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This paper is devoted to the examination of the properties of the string construction for the Birkhoff billiard. Based on purely geometric considerations, string construction is suited to providing a table for the Birkhoff billiard, having the prescribed caustic. Exploiting this framework together with the properties of convex caustics, we give a geometric proof of a result by Innami first proved in 2002 by means of Aubry–Mather theory. In the second part of the paper we show that applying the string construction one can find a new collection of examples of C^2-smooth convex billiard tables with a nonsmooth convex caustic.

1. Introduction

Let Γ be a simple closed C^1-smooth convex curve in the Euclidean plane. We consider a Birkhoff billiard inside Γ. This simple dynamical system creates many geometric and dynamical questions and reflects many difficulties appearing in general Hamiltonian systems. Readers may refer to any textbook among the wide variety written on the subject (e.g., [Katok et al. 1986; Kozlov and Treshchëv 1991; Mather and Forni 1994; Tabachnikov 2005]).

We will use the following nonstandard notations: the interior of the set bounded by the simple closed curve γ will be denoted by γ°, while $\bar{\gamma}$ denotes the compact $\gamma^\circ \cup \gamma$. The length of the curve is denoted by $\mathsf{Length}(\gamma)$. The convex hull of γ is denoted by $\mathsf{Conv}(\gamma)$.

Definition 1. A simple closed curve $\gamma \subset \Gamma^\circ$ is called a *convex caustic* for Γ if $\bar{\gamma}$ is a convex set and any supporting line for $\bar{\gamma}$ remains a supporting line for $\bar{\gamma}$ after billiard reflection in Γ.

Every convex caustic γ corresponds to an invariant curve r_γ of the billiard ball map. The curve $r_\gamma \subset \mathbb{R}_+ \times \mathbb{S}^1$ consists of all supporting lines to γ. This curve winds once around the phase cylinder and therefore is called rotational. We shall denote its rotation number by ρ_γ.

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In the original Birkhoff paper [1917] there was posed a conjecture that the existence of a continuous set of caustics, being a very restrictive property, actually provides an extreme rigidity on the shape of the curve Γ. The first result in this direction was achieved in [Bialy 1993]. Our paper is motivated by recent progress in the Birkhoff conjecture solution achieved in [Avila et al. 2016; Kaloshin and Sorrentino 2016]. The crucial assumption in these papers consists in the existence of convex caustics such that the rotation numbers of the corresponding invariant curves form a rational sequence in the interval $\left(0; \frac{1}{3}\right]$, converging to 0. It seems natural to compare such a result with one proved by N. Innami [2002].

Theorem 2 [Innami 2002]. *Assume that there exists a sequence of convex caustics γ_n inside Γ such that the rotation numbers ρ_n of the corresponding invariant curves tend to $\frac{1}{2}$. Then Γ is an ellipse.*

Originally, Innami's arguments were based on the Aubry–Mather variational theory. In the next section we present a simple geometric proof using string construction. Yet, it remains a challenging question whether one can prove a more general statement relaxing the requirement of convexity of the caustics.

Let us recall the string construction framework. Given a convex compact set $\bar{\gamma}$ bounded by γ, and a number $S > \mathsf{Length}(\gamma)$, define the curve Γ as a union of those points P such that the *cap-body* $\mathsf{Conv}(P \cup \bar{\gamma})$ has boundary of length S. Geometrically such a construction gives the set of all points traversed by the tip of a nonelastic string of length $S > \mathsf{Length}(\gamma)$ wrapped around γ and stretched to its full extent. The curve Γ provided by such construction has γ as its billiard caustic. We shall refer to S as a string parameter of the caustic. A closely related so-called Lazutkin parameter is defined as $L = S - \mathsf{Length}(\gamma)$.

The string construction is widely known and can be easily proved to provide Γ for smooth enough γ. In fact it remains valid also in the more general case as it is stated in the following theorem.

Theorem 3 [Stoll 1930; Turner 1982].

(1) *For a given compact convex set $\bar{\gamma}$ and for every $S > \mathsf{Length}(\gamma)$ the string construction determines a C^1-smooth convex closed curve Γ such that γ is a billiard caustic for Γ.*

(2) *If γ is a convex billiard caustic for a C^1 curve Γ then Γ can be obtained from γ by the string construction for some S.*

Let us emphasize that the string construction is highly nonexplicit and makes calculations difficult. A very important consequence of KAM theory, proved by Lazutkin [1973; 1981] and Douady [1982], states the existence of convex caustics near the boundary of a sufficiently smooth (at least C^6) billiard table. On the other hand, applying string construction to the triangle, one gets a billiard table which is

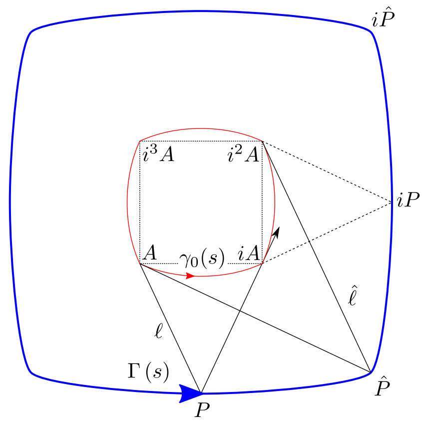

Figure 1. A switched caustic string construction.

piecewise C^2 with jumps of the curvature and hence by [Hubacher 1987] cannot have caustics near the boundary.

The scenario of destruction of caustics when one moves away from the boundary towards the interior could be understood in principle by the analogy with wave front propagation inside a convex curve [Mather and Forni 1994]. For example, take the ellipse and consider the wave fronts as in the famous picture [Arnold 1990, Figure 36]. For small distances the fronts remain smooth, but starting from some critical value they start to develop singularities. However, nobody has observed such a bifurcation in practice for caustics of convex billiards due to the lack of integrable examples. On the other hand, nonconvex caustics exist, for instance, for convex bodies of constant width, and were studied in [Knill 1998].

Motivated by the above discussion, the natural question about the existence of nonsmooth convex caustics arises. More generally, it is natural to study how irregular the convex caustic can be. In [Fetter 2012] a billiard table of class C^2 was constructed which has a caustic of a regular hexagon. In this paper we were able to construct the whole functional family of the examples of C^2 billiard tables having nonsmooth convex caustics.

Theorem 4. *There exist a one-parametric family of strictly convex nonsmooth compact sets $\bar{\gamma}$ and values of the string parameter S such that the curves Γ obtained by the string construction are C^2-smooth.*

We will use the following geometric idea (we use the complex notation $x+iy$ for points (x, y) in the plane). Start with a curve $\gamma_0(t) : [-1, 1] \to \mathbb{C}$ such that $\gamma_0(-1) = A = -1-i,\ \gamma_0(1) = iA = 1-i$ and $\gamma_0(t)$ is symmetric with respect to the vertical axis (i.e., $i\gamma_0(-t) = \overline{i\gamma_0(t)}$) (see Figure 1). Construct γ as a concatenation of $\{i^k\gamma_0\}_{k=0}^3$. Parametrize γ by the arc-length parameter s and choose the initial point in such a way that $\gamma(0) = A$. We will denote the total length of γ by $4\boldsymbol{S}$. Then $\gamma(\boldsymbol{S}) = iA$.

The main idea is to choose the curve γ and string parameter S in such a way that the string construction will have the following properties:

- At the beginning (point P in Figure 1), the left part AP of the string remains fixed at point A while the right part of the string unwinds from the arc $(\widehat{iA, i^2A})$.
- At the moment when the left part of the string becomes tangent to γ at the point A (this corresponds to the point $\hat{P}$ on Γ) the right part reaches the point i^2A and remains fixed after that. We will call this moment the *switching of the first kind*.
- While the left part of the string winds around the arc $(\widehat{A, iA})$ the right part remains fixed at i^2A (see Figure 1) until the moment when the vertex of the string reaches the point iP. We will call this the *switching of the second kind*.
- D_4 symmetry provides the whole picture.

Let us reemphasize, that the string construction, being a nonexplicit procedure, typically does not provide any analytic expression for the table Γ from a given γ. In the example [Fetter 2012], the construction is made explicit by fixing two end-points on the string. The disadvantage of such a situation is the complete loss of any flexibility, since the corresponding table may consist only of the elliptic arcs. We propose another, more flexible yet explicit construction, fixing only one end-point of the string and allowing another point to slide along the given curve γ.

Structure of the paper. In the next section we will provide geometric arguments for the proof of Theorem 2. Section 3 is devoted to the construction of the C^2 tables with nonsmooth caustics. In Section 4 we will pose some open questions arising in our considerations.

2. Geometric proof of Innami's result

We will start with the following simple remarks.

Remark 5. If the billiard in Γ has a convex caustic γ with $\gamma^\circ = \varnothing$ then Γ is either an ellipse or a circle.

Indeed, the condition $\gamma^\circ = \varnothing$ for convex γ means that γ is either a point or a segment. The rest follows from the string construction.

Remark 6. Recall that for any point P and for any convex body with nonempty interior there exist exactly two supporting lines to the body passing through P. Moreover if the convex caustic γ has nonempty interior, then every supporting line to $\bar{\gamma}$ after reflection in Γ at point P becomes the second supporting line to $\bar{\gamma}$ from P.

Indeed, assume that there exists a supporting line l to $\bar{\gamma}$ which is reflected to itself at a point $P \in \Gamma$. This means that l is orthogonal to Γ at P. Let l' be the other supporting line to $\bar{\gamma}$ passing through P. Then by the definition of convex caustic, the line l' is also reflected to itself at the point P and hence is also orthogonal to Γ at P. Thus l and l' coincide, which contradicts the assumption that l and l' are two different supporting lines to $\bar{\gamma}$.

Lemma 7. *Let γ be a convex caustic for Γ. Then $\gamma^\circ \neq \varnothing$ if and only if the rotation number of the corresponding invariant curve is strictly less then $\frac{1}{2}$.*

Proof. If a convex caustic γ has empty interior then, by the Remark 5, Γ is necessarily an ellipse (or a circle) and the invariant curve corresponding to γ has rotation number $\frac{1}{2}$ since it contains a diameter. Vice versa, any convex caustic with nonempty interior has a rotation number strictly less than $\frac{1}{2}$, since otherwise the invariant curve corresponding to the caustic would have a 2-periodic orbit, i.e., a diameter, which is not possible due to Remark 6. □

Let γ_n be a sequence of convex caustics for Γ with the rotation numbers $\rho_n \in \left(0; \frac{1}{2}\right]$ of corresponding invariant curves. By Lemma 7 we may assume that $\rho_n < \frac{1}{2}$ since otherwise γ_n has empty interior and then Γ must be an ellipse by the Remark 5. Passing to a subsequence we can assume with no loss of generality that ρ_n is strictly increasing, $\rho_n \nearrow \frac{1}{2}$.

Lemma 8. *Let γ_1 and γ_2 be two convex caustics for Γ. If the corresponding invariant curves have rotation numbers $\rho_1 < \rho_2$, then $\bar{\gamma}_2 \subset \gamma_1^\circ$.*

Proof. Assume that $\bar{\gamma}_2$ is not a subset of γ_1°. Then there are only three possibilities: (1): $\bar{\gamma}_1 \cap \bar{\gamma}_2 = \varnothing$; (2): $\gamma_1 \cap \gamma_2 \neq \varnothing$ or (3): $\bar{\gamma}_1 \subset \gamma_2^\circ$.

In the third case one obviously has $\rho_1 \geq \rho_2$ contrary to the assumption of the lemma. In the first and the second cases there necessarily exists a supporting line to both $\bar{\gamma}_1$ and $\bar{\gamma}_2$. Therefore, all billiard reflections in Γ of this line are also supporting lines for both $\bar{\gamma}_1$ and $\bar{\gamma}_2$. This means that there exists a whole infinite orbit lying in the intersection of the invariant curves r_1 and r_2 corresponding to γ_1 and γ_2. But then ρ_1 must be equal to ρ_2, since the rotation number is completely determined by one orbit. □

Remark 9. The statement of Lemma 8 holds true also in the opposite direction which will not be used below. Namely, $\bar{\gamma}_2 \subset \gamma_1^\circ$ implies $\rho_1 < \rho_2$. As we already mentioned in the proof, it is obvious that $\rho_1 \leq \rho_2$. In addition ρ_1 cannot be equal to ρ_2. Otherwise there exist two disjoint graphs of r_1 and r_2 with the same rotation number, invariant under the billiard map of the cylinder, which is impossible since a billiard map is a twist map (see [Katok and Hasselblatt 1995, p. 428]).

Let $\{S_n\}$ be the sequence of string parameters corresponding to the caustics γ_n. Then by Lemma 8, S_n is decreasing. Denote $S = \lim_{n\to\infty} S_n$.

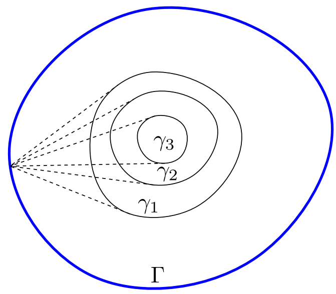

Figure 2. A family of nested convex caustics with decreasing string parameter.

Lemma 10. *The boundary of the intersection set*

$$C = \bigcap_{n=1}^{\infty} \bar{\gamma}_n$$

is a convex caustic for Γ *with string parameter* S.

Proof. The intersection set C is compact and convex. Moreover, it is easy to see that ∂_C is also a caustic with string parameter S. Indeed, this follows from the following geometric consideration (see Figure 2). Fix a point P on Γ and consider the cap-bodies

$$K_n = \mathsf{Conv}(P \cup \bar{\gamma}_n), \qquad K = \mathsf{Conv}(P \cup C).$$

Then, obviously,

$$K_n \subseteq K, \qquad K = \bigcap_{n=1}^{\infty} K_n,$$

and moreover

$$\mathsf{Length}(\partial_{K_n}) = S_n \to S = \mathsf{Length}(\partial_K).$$

In addition, since γ_n is a caustic then S_n does not depend on $P \in \Gamma$ (by Theorem 3). Therefore, S also does not depend on P, and hence C reconstructs Γ via string construction. Thus ∂_C is a caustic by Theorem 3. □

The last step in the proof of Theorem 2 consists in the following Lemma.

Lemma 11. *The limit caustic* ∂_C *has empty interior.*

Proof. First notice that it follows from continuity of the invariant curves and their rotation numbers that the invariant curve corresponding to C has rotation number $\frac{1}{2}$. Then from Lemma 7 we conclude that ∂_C has empty interior. □

3. Nonsmooth caustic

The main idea of the proof of our result is to carefully choose the Lazutkin parameter and the germ of the function γ at the point A. While a vertex of the string slides in

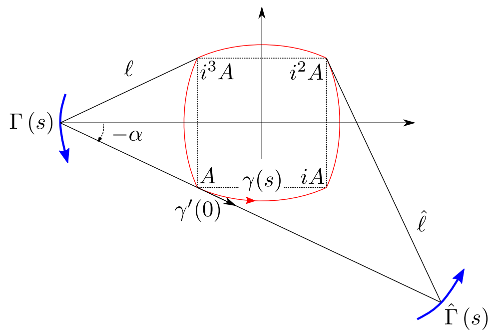

Figure 3. A switched caustic string construction.

the regime corresponding to the unwinding from $\gamma(s)$, its trajectory corresponds to the smooth curve. Thus we have to take care of the smoothness of Γ near only two points corresponding to the switching moments of the first and second kinds respectively. We will denote by $\Gamma(s)$ the part of Γ corresponding to the switching of the second kind about the point A. The part of Γ corresponding to the switching of the first kind about the point A will be denoted by $\hat{\Gamma}$. The smoothness conditions read as follows: all odd terms in the germs of Γ and $\hat{\Gamma}$ have to be orthogonal to the axis of the symmetry while all the even terms must be collinear with the axis of symmetry. Indeed, let $\Gamma(s)$ be the curve symmetric with respect to the line l and intersecting l at the point $\Gamma(0)$. Let R_l be the reflection of the plane in the line l. Differentiating the identity

$$R_l\Gamma(s) = \Gamma(-s)$$

n times, at $s = 0$, we get

$$R_l(\Gamma^{(n)}(0)) = (-1)^n\Gamma^{(n)}(0).$$

Coordinate formulation. Parametrize the curve γ by the arc-length parameter s, so that $|\gamma'(s)| = 1$. Choose the initial point such that $\gamma(0) = A$. Denote by α the angle between $\gamma'(0)$ and the horizontal axis. Then one easily obtains a parametrization for Γ and $\hat{\Gamma}$ (see Figure 3):

$$\begin{aligned} \Gamma(s) &= \gamma(s) - t(s)\gamma'(s), \\ \hat{\Gamma}(s) &= \gamma(s) + \hat{t}(s)\gamma'(s), \end{aligned} \tag{1}$$

where $t(s)$ and $\hat{t}(s)$ are some functions of s denoting the length of the right part of the string near the point $\Gamma(s)$ and the left part of the string near the point $\hat{\Gamma}(s)$ correspondingly. Functions t and $\hat{t}$ can be found from the condition of the string to be unstretchable. We will denote $iA = B$.

$$\begin{aligned} |\Gamma(s) + B| + |t\gamma'(s)| - s &= 2\ell, \\ |\hat{\Gamma}(s) + A| + |\hat{t}\gamma'(s)| + s &= 2\hat{\ell}, \end{aligned} \tag{2}$$

where $\ell = 1/\sin\alpha$ and $\hat{\ell} = \sqrt{2}/\sin(\pi/4-\alpha)$. Simple computations yield:

$$
\begin{aligned}
t(s) &= \frac{p(s)}{p'(s)}, \text{ with } p(s) = \tfrac{1}{2}\big((s+2\ell)^2 - |\gamma(s)+B|^2\big),\\
\hat{t}(s) &= -\frac{\hat{p}(s)}{\hat{p}'(s)}, \text{ with } \hat{p}(s) = \tfrac{1}{2}\big((s-2\hat{\ell})^2 - |\gamma(s)+A|^2\big).
\end{aligned} \tag{3}
$$

Finally, introducing (3) into (1) we get

$$
\Gamma(s) = \gamma(s) - \frac{p(s)}{p'(s)}\gamma'(s), \qquad \hat{\Gamma}(s) = \gamma(s) - \frac{\hat{p}(s)}{\hat{p}'(s)}\gamma'(s). \tag{4}
$$

Orient the curve γ as it is shown in Figure 3. We will use the complex notation for the coordinates of the points. Then smoothness conditions for the n-th derivative of Γ read

$$
\Re(i^{n-1}\Gamma^{(n)}(0)) = 0, \qquad \Re(i^{n-1}\hat{\Gamma}^{(n)}(0)) = \Im(i^{n-1}\hat{\Gamma}^{(n)}(0)). \tag{5}
$$

Here $\Re$ and $\Im$ stand for the real and imaginary part of the complex number. For the curve $\gamma(s)$ we get the following parametrization:

$$
\gamma(s) = A + \int_0^s \exp\{i(\varphi(t)-\alpha)\}\,dt, \qquad \text{where } \varphi(t) = \sum_{n=0}^{\infty}\varphi_n t^n. \tag{6}
$$

Thus $\varphi_0 = 0$, and φ_n corresponds to the $(n-1)$-st derivative of the curvature κ.

Lemma 12. *The smoothness conditions in* (5) *for $n=1$ are always satisfied.*

This lemma follows from the fact that any C^0 caustic produces a C^1 table via string construction. However, we present a more analytic proof of this result for the sake of completeness.

Proof. Switching of the second kind. From (4) we get

$$
\Gamma' = \Big(1 - \Big(\frac{p}{p'}\Big)'\Big)\gamma' - \frac{p}{p'}\gamma''.
$$

Therefore the conditions in (5) read $\Re(p''\gamma' - p'\gamma'') = 0$. We will denote $z_1 \cdot z_2 := \frac{1}{2}\Re(z_1\bar{z}_2)$. Using (3) we get

$$
p' = -(A+B)\cdot\gamma' + 2\ell, \qquad p'' = -(A+B)\cdot\gamma''.
$$

From (6) it follows that $\gamma'' = i\kappa\gamma'$ thus $p''\gamma' - p'\gamma''$ can be written as

$$
\begin{aligned}
p''\gamma' - p'\gamma'' &= \tfrac{1}{2}\big(-\Re((A+B)\overline{i\kappa\gamma'})\gamma' + \Re((A+B)\bar{\gamma}')(i\kappa\gamma') - 4\ell i\kappa\gamma'\big)\\
&= i\kappa(A+B-2\ell\gamma').
\end{aligned}
$$

Thus

$$
\Re(p''\gamma' - p'\gamma'') = \kappa\Im(A+B-2\ell\gamma').
$$

The latter is identically zero since $\ell\gamma'(0) = \Gamma(0) - \gamma(0)$ and so $\Im(\ell\gamma') = \Im(A)$ (see Figure 3).

Switching of the first kind. Similarly, the smoothness conditions in (5) read

$$\Re(\hat{p}''\gamma' - \hat{p}'\gamma'') = \Im(\hat{p}''\gamma' - \hat{p}'\gamma''),$$

where

$$\hat{p}' = -(2A)\cdot\gamma' - 2\hat{\ell}, \qquad \hat{p}'' = -(2A)\cdot\gamma''$$

and so

$$\hat{p}''\gamma' - \hat{p}'\gamma'' = \big(\Re(Ai\kappa\bar{\gamma}')\gamma' + \Re(A\bar{\gamma}')(i\kappa\gamma') + 2\hat{\ell}i\kappa\gamma'\big) = 2i\kappa(A + \hat{\ell}\gamma').$$

The real part of the right-hand side of the latter is always equal to the imaginary part by the definition of $\hat{\ell}$. □

The two conditions in (5) for $n = 2$ provide, via computations similar to the above, two equations for parameters φ_1 and φ_2 with coefficients depending on α:

$$\frac{\varphi_1^2\sin\alpha - \varphi_1\sin\alpha\cos\alpha - \varphi_2\cos\alpha}{\sin\alpha\cos^2\alpha} = 0,$$

$$\frac{\varphi_1(\cos 2\alpha + 2(\sin\alpha - \cos\alpha)\varphi_1) - 2(\cos\alpha + \sin\alpha)\varphi_2}{(\cos\alpha - \sin\alpha)(1 + \sin 2\alpha)} = 0.$$

The latter system has a solution,

$$\varphi_1 = \tfrac{1}{2}\cos\alpha(1 + \sin 2\alpha), \qquad \varphi_2 = -\tfrac{1}{8}\cos^2 2\alpha\sin 2\alpha, \tag{7}$$

which provides a family of germs for γ, depending on the parameter α, guaranteeing the C^2-smoothness for the table Γ.

Next we will need to construct the whole curve γ providing the needed phenomenon in the string construction. Recall that our geometric idea was based on the construction of the curve γ_0 (see Figure 1). Thus we need to present a convex curve of length $\boldsymbol{S}$, starting at A and ending at iA, having tangent slope $-\alpha$ at the left end and being symmetric with respect to the vertical axis. We define γ from φ through (6). In order to finish the construction we have to prove the following theorem.

Theorem 13. *There exists a strictly monotonically increasing function $\varphi(s)$ satisfying the following three conditions*: (1) *$\varphi(s)$ has the given germ* (7) *at $s = 0$,* (2) *$\varphi_0(\boldsymbol{S}/2) = \alpha$ and $\varphi_{2n}(\boldsymbol{S}/2) = 0$ for $n \geqslant 1$, and* (3) $\int_0^{\boldsymbol{S}/2}\cos\varphi(s)\,ds = 1$.

Proof. The Borel theorem states that every power series is the Taylor series of some smooth function. Obviously, using cutting off, one can find a smooth function having a given Taylor series at two given points. Thus there exists a nonempty set Ψ of C^∞ functions having given germs at $s = 0$ and $s = \boldsymbol{S}/2$. Since for $\alpha < \frac{\pi}{2}$ the term φ_1 in (7) is positive, one may assume without loss of generality that Ψ consists of strictly monotonically increasing functions. Therefore the only condition which

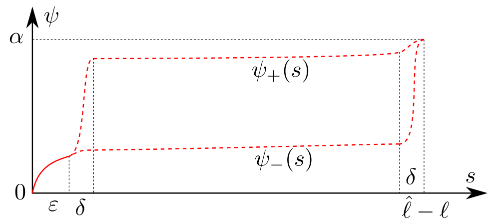

Figure 4. Construction of the solution.

has to be satisfied is Theorem 13(3). Taking a small enough ε-step in s we can ensure $\psi(\varepsilon) < \frac{\alpha}{100}$ for all $\psi \in \Psi$. Next we choose two functions ψ_- and ψ_+ from the set Ψ as in Figure 4. That is, $\psi_+(s)$ is almost equal to α for $s \in (\varepsilon+\delta, \boldsymbol{S}/2-\delta)$ and $\psi_-(s)$ is almost equal to $\psi(\varepsilon)$ for $s \in (\varepsilon, \boldsymbol{S}/2-\delta)$ for small enough δ. We will look for φ as a convex combination $\varphi(s) = l\psi_-(s) + (1-l)\psi_+(s)$. Therefore $\varphi(s)$ obviously satisfies conditions 1 and 2. If we may choose $\psi_\pm$ in such a way that

$$(8)\quad (\boldsymbol{S}/2)\cos\alpha < \int_0^{\boldsymbol{S}/2} \cos(\psi_-(s)-\alpha)\,ds < 1 \quad\text{and}\quad \boldsymbol{S}/2 > \int_0^{\boldsymbol{S}/2} \cos(\psi_+(s)-\alpha)\,ds > 1$$

then there exists l such that $\int_0^{\boldsymbol{S}/2} \cos(\varphi(s))\,ds = 1$, thus satisfying condition Theorem 13(3). Hence it is sufficient to check that the conditions in (8) can be satisfied for an open set of parameters α. Recall that by the construction $\boldsymbol{S} = 2\hat\ell - 2\ell$. From the first inequality in (8) we obtain, since $\alpha < \frac{\pi}{4}$,

$$\hat\ell - \ell = \frac{2}{\cos\alpha - \sin\alpha} - \frac{1}{\sin\alpha} < \frac{1}{\cos\alpha}.$$

This condition can be interpreted as follows: *the length of the curve γ cannot exceed the sum of the lengths of the segments of the two tangent lines from point P to γ* (*see* Figure 1). The latter inequality is satisfied whenever $\tan 2\alpha < 1$ or

$$(9)\qquad \alpha < \frac{\pi}{8}.$$

The second condition in (8) has the following geometric interpretation: *the length of γ cannot be less than the distance between points A and B.* This yields:

$$3\sin\alpha - \cos\alpha > \cos\alpha\sin\alpha - \sin^2\alpha.$$

Since the latter is satisfied for $\alpha = \frac{\pi}{8}$ we have found an open set of α for which one can find appropriate functions ψ_- and ψ_+ shown in Figure 4. □

Remark 14. Since the conditions in (5) provide two conditions on φ_n to obtain C^3 of Γ one gets four equations for φ_1, φ_2, φ_3 and α. Although the number of parameters matches the number of equations, the corresponding value of α violates (9). Since (9) arises from the construction based on square symmetry, there

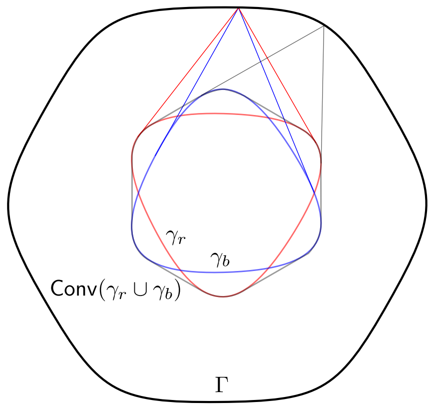

Figure 5. The convex hull of two intersecting caustics is also a caustic.

is a hope that starting from other regular polygons one can obtain an inequality which can be satisfied. However, we haven't found any such examples.

4. Open problems

Here we want to highlight some general questions which are ultimately related to the string construction. Since the string construction is implicit these questions turn out to be nontrivial.

Question 15. Is it possible to have two convex caustics γ_1 and γ_2 of Γ such that neither of them is a subset of the interior of the other?

In such a case γ_1 and γ_2 must have the same rotation number since there is a line tangent to both of the caustics. Moreover it is obvious that $\bar{\gamma}_1$ and $\bar{\gamma}_2$ cannot be disjoint. So the question is if it is possible that two convex caustics have nontrivial intersection. In such a case their convex hull is also a caustic. One can strengthen the question:

Question 16. Is it possible for a Γ which is symmetric with respect to a certain axis to have a convex caustic C which is not symmetric with respect to this axis?

For example one could imagine two caustics forming a rounded Star of David (Figure 5). The answer to the quantum analog of this question is positive: for a symmetric domain the Dirichlet eigenfunction can be nonsymmetric. We could not however decide if such a counterexample would be possible in the original setting.

Question 17. How irregular a convex caustic can be compared to a regular boundary curve Γ?

Question 18. Let Γ be a billiard table different from a circle and having a convex caustic γ. For every point $P \in \Gamma$, denote by P_-, and P_+ the tangency points of the caustic γ with tangent lines to γ passing through P. Is it possible that the length of the arc of γ between P_- and P_+ does not depend on P?

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