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Let $G = \mathrm{GL}_2(D)$ where D is a quaternion division algebra over a number field F and $H = \mathrm{Sp}_2(D)$ is the unique inner form of $\mathrm{Sp}_4(F)$. We study the period of an automorphic form on $G(\mathbb{A})$ relative to $H(\mathbb{A})$ and we provide a formula, similar to the split case, for an automorphic form in the residual spectrum. We confirm the conjecture due to Dipendra Prasad for noncuspidal automorphic representations, which says that symplectic period is preserved under the global Jacquet–Langlands correspondence.

1. Introduction

Let F be a number field and $\mathbb{A}$ its ring of adeles. Let G be a connected reductive group defined over F and H be the fixed point subgroup of an involution on G. For Q an algebraic group defined over F, we let $Y(Q)$ be the group of F-rational characters of Q and

$$Q(\mathbb{A})^1 = \{q \in Q(\mathbb{A}) : |\chi(q)| = 1, \forall \chi \in Y(Q)\}.$$

Let ϕ be an automorphic form on $G(\mathbb{A})$. If ϕ is a cusp form, the period integral $P^H(\phi)$ is defined by the convergent integral

$$\int_{H(F)\backslash(H(\mathbb{A})\cap G(\mathbb{A})^1)} \phi(h)\, dh. \tag{1}$$

We say ϕ is distinguished by H if $P^H(\phi) \neq 0$. A cuspidal automorphic representation π of G is said to be distinguished by H if there exists a $\phi \in \pi$ distinguished by H. It is reasonable to ask for a characterization of H-distinguished cuspidal representations or more generally representations in the discrete spectrum. For a more general automorphic form the period integral may not converge and it is of interest to study the convergence of $P^H(\phi)$.

Let D be a quaternion division algebra over F with involution $\bar{\cdot}$. Let G be the group $\mathrm{GL}_n(D)$ and $H = \mathrm{Sp}_n(D)$ the nonsplit inner form of $\mathrm{Sp}_{2n}(F)$ which we can define as

$$\mathrm{Sp}_n(D) = \{A \in \mathrm{GL}_n(D) : AJ\,{}^t\bar{A} = J\},$$

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where ${}^t\bar{A} = (\bar{a}_{ji})$ for $A = (a_{ij})$ and

$$J = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ 1 & & & & \end{pmatrix}$$

Similarly when $D = F$ we denote these group by $G' = \mathrm{GL}_{2n}(F)$ and $H' = \mathrm{Sp}_{2n}(F)$. Set $D_{\mathbb{A}} = D \otimes_F \mathbb{A}$. Given an automorphic representation π' of $G'(\mathbb{A})$, for some automorphic form $\phi \in \pi'$, when the above integral (1) is nonzero, we say that π' has symplectic period. When $D = F$, H. Jacquet and S. Rallis [1992] proved that a cuspidal automorphic representation of $G'(\mathbb{A})$ cannot have symplectic period. Further they computed the symplectic period for the most cuspidal elements in the residual spectrum and showed that it is nonzero. Offen [2006a; 2006b] considered H'-distinguished representations of the group G'.

The global Jacquet–Langlands correspondence associates to each automorphic representation π of $\mathrm{GL}_n(D_{\mathbb{A}})$ in the discrete spectrum an automorphic representation π' of $\mathrm{GL}_{2n}(\mathbb{A})$ such that $\pi_v \equiv \pi'_v$ at all the places where $\mathrm{GL}_n(D_v) \equiv \mathrm{GL}_{2n}(F_v)$. In [Verma 2014] we studied the symplectic period for the pair $(\mathrm{GL}_n(D), \mathrm{Sp}_n(D))$ both locally and globally. In this paper Dipendra Prasad suggested the conjecture that π' is distinguished if and only if π is distinguished. We will go into more details about the conjecture and partial results in Section 2.

For an algebraic group X defined over F, we will also write X for the group of F-points. Let P be a minimal parabolic F-subgroup of $\mathrm{GL}_2(D)$ (which is unique up to conjugacy) consisting of upper triangular matrices in $\mathrm{GL}_2(D)$ with Levi decomposition $P = MU$, where M is the diagonal subgroup and U is the unipotent subgroup. We will denote by δ_P the modulus function of $P(\mathbb{A})$. Let K be the maximal compact subgroup of $\mathrm{GL}_2(D_{\mathbb{A}})$ such that $\mathrm{GL}_2(D_{\mathbb{A}}) = U(D_{\mathbb{A}})M(D_{\mathbb{A}})K$ is the Iwasawa decomposition. We define a function H on $M(D_{\mathbb{A}})$ by

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = |\mathrm{nrd}\, m_1|^{1/2} |\mathrm{nrd}\, m_2|^{-1/2}.$$

Here nrd is the reduced norm map from $\mathrm{GL}_2(D_{\mathbb{A}})$ to $\mathbb{A}^\times$. Using the Iwasawa decomposition, we extend trivially H on $\mathrm{GL}_2(D_{\mathbb{A}})$ by

$$H(g) = H(m),$$

where $g = umk$ with $k \in K, m \in M(D_{\mathbb{A}})$ and $u \in U(D_{\mathbb{A}})$.

Let σ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_1(D_{\mathbb{A}})$. Assume that σ is trivial on the center. Then $\sigma \otimes \sigma$ is a cuspidal automorphic representation of $M(D_{\mathbb{A}})$. We first realize the cuspidal automorphic representation $\sigma \otimes \sigma$

in the space of square integrable automorphic functions $L^2(Z_M(D_{\mathbb{A}})M \setminus M(D_{\mathbb{A}}))$ with Z_M being the center of M. Let ϕ be a $K \cap M(D_{\mathbb{A}})$-finite automorphic form in the space of $\sigma \otimes \sigma$ which is extended to a function on $\mathrm{GL}_2(D_{\mathbb{A}})$, so that for $g = muk \in \mathrm{GL}_2(D_{\mathbb{A}})$

$$\phi(g) = \delta_P(m)^{1/2}\phi(mk)$$

and for any fixed $k \in K$, the function

$$m \mapsto \phi(mk)$$

is a $K \cap M(D_{\mathbb{A}})$-finite automorphic form in the space $\sigma \otimes \sigma$. We define

$$F(g, \phi, s) = \phi(g)H(g)^s.$$

Then the Eisenstein series is given by

$$E(g, \phi, s) = \sum_{\gamma \in P\backslash \mathrm{GL}_2(D)} F(\gamma g, \phi, s),$$

which converges absolutely for $\mathrm{Re}(s)$ large. The Eisenstein series $E(g, \phi, s)$ can be analytically continued to a meromorphic function of s. It has a simple pole inside $\mathrm{Re}(s) \geq 0$ and depending on σ the only possible pole in that region is either at $s = 1$ or $s = 2$ [Badulescu 2008]. For $s_0 \in \{1, 2\}$, we define the residue $E_{-s_0}(\phi)$ by

$$E_{-s_0}(g, \phi) = \lim_{s \to s_0} (s - s_0)E(g, \phi, s).$$

The functions $E_{-s_0}(\phi)$ are L^2-automorphic forms. As ϕ ranges above, the multiresidue $E_{-s_0}(\phi)$ generates an irreducible representation of $\mathrm{GL}_2(D_{\mathbb{A}})$ in the residual spectrum corresponding to σ which we will denote by $J(2, \sigma)$.

To compute symplectic periods of noncuspidal automorphic representations in the discrete spectrum, we are therefore by Theorem 5 and Remark 12 reduced to the study of the symplectic period of $E_{-1}(\phi)$, residue of the Eisenstein series corresponding to $s_0 = 1$ constructed from an automorphic representation σ of $\mathrm{GL}_1(D_{\mathbb{A}})$ which is not one-dimensional. Now we state the main theorems of this article. We need the notation H for $\mathrm{Sp}_2(D)$ as an algebraic group defined over F.

Theorem 1. *The function $E_{-1}(g, \phi)$ is integrable over $\mathrm{Sp}_2(D) \setminus \mathrm{Sp}_2(D_{\mathbb{A}})$ and*

$$\int_{\mathrm{Sp}_2(D)\backslash \mathrm{Sp}_2(D_{\mathbb{A}})} E_{-1}(\phi, h)\, dh = \int_{K\cap \mathrm{Sp}_2(D_{\mathbb{A}})} \int_{(M\cap H)\backslash (M\cap H)(D_{\mathbb{A}})^1} \phi(mk)\, dm\, dk. \tag{2}$$

Moreover, there exists a choice of ϕ such that the above integral is nonzero.

As a consequence of the above theorem we have the following result.

Theorem 2. *Any noncuspidal automorphic representation in the discrete spectrum of $\mathrm{GL}_2(D_{\mathbb{A}})$ is of the form $\pi = J(2, \sigma)$ and is distinguished by $\mathrm{Sp}_2(D_{\mathbb{A}})$. Further,* *its image under the global Jacquet–Langlands correspondence is distinguished by* $\mathrm{Sp}_4(\mathbb{A})$.

Structure of the paper. In Section 2 we will recall the global Jacquet–Langlands correspondence and state it explicitly for $n = 2$. This explicit description reduces to consider residual Eisenstein series described in Section 1 for $s_0 = 1$. We have described the existence of distinguished cuspidal automorphic representations. We introduce Arthur's truncation operator in Section 3 and proved the convergence of the period integral of $E_{-1}(g, \phi)$. Section 4 gives a description of double cosets $P \backslash G/H$ which is required to compute the formula for the period integral (2). In Section 5 we compute the contribution to the period integral associated to double cosets. Finally we show the nonvanishing of (2) by constructing an automorphic form Φ on $\mathrm{GL}_2(D_{\mathbb{A}})$ by choosing suitable Φ_v at the every place v of F.

2. Discrete spectrum and the global Jacquet–Langlands

For the first half of this section we take D to be an arbitrary division algebra of degree d over F. An irreducible representation of $G(\mathbb{A})$ is called a discrete automorphic representation of G if it occurs as a direct summand in the space $L^2(G(F)\backslash G(\mathbb{A})^1)$. The discrete spectrum of $\mathrm{GL}_n(\mathbb{A})$ is described by Moeglin and Waldspurger [1995] and a similar description for $\mathrm{GL}_n(D_{\mathbb{A}})$ is given by Badulescu [2008].

We first recall the original global Jacquet–Langlands correspondence which is carried out in [Deligne et al. 1984; Heumos and Rallis 1990; Jacquet and Langlands 1970]. Note that all irreducible automorphic representations of $D^\times(\mathbb{A})$ are cuspidal. Originally the correspondence is a bijection between (cuspidal) automorphic representations of $D^\times(\mathbb{A})$ which are not one-dimensional and so called compatible automorphic representations of $\mathrm{GL}_d(\mathbb{A})$. This is extended in [Badulescu 2008; Badulescu and Renard 2010] to one-dimensional automorphic representations of $D^\times(\mathbb{A})$. These correspond to the residual representations of $\mathrm{GL}_d(\mathbb{A})$, which are all one-dimensional as well.

The global correspondence between discrete spectrum of a general linear group $\mathrm{GL}_{nd}(\mathbb{A})$ and its inner form $\mathrm{GL}_n(D_{\mathbb{A}})$ is defined and proved in [Badulescu 2008; Badulescu and Renard 2010].

Theorem 3. *There is a unique map* JL *from the set of irreducible constituents of* $L^2_{\mathrm{disc}}(\mathrm{GL}_n(D)\backslash \mathrm{GL}_n(D_{\mathbb{A}}))$ *to the set of irreducible constituents of* $L^2_{\mathrm{disc}}(\mathrm{GL}_{dn}(F)\backslash \mathrm{GL}_{dn}(\mathbb{A}))$, *such that if* $\mathrm{JL}(\pi) = \pi'$ *then* π' *is compatible* (*with respect to* D), $\pi'_v \equiv \pi_v$ *where places* v *at which* D *splits and* π_v *corresponds to* π'_v *by the local Jacquet–Langlands correspondence for places* v *at which* D *does not split. The map* JL *is injective*, *and the image consists of all compatible constituents of* $L^2_{\mathrm{disc}}(\mathrm{GL}_{dn}(F)\backslash \mathrm{GL}_{dn}(\mathbb{A}))$ *with respect to* D.

The following more precise description of the global correspondence is also proved in [Badulescu 2008]. For a positive integer l, let R_l be the standard parabolic F-subgroup of $\mathrm{GL}_{ln}(F)$ consisting of block upper triangular matrices corresponding to the partition $(n, n, \dots, n)$ of ln. Its Levi factor L_{R_l} is isomorphic to the direct product of l copies of $\mathrm{GL}_n(F)$. Let π' be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. Then, we denote by $I(l, \pi')$ the representation of $\mathrm{GL}_{ln}(\mathbb{A})$ induced from the representation

$$\pi'|\det|^{(l-1)/2} \otimes \cdots \otimes \pi'|\det|^{(1-l)/2}$$

of the Levi factor $L_{R_l}(\mathbb{A})$. This representation has a unique irreducible quotient which we denote by $J'(l, \pi')$. It is a residual representation of $\mathrm{GL}_{ln}(\mathbb{A})$ if $l > 1$. For $l = 1$, we have by definition $J'(1, \pi') = \pi'$. All residual representations of $\mathrm{GL}_N(\mathbb{A})$, for $N > 1$, are obtained in this way for some divisor $l > 1$ of N.

Now we describe the discrete spectrum of $\mathrm{GL}_n(D_{\mathbb{A}})$. The notation $J(m, \pi)$ for inner forms is in analogy with the split case which will be obvious from the theorem below.

Theorem 4. *Let π' be an irreducible cuspidal automorphic representation of the group $\mathrm{GL}_n(\mathbb{A})$. There is a unique positive integer $s_{\pi',D}$, depending only on π' and the division algebra D, which is defined by the condition that $J'(l, \pi')$ is globally compatible (with respect to D) if and only if $s_{\pi',D}$ divides l. Moreover, $s_{\pi',D}$ divides the degree d of the division algebra.*

A representation of the form $J'(s_{\pi',D}, \pi')$ of $\mathrm{GL}_{ns_{\pi',D}}(\mathbb{A})$ corresponds to a cuspidal automorphic representation π of the inner form. A representation of the form $J'(ms_{\pi',D}, \pi')$, with $m > 1$, corresponds to a residual representation $J(m, \pi)$ of the inner form, which is unique irreducible quotient of the representation induced from the representation

$$\pi|\mathrm{nrd}|^{(s_{\pi',D}(m-1))/2} \otimes \cdots \otimes \pi|\mathrm{nrd}|^{(s_{\pi',D}(1-m))/2}.$$

From now on let D be the quaternion division algebra over F and we will describe the discrete spectrum of $\mathrm{GL}_2(D_{\mathbb{A}})$ more explicitly [Grbac and Schwermer 2011]. In this case the only possibilities for $s_{\pi',D}$ are 1 and 2. To simplify the notations, we will write $[G]$ for $\mathrm{GL}_2(D) \setminus \mathrm{GL}_2(D_{\mathbb{A}})$ in the following theorem.

Theorem 5. *The discrete spectrum $L^2_{\mathrm{disc}}([G])$ decomposes into*

$$L^2_{\mathrm{disc}}([G]) = L^2_{\mathrm{cusp}}([G]) \oplus L^2_{\mathrm{res}}([G])$$

where $L^2_{\mathrm{cusp}}([G])$ is the cuspidal spectrum consisting of cuspidal elements, and $L^2_{\mathrm{res}}([G])$ is its orthogonal complement called the residual spectrum. The cuspidal part $L^2_{\mathrm{cusp}}([G])$ decomposes into a Hilbert space direct sum of irreducible cuspidal automorphic representations, each appearing with multiplicity one, and obtained by the global Jacquet–Langlands correspondence either from a cuspidal automorphic *representation of* $G' = \mathrm{GL}_4(\mathbb{A})$, *or from a residual automorphic representation* $J'(ms_{\pi',D}, \pi')$ *with* $m = 1$ *and* $s_{\pi',D} = 2$. *The residual part* $L^2_{\mathrm{res}}([G])$ *decomposes into a Hilbert space direct sum*

$$L^2_{\mathrm{res}}([G]) = \bigoplus_{\mu} \mu \circ \mathrm{nrd} \oplus \bigoplus_{\sigma} J(2, \sigma)$$

where the first sum ranges over all unitary characters μ *of* $\mathbb{A}^\times$ *and* $\mu \circ \mathrm{nrd}$ *is obtained by the Jacquet–Langlands correspondence from* $\mu \circ \det$ *and corresponds to* $m = 2$ *and* $s_{\pi',D} = 2$, *while the second sum, which corresponds to* $m = 2$ *and* $s_{\pi',D} = 1$, *ranges over all cuspidal automorphic representations* σ *of* $D^\times(\mathbb{A})$ *which are not one-dimensional.*

One aims to give a complete classification of distinguished automorphic representations in the discrete spectrum of $\mathrm{GL}_n(D_{\mathbb{A}})$. Dipendra Prasad made a conjecture regarding global distinction.

Conjecture 6 [Verma 2014]. *An automorphic representation* π *in the discrete spectrum of* $\mathrm{GL}_n(D_{\mathbb{A}})$ *is distinguished by* $\mathrm{Sp}_n(D_{\mathbb{A}})$ *if and only if its Jacquet–Langlands lift* $\mathrm{JL}(\pi)$, *an automorphic representation of* $\mathrm{GL}_{2n}(\mathbb{A})$, *is distinguished by* $\mathrm{Sp}_{2n}(\mathbb{A})$.

Remark 7. Suppose $\pi = \otimes \pi_v$ is an automorphic representation of $\mathrm{GL}_1(D_{\mathbb{A}})$ which is distinguished by $\mathrm{Sp}_1(D_{\mathbb{A}})$ then π_v as a representation of $\mathrm{GL}_1(D_v)$ is $\mathrm{Sp}_1(D_v)$-distinguished at all the places v. Then by Lemma 4.1 of [Verma 2014], π_v is one-dimensional for all v and so π is one-dimensional. Its Jacquet–Langlands lift $\mathrm{JL}(\pi)$ lies in residual spectrum which is one-dimensional. This verifies that the above conjecture is true for $n = 1$.

Then we have the following theorem from [Verma 2014] which proves the above conjecture partially.

Theorem 8. *If an automorphic representation* π *of* $\mathrm{GL}_n(D_{\mathbb{A}})$ *occurs in the discrete spectrum and is distinguished by* $\mathrm{Sp}_n(D_{\mathbb{A}})$, *then* $\mathrm{JL}(\pi)$ *is distinguished by* $\mathrm{Sp}_{2n}(\mathbb{A})$.

When $s_{\pi',D}$ equals 1, cuspidal representations correspond to cuspidal representations under Jacquet–Langlands and the symplectic period vanishes on both sides. When $s_{\pi',D}$ equals 2, we have the above conjecture, which gives more precise information about distinguished cuspidal representation of $\mathrm{GL}_n(D_{\mathbb{A}})$.

Conjecture 9. *A cuspidal automorphic representation* π *of* $\mathrm{GL}_n(D_{\mathbb{A}})$ *is distinguished by* $\mathrm{Sp}_n(D_{\mathbb{A}})$ *if and only if* $\mathrm{JL}(\pi) = J'(2, \sigma')$ *for some cuspidal automorphic representation* σ' *of* $\mathrm{GL}_n(\mathbb{A})$.

Remark 10. For $n = 2$, in the above conjecture, the places v of F where D splits (which happens at almost all places), the local component π_v of π is locally distinguished. At the remaining finitely many places where D does not split, the local component π_v of π at the nonsplit places v is either a tempered representation of $\mathrm{GL}_2(D_v)$ which is fully induced from a tensor product of two unitary characters of $D_v^\times$ or a complementary series representation of $\mathrm{GL}_2(D_v)$, attached to a unitary character of $D_v^\times$ and a real number $0 < \alpha < \frac{1}{2}$ [Grbac and Schwermer 2011]. These π_v are distinguished by $\mathrm{Sp}_2(D_v)$ by result of [Verma 2014]. This shows that π is locally distinguished at all the places but we can not conclude that π is globally distinguished.

Further thanks to Dipendra, we have constructed $\mathrm{Sp}_n(D_v)$-distinguished supercuspidal representations of $\mathrm{GL}_n(D_v)$ for n odd. Then by the globalization result of [Prasad and Schulze-Pillot 2008], we have the following theorem.

Theorem 11. *There exists a cuspidal automorphic representation π of $\mathrm{GL}_n(D_{\mathbb{A}})$ for $n \geq 1$ odd, and an automorphic form $\phi \in \pi$ such that*

$$\int_{\mathrm{Sp}_n(D)\backslash \mathrm{Sp}_n(D_{\mathbb{A}})} \phi(h)\, dh \neq 0.$$

Remark 12. In the Theorem 5 first sum ranges over $\mu \circ \mathrm{nrd}$ where μ is the unitary character of $\mathbb{A}^\times$. Any one-dimensional automorphic representation of $\mathrm{GL}_2(D_{\mathbb{A}})$ factors through the reduced norm and they are all $\mathrm{Sp}_2(D_{\mathbb{A}})$-distinguished. Therefore the study of the distinguished residual spectrum of $\mathrm{GL}_2(D_{\mathbb{A}})$ reduces to the study of an automorphic representation of the form $J(2, \sigma)$ as in Theorem 5 corresponding to $m = 2$ and $s_{\pi',D} = 1$. An automorphic representation of the form $J(2, \sigma)$ is generated by the residue of the Eisenstein series $E_{-1}(\phi)$.

3. The truncation of the Eisenstein series

From this section onwards since we are dealing only groups defined over D, we will write $\mathbb{A}$ instead of $D_{\mathbb{A}}$ for simplicity. For the rest of the paper we fix $G = \mathrm{GL}_2(D)$ and $H = \mathrm{Sp}_2(D)$. We will write $[H]$ for $\mathrm{Sp}_2(D) \setminus \mathrm{Sp}_2(\mathbb{A})$. We also recall from Section 1 that P, M, and U denote the F-points of algebraic groups denoted by the same letters. We begin by recalling a special case of the Arthur's truncation method and applying it to our study of the period integral. Let $c > 1$ and denote by τ_c the characteristic function of the set of real numbers greater than c. The truncation operator on the space of automorphic forms on $G(\mathbb{A})$ is defined by

$$\Lambda^c \phi(g) = \phi(g) - \sum_{\gamma \in P\backslash G} \phi_P(\gamma g)\tau_c(H(\gamma g)),$$

where ϕ_P is the constant term of ϕ along P defined as

$$\phi_P(g) = \int_{U\backslash U(\mathbb{A})} \phi(ng)\, dn,$$

for all $g \in G$. For fixed $g \in G$ and c, the above sum is finite.

If ϕ is an automorphic form on $G(\mathbb{A})$ then $\Lambda^c\phi$ is a rapidly decreasing function on $G(\mathbb{A})$. In particular, $\Lambda^c E(\phi, s)$ may be viewed as a meromorphic function of s, with values in the space of rapidly decreasing functions, with residue at $s = s_0$ equal to $\Lambda^c E_{-s_0}(\phi)$. The constant term of E along P is given by

$$E_P(g, \phi, s) = F(g, \phi, s) + F(g, M(s)\phi, -s),$$

where $M(s)$ denotes the standard intertwining operator [Moeglin and Waldspurger 1995]. After applying the truncation operator to the Eisenstein series, we have

$$\Lambda^c E(g, \phi, s) = E(g, \phi, s) - \sum_{\gamma\in P\backslash G} E_P(\gamma g, \phi, s)\tau_c(H(\gamma g)).$$

Whenever the Eisenstein series converges, we can write this as

$$\sum_{\gamma\in P\backslash G} F(\gamma g, \phi, s) - \sum_{\gamma\in P\backslash G} (F(\gamma g, \phi, s) + F(\gamma g, M(s)\phi, -s))\tau_c(H(\gamma g)) := \mathcal{E}_1 - \mathcal{E}_2,$$

where

$$\mathcal{E}_1 = \sum_{\gamma\in P\backslash G} H(\gamma g)^s \phi(\gamma g)(1 - \tau_c(H(\gamma g)))$$

and

$$\mathcal{E}_2 = \sum_{\gamma\in P\backslash G} H(\gamma g)^{-s} M(s)\phi(\gamma g)\tau_c(H(\gamma g)).$$

Let s_0 be a positive real number. Assume that the Eisenstein series $E(g, \phi, s)$ has a simple pole at $s = s_0$. We denote by $E_{-s_0}(g, \phi)$ the nonzero residue of $E(g, \phi, s)$ at s_0.

The truncation of the residue $\Lambda^c E_{-s_0}(g, \phi)$ is

$$\Lambda^c E_{-s_0}(g, \phi) = E_{-s_0}(g, \phi) - \mathcal{E}_3 \tag{3}$$

where $\mathcal{E}_3 = \sum_{\gamma\in P\backslash G} F(\gamma g, M_{-s_0}\phi(\gamma g), -s)\tau_c(H(\gamma g))$. Here M_{-s_0} is the residue of $M(s)$ at $s = s_0$. Consider the period integral $\int_{[H]} \Lambda^c E_{-s_0}(h, \phi)\, dh$ which converges absolutely because of the rapid decay of $\Lambda^c E_{-s_0}(g, \phi)$. By (3), we have

$$\int_{[H]} E_{-s_0}(h, \phi)\, dh = \int_{[H]} \mathcal{E}_3\, dh + \int_{[H]} \Lambda^c E_{-s_0}(h, \phi)\, dh.$$

Since $\Lambda^c E(h, s, \phi)$ is rapidly decreasing, the period

$$\int_{[H]} \Lambda^c E(h, s, \phi)\, dh$$

converges absolutely, the period integral $\int_{[H]} \Lambda^c E(h, s, \phi)\, dh$ defines a meromorphic function in s with possible poles contained in the set of possible poles of the

Eisenstein series $E(g,\phi,s)$ and hence in that of the global intertwining operator $M(s)$. It follows that

$$\mathrm{Res}_{s=s_0}\int_{[H]}\Lambda^c E(h,s,\phi)\,dh=\int_{[H]}\Lambda^c E_{-s_0}(h,\phi)\,dh.$$

Proposition 13. *The periods $\int_{[H]}\mathcal{E}_i\,dh$, for $i=1,2$, converge absolutely for large* $\mathrm{Re}(s)$ *and have meromorphic continuation to the whole complex plane. Also period $\int_{[H]}\mathcal{E}_3\,dh$ converges absolutely.*

We will prove the above proposition in Section 5 during the course of computing those periods. By meromorphic continuation, we have

$$\int_{[H]}\Lambda^c E(h,s,\phi)\,dh=\int_{[H]}\mathcal{E}_1\,dh-\int_{[H]}\mathcal{E}_2\,dh$$

for all s. Hence we have, at $s_0=1$, which is the only point of interest

$$\mathrm{Res}_{s=1}\int_{[H]}\Lambda^c E(h,s,\phi)\,dh=\mathrm{Res}_{s=1}\int_{[H]}\mathcal{E}_1\,dh-\mathrm{Res}_{s=1}\int_{[H]}\mathcal{E}_2\,dh.$$

Therefore

$$\int_{[H]}E_{-1}(h,\phi)\,dh=\mathrm{Res}_{s=1}\left[\int_{[H]}\mathcal{E}_1\,dh-\int_{[H]}\mathcal{E}_2\,dh\right]+\int_{[H]}\mathcal{E}_3\,dh. \tag{4}$$

This shows that $E_{-1}(g,\phi)$ is integrable over $[H]$.

4. Double cosets

From Section 3 we have the task of integrating $\mathcal{E}_i$ over $[H]$. More generally, let F be a function on $G(\mathbb{A})$ which is left invariant by P and $U(\mathbb{A})$ on the left. Consider the series

$$\theta(g)=\sum_{\gamma\in P\backslash G}F(\gamma g).$$

Let $\{\xi\}$ be the finite set of representatives for the double cosets $P\backslash G/H$. Then the integral of θ over $[H]$ can be written as

$$\begin{aligned}\int_{[H]}\theta(h)\,dh&=\int_{H\backslash H(\mathbb{A})}\sum_{\gamma\in P\backslash G}F(\gamma h)\,dh\\&=\sum_{\xi}\int_{P\cap\xi H\xi^{-1}\backslash\xi H(\mathbb{A})\xi^{-1}}F(h\xi)\,dh.\end{aligned}$$

Therefore we will now describe the double cosets $P\backslash G/H$.

Let V be a 2-dimensional Hermitian right D-vector space with a basis $\{e_1,e_2\}$ of V with $(e_1,e_1)=(e_2,e_2)=0$ and $(e_1,e_2)=1$. The one-dimensional subspace generated by a vector v is called isotropic if $(v, v) = 0$; otherwise, it is called anisotropic. For a right D-vector space, let $\mathrm{GL}_D(V)$ be the group of all invertible linear transformations on V. Similarly, let $\mathrm{Sp}_D(V)$ be the group of all invertible linear transformations on V which preserve the Hermitian form on V. Let X be the set of all 1-dimensional D-subspaces of V. The group $G = \mathrm{GL}_D(V)$ acts naturally on V, and induces a transitive action on X, realizing X as homogeneous space for G. The stabilizer of a line W in G is a parabolic subgroup P of G, with $X \simeq G/P$. Using the above basis, $\mathrm{GL}_D(V)$ can be identified with $\mathrm{GL}_2(D)$. For $W = \langle e_1 \rangle$, P is the parabolic subgroup consisting of upper triangular matrices in $\mathrm{GL}_2(D)$. As we have a Hermitian structure on V, $H = \mathrm{Sp}_D(V) \subset \mathrm{GL}_D(V)$.

We want to understand the space $H\backslash G/P$. This space can be seen as the orbit space of H on the flag variety X. This action has two orbits. One of them, say O_1, consists of all 1-dimensional isotropic subspaces of V and the other, say O_2, consists of all 1-dimensional anisotropic subspaces of V.

Theorem 14 (Witt's Theorem). *Let V be a nondegenerate quadratic space and $W \subset V$ any subspace. Then any isometric embedding $f : W \to V$ extends to an isometry of V.*

The fact that $\mathrm{Sp}_D(V)$ acts transitively on O_1 and O_2 follows from Witt's theorem, together with the well-known theorem that the reduced norm $N_{D/F} : D^\times \to F^\times$ is surjective, and the result that if a vector $v \in V$ is anisotropic, in the line $\langle v \rangle = \langle v \cdot D \rangle$ generated by v, there exists a vector v' such that $(v', v') = 1$.

It is easily seen that the stabilizer of the line $\langle e_1 \rangle$ in $\mathrm{Sp}_D(V)$ is

$$P \cap H = P_H = \left\{ \begin{pmatrix} a & b \\ 0 & \bar{a}^{-1} \end{pmatrix} : a \in D^\times, b \in D, a\bar{b} + b\bar{a} = 0 \right\}.$$

The parabolic subgroup P_H of $\mathrm{Sp}_2(D)$ has a Levi decomposition $P_H = M_H U_H$ with Levi subgroup

$$M \cap H = M_H = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} : a \in D^\times \right\}.$$

Now we consider the line $\langle e_1 + e_2 \rangle$ inside O_2. To calculate the stabilizer of this line in $\mathrm{Sp}_D(V)$, note that if an isometry of V stabilizes the line generated by $e_1 + e_2$, it also stabilizes its orthogonal complement which is the line generated by $e_1 - e_2$. Hence, the stabilizer of the line $\langle e_1 + e_2 \rangle$ in $\mathrm{Sp}_D(V)$ stabilizes the orthogonal decomposition of V as

$$V = \langle e_1 + e_2 \rangle \oplus \langle e_1 - e_2 \rangle,$$

and also acts on the vectors $\langle e_1 + e_2 \rangle$ and $\langle e_1 - e_2 \rangle$ by scalars. Thus the stabilizer in $\mathrm{Sp}_D(V)$ of the line $\langle e_1 + e_2 \rangle$ is $D_1 \times D_1$ sitting in a natural way in the Levi

$D^\times \times D^\times$ of the parabolic P in $\mathrm{GL}_2(D)$. Here D_1 is the subgroup of $D^\times$ consisting of reduced norm 1 elements in $D^\times$. The above description of the orbits O_1 and O_2 suggests representatives in $\mathrm{GL}_2(D)$ for double cosets $P \setminus G/H$ which we can take, respectively, to be the 2×2 identity matrix and

$$\xi = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Now we describe the parabolic subgroup of $H_\xi = \xi H \xi^{-1}$ which is conjugate to H and defined by the form $\xi J \xi^{-1} = J'$. Therefore

$$H_\xi = \{g \in \mathrm{GL}_2(D) : gJ'g^T = J'\}.$$

Then parabolic subgroup P_ξ of H_ξ is described by

$$P_\xi = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in D_1 \right\}.$$

Also $P \cap H_\xi = D_1 \times D_1$.

5. Computation of integrals $\mathcal{E}_i$

The task at hand is now to compute the contribution of both orbits to $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ separately and to analyze their absolute convergence. We begin by computing formally, but the computation will be justified by the absolute convergence of the final integral. We recall that

$$\mathcal{E}_1 = \sum_{\gamma \in P \setminus G} H(\gamma g)^s \phi(\gamma g)(1 - \tau_c(H(\gamma g))),$$

$$\mathcal{E}_2 = \sum_{\gamma \in P \setminus G} H(\gamma g)^{-s} M(s)\phi(\gamma g)\tau_c(H(\gamma g)), \quad \text{and}$$

$$\mathcal{E}_3 = \sum_{\gamma \in P \setminus G} F(\gamma g, M_{-s_0}\phi(\gamma g), -s)\tau_c(H(\gamma g)).$$

We can write

$$\int_{[H]} \mathcal{E}_1 \, dh = I_{11} + I_{12}$$

where

$$I_{11} = \int_{P \cap H \setminus H(\mathbb{A})} \phi(h\xi) H(h\xi)^s (1 - \tau_c(H(h)))\, dh$$

and

$$I_{12} = \int_{P \cap \xi H \xi^{-1} \setminus \xi H(\mathbb{A}) \xi^{-1}} \phi(h\xi) H(h\xi)^s (1 - \tau_c(H(h\xi)))\, dh.$$

We will use notation H_ξ for $\xi H \xi^{-1}$.

To compute I_{11}, we choose the Haar measure so that following integration formula is true on $H(\mathbb{A})$.

$$\int_{H(\mathbb{A})} f(h)\,dh = \iiiint f(umak)\delta_{P\cap H}^{-1/2}(a)\,du\,dm\,\frac{dt}{t}\,dk.$$

Here u is integrated over $(U\cap H)(\mathbb{A})$, m over $(M\cap H)(\mathbb{A})^1$, t over $\mathbb{R}^{\times +}$ with $t=|a|$ and k over $K\cap H(\mathbb{A})$.

Then

$$\begin{aligned} I_{11} &= \int_{K\cap H(\mathbb{A})}\int_{M_H\backslash M_H(\mathbb{A})^1}\int_0^c \phi(mk)\delta_P^{1/2}(a)H(a)^s\delta_{H\cap P}^{-1/2}(a)\,\frac{dt}{t}\,dm\,dk \\ &= \int_{K\cap H(\mathbb{A})}\int_{M_H\backslash M_H(\mathbb{A})^1}\int_0^c \phi(mk)|a||a|^s|a|^{-2}\,\frac{dt}{t}\,dm\,dk \\ &= \int_{K\cap H(\mathbb{A})}\int_{M_H\backslash M_H(\mathbb{A})^1}\int_0^c \phi(mk)|a|^{s-2}\,dt\,dm\,dk. \end{aligned}$$

The inner integral converges for $\mathrm{Re}(s)>1$ and its range of integration is 0 to c. Since ϕ is a cusp form in the space $\sigma\otimes\sigma$, the middle integral is bounded and therefore converges. Thus we obtain:

$$I_{11} = \frac{c^{s-1}}{s-1}\int_{K\cap H(\mathbb{A})}\int_{M_H\backslash M_H(\mathbb{A})^1}\phi(mk)\,dm\,dk.$$

At this point we begin the computation for I_{12} and for that purpose one needs the Jacquet–Friedberg [1993] result which is about a majorization of a cusp form.

Lemma 15. *Let ϕ be a cusp form on a reductive group $G(\mathbb{A})$ which is invariant under the connected component of the center of $G(\mathbb{A})$. Let R be the maximal parabolic subgroup of $G(\mathbb{A})$ and δ_R be the module of the group $R(\mathbb{A})$. Let Ω be any compact subset of $G(\mathbb{A})$. Then for every $N\geq 0$, there exist a constant $D>0$ such that*

$$|\phi(rk)|\leq D\delta_R(r)^{-N},$$

for every $r\in R$ and $k\in\Omega$.

Using the Iwasawa decomposition we can write

$$I_{12} = \int_{K\cap H_\xi(\mathbb{A})}\int_{P_\xi\backslash P_\xi(\mathbb{A})}\phi(pk\xi)H(pk\xi)^{-s}(1-\tau_c(H(pk\xi)))\,dp\,dk.$$

We replace $1-\tau_c$ by 1 and by the lemma above we can majorize by a constant multiple of $\delta_P(p)^N$. Since $D_1\setminus D_1(\mathbb{A})$ has finite volume, the above integral converges. Since $\xi\in K$ and the function H takes value 1 on the subgroup $P_\xi(\mathbb{A})$, we can write the above integral

$$I_{12} = \int_{K\cap H_\xi(\mathbb{A})} \int_{D_1\times D_1\backslash D_1(\mathbb{A})\times D_1(\mathbb{A})} \phi(pk)\,dp\,dk.$$

The inner integral, which is defined over $D_1 \times D_1 \setminus D_1(\mathbb{A}) \times D_1(\mathbb{A})$, vanishes because ϕ is a vector in the space $\sigma\otimes\sigma$ and σ is not one-dimensional. Therefore,

$$\int_{[H]} \mathcal{E}_1 dh = \frac{c^{s-1}}{s-1} \int_{K\cap H(\mathbb{A})} \int_{M_H\backslash M_H(\mathbb{A})^1} \phi(mk)\,dm\,dk.$$

Now we show that the period integral of $\mathcal{E}_2$ converges and get a simplified expression for it. Write

$$\int_{[H]} \mathcal{E}_2\,dh = I_{21} + I_{22}$$

where

$$I_{21} = \int_{P\cap H\backslash H(\mathbb{A})} M(s)\phi(h\xi)H(h\xi)^{-s}\tau_c(H(h))\,dh$$

and

$$I_{22} = \int_{P\cap H_\xi\backslash H(\mathbb{A})_\xi} M(s)\phi(h\xi)H(h\xi)^{-s}\tau_c(H(h\xi))\,dh.$$

Similar to the computation done above for I_{11} we have

$$I_{21} = \int_{P\cap H\backslash H(\mathbb{A})} \int_{P\cap H_\xi\backslash H(\mathbb{A})_\xi} \int_c^\infty M(s)\phi(mk)|a|^{-s-1}\,\frac{dt}{t}\,dm\,dk.$$

The integral I_{21} converges for $\mathrm{Re}(s) > -1$ and

$$I_{21} = \frac{c^{-(s+1)}}{s+1} \int_{K\cap H(\mathbb{A})} \int_{M_H\backslash M_H(\mathbb{A})^1} M(s)\phi(mk)\,dm\,dk.$$

Further, using Iwasawa decomposition, one can show that I_{22} converges absolutely for all s and vanishes identically because $c > 1$ and the support of τ_c is empty. Therefore,

$$\int_{[H]} \mathcal{E}_2\,dh = I_{21}.$$

The explicit computations of $\int_{[H]} \mathcal{E}_1\,dh$ and $\int_{[H]} \mathcal{E}_2\,dh$ also proves Proposition 13.

Similar computation and the argument that ϕ is a cusp form in the space $\sigma\otimes\sigma$ shows that $\int_{[H]} \mathcal{E}_3\,dh$ converges and

$$\int_{[H]} \mathcal{E}_3\,dh = \frac{c^{-2}}{2} \int_{K\cap H(\mathbb{A})} \int_{M_H\backslash M_H(\mathbb{A})^1} M_{-1}\phi(mk)\,dm\,dk.$$

Then by (4), we have completed the first half of the proof of Theorem 1.

6. Nonvanishing of the period integral

After proving the convergence of the period integral of the residue of the Eisenstein series and a nice formula to compute it, it remains in Theorem 1 to find out a suitable function ϕ such that the right-hand side of the formula is nonzero. We will achieve this by considering analogous local integrals at every place v of F.

Theorem 16. *There is a K-finite automorphic form Φ on $U(\mathbb{A})M \setminus \mathrm{GL}_2(D_{\mathbb{A}})$ such that the right-hand side of (2) is nonzero.*

Proof. We follow the proof of Jacquet–Rallis [1992] given for the split case. Define a linear form b on the space of smooth vectors in $\sigma \otimes \sigma \subset L^2(M \setminus M(\mathbb{A}))$ given by

$$b(\phi) = \int_{M \setminus M(\mathbb{A})} \phi\left(\begin{pmatrix} g & 0 \\ 0 & \bar{g}^{-1} \end{pmatrix}\right) dg.$$

Since ϕ is a cusp form, the integral is well defined. Consider a function $\Phi : K \to \sigma \otimes \sigma$ such that

$$\Phi(pk) = \sigma \otimes \sigma(p)\Phi(k)$$

when $p \in K \cap P(\mathbb{A})$. Then set

$$I(\Phi) = \int_{K \cap H(\mathbb{A})} b(\Phi(k))\, dk$$

which is equal to the right hand side of (2). We have to choose a K-finite function Φ such that the integral $I(\Phi)$ is nonzero. The linear form b is nonzero because it is a pairing (unique up to scalar) between σ and its contragredient. The linear form b has the following property:

$$b(\sigma(g) \otimes \sigma(\bar{g}^{-1})u) = b(u),$$

for all $g \in D^\times(\mathbb{A})$. Since b is not zero, we can choose a $K \cap M(\mathbb{A})$-finite vector $w = \otimes w_v$ in the space of $\sigma \otimes \sigma$. Then we can write $b = \otimes b_v$ and b_v have same property as b with $b_v(w_v) \neq 0$. Define the local integral

$$I(\Phi_v) = \int_{K_v \cap H_v} b_v(\Phi_v(k_v))\, dk_v.$$

Now we claim that $I(\Phi_v)$ is nonzero for some K_v-finite function $\Phi_v : K_v \to \sigma_v \otimes \sigma_v$ which satisfies

$$\Phi_v(pk) = (\sigma_v \otimes \sigma_v)(p)\Phi_v(k)$$

for $p \in K_v \cap P_v$.

At the finite places v where w_v is $K_v \cap M_v$-invariant, define $\Phi(k_v) = w_v$ for all $k_v \in K_v$. At all of the other remaining finite places, we choose an open compact subgroup Ω_v of $\overline{U_v}$ so small that the points of the form $m_v u_v \omega_v$ with $m_v \in K_v \cap M_v$, $u_v \in U_v \cap K_v$ and $\omega_v \in \Omega_v$ form an open subset of K_v. Then we take Φ_v with support in that set with the property that

$$\Phi_v(m_v u_v \omega_v) = (\sigma_v \otimes \sigma_v)(m_v) w_v.$$

At an infinite place v, by continuity it is enough to choose a smooth function Φ_v such that $I(\Phi_v)$ is not zero. Then Φ_v is any smooth function on K_v such that

$$\Phi_v(m_v k_v) = \sigma_v \otimes \sigma_v(m_v) w_v \Phi_v(k_v)$$

if $m_v \in M_v \cap K_v$. We choose a complement of the Lie algebra of $M_v \cap H_v \cap K_v$ in the Lie algebra of K_v and a small neighborhood of zero in this complement. Let Ω_v be the image of this under exponential map. We also choose a smooth function of compact support $f_v \geq 0$ on Ω_v with $f_v(1) > 0$. Then we define Φ_v by the condition that its support be contained in $(M_v \cap K_v)\Omega_v$ and equal to

$$\sigma_v \otimes \sigma_v(m_v) w_v f_v(\omega_v)$$

where $m_v \in M_v \cap K_v$ and $\omega_v \in \Omega_v$.

The product of the functions Φ_v has then required property. □

Proof of Theorem 2. We recall from [Badulescu 2008] that the automorphic representation $J(2, \sigma)$ is generated by $E_{-1}(g, \phi)$. When σ is a one-dimensional automorphic representation of $GL_1(D_{\mathbb{A}})$, then $J(2, \sigma)$ is also one-dimensional. Under the global Jacquet–Langlands one-dimensional representations of $GL_2(D_{\mathbb{A}})$ corresponds to one-dimensional representations of $GL_4(\mathbb{A})$. When σ is not one-dimensional then $J(2, \sigma)$ is distinguished by the above theorem and under the global Jacquet–Langlands this corresponds to the automorphic representation $J'(2, \sigma')$ of $GL_4(\mathbb{A})$ which is also distinguished by $Sp_4(\mathbb{A})$ [Offen 2006b].

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