mathfrak{p}, \mathcal{O}}([V]) \rangle_{A(\mathfrak{p})}.$$

for all finitely generated  $A^K$ -modules  $V$  and all finitely generated projective  $A(\mathfrak{p})$ -modules  $\bar{P}$ ; see [Lemma A.6](#). Brauer reciprocity is about the existence of this adjoint.

**Theorem 6.2.** *The (unique) adjoint  $e_A^{\mathfrak{p}, \mathcal{O}}$  of  $d_A^{\mathfrak{p}, \mathcal{O}}$  exists. Moreover, the diagram*

$$(55) \quad \begin{array}{ccc} \mathbf{K}_0(A^K) & \xrightarrow{c_{A^K}} & \mathbf{G}_0(A^K) \\ e_A^{\mathfrak{p}, \mathcal{O}} \uparrow & & \downarrow d_A^{\mathfrak{p}, \mathcal{O}} \\ \mathbf{K}_0(A(\mathfrak{p})) & \xrightarrow{c_{A(\mathfrak{p})}} & \mathbf{G}_0(A(\mathfrak{p})) \end{array}$$

commutes, where the horizontal morphisms are the canonical ones (Cartan maps) mapping a class  $[P]$  of a projective module  $P$  to its class  $[P]$  in the Grothendieck group. If  $R$  is normal, the morphism  $e_A^{\mathfrak{p}, \mathcal{O}}$  does not depend on the choice of  $\mathcal{O}$  and we denote it by  $e_A^{\mathfrak{p}}$ .

*Proof.* Since  $\langle \cdot, \cdot \rangle_{A^K}$  is nondegenerate by [Lemma A.6](#), it follows that  $d_A^{\mathfrak{p}, \mathcal{O}}$  has at most one adjoint  $e_A^{\mathfrak{p}, \mathcal{O}}$ , characterized by (54); see [[Scheja and Storch 1988](#), Satz 78.1]. By assumption there is a perfect  $A$ -gate  $\mathcal{O}$  in  $\mathfrak{p}$ . Let  $\mathfrak{m}$  be the maximal ideal of  $\mathcal{O}$ . Since  $A^K$  splits by assumption, [Corollary A.14](#) implies that  $A^\mathcal{O}$  is semiperfect. The morphism  $\mathbf{K}_0(A^\mathcal{O}) \rightarrow \mathbf{K}_0(A^\mathcal{O}(\mathfrak{m}))$  induced by the quotient map  $A^\mathcal{O} \rightarrow A^\mathcal{O}(\mathfrak{m})$  is thus an isomorphism by lifting of idempotents. Furthermore, by assumption the morphism  $d_A^{\mathfrak{p}, \mathcal{O}} : \mathbf{G}_0(A(\mathfrak{p})) \rightarrow \mathbf{G}_0(A^\mathcal{O}(\mathfrak{m}))$  is an isomorphism and then the proof of [Theorem 4.1](#) shows that the canonical morphism  $e_A^{\mathfrak{p}, \mathcal{O}} : \mathbf{K}_0(A(\mathfrak{p})) \rightarrow \mathbf{K}_0(A^\mathcal{O}(\mathfrak{m}))$  is also an isomorphism. We can thus define the morphism  $e_A^{\mathfrak{p}, \mathcal{O}} : \mathbf{K}_0(A(\mathfrak{p})) \rightarrow \mathbf{K}_0(A^K)$  as the following composition:

$$(56) \quad \begin{array}{ccccc} \mathbf{K}_0(A(\mathfrak{p})) & \xrightarrow{\simeq} & \mathbf{K}_0(A^\mathcal{O}(\mathfrak{m})) & \xrightarrow{\simeq} & \mathbf{K}_0(A^\mathcal{O}) & \longrightarrow & \mathbf{K}_0(A^K) \\ & & \searrow & & \nearrow & & \\ & & & e_A^{\mathfrak{p}, \mathcal{O}} & & & \end{array}$$

We will now show that  $e_A^{\mathfrak{p}, \mathcal{O}}$  is indeed an adjoint of  $d_A^{\mathfrak{p}, \mathcal{O}}$ . The arguments in the proof of [[Curtis and Reiner 1981](#), 18.9] can, with some refinements, be transferred to our more general situation and this is what we will do. Let  $\bar{P}$  be a finitely generated projective  $A(\mathfrak{p})$ -module and let  $V$  be a finitely generated  $A^K$ -module. Since  $\mathbf{K}_0(A^\mathcal{O}) \simeq \mathbf{K}_0(A^\mathcal{O}(\mathfrak{m}))$ , there exists a finitely generated projective  $A^\mathcal{O}$ -module  $P$  such that  $(e_A^{\mathfrak{p}, \mathcal{O}})^{-1}([P/\mathfrak{m}P]) = [\bar{P}]$  and then we have  $e_A^{\mathfrak{p}, \mathcal{O}}([\bar{P}]) = [P^K]$ . Let  $\tilde{V}$  be an  $A^\mathcal{O}$ -lattice in  $V$ . Then by the definition of  $d_A^{\mathfrak{p}, \mathcal{O}}$ , see [[Thiel 2016](#), Corollary 1.14], we have  $d_A^{\mathfrak{p}, \mathcal{O}}([V]) = (d_A^{\mathfrak{p}, \mathcal{O}})^{-1}([\tilde{V}(\mathfrak{m})])$ . We denote by  $\tilde{V}$  a representative of  $d_A^{\mathfrak{p}, \mathcal{O}}([V])$ . Since  $P$  is a finitely generated projective  $A^\mathcal{O}$ -module, we can write  $P \oplus Q = (A^\mathcal{O})^n$  for some finitely generated projective  $A^\mathcal{O}$ -module  $Q$  and some  $n \in \mathbb{N}$ . Since  $\mathrm{Hom}_{A^\mathcal{O}}$  is additive, we get

$$\begin{aligned} \mathrm{Hom}_{A^\mathcal{O}}(P, \tilde{V}) \oplus \mathrm{Hom}_{A^\mathcal{O}}(Q, \tilde{V}) &= \mathrm{Hom}_{A^\mathcal{O}}(P \oplus Q, \tilde{V}) = \mathrm{Hom}_{A^\mathcal{O}}((A^\mathcal{O})^n, \tilde{V}) \\ &= (\mathrm{Hom}_{A^\mathcal{O}}(A^\mathcal{O}, \tilde{V}))^n \simeq \tilde{V}^n. \end{aligned}$$

This shows that  $\text{Hom}_{A^\theta}(P, \tilde{V})$  is a direct summand of  $\tilde{V}^n$  and as  $\tilde{V}^n$  is  $\theta$ -free, we conclude that  $\text{Hom}_{A^\theta}(P, \tilde{V})$  is  $\theta$ -projective and thus even  $\theta$ -free since  $\theta$  is a discrete valuation ring. Since  $P$  is a finitely generated projective  $A^\theta$ -module, it follows from [Lemma B.3](#) that there is a canonical  $K$ -vector space isomorphism

$$K \otimes_{\theta} \text{Hom}_{A^\theta}(P, \tilde{V}) \simeq \text{Hom}_{A^K}(P^K, V)$$

and a canonical  $k(\mathfrak{m})$ -vector space isomorphism

$$k(\mathfrak{m}) \otimes_{\theta} \text{Hom}_{A^\theta}(P, \tilde{V}) \simeq \text{Hom}_{A^\theta(\mathfrak{m})}(P/\mathfrak{m}P, \tilde{V}/\mathfrak{m}\tilde{V}).$$

Combining all results and the fact that both  $e_A^{\mathfrak{p}, \theta}$  and  $d_A^{\mathfrak{p}, \theta}$  preserve dimensions by construction, we can now conclude that

$$\begin{aligned} \langle e_A^{\mathfrak{p}, \theta}([\bar{P}]), [V] \rangle_{A^K} &= \dim_K \text{Hom}_{A^K}(P^K, V) = \dim_{\theta} \text{Hom}_{A^\theta}(P, \tilde{V}) \\ &= \dim_{k(\mathfrak{m})} \text{Hom}_{A^\theta(\mathfrak{m})}(P/\mathfrak{m}P, \tilde{V}/\mathfrak{m}\tilde{V}) \\ &= \dim_{k(\mathfrak{p})} \text{Hom}_{A(\mathfrak{p})}(\bar{P}, \bar{V}) \\ &= \langle [\bar{P}], d_A^{\mathfrak{p}, \theta}([V]) \rangle_{A(\mathfrak{p})}. \end{aligned}$$

Proving the commutativity of diagram (55) amounts to proving that  $c_{A(\mathfrak{p})}([\bar{P}]) = d_A^{\mathfrak{p}, \theta} \circ c_{A^K} \circ e_A^{\mathfrak{p}, \theta}([\bar{P}])$  for every finitely generated projective  $A(\mathfrak{p})$ -module  $\bar{P}$ . To prove this, note that the diagram

$$\begin{array}{ccc} K_0(A^\theta(\mathfrak{m})) & \xrightarrow{c_{A^\theta(\mathfrak{m})}} & G_0(A^\theta(\mathfrak{m})) \\ e_A^{\mathfrak{p}, \mathfrak{m}} \uparrow & & \uparrow d_A^{\mathfrak{p}, \mathfrak{m}} \\ K_0(A(\mathfrak{p})) & \xrightarrow{c_{A(\mathfrak{p})}} & G_0(A(\mathfrak{p})) \end{array}$$

commutes. As above we know that there exists a finitely generated projective  $A^\theta$ -module  $P$  such that  $(e_A^{\mathfrak{p}, \mathfrak{m}})^{-1}([P/\mathfrak{m}P]) = [\bar{P}]$  and  $e_A^{\mathfrak{p}, \theta}([\bar{P}]) = [P^K]$ . Since  $P$  is a finitely generated projective  $A^\theta$ -module and  $A$  is a finite  $\theta$ -module, it follows that  $P$  is also a finitely generated projective  $\theta$ -module. As  $\theta$  is a discrete valuation ring, we conclude that  $P$  is actually  $\theta$ -free of finite rank. Hence,  $P$  is an  $A^\theta$ -lattice in  $P^K$  and therefore

$$\begin{aligned} d_A^{\mathfrak{p}, \theta} \circ c_{A^K} \circ e_A^{\mathfrak{p}, \theta}([\bar{P}]) &= d_A^{\mathfrak{p}, \theta}([P^K]) = (d_A^{\mathfrak{p}, \mathfrak{m}})^{-1}([P/\mathfrak{m}P]) \\ &= (d_A^{\mathfrak{p}, \mathfrak{m}})^{-1} \circ c_{A(\mathfrak{m})}([P/\mathfrak{m}P]) \\ &= c_{A(\mathfrak{p})} \circ (e_A^{\mathfrak{p}, \mathfrak{m}})^{-1}([P/\mathfrak{m}P]) = c_{A(\mathfrak{p})}([\bar{P}]). \end{aligned}$$

If  $R$  is normal, then the independence of  $e_A^{\mathfrak{p}, \theta}$  from the choice of  $\theta$  follows from the independence of  $d_A^{\mathfrak{p}, \theta}$  from the choice of  $\theta$  and the fact that  $d_A^{\mathfrak{p}, \theta}$  has at most one adjoint.  $\square$

**6B. Preservation of simple modules vs. preservation of blocks.** In [Thiel 2016] we studied the set

$$(57) \quad \text{DecGen}(A) := \{\mathfrak{p} \in \text{Spec}(R) \mid d_A^{\mathfrak{p}, \mathcal{O}} \text{ is trivial for any } A\text{-gate in } \mathfrak{p}\},$$

where  $d_A^{\mathfrak{p}, \mathcal{O}}$  being *trivial* means that it induces a bijection between simple modules. We have proven in [Thiel 2016, Theorem 2.3] that  $\text{DecGen}(A)$  is open if  $R$  is noetherian and  $A$  has split fibers. Brauer reciprocity implies that  $e_A^{\mathfrak{p}, \mathcal{O}}$  is trivial if and only if  $d_A^{\mathfrak{p}, \mathcal{O}}$  is trivial, so we deduce that the locus of all  $\mathfrak{p}$  such that  $e_A^{\mathfrak{p}, \mathcal{O}}$  is trivial for any  $\mathcal{O}$  is an open subset of  $\text{Spec}(R)$ .

If  $\mathfrak{p} \in \text{DecGen}(A)$ , then the simple modules of  $A^K$  and  $A(\mathfrak{p})$  are “essentially the same”; in particular their dimensions are the same. This is why explicit knowledge about  $\text{DecGen}(A)$  is quite helpful to understand the representation theory of the fibers of  $A$ ; see [Thiel 2016]. So far, we do not have an explicit description of  $\text{DecGen}(A)$ , however. Brauer reciprocity enables us to prove the following relation between decomposition maps and blocks.

**Theorem 6.3.** *We have the inclusion*

$$(58) \quad \text{DecGen}(A) \subseteq \text{BGen}(A).$$

*Proof.* Let  $\mathfrak{p} \in \text{Spec}(R)$  be nonzero. By assumption there is a perfect  $A$ -gate  $\mathcal{O}$  in  $\mathfrak{p}$ . If  $\mathfrak{p} \in \text{DecGen}(A)$ , then by definition  $d_A^{\mathfrak{p}, \mathcal{O}}$  is trivial, so the matrix  $D_A^{\mathfrak{p}, \mathcal{O}}$  of this morphism in bases given by isomorphism classes of simple modules of  $A^K$  and  $A(\mathfrak{p})$ , respectively, is equal to the identity matrix when ordering the bases appropriately. It now follows from Brauer reciprocity, Theorem 6.2, that  $C_{A(\mathfrak{p})} = C_{A^K}$  in appropriate bases, where  $C_{A(\mathfrak{p})}$  is the matrix of the Cartan map  $c_{A(\mathfrak{p})}$  and  $C_{A^K}$  is the matrix of the Cartan map  $c_{A^K}$ . Due to the linkage relation explained in Section 2, the families of  $A^K$  and of  $A(\mathfrak{p})$  are determined by the respective Cartan matrices. Since  $C_{A(\mathfrak{p})} = C_{A^K}$ , it follows that  $\#\text{Bl}(A(\mathfrak{p})) = \#\text{Bl}(A^K)$ , so  $\mathfrak{p} \in \text{BGen}(A)$ .  $\square$

**Remark 6.4.** Suppose that  $A$  has split fibers and that  $R$  is noetherian. Then the fact that  $\#\text{Bl}(A(\mathfrak{p})) = \#\text{Bl}(Z(\mathfrak{p}))$  by Theorem 5.8 together with Lemma A.9 yields the equivalence

$$(59) \quad \mathfrak{p} \in \text{BGen}(A) \iff \dim_K(Z^K + \text{Rad}(A^K)) = \dim_{k(\mathfrak{p})}(Z(\mathfrak{p}) + \text{Rad}(A(\mathfrak{p}))).$$

Let  $\mathcal{O}$  be a perfect  $A$ -gate in  $\mathfrak{p}$ . This exists by Lemma 6.1(b). Suppose that  $\mathfrak{p} \in \text{DecGen}(A)$ . In [Thiel 2016, Theorem 2.2] we have proven that this implies

$$\dim_K \text{Rad}(A^K) = \dim_{k(\mathfrak{p})} \text{Rad}(A(\mathfrak{p})).$$

Let  $X := Z + J$ , where  $J := \text{Rad}(A^K) \cap A^{\mathcal{O}}$ . The arguments in [Thiel 2016] show that  $X$  is an  $A^{\mathcal{O}}$ -lattice of  $Z^K + \text{Rad}(A^K)$  and that the reduction in the maximal

ideal  $\mathfrak{m}$  of  $\mathcal{O}$  is equal to  $Z^{\mathcal{O}}(\mathfrak{m}) + \text{Rad}(A^{\mathcal{O}}(\mathfrak{m}))$ . We thus have

$$\dim_K(Z^K + \text{Rad}(A^K)) = \dim_{k(\mathfrak{m})}(Z^{\mathcal{O}}(\mathfrak{m}) + \text{Rad}(A^{\mathcal{O}}(\mathfrak{m}))).$$

Since  $A(\mathfrak{p})$  splits, the  $k(\mathfrak{m})$ -dimension of  $Z^{\mathcal{O}}(\mathfrak{m}) + \text{Rad}(A^{\mathcal{O}}(\mathfrak{m}))$  is equal to the  $k(\mathfrak{p})$ -dimension of  $Z(\mathfrak{p}) + \text{Rad}(A(\mathfrak{p}))$ . Hence, we have  $\mathfrak{p} \in \text{BIGen}(A)$  by (59). This yields another proof of the inclusion  $\text{DecGen}(A) \subseteq \text{BIGen}(A)$  in the case  $A$  has split fibers.

**Example 6.5.** The following example due to C. Bonnafé shows that in the generality of Theorem 6.3 we do not have equality in (58). Let  $R$  be a discrete valuation ring with fraction field  $K$  and uniformizer  $\pi$ ; i.e.,  $\mathfrak{p} := (\pi)$  is the maximal ideal of  $R$ . Denote by  $k := R/\mathfrak{p}$  the residue field in  $\mathfrak{p}$ . Let

$$A := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}_2(R) \mid b, c \in \mathfrak{p} \right\}.$$

This is an  $R$ -subalgebra of  $\text{Mat}_2(R)$  and it is  $R$ -free with basis

$$(60) \quad e := E_{11}, \quad f := E_{22}, \quad x := \pi E_{12}, \quad y := \pi E_{21},$$

where  $E_{ij} = (\delta_{i,k}\delta_{j,l})_{kl}$  is the elementary matrix. Clearly,  $A^K = \text{Mat}_2(K)$ , so the generic fiber of  $A$  is split semisimple. In particular,  $A^K$  has just one block, and this block contains just one simple module we denote by  $S$ . Now, consider the specialization  $\bar{A} := A(\mathfrak{p}) = A/\mathfrak{p}A$ . We know from Corollary A.14 that the quotient map  $A \rightarrow \bar{A}$ ,  $a \mapsto \bar{a}$ , is block bijective, so we must have  $\#\text{BI}(A(\mathfrak{p})) \leq \text{BI}(A^K)$  and therefore  $\#\text{BI}(A(\mathfrak{p})) = 1$ , so  $\mathfrak{p} \in \text{BIGen}(A)$ . Let  $\bar{J}$  be the  $k$ -subspace of  $\bar{A}$  generated by  $\bar{x}$  and  $\bar{y}$ . This is in fact a two-sided ideal of  $\bar{A}$  since it is stable under multiplication by the generators (60). Moreover, we have  $\bar{x}^2 = 0 = \bar{y}^2$ , so  $\bar{J}$  is a nilpotent ideal of  $\bar{A}$ . Hence,  $\dim_k \text{Rad}(\bar{A}) \geq 2$ . The number of simple modules of  $\bar{A}$  is by [Lam 1991, Theorem 7.17] equal to  $\dim_k \bar{A}/(\text{Rad}(\bar{A}) + [\bar{A}, \bar{A}])$ , so  $\#\text{Irr } \bar{A} \leq 2$  since  $\dim_k \bar{A} = \dim_K A^K = 4$ . The two elements  $\bar{e}$  and  $\bar{f}$  are orthogonal idempotents and so the constituents of the two  $\bar{A}$ -modules  $\bar{A}\bar{e}$  and  $\bar{A}\bar{f}$  are nonisomorphic. So, we have  $\#\text{Irr } \bar{A} \geq 2$  and due to the aforementioned we conclude that  $\#\text{Irr } \bar{A} = 2$ . Let  $\bar{S}_1$  and  $\bar{S}_2$  be these two simple modules. Since  $R$  is a discrete valuation ring, reduction modulo  $\mathfrak{p}$  yields the well-defined decomposition map  $d_A^{\mathfrak{p}} : G_0(A^K) \rightarrow G_0(A(\mathfrak{p}))$ ; see [Thiel 2016, Corollary 1.14]. It is an elementary fact that all simple  $\bar{A}$ -modules must be constituents of  $d_A^{\mathfrak{p}}([S]) = [S/\mathfrak{p}S]$ . Since  $\dim_K S = 2$ , the only possibility is that  $d_A^{\mathfrak{p}}([S]) = [\bar{S}_1] + [\bar{S}_2]$  and  $\dim_k \bar{S}_i = 1$ . In particular,  $\mathfrak{p} \notin \text{DecGen}(A)$ , so  $\mathfrak{p} \in \text{BIGen}(A) \setminus \text{DecGen}(A)$ . Finally, we note that  $\bar{A}$  also splits since  $\#\text{Irr}(\bar{A}) = 2$  implies by the above formula that  $\dim_k \text{Rad}(\bar{A}) = 2$  and we have

$$\dim_k \bar{A} = \dim_k \text{Rad}(\bar{A}) + \sum_{i=1}^2 (\dim_k \bar{S}_i)^2,$$

so  $\bar{A}$  is split by [Lam 1991, Corollary 7.8].



**Lemma 6.6.** *Assume that the  $A^K$ -families are singletons, and that  $\#\text{Irr } A(\mathfrak{p}) \leq \#\text{Irr } A^K$  for all  $\mathfrak{p} \in \text{Spec}(R)$ . Then*

$$\text{BlGen}(A) \setminus \text{DecGen}(A) = \{\mathfrak{p} \in \text{Spec}(R) \mid D_A^{\mathfrak{p}} \text{ is diagonal but not the identity}\}.$$

*Proof.* Since the  $A^K$ -families are singletons, we have  $\#\text{Irr}(A^K) = \#\text{Bl}(A^K)$ . We clearly have  $\#\text{Irr } A(\mathfrak{p}) \geq \#\text{Bl}(A(\mathfrak{p}))$  for all  $\mathfrak{p} \in \text{Spec}(R)$ . Assume that  $\mathfrak{p} \in \text{BlGen}(A)$ . Then we have  $\#\text{Irr } A(\mathfrak{p}) \geq \#\text{Irr}(A^K)$ , so  $\#\text{Irr } A(\mathfrak{p}) = \#\text{Irr } A^K$  by our assumption. Hence, the decomposition matrix  $D_A^{\mathfrak{p}}$  is quadratic. By [Theorem 6.9](#) the  $\mathfrak{p}$ -families are equal to the Brauer  $\mathfrak{p}$ -families. Since  $\mathfrak{p} \in \text{BlGen}(A)$  and the  $A^K$ -families are singletons, it follows that  $D_A^{\mathfrak{p}}$  is a diagonal matrix. The claim is now obvious.  $\square$

**Lemma 6.7.** *Let  $R$  be a noetherian integral domain with fraction field  $K$  and let  $A$  be a cellular  $R$ -algebra of finite dimension such that  $A^K$  is semisimple. Then  $\text{DecGen}(A) = \text{BlGen}(A)$ .*

*Proof.* First of all, specializations of  $A$  are again cellular by [\[Graham and Lehrer 1996, 1.8\]](#). Moreover, it follows from Proposition 3.2 of the same paper that  $A$  has split fibers, so  $A$  satisfies [Lemma 6.1\(b\)](#) and therefore our basic assumption in this paragraph. Let  $\Lambda$  be the poset of the cellular structure of  $A^K$ . Since  $A^K$  is semisimple, each cell module  $M_\lambda$  has simple head  $S_\lambda$  and  $\#\text{Irr } A^K = \#\Lambda$ . Let  $\mathfrak{p} \in \text{Spec}(R)$ . The poset for the cellular structure of  $A(\mathfrak{p})$  is again  $\Lambda$ . Denote by  $M_\lambda^{\mathfrak{p}}$  the corresponding cell modules of  $A(\mathfrak{p})$ . There is a subset  $\Lambda'$  of  $\Lambda$  such that  $M_\lambda^{\mathfrak{p}}$  has simple head  $S_\lambda^{\mathfrak{p}}$  for all  $\lambda \in \Lambda'$  and that these heads are precisely the simple  $A(\mathfrak{p})$ -modules. In particular, we have  $\#\text{Irr } A(\mathfrak{p}) \leq \#\text{Irr } A^K$ . Now, assume that  $\mathfrak{p} \in \text{BlGen}(A)$ . By [Lemma 6.6](#) we just need to show that the decomposition matrix  $D_A^{\mathfrak{p}}$ , which is square by the proof of [Lemma 6.6](#), cannot be a nonidentity diagonal matrix. By [\[Graham and Lehrer 1996, Proposition 3.6\]](#) we know that  $[M_\lambda : S_\lambda] = 1$  and  $[M_\lambda^{\mathfrak{p}} : S_\lambda^{\mathfrak{p}}] = 1$ . By construction, it is clear that  $d_A^{\mathfrak{p}}([M_\lambda]) = [M_\lambda^{\mathfrak{p}}]$ . Hence, if  $d_A^{\mathfrak{p}}([S_\lambda]) = n_\lambda [S_\lambda^{\mathfrak{p}}]$ , we have  $n_\lambda = [M_\lambda^{\mathfrak{p}} : S_\lambda^{\mathfrak{p}}] = 1$ . Hence,  $D_A^{\mathfrak{p}}$  is the identity matrix, so  $\mathfrak{p} \in \text{BlGen}(A)$ .  $\square$

**6C. The Brauer graph.** Geck and Pfeiffer [\[2000\]](#) introduced the so-called Brauer  $\mathfrak{p}$ -graph of  $A$  in our general context but assumed that  $A^K$  is semisimple so that the  $A^K$ -families are singletons. For general  $A$  this definition seems not to be the correct one. We introduce the following generalization of this concept.

**Definition 6.8.** Suppose that  $R$  is normal so that we have unique decomposition maps. The *Brauer  $\mathfrak{p}$ -graph* of  $A$  is the graph with vertices the simple  $A^K$ -modules and an edge between  $S$  and  $T$  if and only if in the  $A^K$ -family of  $S$  there is some  $S'$  and in the  $A^K$ -family of  $T$  there is some  $T'$  such that  $d_A^{\mathfrak{p}}([S'])$  and  $d_A^{\mathfrak{p}}([T'])$  have a common constituent. The connected components of this graph are called the *Brauer  $\mathfrak{p}$ -families* of  $A$ .

If the  $A^K$ -families are singletons, we have an edge between  $S$  and  $T$  if and only if  $d_A^{\mathfrak{p}}([S])$  and  $d_A^{\mathfrak{p}}([T])$  have a common constituent, so this indeed generalizes the Brauer  $\mathfrak{p}$ -graph from [Geck and Pfeiffer 2000] for  $A^K$  semisimple. Our final theorem shows that decomposition maps are compatible with  $\mathfrak{p}$ -families and  $A(\mathfrak{p})$ -families, and relates the Brauer  $\mathfrak{p}$ -families to the  $\mathfrak{p}$ -families.

**Theorem 6.9.** *Assume that  $R$  is normal. The following hold:*

- (a) *A finite-dimensional  $A^K$ -module  $V$  belongs to a  $\mathfrak{p}$ -block of  $A$  if and only if  $d_A^{\mathfrak{p}}([V])$  belongs to a block of  $A(\mathfrak{p})$ .*
- (b) *Two finite-dimensional  $A^K$ -modules  $V$  and  $W$  lie in the same  $\mathfrak{p}$ -block if and only if  $d_A^{\mathfrak{p}}([V])$  and  $d_A^{\mathfrak{p}}([W])$  lie in the same block of  $A(\mathfrak{p})$ .*
- (c) *If  $\mathcal{F} \in \text{Fam}_{\mathfrak{p}}(A)$  is a  $\mathfrak{p}$ -family, then*

$$d_A^{\mathfrak{p}}(\mathcal{F}) := \{T \mid T \text{ is a constituent of } d_A^{\mathfrak{p}}([S]) \text{ for some } S \in \mathcal{F}\}$$

*is a family of  $A(\mathfrak{p})$ , and all families of  $A(\mathfrak{p})$  are obtained in this way.*

- (d) *The Brauer  $\mathfrak{p}$ -families are equal to the  $\mathfrak{p}$ -families.*

*Proof.* (a) By assumption there is a perfect  $A$ -gate  $\mathcal{O}$  in  $\mathfrak{p}$ . Let  $\mathfrak{m}$  be the maximal ideal of  $\mathcal{O}$ . We have the following commutative diagram of canonical morphisms which are all idempotent stable:

$$(61) \quad \begin{array}{ccccc} A_{\mathfrak{p}} & \hookrightarrow & A^{\mathcal{O}} & \hookrightarrow & A^K \\ & & \downarrow & & \\ & & \Downarrow & & \\ & & A(\mathfrak{p}) & \hookrightarrow & A^{\mathcal{O}}(\mathfrak{m}) \end{array}$$

Since  $R$  is assumed to be normal, it follows from Proposition 4.2 that  $A_{\mathfrak{p}} \twoheadrightarrow A(\mathfrak{p})$  is block bijective. By assumption the morphism  $d_A^{\mathfrak{p}, \mathfrak{m}} : G_0(A(\mathfrak{p})) \rightarrow G_0(A^{\mathcal{O}}(\mathfrak{m}))$  is an isomorphism and therefore  $A(\mathfrak{p}) \hookrightarrow A^{\mathcal{O}}(\mathfrak{m})$  is block bijective by Theorem 4.1. Furthermore, by assumption the generic fiber  $A^K$  is split and therefore  $A^{\mathcal{O}} \twoheadrightarrow A^{\mathcal{O}}(\mathfrak{m})$  is block bijective by Corollary A.14. Because of (5) it thus follows that  $A_{\mathfrak{p}} \hookrightarrow A^{\mathcal{O}}$  is block bijective.

Now, let  $V$  be a finite-dimensional  $A^K$ -module and let  $\tilde{V}$  be an  $A^{\mathcal{O}}$ -lattice of  $V$ . Suppose that  $V$  belongs to an  $A_{\mathfrak{p}}$ -block of  $A^K$ . Since  $A_{\mathfrak{p}} \hookrightarrow A^{\mathcal{O}}$  is block bijective, the  $A_{\mathfrak{p}}$ -blocks of  $A^K$  coincide with the  $A^{\mathcal{O}}$ -blocks of  $A^K$  and therefore  $V$  belongs to an  $A^{\mathcal{O}}$ -block of  $A^K$ . Since  $\tilde{V}$  is  $\mathcal{O}$ -free, it follows from Lemma A.1 that  $\tilde{V}$  belongs to a block of  $A^{\mathcal{O}}$ . Again by Lemma A.1 and the fact that  $A^{\mathcal{O}} \twoheadrightarrow A^{\mathcal{O}}(\mathfrak{m})$  is block bijective, it follows that  $\tilde{V}/\mathfrak{m}\tilde{V}$  belongs to a block of  $A^{\mathcal{O}}(\mathfrak{m})$ . Since  $A(\mathfrak{p}) \hookrightarrow A^{\mathcal{O}}(\mathfrak{m})$  is block bijective, Lemma A.1 shows that  $d_A^{\mathfrak{p}}([V])$  belongs to a block of  $A(\mathfrak{p})$ .

Conversely, suppose that  $d_A^{\mathfrak{p}}([V])$  belongs to a block of  $A(\mathfrak{p})$ . Then  $\tilde{V}/\mathfrak{m}\tilde{V}$  belongs to a block of  $A^{\mathcal{O}}(\mathfrak{m})$  and therefore  $\tilde{V}$  belongs to a block of  $A^{\mathcal{O}}$  by Lemma A.1.

But then  $V$  belongs to an  $A^{\mathcal{O}}$ -block of  $A^K$  and thus to an  $A_{\mathfrak{p}}$ -block of  $A^K$  by Lemma A.1.

(b) This follows now from part (a).

(c) Fix a  $\mathfrak{p}$ -family  $\mathcal{F}$  of  $A^K$ . If  $S \in \mathcal{F}$ , then  $d_A^{\mathfrak{p}}([S])$  belongs to an  $A(\mathfrak{p})$ -block by (a) and therefore all constituents of  $d_A^{\mathfrak{p}}([S])$  belong to a fixed family  $\overline{\mathcal{F}}_S$ . If  $S' \in \mathcal{F}$  is another simple module, then by (b) the constituents of  $d_A^{\mathfrak{p}}([S'])$  also lie in  $\overline{\mathcal{F}}_S$ . Hence,  $d_A^{\mathfrak{p}}(\mathcal{F})$  is contained in a fixed  $A(\mathfrak{p})$ -family  $\overline{\mathcal{F}}$ . Let  $T \in \overline{\mathcal{F}}$  be arbitrary. Due to the properties of decomposition maps there is some  $S \in \text{Irr } A^K$  such that  $T$  is a constituent of  $d_A^{\mathfrak{p}}([S])$ . Since  $T$  and  $d_A^{\mathfrak{p}}([S])$  lie in the same  $A(\mathfrak{p})$ -block by (a) and (b), we must have  $S \in \mathcal{F}$  by (b). Hence,  $\overline{\mathcal{F}} = d_A^{\mathfrak{p}}(\mathcal{F})$  is an  $A(\mathfrak{p})$ -family. Since every simple  $A(\mathfrak{p})$ -module is a constituent of  $d_A^{\mathfrak{p}}([S])$  for some simple  $A^K$ -module  $S$ , it is clear that any  $A(\mathfrak{p})$ -family is of the form  $d_A^{\mathfrak{p}}(\mathcal{F})$  for a  $\mathfrak{p}$ -family  $\mathcal{F}$ .

(d) Let  $S$  and  $T$  be simple  $A^K$ -modules contained in the same Brauer  $\mathfrak{p}$ -family; i.e., in the  $A^K$ -family of  $S$  there is some  $S'$  and in the  $A^K$ -family of  $T$  there is some  $T'$  such that  $d_A^{\mathfrak{p}}([S'])$  and  $d_A^{\mathfrak{p}}([T'])$  have a common constituent. It follows from part (b) that  $S'$  and  $T'$  lie in the same  $\mathfrak{p}$ -family of  $A^K$ . Since  $S'$  is in the same  $A^K$ -family as  $S$ , it is also in the same  $\mathfrak{p}$ -family as  $S$  because the  $\mathfrak{p}$ -families are unions of  $A^K$ -families. Similarly,  $T'$  is in the same  $\mathfrak{p}$ -family as  $T$ . Hence,  $S$  and  $T$  lie in the same  $\mathfrak{p}$ -family.

Conversely, suppose that  $S$  and  $T$  lie in the same  $\mathfrak{p}$ -family. We have to show that they lie in the same Brauer  $\mathfrak{p}$ -family. Let  $(S_i)_{i=1}^n$  be a system of representatives of the isomorphism classes of simple  $A^K$ -modules and let  $(U_j)_{j=1}^m$  be a system of representatives of the isomorphism classes of simple  $A(\mathfrak{p})$ -modules. Let  $\mathcal{Q} := (Q_i)_{i=1}^n$  with  $Q_i$  being the projective cover of  $S_i$ , and let  $\mathcal{P} := (P_j)_{j=1}^m$  with  $P_j$  being the projective cover of  $U_j$ . Let  $C_{A(\mathfrak{p})}$  be the matrix of the Cartan map  $c_{A(\mathfrak{p})}$  with respect to the chosen bases, and similarly let  $C_{A^K}$  be the matrix of  $c_{A^K}$ . Furthermore, let  $D_A^{\mathfrak{p}}$  be the matrix of  $d_A^{\mathfrak{p}}$  with respect to the chosen bases. Since

$$C_{A(\mathfrak{p})} = D_A^{\mathfrak{p}} C_{A^K} (D_A^{\mathfrak{p}})^T$$

by Brauer reciprocity, Theorem 6.2, we have

$$(62) \quad (C_{A(\mathfrak{p})})_{p,q} = (D_A^{\mathfrak{p}} C_{A^K} (D_A^{\mathfrak{p}})^T)_{p,q} = \sum_{k,l=1}^n (D_A^{\mathfrak{p}})_{p,k} (C_{A^K})_{k,l} (D_A^{\mathfrak{p}})_{q,l}$$

for all  $p, q$ . Let  $U$  be a constituent of  $d_A^{\mathfrak{p}}([S])$  and let  $V$  be a constituent of  $d_A^{\mathfrak{p}}([T])$ . Since  $S$  and  $T$  lie in the same  $\mathfrak{p}$ -family of  $A^K$ , both  $d_A^{\mathfrak{p}}([S])$  and  $d_A^{\mathfrak{p}}([T])$  lie in the same block of  $A(\mathfrak{p})$  by (b), and therefore  $U$  and  $V$  lie in the same family of  $A(\mathfrak{p})$ . As the families of  $A(\mathfrak{p})$  are equal to the  $\mathcal{P}$ -families of  $A(\mathfrak{p})$  by Section 2, there exist functions  $f : [1, r] \rightarrow [1, m]$ ,  $g : [1, r - 1] \rightarrow [1, m]$  with the following properties:  $U_{f(1)} = U$ ,  $U_{f(r)} = V$ , and for any  $j \in [1, r - 1]$  both  $P_{f(j)}$  and  $P_{f(j+1)}$  have  $U_{g(j)}$

as a constituent. We can visualize the situation as follows:

$$\begin{array}{ccccc}
 P_{f(j)} & & & & P_{f(j+1)} \\
 \uparrow & \swarrow & & \nearrow & \uparrow \\
 U_{f(j)} & & U_{g(j)} & & U_{f(j+1)}
 \end{array}$$

where an arrow  $U \rightarrow P$  signifies that  $U$  is a constituent of  $P$ . For any  $j \in [1, r-1]$  we have  $(C_{A(\mathfrak{p})})_{g(j), f(j)} \neq 0$  and so it follows from (62) that there are indices  $k(j)$  and  $l(j)$  such that

$$(D_A^{\mathfrak{p}})_{g(j), k(j)} \neq 0, \quad (C_{A^K})_{k(j), l(j)} \neq 0, \quad (D_A^{\mathfrak{p}})_{f(j), l(j)} \neq 0.$$

Similarly, since  $(C_{A(\mathfrak{p})})_{g(j), f(j+1)} \neq 0$ , there exist indices  $k'(j)$  and  $l'(j)$  such that

$$(D_A^{\mathfrak{p}})_{g(j), k'(j)} \neq 0, \quad (C_{A^K})_{k'(j), l'(j)} \neq 0, \quad (D_A^{\mathfrak{p}})_{f(j+1), l'(j)} \neq 0.$$

This can be visualized as follows:

$$\begin{array}{ccccccc}
 d_A^{\mathfrak{p}}([S_{l(j)}]) & \text{---} & d_A^{\mathfrak{p}}([S_{k(j)}]) & & d_A^{\mathfrak{p}}([S_{k'(j)}]) & \text{---} & d_A^{\mathfrak{p}}([S_{l'(j)}]) \\
 \uparrow & & \swarrow & & \nearrow & & \uparrow \\
 U_{f(j)} & & & U_{g(j)} & & & U_{f(j+1)}
 \end{array}$$

Here, the dashed edges in the upper row signify that the respective simple  $A^K$ -modules lie in the same  $A^K$ -family. Since  $U_{f(1)} = U$  and  $U_{f(r)} = V$ , this shows that  $S$  and  $T$  lie in the same Brauer  $\mathfrak{p}$ -family of  $A^K$ .  $\square$

## 7. Semicontinuity of blocks in the case of a nonsplit generic fiber

Let  $A$  be a finite flat algebra over an integral domain  $R$  with fraction field  $K$ . We have the map

$$(63) \quad \# \text{Bl}'_A : \text{Spec}(R) \rightarrow \mathbb{N}, \quad \mathfrak{p} \mapsto \# \text{Bl}(A(\mathfrak{p}));$$

i.e.,  $\# \text{Bl}'_A(\mathfrak{p})$  is the number of blocks of the *specialization*  $A(\mathfrak{p})$ . Recall that in (21) we considered the map  $\# \text{Bl}_A$  with  $\# \text{Bl}_A(\mathfrak{p}) = \# \text{Bl}(A_{\mathfrak{p}})$  being the number of blocks of the *localization*  $A_{\mathfrak{p}}$ . In the case  $R$  is normal and  $A^K$  splits, we know from Proposition 4.2 that  $\# \text{Bl}'_A = \# \text{Bl}_A$ . In particular, the map  $\# \text{Bl}'_A$  is lower semicontinuous and thus defines a stratification of  $\text{Spec}(R)$ , the block number stratification; see Section 3B.

In the case  $A^K$  does not split, it still makes perfect sense to consider the map (63) and ask if it is lower semicontinuous so that we have a stratification of  $\text{Spec}(R)$  by the number of blocks of specializations. But since we do not have the connection from Proposition 4.2 between blocks of localizations and blocks of specializations anymore, we cannot directly apply the results from Section 3.

In this section, we will establish a setting where the map  $\# \text{Bl}'_A$  is still lower semicontinuous without assuming that the generic fiber  $A^K$  splits, the main result being [Corollary 7.3](#). To achieve this, however, we have to restrict this map to a subset of  $\text{Spec}(R)$ . As we will see, in general it is not possible to have lower semicontinuity on all of  $\text{Spec}(R)$ .

First of all, because of the difference between blocks of localizations and blocks of specializations, we introduce the sets

$$(64) \quad \beta(A) := \max\{\# \text{Bl}(A(\mathfrak{p})) \mid \mathfrak{p} \in \text{Spec}(R)\},$$

$$(65) \quad \text{BlEx}'(A) := \# \text{Bl}'^{-1}_A(\leq \beta(A) - 1),$$

$$(66) \quad \text{BlGen}'(A) := \text{Spec}(R) \setminus \text{BlEx}'(A).$$

Note that if  $R$  is normal and  $A^K$  splits, then  $\beta(A) = \# \text{Bl}(A^K)$ , so  $\text{BlEx}'(A) = \text{BlEx}(A)$  and  $\text{BlGen}'(A) = \text{BlGen}(A)$ , as defined in [\(19\)](#) and [\(20\)](#).

Now, assume that  $R'$  is an integral extension of  $R$  which is also an integral domain. Let  $K'$  be the fraction field of  $R'$  and let  $\psi : \text{Spec}(R') \rightarrow \text{Spec}(R)$  be the morphism induced by  $R \subseteq R'$ . The scalar extension  $A' := R' \otimes_R A$  is again a finitely presented flat  $R'$ -algebra (using [Remark 3.2](#)). For any  $\mathfrak{p} \in \text{Spec}(R)$  and any  $\mathfrak{p}' \in \text{Spec}(R')$  lying over  $\mathfrak{p}$  we have the diagram

$$(67) \quad \begin{array}{ccc} & & A'_{\mathfrak{p}'} \\ & & \downarrow \\ & & \Downarrow \\ A(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}A_{\mathfrak{p}} & \hookrightarrow & A'(\mathfrak{p}') = A'_{\mathfrak{p}'}/\mathfrak{p}'_{\mathfrak{p}'}A'_{\mathfrak{p}'} \end{array}$$

and it then follows from [\(5\)](#) that

$$(68) \quad \# \text{Bl}(A(\mathfrak{p})) \leq \# \text{Bl}(A'(\mathfrak{p}')) \geq \# \text{Bl}(A'_{\mathfrak{p}'}).$$

Let  $X$  be a set contained in

$$(69) \quad \begin{aligned} X_{R'}(A) \\ := \{\mathfrak{p} \in \text{Spec}(R) \mid \# \text{Bl}(A(\mathfrak{p})) = \# \text{Bl}(A'(\mathfrak{p}')) = \# \text{Bl}(A'_{\mathfrak{p}'}) \text{ for all } \mathfrak{p}' \in \psi^{-1}(\mathfrak{p})\}. \end{aligned}$$

We have seen in [Corollary 4.3](#) that in the case  $R$  is normal and  $A^K$  splits we can choose  $R = R'$  and have  $X = \text{Spec}(R)$ . In general  $X$  will be a proper subset of  $\text{Spec}(R)$  and we have to choose  $R'$  appropriately to enlarge it a bit more. Let us first concentrate on what we can say when restricting to  $X$ . We introduce the following restricted versions of our invariants:

$$(70) \quad \# \text{Bl}'_{A,X} := \# \text{Bl}'_A|_X : X \rightarrow \mathbb{N},$$

$$(71) \quad \# \text{Bl}'^{-1}_{A,X}(\leq n) := \# \text{Bl}'^{-1}_A(\leq n) \cap X = \psi(\# \text{Bl}'^{-1}_A(\leq n)) \cap X,$$

$$(72) \quad \# \text{Bl}'_{A,X}{}^{-1}(n) := \# \text{Bl}'_{A'}{}^{-1}(n) \cap X = \psi(\# \text{Bl}'_{A'}{}^{-1}(n)) \cap X,$$

$$(73) \quad \beta_X(A) := \max\{\# \text{Bl}(A(\mathfrak{p})) \mid \mathfrak{p} \in X\},$$

$$(74) \quad \text{BlEx}'_X(A) := \text{Bl}'_{A,X}{}^{-1}(\leq \beta_X(A) - 1),$$

$$(75) \quad \text{BlGen}'_X(A) := X \setminus \text{BlEx}'_X(A).$$

**Corollary 7.1.** *The map  $X \rightarrow \mathbb{N}$ ,  $\mathfrak{p} \mapsto \# \text{Bl}(A(\mathfrak{p}))$ , is lower semicontinuous on  $X$ , so  $X = \coprod_{n \in \mathbb{N}} \text{Bl}'_{A,X}{}^{-1}(n)$  is a stratification of  $X$  into locally closed subsets. Moreover,*

$$(76) \quad \beta_X(A) \leq \# \text{Bl}(A^{K'}).$$

*Proof.* Since  $\psi$  is a closed morphism and  $\# \text{Bl}'_{A'}{}^{-1}(\leq n)$  is closed in  $\text{Spec}(R')$  by (26), it follows that  $\psi(\# \text{Bl}'_{A'}{}^{-1}(\leq n))$  is closed in  $\text{Spec}(R)$ , hence  $\# \text{Bl}'_{A,X}{}^{-1}(\leq n)$  is closed in  $X$  by (71). Since

$$\# \text{Bl}'_{A,X}{}^{-1}(n) = \# \text{Bl}'_{A,X}{}^{-1}(\leq n) \setminus \# \text{Bl}'_{A,X}{}^{-1}(\leq n - 1),$$

it is clear that  $\# \text{Bl}'_{A,X}{}^{-1}(n)$  is locally closed in  $X$ . We have seen in (27) that

$$\overline{\# \text{Bl}'_{A'}{}^{-1}(n)} \subseteq \bigcup_{m \leq n} \text{Bl}'_{A'}{}^{-1}(m).$$

Hence, since  $\psi$  is closed, we obtain

$$\overline{\# \text{Bl}'_{A,X}{}^{-1}(n)} = \overline{\psi(\# \text{Bl}'_{A'}{}^{-1}(n)) \cap X} \subseteq \bigcup_{m \leq n} \psi(\# \text{Bl}'_{A'}{}^{-1}(m)) \cap X = \bigcup_{m \leq n} \# \text{Bl}'_{A,X}{}^{-1}(m). \quad \square$$

Note that in (76) we could only bound  $\beta_X(A)$  above by  $\# \text{Bl}(A^{K'})$ , and not by  $\# \text{Bl}(A^K)$ . In fact, we will see in Example 7.4 that we may indeed have  $\beta_X(A) > \# \text{Bl}(A^K)$  in general. This is an important difference to blocks of localizations where we always have the maximal number of blocks in the generic point.

In the following lemma we describe a situation where we have  $\beta_X(A) = \# \text{Bl}(A^{K'})$ . We recall that  $X$  is called *very dense* if the embedding  $X \hookrightarrow \text{Spec}(R)$  is a *quasihomomorphism*; i.e., the map  $Z \mapsto Z \cap X$  is a bijection between the closed (equivalently, open) subsets of the two spaces. This notion was introduced by Grothendieck [1966, §10].

**Lemma 7.2.** *Suppose that  $X$  is very dense in  $\text{Spec}(R)$ , that  $R$  is noetherian, and that  $\psi$  is finite. Then  $\beta_X(A) = \# \text{Bl}(A^{K'})$ , and thus  $\text{BlEx}'_X(A) = \psi(\text{BlEx}(A')) \cap X$ . If moreover  $R'$  is normal and  $R$  is universally catenary, then  $\text{BlEx}'_X(A)$  is a reduced Weil divisor in  $X$ .*

*Proof.* The assumptions imply that  $R'$  is noetherian, too. We know from Theorem 3.3 that  $\text{BlGen}(A')$  is a nonempty open subset of  $\text{Spec}(R')$ . In particular, it is constructible. Since  $\text{Spec}(R)$  is quasicompact, the morphism  $\psi$  is quasicompact by

[Görtz and Wedhorn 2010, Remark 10.2.(1)]. It thus follows from Chevalley's constructibility theorem, see [Görtz and Wedhorn 2010, Corollary 10.71], that  $\psi(\text{BGen}(A'))$  is constructible in  $\text{Spec}(R)$ . Since  $X$  is very dense in  $\text{Spec}(R)$ , we conclude that  $\psi(\text{BGen}(A')) \cap X \neq \emptyset$  by [Grothendieck 1966, Proposition 10.1.2]. Hence, there is  $\mathfrak{p} \in X$  and  $\mathfrak{p}' \in \text{BGen}(A')$  with  $\psi(\mathfrak{p}') = \mathfrak{p}$ . But then we have  $\#\text{Bl}(A(\mathfrak{p})) = \#\text{Bl}(A'(\mathfrak{p}')) = \#\text{Bl}(A^{K'})$ , so  $\beta_X(A) = \#\text{Bl}(A^{K'})$ . Now, assume that  $R'$  is normal and  $R$  is universally catenary. We know that  $\text{BlEx}(A')$  is either empty or pure of codimension 1 in  $\text{Spec}(R')$  by Corollary 3.5. In [Huneke and Swanson 2006, Theorem B.5.1] it is shown that the extension  $R \subseteq R'$  satisfies the dimension formula; hence  $\psi(\text{BlEx}(A'))$  is either empty or pure of codimension 1. Since  $X$  is very dense in  $\text{Spec}(R)$ , the same is also true for

$$X \cap \psi(\text{BlEx}(A')) = \text{BlEx}'_X(A). \quad \square$$

**Corollary 7.3.** *Suppose that  $R$  is a finite-type algebra over an algebraically closed field. Let  $X$  be the set of closed points of  $\text{Spec}(R)$ . Then the map  $X \rightarrow \mathbb{N}$ ,  $\mathfrak{m} \mapsto \#\text{Bl}(A(\mathfrak{m}))$ , is lower semicontinuous and so  $X = \coprod_{n \in \mathbb{N}} \#\text{Bl}'_{A,X}{}^{-1}(n)$  is a stratification of  $X$ . Moreover,  $\beta_X(A) = \#\text{Bl}(A^{\bar{K}})$ , where  $\bar{K}$  is an algebraic closure of  $K$ . If  $R$  is also universally catenary, then  $\text{BlEx}'_X(A)$  is a reduced Weil divisor in  $X$ .*

*Proof.* Let  $K'$  be a finite extension of  $K$  such that  $A^{K'}$  splits (this is always possible, see [Curtis and Reiner 1981, Proposition 7.13]) and let  $R'$  be the integral closure of  $R$  in  $K'$ . Now,  $\#\text{Bl}(A'(\mathfrak{p}')) = \#\text{Bl}(A'_{\mathfrak{p}'})$  for all  $\mathfrak{p}' \in \text{Spec}(R)$  by Proposition 4.2. Since  $R$  is a finite-type algebra over an algebraically closed field  $k$ , the residue field in a closed point  $\mathfrak{m}$  of  $\text{Spec}(R)$  is just  $k$ . Hence, the specialization  $A(\mathfrak{m})$  is a finite-dimensional algebra over an algebraically closed field, thus splits and we therefore have  $\#\text{Bl}(A(\mathfrak{m})) = \#\text{Bl}(A'(\mathfrak{m}'))$  for any  $\mathfrak{m}' \in \psi^{-1}(\mathfrak{m})$  by Lemma A.3. Hence,  $X \subseteq X_{R'}(A)$ . The claim about semicontinuity and the stratification thus follows from Corollary 7.1. It is shown in [Görtz and Wedhorn 2010, Proposition 3.35] that  $X$  is very dense in  $\text{Spec}(R)$ . Since  $R$  is a finite-type algebra over a field, it is Japanese, so  $\psi$  is a finite morphism. Hence,  $\beta_X(A) = \#\text{Bl}(A^{K'}) = \#\text{Bl}(A^{\bar{K}})$  by Lemma 7.2. The claim that  $\text{BlEx}'_X(A)$  is a reduced Weil divisor if  $R$  is universally catenary also follows from Lemma 7.2.  $\square$

**Example 7.4.** The following example due to K. Brown shows that in the setting of Corollary 7.3 we may indeed have  $\beta_X(A) > \#\text{Bl}(A^K)$  so that the map  $\mathfrak{p} \mapsto \#\text{Bl}(A(\mathfrak{p}))$  will not be lower semicontinuous on the whole of  $\text{Spec}(R)$ . Let  $k$  be an algebraically closed field of characteristic zero, let  $X$  be an indeterminate over  $k$ , let  $R := k[X^n]$  for some  $n > 1$ , and let  $A := k[X]$ . Let  $C_n$  be the cyclic group of order  $n$ . We fix a generator of  $C_n$  and let it act on  $X$  by multiplication with a primitive  $n$ -th root of unity. Then  $R = k[X]^{C_n}$ , so  $A$  is free of rank  $n$  over  $R$ . Moreover,  $\text{Frac}(A) = k(X)$  is a Galois extension of degree  $n$  of  $K := \text{Frac}(R)$

by [Benson 1993, Proposition 1.1.1], so in particular  $K \neq k(X)$  since  $n > 1$ . By [Goodearl and Warfield 2004, Exercise 6R] we have

$$A^K = A \otimes_R K = A[(R \setminus \{0\})^{-1}] = \text{Frac}(A) = k(X),$$

so the  $K$ -algebra  $A^K = Z(A^K)$  is *not* split (and thus also not block-split by Lemma A.3). It is clear that

$$(77) \quad \# \text{Bl}(A^K) = 1.$$

Now, let  $\mathfrak{m} := (X^n - 1) \in \text{Max}(R)$ . Then  $k(\mathfrak{p}) = k$  and since  $k$  is algebraically closed, we have  $A(\mathfrak{m}) = A/\mathfrak{m}A \simeq k^n$  as  $k$ -algebras. In particular,

$$(78) \quad \# \text{Bl}(A(\mathfrak{m})) = n > 1 = \# \text{Bl}(A^K).$$

We close with a setting where our base ring is not necessarily normal but we still get a global result on  $\text{Spec}(R)$ .

**Lemma 7.5.** *Suppose that  $A$  has split fibers; i.e.,  $A(\mathfrak{p})$  splits for all  $\mathfrak{p} \in \text{Spec}(R)$ . Then the map  $\text{Spec}(R) \rightarrow \mathbb{N}$ ,  $\mathfrak{p} \mapsto \# \text{Bl}(A(\mathfrak{p}))$ , is lower semicontinuous and so  $\text{Spec}(R) = \coprod_{n \in \mathbb{N}} \text{Bl}_A^{-1}(n)$  is a partition into locally closed subsets. Moreover,  $\beta(A) = \# \text{Bl}(A^K)$ . If  $R$  is also universally catenary, Japanese, and noetherian, then  $\text{BlEx}'(A)$  is a reduced Weil divisor in  $\text{Spec}(R)$ .*

*Proof.* Let  $R'$  be the integral closure of  $R$  in  $K$ . Then  $\# \text{Bl}(A'(\mathfrak{p}')) = \# \text{Bl}(A'_{\mathfrak{p}'})$  for all  $\mathfrak{p}' \in \text{Spec}(R')$  by Proposition 4.2. Since  $A(\mathfrak{p})$  splits, we moreover have  $\# \text{Bl}(A(\mathfrak{p})) = \# \text{Bl}(A'(\mathfrak{p}'))$  for all  $\mathfrak{p} \in \text{Spec}(R)$   $\mathfrak{p}' \in \psi^{-1}(\mathfrak{p})$  by Lemma A.3. Hence,  $X_{R'}(A) = \text{Spec}(R)$ . The claim about semicontinuity and the partition follows from Corollary 7.1. Now, assume that  $R$  is universally catenary, Japanese, and noetherian. Since  $R$  is Japanese, it follows by definition that  $\psi$  is finite. The claim about  $\text{BlEx}'(A)$  being a reduced Weil divisor now follows from Lemma 7.2.  $\square$

## Appendix A: More on base change of blocks

In this appendix we collect several facts about base change of blocks. Some results here should also be of independent interest.

**Block compatibility of scalar extension of modules.** Recall the decomposition of the module category of a ring  $A$  relative to a decomposition of  $1 \in A$  into pairwise orthogonal central idempotents described in Section 2. We have the following compatibility.

**Lemma A.1.** *Let  $\phi : R \rightarrow S$  be a morphism of commutative rings and let  $A$  be an  $R$ -algebra. Suppose that  $\phi_A$  is central idempotent stable and let  $V$  be a nonzero  $A$ -module. In any of the following cases the  $A$ -module  $V$  belongs to the block  $c_i$  if and only if the  $A^S$ -module  $V^S$  belongs to the  $\phi$ -block  $\phi_A(c_i)$ :*



- (a)  $\phi$  is injective and  $V$  is  $R$ -projective.
- (b)  $\phi$  is faithfully flat.
- (c)  $R$  is local or a principal ideal domain and  $V$  is  $R$ -free.

*Proof.* As  $c_j V$  is a direct summand of  $V$ , it follows that we have a canonical isomorphism between isomorphisms  $\phi_A^*(c_j V) \simeq \phi_A(c_j)\phi_A^* V$  of  $A^S$ -modules for all  $j$ . The claim thus holds if we can show that no nonzero direct summand  $V'$  of  $V$  is killed by  $\phi_A^*$ , i.e.,  $\phi_A^* V' \neq 0$ . But this is implied by the assumptions in each case. Namely, in the first two cases it follows from [Lemma 2.1](#) that  $\phi_V$  is injective, which implies that  $\phi_{V'}$  is also injective, so  $\phi_A^* V'$  cannot be zero for nonzero  $V'$ . In the third case neither  $\phi$  nor  $\phi_V$  needs to be injective, so this needs extra care. First of all, since  $V$  is assumed to be  $R$ -free, the assumptions on  $R$  imply that a direct summand  $V'$  of  $V$ , which a priori is only  $R$ -projective, is already  $R$ -free, too. In the case  $R$  is local, this follows from Kaplansky's theorem [\[1958\]](#) and in the case  $R$  is a principal ideal domain, this is a standard fact. Now, if  $V$  is  $R$ -free with basis  $(v_\lambda)_{\lambda \in \Lambda}$ , then it is a standard fact, see [\[Bourbaki 1989, II, §5.1, Proposition 4\]](#), that  $\phi_A^* V$  is  $S$ -free with basis  $(\phi_V(v_\lambda))_{\lambda \in \Lambda}$ . This shows that  $\phi_A^* V \neq 0$  for any nonzero  $R$ -free  $A$ -module  $V$ . This applied to direct summands of  $V$ , which are  $R$ -free as shown, proves the claim.  $\square$

**Field extensions.** Throughout this paragraph let  $A$  be a finite-dimensional algebra over a field  $K$ . From [\(5\)](#) we know that  $\# \text{Bl}(A) \leq \# \text{Bl}(A^L)$  for any extension field  $L$  of  $K$ .

**Definition A.2.** We say that  $A$  is *block-split* if  $\# \text{Bl}(A) = \# \text{Bl}(A^L)$  for any extension field  $L$  of  $K$ .

Our aim is to show the following lemma.

**Lemma A.3.** *If  $Z(A)$  is a split  $K$ -algebra (e.g., if  $A$  itself splits), then  $A$  is block-split. The converse holds if  $K$  is perfect.*

The first assertion of the lemma is essentially obvious since  $Z(A)$  is semiperfect and therefore

$$(79) \quad \# \text{Bl}(A) = \# \text{Bl}(Z(A)) = \text{rk}_{\mathbb{Z}} K_0(Z(A)) = \# \text{rk}_{\mathbb{Z}} G_0(Z(A)) = \# \text{Irr } Z(A),$$

where the second equality follows from the fact that idempotents in a commutative ring are isomorphic if and only if they are equal; see [\[Lam 1991, §22, Exercise 2\]](#). The same equalities of course also hold for  $Z(A)^L = Z(A^L)$ , where  $L$  is an extension field of  $K$ . Hence, if  $Z(A)$  is split, then  $A$  is block-split. If  $A$  itself is split, it is a standard fact that its center splits, so  $A$  is block-split.

We will prove the converse (assuming that  $K$  is perfect) from a more general point of view as the results might be of independent interest and we reuse some

of them in the last section. First of all, the field extension  $K \subseteq L$  induces natural group morphisms

$$(80) \quad d_A^L : G_0(A) \rightarrow G_0(A^L) \quad \text{and} \quad e_A^L : K_0(A) \rightarrow K_0(A^L).$$

Without any assumptions on the field  $K$  we have the following property.

**Lemma A.4.** *The morphisms  $d_A^L$  and  $e_A^L$  are injective.*

*Proof.* Let  $(S_i)_{i \in I}$  be a system of representatives of the isomorphism classes of simple  $A$ -modules. For each  $i$  let  $(T_{ij})_{j \in J_i}$  be a system of representatives of the isomorphism classes of simple  $A^L$ -modules which occur as constituents of  $S_i^L$ . Then by [Lam 1991, Proposition 7.13] the set  $(T_{ij})_{i \in I, j \in J_i}$  is a system of representatives of the isomorphism classes of simple  $A^L$ -modules. Hence, the matrix  $D_A^L$  of  $d_A^L$  in bases given by the isomorphism classes of simple modules is in column-echelon form, has no zero columns, and no zero rows. In particular,  $d_A^L$  is injective.

For each  $i \in I$  let  $P_i$  be the projective cover of  $S_i$  and for each  $j \in J_i$  let  $Q_{ij}$  be the projective cover of  $T_{ij}$ . By the above,  $(Q_{ij})_{i \in I, j \in J_i}$  is a system of representatives of the isomorphism classes of projective indecomposable  $A^L$ -modules. We claim that in the direct sum decomposition of the finitely generated projective  $A^L$ -module  $P_i^L$  into projective indecomposable  $A^L$ -modules, only the  $Q_{ij}$  with  $j \in J_i$  occur. With the same argument as above, this implies that  $e_A^L$  is injective. So, let us write  $P_i^L = \bigoplus_{\lambda \in \Lambda} U_\lambda$  for (not necessarily nonisomorphic) projective indecomposable  $A^L$ -modules  $U_\lambda$ . The  $U_\lambda$  are the, up to isomorphism, unique projective indecomposable  $A^L$ -modules occurring as direct summands of  $P_i^L$ . As the radical is additive by [Lam 1991, Proposition 24.6(ii)], we have  $\text{Rad}(P_i^L) = \bigoplus_{\lambda \in \Lambda} \text{Rad}(U_\lambda)$ , so

$$(81) \quad S_i^L = (P_i / \text{Rad}(P_i))^L = P_i^L / \text{Rad}(P_i)^L = \bigoplus_{\lambda \in \Lambda} U_\lambda / (\text{Rad}(P_i)^L \cap U_\lambda).$$

Moreover, we have  $\text{Rad}(P_i)^L \subseteq \text{Rad}(P_i^L)$ . This follows from the fact that  $\text{Rad}(A)^L \subseteq \text{Rad}(A^L)$  by [Lam 1991, Theorem 5.14] and the fact that  $\text{Rad}(P_i) = \text{Rad}(A)P_i$  and  $\text{Rad}(P_i^L) = \text{Rad}(A^L)P_i^L$  by [Lam 1991, Theorem 24.7] since  $P_i$  and  $P_i^L$  are projective. For each  $\lambda \in \Lambda$  the radical of  $U_\lambda$  is a proper submodule of  $U_\lambda$  and therefore

$$\text{Rad}(P_i)^L \cap U_\lambda \subseteq \text{Rad}(P_i^L) \cap U_\lambda = \text{Rad}(U_\lambda) \subsetneq U_\lambda.$$

Hence, the head of  $U_\lambda$  is a constituent of  $U_\lambda / (\text{Rad}(P_i^L) \cap U_\lambda)$ , and since all constituents of the latter are constituents of  $S_i^L$ , we must have  $\text{Hd}(U_\lambda) \simeq S_{ij_\lambda}$  for some  $j_\lambda \in J_i$  by the above. This implies that  $U_\lambda = Q_{ij_\lambda}$ , thus proving the claim.  $\square$

**Lemma A.5.** *The following hold:*

- (a) *The morphism  $d_A^L$  is an isomorphism if and only if it induces a bijection between isomorphism classes of simple modules. Similarly, the morphism  $e_A^L$*

is an isomorphism if and only if it induces a bijection between isomorphism classes of projective indecomposable modules.

(b) If  $d_A^L$  is an isomorphism, so is  $e_A^L$ . The converse holds if  $K$  is perfect.

For the proof of [Lemma A.5](#) we will need the following well-known elementary lemma that is also used in the last section. Recall from [\(52\)](#) the intertwining form  $\langle \cdot, \cdot \rangle_A$  of  $A$ .

**Lemma A.6.** *Let  $P$  be a projective indecomposable  $A$ -module and let  $V$  be a finitely generated  $A$ -module. Then*

$$(82) \quad \langle [P], [V] \rangle_A = [V : \text{Hd}(P)] \cdot \dim_K \text{End}_A(\text{Hd}(P)),$$

where  $\text{Hd}(P) = P / \text{Rad}(P)$  is the head of  $P$ . In particular,  $\langle \cdot, \cdot \rangle_A$  is nondegenerate.

*Proof.* We first consider the case  $V = \text{Hd}(P)$ . Let  $f \in \text{Hom}_A(P, \text{Hd}(P))$  be nonzero. Since  $\text{Hd}(P)$  is simple, this morphism is already surjective and thus induces an isomorphism  $P / \text{Ker}(f) \cong \text{Hd}(P)$ . But as  $\text{Rad}(P)$  is the unique maximal submodule of  $P$ , we must have  $\text{Ker}(f) = \text{Rad}(P)$  and thus get an induced morphism  $\text{Hd}(P) \rightarrow \text{Hd}(P)$ . This yields a  $K$ -linear morphism  $\Phi : \text{Hom}_A(P, \text{Hd}(P)) \rightarrow \text{End}_A(\text{Hd}(P))$ . On the other hand, if  $f \in \text{End}_A(\text{Hd}(P))$ , then composing it with the quotient morphism  $P \rightarrow P / \text{Rad}(P) = \text{Hd}(P)$  yields a morphism  $P \rightarrow \text{Hd}(P)$ . In this way we also get a  $K$ -linear morphism  $\Psi : \text{End}_A(\text{Hd}(P)) \rightarrow \text{Hom}_A(P, \text{Hd}(P))$ . By construction,  $\Phi$  and  $\Psi$  are pairwise inverse; hence

$$\langle [P], [\text{Hd}(P)] \rangle_A = \dim_K \text{Hom}_A(P, \text{Hd}(P)) = \dim_K \text{End}_A(\text{Hd}(P))$$

as claimed.

Now, suppose that  $V$  is a simple  $A$ -module not isomorphic to  $\text{Hd}(P)$ . We can write  $P = Ae$  for some primitive idempotent  $e \in A$ . Since  $A$  is artinian,  $e$  is already local and now it follows from [\[Lam 1991, Proposition 21.19\]](#) that  $\text{Hom}_A(Ae, V)$  is nonzero if and only if  $V$  has a constituent isomorphic to  $\text{Hd}(Ae)$ . This is not true by assumption, and therefore  $\text{Hom}_A(P, V) = 0$ , so  $\langle [P], [V] \rangle_A = 0$ .

Finally, for  $V$  general we have  $[V] = \sum_{S \in \text{Irr } A} [V : S][S]$  in  $G_0(A)$ . By the above we get

$$\begin{aligned} \langle [P], [V] \rangle_A &= \sum_{S \in \text{Irr } A} [V : S] \langle [P], [S] \rangle_A = [V : \text{Hd}(P)] \langle [P], [\text{Hd}(P)] \rangle_A \\ &= [V : \text{Hd}(P)] \cdot \dim_K \text{End}_A(\text{Hd}(P)) \end{aligned}$$

as claimed. It follows that the Gram matrix  $\mathcal{G}$  of  $\langle \cdot, \cdot \rangle$  with respect to the basis  $(P(S))_{S \in \text{Irr } A}$  of  $K_0(A)$  and the basis  $(S)_{S \in \text{Irr } A}$  of  $G_0(A)$  is diagonal with positive diagonal entries. The determinant of  $\mathcal{G}$  is thus a nonzero divisor on  $\mathbb{Z}$  and since both  $K_0(A)$  and  $G_0(A)$  are  $\mathbb{Z}$ -free of the same finite dimension, it follows that  $\langle \cdot, \cdot \rangle_A$  is nondegenerate; see [\[Scheja and Storch 1988, Satz 70.5\]](#).  $\square$

*Proof of Lemma A.5.* We use the same notation as in the proof of Lemma A.4. Since  ${}_A A$  is a projective  $A$ -module, there is a decomposition  ${}_A A = \bigoplus_{i \in I} P_i^{r_i}$  for some  $r_i \in \mathbb{N}$ . Using Lemma A.6 we see that

$$\begin{aligned} \dim_K \text{Hd}(P_j) &= \langle [{}_A A], [\text{Hd}(P_j)] \rangle_A \\ &= \sum_{i \in I} r_i \langle [P_i], [\text{Hd}(P_j)] \rangle_A = r_j \langle [P_j], [\text{Hd}(P_j)] \rangle_A \\ &= r_j \dim_K \text{End}_A(\text{Hd}(P_i)). \end{aligned}$$

Hence,  $r_i = n_i/m_i$ , where  $n_i := \dim_K S_i$  and  $m_i := \dim_K \text{End}_A(S_i)$ . In particular,

$$(83) \quad \dim_K A = \sum_{i \in I} \frac{n_i}{m_i} \dim_K P_i.$$

Now, suppose that  $d_A^L$  is an isomorphism. Then clearly  $\# \text{Irr } A = \# \text{Irr } A^L$ . The properties of the matrix  $D_A^L$  of the morphism  $d_A^L$  derived in the proof of Lemma A.4 immediately imply  $D_A^L$  is diagonal. Since it is invertible with natural numbers on the diagonal, it must already be the identity matrix; i.e.,  $d_A^L$  induces a bijection between the isomorphism classes of simple modules. In particular,  $(S_i^L)_{i \in I}$  is a system of representatives of the isomorphism classes of simple  $A^L$ -modules. The properties of the matrix  $E_A^L$  of  $e_A^L$  derived in the proof of Lemma A.4 now imply that we must have  $P_i^L \simeq Q_i^{s_i}$  for some  $s_i \in \mathbb{N}$ . We argue that  $s_i = 1$ . This shows that  $e_A^L$  is an isomorphism inducing a bijection between the isomorphism classes of projective indecomposable modules. In the same way we deduced (83) we now get

$$(84) \quad \dim_K A = \dim_L A^L = \sum_{i \in I} \frac{n'_i}{m'_i} \dim_L Q_i$$

with

$$n'_i = \dim_L \text{Hd}(Q_i) = \dim_L S_i^L = \dim_K S_i = n_i$$

and

$$m'_i = \dim_L \text{End}_{A^L}(\text{Hd}(Q_i)) = \dim_L \text{End}_{A^L}(S_i^L) = \dim_K \text{End}_K(S_i) = m_i,$$

using the fact that  $L \otimes_K \text{End}_A(S_i) \simeq \text{End}_{A^L}(S_i^L)$ ; see [Reiner 2003, Theorem 2.38]. Since  $\dim_L Q_i \leq \dim_L P_i^L = \dim_K P_i$ , (83) and (84) imply that  $\dim_L Q_i = \dim_K P_i$ , so  $Q_i = P_i^L$ .

Conversely, suppose that  $e_A^L$  is an isomorphism. With the properties of the matrix  $E_A^L$  of  $e_A^L$  established in the proof of Lemma A.4, we see much as above that  $e_A^L$  already induces a bijection between the projective indecomposable modules. In particular,  $P_i^L \simeq Q_i$ . Due to the properties of the matrix  $D_A^L$  of  $d_A^L$  established in the proof of Lemma A.4, the only constituent of  $S_i^L$  is  $T_i$ . Since  $P_i$  is the projective cover of  $P_i$ , we have a surjective morphism  $\phi : P_i \twoheadrightarrow S_i$  with  $\text{Ker}(\phi) = \text{Rad}(P_i)$ . Scalar extension induces a surjective morphism  $\phi^L : P_i^L \twoheadrightarrow S_i^L$  with  $\text{Ker}(\phi^L) =$

$\text{Ker}(\phi)^L = \text{Rad}(P_i)^L \subseteq \text{Rad}(P_i^L)$ . It thus follows from [Curtis and Reiner 1981, Corollary 6.25(i)] that  $P_i^L$  is the projective cover of  $S_i^L$ . Now, we assume that  $K$  is perfect. Then by [loc. cit., Theorem 7.5] all simple  $A$ -modules are separable, so  $S_i^L = T_i^{s_i}$  for some  $s_i$ . Since projective covers are additive, we get  $P_i^L = Q_i^{s_i}$ . As  $P_i^L = Q_i$ , this implies that  $s_i = 1$ , so  $S_i^L = T_i$  is simple. Hence,  $d_A^L$  induces a bijection between the isomorphism classes of simple modules.  $\square$

**Remark A.7.** With the same arguments as in the proof of Lemma A.5 we can show that the converse in Lemma A.5(b) still holds when we only assume that all simple  $A$ -modules are separable, i.e., they remain semisimple under field extension. This holds for example when  $A$  splits or if  $A$  is a group algebra (over any field). We do not know whether it holds more generally.

*Proof of Lemma A.3.* Let  $Z := Z(A)$ . Suppose that  $L$  is an extension field of  $K$  with  $\# \text{Bl}(A) = \# \text{Bl}(A^L)$ . By (79) we know that  $\# \text{Irr } Z = \# \text{Irr } Z^L$ . The arguments in the proof of Lemma A.4 thus imply that the matrix  $D_A^L$  of the morphism  $d_Z^L : G_0(Z) \rightarrow G_0(Z^L)$  must be a diagonal matrix. We claim that it is the identity matrix. Since this holds for any  $L$ , it means that the simple modules of  $Z$  remain simple under any field extension, so  $Z$  splits. Our assumption implies that  $Z$  and  $Z^L$  have the same number of primitive idempotents, so every primitive idempotent  $e \in Z$  remains primitive in  $Z^L$ . This shows that  $e_A^L : K_0(Z) \rightarrow K_0(Z^L)$  induces a bijection between projective indecomposable modules. In particular, it is an isomorphism. Now, Lemma A.5 shows that also  $d_A^L$  is an isomorphism. Since its matrix  $D_A^L$  is invertible with natural numbers on the diagonal, it must be the identity.  $\square$

**Remark A.8.** In the proof of Lemma A.3 we have deduced that for a commutative finite-dimensional  $K$ -algebra  $Z$  the condition  $\text{rk}_{\mathbb{Z}} K_0(Z) = \text{rk}_{\mathbb{Z}} K_0(Z^L)$  already implies that  $e_Z^L$  induces a bijection between projective indecomposable modules. This follows from the fact that idempotents in a commutative ring are isomorphic if and only if they are equal. This is not true for a noncommutative ring  $A$ . Here, we can have  $\text{rk}_{\mathbb{Z}} K_0(A) = \text{rk}_{\mathbb{Z}} K_0(A^L)$  but still a primitive idempotent  $e \in A$  can split into a sum of *isomorphic* orthogonal primitive idempotents of  $A^L$ . Then the matrix  $E_A^L$  of  $e_A^L$  is diagonal but not the identity.

Let us record the following additional fact:

**Lemma A.9.** *If  $Z(A)$  splits, then*

$$(85) \quad \begin{aligned} \# \text{Bl}(A) &= \dim_K Z(A) - \dim_K \text{Rad}(Z(A)) \\ &= \dim_K Z(A) - \dim_K (Z(A) \cap \text{Rad}(A)). \end{aligned}$$

*Proof.* This is an immediate consequence of (79) and the fact that  $\text{Rad}(Z(A)) = Z(A) \cap \text{Rad}(A)$  since  $Z(A) \subseteq A$  is a finite normalizing extension; see [Lorenz 1981, Theorem 1.5].  $\square$

**Reductions.** Now, we consider a situation which in a sense is opposite to the one considered in the last paragraph; namely we consider the quotient morphism  $\phi: R \rightarrow R/\mathfrak{m} =: S$  for a local commutative ring  $R$  with maximal ideal  $\mathfrak{m}$  and a finitely generated  $R$ -algebra  $A$ . By [Lemma 2.3\(b\)](#) the morphism  $\phi_A: A \rightarrow A^S \simeq A/\mathfrak{m}A =: \bar{A}$  is idempotent stable. We say that  $\phi$  is *idempotent surjective* if for each idempotent  $e' \in A^S$  there is an idempotent  $e \in A$  with  $\phi_A(e) = e'$ . We say that  $\phi_A$  is *primitive idempotent bijective* if it induces a bijection between the *isomorphism classes* of primitive idempotents of  $A$  and the isomorphism classes of primitive idempotents of  $A^S$ . The question of whether  $\phi_A$  is idempotent surjective is precisely the question of whether idempotents of  $\bar{A}$  can be lifted to  $A$ , and this is a classical topic in ring theory. The following lemma is standard; we omit the proof.

**Lemma A.10.** *If  $\phi_A: A \rightarrow \bar{A}$  is idempotent surjective, it is primitive idempotent bijective and block bijective.*

**Theorem A.11** [[Neunhöffer 2003](#), Proposition 5.10]. *The morphism  $\phi_A: A \rightarrow \bar{A}$  is idempotent surjective if and only if  $A$  is semiperfect.*

We recall two standard situations of idempotent surjective reductions.

**Lemma A.12.** *In the following two cases the morphism  $\phi_A: A \rightarrow \bar{A}$  is idempotent surjective:*

- (a)  *$R$  is noetherian and  $\mathfrak{m}$ -adically complete.*
- (b)  *$R$  is henselian.*

*Proof.* For a proof of the first case, see [[Lam 1991](#), Proposition 21.34]. For a proof of the second case assuming that  $A$  is commutative, see [[Raynaud 1970](#), I, §3, Proposition 2]. To give a proof for noncommutative  $A$  let  $\bar{e} \in \bar{A}$  be an idempotent. Let  $k := R/\mathfrak{m}$  and let  $\bar{B} := k[\bar{e}]$  be the  $k$ -subalgebra of  $\bar{A}$  generated by  $\bar{e}$ . Since  $\bar{A}$  is a finite-dimensional  $k$ -algebra, also  $\bar{B}$  is finite-dimensional. Moreover,  $\bar{B}$  is commutative. Let  $e \in A$  be an arbitrary element with  $\phi_A(e) = \bar{e}$ . Let  $B := R[e]$ , a commutative subalgebra of  $A$ . Note that  $\bar{B} = B/\mathfrak{m}B$ . Since  $A$  is a finitely generated  $R$ -module, the Cayley–Hamilton theorem implies that  $B$  is a finitely generated  $R$ -algebra. Now, by the commutative case, the map  $\phi_B: B \rightarrow \bar{B}$  is idempotent surjective and so there is an idempotent  $e' \in B \subseteq A$  with  $\phi_A(e') = \phi_B(e') = \bar{e}$ . This shows that  $\phi_A$  is idempotent surjective.  $\square$

The next theorem was again proven by Neunhöffer [[2003](#), Proposition 6.2]. It is one of our key ingredients in proving Brauer reciprocity for decomposition maps in a general setting.

**Theorem A.13** (M. Neunhöffer). *Suppose that  $R$  is a valuation ring with fraction field  $K$  and that  $A$  is a finite flat  $R$ -algebra with split generic fiber  $A^K$ . If  $\widehat{R} \otimes_R A$  is semiperfect, where  $\widehat{R}$  is the completion of  $R$  with respect to the topology defined by a valuation on  $K$  defining  $R$ , then also  $A$  is semiperfect.*

**Corollary A.14** (J. Müller, M. Neunhöffer). *Suppose that  $R$  is a discrete valuation ring and that  $A$  is a finite flat  $R$ -algebra with split generic fiber. Then  $A$  is semiperfect. In particular,  $\phi_A : A \rightarrow \bar{A}$  is primitive idempotent bijective and block bijective.*

*Proof.* Since  $R$  is a discrete valuation ring, its valuation topology coincides with its  $\mathfrak{m}$ -adic topology so that the topological completion  $\widehat{R}$  is  $\widehat{\mathfrak{m}}$ -adically complete, where  $\mathfrak{m}$  denotes the maximal ideal of  $R$  and  $\widehat{\mathfrak{m}}$  denotes the maximal ideal of  $\widehat{R}$ . Hence,  $\widehat{R} \otimes_R A$  is semiperfect by [Lemma A.12\(a\)](#) and [Theorem A.11](#). Now, [Theorem A.13](#) shows that  $A$  is semiperfect, too.  $\square$

**Remark A.15.** One part of [Corollary A.14](#), the fact that idempotents lift, was also stated earlier by Curtis and Reiner [[1981](#), Exercise 6.16] in the special case where  $A^K$  is assumed to be semisimple. The semisimplicity assumption was later removed by J. Müller [[1995](#), Satz 3.4.1] using the Wedderburn–Malcev theorem (this can be applied without a perfectness assumption on the base field if  $A^K$  splits since then  $A^K / \text{Rad}(A^K)$  is separable; see [[Curtis and Reiner 1962](#), Theorem 72.19]).

## Appendix B: Further elementary facts

Here, we prove three further elementary facts that we used in the paper.

**Lemma B.1.** *A finitely generated module  $M$  over an integral domain  $R$  is flat if and only if it is faithfully flat. In particular, if  $M \neq 0$ , we have  $0 \neq \mathfrak{k}(\mathfrak{p}) \otimes_R M = M(\mathfrak{p})$  for all  $\mathfrak{p} \in \text{Spec}(R)$ .*

*Proof.* We can assume that  $M \neq 0$ . Since  $M$  is flat, it is torsion-free and so the localization map  $M \rightarrow M_{\mathfrak{p}}$  is injective; see [Lemma 2.1\(c\)](#). Hence,  $M_{\mathfrak{p}} \neq 0$ . Since  $M$  is a finitely generated  $R$ -module, also  $M_{\mathfrak{p}}$  is a finitely generated  $R_{\mathfrak{p}}$ -module and now Nakayama’s lemma implies that  $0 \neq M_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}M_{\mathfrak{p}} = \mathfrak{k}(\mathfrak{p}) \otimes_R M$ . Hence,  $M$  is faithfully flat by [[Matsumura 1986](#), Theorem 7.2].  $\square$

**Lemma B.2.** *Let  $A$  be a finite flat algebra over an integral domain  $R$ . Then the structure map  $R \rightarrow A$ ,  $r \mapsto r \cdot 1_A$ , is injective. Hence, we can identify  $R \subseteq Z(A)$ . If  $R$  is noetherian, the induced map  $\Upsilon : \text{Spec}(Z(A)) \rightarrow \text{Spec}(R)$  is finite, closed, and surjective.*

*Proof.* It follows from [Lemma B.1](#) that  $A$  is already faithfully flat. Let  $\phi : R \rightarrow A$  be the structure map. This is an  $R$ -module map and applying  $-\otimes_R A$  yields a map

$$A \simeq R \otimes_R A \xrightarrow{\phi \otimes_R A} A \otimes_R A$$

of right  $A$ -modules, mapping  $a$  to  $1 \otimes a$ . This map has an obvious section mapping  $a \otimes a'$  to  $aa'$ ; hence it is injective. Since  $A$  is faithfully flat, the original map  $\phi$  has to be injective, too. As the image of  $\phi$  is contained in the center  $Z$  of  $A$ , the structure map is actually an injective map  $R \hookrightarrow Z$ . Now, assume that  $R$  is noetherian. Since

$A$  is a finitely generated  $R$ -module, also  $Z$  is a finitely generated  $R$ -module. Hence,  $R \subseteq Z$  is a finite ring extension and now it is an elementary fact that  $\Upsilon$  is closed and surjective.  $\square$

The following lemma about base change of homomorphism spaces is well known but we could not find a reference in this generality; see [Bourbaki 1972, II, §5.3] for a proof in the case of a commutative base ring.

**Lemma B.3.** *Let  $A$  be an algebra over a commutative ring  $R$  and let  $\phi : R \rightarrow S$  be a morphism into a commutative ring  $S$ . Let  $V$  and  $W$  be  $A$ -modules. If  $V$  is finitely generated and projective as an  $A$ -module, then there is a canonical  $S$ -module isomorphism:*

$$(86) \quad S \otimes_R \operatorname{Hom}_A(V, W) \simeq \operatorname{Hom}_{A^S}(V^S, W^S).$$

*Proof.* We can define a map  $\gamma : S \otimes_R \operatorname{Hom}_A(V, W) \rightarrow \operatorname{Hom}_{A^S}(V^S, W^S)$  by mapping  $s \otimes f$  with  $s \in S$  and  $f \in \operatorname{Hom}_A(V, W)$  to  $s_r \otimes f$ , where  $s_r$  denotes right multiplication by  $s$ . It is a standard fact that this is an  $S$ -module morphism; see [Reiner 2003, (2.36)]. Recall that  $\operatorname{Hom}_A(-, W)$  commutes with finite direct sums by [Bourbaki 1989, II, §1.6, Corollary 1 to Proposition 6]. This shows that the canonical isomorphism  $\operatorname{Hom}_A(A, W) \simeq W$  induces a canonical isomorphism  $\operatorname{Hom}_A(A^n, W) \simeq W^n$  for any  $n \in \mathbb{N}$  and now we conclude that there is a canonical isomorphism

$$S \otimes_R \operatorname{Hom}_A(A^n, W) \simeq S \otimes_R W^n \simeq (S \otimes_R W)^n \simeq \operatorname{Hom}_{A^S}((A^S)^n, W^S),$$

which is easily seen to be equal to  $\gamma$ . The assertion thus holds for finitely generated free  $A$ -modules. Now, the assumption on  $V$  allows us to write without loss of generality  $A^n = V \oplus X$  for some  $A$ -module  $X$ . It is not hard to see that we get a commutative diagram

$$\begin{array}{ccc} S \otimes_R \operatorname{Hom}_A(A^n, W) & \xrightarrow{\cong} & (S \otimes_R \operatorname{Hom}_A(V, W)) \oplus (S \otimes_R \operatorname{Hom}_A(X, W)) \\ \cong \downarrow & & \downarrow \qquad \qquad \downarrow \\ \operatorname{Hom}_{A^S}((A^S)^n, W^S) & \xrightarrow{\cong} & (\operatorname{Hom}_{A^S}(V^S, W^S)) \oplus (\operatorname{Hom}_{A^S}(X^S, W^S)) \end{array}$$

where the horizontal morphisms are obtained by the projections and the vertical morphisms are the morphisms  $\gamma$  in the respective situation. The commutativity of this diagram implies that the morphism  $S \otimes_R \operatorname{Hom}_A(V, W) \rightarrow \operatorname{Hom}_{A^S}(V^S, W^S)$  also has to be an isomorphism.  $\square$

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## References

- [Benson 1993] D. J. Benson, *Polynomial invariants of finite groups*, London Mathematical Society Lecture Note Series **190**, Cambridge University Press, 1993. [MR](#) [Zbl](#)
- [Bonnafé and Rouquier 2017] C. Bonnafé and R. Rouquier, “Cherednik algebras and Calogero–Moser cells”, preprint, 2017. [arXiv](#)
- [Bourbaki 1972] N. Bourbaki, *Elements of mathematics: commutative algebra*, Hermann, Paris, 1972. [MR](#) [Zbl](#)
- [Bourbaki 1989] N. Bourbaki, *Algebra, I: Chapters 1–3*, Springer, 1989. [MR](#) [Zbl](#)
- [Brauer and Nesbitt 1941] R. Brauer and C. Nesbitt, “On the modular characters of groups”, *Ann. of Math. (2)* **42** (1941), 556–590. [MR](#) [Zbl](#)
- [Brown and Goodearl 2002] K. A. Brown and K. R. Goodearl, *Lectures on algebraic quantum groups*, Birkhäuser, Basel, 2002. [MR](#) [Zbl](#)
- [Brown and Gordon 2001] K. A. Brown and I. Gordon, “The ramification of centres: Lie algebras in positive characteristic and quantised enveloping algebras”, *Math. Z.* **238**:4 (2001), 733–779. [MR](#) [Zbl](#)
- [Brown and Gordon 2002] K. A. Brown and I. Gordon, “The ramifications of the centres: quantised function algebras at roots of unity”, *Proc. London Math. Soc. (3)* **84**:1 (2002), 147–178. [MR](#) [Zbl](#)
- [Chlouveraki 2009] M. Chlouveraki, *Blocks and families for cyclotomic Hecke algebras*, Lecture Notes in Mathematics **1981**, Springer, 2009. [MR](#) [Zbl](#)
- [Curtis and Reiner 1962] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics **11**, Interscience, New York, 1962. [MR](#) [Zbl](#)
- [Curtis and Reiner 1981] C. W. Curtis and I. Reiner, *Methods of representation theory, I: With applications to finite groups and orders*, Wiley, New York, 1981. [MR](#) [Zbl](#)
- [Donkin and Tange 2010] S. Donkin and R. Tange, “The Brauer algebra and the symplectic Schur algebra”, *Math. Z.* **265**:1 (2010), 187–219. [MR](#) [Zbl](#)
- [Eisenbud 1995] D. Eisenbud, *Commutative algebra: with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer, 1995. [MR](#) [Zbl](#)
- [Gabriel 1975] P. Gabriel, “Finite representation type is open”, pp. 132–155 in *Proceedings of the International Conference on Representations of Algebras*, edited by V. Dlab and P. Gabriel, Lecture Notes in Mathematics **488**, Springer, 1975. [MR](#) [Zbl](#)
- [Geck and Pfeiffer 2000] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori–Hecke algebras*, London Mathematical Society Monographs. New Series **21**, Oxford Univ. Press, New York, 2000. [MR](#) [Zbl](#)
- [Geck and Rouquier 1997] M. Geck and R. Rouquier, “Centers and simple modules for Iwahori–Hecke algebras”, pp. 251–272 in *Finite reductive groups* (Luminy, 1994), edited by M. Cabanes, Progr. Math. **141**, Birkhäuser, Boston, 1997. [MR](#) [Zbl](#)

- [Goodearl and Warfield 2004] K. R. Goodearl and R. B. Warfield, Jr., *An introduction to noncommutative Noetherian rings*, 2nd ed., London Mathematical Society Student Texts **61**, Cambridge Univ. Press, 2004. [MR](#) [Zbl](#)
- [Gordon 2001] I. Gordon, “Representations of semisimple Lie algebras in positive characteristic and quantum groups at roots of unity”, pp. 149–167 in *Quantum groups and Lie theory* (Durham, 1999), edited by A. Pressley, London Mathematical Society Lecture Note Series **290**, Cambridge Univ. Press, 2001. [MR](#) [Zbl](#)
- [Görtz and Wedhorn 2010] U. Görtz and T. Wedhorn, *Algebraic geometry, I: Schemes with examples and exercises*, Vieweg + Teubner, Wiesbaden, 2010. [MR](#) [Zbl](#)
- [Graham and Lehrer 1996] J. J. Graham and G. I. Lehrer, “Cellular algebras”, *Invent. Math.* **123**:1 (1996), 1–34. [MR](#) [Zbl](#)
- [Grothendieck 1965] A. Grothendieck, “Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, II”, *Inst. Hautes Études Sci. Publ. Math.* **24** (1965), 5–231. [MR](#) [Zbl](#)
- [Grothendieck 1966] A. Grothendieck, “Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, III”, *Inst. Hautes Études Sci. Publ. Math.* **28** (1966), 5–255. [MR](#) [Zbl](#)
- [Grothendieck 1967] A. Grothendieck, “Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, IV”, *Inst. Hautes Études Sci. Publ. Math.* **32** (1967), 5–361. [MR](#) [Zbl](#)
- [Huneke and Swanson 2006] C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series **336**, Cambridge Univ. Press, 2006. [MR](#) [Zbl](#)
- [Jategaonkar 1986] A. V. Jategaonkar, *Localization in Noetherian rings*, London Mathematical Society Lecture Note Series **98**, Cambridge Univ. Press, 1986. [MR](#) [Zbl](#)
- [Kaplansky 1958] I. Kaplansky, “Projective modules”, *Ann. of Math (2)* **68** (1958), 372–377. [MR](#) [Zbl](#)
- [Lam 1991] T. Y. Lam, *A first course in noncommutative rings*, Graduate Texts in Mathematics **131**, Springer, 1991. [MR](#) [Zbl](#)
- [Lam 1999] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics **189**, Springer, 1999. [MR](#) [Zbl](#)
- [Lorenz 1981] M. Lorenz, “Finite normalizing extensions of rings”, *Math. Z.* **176**:4 (1981), 447–484. [MR](#) [Zbl](#)
- [Matsumura 1986] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge Univ. Press, 1986. [MR](#) [Zbl](#)
- [McConnell and Robson 2001] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Graduate Studies in Mathematics **30**, American Mathematical Society, Providence, RI, 2001. [MR](#) [Zbl](#)
- [Müller 1976] B. J. Müller, “Localization in non-commutative Noetherian rings”, *Canad. J. Math.* **28**:3 (1976), 600–610. [MR](#) [Zbl](#)
- [Müller 1985] B. J. Müller, “Affine Noetherian PI-rings have enough clans”, *J. Algebra* **97**:1 (1985), 116–129. [MR](#) [Zbl](#)
- [Müller 1995] J. Müller, *Zerlegungszahlen für generische Iwahori–Hecke-Algebren von exzeptionellen Typ*, Ph.D. thesis, RWTH Aachen University, 1995, [www.math.rwth-aachen.de/~Juergen.Mueller/preprints/jm3.ps](http://www.math.rwth-aachen.de/~Juergen.Mueller/preprints/jm3.ps). [Zbl](#)

- [Neunhöffer 2003] M. Neunhöffer, *Untersuchungen zu James' Vermutung über Iwahori–Hecke-Algebren vom Typ A*, Ph.D. thesis, RWTH Aachen University, 2003, <http://www.math.rwth-aachen.de/~Max.Neunhoeffer/Publications/pdf/phd.pdf>.
- [Neunhöffer and Scherotzke 2008] M. Neunhöffer and S. Scherotzke, “Formulas for primitive idempotents in Frobenius algebras and an application to decomposition maps”, *Represent. Theory* **12** (2008), 170–185. [MR](#) [Zbl](#)
- [Raynaud 1970] M. Raynaud, *Anneaux locaux henséliens*, Lecture Notes in Mathematics **169**, Springer, 1970. [MR](#) [Zbl](#)
- [Reiner 2003] I. Reiner, *Maximal orders*, London Mathematical Society Monographs. New Series **28**, Oxford University Press, 2003. [MR](#) [Zbl](#)
- [Rui 2005] H. Rui, “A criterion on the semisimple Brauer algebras”, *J. Combin. Theory Ser. A* **111**:1 (2005), 78–88. [MR](#) [Zbl](#)
- [Scheja and Storch 1988] G. Scheja and U. Storch, *Lehrbuch der Algebra, II: Unter Einschluss der linearen Algebra*, B. G. Teubner, Stuttgart, 1988. [MR](#) [Zbl](#)
- [Stacks 2005–] P. Belmans, A. J. de Jong, et al., “The Stacks project”, electronic reference, Columbia University, 2005–, <http://stacks.math.columbia.edu>.
- [Thiel 2016] U. Thiel, “Decomposition matrices are generically trivial”, *Int. Math. Res. Not.* **2016**:7 (2016), 2157–2196. [MR](#)

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